2.4 (a) For equation (2), we can define x' explicitly by the following:

$$x' = \pm\sqrt{1+x^2}.$$

Therefore taking the absolute value of x' yields:

$$|x'| = \sqrt{1+x^2} \leq \sqrt{x^2+x^2} = \sqrt{2x^2} = \sqrt{2}|x|.$$

Since (2) has a lipschitz constant of $\sqrt{2}$, then by theorem 5 (2) has unique solutions for all $x_0 \in (-1,1)$.
Turning our attention to (1), then we have the equation

$$x' = \pm\sqrt{1-x^2}$$

Taking the absolute value of the derivative of x' yields

$$x'' = \frac{|x|}{\sqrt{1-x^2}}.$$

Taking the limit as x approaches -1 yields:

$$\lim_{x\to-1} |x''| = \frac{1}{\sqrt{1-1}} = \frac{1}{0} = \infty.$$

Since $|x'|$ is continuous on $(-1,1)$ and is not lipschitz then (1) has infinite solutions.

(b) Since (1) does not have a unique solutions we must show an infinite number of solutions to $(x')^2 + x^2 = 1, x(0) = x_0$. Since $x(t) = 1$ solves the equation as

$$(x_0')^2 + x_0 = 0^2 + 1^2 = 1$$

but not the initial value of $x_0 \in (-1,1)$, we must solve the differential equations by other means to give another solution to interpolate with.
Solving for non-steady state:

$$\begin{aligned}
x' &= \pm\sqrt{1-x^2} \\
1 &= \frac{\pm x'}{\sqrt{1-x^2}} \\
\int_{t_0}^{t} dt &= \pm\int_{x_0}^{x} \frac{dz}{\sqrt{1-z^2}} \\
t - t_0 &= \pm(\arcsin(x) - \arcsin(x_0)) \\
\pm(t - t_0 + \arcsin(x_0)) &= \arcsin(x) \\
x(t) &= \pm\sin(t - t_0 + arcsin(x_0))
\end{aligned}$$

Since $x(t) = 1, x(t) = \sin(t - t_0 + arcsin(x_0))$ both solve the differential equation, then we may create a new solution

$$x(t) = \begin{cases} 1 & t > t_0 - arcsin(x_0) + 2\pi a + \frac{\pi}{2} \\ \sin(t - t_0 + arcsin(x_0)) & t \leq t_0 - arcsin(x_0) + 2\pi a + \frac{\pi}{2} \end{cases}$$

where $a \in \mathbb{N} \cup \{0\}$. Note that at $t = t_0 - arcsin(x_0 + 2\pi a + \frac{\pi}{2}$, that for sin we have:

$$\sin(t - t_0 + arcsin(x_0)) = \sin(2\pi a + \frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$$

and for the derivative

$$cos(\frac{\pi}{2}) = 0$$

Which exactly aligns with the value and derivative of the constant function $x(t) = 1$. Therefore since our solutions are continuous, and there exist one for each natural number, then we have found an infinite number of solutions.

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