We say that A *is a neighbor of* B if $A \triangle B$ consists of exactly one element.

A list of sets is a *neighborly list of sets* if each set is a neighbor of the set following it in the list, and the last set is a neighbor of the first set. Prove that for all positive integers n, there is a list consisting of subsets of $\{1, \ldots, n\}$ such that:
(1) every subset of $\{1, \ldots, n\}$ appears *precisely once* on the list and
(2) the list is neighborly.
We must show that for all $n \in \mathbb{N}$, there exists a list of subsets of $\{1, \ldots, n\}$ which contains all of the subsets exactly once and is neighborly. Suppose $n \in \mathbb{N}$. We must show there exists a list of subsets of $\{1, \ldots, n\}$ which contains all of the subsets exactly once and is neighborly. By the principal of mathematical induction for all $k \in \mathbb{N}$ if $k < n$ then there exists a list of the subsets of $\{1, \ldots, k\}$ such that each subset occurs exactly once and the list is neighborly. We now have two cases:

- Assume $n = 1$. Then the list $(\emptyset, \{1\})$ is a neighborly list, and every subset of $\{1\}$ occurs exactly once.

- Assume $n > 1$. Since $n - 1 < n$, by the induction hypothesis there exists a list of every subset of $\{1, \ldots, n-1\}$ exactly once and is neighborly. Let this list be denoted $(S_1, \ldots, S_{2^n})$. We claim that the list satisfying the requirements for $\{1, \ldots, n\}$ is given by $(S_1, \ldots, S_{2^n}, \{n\} \cup S_{2^n}, \ldots, \{n\} \cup S_1)$. We must show this list satisfies the requirements.

  1. Since the only difference between $\{1, \ldots, n-1\}$ and $\{1, \ldots, n\}$ is the inclusion of n, and $(S_1, \ldots, S_{2^n})$ contains every subset of $\{1, \ldots, n-1\}$ exactly once, then $(S_1, \ldots, S_{2^n}, \{n\} \cup S_{2^n}, \ldots, \{n\} \cup S_1)$ contains all possible subsets that don't include n and do include n. Therefore every subset appears exactly once.
  2. Since $(S_1, \ldots, S_{2^n})$ is already neighborly, then $(\{n\} \cup S_{2^n}, \ldots, \{n\} \cup S_1)$ would be neighborly by virtual of the symmetric difference being symmetric, and the set difference between two sets both containing n would not have n by definition. Then for the boundary interaction $S_{2^n} \triangle \{n\} \cup S_{2^n} = (S_{2^n} \backslash \{n\} \cup S_{2^n}) \cup (\{n\} \cup S_{2^n} \backslash S_{2^n}) = \emptyset \cap \{n\} = \{n\}$. Therefore since all the possible symmetric differences between neighboring subsets have exactly one subset, then the list is neighborly.