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A set $\mathcal{S}$ of sets is said to be intersecting if for any two members A and B of $\mathcal{S}$, we have $A \cap B \neq \emptyset$. Prove that for any nonempty set U and for any intersecting collection $\mathcal{S}$ of subsets of U and for any $X \subseteq U$ at least one of the two collections $\mathcal{S} \cup \{X\}$ and $\mathcal{S} \cup \{U \backslash X\}$ is intersecting.
Proof: We must show that $\mathcal{S} \cup \{X\}$ or $\mathcal{S} \cup \{U \backslash X\}$ is intersecting. Assume X is an arbitrary subset of U. Assume $\mathcal{S} \cup \{X\}$ is not intersecting. We must show $\mathcal{S} \cup \{U \backslash X\}$ is intersecting. By definition of being not intersecting, there exist $A, B \in \mathcal{S} \cup \{X\}$ such that $A \cap B = \emptyset$. Since $\mathcal{S}$ was defined to be previously intersecting, then without loss of generality $A = X$. Since $B \subseteq U$, and A, B were disjoint, then $B \subseteq U \backslash X$, and therefore $B \cap (U \backslash X) \neq \emptyset$. Since $B \subseteq (U \backslash X)$ and $\mathcal{S}$ is intersecting, then all the other sets of $\mathcal{S}$ also have non-empty intersections. Therefore $\mathcal{S} \cup \{U \backslash X\}$ is intersecting.