1.5.7
- $\{a, b, c\}$
- $\{a, b, c\}$
- $\{1, 2, 3, 4\}$

1.5.8 (a) Suppose $a' \in B$. Then by definition of B $a' \notin f(a') = B$. This is a contradiction

(b) Suppose $a' \notin B$. This means by the construction of B, $a' \in f(a')$. However $f(a') = B$. This is a contradiction

1.5.9 (a) We claim that the set of all functions from $\{0, 1\}$ to $\mathbb{N}$ is countable. Each function can be represented as $\{(0, a), (1, b)\}$ where $a, b \in \mathbb{N}$. Note that there is an obvious bijection between the set $\{(0, a), (1, b)\}$ and (a, b). Therefore the set of all functions from $\{0, 1\}$ to $\mathbb{N}$ has the same cardinality as $\mathbb{N}^2$. Since $\mathbb{N}^2$ has the same cardinality as $\mathbb{N}$, then we have shown that the set of all function from $\{0, 1\}$ to $\mathbb{N}$ is countable.

(b) We claim that the set of all functions from $\mathbb{N}$ to $\{0, 1\}$ is uncountable. We claim that there is a bijection between the set of all functions from $\mathbb{N}$ to $\{0, 1\}$ and the set $S = \{(a_1, a_2, \ldots) : a_n = 0 \text{ or } a_n = 1\}$ as defined in exercise 1.5.4 which was proven to be uncountable. Let $f : \{0, 1\}^{\mathbb{N}} \to S$ be given by $g(f) = (f(1), f(2), f(3), \ldots)$. Suppose $f_1, f_2 \in \{0, 1\}^{\mathbb{N}}, g(f_1) = g(f_2)$. We must show that $f_1 = f_2$. Since f_1, f_2 are over the natural numbers, it suffices to show that for all $n \in \mathbb{N}, f_1(n) = f_2(n)$. Since $g(f_1) = g(f_2)$, then $f_1(n) = f_2(n)$ for all $n \in \mathbb{N}$. Therefore $f_1 = f_2$. Since $\{0, 1\}^{\mathbb{N}}$ injects into S, and S is uncountable

(c) Let the set S be as described in problem 1.5.4, the set of all binary sequences. Let the function $f : \mathbb{N} \times S \to \mathbb{N}$ be given by

$$f(n, s) := \begin{cases} 2n & s_n = 0 \\ 2n - 1 & s_n = 1. \end{cases}$$

For a sequence $s \in S$ let $A_s = \{f(n, s) : n \in \mathbb{N}\}$. We claim that the set $K = \{A_s : s \in S\}$ is an uncountable antichain.

- From 1.5.4 we know that S is uncountable. We claim that there is a 1-1 correspondence with S and K. Let the function $g : S \to K$ be given by $g(s) = A_s$. We claim that g is 1-1. Suppose $r, s \in S, g(r) = g(s)$. We must show that $r = s$. To show that $r = s$, it suffices to show for all $n \in \mathbb{N}$, $r_n = s_n$. Suppose $n \in \mathbb{N}, s_n = 1$. Then $2n - 1 \in g(s), g(r)$. Since $2n - 1 \in g(r)$, then $r_n = 1$ as this is the only condition under which $2n - 1 \in g(r)$. A similar proof exists for $s_n = 0$. Since $s_n = r_n$ for arbitrary n, then $s = r$.
- We claim that for arbitrary distinct $r, s \in S$ that $A_s \not\subset A_r, A_r \not\subset A_s$. Suppose $r, s \in S, r \neq s$. Then by definition there must exists $n \in \mathbb{N}$ such that $r_n \neq s_n$. Suppose WLOG $r_n = 1$. Therefore $s_n = 0, 2n-1 \in A_r, 2n \in A_s$. Since $r_n = 1$ then $2n \notin A_r$ as if it was then that would contradict $2n - 1 \in A_r$. Similarly, there would be a contradiction if $2n - 1 \in A_s$. Therefore $A_s \not\subset A_r, A_r \not\subset A_s$.

Since there is a 1-1 correspondence with an uncountable set, and K is an antichain then we have satisfied finding an uncountable subset of $\mathcal{P}(\mathbb{N})$.