8.16 (rudin) Let $\sigma_N[f](x) = \int_{-\pi}^{\pi} f(x-t)K_N(t)dt$. We want to show that $\sigma_N[f](x) \to \frac{f(x-)+f(x+)}{2}$. Let $\epsilon > 0$ be given. By using the evenness of K_N and that $\frac{1}{2\pi}\int_0^{\pi} K_N(t)dt = 1/2$ we have that,

$$\begin{aligned}
&|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)dt - \frac{f(x-)+f(x+)}{2}| \\
&= |\frac{1}{2\pi}\int_{0}^{\pi} [f(x-t)+f(x+t)]K_N(t)dt - \frac{f(x-)+f(x+)}{2}| \\
&= |\int_{0}^{\pi} [f(x-t)+f(x+t)]K_N(t)dt - \frac{1}{2\pi}\int_0^{\pi} f(x-)K_N(t)dt - \frac{1}{2\pi}\int_0^{\pi} f(x+)K_N(t)dt| \\
+ &\leq |\frac{1}{2\pi}\int_0^{\pi} |f(x-t)-f(x-)|K_N(t)dt + \frac{1}{2\pi}\int_0^{\pi} |f(x+t)-f(x+)|K_N(t)dt.
\end{aligned}$$

Looking just at the integral $\frac{1}{2\pi}\int_0^{\pi} |f(x-t)-f(x-)|K_N(t)dt$, we know that $\lim_{t\to x^-} f(t)$ exists, therefore there exists $\delta > 0$ such that $|f(x-t)-f(x-)| < \epsilon$ for all $t \in (0,\delta)$. Note that if we split the integral into $\frac{1}{2\pi}\int_0^{\delta}[f(x-t)-f(x-)]K_N(t)dt + \frac{1}{2\pi}\int_{\delta}^{\pi} K_N(t)dt$ that the first integral will be bounded above by $\frac{\epsilon\delta}{2\pi}$, and by the proof of fejer's theorem in your (Han's) notes the second integral is bounded above by $M\epsilon$ for a sufficently large N, and the bound $|f(t)| < M$ on $[-\pi,\pi]$, as f is Riemann integrable. Note that all the previous steps can be applied to the other integral approaching x from the left. Therefore the proposition holds.

4.3.6 The fourier sine series of $g(x) = 1$ is given by the fourier coefficents $\frac{1}{\pi}\int_0^{\pi}\sin(nx)dx = \frac{1}{\pi}\frac{\sin(n\pi)-\sin(0)}{n} = 0$, and $\frac{1}{\pi}\int_0^{\pi} dx = 1$. Thus the fourier series of $g(x)$ is $g(x)$, it is uniformly convergent.

4.3.8 Let $g \approx a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{l}x\right) + b_n\sin\left(\frac{n\pi}{l}x\right)$, by the conditions stated in the problem we know this holds on $[-l,l]$. Let $f(x) = g(\frac{l}{\pi}x)$. Then $f \approx a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$ on $[-\pi,\pi]$ With f' defined on all but a finite number of points and is piecewise continuous. Then by theorem 4.3.2 $f' \approx \sum_{n=1}^{\infty} n[a_n'\cos\left(\frac{n\pi}{l}x\right) + b_n'\sin\left(\frac{n\pi}{l}x\right)]$ where $a_n' = nb_n, b_n' = -na_n$. Therefore by parseval's equality we have that $\int_{-\pi}^{\pi}|f'(t)|^2dt = \pi\sum_{n=1}^{\infty}[{a_n'}^2+{b_n'}^2] = \pi\sum_{n=1}^{\infty} n^2[{a_n}^2+{b_n}^2]$. Observe that $\int_{-l}^{l}|g'(t)|^2dt = \int_{-\pi}^{\pi}|g'(\frac{l}{\pi}t)|^2\frac{l}{\pi}dt = \int_{-\pi}^{\pi}\frac{l}{\pi}|\frac{\pi}{l}f'(t)|^2dt = \frac{\pi}{l}\int_{-\pi}^{\pi}|f'(t)|^2dt = \frac{\pi^2}{l}\sum_{n=1}^{\infty} n^2[{a_n}^2+{b_n}^2] = \sum_{n=1}^{\infty}\left(\frac{n\pi}{l}\right)^2 l[{a_n}^2+{b_n}^2]$. This demonstrates the desired result.

4.3.9 By the conditions gave in the problem then there exists a fourier series such that $g(x) \approx a_0 + \sum_{i=1}^{n} a_n\cos(\frac{n\pi}{L}x) + \sin(\frac{n\pi}{L}x)$. Therefore $g(x) - \bar{g} \approx \sum_{i=1}^{n} a_n\cos(\frac{n\pi}{L}x)$. Thus applying parseval's equality yields $\int_0^L |g-\bar{g}|^2dx = L\sum_{i=1}^{n}[a_n^2+b_n^2]$. Furthermore, by the previous problem we know that $\int_0^L |g'(x)|^2dx = \sum_{n=1}^{\infty}\left(\frac{n2\pi}{L}\right)^2 L[{a_n}^2+{b_n}^2]$. Therefore if we multiply this series by $\left(\frac{L}{2\pi}\right)^2 2$ we get that $\left(\frac{L}{2\pi}\right)^2\int_0^L |g'(x)|^2dx = \sum_{n=1}^{\infty} n^2L[{a_n}^2+{b_n}^2]$. Note that term by term in the series representation we have $L[{a_n}^2+{b_n}^2]$ and $Ln^2[{a_n}^2+{b_n}^2]$. Note that $n^2 \geq 1$, therefore for each term the inequality $L[{a_n}^2+{b_n}^2] \leq Ln^2[{a_n}^2+{b_n}^2]$ holds. Therefore, since each $L[{a_n}^2+{b_n}^2] > 0$, the sums maintain the inequality, therefore $\int_0^L |g-\bar{g}|^2dx \leq \left(\frac{L}{2\pi}\right)^2\int_0^L |g'(x)|^2dx$. Furthermore if $g = \bar{g} +$

$a_1 \cos\left(\frac{2\pi}{L}x\right) + b_1 \sin\left(\frac{2\pi}{L}x\right)$, then additionally $g' = \frac{2\pi}{L}[b_1 \cos\left(\frac{2\pi}{L}x\right) - a_1 \sin\left(\frac{2\pi}{L}x\right)]$, which gives us the identity proved previously that $\int_0^L |g'(x)|^2 dx = \frac{(2\pi)^2}{L^2} L[a_1^2 + a_2^2]$. Therefore

$$\begin{aligned}
&\int_0^L |g-\bar{g}|^2 dx \\
&= \int_0^L a_1^2 \sin\left(\frac{2\pi}{L}x\right) + b_1^2 \cos\left(\frac{2\pi}{L}x\right) + a_1 b_2 |\sin\left(\frac{2\pi}{L}x\right)\cos\left(\frac{2\pi}{L}x\right)|dx \\
&= L[a_1^2 + b_1^2] \\
&= \left(\frac{L}{2\pi}\right)^2 \int_0^L |g'(x)|^2 dx
\end{aligned}$$

Note that in the only if direction, since the terms in $\frac{L^2}{4\pi^2}\int_0^L |f'|dx$ have a factor of n^2, then equality can only occur if $n = 1$, and this can only occur if $g = c + a_1\cos(\frac{2\pi}{L}x) + b_1 \sin(\frac{2\pi}{L}x)$, and furthermore we need $c = \bar{g}$ to have it be properly subtracted.

4.3.10 In this scenario we consider g approximated by a cosine fourier series, $g(x) \approx a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{L}x\right)$. Note that by definition $a_0 = \bar{g}$, thus allowing us to apply parseval's identity and get us $\int_0^L |g-\bar{g}|^2 dx = \frac{L}{2}\sum_{i=1}^n a_n^2$. Note that g' is piecewise continuous, therefore there exists a sine series such that $g' \approx \sum_{i=1}^n a_n' \sin\left(\frac{n\pi}{L}x\right)$ where $a_n' = -\frac{n\pi}{L}a_n$. Note that applying parseval's for this series we get that $\int_0^L |g'|^2 dx = \frac{L}{2}\sum_{i=1}^n \left(\frac{n\pi}{L}\right)^2 a_n^2$. Note that term by term in the respective series we have $a_n^2 \leq n^2 a_n^2$. Thus $\int_0^L |g-\bar{g}|^2 dx = \frac{L}{2}\sum_{i=1}^n a_n^2 \leq \frac{L}{2}\sum_{i=1}^n n^2 a_n^2 = \left(\frac{L}{\pi}\right)^2 \frac{L}{2}\sum_{i=1}^n \left(\frac{n\pi}{L}\right)^2 a_n^2 = \left(\frac{L}{\pi}\right)^2 \int_0^L |g'|dx$. Additionally by nearly the same proof as above, we have equality iff $g = \bar{g} + a_1\cos(\frac{\pi}{L}x)$.