4.2.5 Let f and g be functions defined on the domain $A \subseteq \mathbb{R}$, and assume $\lim_{x\to c} f(x) = L, \lim_{x\to c} g(x) = M$ for some limit point c of A.

(a) Show that $\lim_{x\to c}[f(x)+g(x)] = L+M$, assuming theorem 4.2.3 and the algebraic limit theorem for sequences.
Since both f and g converge at c, then by theorem 4.2.3 for any sequence which converges to c, x_n, $\lim f(x_n) \to L, \lim g(x_n) \to M$. Therefore by the algebraic limit theorem for sequences, $f(x_n)+g(x_n) \to L+M$. Therefore using the converse of theorem 4.2.3, this implies that $\lim_{x\to c}[f(x)+g(x)] = L+M$

(b) Show that $\lim_{x\to c}[f(x)+g(x)] = L+M$, without assuming theorem 4.2.3.
Let $\epsilon > 0$. Since f converges at c then there exists $\delta_1 > 0$ such that $|x-c| < \delta_1$ implies $|f(x)-L| < \frac{\epsilon}{2}$. Similarly since g converges at c there exists $\delta_2 > 0$ such that $|x-c| < \delta_2$ implies $|g(x)-M| < \frac{\epsilon}{2}$. Therefore, if we let $\delta = \min\{\delta_1, \delta_2\}$ then $|x-c| < \delta$ implies

$$\begin{aligned}
|f(x)+g(x)-(L+M)| &= |(f(x)-L)+(g(x)-M)| \\
&\leq |f(x)-L| + |g(x)-M| && \text{triangle inequality} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} && \text{convergence definitions} \\
&= \epsilon
\end{aligned}$$

(c) Show that $\lim_{x\to c} f(x)g(x) = LM$, assuming theorem 4.2.3 and the algebraic limit theorem for sequences.
Since both f and g converge at c, then by theorem 4.2.3 for any sequence which converges to c, x_n, $\lim f(x_n) \to L, \lim g(x_n) \to M$. Therefore by the algebraic limit theorem for sequences, $f(x_n)g(x_n) \to LM$. Therefore using the converse of theorem 4.2.3, this implies that $\lim_{x\to c} f(x)g(x) = LM$

(d) Show that $\lim_{x\to c} f(x)g(x) = LM$, without assuming theorem 4.2.3.
Let $\epsilon > 0$. Since f converges at c then there exists $\delta_1 > 0$ such that $|x-c| < \delta_1$ implies $|f(x)-L| < \frac{\epsilon}{2|M|}$. Note that by manipulating $|f(x)-L| < \frac{\epsilon}{2|M|}$ and applying the reverse triangle inequality we find that $|f(x)| < \frac{\epsilon}{2|M|} + |L|$. Let $B = \frac{\epsilon}{2|M|} + |L|$ Similarly since g converges at c then there exists $\delta_2 > 0$ such that $|x-c| < \delta_2$ implies $|g(x)-M| < \frac{\epsilon}{2|B|}$. Therefore, if we let $\delta = \min\{\delta_1, \delta_2\}$ then $|x-c| < \delta$ implies

$$\begin{aligned}
|f(x)g(x)-LM| &= |f(x)g(x) - Mf(x) + Mf(x) - LM| \\
&= |f(x)(g(x)-M) + M(f(x)-L)| \\
&\leq |f(x)(g(x)-M)| + |M(f(x)-L)| && \text{triangle inequality} \\
&= |f(x)||g(x)-M| + |M||f(x)-L| && \text{definition of absolute value} \\
&< |B||g(x)-M| + |M||f(x)-L| \\
&< |B|\frac{\epsilon}{2|B|} + |M|\frac{\epsilon}{2|M|} \\
&= \epsilon
\end{aligned}$$