For each of the following relations, determine which of the properties reflexive, anti-reflexive, transitive, symmetric, and anti-symmetric it satisfies. If the property is not satisfied, give a counterexample; if it's satisfied provide a proof.

(a) Let $\mathcal{S}$ be a collection of non-empty subsets of a set $X$ and let $R$ be the relation on $\mathcal{S}$ with pairs($R$) consisting of all pairs $(S,T) \in \mathcal{S} \times \mathcal{S}$ satisfying $S \cap T = \emptyset$.

- Proof of anti-reflexivity. Suppose $S \in X$. We must show that $S \cap S \neq \emptyset$. By definition of set intersection, $S \cap S = S$. Therefore since $S$ is non-empty, then $S \cap S \neq \emptyset$.
- Proof of symmetry. Suppose $S, T \in X, S \cap T = \emptyset$. We must show that $T \cap S = \emptyset$. By the definition of set intersection $S \cap T = \{x : x \in S \wedge x \in T\}$. By the definition of and commutativity $\{x : x \in T \wedge x \in S\}$. By the definition of set intersection $S \cap T = T \cap S$. Therefore since $S \cap T = \emptyset$, then $T \cap S = \emptyset$.
- Counterexample to transitivity. Suppose $A = \{1\}, B = \{2\}, C = \{1, 3\}$. $A \cap B = \emptyset$ and $B \cap C = \emptyset$. However, $A \cap C = \{1\}$, therefore the relation is not transitive.

(b) Let $\mathcal{S}$ be a collection of non-empty subsets of a set $X$ and let $R$ be the relation on $\mathcal{S}$ with pairs ($R$) consisting of all pairs $(S,T) \in \mathcal{S} \times \mathcal{S}$ satisfying $S \cap T \neq \emptyset$.

- Proof of reflexivity. Suppose $S \in X$. We must show that $S \cap S \neq \emptyset$. By definition of set intersection, $S \cap S = S$. Therefore since $S$ is non-empty, then $S \cap S \neq \emptyset$.
- Proof of symmetry. Suppose $S, T \in X, S \cap T \neq \emptyset$. We must show that $T \cap S \neq \emptyset$. By the definition of set intersection $S \cap T = \{x : x \in S \wedge x \in T\}$. By the definition of and commutativity $\{x : x \in T \wedge x \in S\}$. By the definition of set intersection $S \cap T = T \cap S$. Therefore since $S \cap T \neq \emptyset$, then $T \cap S \neq \emptyset$.
- Counterexample to transitivity. Suppose $A = \{1\}, B = \{1, 2\}, C = \{2\}$. Therefore $A \cap B \neq \emptyset, B \cap C \neq \emptyset$. However $A \cap C = \{1\} \cap \{2\} = \emptyset$.

(c) Let $R$ be a relation on $\mathbb{Z}$ defined so that for $m, n \in \mathbb{Z}, (m, n) \in R$ provided there are odd integer $r$ and $s$ so that $mr = ns$.

- Proof of reflexivity. Suppose $m \in \mathbb{Z}$ We must show that there exist odd integers $r, s$ so that $mr = ms$. Suppose $r = s = 1$. Then $m = m$.
- Proof of symmetry. Suppose $m, n \in \mathbb{Z}$, and there exist odd integers $mr = ns$. We must show that there exist odd integers $t, u$ such that $nt = mu$. Let $t = s, u = r$. Therefore $ns = mr$.
- Proof of transitivity. Suppose $l, m, n \in \mathbb{Z}$ and there exist odd integers $r, s, t, u$ such that $lr = ms$ and $mt = nu$. We must show that there exist odd integers $v, w$ such that $lv = nw$. Since $lr = ms$, then by definition $\frac{lr}{s} = m$. Substituting that definition of $m$ into $mt = nu$ yields $\frac{lr}{s}t = nu$. Multiplying both sides by $s$ yields $lrt = nus$. Since the product of two odd numbers is odd, then $v = rt$ and $w = us$. Thus $lv = nw$.

(d) Let $S = \mathbb{R} \times \mathbb{R}$ and let $R$ be the relation defined as follows for $(x_1, x_2) \in S$ and $(y_1, y_2) \in S$, we have $(x_1, x_2)\, R\, (y_1, y_2)$ if $x_1 \leq y_1$ and $x_2 > y_2$. (Careful, this one may be confusing because the set $S$ consists of ordered pairs, so pairs($R$) is a set of ordered pairs, and for each ordered pair in pairs ($R$) each of its coordinates is an ordered pair.)

- Proof of anti-reflexivity. Suppose $(x, y) \in \mathbb{R}^2$. We must show for all $(x, y) \in \mathbb{R}^2$ that $(x, y)\not R(x, y)$. By definition we must show that $x \leq x \wedge y > y$ is false. Since by the definition of greater than $y > y$ is always false, $R$ is anti-reflexive.
- Proof of anti-symmetry. Suppose $(x, y), (a, b) \in \mathbb{R}^2, (x, y)R(a, b)$. We must show $(a, b)\not R(x, y)$, and $(a, b) \neq (x, y)$. Since the relation is anti-reflexive, we can assume $(x, y) \neq (a, b)$. By definition of this relation, $x \leq a$ and $y > b$. Therefore by definition of the relation we must show that $\neg(a \leq x \wedge b > y)$. Therefore we must show $a > x$ or $y \leq b$. Choose $a > x$. Since we already have $(a, b) \neq (x, y)$, then by definition of ordered pair $a \neq x$. Therefore since we already have $a \geq x$ and $a \neq x$, then by definition $a > x$.
- Transitivity proof. Suppose $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathbb{R}^2, (x_1, x_2)R(y_1, y_2), (y_1, y_2)R(z_1, z_2)$. We must show for all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathbb{R}^2$ that $(x_1, x_2)R(z_1, z_2)$. By definition of the relation we must show $x_1 \leq z_1$ and $x_2 > z_2$. By definition of the relation we have $x_1 \leq y_1$, $x_2 > y_2$, $y_1 \leq z_1$, $y_2 > z_2$. By the composition of inequalities we have $x_2 > y_2 > z_2$ and $x_1 \leq y_1 \leq z_1$. Therefore by the transitivity of greater than and less than or equal to $x_2 > z_2$ and $x_1 \leq z_1$.