3.66 Let $X \sim \mathcal{N}(8, 3), Z \sim \mathcal{N}(1, 0)$. Therefore

$$\begin{aligned}
\mathbb{P}(X > \alpha) &= 0.15 \\
1 - \mathbb{P}(X \leq \alpha) &= 0.15 \\
\mathbb{P}(X \leq \alpha) &= 0.85 \\
\mathbb{P}(\sqrt{3}Z + 8 \leq \alpha) &= 0.85 \\
\mathbb{P}(Z \leq \frac{\alpha - 8}{\sqrt{3}}) &= 0.85 \\
\Phi(\frac{\alpha - 8}{\sqrt{3}}) &= 0.85 \\
\frac{\alpha - 8}{\sqrt{3}} &= \Phi^{-1}(0.85) \\
\frac{\alpha - 8}{\sqrt{3}} &\approx 1.04 \\
\alpha &\approx 9.8
\end{aligned}$$

3.67 Let $X \sim \mathcal{N}(\mu, \sigma^2), Z \sim \mathcal{N}(1, 0)$

(a) Since x^3 is odd and $\phi(x)$ is even then $x^3\phi(x)$ is odd. Thus the integral across all of $\mathbb{R}$ is 0.

(b)

$$\begin{aligned}
\mathbb{E}[X^3] &= \mathbb{E}[(\sigma Z + \mu)^3] \\
&= \sigma^3\mathbb{E}[Z^3] + 3\mu\sigma^2\mathbb{E}[Z^2] + 3\mu^2\sigma\mathbb{E}[Z] + \mu^3\mathbb{E}[1] \\
&= \sigma^3 * 0 + 3\mu\sigma^2(1 + 0^2) + 3\mu^2\sigma * 0 + \mu^3 * 1 \\
&= \mu\sigma^2 + \mu^3
\end{aligned}$$

3.68 (a)

$$\begin{aligned}
\int_{-\infty}^{\infty} x^4 e^{-x^2/2} dx &= -x^3 e^{-x^2/2}]_{-\infty}^{\infty} + 3\int_{-\infty}^{\infty} x^2 e^{-x^2/2} dx \\
&= 3(-xe^{-x^2/2}]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2} dx) \\
&= 3\sqrt{2\pi}
\end{aligned}$$

Thus $\mathbb{E}[Z^4] = 3$.

(b) Same X and Z as stated in the previous problem

$$\begin{aligned}
\mathbb{E}[X^4] &= \mathbb{E}[(\sigma Z + \mu)^4] \\
&= \sigma^4\mathbb{E}[Z^4] + 4\sigma^3\mu\mathbb{E}[Z^3] + 6\sigma^2\mu^2\mathbb{E}[Z^2] + \sigma\mu^3\mathbb{E}[Z] + \mu^4 \\
&= 3\sigma^4 + 0 + 6\sigma^2\mu^2 1 + 0 + \mu^4 \\
&= 3\sigma^4 + 6\sigma^2\mu^2 + \mu^4
\end{aligned}$$

4.4 Let X_{90} be the total steps taken after 90 rolls. Let S_{90} be the number of rolls yielding 1 step forward. Then our desired probability can be written as $\mathbb{P}(X_{90} \geq 160) = \mathbb{P}(S_{90} \leq 20)$. Since each roll has probability 1/3 of hitting a number which advances us one tile, then clearly $S_{90} \sim Bin(90, 1/3)$. Therefore if we normalize we find that our probability becomes $\mathbb{P}(\frac{S_{90}-30}{\sqrt{20}} \leq -\sqrt{5})$. Thus by the central limit theorem this can be approximated by $\Phi(-\sqrt{5}) = 1 - \Phi(\sqrt{5})$. Thus the probability is approximately 0.013.

4.18 Since we can hit a point uniformly on the dart board, then the probability of hitting the center is simply dividing the areas yielding $p = \frac{\pi 1^2}{\pi 5^2} = \frac{1}{25}$. Therefore the variable H_{2000} denoting the number of bullseyes is a binomial variable $H_{2000} \sim Bin(2000, \frac{1}{25})$ with $\mathbb{E}H_{2000} = 80, Var(H_{2000}) = 76.8$.

Therefore our desired probability is

$$\mathbb{P}(H_{2000} \geq 100) = 1 - \mathbb{P}(H_{2000} < 100) = 1 - \mathbb{P}(\frac{H_{2000} - 80}{\sqrt{76.8}} < \frac{20}{\sqrt{76.8}}) \approx 1 - \Phi(2.282) \approx 0.0113$$

AE1 $\mathbb{E}[e^{cZ}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{cx} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{cx - \frac{x^2}{2}} dx$. Thus by homework 1 problem 1a we have that the integral evaluates to $e^{-\frac{c^2}{2}}$

AE2 The only c which works is $c < \frac{1}{2}$, because if $c \geq \frac{1}{2}$ then $c - \frac{1}{2} \geq 0$, and since for every $|x| > 1$ $e^{(c-\frac{1}{2})|x|} < e^{(c-\frac{1}{2})x^2}$, then for a sufficiently large a, $\int_{-a}^{a} e^{(c-\frac{1}{2})|x|} dx < \int_{-a}^{a} e^{(c-\frac{1}{2})x^2} dx$. Since $\int_{-\infty}^{\infty} e^{(c-\frac{1}{2})|x|} dx = \infty$, then $\int_{-\infty}^{\infty} e^{(c-\frac{1}{2})x^2} = \infty$. (Note if $c = \frac{1}{2}$ then we have the integral $\int_{-\infty}^{\infty} 1 dx$, which clearly diverges.)

AE3 (a) Let $X = \sigma Z + \mu$ where $Z \sim \mathcal{N}(0, 1)$

$$\begin{aligned}
\mathbb{P}(Y \geq K) &= \mathbb{P}(e^X \geq K) \\
&= \mathbb{P}(X \geq \log(K)) \\
&= \mathbb{P}(Z \geq \frac{\log(K) - \mu}{\sigma}) \\
&= 1 - \mathbb{P}(Z \leq \frac{\log(K) - \mu}{\sigma}) \\
&= 1 - \Phi(\frac{\log(K) - \mu}{\sigma}) \\
&= \Phi(\frac{\mu - \log(K)}{\sigma})
\end{aligned}$$

(b)

$$
\begin{aligned}
\mathbb{E}[\max(Y-K,0)] &= \mathbb{E}[(Y-K)\mathbb{I}_{X\geq \log(K)}] \\
&= \mathbb{E}[Y\mathbb{I}_{X\geq \log(K)}] - \mathbb{E}[K\mathbb{I}_{X\geq \log(K)}] \\
&= \mathbb{E}[Y\mathbb{I}_{X\geq \log(K)}] - K\mathbb{E}[\mathbb{I}_{X\geq \log(K)}] \\
&= \mathbb{E}[Y\mathbb{I}_{X\geq \log(K)}] - K\Phi(\frac{\mu-\log(K)}{\sigma}) \\
&\text{ by the hint} \\
&= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{\log(K)}^{\infty} e^r e^{\frac{(r-\mu)^2}{2\sigma^2}}dr - K\Phi(\frac{\mu-\log(K)}{\sigma}) \\
&= \frac{e^\mu}{\sqrt{2\pi}}\int_{\frac{\log(K)-\mu}{\sigma}}^{\infty} e^{\sigma l-\frac{l^2}{2}}dl - K\Phi(\frac{\mu-\log(K)}{\sigma}) \\
&\text{Let } l = \frac{r-\mu}{\sigma}, dl = \frac{dr}{\sigma} \\
&= \frac{e^\mu}{\sqrt{2\pi}}(\sqrt{2\pi}e^{\frac{\sigma^2}{2}} - \int_{-\infty}^{\frac{\log(K)-\mu}{\sigma}} e^{\sigma l-\frac{l^2}{2}}dl) - K\Phi(\frac{\mu-\log(K)}{\sigma}) \\
&\text{By 1a on hw 1} \\
&= \frac{e^\mu}{\sqrt{2\pi}}(\sqrt{2\pi}e^{\frac{\sigma^2}{2}} - \int_{-\infty}^{\frac{\log(K)-\mu}{\sigma}} e^{-\frac{(l-\sigma)^2}{2}+\frac{\sigma^2}{2}}dl) - K\Phi(\frac{\mu-\log(K)}{\sigma}) \\
&\text{completing the square} \\
&= \frac{e^{\mu+\frac{\sigma^2}{2}}}{\sqrt{2\pi}}(\sqrt{2\pi} - \int_{-\infty}^{\frac{\log(K)-\mu}{\sigma}-\sigma} e^{-\frac{u^2}{2}}du) - K\Phi(\frac{\mu-\log(K)}{\sigma}) \\
&\text{Let } u = l-\sigma, du = dl \\
&= e^{\mu+\frac{\sigma^2}{2}}(1-\Phi(\frac{\log(K)-\mu}{\sigma}-\sigma)) - K\Phi(\frac{\mu-\log(K)}{\sigma}) \\
&\text{Definition of } \Phi \\
&= e^{\mu+\frac{\sigma^2}{2}}(\Phi(\frac{\mu-\log(K)}{\sigma}+\sigma)) - K\Phi(\frac{\mu-\log(K)}{\sigma})
\end{aligned}
$$