1. Email sent!

2. (a) $\bigcap_{n\in\mathbb{N}}(\frac{-2}{\sqrt{n}}, \frac{3}{n^2})$
   Since these end points both converge to 0, and since the end points aren't included, then this set is empty.

   (b) $\bigcap_{n\in\mathbb{N}}[\frac{-4}{n}, \frac{4}{n}]$
   Since both end points go to 0, and since the end points included, means this set exactly contains the point 0.

   (c) $\bigcup_{n\in\mathbb{N}}(\frac{-1}{5n}, \frac{1}{n^2+n})$
   Since the end points are monotonically increasing and decreasing respectively, all intervals being unioned together fit inside the first interval when $n = 1$. Therefore it evaluates to the interval $(\frac{-1}{5}, \frac{1}{3})$.

   (d) $\bigcup_{n\in\mathbb{N}}[\frac{-7}{n}, \frac{8}{n}]$
   Same as above, since the end points are monotonically increasing and decreasing respectively, then the union is equivalent to $[-7, 8]$.

3. For $n \in \mathbb{N}$ compute $\sum_{k=1}^{n} k^2$ and $\sum_{k=1}^{n} k^3$.
   We will be using the fact that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2} = S_1$ without proof.

   - $\sum_{k=1}^{n} k^2 = S_2$
     Consider the telescoping series $n^3 = \sum_{k=1}^{n} k^3 - (k-1)^3$. If we expand the sum on the right hand side we get

     $$n^3 = \sum_{k=1}^{n} k^3 - (k-1)^3 = \sum_{k=1}^{n} k^3 - k^3 + 3k^2 - 3k + 1 = 3 * S_2 - 3 * S_1 + n.$$

     Therefore if we do some rearranging:

     $$\begin{aligned} n^3 &= 3 * S_2 - 3 * S_1 + n \\ n^3 + \frac{3}{2}n(n+1) - n &= 3 * S_2 \\ \frac{n^3}{3} + \frac{n(n+1)}{2} - \frac{n}{3} &= S_2 \\ \frac{2n^3 + 3n^2 + 3n - 2n}{6} &= \\ \frac{n(2n^2 + 3n + 1)}{6} &= \\ \frac{n(2n^2 + 3n + 1)}{6} &= \\ \frac{n(n+1)(2n+1)}{6} & \end{aligned}$$

   - $\sum_{k=1}^{n} k^3 = S_3$
     Consider the telescoping series $n^4 = \sum_{k=1}^{n} k^4 - (k-1)^4$. If we expand the sum

on the right hand side we get

$$n^4 = \sum_{k=1}^{n} k^4 - (k-1)^4 = \sum_{k=1}^{n} k^4 - k^4 + 4k^3 - 6k^2 + 4k - 1 = 4S_3 - 6S_2 + 4S_1 - n.$$

Therefore if we do some rearranging:

$$\begin{aligned}
4S_3 - 6S_2 + 4S_1 - n &= n^4 \\
4S_3 &= n^4 + 6S_2 - 4S_1 + n \\
S_3 &= \frac{n^4}{4} + \frac{3}{2}S_2 - S_1 + \frac{n}{4} \\
&= \frac{n^4 + 2n^3 + 3n^2 + n - n^2 - n + n}{4} \\
&= \frac{n^2(n^2 + 2n + 1)}{4} \\
&= \frac{n^2(n+1)^2}{4} \\
&= \left(\frac{n(n+1)}{2}\right)^2
\end{aligned}$$

4. *Prove that 1 is an upper bound of the following set*

$$\{\frac{nm}{m^2 + 4n^2} : n, m \in \mathbb{N}\}$$

We will show by cases:

- Assume $n = m$. Therefore our fraction evaluates to $\frac{1}{5}$. Clearly $\frac{1}{5} < 1$.
- Assume $n > m$. Therefore $\frac{n}{m} > 1$. Thus

$$n < \frac{n^2}{m} < 4\frac{n^2}{m} < 4\frac{n^2}{m} + m.$$

Therefore $1 > \frac{n}{4\frac{n^2}{m} + m} = \frac{nm}{n^2 + 4m^2}$.

- Assume $n < m$. Therefore $\frac{m}{n} > 1$. Thus

$$m < \frac{m^2}{n} < \frac{m^2}{n} + 4n.$$

Therefore $1 > \frac{m}{\frac{m^2}{n} + 4n} = \frac{nm}{m^2 + 4n^2}$

Therefore our set is bounded above by 1.

5. *Prove that 1 is an upper bound of the following set* $\{\frac{nmk^2}{3n^3 + 9m^3 + k^6} : n, m, k \in \mathbb{N}\}$
We will work by cases, and using the substitution $q = k^2$:

- Assume $n, m \leq q$. Then $1 \leq \frac{q}{m}, \frac{q}{n}$. Therefore

$$q \leq q(\frac{q}{n})(\frac{q}{m}) \leq q(\frac{q}{n})(\frac{q}{m}) + 9(\frac{m^2}{qn}) + 3\frac{n^2}{mq}.$$

  Therefore $1 \geq \frac{q}{q(\frac{q}{n})(\frac{q}{m})+9(\frac{m^2}{qn})+3\frac{n^2}{mq}} = \frac{nmq}{3n^3+9m^3+q^3}$ for the given condition.

- Assume $n, q \leq m$. Then $1 \leq \frac{m}{q}, \frac{m}{n}$. Therefore

$$m \leq 9m\frac{m^2}{qn} \leq 9m\frac{m^2}{qn} + 3\frac{n^2}{qm} + \frac{q^2}{mn}.$$

  Therefore $\frac{m}{9m\frac{m^2}{qn}+3\frac{n^2}{qm}+\frac{q^2}{mn}} = \frac{nmq}{3n^3+9m^3+q^3} \leq 1$ for the given condition.

- Assume $m, q \leq n$. Then $\frac{n}{q}, \frac{n}{m} \geq 1$. Therefore

$$n \leq 3n\frac{n}{q}\frac{n}{m} \leq 3n\frac{n}{q}\frac{n}{m} + 9\frac{m^2}{nq} + \frac{q^2}{mn}.$$

  Therefore $1 \geq \frac{n}{3n\frac{n}{q}\frac{n}{m}+9\frac{m^2}{nq}+\frac{q^2}{mn}} = \frac{mnq}{3n^3+9m^3+q^3}$

Since all of the conditions encompass all possible elements in the set, 1 is an upper bound.

6. *Let $A = \{1, 2, 3, 4, 5\}$. Verify if the following relations are equivalence relations:*

   (a) $R_1 = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\}$

   R_1 is not an equivalence relation because no element is related to itself.

   (b) $R_2 = \{(1, 1), (1, 3), (1, 4), (2, 2), (2, 5), (3, 1), (3, 3), (3, 4), (4, 1), (4, 3), (4, 4), (5, 2), (5, 5)\}$

   R_2 is an equivalence relation, every element is related to itself, and it is transitive. The equivalence classes are $[1] \equiv [3] \equiv [4]$ and $[5] \equiv [2]$.

7. *Suppose R_1 and R_2 are equivalence relations on a set A. Define the relation R on A by xRy if xR_1y and xR_2y. Prove that R is an equivalence relation.*

   - For all $x \in A$, xR_1x and xR_2x. Therefore xRx
   - Suppose xRy. Therefore xR_1y and xR_2y. Since both are equivalence relations then yR_1x and yR_2x. Therefore yRx.
   - Suppose xRy and yRz. Therefore xR_1y, yR_1z and xR_2y, yR_2z. Therefore xR_1z, xR_2z since both relations are equivalence relations. Therefore by the definition of R, xRz.

8. Note that the first few elements are given by $a_0 = 0, a_1 = 1, a_2 = 3, a_3 = 7, a_4 = 15$ We want to show by induction that $2^n - 1 = a_n$ for all $n \in \mathbb{N}$. Note that for the base case $n = 0$ we have $2^0 - 1 = 1 - 1 = 0$. Thus by the principle of mathematical induction for $k \in \mathbb{N}$ if $k \leq n$ then $a_k = 2^k - 1$. Therefore if we want to find a_{n+1} we can apply the recursion formula and get $3a_n - 2a_{n-1}$. Therefore by the induction hypothesis we have that

$$a_n = 3(2^n - 1) - 2(2^{n-1} = (2+1)2^n - 3 - 2^n + 2 = 2^{n+1} + 2^n - 2^n - 1 = 2^{n+1}.$$

Therefore $a_n = 2^n$ for all $n \in \mathbb{N}$.

9. *Prove that* $133 \mid 11^{n+1} + 12^{2n-1}$. Note that for $n = 1$, $11^2 + 12^{2-1} = 121 + 12 = 133$. Therefore the base case holds. Therefore by the principle of mathematical induction for all $k \in \mathbb{N}$ if $k \leq n$ then $133 | 11^{k+1} + 12^{2k-1}$. We want to show that the condition holds for $n + 1$. Note that since

$$\begin{aligned} 11^{n+2} + 12^{2n+1} &= 11^{n+2} + 12^{2n+1} \\ &= 11 \cdot 11^{n+1} + 12^2 \cdot 12^{2n-1} \\ &= 11 \cdot 11^{n+1} + (133 + 11) \cdot 12^{2n-1} \\ &= 11(11^{n+1} + 12^{2n-1}) + 133 \cdot 12^{2n-1} \end{aligned}$$

then we only need to know if $133 \mid 11^{n+1} + 12^{2n-1}$. Since $n \leq n$ then by the induction hypothesis $133 \mid 11^{n+1} + 12^{2n-1}$ holds.

10. The proof that horses are all the same color fails is when one considers the $n = 2$ case. If we have two horses in a set, $\{h_1, h_2\}$, we know that the singleton sets of the horses have one color trivially. However this does not imply that the singleton sets of horses have the same color. Therefore the set of two horses is not guaranteed to have all the same color.

11. It is not a mathematical proof because the contradiction is of the form $n \notin I \Rightarrow n \in I$. This, logically speaking, is not well defined. Therefore there isn't a logically meaningful contradiction, invalidating the proof.