1. - Show that $x(t)$ satisifes $\|A^{-1}x(t)\|^2 = 1$.

$$\begin{aligned}
\|A^{-1}x(t)\|^2 &= \|A^{-1}Au(t)\|^2 \\
&= \|u(t)\|^2 \\
&= \| \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \|^2 \\
&= \cos^2(t) + \sin^2(t) \\
&= 1.
\end{aligned}$$

   - Show that the equation above may be written as $x \cdot Mx = 1$.

$$\begin{aligned}
x \cdot Mx &= x^T Mx \\
&= x^T (A^{-1})^T A^{-1} x \\
&= (A^{-1}x)^T A^{-1} x \\
&= A^{-1}x \cdot A^{-1}x \\
&= \|A^{-1}x\|^2 \\
&= 1.
\end{aligned}$$

   - Show that M is symmetric.

$$\begin{aligned}
M^T &= ((A^{-1})^T A^{-1})^T \\
&= (A^{-1})^T ((A^{-1})^T)^T \\
&= (A^{-1})^T A^{-1} \\
&= M
\end{aligned}$$

   - Suppose $M = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$. We must show that $x \cdot Mx$ can be written as $ax^2 + by^2 + 2cxy$.

$$\begin{aligned}
1 &= x \cdot Mx \\
&= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c \\ c & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} ax + cy \\ cx + by \end{bmatrix} \\
&= ax^2 + by^2 + 2cxy.
\end{aligned}$$

2. - Show that both λ_1 and λ_2 are positive. Since $x \cdot Mx = \|A^{-1}x\|^2$, then all outputs of $x \cdot Mx$ are strictly positive. Suppose $x = u_1$, then $u_1 \cdot Mu_1 = u_1 \cdot \lambda_1 u_1 = \lambda_1 \|u_1\|^2 = \lambda_1 > 0$. A similar proof exists for u_2. Therefore the eigenvalues are strictly positive.
   - Suppose $\lambda_1 = \lambda_2$. We must show that $\|x(t)\| = \frac{1}{\sqrt{\lambda_1}}$. Let V be the matrix where u_1 and u_2 are columns. Since u_1, u_2 are orthonormal, then V is an orthogonal

matrix. Therefore $V^{-1} = V^T$. Let $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, and let $q = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = V^T x$. Therefore we may rewrite $x \cdot Mx$ as follows:

$$1 = x \cdot Mx = q \cdot Dq = \lambda_1 q_1^2(t) + \lambda_2 q_2^2(t).$$

This equation defines an ellipse paramaterized by $q_1(t) = \pm\frac{1}{\lambda_1}\cos(t), q_2(t) = \pm\frac{1}{\lambda_2}\sin(t)$. Since $q = V^T x$, then we may explicitly solve for x via $x = Vq = \pm(\frac{1}{\lambda_1}\cos(t)u_1 + \frac{1}{\lambda_2}\sin(t)u_2)$. Therefore if $\lambda_1 = \lambda_2$, then $\|x\| = \sqrt{\frac{\cos^2(t)}{\lambda_1} + \frac{\sin^2(t)}{\lambda_2}} = \sqrt{\frac{\cos^2(t)}{\lambda_1} + \frac{\sin^2(t)}{\lambda_1}} = \frac{1}{\sqrt{\lambda_1}}$.

- Note that $\lambda_1 > \lambda_2$
  - Suppose $x(t) = \pm\frac{1}{\lambda_2}u_2$. We must show that $\|x(t)\|$ is maximal.

$$\|x(t)\| = \sqrt{\frac{\cos^2(t)}{\lambda_1} + \frac{\sin^2(t)}{\lambda_2}} \leq \sqrt{\frac{\cos^2(t)}{\lambda_2} + \frac{\sin^2(t)}{\lambda_2}} = \frac{1}{\sqrt{\lambda_2}} = \| \pm \frac{1}{\sqrt{\lambda_2}} u_2 \|$$