4.4.11 Show that g is continuous iff for all open sets $O \subseteq \mathbb{R}$, $g^{-1}(0)$ is open

$\Rightarrow$ Suppose g is continuous, O is an open set, if $g^{-1}(O)$ is empty then it is vacuously open, therefore we assume $g^{-1}(O)$ is nonempty. Let $x \in g^{-1}(O)$. Therefore $f(x) \in O$. Thus by the definition of open set there exists $\epsilon > 0, V_\epsilon(f(x)) \subseteq O$. Since g is continuous then there exists $\delta > 0$ such that $y \in V_\delta(x)$ implies $f(y) \in V_\epsilon(f(x))$. Therefore $V_\delta(x) \subseteq g^{-1}(O)$. Thus $g^{-1}(O)$ is open.

$\Leftarrow$ Suppose $O \subseteq \mathbb{R}$ is an open set implies $g^{-1}(O)$ is open. We will show that g is continuous. Let $c \in \mathbb{R}, \epsilon > 0$. Since $V_\epsilon(f(c))$ is open then by our assumption, $g^{-1}(V_\epsilon(f(c)))$ is open. Since $g^{-1}(V_\epsilon(f(c)))$ is open and nonempty then there exists $\delta > 0$ such that $V_\delta(c) \subseteq g^{-1}(V_\epsilon(f(c)))$. Therefore $y \in V_\delta(c)$ implies $f(y) \in V_\epsilon(f(c))$. Therefore f is continuous.