1. Prove that if $W_1, W_2$ are finite dimensional subspaces of a vector space $V$, then the subspace $W_1 + W_2$ is finite-dimensional, and $dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2)$.

   - We must show that $W_1 + W_2$ is finite dimensional. Since $W_1 \cap W_2$ is a finite dimensional subspace then let $\vec{u}_1, \ldots, \vec{u}_k$ be a basis of $W_1 \cap W_2$. Since $W_1 \cap W_2$ is a subspace of $W_1$, then the basis of $W_1 \cap W_2$ may be extended to a full basis of $W_1$ given by $\vec{u}_1, \ldots, \vec{u}_k, \vec{v}_1, \ldots, \vec{v}_m$. A similar process may be performed for $W_2$ yielding $\vec{u}_1, \ldots, \vec{u}_k, \vec{w}_1, \ldots, \vec{w}_p$ as $W_2$'s basis. We claim that $W_1 + W_2$ has a basis of $\vec{u}_1, \ldots, \vec{u}_k, \vec{v}_1, \ldots, \vec{v}_m, \vec{w}_1, \ldots, \vec{w}_p$.

     (a) We must show that $\vec{u}_1, \ldots, \vec{u}_k, \vec{v}_1, \ldots, \vec{v}_m, \vec{w}_1, \ldots, \vec{w}_p$ generates $W_1 + W_2$. Suppose $\vec{y} \in W_1 + W_2$. We must show that $\vec{y} \in Span(\{\vec{u}_1, \ldots, \vec{u}_k, \vec{v}_1, \ldots, \vec{v}_m, \vec{w}_1, \ldots, \vec{w}_p\})$. Since $W_1 + W_2$ is the linear combination of vectors from $W_1, W_2$, then there exists vectors $\vec{x}_1 \in W_1, \vec{x}_2 \in W_2$ such that $\vec{y} = \vec{x}_1 + \vec{x}_2$. Therefore there exists $a_1, \ldots, a_k, b_1, \ldots, b_m, c_1, \ldots, c_k, d_1, \ldots, d_p \in F$ such that $x_1 = \sum_{\alpha=1}^{k} a_\alpha \vec{u}_\alpha + \sum_{\beta=1}^{m} b_\beta \vec{v}_\beta$, $x_2 = \sum_{\gamma=1}^{k} c_\gamma \vec{u}_\gamma + \sum_{\delta=1}^{p} d_\delta \vec{w}_\delta$. Therefore:

     $$\begin{aligned}
     \vec{y} &= \vec{x}_1 + \vec{x}_2 \\
     &= \sum_{\alpha=1}^{k} a_\alpha \vec{u}_\alpha + \sum_{\beta=1}^{m} b_\beta \vec{v}_\beta + \sum_{\gamma=1}^{k} c_\gamma \vec{u}_\gamma + \sum_{\delta=1}^{p} d_\delta \vec{w}_\delta \\
     &= \sum_{i=1}^{k} (a_i + c_i) \vec{u}_i + \sum_{\beta=1}^{m} b_\beta \vec{v}_\beta + \sum_{\delta=1}^{p} d_\delta \vec{w}_\delta \\
     &\text{due to the closure of F under addition,} \\
     &\text{let } e_i = a_i + c_i, \text{ for all } i \in [k] \\
     &= \sum_{i=1}^{k} e_i \vec{u}_i + \sum_{\beta=1}^{m} b_\beta \vec{v}_\beta + \sum_{\delta=1}^{p} d_\delta \vec{w}_\delta.
     \end{aligned}$$

     Thus $\vec{y} \in Span(\{\vec{u}_1, \ldots, \vec{u}_k, \vec{v}_1, \ldots, \vec{v}_m\})$.

     (b) We must show that $\vec{u}_1, \ldots, \vec{u}_k, \vec{v}_1, \ldots, \vec{v}_m, \vec{w}_1, \ldots, \vec{w}_p$ is linearly independent. Suppose for contradiction that they aren't. Since $\{\vec{u}_1, \ldots, \vec{u}_k, \vec{v}_1, \ldots, \vec{v}_m, \vec{w}_1, \ldots\}$ and $\{\vec{u}_1, \ldots, \vec{u}_k, \vec{w}_1, \ldots, \vec{w}_p\}$ are linearly independent, then $\{\vec{v}_1, \ldots, \vec{v}_m, \vec{w}_1, \ldots, \vec{w}_p\}$ are linearly dependent. Suppose WLOG $w_1 \in Span(\{\vec{v}_1, \ldots, \vec{v}_m\})$. Then $w_1 \in W_1$. Therefore $w_1 \in W_1 \cap W_2$. Therefore $w_1 \in Span(\{\vec{u}_1, \ldots, \vec{u}_k\})$. This is a contradiction as $\{\vec{u}_1, \ldots, \vec{u}_k, \vec{w}_1, \ldots, \vec{w}_p\}$ are linearly independent.

     Therefore since $\vec{u}_1, \ldots, \vec{u}_k, \vec{v}_1, \ldots, \vec{v}_m, \vec{w}_1, \ldots, \vec{w}_p$ is a basis of $W_1 + W_2$, and the basis has a finite number of vectors, then $W_1 + W_2$ is finite dimensional.

   - We must show that $dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2)$. Note that $dim(W_1 \cap W_2) = k, dim(W_1) = k + m, dim(W_2) = k + p$, and as we showed above $dim(W_1 + W_2) = k + m + p$.

Therefore:

$$\begin{aligned} dim(W_1 + W_2) &= k + m + p \\ &= k + m + k + p - k \\ &= dim(W_1) + dim(W_2) - dim(W_1 \cap W_2). \end{aligned}$$

2. Let $W_1, W_2$ be subspaces of a vector spaces of $V$, where $dim(W_1) = m, dim(W_2) = n, n \leq m$.

   (a) Prove that $dim(W_1 \cap W_2) \leq n$. Since $W_1 \cap W_2$ is a subspace of $W_2$, and a subset of linearly independent vectors of $W_2$ can have up to $dim(W_2)$ linearly independent vectors, then $dim(W_1 \cap W_2) \leq dim(W_2)$. Therefore $dim(W_1 \cap W_2) \leq n$.

   (b) Prove that $dim(W_1 + W_2) \leq n + m$. By the previous problem, we have that $dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2)$. Note that since $dim(W_1 \cap W_2)$ can be 0, then $dim(W_1) + dim(W_2) - dim(W_1 \cap W_2) \leq dim(W_1) + dim(W_2) = m + n$. Therefore $dim(W_1 + W_2) \leq m + n$.

3. Let $V$ be the set of real numbers regarded as a vector space over the field of rational numbers. Prove that $V$ is infinite-dimensional. By theorem 1.19 we must show that $V$ has an infinite linearly independent subset. Note that $\pi \in \mathbb{R}$, and $\pi$ is transcendental. By definition of transcendental, there does not exist a non-zero polynomial with rational coefficients such that it has a root at $\pi$. Therefore for all $f \in P(\mathbb{Q})\backslash\{0\}$ where $f = \sum_{i=0}^{n} a_i x^i, a_0, \ldots, a_n \in \mathbb{Q}, f(\pi) = \sum_{i=0}^{n} a_i \pi^i \neq 0$. Therefore for each $n$, the only polynomial $f \in P_n(\mathbb{Q})$ such that $f(\pi) = 0$ is the polynomial $f = 0$. Therefore each $P_n(\mathbb{Q})$ has the set $\{1, \pi, \pi^2, \ldots, \pi^n\}$ satisfying the definition of linear independence. Therefore let $S \subset V$ be the collection of the sets of the powers of $\pi$ such that $\{\pi^i$ : for all $i \in [n] \cup \{0\}\}$ is linearly independent. Therefore by the maximal principle $S$ contains the maximal element $\{1, \pi, \pi^2, \ldots\}$. Therefore $\mathbb{R}$ has an infinite linearly independent subset. Thus $V$ is infinite dimensional.

4. Let $S_1, S_2$ be subsets of the vector space $V$, $S_1 \subseteq S_2$. If $S_1$ is linearly independent, and $Span(S_2) = V$, then there exists a basis $\beta$ of $V$ such that $S_1 \subseteq \beta \subseteq S_2$. Let $\beta$ be the maximal element be the maximal set of all subsets of $S_2$ containing $S_1$ and linearly independent. Thus since $\beta$ is a maximal and linearly independent subset of $S_2$, then by theorem 1.12, $\beta$ is a basis of $V$.

5. Define $T : P(\mathbb{R}) \to P(\mathbb{R})$ by $T(f(x)) = \int_0^x f(t)dt$. Prove that $T$ is linear, 1-1, and not onto.

   (a) We must show that $T$ is linear. Suppose $f, g \in P(\mathbb{R}), c \in \mathbb{R}$. Therefore:

$$\begin{aligned} T(cf+g) &= \int_0^x cf(t)+g(t)dt \\ &= c\int_0^x f(t)dt + \int_0^x g(t)dt \\ &= cT(f)+T(g). \end{aligned}$$

   (b) We must show that $T$ is 1-1. Suppose $f, f^* \in P(\mathbb{R}), T(f) = T(f^*)$. We must show $f = f^*$. Since $T$ is linear, then $T(f - f^*) = 0$ implies $\int_0^x f(t) - f^*(t)dt = 0$, however since the only function which integrates to 0 is $f = 0$, then $f(t)-f^*(t)-0$, thus $f(t) = f^*(t)$.

   (c) We must show that $T$ is not onto. We claim that for all $f \in P(\mathbb{R})$ that $T(f) = c, c \in \mathbb{R}$ is impossible. Suppose for contradiction that there exists $f \in P(\mathbb{R})$ such that $\int_0^x f(t)dt = c$. Let $F : \mathbb{R} \to \mathbb{R}$ be given by $F' = f$. Then evaluating the integral above yields $F(x)-F(0) = c$. Therefore $F(x) = F(0)+c$. Differentiating both sides yields $f(x) = 0$. This is a contradiction as $\int_0^x 0 = 0 \neq c$. Therefore $T$ is not onto.