1. - Show that $x(t)$ satisifes $\|A^{-1}x(t)\|^2 = 1$.

$$\begin{aligned}
\|A^{-1}x(t)\|^2 &= \|A^{-1}Au(t)\|^2 \\
&= \|u(t)\|^2 \\
&= \| \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \|^2 \\
&= \cos^2(t) + \sin^2(t) \\
&= 1.
\end{aligned}$$

   - Show that the equation above may be written as $x \cdot Mx = 1$.

$$\begin{aligned}
x \cdot Mx &= x^T Mx \\
&= x^T (A^{-1})^T A^{-1} x \\
&= (A^{-1}x)^T A^{-1} x \\
&= A^{-1}x \cdot A^{-1}x \\
&= \|A^{-1}x\|^2 \\
&= 1.
\end{aligned}$$

   - Show that M is symmetric.

$$\begin{aligned}
M^T &= ((A^{-1})^T A^{-1})^T \\
&= (A^{-1})^T ((A^{-1})^T)^T \\
&= (A^{-1})^T A^{-1} \\
&= M
\end{aligned}$$

   - Suppose $M = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$. We must show that $x \cdot Mx$ can be written as $ax^2 + by^2 + 2cxy$.

$$\begin{aligned}
1 &= x \cdot Mx \\
&= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c \\ c & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} ax + cy \\ cx + by \end{bmatrix} \\
&= ax^2 + by^2 + 2cxy.
\end{aligned}$$

2. - Show that both λ_1 and λ_2 are positive. Since $x \cdot Mx = \|A^{-1}x\|^2$, then all outputs of $x \cdot Mx$ are strictly positive. Suppose $x = u_1$, then $u_1 \cdot Mu_1 = u_1 \cdot \lambda_1 u_1 = \lambda_1 \|u_1\|^2 = \lambda_1 > 0$. A similar proof exists for u_2. Therefore the eigenvalues are strictly positive.
   - Suppose $\lambda_1 = \lambda_2$. We must show that $\|x(t)\| = \frac{1}{\sqrt{\lambda_1}}$. Let U be the matrix where u_1 and u_2 are columns. Since u_1, u_2 are orthonormal, then U is an orthogonal

matrix. Therefore $U^{-1} = U^T$. Let $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, and let $q = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = U^T x$. Since M is diagonalizable we may rewrite $x \cdot Mx$ as follows:

$$1 = x \cdot Mx = x \cdot UMU^T x = q^T U^T UMU^T Uq = q \cdot Dq = \lambda_1 q_1^2(t) + \lambda_2 q_2^2(t).$$

This equation defines an ellipse paramaterized by $q_1(t) = \pm\frac{1}{\sqrt{\lambda_1}}\cos(t), q_2(t) = \pm\frac{1}{\sqrt{\lambda_2}}\sin(t)$. Since $q = U^T x$, then $x = Uq$, therefore we may explicitly solve for x via $x = Uq = \pm(\frac{1}{\lambda_1}\cos(t)u_1 + \frac{1}{\lambda_2}\sin(t)u_2)$. Therefore if $\lambda_1 = \lambda_2$, then $\|x\| = \sqrt{\frac{\cos^2(t)}{\lambda_1} + \frac{\sin^2(t)}{\lambda_2}} = \sqrt{\frac{\cos^2(t)}{\lambda_1} + \frac{\sin^2(t)}{\lambda_1}} = \frac{1}{\sqrt{\lambda_1}}$.

- Consider the case where $\lambda_1 > \lambda_2$. Note that $\|x\|^2 = \frac{\cos^2(t)}{\lambda_1} + \frac{\sin^2(t)}{\lambda_2} = u \cdot \begin{bmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{bmatrix} u$, let the matrix in the quadratic function above be denoted S^2. By lemma 19 in the previous carlen multivarible textbook, $\|x\|^2$ on the unit circle is maximized and minimized by the eigenvalues of S^2. For the matrix S^2 the eigenvalues are obviously $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}$. Since $\lambda_1 > \lambda_2$, then $\frac{1}{\lambda_1} < \frac{1}{\lambda_2}$, therefore making $\frac{1}{\lambda_2}$ the maximum of $\|x\|^2$, and therefore forcing $\frac{1}{\lambda_1}$ to be the minimum. Since $\sqrt{}$ is a monotonically increasing function on $\mathbb{R}_+$, then $\|x\|$ has a maximum of $\frac{1}{\sqrt{\lambda_2}}$ and a minimum of $\frac{1}{\sqrt{\lambda_1}}$. We must show that $\|x\|$ is maximal if and only if $x(t) = \pm\frac{1}{\lambda_2}u_2$.
  - Suppose $x(t) = \pm\frac{1}{\lambda_2}u_2$. We must show that $\|x(t)\|$ is maximal.

$$\|x(t)\| = \sqrt{\frac{\cos^2(t)}{\lambda_1} + \frac{\sin^2(t)}{\lambda_2}} \leq \sqrt{\frac{\cos^2(t)}{\lambda_2} + \frac{\sin^2(t)}{\lambda_2}} = \frac{1}{\sqrt{\lambda_2}} = \| \pm \frac{1}{\sqrt{\lambda_2}}u_2\|$$

  - Suppose $\|x(t)\|$ is maximal. We must show that $x(t) = \pm\frac{1}{\sqrt{\lambda_2}}u_2$. Since $\|x(t)\|$ is maximal and $\|x(t)\|$ has a single maximum then $\|x(t)\| = \frac{1}{\sqrt{\lambda_2}}$. Therefore:

$$\begin{aligned} \|x(t)\|^2 &= \frac{1}{\lambda_2} \\ \frac{\cos^2(t)}{\lambda_1} + \frac{\sin^2(t)}{\lambda_2} &= \frac{1}{\lambda_2} \\ \frac{\cos^2(t)}{\lambda_1} &= \frac{1}{\lambda_2}(1 - \sin^2(t)) \\ \frac{\cos^2(t)}{\lambda_1} &= \frac{\cos^2(t)}{\lambda_2} \\ \cos^2(t)(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}) &= 0. \end{aligned}$$

  Since $\lambda_1 > \lambda_2$, then the only solution to that equation is when $\cos^2(t) = 0$, thus when $\cos(t) = 0$. Therefore evaluating $x(t)$ when $cos(t) = 0$ yields $x(t) = \pm(\frac{1}{\lambda_1}\cos(t)u_1 + \frac{1}{\lambda_2}\sin(t)u_2) = \pm\frac{1}{\lambda_2}\sin(t)u_2 = \pm\frac{1}{\lambda_2}\sqrt{1-\cos^2(t)}u_2 = \pm\frac{1}{\lambda_2}u_2$. The proof for $x(t) = \pm\frac{1}{\lambda_1}u_1$ if and only if $\|x(t)\|$ is minimal is nearly identical to the one above, simply replace maximal with minimal, finding $\sin(t) = 0$, changing the direction of an inequality with substituting λ_2 for λ_1.

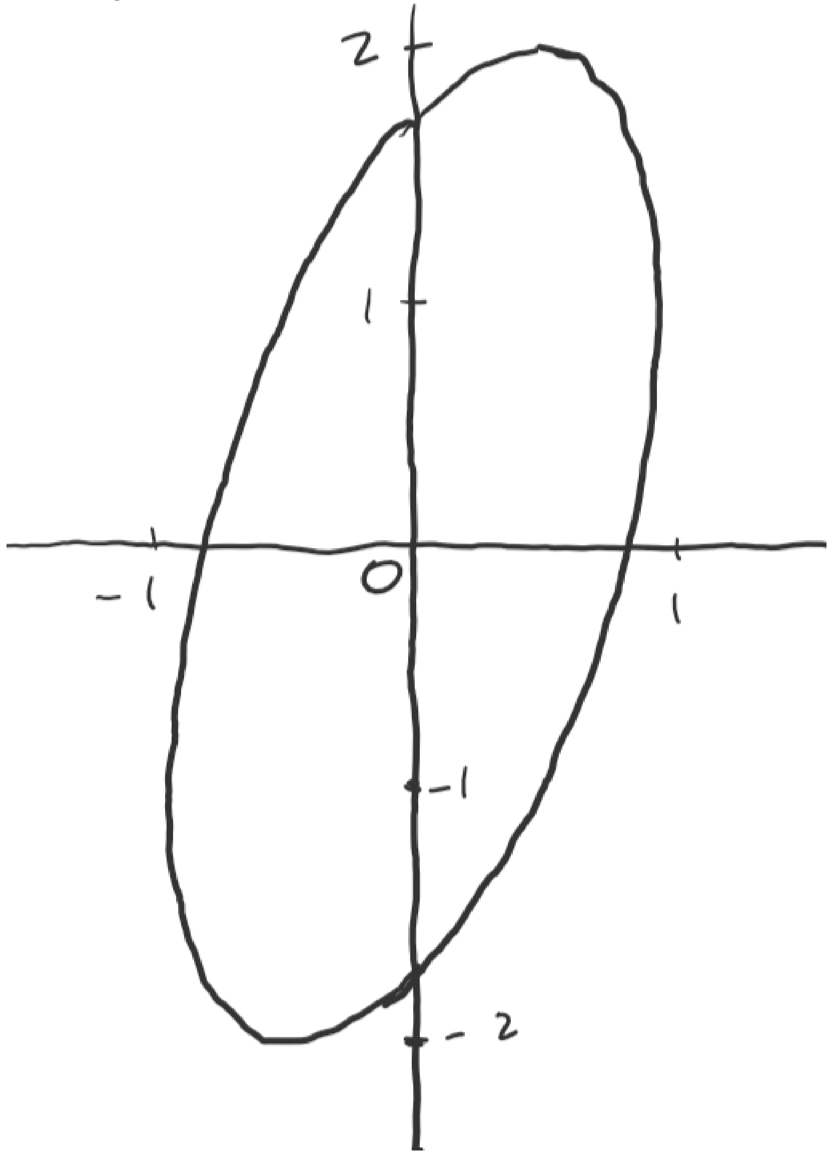
3. Let
$$v_1 = \frac{1}{\sqrt{\lambda_1}} A^{-1} u_1, v_2 = \frac{1}{\sqrt{\lambda_2}} A^{-1} u_2.$$

- Show that $\{v_1, v_2\}$ is an orthonormal basis for $\mathbb{R}^2$. By definition of orthonormal basis we must show that $v_1 \cdot v_2 = 0, v_1 \cdot v_1 = v_2 \cdot v_2 = 1$.
  -
$$\begin{aligned} v_1 \cdot v_2 &= v_1^T v_2 \\ &= \frac{1}{\sqrt{\lambda_1}} (A^{-1} u_1)^T \frac{1}{\sqrt{\lambda_2}} A^{-1} u_2 \\ &= \frac{1}{\sqrt{\lambda_1 \lambda_2}} u_1^T (A^{-1})^T A^{-1} u_2 \\ &= \frac{1}{\sqrt{\lambda_1 \lambda_2}} \lambda_2 u_1^T u_2 \\ &= 0 \end{aligned}$$
  - Let $i \in \{1, 2\}$
$$\begin{aligned} v_i \cdot v_i &= v_i^T v_i \\ &= \frac{1}{\sqrt{\lambda_i}} (A^{-1} u_i)^T \frac{1}{\sqrt{\lambda_i}} A^{-1} u_i \\ &= \frac{1}{\lambda_i} u_i^T (A^{-1})^T A^{-1} u_i \\ &= \frac{1}{\lambda_i} u_i^T M u_i \\ &= \frac{\lambda_i}{\lambda_i} u_i^T u_i \\ &= 1. \end{aligned}$$
- We must show that $\|x(t)\|$ is maximal if and only if $(\cos(t), \sin(t)) = \pm v_2$.
  - Suppose $\|x(t)\|$ is maximal. We must show that $(\cos(t), \sin(t)) = \pm v_2$. We know from exercise 3 that $x(t) = \pm \frac{1}{\sqrt{\lambda_2}} u_2$. Therefore
$$\begin{aligned} x(t) &= \pm \frac{1}{\lambda_2} u_2 \\ A(\cos(t), \sin(t)) &= \pm \frac{1}{\sqrt{\lambda_2}} u_2 \\ (\cos(t), \sin(t)) &= \pm \frac{1}{\sqrt{\lambda_2}} A^{-1} u_2 \\ (\cos(t), \sin(t)) &= \pm v_2. \end{aligned}$$
  - Suppose $(\cos(t), \sin(t)) = \pm v_2$. We must show that $\|x(t)\|$ is maximal. Therefore
$$x(t) = A(\cos(t), \sin(t)) = \pm A v_2 = \pm \frac{1}{\sqrt{\lambda_2}} A A^{-1} u_2 = \pm \frac{1}{\sqrt{\lambda_2}} u_2.$$

Thus by exercise 3 since $x(t) = \pm\frac{1}{\sqrt{\lambda_2}}u_2$ then $\|x(t)\|$ is maximal.

The proof is identical for showing $\|x(t)\|$ is minimal if and only if $(\cos(t), \sin(t)) = \pm v_1$ by swapping 2 for 1 and using the minimal portion of what was proved in exercise 3.

4. Let $A = \begin{bmatrix} 1 & 0 \\ 1 & \sqrt{3} \end{bmatrix}$. Let $M = (A^{-1})^T A^{-1}$ This has the corresponding characteristic polynomial of $(\lambda - \frac{1}{6}(5-\sqrt{13}))(\lambda - \frac{1}{6}(5+\sqrt{13})) = 0$, thus the eigenvalues $\mu_1 = \frac{1}{6}(5+\sqrt{13}), \mu_2 = \frac{1}{6}(5-\sqrt{13})$ Therefore $\sigma_1 = \sqrt{\frac{6}{5+\sqrt{13}}}, \sigma_2 = \sqrt{\frac{6}{5-\sqrt{13}}}$. This gives us the major axis of length of $\sigma_2 = 2.07431$ and the minor axis length of $\sigma_1 = 0.835$. These correspond to $u_1 = (0.957092, 0.289784)$ and $u_2 = (0.289784, -0.957092)$. The angle between u_1 and the x-axis is 0.294001 radians. The ellipse looks like:



5. Let $V = \begin{bmatrix} v_1 & v_2 \end{bmatrix}, S = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}, U = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$ We must show that $\sigma_1(x \cdot v_1)u_1 + \sigma_2(x \cdot$

$v_2)u_2 = USV^Tx$. Therefore

$$\begin{aligned}\sigma_1(x \cdot v_1)u_1 + \sigma_2(x \cdot v_2)u_2 &= U(\sigma_1(x \cdot v_1), \sigma_2(x \cdot v_2)) \\ &= US(v_1 \cdot x, v_2 \cdot x) \\ &= USV^Tx\end{aligned}$$

Therefore $A = USV^T$.

6. Let $A = \begin{bmatrix} 11 & -5 \\ 2 & -10 \end{bmatrix}$ We must find orthogonal matrices U, V and diagonal matrix S such that $A = USV^T$. Note that A^TA has eigenvalues of 200 and 50 and these correspond with the eigenvectors $(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Since $A^TA = VS^2V^T$ then $S = \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix}, V = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. Note that AA^T has eigenvectors of $(\frac{3}{5}, -\frac{4}{5})$ and $(-\frac{4}{5}, -\frac{3}{5})$ and corresponding to the same eigenvalues of 50 and 200. Note that $AA^T = US^2U^T$, thus we have found $U = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{4}{5} \end{bmatrix}$. Verification:

$$\begin{aligned}USV^T &= \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{4}{5} \end{bmatrix} \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{4}{5} \end{bmatrix} \begin{bmatrix} -10 & 10 \\ 5 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 11 & -5 \\ 2 & -10 \end{bmatrix} \\ &= A.\end{aligned}$$