Alex Valentino

Homework 7
350H

1.

2. Suppose $A \in M_{m\times n}(F), rank(A) = m$. We must show that there exists $B \in M_{n\times m}(F)$ such that $\mathbb{I}_m = AB$.
Proof: We know from the first corollary of theorem 3.6 that there exists $L \in GL_m(F), R \in GL_n(F)$ such that $LAR = \begin{bmatrix} \mathbb{I}_r & O_1 \\ O_2 & O_3 \end{bmatrix}$ where $r = rank(A)$ and O_1, O_2, O_3 are zero matrices. Since $rank(A) = m$, and the matrix LAR is $m \times n$ then $LAR = \left[\begin{array}{c|c} \mathbb{I} & O \end{array}\right]$ where O is a $m\times(n-m)$ 0 matrix. Therefore left multplying by L^{-1} yields $AR = \left[\begin{array}{c|c} L^{-1} & O \end{array}\right]$. Let $L' \in M_{n\times m}$ be the matrix given by for all $i \in [n], j \in [m], (L')_{ij} = L_{ij}$ if $j \leq n$ otherwise $(L')_{ij} = 0$. We claim that $RL' = B$. Since $L' \in M_{n\times m}$ and $R \in GL_n(F)$ then $RL' \in M_{n\times m}$. Therefore $AB = ARL' = \left[\begin{array}{c|c} L^{-1} & O \end{array}\right] L'$. Note that by the definition of matrix multiplication and the identity matrix $\delta_{ij} = \sum_{k=1}^{m}(L^{-1})_{ik}L_{kj}$. Therefore each entry in the new matrix D is given by $D_{ij} = \sum_{k=1}^{n}\left[\begin{array}{c|c} L^{-1} & O \end{array}\right]_{ik} L'_{kj}$, since for $\left[\begin{array}{c|c} L^{-1} & O \end{array}\right]_{ik}$ if $k > m$ then the entry is 0 and similarly $L'_{kj} = 0$ by definition means that the matrix multiply reduces to $D_{ij} = \sum_{k=1}^{m}(L^{-1})_{ik}L_{kj} = \delta_{ij}$. Therefore $AB = \mathbb{I}_m$.

3.

4.