Suppose that X, Y are disjoint subsets of $\mathbb{R}$ that are gap-free. Prove that for all $(x, y) \in X \times Y, x < y$, or for all $(x, y) \in X \times Y, y < x$.

Proof: Suppose there exists $(x_0, y_0) \in X \times Y$ such that $x_0 \leq y_0$. We must show that for all $(x, y) \in X \times Y, x < y$. Suppose $(x, y) \in X \times Y$. We must show that $x < y$. Since $x_0 \in X, y_0 \in Y$, and X, Y are disjoint, then $x_0 \neq y_0$. Since $x_0 \leq y_0, x_0 \neq y_0$, then $x_0 < y_0$. We have four cases.

- Assume $x \leq x_0, y_0 \leq y$. Therefore by chaining the inequalities we have $x < y$.

- Assume $x_0 \leq x, y_0 \leq y$.