4.3 a Let the Klein 4 group be defined as $V_4 =< a, b|a^2 = b^2 = (ab)^2 = e >$, then we have the character table:

	$\{e\}$	$\{a, a^2\}$	$\{b, b^2\}$	$\{ab, (ab)^2\}$
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

c Let $D_4 =< r, s|r^4 = s^2 = e, srs = r^{-1} >$ and it's character table as follows:

	$\{e\}$	$\{r^2\}$	$\{r, r^3\}$	$\{s, r^2s\}$	$\{rs, r^3s\}$
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

4.7 Let G be a finite group with $|G| = nm, N \leq G, G' = G/N = \{r_1, \cdots, r_m\}$, with the canonical projection map $\pi : G \to G'$, and let $\rho' : G' \to GL(V)$ be an irreducible representation of G', and $\rho = \rho' \circ \pi$. To directly show that ρ is an irreducible representation of G, assume for contradiction that W is a G-invariant subspace. Then for all $g \in G, v \in W, \rho_g(v) = v$. However, consider for each g, there exists r_i such that $\pi(g) = r_i$. This now implies that $\rho_g(v) = \rho'_{r_i}(v) = v$. This implies that W is a $G' - invariant$ subspace, contradicting that ρ' is irreducible. Now by using theorem 10.4.6, if we consider χ' to be the character of ρ' and χ to be the character of ρ, then

$$\begin{aligned}
< \chi, \chi > &= \frac{1}{nm} \sum_{g \in G} \overline{\chi(g)} \chi(g) \\
&= \frac{1}{nm} \sum_{g \in G} \overline{\chi'(\pi(g))} \chi'(\pi(g)) \\
&= \frac{1}{nm} \sum_{i=1}^{m} n \overline{\chi'(r_i)} \chi'(r_i) \\
&\text{since } |gN| = n \\
&= \frac{1}{m} \sum_{i=1}^{m} \overline{\chi'(r_i)} \chi'(r_i) \\
&=< \chi', \chi' > \\
&= 1
\end{aligned}$$

Since the inner product of χ with itself is 1, then it is necessarily irreducible.

5.5 Let G be a group. Let χ_1, χ_2 by characters for G. If we let $\chi = \chi_1\chi_2$, then for any $g, h \in G$ then $\chi(gh) = \chi_1(gh)\chi_2(gh) = \chi_1(g)\chi_2(h)\chi_1(g)\chi_2(h) = \chi_1(g)\chi_2(g)\chi_1(h)\chi_2(h) = \chi(g)\chi(h)$. Since the set of characters is just the set of homomorphisms from $G \to C^\times$ then χ is a homomorphism as well. Additionally, for every $\chi \in \hat{G}, g \in G$, $\chi(g) \neq 0$, therefore $\frac{1}{\chi(g)}$ exists, giving us that, $\frac{1}{\chi(g)}\frac{1}{\chi(h)} = \frac{1}{\chi(g)\chi(h)} = \frac{1}{\chi(gh)}$, therefore the inverse of

a character is a character as well. Thus the set of characters and their inverses is closed under multiplication. Finally, since the trivial representation exists and the characters are functions over complex numbers the $\hat{G}$ has an identity and associativity. Thus the dual group is a group.
If G is abelian and finite then by the structure theorem we have that $G \cong \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_k}$ where $d_1, \cdots, d_k$ are prime powers. Note for cyclic groups, if g generates $\mathbb{Z}_n$, then $1 = \chi(1) = \chi(g^n) = \chi(g)^n$, which implies that $\chi(g) = e^{\frac{2\pi ikg}{n}}$ where k is some number between 0 and $n-1$. Furthermore since the roots of unity for n are cyclic implies that $\hat{Z}_n$ is isomorphic to Z_n. Therefore $\mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_k} \cong \{e^{\frac{2\pi ij}{d_1}}\}_{j=1}^{d_1} \oplus \cdots \oplus \{e^{\frac{2\pi ij}{d_k}}\}_{j=1}^{d_k}$. Furthermore since $d_1, \cdots, d_k$ are prime powers there's an isomorphism from $\{e^{\frac{2\pi ij}{d_1}}\}_{j=1}^{d_1} \oplus \cdots \oplus \{e^{\frac{2\pi ij}{d_k}}\}_{j=1}^{d_k}$ to $\{e^{\frac{2\pi ij_1}{d_1} + \cdots + \frac{2\pi ij_k}{d_k}}\}_{j_1, \cdots, j_k=1}^{d_1, \cdots, d_k}$, as the sum of fractions with prime powers can uniquely be written that way, thus each element is uniquely mapped. Thus we have shown that G is isomorphic to it's dual group.

5.7 (a) Note since G, G' are finite abelian groups then their homomorphism is isomorphic to a linear transformation from $\mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_k}$ to $\mathbb{Z}_{d'_1} \oplus \cdots \oplus \mathbb{Z}_{d'_l}$. This can be given as a $l \times k$ matrix. Therefore the induced homomorphism is simply the d'_i component in $\mathbb{Z}_{d'_1} \oplus \cdots \oplus \mathbb{Z}_{d'_l}$ after applying ϕ to $(a_1, \cdots, a_k) \in \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_k}$. This is equivalent to $e^{\frac{2\pi i\phi(a_1, \cdots, a_k)_1}{d'_1} + \cdots + \frac{2\pi i\phi(a_1, \cdots, a_k)_l}{d'_l}}$.

(b) We know from linear algebra that the dual $\hat{\phi}$ corresponds to the transpose of the transpose of the linear operation ϕ. Therefore if ϕ is injective implies that ϕ^T is surjective, and ϕ is surjective implies that ϕ^T is injective. Since one can create an isomorphism from ϕ^T and $\hat{\phi}$ implies the desired result.