Let T be a finite tournament on the ground set X (Recall that a tournament is a relation that is anti-reflexive, anti-symmetric and full. Thus for any $x, y \in X$ if $x \neq y$ then exactly one of xTy and yTx holds.) Prove that T is transitive if and only if T has no cycles.

We must show that T is transitive if and only if T has no cycles. Therefore we have two cases.

- Suppose T is transitive. We must show that T has no cycles. Suppose for contradiction that T has a cycle. Since T is transitive, then for all $a, b, c \in X$ if aTb, bTc, then aTc. Since T is a cycle, then let C be the cycle in T. Since C is a cycle, then $|C| \geq 3$. We have two cases:
  - Assume $|C| = 3$. Then there exists $x, y, z \in C$ such that xTy, yTz, zTx, which contradicts the fact that T is transitive.
  - Assume $|C| > 3$. Then there exists $x, y, z \in C$ such that x, y are adjacent and z is not adjacent to y (This is to ensure that x, y, z are not in a row, if they were then by the transitivity of T there would not be a cycle). Note that since x, y are adjacent then xTy. Since T is transitive, and x, y, z are in a cycle together where z is after y, then yTz, and zTx. This is a contradiction since if xTy, yTz, then xTz.
- Suppose T has no cycles. We must show that T is transitive. Suppose $a, b, c \in X, aTb, bTc$. We must show that aTc. Since $a, c \in X$ and T is defined on X, then we have aTc or cTa. However if cTa then we have a cycle, which contradicts the fact that T has no cycles. Therefore aTc.