2.4.1

2.4.6 (a) Suppose (a_n) is a bounded sequence. Prove that the sequence $y_n = \sup\{a_k : k \geq n\}$ converges.
Proof: We claim that (y_n) is a decreasing and bounded sequence.

- We claim that (y_n) is decreasing. Suppose $n \in \mathbb{N}$. We must show that $y_n \geq y_{n+1}$. We have two cases, $y_n = a_n, y_n \neq a_n$.
  - Suppose $a_n = y_n$. Therefore for all other elements in the sequence after a_n, $a_n \geq a_k$ where $k > n$. Therefore a_n is an upper bound on $\{a_k : k \geq n+1\}$. Since y_{n+1} is the supremum of the set mentioned before, and we have established that a_n is an upper bound, then by definition $y_n = a_n \geq y_{n+1}$.
  - Suppose $a_n \neq y_n$. Since a_n is already not a suprememum of the set $\{a_k : k \geq n\}$ then computing the supremum of the set excluding a_n, $\{a_k : k \geq n+1\}$ should not change the supremum value. Therefore $y_n = y_{n+1}, y_n \geq y_{n+1}$.

  Therefore (y_n) is a decreasing sequence.
- We claim that (y_n) is bounded. Since (y_n) is decreasing we simply need to show that there is a quantity larger than y_1. Since $y_1 = \sup\{a_k : k \geq 1\}$ then y_1 is the supremum for (a_n). Since (a_n) is bounded, suppose for some quantity $M \in \mathbb{R}$, then by definition of supremum $y_1 \leq M$. Therefore (y_n) is bounded

Therefore by the monotone convergences theorem (y_n) converges.

(b) Let $z_n = \inf\{a_k : k \geq n\}$. Then $\lim z_n = \liminf a_n$. This should converge since (z_n) can easily be proved to be increasing and bounded.

(c)
- Prove that $\liminf a_n \leq \limsup a_n$
  Proof: Suppose $n \in \mathbb{N}$. For an arbitrary element $e \in \{a_k : k \geq n\}$, $e \leq y_n, e \geq z_n$ by the respective definitions of supremum and infimum. Therefore for all $n \in \mathbb{N}, z_n \leq y_n$. Since we know that $\liminf a_n, \limsup a_n$ exists, then by the algebraic order theorem $\liminf a_n \leq \limsup a_n$.
- An example of a strict inequality between $\liminf a_n$ and $\limsup a_n$ is the sequence $a_n = \frac{1}{n} + (-1)^{n+1}$. This is because $\liminf a_n = \lim \frac{1}{n} - 1 = -1 < 1 = \lim \frac{1}{n} + 1 = \limsup a_n$

(d) We must prove that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists

$\Rightarrow$ Suppose $\liminf a_n = \limsup a_n$. We must show that $\lim a_n$ exists. Note that by definition of infimum and supremum for all $k \in \mathbb{N}, z_k \leq a_k, a_k \leq y_k$. Therefore for all $k \in \mathbb{N}, z_k \leq a_k \leq y_k$. Since $\lim z_n = \lim y_n$, then by the squeeze theorem $\lim a_n = \liminf a_n = \limsup a_n$.

$\Leftarrow$ Suppose $\lim a_n$ exists and suppose for contradiction that $\liminf a_n \neq \limsup a_n$. Since $\liminf a_n \leq \limsup a_n$ then $\liminf a_n < \limsup a_n$. Therefore $0 < \liminf a_n - \limsup a_n$. Let l be the limit point of (a_n). Note that since the inequality between $\liminf a_n$ and $\limsup a_n$ is strict then l can at most converge to either $\liminf a_n$ or $\limsup a_n$, therefore we assume WLOG $l < \limsup a_n$.

Since (a_n) converges to l and $0 < \limsup a_n - l$ then by the definition of convergence there exists $N \in \mathbb{N}$ such that for all $n \geq N, |a_n - l| < \limsup a_n - l$. Furthermore by the definition of convergence there exists $M \in \mathbb{N}$ such that for all $m \geq N + M$, $|a_m - l| < (\limsup a_n - l)/2$. Since there are only finitely many points in the sequence which are closer to $\limsup a_n$ than l for the sequence starting at N then we can upper bound this sequence by taking $M = \max(\{a_N, a_{N+1}, \cdots, a_{N+M-1}, \frac{\limsup a_n + l}{2}\})$. Note that

$$\begin{aligned} l &< \limsup a_n \\ l + \limsup a_n &< 2 * \limsup a_n \\ \frac{l + \limsup a_n}{2} &< \limsup a_n \end{aligned}$$

therefore $M < \limsup a_n$. Since $|a_n - l| < \limsup a_n - l$ for all $n \geq N$, then $a_n < \limsup a_n$. By the definition of supremum, $y_N \leq M$. Therefore $y_N < \limsup a_n$. This is a contradiction as (y_n) is a decreasing sequence, and it would be expected that $y_N \geq \limsup a_n$. Therefore $\liminf a_n = \limsup a_n$.