4.4.11 Suppose g is defined on all of $\mathbb{R}$. Show that g is continuous if and only if $g^{-1}(O)$ is open whenever $O \subseteq \mathbb{R}$ is an open set.

- $\Rightarrow$ Suppose g is continuous, O is an open set. If $g^{-1}(O)$ is empty then it vacuously open, so assume $g^{-1}(O)$ is non-empty. Let $x \in g^{-1}(O)$. Therefore $f(x) \in O$. Thus by the definition of an open set there exists $\epsilon > 0$ such that $V_\epsilon(f(x)) \subseteq O$. Since g is continuous then there exists $\delta > 0$ such that for all $y \in V_\delta(x)$ implies $f(y) \in V_\epsilon(f(x))$. Since every $f(y)$ is contained within O, then $V_\delta(x) \subset g^{-1}(O)$. Therefore $g^{-1}(O)$ is open.

- $\Leftarrow$ Assume if $O \subseteq \mathbb{R}$ and O is open implies $g^{-1}(O)$ is open. We must show that g is continuous. Let $c \in g(O), \epsilon > 0$. Since $V_\epsilon(c)$ is open then by our initial assumption $g^{-1}(V_\epsilon(c))$ is open as well. Since c is defined to be in the image $g(O)$ then there exists $x \in \mathbb{R}$ such that $c = f(x)$. Since $g^{-1}(V_\epsilon(c))$ is open then there exists $\delta > 0$ such that $V_\delta(x) \subseteq g^{-1}(V_\epsilon(c))$. Since $V_\delta(x)$ is a subset of $g^{-1}(V_\epsilon(c))$ then by the definition of $g^{-1}(V_\epsilon(c))$, $y \in V_\delta(x)$ implies $f(y) \in V_\epsilon(c)$. Therefore g is continuous.