Alex Valentino Assignment 4 problem 3
300H

Below several pairs of predicates are given. For each pair do the following:

- Identify the atomic assertions common to each pair of assertions and assign a variable to each of these assertions.
- Find logical expressions for each sentence in terms of the variables.
- Determine whether the first can be logically deduced from the second, and whether the second can be logically deduced from the first. Explain your answers.

1. (a) if $n$ is prime or $n+2$ is prime, then $n^2+2$ is prime or $n^2-2$ is prime

   (b) $n^2+2$ is non-prime and $n^2-2$ is non-prime implies $n$ is non-prime and $n+2$ is non-prime.

   $A :=$ $n$ is prime.
   $B :=$ $n+2$ is prime.
   $C :=$ $n^2+2$ is prime.
   $D :=$ $n^2-2$ is prime.

   (a) $(A \vee B) \Rightarrow (C \vee D)$

   (b) $\neg C \wedge \neg D \Rightarrow \neg A \wedge \neg B$

   $(A \vee B) \Rightarrow (C \vee D) \leftrightarrow \neg(C \vee D) \Rightarrow \neg(A \vee B)$ (Modus Tollens)
   $\neg C \wedge \neg D \Rightarrow \neg A \wedge \neg B \leftrightarrow \neg(C \vee D) \Rightarrow \neg(A \vee B)$ (DeMorgan's rule)
   Therefore the two statements are logically equivalent.

2. (a) For all real numbers $x$, there is a real number $y$ such that $y^2+y+10x=0$ or $x \leq 9$ and there is a real number $z$ such that $z^2+2z+15x=0$.

   (b) For all real numbers $x, x \leq 9$ or there is both a real number $y$ such that $y^2+y+10x=0$ and a real number $z$ such that $z^2+2z+15x=0$.

   $A(x,y) := y^2+y+10x=0$
   $B(x) := x \leq 9$
   $C(x,z) := z^2+2z+15x=0$

   (a) $(\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, (A(x,y) \vee B(x))) \wedge (\exists z \in \mathbb{R}, C(x,z))$

   (b) $(\forall x \in \mathbb{R}, B(x)) \vee (\exists y \in \mathbb{R}, \exists z \in \mathbb{R}, (A(x,y) \wedge C(x,z)))$

   (b) cannot imply (a) as if $A = T, B = T, C = F$ then $T \vee (T \wedge F) = T \vee F = T \not\Rightarrow (T \vee T) \wedge F = T \wedge F = F$. However (a) implies (b) as if $(\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, (A(x,y) \vee B(x))) \wedge (\exists z \in \mathbb{R}, C(x,z))$ is true then $(A \wedge B)$ is true. Since it is an or statement we can choose $\forall x \in RB(x)$ to be true. Since the highest level logical operator in (b) is an or, then $B(x) = T$ satisfies the statement.

3. (a) $f(x) > y$ and $g(y) > x$ implies $f(g(y)) > y$ and $g(f(x)) > x$.

(b) $f(g(y)) \leq y$ implies $f(x) \leq y$, and $g(f(x)) \leq x$ implies $g(y) \leq x$.

$A(x, y) := f(x) > y$
$B(x, y) := g(y) > x$
$C(y) := f(g(y)) > y$
$D(x) := g(f(x)) > x$

(a) $(A(x, y) \wedge B(x, y)) \Rightarrow (C(y) \wedge D(x))$

(b) $(\neg C(y) \Rightarrow \neg A(x, y)) \wedge (\neg D(x) \Rightarrow \neg B(x, y))$

$(a)\neg(A \wedge B) \vee (C \wedge D)$ (Implication definiton)
$(a)((\neg A \vee \neg B) \vee C) \wedge ((\neg A \vee \neg B) \vee D$ (DeMorgan's rule)
$(a)(\neg A \vee (\neg B \vee C)) \wedge (\neg A \vee (\neg B \vee D))$ (or associativity)
$(a)(A \Rightarrow (B \Rightarrow C)) \wedge (A \Rightarrow (B \Rightarrow D))$ $(b)(A \Rightarrow C) \wedge (B \Rightarrow D)$ (Contrapositive)
a does not logically imply b as in the case where $A$ is true and the rest are false then $(T \wedge F) \Rightarrow (F \wedge F) = F \Rightarrow F = T \neq (T \Rightarrow (F \wedge F)) \wedge (F \Rightarrow (F \wedge F)) = (T \Rightarrow F) \wedge (F \Rightarrow F) = F \wedge T = F$. However b implies a, as if $(A \Rightarrow C) \wedge (B \Rightarrow D) = T, (A \Rightarrow C) = T, (B \Rightarrow D) = T$, then since we already know $(A \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$ from problem 2, then we have $T \wedge (A \Rightarrow T) = T \wedge (\neg A \vee T) = T \wedge T = T$.

4. In this pair of assertions, $S, T, V$, and $W$ are all sets.

(a) $S \subseteq T$ if and only if $S \subseteq V$, or $S \subseteq T$ if and only if $W \subseteq T$

(b) $S \subseteq T$ if and only if ($S \subseteq T$ or $W \subseteq T$).

$A(S, T) := S \subseteq T$
$B(S, V) := S \subseteq V$
$C(T, W) := S \subseteq W$

(a) $A(S, T) \Leftrightarrow B(S, V) \vee A(S, T) \Leftrightarrow C(T, W)$

(b) $A(S, T) \Leftrightarrow (B(S, T) \vee C(T, W))$

b doesn't imply a since if $A = F, B = T, C = F$, then $T \Leftrightarrow F \vee F \Leftrightarrow F = F \vee T = T \neq F \Leftrightarrow (T \vee F) = F \Leftrightarrow T = F$. However it does work the other way around, if we take $(b)$ to be true then we have $A \Leftrightarrow (B \vee C)$. Since we have an or statement, then we can just take $B$ out of it, and we're left with the statement $A \Leftrightarrow B$, which if substituted into a as true, then we get $T \vee A \Leftrightarrow C = T$.