1. Let $(W, \leq)$ be a linearly order set.

   - $\Rightarrow$ Suppose W is well ordered. We want to show that there does not exists a descending chain. Suppose for contradiction that there is a sequence $(w_n)_{n\in\mathbb{N}}$ where $w_n > w_{n+1}$. Since W is well order than $\min(w_n)$ exists. Since (w_n) has a minimum then there exists $n' \in \mathbb{N}$ such that for all $n \in \mathbb{N}, w_{n'} \leq w_n$. Note that $w_{n'+1} < w_{n'}$ by the definition of (w_n). Therefore $w_{n'+1} < w_{n'}$ and $w_{n'+1} \geq w_{n'}$. This is a contradiction. Therefore a descending chain does not exists.
   - $\Leftarrow$ Suppose W is not well ordered. We want to show that there exists a descending chain. Since W is not well ordered then there exists a set S where $S \neq \emptyset, S \subseteq W$ where $\min S$ does not exists. Since S is nonempty then there exists $x_1 \in S$. Note that $\min\{x_1\}$ exists, therefore there exists $x_2 \in S$ such that $x_2 < x_1$. Therefore by induction $\{x_1, \cdots, x_n\} \subset S$, since $\min\{x_1, \cdots, x_n\}$ exists. Therefore there exists $x_{n+1} \in S$ such that $x_{n+1} < x_n$. Therefore by induction we have constructed a descending chain.

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10. (a) We claim that $card((\{0,1\}^{\mathbb{N}})^{\mathbb{N}}) = card(\{0,1\}^{\mathbb{N}\times\mathbb{N}})$. Note that $(\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$ is the set of of sequences of infinite binary sequences. Therefore for a given $f \in (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$ we have that $f(n) = (b_{nk})_{k\in\mathbb{N}}$ for all $n \in \mathbb{N}$. If we define $g : (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}) \to \{0,1\}^{\mathbb{N}\times\mathbb{N}}$ by $g(f) = (b_{nk})_{(n,k)\in\mathbb{N}^2}$, then this is clearly a bijection. Thus $card((\{0,1\}^{\mathbb{N}})^{\mathbb{N}}) = card(\{0,1\}^{\mathbb{N}\times\mathbb{N}})$. Therefore:

    $$card(\mathbb{R}^{\mathbb{N}}) = card((\{0,1\}^{\mathbb{N}})^{\mathbb{N}}) = card(\{0,1\}^{\mathbb{N}\times\mathbb{N}}) = card(\{0,1\}^{\mathbb{N}}) = card(\mathbb{R})$$

    .

    (b) Let S be some countable set, and let $X = \{S^n : n \in \mathbb{N}\}$. We want to show that X is countable. Note that S is countable, and therefore S^n is countable by slide 22 of lecture 14. Since S^n and $\mathbb{N}$ is countable, then $\bigcup_{n\in\mathbb{N}}^{\infty} S^n$ is countable. Since $\bigcup_{n=1}^{\infty} S^n = X$, then we're done.

    (c) Note that a polynomial is uniquely determined by it's coefficients. Therefore the set of polynomials over $\mathbb{Z}$ has the same cardinality as all of the finite integer sequences Thus $card(\mathbb{Z}[x]) = card(\{\mathbb{Z}^n : n \in \mathbb{N}\})$. Since $\{\mathbb{Z}^n : n \in \mathbb{N}\}$ is countable then $\mathbb{Z}[x]$ is countable

(d) Note that since $\mathbb{Z}[x]$ is countable and for $p \in \mathbb{Z}[x]$ the set $r(p) = \{p(x) = 0 : x \in \mathbb{C}\}$ is finite, then $\bigcup_{p\in\mathbb{Z}[x]} r(p)$ is countable. Note that this is exactly the set of algebraic numbers.

(e)