6.7
- We want to show that $(2) \cap (x) = (2x)$. Suppose $a \in (2) \cap (x)$. Then there exists $b, c \in \mathbb{Z}[x]$ such that $bx = a, 2c = a$. From the second equivalence of a we know that $2 \mid a$. Therefore $2 \mid cx$. By Euclid's lemma we have that $2 \mid c$ or $2 \mid x$. Since $2 \nmid x$ then $2 \mid c$. Thus there exists $d \in \mathbb{Z}[x]$ such that $a = 2xd$. Thus $a \in (2x)$. The other direction is trivial. Thus $(2) \cap (x) = (2x)$.
- Consider the map $\phi : \mathbb{Z}[x] \to F_2[x] \times \mathbb{Z}$ given by $\varphi(f) = (f \mod 2, f(0))$. Note that $\ker \varphi$ is all of the polynomials which are both divisible by 2 and have x as a factor. This is exactly $(2) \cap (x) = (2x)$, the ideal mentioned above. Since the projection from $\mathbb{Z}[x] \to \mathbb{Z}[x]/(2x)$ is surjective, then by the first isomorphism theorem for rings we have that $\mathbb{Z}[x]/(2x) \cong im\varphi$. Additionally $im\varphi$ satisfies the requirements put forward in the question since trivially if $f(0) = n$ then $\overline{f}(0) \equiv n \mod 2$, since $n \equiv \overline{n} = \overline{f}(n)$.

7.1 Let D be a finite integral domain. Suppose for contradiction that D is not a field. Then there exists $a \in D$ where a is not a unit, $a \neq 0$. Then $aD \subset D$, otherwise there would exists $d \in D$ where $ad = 1$, contradicting the fact that a is not a unit. Since D is finite and aD is not injective then there exists $b, c \in D$ where $b \neq c, ab = ac$. Therefore $a(b-c) = 0$. Since D is a domain and $a \neq 0$ then $b - c = 0$. This contradicts the fact that $b \neq c$, therefore D is a field.

7.2 Let R be a domain. We will show that $R[x]$ is a domain. Suppose for contradiction that $R[x]$ is not a domain. Then there exists $p, q \in R[x], p(x)q(x) = 0$. Since $p, q \in R[x]$ then $p(x) = \sum_{i=0}^{n} p_i x^i, q(x) = \sum_{i=0}^{m} q_i x^i$. Therefore $p_n q_m x^{n+m} = 0$. This implies that $p_n q_m = 0$. Since p, q are of degree n, m respectively then their leading coefficients must be non-zero. This contradicts the fact that R is a domain.

8.1 We will work by cases:

- $(1) = \mathbb{Z}[x]$, and $n \neq 1, (n) \subset (n, x) \subset \mathbb{Z}[x]$, since (n, x) does no contain 1, and entirely contains (n).
- Consider $p(x) \in \mathbb{Z}[x]$. There must exists $r \in \mathbb{Z}$ where $p(r) \neq -1, 0, 1$. Then for any $q(x) \in (p(x), p(r))$, then $p(r)/q(x)$. Thus $1 \notin (p(x), p(r))$.

Thus no principle ideal of $\mathbb{Z}[x]$ is maximal. (credit to brian bi)

8.4 We know that the maximal ideals of $\mathbb{R}[x]$ are the irreducible elements of $\mathbb{R}[x]$. Additionally we have the fact that the only irreducible polynomials in $\mathbb{R}[x]$ are of degree 1 or 2. Therefore we have trivially that there is a bijective correspondence between the polynomials of degree one and the real line. We must look to the quadratic polynomials. The degree 2 polynomials which are irreducible over $\mathbb{R}$ factor over $\mathbb{C}$. Since they factor over $\mathbb{C}$, then they have a complex root. Consider $p(x) \in \mathbb{R}[x], deg(p) = 2, \alpha \in \mathbb{C}, p(\alpha) = 0$. Since $p \in \mathbb{R}[x]$, then it is invariant under complex conjugation, giving us that $\overline{p(x)} = p(\bar{x})$. Thus $\bar{\alpha}$ is the second root of $p(x)$, and $p(x) = (x - \alpha)(x - \bar{\alpha})$. Therefore we can uniquely associate each degree two polynomial to a point in the upper half plane, since it is entirely defined by that single root and it's conjugate.

9.1 Consider the ideal $I = (y^2 + x^3 - 17)$ and the quotient ring $R = \mathbb{C}[x, y]/I$.

- The ideal $(x-1, y-4)$ is maximal since by the theorem 11.9.1 in the textbook, $(x-1, y-4)$ corresponds with the point $(1, 4)$, which satisfies $1+16-17=0$, thus the ideal is maximal in R
- $(x+1, y+4)$ is not maximal since $-1+16-17 \neq 0$, thus the ideal does not correspond to a point on the curve $y^2 = 17 - x^3$
- The ideal (x^3-17, y^2) is not maximal since the zeros of x^3-17 in $\mathbb{C}$ are $\{\sqrt[3]{17}\omega^n : n \in \mathbb{N}\}$ where ω is the third root of unity. Thus it is not maximal since it has a correspondence with 3 points instead of 1.

9.5 Suppose V is the variety for a set of polynomials $\{f_1, \cdots, f_r\}$, and that $I = (f_1, \cdots, f_r)$. Suppose $x \in V$. Then for every $p(x) \in I$, it is a linear combination of polynomials which are all zero at x. Thus $p(x) = 0$. Suppose $x \in \mathbb{C}^n$ has the property that for all $p \in I, p(x) = 0$. Then it is true for the generators that $f_i(x) = 0$ for all $i \in [r]$. Thus $x \in V$. Therefore V and I depend on just each other.