2 Let $z, w \in \mathbb{C}$ with $z = z_1 + iz_2, w = w_1 + iw_2, z_1, z_2, w_1, w_2 \in \mathbb{R}$. Additionally, let $\langle z, w\rangle = z_1w_1 + z_2w_2, (z, w) = z\bar{w}, Re(z, w) = Re(z\bar{w})$. First I must demonstrate that $\frac{1}{2}[(z, w) + (w, z)] = \langle z, w\rangle$:

$$\begin{aligned}
\frac{1}{2}[(z, w) + (w, z)] &= \frac{1}{2}(z\bar{w} + \bar{z}w) \\
&= \frac{1}{2}[(z_1 + iz_2)(w_1 - iw_2) + (z_1 - iz_2)(w_1 + iw_2)] \\
&= \frac{1}{2}(z_1w_1 - iz_1w_2 + iz_2w_1 + z_2w_2 + z_1w_1 + iz_1w_2 - iz_2w_1 + z_2w_2) \\
&= \frac{1}{2}(2z_1w_1 + 2z_2w_2) \\
&= z_1w_1 + z_2w_2 \\
&= \langle z, w\rangle.
\end{aligned}$$

Next I must demonstrate that $Re(z, w) = \langle z, w\rangle$:

$$\begin{aligned}
Re(z, w) &= Re(z\bar{w}) \\
&= Re((z_1 + iz_2)(w_1 - iw_2)) \\
&= Re(z_1w_1 - iz_1w_2 + iz_2w_1 + z_2w_2) \\
&= z_1w_1 + z_2w_2 \\
&= \langle z, w\rangle.
\end{aligned}$$

Thus $\langle z, w\rangle = \frac{1}{2}[(z, w) + (w, z)] = Re(z, w)$

3

7 (a) Let $z, w \in \mathbb{C}$ such that $z = re^{i\theta}$, $\bar{z}w \neq 1$, $r = |z| < 1$, $|w| < 1$. Then we have that

$$\begin{aligned}
1 &\leq \frac{1}{r^2} \\
r^2 + |w|^2 &\leq \frac{1}{r^2}(r^2 + |w|^2) \\
r^2 + |w|^2 &\leq 1 + r^2|w|^2 \\
r^2 - rw - r\bar{w} + |w|^2 &\leq 1 + -rw - r\bar{w} + r^2|w|^2 \\
r^2 - rw - r\bar{w} + (-w)(-\bar{w}) &\leq 1 + -rw - r\bar{w} + (-rw)(-r\bar{w}) \\
(r - w)(r - \bar{w}) &\leq (1 - rw)(1 - r\bar{w})
\end{aligned}$$

Note the inequality is strict if one assumes that $r^2 < 1$ which implies that $r < 1$. Thus

$$\frac{r - w}{1 - r\bar{w}}\frac{r - \bar{w}}{1 - rw} < 1$$

$$\frac{r-w}{1-r\bar{w}}\frac{r-\bar{w}}{1-rw} < 1$$
$$\|\frac{r-w}{1-r\bar{w}}\|^2 < 1$$
$$\|\frac{r-w}{1-r\bar{w}}\| < 1$$
$$\|\frac{r-w}{1-r\bar{w}}\| < 1$$

(b) For a fixed $w \in \mathbb{D}$ let $F(z) = \frac{w-z}{1-\bar{w}z}$

i. We know that F maps from $\mathbb{D} \to \mathbb{D}$ by the proof above. F being holomorphic is equivalent to $\frac{\partial F}{\partial \bar{z}} = 0$ since

$$\frac{\partial F}{\partial \bar{z}} = \frac{0 \cdot (1-\bar{w}z) - 0 \cdot (w-z)}{(1-\bar{w}z)^2} = 0$$

then F is holomorphic.

ii. To show that F swaps 0 and w:
- $F(0) = \frac{w-0}{1-0\bar{w}} = \frac{w}{1} = w$.
- $F(w) = \frac{w-w}{1-\bar{w}w} = 0$. Note that $|w| < 1$ thus $1 - \bar{w}w \neq 0$.

iii. Note by the proof in a equality is attained when $r = 1$, which implies if $|z| = 1$ then $|F(z)| = 1$.

iv. Note that

$$\begin{aligned} F \circ F(z) &= \frac{w - \frac{w-z}{1-\bar{w}z}}{1 - \bar{w}\frac{w-z}{1-\bar{w}z}} \\ &= \frac{w - |w|^2 z - w + z}{1 - |w|^2} \\ &= \frac{z - |w|^2 z}{1 - |w|^2} \\ &= z. \end{aligned}$$

which holds if $|z| \leq 1$ since $|w| < 1$ and $|\bar{w}^{-1}| > 1$, thus ensuring the denominator is never 0. Furthermore this implies that F is bijective since we have found an inverse.

9

10 need to clarify if we can swap the order of the x and y partial derivatives

13 Assume $f : \Omega \to \mathbb{C}$ is holomorphic and Ω is an open set. Let $F : \mathbb{R}^2 \to \mathbb{C}$ where $f(x+iy) = F(x,y) = u(x,y) + iv(x,y)$.

(a) If $Re(f)$ is constant then u is constant. Thus $\partial_x u = \partial_y u = 0$. This implies by Cauchy-Riemann that $\partial_x v = \partial_y v = 0$. This implies that v is constant. Thus f is constant

(b) If $Im(f)$ is constant then v is constant. Thus $\partial_x v = \partial_y v = 0$. This implies by Cauchy-Riemann that $\partial_x u = \partial_y u = 0$. This implies that u is constant. Thus f is constant

(c) If $|f|$ is constant than $u^2 + v^2$ is constant. Thus $\partial_x(u^2 + v^2) = \partial_y(u^2 + v^2) = 0$ giving us the equations

$$2u\partial_x u + 2v\partial_x v = 0$$
$$2u\partial_y u + 2v\partial_y v = 0$$

Note that this can be expressed in the form

$$\begin{bmatrix} \partial_x u & \partial_x v \\ \partial_y u & \partial_y v \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Observe that the matrix being used is the transpose of the jacobian, and we have found an element in it's null space. Thus $\det J_F = 0$. Therefore $|f'(z)| = 0$. Thus f is constant.

14

16 (a)

(b)

(c)

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