5.2.8 (a) If a derivative function is not constant, then the derivative must take on some irrational values.

True. Suppose $f : A \to \mathbb{R}$, $f'(x)$ exists, $f'(x) \neq c$. Since $A$ by definition is an interval for $f'$ to be well defined then if $A$ is a closed interval $[a, b]$ then by the darboux theorem $f'$ attains all values between $f'(a)$ and $f'(b)$. Since $f'(a)$ and $f'(b)$ are real numbers then by the density of the irrationals in $\mathbb{R}$ there exists $i$ such that (WLOG) $f'(a) < i < f'(b)$. Therefore $f'$ always attains an irrational. If the interval is open, then we can find the midpoint, $m = \frac{a+b}{2}$, take the open ball around $m$, $V_\epsilon(m)$ guaranteed by it's entry in an open set, then take take the set $[m - \epsilon/2, m + \epsilon/2]$, which we know contain the end points as $m - \epsilon < m - \epsilon/2$ and $m + \epsilon/2 > m + \epsilon/2$, thus constructing a closed interval on which $f'$ is defined.

(b) If $f'$ exists on an open interval, and there is some point $c$ where $f'(c) > 0$, then there exists a $\delta$-neighborhood $V_\delta(c)$ around $c$ in which $f'(x) > 0$ for all $x \in V_\delta(c)$).

False, consider the function

$$f(x) = \begin{cases} x + x^2 \sin(e^{1/|x|}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

at 0. If we evaluate the derivative manually we find $\lim_{x\to 0} 1 + x \sin(e^{1/|x|})$, which similar to evaluating $1 + x \sin(1/x)$ converges to 1 by the squeeze theorem. However with the actual evaluation of $f'(x) = 1 + 2x \sin(e^{1/|x|}) - e^{1/|x|} \cos(e^{1/|x|})$ clearly the $e^{1/|x|} \cos(e^{1/|x|})$ term will fluctuate wildly when approaching 0, contradicting that an open neighborhood around 0 will be purely positive.

(c) If $f$ is differentiable on an interval containing zero and if $\lim_{x\to 0} f'(x) = L$, then it must be that $L = f'(0)$

True. Suppose for contradiction that $f'(0) \neq L$. Then there exists $\epsilon_0 > 0$ such that $|f'(0) - L| > \epsilon_0$. Since $f'$ converges to $L$, then there exists $\delta_0 > 0$ such that $x \in V_{\delta_0}(0)$ implies $|f(x) - L| < \epsilon_0/2$. Since $f$ is differentiable on $[-\delta_0/2, 0]$, then by darboux's theorem $f'$ attains all values between $f'(-\delta_0/2)$ and $f'(0)$. This is a contradiction as for every x value in $(-\delta_0/2, 0)$, $|f'(x) - L| < |f'(0) - L|/2$, however all values between $f'(-\delta_0/2)$ and $f'(0)$ must be attained.

(d) The question above without the requirement of the limit existing

False, take $f(x) = \frac{x^2 - x}{x}$. Clearly $\lim_{x\to 0} \frac{x^2 - x}{x} = \lim_{x\to 0} \frac{x(x-1)}{x} = -1$, however directly evaluating $f(0)$ is undefined.