Alex Valentino Assignment 7 problem 1
300H

Prove: for any function $f : S \to T$ and $g : T \to S$, $|S| > 1$, if g is a left inverse for f, then g is a right inverse for f if and only if g is the unique left inverse for f.
Suppose $f : S \to T$ and $g : T \to S$ are arbitrary functions on arbitrary sets S, T and that g is a left inverse for f. We must show that g is a right inverse for f if and only if g is the unique left inverse for f. By definition of if and only if, we must show that if g is a right inverse for f then g is the unique left inverse for f and if g is the unique left inverse for f then g is a right inverse for f.

- Assume g is a right inverse for f. Therefore we must show that g is a unique left inverse for f. By definition we must show that if g is a left inverse of f and g' is a left inverse of f then $g = g'$. Suppose $g' : T \to S$ is a left inverse of f. We must show that $g = g'$. By definition of left inverse, for all $g' \circ f = id_S$. Composing $g' \circ f$ with g yields, $(g' \circ f) \circ g = id_S \circ g$. By definition of the identity function $(g' \circ f) \circ g = g$. By the associativity of function composition $g' \circ (f \circ g) = g$. By the definition of right invertibility $g' \circ id_T = g$. By the definition of the identity function $g' = g$.

- Suppose g is not the right inverse of f. We must show that g is not the unique left inverse of f. Suppose that there exist $g' : T \to S$ which is another left inverse of f. By the definition of not being a right inverse, $f \circ g \neq Id_T$. Composing g' with both sides yields: $g' \circ (f \circ g) \neq g' \circ Id_T$. By the associativity of composition and definition of left inverse we have $Id_S \circ g \neq g' \circ Id_T$. By the definition of the identity function, $g \neq g'$. Therefore g is not the unique left inverse of f.