4.8 *Prove that a set $B = (v_1, \cdots, v_n)$ of vectors in $F^n$ is a basis if and only if the matrix obtained by assembling the coordinate vectors of $v_i$ is invertible*

- $\Rightarrow$ Suppose the set $(v_1, \cdots, v_n)$ is a basis of $F^n$. We want to show that the matrix given by $(v_1, \cdots, v_n)$ is invertible. Since $B$ is a basis, then it is linearly independent. Therefore, it has a non-zero determinate. Therefore, since there are $n$ vectors with $n$ elements, then the matrix given by $B$ is invertible
- $\Leftarrow$ Suppose the matrix represented by $B$ in $F^n$ is invertible. Since the matrix is invertible then it has a non-zero determinate. Therefore it's columns are linearly independent. Since there is a set of $n$ linearly independent vectors in $F^n$, then it must span $F^n$. Therefore $B$ is a basis.

5.2 (a) The base change matrix is given by:
$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$

(b) The matrix which swaps all of the basis vectors to the opposite order is given by:

$$\begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix}$$

6.1 *Prove that $M_n(\mathbb{R})$ is the direct sum of the space of symmetry and the space of skew symmetric matrices*

- We will show that the intersection of the set of symmetric matrices and skew symmetric matricies is exactly the 0 matrix. Suppose for contradiction that there is a non-zero matrix where is both skew symmetric and symmetric. Let this matrix be denoted $E$. Then $E = E^t, E = -E^t$. Therefore $2E = 0$. Therefore $E$ is identically the 0 matrix, giving us the desired contradiction. Therefore the only element shared between the two subspaces is the 0 matrix.
- Suppose $A \in M_n(\mathbb{R})$. We will show there exists a symmetric matrix $S$ and a skew symmetric matrix $K$ such that $A = S + K$. We claim that $S = (\frac{a_{ij}+a_{ji}}{2}), K = (\frac{a_{ij}-a_{ji}}{2})$ satisfy the requirements.
  - We claim that $S$ is symmetric. Note that $(S)_{ij} = \frac{a_{ij}+a_{ji}}{2}$ and $(S)_{ji} = \frac{a_{ji}+a_{ij}}{2}$. Therefore $(S)_{ij} = (S)_{ji}$.
  - We claim that $K$ is skew symmetric. Note that $(K)_{ij} = \frac{a_{ij}-a_{ji}}{2}$ and that $(K)_{ji} = \frac{a_{ji}-a_{ij}}{2}$. Therefore $-(K)_{ij} = -\frac{a_{ij}-a_{ji}}{2} = \frac{a_{ji}-a_{ij}}{2} = K_{ji}$. Thus $K$ is skew symmetric.
  - We claim that $A = S + K$. Note that

$$(A)_{ij} = a_{ij} = \frac{2a_{ij}}{2} = \frac{a_{ij}+a_{ji}}{2} + \frac{a_{ij}-a_{ji}}{2} = (S)_{ij} + (K)_{ij}$$

and
$$(A)_{ji} = a_{ji} = \frac{2a_{ji}}{2} = \frac{a_{ji} + a_{ij}}{2} + \frac{a_{ji} - a_{ij}}{2} = (S)_{ji} + (K)_{ji}$$
Therefore $A = S + K$.

1.1 Suppose $A \in M_{l\times m}(F), B \in M_{n\times p}(F)$. We want to show for $M \in M_{m\times n}(F)$ that $T(M) = AMB$ is linear. Suppose $c \in F, N \in M_{m\times n}(F)$. Therefore

$$\begin{aligned} T(cM + N) &= A(cM + N)B \\ &= (AcM + AN)B \\ &= AcMB + ANB \\ &= cAMB + ANB \\ &= cT(M) + T(N). \end{aligned}$$

Thus $T$ is linear.

1.3 *Show using the dimension theorem that the dimension of the solutions to AX=0 is atleast $n - m$.* Note that $AX = 0$ is exactly the kernel of $A$. Therefore by applying the dimension theorem we have that $n = dim(im(T)) + dim(ker(T))$. Note that the rank is maximized by $m$, giving the formula $n \leq m + dim(ker(T))$. Therefore

$$dim(ker(T)) \geq n - m.$$

2.1 *Given two matrices* $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in F^{2\times 2}$ *show that for arbitrary* $M \in F^{2\times 2}$ *the matrix representation of* $AMB$.
The matrix representation on the unit elements of $F^{2\times 2}$ with the ordered basis of $(e_{11}, e_{12}, e_{21}, e_{22})$ is the following:

$$\begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{21}b_{11} & a_{21}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{21}b_{21} & a_{21}b_{22} \\ a_{12}b_{11} & a_{12}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{12}b_{21} & a_{12}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}$$