Alex Valentino Assignment 6 problem 3
300H

Claim: for $n \in \mathbb{Z}_+$, if a is the exponent of the largest power of two that divides n and b is $\lceil \frac{n}{2^{a+1}} \rceil$ then the function $f : \mathbb{Z}_+ \to \mathbb{R}$ given by $f(n) = \frac{a}{b}$ has a range of $\mathbb{Q}_{\geq 0}$
Proof: We must show that f has a range of $Q_{\geq 0}$. Suppose n is an arbitrary natural number. Then by definition it has a prime factorization $n = 2^a p_1^{a_1} \cdots p_k^{a_k}$. By definition of $range(f) = \mathbb{Q}_{\geq 0}$ we must show for all $n \in \mathbb{Z}_+, f(n) = \frac{a}{b}$ where $a \in \mathbb{Z}_{\geq 0}$ and $b \in \mathbb{Z}_+$.

- (Proof of $a \in \mathbb{Z}_{\geq 0}$): We claim that the exponent of 2 in n's prime factorization has a range of $\mathbb{Z}_{\geq 0}$. Since $\mathbb{Z}_+$ contains the sequence $\{2^i\}_{i=0}^{\infty}$, and 2 is prime, then every natural number is a multiple of that series. Thus the range of a is $\mathbb{Z}_+$ since f is defined for every member of that sequence.

- (Proof of $b \in \mathbb{Z}_+$): By substituting in n's prime factorization in b we get $b = \lceil \frac{2^a p_1^{a_1} \cdots p_k^{a_k}}{2^{a+1}} \rceil = \lceil \frac{p_1^{a_1} \cdots p_k^{a_k}}{2} \rceil$. Since $p_1^{a_1} \cdots p_k^{a_k}$ represents every odd natural number, as all primes greater than 2 are odd, then we can say by definition $p_1^{a_1} \cdots p_k^{a_k} = 2l - 1, l \in \mathbb{Z}_+$, therefore b becomes $\lceil \frac{2l-1}{2} \rceil$. Since the ceiling function is the closest integer greater than or equal to it's input, and $\lceil \frac{2l-1}{2} \rceil = \lceil l - \frac{1}{2} \rceil$, then $\lceil l - \frac{1}{2} \rceil = l$, as l is the smallest integer larger than $l - \frac{1}{2}$. Therefore since l is an arbitrary positive integer, the range of b is $\mathbb{Z}_+$.

Therefore since a and b independently range over $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_+$, then $range(f) = \mathbb{Q}_{\geq 0}$.