1. We must solve the equation $u''(x) - \frac{2}{x^2}u = \frac{3}{x^3}, u(1) = u(2) = 0$.

   (a) Since there is no u' term, then $P(x) = 0, p(x) = e^{\int_x^a 0ds} = 1$, thus the equation is already in the form $\mathcal{L}u(x) = \frac{3}{x^3}$ where $\mathcal{L}u(x) = u''(x) - \frac{2}{x^2}u$.

   (b) First we must find solutions to $\mathcal{L}u = 0$. If we suppose $u = x^\alpha$, then we find that $x^2, \frac{1}{x}$ are solutions. If we attempt to solve with the constraint $u(1) = u(2) = 0$, then we get the constant solution. Therefore the solution will be uniquely solved by $u(x) = \int_a^b G(x,y)f(y)dy$. By the super position principle, we can generate u_1, u_2 such that $u_1(1) = u_2(2) = 0$, satisfying the constraints on the simplified Green's function formula. They are the following:

   $$u_1(x) = x^2 - \frac{1}{x}, u_2(x) = x^2 - \frac{8}{x}.$$

   Now we can generate the Green's function:

   $$G(x,y) = \frac{1}{27}\begin{cases} (y^2 - \frac{1}{y})(x^2 - \frac{8}{x}) & y \geq x \\ (x^2 - \frac{1}{x})(y^2 - \frac{8}{y}) & y < x \end{cases}$$

   Therefore we may now compute the final solution $u(x)$:

   $$\begin{aligned} u(x) &= \int_1^2 G(x,y)f(y)dy \\ &= \frac{1}{7}\left(\left(x^2 - \frac{1}{x}\right)\int_1^x \left(y^2 - \frac{8}{y}\right)\frac{1}{y^3}dy + \left(x^2 - \frac{8}{x}\right)\int_x^2 \left(y^2 - \frac{1}{y}\right)\frac{1}{y^3}dy\right) \\ &= \frac{(x^3 - 1)\log(2) - 7\log(x)}{7x}. \end{aligned}$$

2. We must solve the equation $u''(x) - (\frac{x+2}{x})u'(x) + \frac{2}{x}u(x) = x^2e^x, u(1) = u(2) = 0$.

   (a) Since $P(x) = -(1 + \frac{2}{x})$ then $p(x) = e^3x^{-2}e^{-x}$. Therefore our equation in the Sturm–Liouville form is:

   $$(x^{-2}e^{3-x}u)' + \frac{2e^{3-x}}{x^3}u = e^3$$

   (b) We first must solve $\mathcal{L}u = 0$. If we suppose that there is a solution of the form $e^{\alpha x}$ we find that we get the polynomial $(\alpha - 2)(\alpha - 1) = 0$. After testing we find that e^x is a valid solution. To find an additional linearly independent solution to $\mathcal{L}u = 0$, we can apply the variation of constants formula where $v(x) = \int \frac{x^2e^x}{e^{2x}}dx = \int x^2e^{-x}dx = -e^{-x}(x^2 + 2x + 2)$, giving us the solution $-x^2 - 2x - 2$. If we attempt to solve with the constraints $u(1) = u(2) = 0$ we find that we get the constant solution. Therefore we have a unique solution. We can however generate u_1, u_2 such that $u_1(1) = u_2(2) = 0$ to generate a Green's function. They are the following:

   $$u_1(x) = 5e^x - e(x^2 + 2x + 2), u_2(x) = 10e^x - e^2(x^2 + 2x + 2).$$

Now we can generate the Green's function:

$$G(x,y) = \frac{1}{10e^4 - 5e^5}\begin{cases}(5e^y - e(y^2+2y+2))(10e^x - e^2(x^2+2x+2)) & y \geq x \\ (5e^x - e(x^2+2x+2))(10e^y - e^2(y^2+2y+2)) & y < x\end{cases}$$

Therefore we may now compute the final solution $u(x)$:

$$\begin{aligned}u(x) &= \int_1^2 G(x,y)f(y)dy \\ &= \frac{1}{10e-5e^2}\Bigg((5e^x - e(x^2+2x+2))\int_1^x (10e^y - e^2(y^2+2y+2))dy \\ &+ (10e^x - e^2(x^2+2x+2))\int_x^2 (5e^y - e(y^2+2y+2))dy\Bigg) \\ &= \frac{5e^x\left(-2x^3 + e\left(x^3-8\right)+2\right) + 7e^2(x(x+2)+2)}{15(e-2)}\end{aligned}$$

3. We must solve the equation $u''(x) - \frac{6}{x^2}u = x^2, u(1) = u(2) = 0$.

    (a) Since there is no u' term then $P(x) = 0$. Therefore $p(x) = 1$. Thus our equation is already in Sturm-Liouville form, yielding $\mathcal{L}u = x^2$.

    (b) We must first compute solutions to $\mathcal{L}u = 0$. If we guess the solution to take the form x^α we get the polynomial $(\alpha-3)(\alpha+2) = 0$. After testing we find that x^3 and $\frac{1}{x^2}$ are both valid solutions. If we attempt to solve with the constraints $u(1) = u(2) = 0$ we find that only the constant solution satisifes the equation. Therefore the solution will be uniquely solved by $u(x) = \int_a^b G(x,y)f(y)dy$. By the super position principle, we can generate u_1, u_2 such that $u_1(1) = u_2(2) = 0$, satisfying the constraints on the simplified Green's function formula. They are the following:

    $$u_1(x) = x^3 - \frac{1}{x^2}, u_2 = x^3 - \frac{32}{x^2}.$$

    Now we can generate the Green's function:

    $$G(x,y) = \frac{1}{155}\begin{cases}(y^3 - \frac{1}{y^2})(x^3 - \frac{32}{x}) & y \geq x \\ (x^3 - \frac{1}{x^2})(y^3 - \frac{32}{y^2}) & y < x\end{cases}$$

    Therefore we may now compute the final solution $u(x)$:

    $$\begin{aligned}u(x) &= \int_1^2 G(x,y)f(y)dy \\ &= \frac{1}{155}\left((x^3 - \frac{32}{x^2})\int_x^2 (y^3 - \frac{1}{y^2})y^2dy + (x^3 - \frac{1}{x^2})\int_1^x (y^3 - \frac{32}{y^2})y^2dy\right) \\ &= \frac{32 - 63x^5 + 31x^6}{186x^2}\end{aligned}$$

4. We must solve the equation

$$u(x) = \begin{cases} u'' - u' - 6u = e^x \\ u'(0) = 0 \\ u'(1) = 0 \end{cases}$$

In order to apply the variation of constants formula, we need two linearly independent solutions of $u'' - u' - 6u = 0$. If we guess a form of $e^{\alpha x}$ then we find that e^{3x} and e^{-2x} are valid choices of α, and let them be respectively denoted u_1 and u_2. Therefore we may compute u_p.

$$u_p(x) = \frac{-1}{5} \int_0^x e^{3s-2x} - e^{3x-2s} ds = \frac{1}{30}(2e^{-2x} - 5e^x + 3e^{3x}).$$

Therefore $u(x) = a_1 e^{3x} + a_2 e^{-2x} + \frac{1}{30}(2e^{-2x} - 5e^x + 3e^{3x})$. Note that the e^{3x} and e^{-2x} terms of u_p can be rolled into the constants a_1, a_2. Therefore $u(x) = c_1 e^{3x} + c_2 e^{-2x} - \frac{e^x}{6}$ We now must solve for c_1, c_2. Applying the initial conditions we get the system:

$$\begin{aligned} 3c_1 - 2c_2 &= \frac{1}{6} \\ 3e^3 c_1 - 2e^{-2} c_2 &= \frac{e}{6} \end{aligned}$$

Solving yields $c_1 = \dfrac{1 + e + e^2}{18\left(1 + e + e^2 + e^3 + e^4\right)}, c_2 = -\dfrac{e^3(1+e)}{12\left(1 + e + e^2 + e^3 + e^4\right)}$.
Therefore

$$u(x) = \frac{1 + e + e^2}{18\left(1 + e + e^2 + e^3 + e^4\right)} e^{3x} - \frac{e^3(1+e)}{12\left(1 + e + e^2 + e^3 + e^4\right)} e^{-2x} - \frac{e^x}{6}$$