2.6 *Let G be a group. Define an opposite group G^o with law of composition $a * b$ as follows: The underlying set is the same as G, but the law of composition is $a * b = ba$. Prove that G^o is a group.*

- Associativity: For all $a, b, c \in G^o$

$$a * (b * c) = a * (cb) = cba = c(ba) = (ba) * c = (a * b) * c$$

- Identity: For all $a \in G^o$

$$a * e = ea = a = ae = e * a$$

- Inverses: For all $a \in G^o$

$$a * a^{-1} = a^1 a = e = aa^{-1} = a^{-1} * a$$

4.9 *How many elements of order 2 does the symmetric group S_4 contain?*
There are 6 elements of order 2 in S_4, they are all of the transpositions, which is equivalent to $\binom{4}{2} = 6$. Note that for $a, b \in [4]$ if $a \neq b$, then $(ab)(ab)$, when evaluated on a yields $(ab)(ab)a = (ab)b = a$ and on b becomes $(ab)(ab)b = (ab)a = b$. Therefore $(ab)(ab) = 1$.

6.1 *Let G' be the group of real matrices of the form $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$. Is the map $\mathbb{R}^+ \to G'$ that sends x to this matrix an isomorphism?*
Let $f : \mathbb{R}^+ \to G'$ be given by $x \mapsto \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$.

- f is a homomorphism:
$f(x)f(y) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x+y \\ 0 & 1 \end{bmatrix} = f(x+y)$
- f is injective:
Suppose $x, y \in \mathbb{R}^+, f(x) = f(y)$. Then

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}$$

$$x = y.$$

- f is surjective:
Suppose $A \in G'$. Then $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, where $a \in \mathbb{R}$. Therefore $f(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$.

6.4 *Prove that in a group, the products ab and ba are conjugate elements.*
Note that $(ba^{-1})ab(b^{-1}a) = b(a^{-1}a)(bb^{-1})a = ba$ Therefore ab and ba are conjugates of each other by the element ba^{-1}.

7.1 *Let G be a group. Prove that the relation $a \sim b$ if $b = gag^{-1}$ for some g in G is an equivalence relation on G*

- Reflexivity: Suppose $a \in G$. Note that $eae^{-1} = a$. Therefore $a \sim a$.
- Symmetry: Suppose $a \sim b$. Then there exists $g \in G$ such that $gag^{-1} = b$. Then $g^{-1}bg = g^{-1}gag^{-1}g = a$
- Transitivity: Suppose $a, b, c \in G, a \sim b, b \sim c$. Then there exists $g, h \in G$ such that $gag^{-1} = b, hbh^{-1} = c$.
  Then $hgag^{-1}h^{-1} = hbh^{-1} = c$. Thus a c by the element gh.

8.1 *Let H be the cyclic subgroup of the alternating group A_4 generated by the permutation* (123). *Exhibit the left and the right cosets of H explicitly.*

- Right cosets:
  - H acting upon the identity operation (123) or (132): $\{e, (123), (132)\}$
  - H acting upon $(12)(34), (234)$ or (134): $\{(12)(34), (234), (134)\}$
  - H acting upon $(13)(24), (124)$ or (243): $\{(13)(24), (124), (243)\}$
  - H acting upon $(14)(23), (142)$ or (143): $\{(14)(23), (142), (143)\}$
- Left cosets:
  - The identity operation, (123), or (132) acting upon H: $\{e, (123), (132)\}$
  - $(12)(34), (243)$ or (143) acting upon H: $\{(12)(34), (243), (143)\}$
  - $(13)(24), (142)$ or (234) acting upon H: $\{(13)(24), (142), (234)\}$
  - $(14)(23), (124)$ or (134) acting upon H: $\{(14)(23), (124), (134)\}$

8.12 *Let S be a subset of a group G that contains the identity element 1, and such that the left cosets aS, with a in G, partition G. Prove that S is a subgroup of G*

- Identity: We assume this.
- Closure: Suppose $a, b \in S$. We want to show that $ab \in S$. Note that $ab \in aS$. Since the cosets of S partition G, then $a \in aS$ implies that $aS = S$. Therefore $ab \in S$.
- Inverses: Suppose $a \in S$. We want to show that $a^{-1} \in S$. Note that $a^{-1}S$ contains the identity. Since the cosets of S partition G, and $e \in S, a^{-1}S$ then that implies $S = a^{-1}S$. Therefore $a^{-1} \in S$.

9.7 *Determine the order of each of the matrices* $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

- A has order 3

$$A^3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- B has order 8

$$
\begin{aligned}
B^8 &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^7 \\
&= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^6 \\
&= \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^5 \\
&= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^4 \\
&= \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^3 \\
&= \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$