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Let r_n be the sequence defined by the recurrence $r_1 = 1$ and for $n \geq 2$, $r_n = r_{n-1} + 1/r_{n-1}$. Prove that for all $n \geq 1$, $r_n \geq \sqrt{2n-1}$.

Proof: We must show for all $n \geq 1, r_n \geq \sqrt{2n-1}$. By the principle of mathematical induction for all $k \in \mathbb{N}$ if $k > n$ then $r_k \geq \sqrt{2k-1}$. We now have two cases:

- Assume $n = 1$. Then since $r_1 = 1$, we have $r_1 = 1 = \sqrt{1} = \sqrt{2 * 1 - 1}$.
- Assume $n > 1$. Since $n - 1 < 1$ by the induction hypothesis we have $r_{n-1} = \sqrt{2n-3}$. By the definition of r_n we have $r_n = r_{n-1} + 1/r_{n-1}$. Therefore by proposition 10.16 we have $r_{n-1} + 1/r_{n-1} \geq \sqrt{2n-3} + 1/\sqrt{2n-3}$. Therefore:

$$\begin{aligned}
r_n &\geq \sqrt{2n-3} + 1/\sqrt{2n-3} \\
r_n^2 &\geq (\sqrt{2n-3} + 1/\sqrt{2n-3})^2 \\
&= 2n - 1 + 1/(2n-3) \\
&\geq 2n - 1 \\
r_n &\geq \sqrt{2n-1}.
\end{aligned}$$