6.2.11 Assume (f_n) and (g_n) are uniformly convergent sequences of functions on the set A.

(a) Show that $(f_n + g_n)$ is a uniformly convergent sequence of functions
Proof: Let $\epsilon > 0$. Since $f_n \to f$ then there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1, |f(x) - f_n(x)| < \frac{\epsilon}{2}$. Similarly for g_n, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2, |f(x) - f_n(x)| < \frac{\epsilon}{2}$. Therefore if we take $N = \max\{N_1, N_2\}$, then consider $n \geq N$, we have

$$\begin{aligned} |(f(x) - g(x)) - (f_n(x) - g_n(x))| &= |(f(x) - f_n(x)) + (g(x) - g_n(x))| \\ &\leq |f(x) - f_n(x)| + |g(x) - g_n(x)| && \text{triangle inequality} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

(b) Give an example to show that the product $(f_n g_n)$ may not converge uniformly.
Take $f_n = \frac{1}{x} + \frac{1}{n}, g_n = \frac{x^2+nx}{n}$. Since $f_n \to \frac{1}{x}$ and $g_n \to x$ then $f_n g_n \to 1$. Therefore if we compute their diference we find

$$\begin{aligned} |(\frac{1}{x} + \frac{1}{n})(\frac{x^2+nx}{n}) - 1| &= |\frac{x(x+n)(x+n)}{xn^2} - 1| \\ &= |\frac{x^2+2nx+n^2}{n^2} - 1| \\ &= |\frac{x^2+2nx}{n^2}| \\ &= |\frac{x^2}{n^2} + \frac{2x}{n}| \\ &\leq |\frac{x^2}{n} + \frac{2x}{n}| \\ &= \frac{1}{n}(x^2+2x) \end{aligned}$$

Therefore to have N be sufficiently large we must have $\frac{x^2+2x}{\epsilon} < N$. Since N depends on both x and ϵ then $f_n g_n$ is not uniformly continuous.

(c) Prove that if there exists an $M > 0$ such that $|f_n| \leq M$ and $|g_n| \leq M$ for all $n \in \mathbb{N}$, then $(f_n g_n)$ does converge uniformly.
Lemma: If $f_n \to f$ uniformly and $|f_n| \leq M$ for all $n \in \mathbb{N}$ then $|f| \leq M$. Since $f_n \to f$, then for every element in the sequence (k^{-1}) there exists $N_k \in \mathbb{N}$ such that for all $n \geq N_k$, $|f(x) - f_n)| < \frac{1}{k}$. Therefore by the triangle inequality and applying the bound we have that $|f(x)| < M + \frac{1}{k}$. Therefore by the algebraic order theorem $|f(x)| \leq M$
Proof: Let $\epsilon > 0$. Since $f_n \to f$ then there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1, |f(x) - f_n(x)| < \frac{\epsilon}{2M}$. Similarly for g_n, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2, |f(x) - f_n(x)| < \frac{\epsilon}{2M}$. Therefore if we take $N = \max\{N_1, N_2\}$, then

consider $n \geq N$, we have

$$\begin{aligned}
|f_n(x)g_n(x) - f(x)g(x)| &= |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)| \\
&= |f_n(x)(g_n - g(x) + g(x)(f_n(x) - f(x))| \\
&\leq |f_n(x)||(g_n - g(x)| + |g(x)||(f_n(x) - f(x))| && \text{triangle inequality} \\
&\leq M|(g_n - g(x)| + M|(f_n(x) - f(x))| && \text{bound on } f \text{ and } g \\
&< M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} \\
&= \epsilon.
\end{aligned}$$