4.3 a Let the Klein 4 group be defined as $V_4 =< a, b|a^2 = b^2 = (ab)^2 = e >$, then we have the character table:

	$\{e\}$	$\{a, a^2\}$	$\{b, b^2\}$	$\{ab, (ab)^2\}$
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

c Let $D_4 =< r, s|r^4 = s^2 = e, srs = r^{-1} >$ and it's character table as follows:

	$\{e\}$	$\{r^2\}$	$\{r, r^3\}$	$\{s, r^2 s\}$	$\{rs, r^3 s\}$
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

4.7 Let G be a finite group with $|G| = nm, N \leq G, G' = G/N = \{r_1, \cdots, r_m\}$, with the canonical projection map $\pi : G \to G'$, and let $\rho' : G' \to GL(V)$ be an irreducible representation of G', and $\rho = \rho' \circ \pi$. To directly show that ρ is an irreducible representation of G, assume for contradiction that W is a G-invariant subspace. Then for all $g \in G, v \in W, \rho_g(v) = v$. However, consider for each g, there exists r_i such that $\pi(g) = r_i$. This now implies that $\rho_g(v) = \rho'_{r_i}(v) = v$. This implies that W is a $G' - invariant$ subspace, contradicting that ρ' is irreducible. Now by using theorem 10.4.6, if we consider χ' to be the character of ρ' and χ to be the character of ρ, then

$$
\begin{aligned}
< \chi, \chi > &= \frac{1}{nm} \sum_{g \in G} \overline{\chi(g)} \chi(g) \\
&= \frac{1}{nm} \sum_{g \in G} \overline{\chi'(\pi(g))} \chi'(\pi(g)) \\
&= \frac{1}{nm} \sum_{i=1}^{m} n \overline{\chi'(r_i)} \chi'(r_i) \\
&\text{since } |gN| = n \\
&= \frac{1}{m} \sum_{i=1}^{m} \overline{\chi'(r_i)} \chi'(r_i) \\
&=< \chi', \chi' > \\
&= 1
\end{aligned}
$$

Since the inner product of χ with itself is 1, then it is necessarily irreducible.

5.5 Let G be a finite group with $|G| = n$ and k conjugacy classes. Let χ_i, χ_j by characters for irreducible representations of ρ_i, ρ_j for G. If we let $\chi = \chi_i \chi_j$, then for any $g, h \in G$ then $\chi(gh) = \chi_i(gh)\chi_j(gh) = \chi_i(g)\chi_i(h)\chi_j(g)\chi_j(h) = \chi_i(g)\chi_j(g)\chi_i(h)\chi_j(h) = \chi(g)\chi(h)$. Since the set of characters is just the set of homomorphisms from $G \to C^\times$ then χ is a homomorphism as well. Additionally the above proof can be redone with

$\frac{1}{\chi}$. Thus the set of characters and their inverses is closed under multiplication, and since these are functions over complex numbers all other group axioms are satisfied. Thus the dual group is a group. If G is abelian then by the structure theorem we have that $G \cong \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_k} \oplus \mathbb{Z}^n$. We know that the injective homomorphism from $\chi : \mathbb{Z}_{d_i} \to \mathbb{C}^\times$ is $\chi(g) = e^{\frac{2\pi i g}{d_i}}$, and $\chi : \mathbb{Z} \to \mathbb{C}^\times$ is $\chi(n) = e^n$. Since these also are trivially irreducible implies that by the structure theorem we have an injective map from G to $\mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_k} \oplus \mathbb{Z}^n$ and for each respective component we have an irreducible representation implies that we have an injection from G to a product over the characters specified above. If G is finite then since we have an injective map then the sets are isomorphic. Additionally since these are all homomorphisms then G is isomorphic to it's dual.

5.7 (a) Since a given element $g \in G$ with the isomorphism from G to $\mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_k}$ implies that g can map into $e^{\frac{2\pi i v_1}{d_1}} + \cdots + e^{\frac{2\pi i v_k}{d_k}}$. Thus the homomorphism is a homomorphism between direct sums of cyclic groups. Thus one can take g to $\phi(g)$ with G' to $\mathbb{Z}_{d'_1} \oplus \cdots \oplus \mathbb{Z}_{d'_k}$ and get $e^{\frac{2\pi i v'_1}{d'_1}} + \cdots + e^{\frac{2\pi i v'_k}{d'_k}}$

(b) If dual ϕ is surjective then that implies that every possible combination over the module $\mathbb{Z}_{d'_1} \oplus \cdots \oplus \mathbb{Z}_{d'_k}$ is reached. Therefore every $g' \in G'$ has a unique representation in G under ϕ^{-1}