Consider the following two partially ordered sets:

- $D(\mathbb{N})$ consisting of the natural numbers ordered by divisibility.
- The poset **Poly** consisting of all polynomials in the variable x with coefficients in $\mathbb{Z}_{\geq 0}$ with the ordering $q(x) \leq p(x)$ if $degree(q) \leq degree(p)$ and $q_i \leq p_i$ for each $i \in \{0, \ldots, degree(q)\}$. Here q_i means the coefficient of x^i in q.

Prove that there is an isomorphism between $D(\mathbb{N})$ and **Poly**

Note that $p_1 \cdots p_l$ represents the product from the first prime number to the lth prime number.

Proof. We must show that there is an isomorphism between $D(\mathbb{N})$ and **Poly**. By definition of isomorphism we must show there exists a bijection $f : \mathbb{N} \to \mathbb{Z}_{\geq 0}[x]$ such that for all $p, q \in \mathbb{N}$ $q \leq_D p$ iff $f(q) \leq_{\mathbf{Poly}} f(p)$. Suppose f is given by for any $z \in \mathbb{N}$, with unique prime factorization (as guaranteed by the FTA) $z = p_1^{\delta_1} \cdots p_k^{\delta_k}$, then $f(z) = \sum_{i=0}^{k-1} \delta_{i+1} x^i$. We must show that f is a bijection.

- We must show that f is injective. Suppose $p, q \in \mathbb{N}, f(p) = f(q)$. We must show $p = q$. By the FTA we have that p has a unique representation as $p = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$, and q has the unique representation as $q = p_1^{\beta_1} \cdots p_m^{\beta_m}$ By definition of being a member of $\mathbb{Z}_{\geq 0}[x]$ we have $f(p) = \sum_{i=0}^{n-1} \alpha_{i+1} x^i, f(q) = \sum_{i=0}^{m-1} \beta_{i+1} x^i$. Since $f(p) = f(q)$, then $m = n, \alpha_k = \beta_k, k \in [n]$. Since p, q have been shown to have equivalent prime factorizations, then $p = q$.
- we must show that f is surjective. Suppose $y(x) \in \mathbb{Z}_{\geq 0}[x]$. We must show there exists $x_* \in \mathbb{N}$ such that $f(x_*) = y$. Since $y \in \mathbb{Z}_{\geq 0}[x]$ then by definition we have $y(x) = \sum_{i=0}^{n} y_i x^i$. We claim that x_* is given by prime exponents $\gamma_{i+1} = y_i, i \in \{0\} \cup [n]$ such that $x_* = p_1^{\gamma_1} \cdots p_{n+1}^{\gamma_{n+1}}$. Therefore by algebraic manipulation we have,

$$
\begin{aligned}
f(x_*) &= f(p_1^{\gamma_1} \cdots p_{n+1}^{\gamma_{n+1}}) \\
&= \sum_{i=0}^{n} \gamma_{i+1} x^i \\
&= \sum_{i=0}^{n} y_i x^i \\
&= y.
\end{aligned}
$$

We must show for all $p, q \in \mathbb{N}$ $q \leq_D p$ iff $f(q) \leq_{\mathbf{Poly}} f(p)$. Suppose $p, q \in \mathbb{N}$. By the FTA since $p, q \in \mathbb{N}$, then there exists unique non-negative integers $\alpha_1, \ldots \alpha_n, \beta_1, \ldots, \beta_m \in \mathbb{Z}_{\geq 0}$ such that $p = p_1^{\alpha_1} \cdots p_n^{\alpha_n}, q = p_1^{\beta_1} \cdots p_m^{\beta_m}$.

- Suppose $q \leq_D p$. We must show that $f(q) \leq_{\mathbf{Poly}} f(p)$. By definition of $\leq_{\mathbf{Poly}}$ we must show $deg(f(q)) \leq deg(f(p))$ and for all $i \in [deg(q)], q_i \leq p_i$, where q_i is the coefficient of x^i in q. By definition of $q \leq_D p$, $q \mid p$. By definition of divisibility

$\frac{p}{q} = r, r \in \mathbb{N}$. Note that applying our prime factorizations of p, q to r yield $r = p_{m+1}^{\alpha_{m+1}} \cdots p_n^{\alpha_n} \prod_{i=1}^m p_i^{\alpha_i - \beta_i}$. We claim that $n \geq m$. Suppose for contradiction that $m > n$. Therefore there exists primes in the unique factorization of q that don't exists in p. Since p does not contain those prime factors, then there is no way to remove those prime factors from the denominator, therefore $\frac{p}{q} \notin \mathbb{N}$. This is a contradiction. We claim that for all $i \in [m], \alpha_i \geq \beta_i$. Suppose for contradiction that there exists $i_* \in [m]$ such that $\alpha_{i_*} < \beta_{i_*}$. Note that $\alpha_{i_*} - \beta_{i_*} < 0$. Therefore $\alpha_{i_*} - \beta_{i_*}$ is a negative number. Let $-e = \alpha_{i_*} - \beta_{i_*}$. Therefore r can now be written as $r = \frac{p_{m+1}^{\alpha_{m+1}} \cdots p_n^{\alpha_n} \prod_{\substack{i=1 \\ i \neq i_*}}^m p_i^{\alpha_i - \beta_i}}{p_{i_*}^e}$. Since p_{i_*} is prime and explicitly does not occur in the numerator of r, then $p_{i_*}^e$ never has anything to divide out with. Therefore $r \notin \mathbb{N}$. This is a contradiction. By definition of f, $f(q) = \sum_{j=0}^{m-1} \beta_{i+1} x^i, f(p) = \sum_{i=0}^{n-1} \alpha_{i+1} x^i$. Since $m \leq n, 1 \leq 1$, then $m - 1 \leq n - 1$, therefore $deg(f(q)) \leq deg(f(p))$ as $deg(f(q)) = m - 1, deg(f(p)) = n - 1$. Since the coefficients of $f(q), f(p)$ from 0 to $deg(f(q))$ are simply $\alpha_1, \ldots, \alpha_m$, and $\beta_1, \ldots, \beta_m$, and it was shown for all $i \in [m] \alpha_i \geq \beta_i$, then the second requirement is satisfied.

- Suppose $f(q) \leq_{\mathbf{Poly}} f(p)$. We must show that $q \leq_D p$. By definition of f, $f(q) = \sum_{i=0}^{m-1} \beta_{i+1} x^i, \sum_{j=0}^{n-1} \alpha_{i+1} x^j$. By definition of $\leq_{\mathbf{Poly}}$ for all $i \in [m], \beta_i \leq \alpha_i, m-1 \leq n-1$. Therefore $m \leq n$ and for all $i \in [m], 0 \leq \alpha_i - \beta_i$. Therefore exponentiating the inequality with p_i yields $1 \leq p_i^{\alpha_i - \beta_i}$. Since $\alpha_i - \beta_i \in \mathbb{Z}, 0 \leq \alpha_i - \beta_i$ then $p_i^{\alpha_i - \beta_i} \in \mathbb{N}$. Therefore the product $p_{m+1}^{\alpha_{m+1}} \cdots p_n^{\alpha_n} \prod_{i=1}^m p_i^{\alpha_i - \beta_i} \in \mathbb{N}$. Therefore $\frac{p}{q} \in \mathbb{N}$. Thus $q \leq_D p$ by definition.

Thus the requirements have been satisfied.