4.3 To show that $f(x) = x^4 + 6x^3 + 9x + 3$ generates a maximal ideal we need to show that it is irreducible. Note that for the prime $p = 3$, $f(x) \equiv x^4 \mod 3$, and $9 \nmid 3$. Therefore f is irreducible by Eisenstein's criterion. Thus (f) is a maximal ideal over $\mathbb{Q}[x]$.

4.6 $x^5 + 5x + 5$ is irreducible over $\mathbb{Q}$ since it satisfies Eisenstein's criterion for $p = 5$. For $\mathbb{Z}/2$, $x^5 + 5x + 5 = (x^2 + x + 1)(x^3 + x^2 + 1)$

4.7 $f(x) = x^3 + x + 1$

- $p = 2$ $x^3 + x + 1 = x^3 + x + 1$ since for 0 and 1 the polynomial is 1
- $p = 3$ $f(x) = (x - 1)(x^2 + x - 1)$ since $f(1) \equiv 0 \mod 3$
- $p = 5$ $f(x) = x^3 + x + 1$ since for $x = 0, 1, 2, 3, 4, f(x) \neq 0$.

5.1 (a) $1 - 3i = (1 - i)(2 - i)$

(b) $10 = (1 - i)(1 + i)(2 - i)(2 + i)$

(c) $6 + 9i = 3(2 + 3i)$, note that $(2 + 3i)(2 - 3i) = 13$, which is a prime

(d) $7 + i = (2 + i)^2(1 - i)$

5.3 Note that $(2 + i)^2 = 3 + 4i$ and $(2 + i)(3 + 2i) = 4 + 7i$. Since $3 + 2i$ was computed above to be prime then the smallest element which generates both is $(2 + i)$.

5.5 Let π be a gauss prime.

- Suppose π and $\overline{\pi}$ are associates. Then there are four possible units by which π and $\overline{\pi}$ are associates.
  - If $\overline{\pi} = \pi$ then π is invariant under complex conjugation. Therefore π is an integer. Since π is a gauss prime which is an integer then π must be an integer over the primes.
  - If $\overline{\pi} = -\pi$ then π must be purely imaginary since $-(a + bi) = a - bi$ implies $a = -a$, which is only true when $a = 0$. Since we have a purely imaginary Gauss prime and we know that it's norm must correspond to the square of a prime (otherwise implying that the square root of a prime is defined in the integers) implies that π is an associate of an integer prime.
  - If $\overline{\pi} = i\pi$ then $\pi = a + bi$ must satisfy the relation $a = -b$. The only Gauss prime satisfying this requirement is $1 - i$, which is one of the factors which ramifies 2.
  - If $\overline{\pi} = -i\pi$ then we have the conjugate of the prime found above, $1 + i$, which is the other factor of 2 in $\mathbb{Z}[i]$.

  Thus by cases we have shown that either π divides 2 or π is an associate of an integer prime.
- - Suppose $\pi\overline{\pi} = 2$. Then $\pi = 1 - i$. Since $i\pi = i - i^2 = 1 + i = \overline{\pi}$, then π and $\overline{\pi}$ are associates.

- Suppose π is an associate of a prime integer, $p \in \mathbb{Z}$. then if $\pi = \pm p$, the trivially $\overline{\pi} = \pm p$, thus $\pi = \overline{\pi}$, making them associates with 1. If $\pi = \pm ip$, then $\overline{\pi} = \mp ip$. Therefore π and $\overline{\pi}$ are associates by -1. Thus no matter by which unit π is an associate of p, π and $\overline{\pi}$ are associates.