8.3 Suppose $|G| = p^n, n > 1$. If we have an element of the order $|a| = p^k$ where $1 < k < n$ then we can simply take the element $a^{p^{k-1}}$ then we have that $(a^{p^{k-1}})^p = 1$. Therefore we have an element of order p.

8.10
- Suppose for a subgroup H of G that $[G : H] = 2$. Then we have that $\{H, aH\}$ partitions G for some $a \in G \backslash H$. Suppose $r \in aH$. Therefore $r \notin H$. Therefore since $\{H, Ha\}$ partitions G then $r \in HA$. A similar proof exists for the other direction. Thus $aH = Ha$.
- Consider the subgroup of S_3 $\{e, (12)\}$. Clearly the index is 3 since $3 = 6/2$ by Lagrange's theorem. Note that $(123)(12) = (13)$ and $(12)(123) = (23)$. Thus the subgroup is not normal.

9.4
- Note that $2^{-1} \equiv 5 \mod 9$. Thus $25 \equiv 7 \equiv x \mod 9$. For all $n \in \mathbb{Z}$ $x = 9n + 7$
- Since $2x - 5$ is always odd, and 6 is an even number then $6 \nmid 2x - 5$. Thus $2x \not\equiv 5 \mod 6$

10.5
- $ker\phi$ is a subgroup w/ $ker\phi$.
- S_4 is a subgroup w/ $ker\phi$.
- A_4 contains $\ker\phi$.
- The group generated by $(1234), (12)(34)$ contains $\ker\phi$.
- The group generated by $(1324), (13)(24)$ contains $\ker\phi$.
- The group generated by $(1342), (13)(42)$ contains $\ker\phi$.

11.5 We want to show that if Z_1 is the center of G_1 and Z_2 is the center of G_2 then $Z_1 \times Z_2$ is the center of $G_1 \times G_2$. Suppose $g \in G_1 \times G_2$ and $z \in Z_1 \times Z_2$. Then by definition $g = (g_1, g_2)$ and $z = (z_1, z_2)$. Therefore $g \cdot z = (g_1 z_1, g_2 z_2) = (z_1 g_1, z_2 g_2) = z \cdot g$.
To show that $Z_1 \times Z_2$ is uniquely the center of $G_1 \times G_2$ suppose for contradiction that there is an element $h \in G_1 \times G_2 \backslash Z_1 \times Z_2$ such that for all $g \in G_1 \times G_2$, $g \cdot h = h \cdot g$. Therefore by definition of being a member of the product group there exists $h_1 \in G_1, h_2 \in G_2$ such that $h = (h_1, h_2)$. Thus if $h \cdot g = g \cdot h$ then $(h_1 g_1, h_2 g_2) = (g_1 h_1, g_2 h_2)$. However this implies that $h_1 \in Z_1, h_2 \in Z_2$. Thus $h \in Z_1 \times Z_2$. Thus we have found the unique centralizer of $G_1 \times G_2$.

12.2
- H is a subgroup of $GL_3(\mathbb{R})$ since given two elements $A, B \in H$ of the form

$$A = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

, then $AB = \begin{bmatrix} 1 & x+a & y+b+cx \\ 0 & 1 & z+c \\ 0 & 0 & 1 \end{bmatrix}$. This gives us that it's closed under multiplication, and if we set z, y, z, a, b, c to 0 then we have the identity matrix.
If for a given matrix with x, y, z as entries as shown above then if we set $a = -x, c = -z, b = zx - y$ then we have the identity matrix, giving us the inverse.
Thus H is a group

- Clearly K is a subgroup since by the calculation above if we set $x = z = a = c = 0$ then we have two matrices which are in K, and clearly their product is also in K. Similarly the identity is in K.
  Note that if $c = x = z = a = 0$ and $b = -y$ then we get the identity matrix. This makes it a subgroup of H.
- The quotient group of H/K is of the form $\{\begin{bmatrix}1 & l & 0\\0 & 1 & r\\0 & 0 & 1\end{bmatrix} : l, r \in \mathbb{R}\}$ since by our formula above we have that the multiplication of any two of these matrices from this group yields the matrix $\begin{bmatrix}1 & x+a & cx\\0 & 1 & z+c\\0 & 0 & 1\end{bmatrix}$, which we can quotient out by the matrix $\begin{bmatrix}1 & 0 & cx\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ to get the matrix $\begin{bmatrix}1 & x+a & 0\\0 & 1 & z+c\\0 & 0 & 1\end{bmatrix}$, which is in the set I defined to be the quotient group.
- In order to find the centralizer, if we assume $A \in Z$, fixing x, y, z and allow B to vary, we can find the solution to the equation $AB = BA$ by swapping all x, y, z for a, b, c. This will yield the equation $y + b + cx = y + baz$ in top left entry. Since a, c vary and are independent of each other, then the only fixed solution for x, z would be $x = z = 0$. Thus A is in K. Similarly it's clear that all K commute with elements in H, by taking B to have $a = c = 0$

$$AB = \begin{bmatrix}1 & x & y+b\\0 & 1 & z\\0 & 0 & 1\end{bmatrix} = BA$$

  , thus $K = Z$, K is the centralizer.