Alex Valentino Assignment 11 problem 1

300H

Recall that a number is *perfect* if the sum of its proper divisors is equal to the number itself. Prove the following: if n is a positive integer such that 2^n-1 is prime, then $2^{n-1}(2^n-1)$ is perfect.

Proof: Suppose 2^n-1 is prime. We must show that $2^{n-1}(2^n-1)$ is perfect. Let $l = 2^{n-1}(2^n-1)$. By definition of perfect we must show $\sum_{\substack{d|l \\ d\neq l}} d = l$. Since 2^n-1 is prime, then for all $d \in Div(l)$, either $2^n-1 \mid d$ or $2^n-1 \nmid d$. Let the set of all $d \in Div(l), 2^n-1 \nmid d$ be denoted A, and the set of all $d \in Div(l), 2^n-1 \mid d$ be denoted B. Note by the definition of set union $A \cup B = Div(l)$. Since for all $a \in A, 2^n-1 \nmid a, a \mid n$, then $a \mid 2^{n-1}$. Since 2 is prime, then the only possible divisors of 2^{n-1} are $\{2^0, 2^1, \ldots, 2^{n-1}\}$. Therefore $A = \{2^0, 2^1, \ldots, 2^{n-1}\}$. Since the set B is the set of divisors of l who are divisable by 2^n-1, then aside from $1, b \in B$ should have the property $(b/(2^n-1)) \mid 2^{n-1}$. Since $(b/(2^n-1)) \mid 2^{n-1}$, and the only divisors of 2^{n-1} are $\{2^0, \ldots, 2^{n-1}\}$, then all $b \in B \backslash \{1\}$ should have the form $b = 2^k(2^n-1)$, where $k \in \mathbb{Z}, n-1 \geq k \geq 0$. Therefore $B = \{1, 2^n-1, 2(2^n-1), \ldots, 2^{n-1}(2^n-1)\}$. Also note that for any $r \in \mathbb{Z}_{\geq 0}, \sum_{i=0}^{r} 2^i = 2^{r+1}-1$, since $\sum_{i=0}^{r} 2^i$ is geometric series which when evaluated yields $\sum_{i=0}^{r} 2^i = \frac{1-2^{r+1}}{1-2} = \frac{1-2^{r+1}}{-1} = 2^{r+1}-1$. Therefore by algebraic manipulation

$$\begin{aligned}
\sum_{\substack{d|l \\ d\neq l}} d &= \sum_{a\in A} a + \sum_{b \in B\backslash\{1,l\}} b \\
&= \sum_{i=0}^{n-1} 2^i + (2^n-1)\sum_{i=0}^{n-2} 2^i \\
&= 2^n - 1 + (2^n-1)(2^{n-1}-1) \\
&= (2^n-1)(1+2^{n-1}-1) \\
&= 2^{n-1}(2^n-1) \\
&= l.
\end{aligned}$$