1.4 (a) Suppose $S \subset R, S$ is a simple module, $S \neq \emptyset$. Additionally, consider the map $\psi : R \to S$ given by $\psi(r) = rs$, where $s \in S$. We can say this because $S \neq \emptyset$. Note that ψ is surjective because $Im\psi$ is a submodule of S, and if $Im\psi \neq S$ then S would have a proper submodule, contradicting the simplicity of S. Since ψ is surjective then we can apply the correspondence theorem. Since S has no proper submodules, then it contains exactly 2 ideals. This means that S is a field over the ring R. Therefore the kernel of ψ must be maximal. Therefore by the first isomorphism theorem $S \cong R/\ker\psi$

(b) Suppose S, S' are simple modules and $\phi S \to S'$ is a homomorphism. Suppose that ϕ is not the zero map. Then for all $s \in S$, $\phi(s) \neq 0$. We know by the previously proved theorem that S, S' are isomorphic to $R/M, R'/M'$, where M, M' are maximal ideals.

2.1 We claim that $M = (x, y)$ is not a free module over $\mathbb{C}[x, y]$. First we claim that the size of the basis must be greater than 1. Assume for contradiction that a single element $g \in (x, y)$ can generate (x, y). Then there must exists $p, q \in \mathbb{C}[x, y]$ such that $pg = x, qg = y$. Since x, y are irreducible, then g must be a unit. However, the only units of $\mathbb{C}[x, y]$ are $\{-1, 1, i, -i\}$. Since every element in (x, y) is a linear combination of the variables x and y then every element in M is a polynomial. Thus $\{-1, 1, i, -i\} \subset \mathbb{C} \not\subset M$. Thus the basis of (x, y) must contain greater than 1 element. Suppose $b_1, \cdots, b_n \in M$ is a basis of M. Then if n is even we can have for each b_i to be multiplied by $(-1)^i \prod_{j=1, i\neq j}^n$. Thus every element is either the negative product of all the basis elements or positive. Thus their sum is 0. If n is odd we can simply choose to multiply b_n by 0 then repeat the process for the evens. Thus $b_1, \cdots, b_n$ is not linearly independent. Thus M is not a free module since it lacks a basis.

2.3 Let $\varphi : \mathbb{Z}^n \to \mathbb{Z}^m$ be given by $\varphi(x) = Ax$

(a)
- $\Rightarrow$ Suppose φ is injective. We want to show that φ has a trivial kernel. Suppose for contradiction that $\ker\phi \neq \{0\}$. Then there exists $x \in \mathbb{Z}^n, x \neq 0, \phi(x) = 0$. Therefore we can consider that x and $2x$ under ϕ, where $x \neq 2x$ yet $\varphi(x) = 0 = 2 * 0 = 2 * \varphi(x) = \varphi(2x)$. This contradicts the injectivity of φ. Thus $\ker\varphi$ is trivial. Note that this kernel is over $\mathbb{Z}^n$, however this isn't an issue. If there was a rational solution to $\varphi(v) = 0$, with $v = (\frac{a_1}{b_1}, \cdots, \frac{a_n}{b_n})$, then one can multiply both sides of $\varphi(v) = 0$ by $\prod_{i=1}^n b_i$, giving us $\prod_{i=1}^n b_i v \in \mathbb{Z}^n$ and $\varphi(\prod_{i=1}^n b_i v) = 0$. Thus a trivial kernel over $\mathbb{Z}$ is equivalent to a trivial kernel over $\mathbb{Q}$. And since our equations are just multiplying and adding integers then our solutions will always be rational. Thus we have a trivial real kernel for our matrix. Thus by rank nullity $n = rank(A) + nullity(A) = rank(A) + 0 = rank(A)$.
- $\Leftarrow$ Suppose φ is not injective. Then there exists $x, y \in \mathbb{Z}^n$ such that $x \neq y$ and $\varphi(x) = \varphi(y)$. Thus $\varphi(x - y) = 0$ and $x - y \neq 0$. Thus φ has a non-trivial kernel. Thus by rank-nullity $n = rank(A) + nullity(A) \geq rank(A) + 1, n - 1 \geq rank(A)$. Therefore $n \neq rank(A)$.

(b) Note that with the 3 integer row (column) operations (1. row swap, 2. multiplying a row by -1, row addition), the first two simply change the sign, which the gcd is invariant under, and for adding two rows, if the minor contains a portion of both rows then the gcd is unchanged, and if the minor have one of the rows then the gcd has the invariant $gcd(a, b) = gcd(a, b + a)$. Since row and column operations preserve the gcd of the determinates of the minor, then the matrix should have the same gcd of the determinates of minors of the matrix in smith normal form. Note that the matrix being in smith normal form means that exactly one of the m by m minors can be non-zero. Thus having a determinate of an integer matrix with only a diagonal be 1 is exactly equivalent to the diagonal being 1s. Having a diagonal be all 1s is equivalent to the matrix being surjective.

$$\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \xrightarrow{R_1+2R_2} \begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} 1 & 5 \\ 0 & 7 \end{bmatrix} \xrightarrow{C_2-5C_1} \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 7 & 2 \\ 2 & 4 & 6 \end{bmatrix} \xrightarrow{R_1-2R_2} \begin{bmatrix} 0 & -1 & -10 \\ 2 & 4 & 6 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 6 \\ 0 & -1 & -10 \end{bmatrix} \xrightarrow{R_1+4R_2} \begin{bmatrix} 2 & 0 & -36 \\ 0 & -1 & -10 \end{bmatrix}$$

$$\xrightarrow{-1\cdot R_2} \begin{bmatrix} 2 & 0 & -36 \\ 0 & 1 & 10 \end{bmatrix} \xrightarrow{C_3+18C_1} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 10 \end{bmatrix} \xrightarrow{C_3-10C_2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & -4 \\ 2 & -3 & 1 \\ -4 & 6 & -2 \end{bmatrix} \xrightarrow[R_3+2R_2]{R_1-R_2} \begin{bmatrix} 1 & 4 & -5 \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2-2R_1} \begin{bmatrix} 1 & 4 & -5 \\ 0 & -11 & 11 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{C_2-4C_1} \begin{bmatrix} 1 & 0 & -5 \\ 0 & -11 & 11 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{C_3+5C_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -11 & 11 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{C_2+C_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2\cdot -1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

4.1a

4.6

- $\Rightarrow$ Suppose for contradiction that A is singular. Then we know by problem 2.3 that it has a rank which is less than k. This gives us that in smith normal form there are guaranteed to be at least one zero in the diagonal. Let $A' = Q^{-1}AP$. Suppose this zero is at index $i \in [k]$ then if we take the vector $v' = (0, 0, \cdots, 1, 1, \cdots, 1)$ then this vector must be orthogonal to the image of A'. Since P, Q are invertible matrices, then the orthogonality is preserved, thus $Q^{-1}vA$ is orthogonal to the image of A'. Thus we can add subtract arbitrarily from the set imA by $v = Q^{-1}v'A$ then we have an infinite number cosets of imA in $\mathbb{Z}^k$
- $\Leftarrow$ Suppose A is non-singular. Then it has smith normal form equivalent to

$\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & d_k \end{bmatrix}$ Therefore a vector v is in the cokernel of A if there exists $v' \in \mathbb{Z}_{d_1} \cdots \mathbb{Z}_{d_k}$. Note that there is $d_1 \cdots d_k$ possible types of vectors in the cokernel (unscaled). Thus there are $\det A$ possible unscaled vectors in the cokernel. Thus there are a finite number of cosets.

6.1 Note that by Hilbert's theorem $\mathbb{C}[x_1, \cdots, x_n]$ is a noetherian ring. Let us construct an ascending chain of ideals. Start with (f_1). Clearly $(f_1) \subseteq (f_1, f_2)$. Furthermore $(f_1, f_2) \subseteq (f_1, f_2, f_3)$. By induction we have an ascending chain of polynomial ideals. However, since $\mathbb{C}[x_1, \cdots, x_n]$ is Noetherian then there exists $k \in \mathbb{N}$ where $(f_1, \cdots f_k) = (f_1, f_2, \cdots)$. Thus $\mathbb{C}[x_1, \cdots, x_n]/(f_1, \cdots, f_k) = C[x_1, \cdots, x_n]/(f_1, f_2, \cdots)$. Therefore by theorem 11.9.1 gives us that the set of common zeros of the set of infinite polynomials is represented by a finite number of polynomials.

7.1 The matrix can be put into smith normal form as $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. We know that smith normal form corresponds bijectively with an abelian group. Since there are no zeros in the diagonal, then the abelian group is simply $(\mathbb{Z}_2)^3$.

7.2 The abelian group is $C_\infty \oplus C_1$

9.1a We can reduce the matrix $\begin{bmatrix} x^2+1 & x \\ x^2y+x+y & xy+1 \end{bmatrix}$ to the identity matrix without division. Therefore the matrix has rank 2 for every $x, y \in \mathbb{C}$. Thus the module presented by this matrix is free.