

2.4 (a) For equation (2), we can define x' explicitly by the following:

$$x' = \pm\sqrt{1+x^2}.$$

Therefore taking the absolute value of x' yields:

$$|x'| = \sqrt{1+x^2} \leq \sqrt{x^2+x^2} = \sqrt{2x^2} = \sqrt{2}|x|.$$

Since (2) has a lipschitz constant of $\sqrt{2}$, then by theorem 5 (2) has unique solutions for all $x_0 \in (-1, 1)$.

Turning our attention to (1), then we have the equation

$$x' = \pm\sqrt{1-x^2}$$

Taking the absolute value of the derivative of x' yields

$$x'' = \frac{|x|}{\sqrt{1-x^2}}.$$

Taking the limit as x approaches -1 yields:

$$\lim_{x \rightarrow -1} |x''| = \frac{1}{\sqrt{1-1}} = \frac{1}{0} = \infty.$$

Since $|x'|$ is continuous on $(-1, 1)$ and is not lipschitz then (1) has infinite solutions.

(b) Since (1) does not have a unique solutions we must show an infinite number of solutions to $(x')^2 + x^2 = 1, x(0) = x_0$. Since $x(t) = 1$ solves the equation as

$$(x'_0)^2 + x_0 = 0^2 + 1^2 = 1$$

but not the initial value of $x_0 \in (-1, 1)$, we must solve the differential equations by other means to give another solution to interpolate with.

Solving for non-steady state:

$$\begin{aligned} x' &= \pm\sqrt{1-x^2} \\ 1 &= \frac{\pm x'}{\sqrt{1-x^2}} \\ \int_{t_0}^t dt &= \pm \int_{x_0}^x \frac{dz}{\sqrt{1-z^2}} \\ t - t_0 &= \pm(\arcsin(x) - \arcsin(x_0)) \\ \pm(t - t_0 + \arcsin(x_0)) &= \arcsin(x) \\ x(t) &= \pm \sin(t - t_0 + \arcsin(x_0)) \end{aligned}$$

Since $x(t) = 1, x(t) = \sin(t - t_0 + \arcsin(x_0))$ both solve the differential equation, then we may create a new solution

$$x(t) = \begin{cases} 1 & t > t_0 - \arcsin(x_0) + 2\pi a + \frac{\pi}{2} \\ \sin(t - t_0 + \arcsin(x_0)) & t \leq t_0 - \arcsin(x_0) + 2\pi a + \frac{\pi}{2} \end{cases}$$

where $a \in \mathbb{N} \cup \{0\}$. Note that at $t = t_0 - \arcsin(x_0 + 2\pi a + \frac{\pi}{2})$, that for \sin we have:

$$\sin(t - t_0 + \arcsin(x_0)) = \sin(2\pi a + \frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$$

and for the derivative

$$\cos(\frac{\pi}{2}) = 0$$

Which exactly aligns with the value and derivative of the constant function $x(t) = 1$. Therefore since our solutions are continuous, and there exist one for each natural number, then we have found an infinite number of solutions.

- 2.9 (a) Note that different classes of solutions are had for $\alpha = 1$ and $\alpha \neq 1$. Proceeding with the $\alpha \neq 1$ case:

$$x' = x|\ln|x||^\alpha$$

Applying barrow's formula yields:

$$\int_{x_0}^x \frac{dx}{x|\ln|x||^\alpha} = t - t_0$$

Note that x is within the maximal interval $(0, 1)$, therefore $|x| = x, |\ln(x)| = -\ln(x)$. Thus the substitutions of $u = |\ln|x||$ and $du = \frac{-1}{x}dx$ may be made:

$$-\int_{x_0}^x u^{-\alpha} du = t - t_0.$$

Therefore after evaluation we have:

$$\frac{-1}{1-\alpha}(u(x)^{1-\alpha} - u(x_0)^{1-\alpha}) = t - t_0$$

Let $k = 1 - \alpha$, therefore by algebraic manipulation we have

$$\begin{aligned} \frac{-1}{k}(u(x)^k - u(x_0)^k) &= t - t_0 \\ u(x)^k - u(x_0)^k &= k(t_0 - t) \\ u(x)^k &= k(t_0 - t) + u(x_0)^k \\ u(x) &= (k(t_0 - t) + u(x_0)^k)^{\frac{1}{k}} \\ -\ln(x) &= (k(t_0 - t) + u(x_0)^k)^{\frac{1}{k}} \\ \ln(x) &= -(k(t_0 - t) + u(x_0)^k)^{\frac{1}{k}} \\ x &= e^{-(k(t_0 - t) + u(x_0)^k)^{\frac{1}{k}}}. \end{aligned}$$

Some observations: if $\alpha > 1$ then $1 - \alpha = k < 1$. Since k is negative, then $u(0)^k = (-\ln(0))^k = \infty^k = 0, u(1)^k = (-\ln(1))^k = 0^k = \infty$. On the other hand if $\alpha < 1$ then $1 - \alpha = k > 1$. Since k is positive, then $u(1)^k = (-\ln(1))^k = 0^k = 0, u(0)^k = (-\ln(0))^k = \infty^k = \infty$. Therefore evaluating T_0, T_1 for the cases of $\alpha > 1, \alpha < 1$ yields:

i. $T_0, \alpha > 1$:

$$t_0 + \int_{x_0}^0 -u^{-\alpha} = t_0 + \frac{-1}{k}(u(0)^k - u(x_0)^k) = t_0 + \frac{u(x_0)^k}{k} < \infty$$

ii. $T_1, \alpha > 1$:

$$t_0 + \int_{x_0}^1 -u^{-\alpha} = t_0 + \frac{-1}{k}(u(1)^k - u(x_0)^k) = -\infty$$

iii. $T_0, \alpha < 1$:

$$t_0 + \int_{x_0}^0 -u^{-\alpha} = t_0 + \frac{-1}{k}(u(0)^k - u(x_0)^k) = \infty$$

iv. $T_1, \alpha < 1$:

$$t_0 + \int_{x_0}^1 -u^{-\alpha} = t_0 + \frac{-1}{k}(u(1)^k - u(x_0)^k) = t_0 + \frac{u(x_0)^k}{k} < \infty$$

Since T_0 is finite for $\alpha > 1$, then the solution does not hold for all t , however since T_1 is infinite, then we have found the solutions which are valid for $t > t_0$. Similarly for $\alpha < 1$, T_1 is finite, and T_0 is infinite, giving us another partial solution for $t < t_0$. Let us evaluate the $\alpha = 1$ case. The integral in terms of u from barrow's formula is still valid, but it's evaluation is different:

$$\begin{aligned} - \int_{x_0}^x u^{-1} du &= t - t_0 \\ \ln|u(x)| - \ln|u(x_0)| &= t_0 - t \\ \ln|u(x)| &= t_0 - t + \ln|u(x_0)| \\ u(x) &= e^{t_0 - t + \ln|u(x_0)|} \\ -\ln(x) &= e^{t_0 - t + \ln|u(x_0)|} \\ x &= e^{-e^{t_0 - t + \ln|u(x_0)|}} \end{aligned}$$

Evaluating on the endpoints:

i.

$$T_0 = t_0 + \int_{x_0}^0 -u^{-1} du = t_0 - \ln|u(0)| + \ln|u(x_0)| = t_0 - \ln(\infty) + \ln|u(x_0)| = -\infty$$

ii.

$$T_1 = t_0 + \int_{x_0}^1 -u^{-1} du = t_0 - \ln|u(1)| + \ln|u(x_0)| = t_0 - \ln(0) + \ln|u(x_0)| = \infty$$

Since the endpoints take an infinite amount of time to achieve, we have found the unique solution for all t .

(b) We claim that for $x(0) = 0$ unique steady state solution occurs when $\alpha \leq 1$. This gives us two cases to test:

- i. Suppose $\alpha < 1$. Since we know from 2.9 that $k > 0$, and that $|T_0| = \infty$, which is equivalent to evaluating barrow's formula on the neighborhood $(-\delta, 0)$, where $x_0 = -\delta$, and that $u(x) = |ln|x||$, thus we only need to evaluate the following:

$$\lim_{\delta \rightarrow 0} \int_0^\delta |u^{-\alpha}| du = \lim_{\delta \rightarrow 0} \frac{-1}{k} (u(\delta)^k - u(0)^k) = \infty$$

Thus we have found the unique solution to be the steady state for $x(0) = 0$.

- ii. Suppose $\alpha = 1$. Since we know that $|T_0| = \infty$, and that provides our cases for $(-\delta, x_0)$ as we simply can let $x_0 = -\delta$ and due to the absolute values we don't have to worry about sign. Thus we need to only evaluate the following:

$$\lim_{\delta \rightarrow 0} - \int_0^\delta |u^{-1}| du = \lim_{\delta \rightarrow 0} | -ln|u(0)| + ln|u(\delta)|| = \infty.$$

Thus the steady state solution is unique for $\alpha = 1, x(0) = 0$.

We claim that for $x(0) = 1$ unique steady state solution occurs when $\alpha \geq 1$. Thus gives us two cases to test:

- i. Suppose $\alpha > 1$ Since we know that for $\alpha > 1, T_1 = \infty$, then we have evaluated on $(1 - \delta, 1)$. Thus we only need to evaluate the following:

$$\lim_{\delta \rightarrow 0} \int_1^{1+\delta} |u^{-\alpha}| du = | \frac{-1}{k} (u(1 + \delta)^k - u(1)^k) | = \infty$$

- ii. Suppose $\alpha = 1$. Since we know that for $\alpha = 1, T_1 = \infty$ then we have already evaluated on $(1 - \delta, 1)$. Thus we only need to evaluate the following:

$$\lim_{\delta \rightarrow 0} \int_1^{1+\delta} |u^{-1}| du = \lim_{\delta \rightarrow 0} | -ln|u(1)| + ln|u(\delta)|| = \infty.$$

(c) For which values of α is v Lipschitz? Note that since there does not exists solutions for all values of $\alpha \neq 1$, then $x|ln|x||^\alpha$ is not Lipschitz for those values. We must test for $\alpha = 1$. Note that since $x \in (0, 1)$ then $v(x) = x|ln|x|| = -xln(x)$. Now testing the boundedness of $|v'|$ at 0:

$$\lim_{x \rightarrow 0} |v'| = \lim_{x \rightarrow 0} | -1 - ln(x) | = | -1 - ln(0) | = \infty.$$

Since v' is unbounded then the $\alpha = 1$ case is not Lipschitz.

2.10 (a) To show that $H(x', x) = H(v_0, x_0)$, we simply have to show that $\frac{d}{dt}H(x', x) = 0$:

$$\begin{aligned} \frac{d}{dt}H(x', x) &= \frac{d}{dt}(\frac{1}{2}(x')^2 + V(x)) \\ &= x'x'' + \frac{d}{dx}V(x)x' \\ &= x'x'' + -F(x)x' \\ &= x'x'' + -x''x' \\ &= 0. \end{aligned}$$

Therefore $H(x', x)$ is an arbitrary constant, which can be set to $H(v_0, x_0)$.

Therefore to solve $x' = \pm\sqrt{2(H(v_0, x_0) - V(x))}$ we simply apply barrows formula:

$$t(x) = t_0 + \int_{x_0}^x \frac{\pm dz}{\sqrt{2(H(v_0, x_0) - V(z))}}.$$

(b) Let $V(x) = \frac{1}{2}x^2$, and $x_0 = 1, v_0 = 0$. Then by the formula above we have:

$$t = t_0 + \int_1^x \frac{\pm dz}{\sqrt{2(\frac{1}{2} - \frac{1}{2}z^2)}} = t_0 + \int_1^x \frac{\pm dz}{\sqrt{1 - z^2}}.$$

This is solved by $x(t) = \pm \sin(t - t_0 + \frac{\pi}{2})$. Note that this is exactly the equation we found infinite solutions for in 2.4! Therefore the piecewise solution

$$x(t) = \begin{cases} 1 & t > t_0 + 2\pi a \\ \sin(t - t_0 + \frac{\pi}{2}) & t \leq t_0 + 2\pi a \end{cases}$$

provides solutions for all $a \in \mathbb{N}$ which have infinite rest periods at the equilibrium point $x = 1$.