Lemma: $\int_0^\infty xe^{-\lambda x} = \frac{-1}{\lambda}(xe^{-\lambda x})|_0^\infty + \frac{1}{\lambda}\int_0^\infty e^{-\lambda x}dx = 0 + \frac{1}{\lambda}\int_0^\infty e^{-\lambda x}dx = \frac{1}{\lambda^2}$.

6.2
- The respective p.m.fs of X and Y:

$$p_X(1) = \frac{1}{3}, p_X(2) = \frac{1}{2}, p_X(3) = \frac{1}{6}$$

$$p_y(0) = \frac{1}{5}, p_y(1) = \frac{1}{5}, p_y(2) = \frac{1}{3}, p_y(3) = \frac{4}{15}.$$

- Note that the X, Y pairs satisfying the condition are as follows: $(1,0), (2,0), (1,1)$. Thus the probability of $\mathbb{P}(X + Y^2 \leq 2) = \mathbb{P}(X = 1, Y = 0) + \mathbb{P}(X = 2, Y = 0) + \mathbb{P}(X = 1, Y = 1) = \frac{1}{15} + \frac{1}{10} + \frac{1}{15} = \frac{7}{30}$

6.6 (a)

$$f_X(x) = \int_0^\infty xe^{-x(y+1)}dy = xe^{-x}\int_0^\infty e^{-xy}dy = xe^{-x}\frac{1}{x} = e^{-x}$$

by the cdf of the exponential distribution

$$f_Y = \int_0^\infty xe^{-x(y+1)}dx = \frac{1}{(1+y)^2} \text{ by the lemma}$$

(b)

$$\mathbb{E}[XY] = \int_0^\infty \int_0^\infty x^2ye^{-x(y+1)}dydx = \int_0^\infty x^2e^{-x}\int_0^\infty ye^{-xy}dydx = \int_0^\infty e^{-x} = 1 \text{ by the lemma}$$

6.10
- The joint density is given by $f(x,y) = \begin{cases} 1 & \text{if } (x,y) \in (0,1)^2 \\ 0 & \text{otherwise} \end{cases}$, since the area of the interior of the unit square is 1.
- The probability $\mathbb{P}(X < Y)$ is given by the integral $\int_0^1 \int_0^y f(x,y)dxdy$. This evaluates to $\int_0^1 \int_0^y f(x,y)dxdy = \int_0^1 ydy = \frac{1}{2}$.

6.12 We claim that $X \sim Exp(1)$ and $Y \sim Exp(2)$. Note that for $x > 0$ and $y > 0$ that $f_X(x)f_Y(y) = e^{-x}2e^{-2y} = 2e^{-(x+2y)} = f_{X,Y}(x,y)$. Similarly if either $x \leq 0$ or $y \leq 0$ then $f_X(x)f_Y(y) = 0 = f_{X,Y}(x,y)$. Since we have found the marginal distributions for X, Y and we have shown that the joint density function is separable then X is independent of Y.

6.18 Let $f_{X,Y}(a,b)$ be the alleged joint probability mass function.

(a) Note that if $a \in [4]$ then $\frac{1}{4a} \geq 0$, and otherwise $a \geq 0$. Additionally, $\sum_{a=1}^4 \sum_{b=1}^a \frac{1}{4a} = \sum_{a=1}^4 a\frac{1}{4a} = \sum_{a=1}^4 \frac{1}{4} = 1$. Therefore $f_{X,Y}$ is a pmf.

(b) The pmf for X is given by $p_X(a) = \sum_{b=1}^a = \frac{1}{4a} = \frac{1}{4}$. The pmf for Y is given by $p_Y(1) = \frac{25}{48}, p_Y(2) = \frac{13}{48}, p_Y(3) = \frac{7}{48}, p_Y(4) = \frac{1}{16}$.

(c) $\mathbb{P}(X = Y + 1) = \sum_{i=1}^3 \mathbb{P}(Y = i, X = i+1) = \frac{1}{8} + \frac{1}{12} + \frac{1}{16} = \frac{13}{48}$

6.30 Note that $f_X(k) = (1-p)^{k-1}p$, $f_Y(k) = e^{-\lambda}\frac{\lambda^k}{k!}$. Since we defined X, Y to be independent then $f_{X,Y}(m,n) = f_X(m)f_Y(n) = (1-p)^{m-1}pe^{-\lambda}\frac{\lambda^n}{n!}$. Note that since $X+1 = Y$, then we can find the probability by indexing over $f(n+1,n)$, which satisfies $X = Y+1$. Therefore

$$\begin{aligned}\mathbb{P}(X = Y+1) &= \sum_{n=0}^{\infty} f(n+1,n) \\ &= \sum_{n=0}^{\infty}(1-p)^n pe^{-\lambda}\frac{\lambda^n}{n!} \\ &= pe^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda(1-p))^n}{n!} \\ &= pe^{-\lambda}e^{\lambda(1-p)} \\ &= pe^{-\lambda p}\end{aligned}$$

6.36 (a) Note that

$$\begin{aligned}c &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}-\frac{(x-y)^2}{2}}dydx \\ &= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\int_{-\infty}^{\infty} e^{xy-\frac{y^2}{2}}dydx \\ &= \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}}\sqrt{2\pi}e^{x^2} \text{ by HW 1} \\ &= 1\end{aligned}$$

(b)

$$\begin{aligned}f_Y(y) &= e^{\frac{-y^2}{2}}\int_{-\infty}^{\infty} e^{xy-x^2}dx \\ &= \frac{e^{\frac{-y^2}{2}}}{\sqrt{2}}\int_{-\infty}^{\infty} e^{\frac{y}{\sqrt{2}u-\frac{u^2}{2}}}du \\ &= \sqrt{\pi}e^{\frac{-y^2}{4}}\end{aligned}$$

$$\begin{aligned}f_X(x) &= e^{-\frac{x^2}{2}}\int_{-\infty}^{\infty} e^{xy-x^2/2}dx \\ &= \sqrt{2\pi}e^{\frac{-x^2}{2}}\end{aligned}$$

(c) Note that $f_X(x)f_Y(y) = \sqrt{2}\pi e^{-\frac{x^2}{2}-\frac{y^2}{4}} \neq e^{-\frac{x^2}{2}-\frac{(x-y)^2}{2}} = f_{X,Y}(x,y)$. Thus the distributions are not independent.