1. Suppose f is twice differentiable. Then

$$\begin{aligned}
d(df) &= d\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i\right) \\
&= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j \partial x_i} dx_j\right) \wedge dx_i \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial f}{\partial x_j \partial x_i} dx_j \wedge dx_i \\
&= \sum_{i<j}^{n} \frac{\partial f}{\partial x_j \partial x_i} dx_j \wedge dx_i + \sum_{i<j}^{n} \frac{\partial f}{\partial x_i \partial x_j} dx_i \wedge dx_j \\
&= \sum_{i<j}^{n} \frac{\partial f}{\partial x_j \partial x_i} dx_j \wedge dx_i - \sum_{i<j}^{n} \frac{\partial f}{\partial x_j \partial x_i} dx_j \wedge dx_i \\
&= 0
\end{aligned}$$

Now for the map $\gamma : [0,1] \to \mathbb{R}^n$ with $\gamma(0) = \gamma(1)$, we have by the fundamental theorem of calculus that $\int_\gamma df = \int_0^1 \left(\frac{\partial f(\gamma(t))}{\partial x_1}, \cdots, \frac{\partial f(\gamma(t))}{\partial x_n}\right) \cdot \gamma'(t) = \int_0^1 (f(\gamma(t))'dt = f(\gamma(1)) - f(\gamma(0)) = f(\gamma(0)) - f(\gamma(0)) = 0$.

2. Let $\omega = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$

(a)

$$\begin{aligned}
d\omega &= d\left(\frac{-y}{x^2+y^2}\right) \wedge dx + d\left(\frac{x}{x^2+y^2}\right) \wedge dy \\
&= \left(\frac{2xy}{(x^2+y^2)^2} dx + \frac{y^2-x^2}{(x^2+y^2)^2} dy\right) \wedge dx + \left(\frac{y^2-x^2}{(x^2+y^2)^2} dx + \frac{-2xy}{x^2+y^2} dy\right) \wedge dy \\
&= 0 + \frac{y^2-x^2}{(x^2+y^2)^2} dy \wedge dx + \frac{y^2-x^2}{(x^2+y^2)^2} dx \wedge dy + 0 \\
&= \frac{y^2-x^2}{(x^2+y^2)^2} dx \wedge dy - \frac{y^2-x^2}{(x^2+y^2)^2} dx \wedge dy \\
&= 0
\end{aligned}$$

(b) Let $\gamma(t) = (\cos(2\pi t), \sin(2\pi t))$. The integral $\int_\gamma \omega = 2\pi \int_0^1 (-\sin(2\pi t))(-\sin(2\pi t)) + \cos(2\pi t)\cos(2\pi t) dt = 2\pi \int_0^1 dt = 2\pi$. Note that the given integral evaluates to 2π contradicts that there exists a differentiable g such that $dg = \omega$ since we showed above that for any differentiable function g we have that $\int_\gamma dg = 0$ where γ is a closed differentiable loop. Since our given γ is a closed differentiable loop and $0 \neq 2\pi$, then it is impossible to find such a g.

3. Let $x = r\cos(\theta), y = r\sin(\theta)$

(a)

$$
\begin{aligned}
dx \otimes dx + dy \otimes dy &= (\cos(\theta)dr + -r\sin(\theta)d\theta) \otimes (\cos(\theta)dr + -r\sin(\theta)d\theta) \\
&+ (\sin(\theta)dr + r\cos(\theta)d\theta) \otimes (\sin(\theta)dr + r\cos(\theta)d\theta) \\
&= \cos^2(\theta)dr \otimes dr - r\cos(\theta)\sin(\theta)dr \otimes d\theta - r\cos(\theta)\sin(\theta)d\theta \otimes dr \\
&+ r^2\sin^2(\theta)d\theta \otimes d\theta + \sin^2(\theta)dr \otimes dr + r\cos(\theta)\sin(\theta)dr \otimes d\theta \\
&+ r\cos(\theta)\sin(\theta)d\theta \otimes dr + r^2\cos^2(\theta)d\theta \otimes d\theta \\
&= (\cos^2(\theta) + \sin^2(\theta))dr \otimes dr + r^2(\cos^2(\theta) + \sin^2(\theta))d\theta \otimes d\theta \\
&= dr \otimes dr + r^2 d\theta \otimes d\theta
\end{aligned}
$$

(b)

$$
\begin{aligned}
dx \wedge dy &= (\cos(\theta)dr + -r\sin(\theta)d\theta) \wedge (\sin(\theta)dr + r\cos(\theta)d\theta) \\
&= \cos(\theta)\sin(\theta)dr \wedge dr + r\cos^2(\theta)dr \wedge d\theta - \sin^2(\theta)d\theta \wedge dr - r^2\cos(\theta)\sin(\theta)d\theta \wedge d\theta \\
&= r\cos^2(\theta)dr \wedge d\theta - r\sin^2(\theta)d\theta \wedge dr \\
&= r\cos^2(\theta)dr \wedge d\theta + r\sin^2(\theta)dr \wedge d\theta \\
&= r(\cos^2(\theta) + \sin^2(\theta))dr \wedge d\theta \\
&= r dr \wedge d\theta
\end{aligned}
$$

4. (a) Note that

$$dx = \sin(\phi)\cos(\theta)dr + r\cos(\phi)\cos(\theta)d\phi - r\sin(\phi)\sin(\theta)d\theta,$$

$$dy = \sin(\phi)\sin(\theta)dr + r\cos(\phi)\sin(\theta)d\phi + r\sin(\phi)\cos(\theta)d\theta,$$

$$dz = \cos(\phi)dr - r\sin(\phi)d\phi.$$

Therefore,

$$
\begin{aligned}
dx \wedge dy \wedge dz &= (\sin(\phi)\cos(\theta)dr + r\cos(\phi)\cos(\theta)d\phi - r\sin(\phi)\sin(\theta)d\theta) \\
&\wedge (\sin(\phi)\sin(\theta)dr + r\cos(\phi)\sin(\theta)d\phi + r\sin(\phi)\cos(\theta)d\theta) \\
&\wedge (\cos(\phi)dr - r\sin(\phi)d\phi) \\
&= \sin(\phi)\cos(\theta)(r\sin(\phi)\cos(\theta))(r\sin(\phi))dr \wedge d\phi \wedge d\theta \\
&- r\cos(\phi)\cos(\theta)(-r\sin(\phi)\cos(\theta))(\cos(\phi(dr)dr \wedge d\phi \wedge d\theta \\
&- r\sin(\phi)\sin(\theta)(-r\sin^2(\phi)\sin(\theta) - r\cos^2(\phi)\sin(\theta))dr \wedge d\phi \wedge d\theta \\
&= r^2\sin(\phi)(\cos^2(\theta)\sin^2(\phi) + \cos^2(\phi)\cos^2(\theta) \\
&+ \sin^2(\theta)(\cos^2(\phi) + \sin^2(\phi)))dr \wedge d\phi \wedge d\theta \\
&= r^2\sin(\phi)(\cos^2(\theta) + \sin^2(\theta))dr \wedge d\phi \wedge d\theta \\
&= r^2\sin(\phi) \wedge d\phi \wedge d\theta
\end{aligned}
$$

(b)
- $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x}\sin(\phi)\cos(\theta) + \frac{\partial f}{\partial y}\sin(\phi)\sin(\theta) + \frac{\partial f}{\partial z}\cos(\phi)$
- $\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \phi} = \frac{\partial f}{\partial x}r\cos(\phi)\cos(\theta) + \frac{\partial f}{\partial y}r\cos(\phi)\sin(\theta) - \frac{\partial f}{\partial z}r\sin(\phi)$
- $\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \theta} = -\frac{\partial f}{\partial x}r\sin(\phi)\cos(\theta) + \frac{\partial f}{\partial y}r\sin(\phi)\cos(\theta)$

5.

6. Let $\omega = \frac{xdy\wedge dz + ydz\wedge dx + zdx\wedge dy}{(x^2+y^2+z^2)^{3/2}}$, and C be the sphereically-parameterized sphere $x^2+y^2+z^2 = R^2$ with $(\phi,\theta) \in [0,\pi]\times[0,2\pi]$. Note that by example 6.1 in the stokes theorem notes that a two form exactly of the form above (where $P = x/(x^2+y^2+z^2)^{3/2}, Q = y/(x^2+y^2+z^2)^{3/2}, R = z/(x^2+y^2+z^2)^{3/2}$) with a parameterization from $\gamma : (u,v) \in \square \to \mathbb{R}^3$(which is c in our case can be computed via $\int_\square (P\circ\gamma, Q\circ\gamma, R\circ\gamma)\cdot(D_u\gamma \times D_v\gamma)$. Therefore, we first compute $D_\phi \times D_\theta = (R\cos(\phi)\cos(\theta), R\cos(\phi)\sin(\theta), -R\sin(\phi)) \times (R\sin(\phi)\cos(\theta), R\sin(\phi)\cos(\theta), 0) = (R^2\sin^2(\phi)\cos(\theta), R^2\sin^2(\phi)\sin(\theta), R^2\sin(\theta)\cos(\phi))$. Furthermore, the other vector we're dotting is
$(P\circ\gamma, Q\circ\gamma, R\circ\gamma) = (R\sin(\phi)\cos(\theta), R\sin(\phi)\sin(\phi), R\cos(\phi))$

$$\begin{aligned}\int_c \omega &= \int_0^\pi \int_0^{2\pi} (R\sin(\phi)\cos(\theta), R\sin(\phi)\sin(\phi), R\cos(\phi)) \\ &\quad \cdot (R^2\sin^2(\phi)\cos(\theta), R^2\sin^2(\phi)\sin(\theta), R^2\sin(\theta)\cos(\phi)) d\theta d\phi \\ &= \int_0^\pi \int_0^{2\pi} \sin(\phi) d\theta d\phi \\ &= 4\pi\end{aligned}$$