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- Function examples:

  - odd functions: x^3, $sin(x)$,
  - even functions: $x^2, cos(x)$

- Prove that for all functions $f : \mathbb{R} \to \mathbb{R}$, the function $s(x)$ given by the rule $s(x) = f(x) + f(-x)$ is even.
  We must show that $s(x)$ is even. By definition we must show $s(x) = s(-x)$ Suppose x is an arbitrary real number. Therefore,

$$\begin{aligned} s(-x) &= f(-x) + f(-(-x)) & &= f(x) + f(-x) \\ &= s(x). \end{aligned}$$

- Prove that all functions $f : \mathbb{R} \to \mathbb{R}$ can be written as the sum of an odd and even function. Lemma: For all functions $f : \mathbb{R} \to \mathbb{R}$, the function $O(x)$ given by the rule $O(x) = f(x) - f(-x)$ is even.
  Proof: We must show that $O(x)$ is odd. By definition we must show that $O(-x) = -O(x)$. Suppose x is an arbitrary real number. Therefore,

$$\begin{aligned} O(-x) &= f(-x) - f(-(-x)) \\ &= -1 * f(x) + -1 * -f(-x) \\ &= -1 * (f(x) - f(-x)) \\ &= -O(x). \end{aligned}$$

  Proof: We must show that f is the sum of an odd function and an even function. We claim that $f(x) = \frac{s(x)+O(x)}{2}$. Suppose x is an arbitrary real number. Therefore,

$$\begin{aligned} \frac{s(x) + O(x)}{2} &= \frac{f(x) + f(-x) + f(x) - f(-x)}{2} \\ &= \frac{2f(x)}{2} \\ &= f(x). \end{aligned}$$