1.2.1 (a) We must show that the square root of 3 is irrational. Suppose for contradiction that there exists $a, b \in \mathbb{Z}$ such that $\frac{a}{b} = \sqrt{3}$ and a, b are coprime. Therefore $a^2 = 3b^2$, 3 is a divisor of a^2. By Euclid's lemma 3 is a divisor of a. Therefore there exists $c \in \mathbb{Z}$ such that $a = 3c$. Thus $3c^2 = b^2$. By similar reasoning above, 3 is a divisor of b. This is a contradiction, as a, b are coprime. Therefore $\sqrt{3}$ is irrational. A similar argument works for $\sqrt{6}$, one must simply choose a single prime from the prime factorization of 6 and then get the contradiction from a, b not being coprime.

(b) The argument fails due to 4 having no prime factor of an odd power, no way to apply Euclid's lemma.

1.2.2 (a) If $A_1 \supseteq A_2 \supseteq \cdots$ are all sets containing an infinite number of elements, then the intersection of all of the sets must be infinite as well. This is false, if we have the sets $A_n = n, n+1, n+1, \cdots$ then the intersection $\bigcap_{n=1}^{\infty} A_n = \emptyset$

(b) If $A_1 \supseteq A_2 \supseteq \cdots$ are all sets containing a number of reals, then the intersection of all of the sets must be finite and non-empty. This is true

(c) $A \cap (B \cup C) = (A \cap B) \cup C$. This is false. If $A = \{1\}, B = \{2\}, C = \{2, 3\}$. Then $\emptyset = \{1\} \cap (\{2\} \cup \{2, 3\}) = A \cap (B \cup C) \neq (A \cap B) \cup C = \emptyset \cup \{2, 3\} = \{2, 3\}$

(d) true

(e) true

1.2.10 Let $y_1 = 1$ and for each $n \in \mathbb{N}$ define $y_{n+1} = \frac{3y_n + 4}{4}$.

(a) Use induction to prove that the sequence satisfies $y_n < 4$ for all $n \in \mathbb{N}$.
Proof: For the base case, $y_1 = 1 < 4$. By the principle of mathematical induction for all $k \in \mathbb{N}$ if $k < n$ then $y_k < 4$. We must show that $y_n < 4$. Since $n - 1 < n$, then by the induction hypothesis $y_{n-1} < 4$. Therefore,

$$\begin{aligned} y_{n-1} &< 4 \\ 3y_{n-1} &< 12 \\ 3y_{n-1} + 4 &< 16 \\ (3y_{n-1} + 4)/4 &< 4 \\ y_n &< 4. \end{aligned}$$

(b) We must show that $(y_1, y_2, \cdots)$ is increasing. For the base case, $y_1 = 1, y_2 = \frac{3+4}{4} = \frac{7}{4}, 1 < \frac{7}{4}$. By PMI for all $k \in \mathbb{N}$ if $k < n$ then $y_k < y_{k+1}$. Since $n - 1 < n$, by the induction hypothesis $y_{n-1} < y_n$. Therefore,

$$\begin{aligned} y_{n-1} &< y_n \\ 3y_{n-1} + 4 &< 3y_n + 4 \\ \frac{3y_{n-1} + 4}{4} &< \frac{3y_n + 4}{4} \\ y_n &< y_{n+1} \end{aligned}$$

1.3.2 (a) A real number s is a greatest lower bound for a set $A \subseteq \mathbb{R}$ if

i. s is a lower bound for A

ii. if for every lower bound b, $b \leq s$

(b) Lemma 1.3.7 for infimums:
Suppose $s \in \mathbb{R}, A \subseteq \mathbb{R}$ where s is a lower bound for A, then $inf(A) = s$ if and only if for all $\epsilon > 0$ there exists $a \in A$ such that $a < s + \epsilon$.

i. ($\Rightarrow$) Suppose $s = inf(A), \epsilon > 0$. Then the number $s + \epsilon$ is not a lower bound as that would contradict being less than or equal to s. Since it is not a lower bound on A, then there is a smaller element a of A. Therefore $a < s + \epsilon$

ii. ($\Leftarrow$) Suppose for all $\epsilon > 0$ there exists $a \in A$ such that $a < \epsilon + s$. We must show that $s = inf(A)$. Since s is already a lower bound, we simply need to show that any lower bound is less than or equal to s. Suppose b is a lower bound on A. Since we have already shown that for any number greater than s, it can't be a lower bound, then conversely for b, since b is a lower bound then $b \leq s$.

1.3.3 Suppose $A, B \subseteq \mathbb{R}$, $sup(A) < sup(B)$. We must show that there exists $b \in B$ such that b is an upper bound of A. By theorem 1.3.7 for all $\epsilon > 0$ there exists $b \in B$ such that $sup(b) - \epsilon < b$. Since $sup(A) < sup(B)$ then $0 < sup(B) - sup(A)$. Therefore let $\epsilon = sup(B) - sup(A)$. Therefore there exists $b \in B$ such that $sup(A) < b$. By definition of supremum b is an upper bound on A.

1.3.9 (a) A finite, non-empty set always contains its supremum
True

(b) If $a < L$ for every element a in A then $sup(A) < L$.
False, if $L = sup(A)$ and A does not contain it's supremum then we satisfy the preposition, however $sup(A) < sup(A)$ is clearly false.

(c) If A and B are sets with the property that $a < b$ for every $a \in A$ and $b \in B$, then it follows that $sup(A) < inf(B)$.
False, if $A = [0, 1), B = (1, 2]$ then it is true for all $a \in A, b \in B$ that $a < b$. However $sup(A) = 1, inf(B) = 1$ as by theorem 1.3.7 subtracting any $\epsilon > 0$, there exists $a \in A$ such that $1 - \epsilon < a$ and similarly for the result proved in exercise 1.3.2. Therefore $sup(A) = inf(A)$.

(d) True

(e) True, proved in the previous exercise above.

1.4.2 Recall that $\mathbb{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbb{Q}$, then $a+b, ab \in \mathbb{Q}$.
Suppose $a, b \in \mathbb{Q}$. By definition of being members of $\mathbb{Q}$, there exists $c, d, e, f \in \mathbb{Z}$ such that $a = \frac{c}{d}, b = \frac{e}{f}$.

- We will show that $a + b \in \mathbb{Q}$

$$\begin{aligned} a + b &= \frac{c}{d} + \frac{e}{f} \\ &= \frac{cf + ed}{df} \end{aligned}$$

  Since $\mathbb{Z}$ is closed under addition and multiplication, $cf + ed \in \mathbb{Z}, df \in \mathbb{Z}$. Therefore $a + b$ satisfies the definition of a rational number
- We will show that $ab \in \mathbb{Q}$. Since $ab = \frac{ce}{df}$ by definition, and $\mathbb{Z}$ is closed under multiplication then $ce \in \mathbb{Z}, df \in \mathbb{Z}$. Therefore $ab \in \mathbb{Q}$.

(b) Suppose $a \in \mathbb{Q}, t \in \mathbb{I}$.

- We must show that $a+t \in \mathbb{I}$. Suppose for contradiction that $a+t \in \mathbb{Q}$. Then there exists $r \in \mathbb{Q}$ such that $a + t = r$. Since $\mathbb{Q}$ is closed under addition then $t \in \mathbb{Q}$. This is a contradiction as $t \notin \mathbb{Q}$.
- We must show that if $a \neq 0$ then $at \in \mathbb{I}$. Suppose for contradiction that $at \in \mathbb{Q}$. Then there exists $r \in \mathbb{Q}$ such that $at = r$. Since $\mathbb{Q}$ is closed under non-zero division then $t \in \mathbb{Q}$. This is a contradiction as $t \notin \mathbb{Q}$.

(c) Given two irrational numbers $s, t \in \mathbb{I}$ we can say nothing about whether $st \in \mathbb{I}$ or $s + t \in \mathbb{I}$. As $\sqrt{3} * \sqrt{2} = \sqrt{6} \in \mathbb{I}$, however $\sqrt{2} * \sqrt{2} = 4 \in \mathbb{Q}$. Similarly $\frac{\sqrt{2}}{2} \in \mathbb{I}$, $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \in \mathbb{I}$, however $\sqrt{2}, -\sqrt{2} + 2 \in \mathbb{I}$, $\sqrt{2} - \sqrt{2} + 2 = 2 \in \mathbb{Q}$. Therefore you can't say anything conclusive about the product and sum of general irrational numbers.

1.4.6 (a) Let $T = \{x \in \mathbb{R} : x^2 < 2\}, \alpha = \sup T$. Suppose for contradiction that $\alpha^2 > 2$. Suppose $n \in \mathbb{N}$. Then we have that

$$\begin{aligned} \left(\alpha - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n} \end{aligned}$$

By the archimedean principle, we may choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha}$. Therefore if we set $n_0 = n$ we have that $\left(\alpha - \frac{1}{n}\right)^2 > \alpha^2 - \alpha^2 + 2 = 2$. This contradicts the fact that all upper bounds must be greater than or equal to α.

(b) Suppose $b \geq 0$. Let $T = \{x \in \mathbb{R} : x^2 < b\}, \alpha = \sup T$. We will show that $\alpha^2 = b$ by cases. Two notes before proceeding with the proof. First note that we already know that $0^2 = 0 \in \mathbb{R}$, therefore we will operate on the assumption that $b > 0$.

1.5.4 Let S be the set of all infinite binary strings. Suppose for contradiction there exists a bijection $f : \mathbb{N} \to S$. Let the binary string b be defined by $b_n = \begin{cases} f(n)_n + 1 & f(n)_n = 0 \\ f(n)_n - 1 & f(n)_n = 1 \end{cases}$ where $f(n)_n$ is the nth bit in the binary string given by $f(n)$. We claim that for all $n \in \mathbb{N}$ $f(n) \neq b$. For $n = 1$ we have that $f(1)_1 \neq b_1$, thus $f(1) \neq b$. By the principle of mathematical induction for all $k \in \mathbb{N}$ if $k < n$ then $b_k \neq f(k)_k$. Looking at $f(n)_n$, the bit at postition n is exact the inverted bit at b_n. Therefore $f(n) \neq b$. Therefore by induction, for all $m \in \mathbb{N}, f(m) \neq b$. This is a contradiction as f is a bijection from $f : \mathbb{N} \to S$. Therefore f is uncountable.

1.5.5 (a) Let $A = \{a, b, c\}$. Then $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$

(b) We must show that if A is a set and $|A| = n$ then $|\mathcal{P}(A)| = 2^n$. We will show this by induction. Assume WLOG that $A = \{1, 2, \ldots, n\}$. For $n = 1$ we have $A = \{1\}$, therefore $\mathcal{P}(A) = \{\emptyset, \{1\}\}$. There are two elements in $\mathcal{P}(A)$, thus $|\mathcal{P}(A)| = 2 = 2^1$. By PMI for all $k \in \mathbb{N}$ if $k < n$ then for a set B of size k we have that $|\mathcal{P}(B)| = 2^k$. Let $A' = A \backslash \{n\}$. Since $|A'| = n - 1 < n$ then by the induction hypothesis $|\mathcal{P}(A')| = 2^{n-1}$. We claim that $|\mathcal{P}(A) \backslash \mathcal{P}(A')| = 2^{n-1}$. Note that by definition if a subset $S \subseteq A$ is not in $\mathcal{P}(A')$ implies that $n \in S$. If we consider the set $S \backslash \{n\}$ since S is already a subset of A then $S \backslash \{n\} \subseteq A'$. Since all the sets in $\mathcal{P}(A) \backslash \mathcal{P}(A')$ are simply modified copies of sets from $\mathcal{P}(A')$ and $|\mathcal{P}(A')| = 2^{n-1}$ then $|\mathcal{P}(A) \backslash \mathcal{P}(A')| = 2^{n-1}$. Therefore

$$|\mathcal{P}(A)| = |\mathcal{P}(A') \cup \mathcal{P}(A) \backslash \mathcal{P}(A')| = |\mathcal{P}(A')| + |\mathcal{P}(A) \backslash \mathcal{P}(A')| = 2^{n-1} + 2^{n-1} = 2^n.$$

1.5.6 (a)
- Let $f : A \to \mathcal{P}(A)$ be given by:

$$f(a) = \{a\}, f(b) = \{b\}, f(c) = \{c\}.$$

- Let $h : A \to \mathcal{P}(A)$ be given by:

$$h(a) = \{a, b\}, h(b) = \{b, c\}, h(c) = \{a, c\}.$$

(b) Let $B = \{1, 2, 3, 4\}$. Let $g : B \to \mathcal{P}(B)$ be given by:

$$g(1) = \{1\}, g(2) = \{2\}, g(3) = \{3\}, g(4) = \{4\}.$$

(c) It is impossible to have an onto mapping for the previous two parts because for any arbitrary set A with size n, any arbitrary function between the two is mapping from n elements to 2^n elements. By definition a function must have a single output for every input

1.5.7
- $\{a, b, c\}$
- $\{a, b, c\}$
- $\{1, 2, 3, 4\}$

1.5.8 (a) Suppose $a' \in B$. Then by definition of B $a' \notin f(a') = B$. This is a contradiction

(b) Suppose $a' \notin B$. This means by the construction of B, $a' \in f(a')$. However $f(a') = B$. This is a contradiction

1.5.9 (a) We claim that the set of all functions from $\{0, 1\}$ to $\mathbb{N}$ is countable. Each function can be represented as $\{(0, a), (1, b)\}$ where $a, b \in \mathbb{N}$. Note that there is an obvious bijection between the set $\{(0, a), (1, b)\}$ and (a, b). Therefore the set of all functions from $\{0, 1\}$ to $\mathbb{N}$ has the same cardinality as $\mathbb{N}^2$. Since $\mathbb{N}^2$ has the same cardinality as $\mathbb{N}$, then we have shown that the set of all function from $\{0, 1\}$ to $\mathbb{N}$ is countable.

(b) We claim that the set of all functions from $\mathbb{N}$ to $\{0, 1\}$ is uncountable. We claim that there is a bijection between the set of all functions from $\mathbb{N}$ to $\{0, 1\}$ and the set $S = \{(a_1, a_2, \ldots) : a_n = 0 \text{ or } a_n = 1\}$ as defined in exercise 1.5.4 which was proven to be uncountable. Let $f : \{0, 1\}^{\mathbb{N}} \to S$ be given by $g(f) = (f(1), f(2), f(3), \ldots)$. Suppose $f_1, f_2 \in \{0, 1\}^{\mathbb{N}}, g(f_1) = g(f_2)$. We must show that $f_1 = f_2$. Since f_1, f_2 are over the natural numbers, it suffices to show that for all $n \in \mathbb{N}, f_1(n) = f_2(n)$. Since $g(f_1) = g(f_2)$, then $f_1(n) = f_2(n)$ for all $n \in \mathbb{N}$. Therefore $f_1 = f_2$. Since $\{0, 1\}^{\mathbb{N}}$ injects into S, and S is uncountable

(c) Let the set S be as described in problem 1.5.4, the set of all binary sequences. Let the function $f : \mathbb{N} \times S \to \mathbb{N}$ be given by

$$f(n, s) := \begin{cases} 2n & s_n = 0 \\ 2n - 1 & s_n = 1. \end{cases}$$

For a sequence $s \in S$ let $A_s = \{f(n, s) : n \in \mathbb{N}\}$. We claim that the set $K = \{A_s : s \in S\}$ is an uncountable antichain.

- From 1.5.4 we know that S is uncountable. We claim that there is a 1-1 correspondence with S and K. Let the function $g : S \to K$ be given by $g(s) = A_s$. We claim that g is 1-1. Suppose $r, s \in S, g(r) = g(s)$. We must show that $r = s$. To show that $r = s$, it suffices to show for all $n \in \mathbb{N}$, $r_n = s_n$. Suppose $n \in \mathbb{N}, s_n = 1$. Then $2n - 1 \in g(s), g(r)$. Since $2n - 1 \in g(r)$, then $r_n = 1$ as this is the only condition under which $2n - 1 \in g(r)$. A similar proof exists for $s_n = 0$. Since $s_n = r_n$ for arbitrary n, then $s = r$.
- We claim that for arbitrary distinct $r, s \in S$ that $A_s \not\subset A_r, A_r \not\subset A_s$. Suppose $r, s \in S, r \neq s$. Then by definition there must exists $n \in \mathbb{N}$ such that $r_n \neq s_n$. Suppose WLOG $r_n = 1$. Therefore $s_n = 0, 2n-1 \in A_r, 2n \in A_s$. Since $r_n = 1$ then $2n \notin A_r$ as if it was then that would contradict $2n - 1 \in A_r$. Similarly, there would be a contradiction if $2n - 1 \in A_s$. Therefore $A_s \not\subset A_r, A_r \not\subset A_s$.

Since there is a 1-1 correspondence with an uncountable set, and K is an antichain then we have satisfied finding an uncountable subset of $\mathcal{P}(\mathbb{N})$.

2.2.1 (a) Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ with $N > \sqrt{\frac{\frac{1}{\epsilon}-1}{6}}$. To verify that our choice of N is appropriate, let $n \in \mathbb{N}$ satisfy $n > N$. Therefore

$$\begin{aligned} n &> \sqrt{\frac{\frac{1}{\epsilon}-1}{6}} \\ n^2 &> \frac{\frac{1}{\epsilon}-1}{6} \\ 6n^2+1 &> \frac{1}{\epsilon} \\ \frac{1}{6n^2+1} &< \epsilon. \end{aligned}$$

Since $n^2 > 0$ for all $n \in \mathbb{N}$ then we have just proved $|\frac{1}{6n^2+1}| < \epsilon$.

2.2.7 (a) The convergence to infinity definition is thus: For all $\epsilon > 0$ there exists $B]in\mathbb{N}$ such that $n > N$ implies $|x_n| > \epsilon$.
Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > \epsilon^2$. To verify that our choice of N is correct, let $n \in \mathbb{N}$ satisfy $n > N$. Therefore $n > \epsilon^2 \Rightarrow \sqrt{n} > \epsilon$. Since $\sqrt{n} > 0$ for all $n \in \mathbb{N}$ then we have shown that $\lim_{n\to\infty} |\sqrt{n}| = \infty$

(b) The sequence $(1, 0, 2, 0, 3, 0, 4, \ldots)$ does not converge to infinity as choosing $\epsilon = 0.5$ does not satisfy the condition. As for all $n \in \mathbb{N}, x_{2n} = 0$ thus no matter how large N is choosen to be $x_{2n} < 0.5$.

2.2.8 (a) The sequence $(-1)^n$ is frequently in the set $\{1\}$ as given an arbitrary $N \in \mathbb{N}$ if N is even then the choice $n = N$ yields $(-1)^{2N} = 1$ and if N is odd where $N = 2l-1$ then $n = N+1$ yields $(-1)^n = (-1)^{2l+2} = ((-1)^2)^{l+1} = 1^{l+1} = 1$.
We claim that the sequence is not eventually in the set $\{1\}$ Supposes $N \in \mathbb{N}$. If N is odd then there exists $l \in \mathbb{Z}_{\leq 0}$ such that $N = 2l+1$. Since $N \leq N$ then $(-1)^N = (-1)^{2l+1} = -1 \notin \{1\}$. If N is even then we take $N+1 > N$, therefore $(-1)^{N+1} = (-1)^{2l+1} = -1 \notin \{1\}$. Since every choice of N results in the sequence not being 1 after a certain point then it is not eventually in $\{1\}$.

(b) Eventually is a stronger definition then frequently. As shown above we can have sequences frequently dance in and out of sets, but they are not guaranteed to stay inside after a certain point. We claim that eventually implies frequently and that the converse is not true:

- Suppose (a_n) is eventually in the set A. Therefore there exists $N_e \in \mathbb{N}$ such that for all $n \geq N_e, a_n \in A$. We must show that (a_n) is frequently in A. Suppose $N \in \mathbb{N}$. We must show there exists $n \geq N$ such that $a_n \in A$. We claim that all $n \geq N_e$ satisfy the definition. If $N \leq N_e$ then the definition is satisfied by $n = N_e$. If $N > N_e$ then any natural number k such that $k > N$ would suffice as $k > N > N_e$, therefore $a_k \in A$. Therefore (a_n) is frequently in A.
- Frequently does not imply eventually as part (a) of this problem serves as a counterexample.

2.3.10 If $(a_n) \to 0$ and $|b_n - b| \leq a_n$, then show that $(b_n) \to b$.
Proof: Since $(a_n) \to 0$ then for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N, |a_n| < \epsilon$. Therefore $a_n < \epsilon$. Thus $|b_n - b| < \epsilon$.

2.4.1

2.4.6 (a) Suppose (a_n) is a bounded sequence. Prove that the sequence $y_n = \sup\{a_k : k \geq n\}$ converges.

Proof: We claim that (y_n) is a decreasing and bounded sequence.

- We claim that (y_n) is decreasing. Suppose $n \in \mathbb{N}$. We must show that $y_n \geq y_{n+1}$. We have two cases, $y_n = a_n, y_n \neq a_n$.
  - Suppose $a_n = y_n$. Therefore for all other elements in the sequence after a_n, $a_n \geq a_k$ where $k > n$. Therefore a_n is an upper bound on $\{a_k : k \geq n+1\}$. Since y_{n+1} is the supremum of the set mentioned before, and we have established that a_n is an upper bound, then by definition $y_n = a_n \geq y_{n+1}$.
  - Suppose $a_n \neq y_n$. Since a_n is already not a suprememum of the set $\{a_k : k \geq n\}$ then computing the supremum of the set excluding a_n, $\{a_k : k \geq n+1\}$ should not change the supremum value. Therefore $y_n = y_{n+1}, y_n \geq y_{n+1}$.

  Therefore (y_n) is a decreasing sequence.
- We claim that (y_n) is bounded. Since (y_n) is decreasing we simply need to show that there is a quantity larger than y_1. Since $y_1 = \sup\{a_k : k \geq 1\}$ then y_1 is the supremum for (a_n). Since (a_n) is bounded, suppose for some quantity $M \in \mathbb{R}$, then by definition of supremum $y_1 \leq M$. Therefore (y_n) is bounded

Therefore by the monotone convergences theorem (y_n) converges.

(b) Let $z_n = \inf\{a_k : k \geq n\}$. Then $\lim z_n = \liminf a_n$. This should converge since (z_n) can easily be proved to be increasing and bounded.

(c)
- Prove that $\liminf a_n \leq \limsup a_n$

  Proof: Suppose $n \in \mathbb{N}$. For an arbitrary element $e \in \{a_k : k \geq n\}$, $e \leq y_n, e \geq z_n$ by the respective definitions of supremum and infimum. Therefore for all $n \in \mathbb{N}, z_n \leq y_n$. Since we know that $\liminf a_n, \limsup a_n$ exists, then by the algebraic order theorem $\liminf a_n \leq \limsup a_n$.
- An example of a strict inequality between $\liminf a_n$ and $\limsup a_n$ is the sequence $a_n = \frac{1}{n} + (-1)^{n+1}$. This is because $\liminf a_n = \lim \frac{1}{n} - 1 = -1 < 1 = \lim \frac{1}{n} + 1 = \limsup a_n$

(d) We must prove that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists

$\Rightarrow$ Suppose $\liminf a_n = \limsup a_n$. We must show that $\lim a_n$ exists. Note that by definition of infimum and supremum for all $k \in \mathbb{N}, z_k \leq a_k, a_k \leq y_k$. Therefore for all $k \in \mathbb{N}, z_k \leq a_k \leq y_k$. Since $\lim z_n = \lim y_n$, then by the squeeze theorem $\lim a_n = \liminf a_n = \limsup a_n$.

$\Leftarrow$ Suppose $\lim a_n$ exists and suppose for contradiction that $\liminf a_n \neq \limsup a_n$. Since $\liminf a_n \leq \limsup a_n$ then $\liminf a_n < \limsup a_n$. Therefore $0 < \liminf a_n - \limsup a_n$. Let l be the limit point of (a_n). Note that since the inequality between $\liminf a_n$ and $\limsup a_n$ is strict then l can at most converge to either $\liminf a_n$ or $\limsup a_n$, therefore we assume WLOG $l < \limsup a_n$.

2.5.6 Note that $\sup S$ exists as (a_n) is bounded above by M. Therefore let $\sup S = s$ We claim for arbitrary $\epsilon > 0$ there are an infinite number of elements from the sequence (a_n) contained within $V_\epsilon(s)$. By definition of suprememum, for arbitrary $\epsilon > 0, s - \epsilon \in S$. Therefore by definition there are an infinite number of elements of (a_n) above $s - \epsilon$. Similiarly, by definition of supremum $s + \epsilon \notin S$. Therefore there are only finitely many elements of (a_n) above $s + \epsilon$. Since (a_n) is infinite then there must be infinitely many terms less than $s+\epsilon$. Since there are infinitely many terms greater than $s-\epsilon$ and above $s+\epsilon$ there are only finitely many terms then there must be an infinite number of terms between $s - \epsilon$ and $s + \epsilon$. Therefore $V_\epsilon(s)$ has an infinite number of terms from (a_n). We will now construct by induction a subsequence of (a_n) which converges to s, where each a_{n_k} sits within $V_{2^{-k}}(s)$. For $k = 1$, since $0 < 1$, we have an infinite number of terms in $V_1(s)$. Choose $n_1 \in \mathbb{N}$ satisfying $a_{n_1} \in V_1(s)$. By the principle of mathematical induction for all $j \in \mathbb{N}$ if $j < k$ then there exists $n_j \in \mathbb{N}$ satisfying $n_j > n_{j-1} > \cdots > n_1$ and $a_{n_j} \in V_{2^{-j}}(s)$. By the induction hypothesis $a_{n_{k-1}} \in V_{2^{1-k}}(s)$. As proved above there is an infinite number of terms in $V_{2^{-k}}(s)$, therefore we can go out far enough and select a $n_k \in \mathbb{N}$ such that $n_k > n_{k-1}, a_{n_k} \in V_{2^{-k}}(s)$. Now, to prove convergence, suppose $\epsilon > 0$. Since (2^{-n}) goes to 0 then we can find sufficient $k \in \mathbb{N}$ such that $2^{-k} < \epsilon$. Since (2^{-n}) is a decreasing sequence then for any $m > n, V_{2^{-n}}(s) \supseteq V_{2^{-m}}(s)$. Therefore for all $l \geq k, a_{n_l} \in V_{2^{-k}}(s)$. Therefore (a_{n_k}) converges to s.

2.6.4 Assume $(a_n), (b_n)$ are Cauchy. We must show that the series formed by $c_n = |a_n - b_n|$ is Cauchy. Let $\epsilon > 0$. Since $(a_n), (b_n)$ are Cauchy then there exists $N_1, N_2 \in \mathbb{N}$ such that for all $n_1, m_1 \geq N_1, |a_{n_1} - a_{m_1}| < \frac{\epsilon}{2}$ and for all $n_2, m_2 \geq N_2, |b_{n_2} - b_{m_2}| < \frac{\epsilon}{2}$. Let $N = \max(N_1, N_2)$. Therefore, for all $m, n \geq N$,

$$\begin{aligned}
|c_n - c_m| &= ||a_n - b_n| - |a_m - b_m|| \text{ definition of } c_n \\
&\leq |a_n - b_n - a_m + b_n| \text{reverse triangle inequality} \\
&= |a_n - a_m + b_m - b_n| \\
&\leq |a_n - a_m| + |b_n - b_m| \text{trianlge inequality} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

2.7.13 (a)

$$
\begin{aligned}
|\sum_{j=m+1}^{n} x_n y_n| &= |s_n y_{n+1} - s_m y_{m+1} + \sum_{j=m+1}^{n} s_j (y_j - y_{j+1})| && \text{Exercise 2.7.12} \\
&\leq M|y_{n+1} - y_{m+1} + \sum_{j=m+1}^{n} (y_j - y_{j+1})| && \text{Upper bound on } (s_n) \\
&\leq M|y_{n+1} + y_{m+1} + \sum_{j=m+1}^{n} (y_j - y_{j+1})| \\
&= M|y_{n+1} + y_{m+1} + y_{m+1} - y_{n+1}| && \text{Expanding the telescoping series} \\
&= 2M|y_{m+1}|
\end{aligned}
$$

(b) Dirichlet's Test proof:
We will show that the series $t_m = \sum_{j=1}^{m} x_j y_j$ converges by the Cauchy Criterion for Series. Let $\epsilon > 0$. Since (y_n) converges to 0 then there exists $N \in \mathbb{N}$ such that for all $n \geq N$ $|y_n| < \frac{\epsilon}{2M}$. Therefore for all $n > m \geq N$

$$
|t_n - t_m| = |\sum_{j=m+1}^{n} x_j y_j| \leq 2M|y_{m+1}| \leq 2M|y_N| < 2M\frac{\epsilon}{2M} = \epsilon.
$$

Therefore by the Cauchy Criterion for Series, (t_m) converges.

(c) The Alternating Series Test is simply the cases where $x_n = (-1)^{n+1}$, as it is a sequence bounded above and below by 1. The requirement on y_n is the exact same as in the Alternating Series Test

3.2.10 (a) Prove for a collection of sets $\{E_\lambda : \lambda \in \Lambda\}$ that

$$(\cup_{\lambda\in\Lambda} E_\lambda)^c = \cap_{\lambda\in\Lambda} E_\lambda^c \text{ and } (\cap_{\lambda\in\Lambda} E_\lambda)^c = \cup_{\lambda\in\Lambda} E_\lambda^c$$

Proof: Suppose $x \in (\cup_{\lambda\in\Lambda} E_\lambda)^c$. We must show that $x \in \cap_{\lambda\in\Lambda} E_\lambda^c$. By definition of set compliment, $x \notin \cup_{\lambda\in\Lambda} E_\lambda$. Since x does not belong to the union of E_λ for every $\lambda \in \Lambda$, then $x \notin E_\lambda$ for each $\lambda \in \Lambda$. Therefore by definition of set compliment, $x \in E_\lambda^c$ for every $\lambda \in \Lambda$. Therefore by definition of set intersection, $x \in \cap_{\lambda\in\Lambda} E_\lambda^c$.
Next, suppose $x \in \cap_{\lambda\in\Lambda} E_\lambda^c$. We must show that $x \in (\cup_{\lambda\in\Lambda} E_\lambda)^c$. By definition of set intersection, $x \in E_\lambda^c$ for every $\lambda \in \Lambda$. Therefore by definition of set compliment, $x \notin E_\lambda$, for every λ. Since x does not belong to any of E_λ's individually, then x does not belong to the union. Therefore $x \notin \cup_{\lambda\in\Lambda} E_\lambda$. Therefore by the definition of set compliment, $x \in (\cup_{\lambda\in\Lambda} E_\lambda)^c$.
Since we have established that $(\cup_{\lambda\in\Lambda} E_\lambda)^c = \cap_{\lambda\in\Lambda} E_\lambda^c$, by defining $G_\lambda = E_\lambda^c$ for every $\lambda \in \Lambda$, then we have that $(\cup_{\lambda\in\Lambda} G_\lambda)^c = \cap_{\lambda\in\Lambda} G_\lambda^c$. If we take the compliment of the left hand side then we have that

$$\cup_{\lambda\in\Lambda} E_\lambda^c = ((\cup_{\lambda\in\Lambda} G_\lambda)^c)^c = (\cap_{\lambda\in\Lambda} G_\lambda^c)^c = (\cap_{\lambda\in\Lambda} (E_\lambda^c)^c)^c = (\cap_{\lambda\in\Lambda} E_\lambda)^c.$$

Therefore $(\cap_{\lambda\in\Lambda} E_\lambda)^c = \cup_{\lambda\in\Lambda} E_\lambda^c$

(b) i. The union of a finite number of closed sets is closed
Let $\{E_1, \cdots, E_n\}$ be a finite set of closed sets. If we consider that E_i^c is open for all $i \in [n]$, and take their intersection then we know by theorem 3.2.3 that $\cap_{i=1}^n E_i^c$ is open. Therefore if we take the compliment and and apply DeMorgan's law then we have that $(\cap_{i=1}^n E_i^c)^c = ((\cup_{i=1}^n E_i)^c)^c = \cup_{i=1}^n E_i$ is closed by theorem 3.2.13.

ii. The intersection of an arbitrary number of closed sets is closed. Let $\{E_\lambda : \lambda \in \Lambda\}$ be a collection of closed sets. Noting that E_λ^c is open for all $\lambda \in \Lambda$ then we know by theorem 3.2.3 that $\cup_{\lambda\in\Lambda} E_\lambda^c$ is open. Therefore by taking the compliment and applying DeMorgan's law we have that $(\cup_{\lambda\in\Lambda} E_\lambda^c)^c = ((\cap_{\lambda\in\Lambda} E_\lambda)^c)^c = \cap_{\lambda\in\Lambda} E_\lambda$ is closed by theorem 3.2.13.

3.3.10 The only sets which are "clompact" are finite collections of real number singletons and the empty set. Otherwise if a set has a non-zero length then one can scale a copy of $\cup_{n=0}^{\infty}[2^{-(n+1)}, 2^{-n}]$ to be shorter than whatever length given and translated to be inside the set. Once inside the set, there is no finite sub cover of the sets given which can be chosen to still cover the set, as they fit together exactly to form the interval $[0, 1/2]$.

3.4.7 (a) Find a disconnected set whose closure is connected:
$(1,2)\cup(2,3)$ is a disconnected set as proven by exercise 3.4.5. Clearly $[1,2]\cup[2,3]$ are not disconnected as they form the interval $[1,3]$, which by theorem 3.4.7 is in fact connected.

(b) If A is connected, is $\overline{A}$ connected? If A is perfect is $\overline{A}$ perfect too?

- As shown by theorem 3.4.7, all connected sets correspond with intervals. Therefore the only interval which will gain points would be an open interval, say $A=(a,b)$. Since $\overline{A}=[a,b]$ is still an interval, then $\overline{A}$ is connected
- If A is perfect, then by definition it is closed. Therefore $\overline{A}$ doesn't gain any new points, as all limit points of A are already self contained. Therefore $\overline{A}$ is still perfect.

3.5.3 (a) Show that a closed interval $[a, b]$ is a G_δ set.
Suppose $r \in (0, 1)$. We claim that $[a, b] = \bigcap_{n=1}^{\infty}(a - r^n, b + r^n)$.

- We want to show that $[a, b] \subseteq \bigcap_{n=1}^{\infty}(a - r^n, b + r^n)$.
Since for all $n \in \mathbb{N}, [a, b] \subseteq (a - r^n, b + r^n)$, then by the definition of set intersection, $[a, b] \subseteq \bigcap_{n=1}^{\infty}(a - r^n, b + r^n)$.
- We want to show that $\bigcap_{n=1}^{\infty}(a - r^n, b + r^n) \subseteq [a, b]$. Suppose $x \in \bigcap_{n=1}^{\infty}(a - r^n, b + r^n)$. We must show that $a \leq x \leq b$. First to show that $a \leq x$. We know that for all $n \in \mathbb{N}, a - r^n < x$, therefore x is an upper bound on $\{a - r^n : n \in \mathbb{N}\}$. Therefore all we need to prove is that $a \in \bigcap_{n=1}^{\infty}(a - r^n, b + r^n)$. Since a is the limit point of the sequence $(a - r^n)_{n=1}^{\infty}$, then there is no set $(a - r^n, b + r^n)$ which excludes it. Therefore $a \in \bigcap_{n=1}^{\infty}(a - r^n, b + r^n)$. Thus, since x is an upper bound on $(a - r^n)_{n=1}^{\infty}$, and our set contains a, then $a \leq x$. Next we must show that $x \leq b$. We know that for all $n \in \mathbb{N}, x < b + r^n$. Thus x is a lower bound on $(b + r^n)$. Therefore, similar to above, $x \leq b$. Therefore $a \leq x \leq b$. Thus $x \in [a, b]$

(b) Show that the half open interval $(a, b]$ is both G_δ and F_σ set

- Show that $(a, b]$ is a G_δ set
We claim that $\cap_{n=1}^{\infty}(a, b + r^n) = (a, b]$. As shown in the proof above, for the closed end point, the set above converges, and for the open right endpoint, it is trivial.
- Show that $(a, b]$ is a F_σ set
We claim that $\cup_{n=1}^{\infty}[a + \frac{1}{n}, b] = (a, b]$.

  – We must show that $\cup_{n=1}^{\infty}[a - \frac{1}{n}, b] \subseteq (a, b]$. Suppose $x \in \cup_{n=1}^{\infty}[a + \frac{1}{n}, b]$ We must show that $a < x \leq b$. Since $x \leq b$ is trivial, we must show that $a < x$. Since x is in the union, there must be a smallest $n_0 \in \mathbb{N}$ in which $x \in [a + \frac{1}{n_0}, b]$. Therefore $a < a + \frac{1}{n_0} \leq x$.
  – We must show that $(a, b] \subseteq \cup_{n=1}^{\infty}[a - \frac{1}{n}, b]$. Since $\{b\}$ is trivially in both, we must show that $(a, b) \subseteq \cup_{n=1}^{\infty}[a - \frac{1}{n}, b]$. Suppose $x \in (a, b)$. Therefore there exists $\epsilon > 0$ such that $V_\epsilon(x) \subset (a, b)$. Since the neighborhood is contained within (a, b) then $a < x - \epsilon$. Therefore $0 < x - \epsilon - a$. Thus by the archimedean principle there exists $n' \in \mathbb{N}$ such that $\frac{1}{n'} < x - \epsilon - a$. Therefore $a + \frac{1}{n'} < x - \epsilon$, and therefore $V_\epsilon(x) \subset [a + \frac{1}{n'}, b] \subseteq \cup_{n=1}^{\infty}[a + \frac{1}{n}, b]$. Thus $(a, b] \subseteq \cup_{n=1}^{\infty}[a + \frac{1}{n}, b]$

Making $(a, b]$ a F_σ set.

(c)
- Show that $\mathbb{Q}$ is a F_σ set.
Since $\mathbb{Q}$ is countable, then there exists a bijection between $\mathbb{N}$ and $\mathbb{Q}$. Therefore let $f : \mathbb{N} \to \mathbb{Q}$ be a bijection. Then we claim that $\bigcup_{n=1}^{\infty}[f(n), f(n)]$ is $\mathbb{Q}$. Since we're guaranteed to uniquely attain every rational number, then we have exactly $\mathbb{Q}$. Since we have a countable union of closed intervals, then we have satisifed the definition of F_σ
- Show that the set of irrationals forms a G_δ set.

4.2.5 Let f and g be functions defined on the domain $A \subseteq \mathbb{R}$, and assume $\lim_{x\to c} f(x) = L, \lim_{x\to c} g(x) = M$ for some limit point c of A.

(a) Show that $\lim_{x\to c}[f(x)+g(x)] = L+M$, assuming theorem 4.2.3 and the algebraic limit theorem for sequences.
Since both f and g converge at c, then by theorem 4.2.3 for any sequence which converges to c, x_n, $\lim f(x_n) \to L, \lim g(x_n) \to M$. Therefore by the algebraic limit theorem for sequences, $f(x_n)+g(x_n) \to L+M$. Therefore using the converse of theorem 4.2.3, this implies that $\lim_{x\to c}[f(x)+g(x)] = L+M$

(b) Show that $\lim_{x\to c}[f(x)+g(x)] = L+M$, without assuming theorem 4.2.3.
Let $\epsilon > 0$. Since f converges at c then there exists $\delta_1 > 0$ such that $|x-c| < \delta_1$ implies $|f(x)-L| < \frac{\epsilon}{2}$. Similarly since g converges at c there exists $\delta_2 > 0$ such that $|x-c| < \delta_2$ implies $|g(x)-M| < \frac{\epsilon}{2}$. Therefore, if we let $\delta = \min\{\delta_1, \delta_2\}$ then $|x-c| < \delta$ implies

$$\begin{aligned}
|f(x)+g(x)-(L+M)| &= |(f(x)-L)+(g(x)-M)| \\
&\leq |f(x)-L| + |g(x)-M| && \text{triangle inequality} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} && \text{convergence definitions} \\
&= \epsilon
\end{aligned}$$

(c) Show that $\lim_{x\to c} f(x)g(x) = LM$, assuming theorem 4.2.3 and the algebraic limit theorem for sequences.
Since both f and g converge at c, then by theorem 4.2.3 for any sequence which converges to c, x_n, $\lim f(x_n) \to L, \lim g(x_n) \to M$. Therefore by the algebraic limit theorem for sequences, $f(x_n)g(x_n) \to LM$. Therefore using the converse of theorem 4.2.3, this implies that $\lim_{x\to c} f(x)g(x) = LM$

(d) Show that $\lim_{x\to c} f(x)g(x) = LM$, without assuming theorem 4.2.3.
Let $\epsilon > 0$. Since f converges at c then there exists $\delta_1 > 0$ such that $|x-c| < \delta_1$ implies $|f(x)-L| < \frac{\epsilon}{2|M|}$. Note that by manipulating $|f(x)-L| < \frac{\epsilon}{2|M|}$ and applying the reverse triangle inequality we find that $|f(x)| < \frac{\epsilon}{2|M|} + |L|$. Let $B = \frac{\epsilon}{2|M|} + |L|$ Similarly since g converges at c then there exists $\delta_2 > 0$ such that $|x-c| < \delta_2$ implies $|g(x)-M| < \frac{\epsilon}{2|B|}$. Therefore, if we let $\delta = \min\{\delta_1, \delta_2\}$ then $|x-c| < \delta$ implies

$$\begin{aligned}
|f(x)g(x)-LM| &= |f(x)g(x) - Mf(x) + Mf(x) - LM| \\
&= |f(x)(g(x)-M) + M(f(x)-L)| \\
&\leq |f(x)(g(x)-M)| + |M(f(x)-L)| && \text{triangle inequality} \\
&= |f(x)||g(x)-M| + |M||f(x)-L| && \text{definition of absolute value} \\
&< |B||g(x)-M| + |M||f(x)-L| \\
&< |B|\frac{\epsilon}{2|B|} + |M|\frac{\epsilon}{2|M|} \\
&= \epsilon
\end{aligned}$$

4.2.8 Assume $f(x) \geq g(x)$ for all $x \in A \subseteq \mathbb{R}$ on which f, g are defined. Show that for any limit point $c \in A$ we must have

$$\lim_{x\to c} f(x) \geq \lim_{x\to c} g(x).$$

Let $\lim_{x\to c} f(x) = L, \lim_{x\to c} g(x) = M$. Suppose $(x_n) \subseteq A, (x_n) \to c$. Therefore by theorem 4.2.3, $\lim f(x_n) \to L, \lim g(x) \to M$. Since $(x_n) \subseteq A$ then $f(x_n) \geq g(x_n)$ for all $n \in \mathbb{N}$. Therefore by the order limit theorem, since $f(x_n) \geq g(x_n)$ then $L \geq M$. Therefore $\lim_{x\to c} f(x) \geq \lim_{x\to c} g(x)$

4.3.6 (a) Suppose $c \in \mathbb{Q}$. Thus $h(c) = 1$. Consider $\pi^{-n} + c = y_n$. Since π is transcendental then every y_n is irrational. Therefore $h(y_n) \to 0$.

Suppose $c \notin \mathbb{Q}$. Therefore $h(c) = 0$. Consider $x_n = \frac{p_n}{q_n}$, which is the nth decimal expansion of c. Since each $x_n \in \mathbb{Q}$, then $h(x_n) \to 1$.

Since every possible $\mathbb{R}$ value is disconntinous, then h is a nowhere continuous function.

(b) Suppose $c \in \mathbb{Q}$. Consider $x_n = c - \pi^{-n}$. As shown above, each x_n is irrational, thus $t(x_n) \to 0$, not to the non-zero value of $t(c)$.

(c) Let $\epsilon > 0, i \in \mathbb{R}\backslash\mathbb{Q}$, and consider the set $T = \{x \in \mathbb{R} : t(x) \geq \epsilon\}$. Since all elements in T are rational, then we know something about their denominators, in particular that they are bounded below by ϵ. Looking at $[i - \frac{1}{2}, i + \frac{1}{2}] \cap T$, we have a finite number of elements, therefore looking to $\min t([i - \frac{1}{2}, i + \frac{1}{2}] \cup T) = \frac{1}{m}$, then we have for each $x \in [i - \frac{1}{2}, i + \frac{1}{2}] \cap T$, $V_{\frac{1}{2m}}(x) \cap T = \{x\}$, otherwise multiple points would imply denominators smaller than $\frac{1}{m}$. Therefore if we choose $\delta < \frac{1}{2m}$, then for all $x \in [i - \frac{1}{2}, i + \frac{1}{2}] \cap T$, $x \notin V_\delta(c)$. Therefore if $y \in V_\delta(c), y \notin T$, thus $t(y) < \epsilon, y \in V_\epsilon(0)$. Thus we have convergence for $t(i)$.

4.3.11 (a) For $A = \mathbb{Z}$, the floor function, $[x]$

(b) For $A = (0, 1)$, the function $f(x) = \begin{cases} \infty & \text{if } x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$

(c) For $A = [0, 1]$, the function $f(x) = \begin{cases} \infty & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$

(d) For $A = \{\frac{1}{n} : n \in \mathbb{N}\}$, the function $f(x) = \begin{cases} [\frac{1}{x}] & \text{if } x \in (0,1) \\ 0 & \text{if } x \notin (0,1) \end{cases}$.

4.4.6 (a) Consider $h(x) = \sin(\frac{1}{x})$, and $x_n = \begin{cases} \frac{1}{\frac{\pi}{2}+2\pi n} & n \text{ even} \\ \frac{1}{\frac{3\pi}{2}+2\pi n} & n \text{ odd} \end{cases}$. Clearly $x_n \to 0$, however $h(x_n)$ fluctuates between 1 and -1.

(b) This is impossible, as by theorem 4.4.2 the image of $f([0,1])$ is a compact set, and since the Cauchy sequence converges, then it must converge in it's image.

(c) Similar to (b), since the sequence is Cauchy then it is bounded, thus we are mapping a compact set to another compact set, above logic applies.

(d) Consider the function $f(x) = \frac{1}{2} - |x - \frac{1}{2}|$. At $\frac{1}{2}$ clearly f attains a maximum. f has a 0 value at $x = 0, 1$. However since those points aren't attained, then the minimum can never be attained.

4.4.11 Show that g is continuous iff for all open sets $O \subseteq \mathbb{R}$, $g^{-1}(0)$ is open

$\Rightarrow$ Suppose g is continuous, O is an open set, if $g^{-1}(O)$ is empty then it is vacuously open, therefore we assume $g^{-1}(O)$ is nonempty. Let $x \in g^{-1}(O)$. Therefore $f(x) \in O$. Thus by the definition of open set there exists $\epsilon > 0, V_\epsilon(f(x)) \subseteq O$. Since g is continuous then there exists $\delta > 0$ such that $y \in V_\delta(x)$ implies $f(y) \in V_\epsilon(f(x))$. Therefore $V_\delta(x) \subseteq g^{-1}(O)$. Thus $g^{-1}(O)$ is open.

$\Leftarrow$ Suppose $O \subseteq \mathbb{R}$ is an open set implies $g^{-1}(O)$ is open. We will show that g is continuous. Let $c \in \mathbb{R}, \epsilon > 0$. Since $V_\epsilon(f(c))$ is open then by our assumption, $g^{-1}(V_\epsilon(f(c)))$ is open. Since $g^{-1}(V_\epsilon(f(c)))$ is open and nonempty then there exists $\delta > 0$ such that $V_\delta(c) \subseteq g^{-1}(V_\epsilon(f(c)))$. Therefore $y \in V_\delta(c)$ implies $f(y) \in V_\epsilon(f(c))$. Therefore f is continuous.

4.5.6 Let $f : [a, b] \to \mathbb{R}$ be continuous, $f(a) < 0 < f(b)$. We must show there exists $c \in [a, b]$ such that $f(c) = 0$. Let $I_0 = [a, b]$, consider the midpoint $z_1 = \frac{a+b}{2}$. If $f(z_1) > 0$, then set $b_1 = z_1, a_1 = a, I_1 = [a_1, b_1]$, and if $f(z_1) \leq 0$ then set $a_1 = z_1, b_1 = b$. This lends itself to a general formula for I_n, where for I_{n-1} with $z_{n-1} = \frac{a_{n-1}+b_{n-1}}{2}$, then if $f(z_{n-1}) > 0$, $I_n = [a_{n-1}, z_{n-1}]$, otherwise $I_n = [z_{n-1}, b_{n-1}]$. Since we have $I_0 \supseteq I_1 \supseteq \cdots$, then there exists $x \in \bigcap_{n=0}^{\infty} I_n$. We claim that (a_n) converges to x. Since each interval is half the length of the previous, then for an arbitrary $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - x| \leq 2^{-n}(b - a) < \epsilon$. A similar argument exists for $b_n \to x$. Since f is continuous then $\lim f(a_n) = \lim f(b_n) = f(x)$. We know by the algebraic limit theorem that since $f(a_n) \leq 0$ that $f(x) \leq 0$, and similarly $f(b_n) > 0$ implies that $f(x) \geq 0$. Therefore $f(x) = 0$.

5.2.8 (a) If a derivative function is not constant, then the derivative must take on some irrational values.

True. Suppose $f : A \to \mathbb{R}$, $f'(x)$ exists, $f'(x) \neq c$. Since A by definition is an interval for f' to be well defined then if A is a closed interval $[a, b]$ then by the darboux theorem f' attains all values between $f'(a)$ and $f'(b)$. Since $f'(a)$ and $f'(b)$ are real numbers then by the density of the irrationals in $\mathbb{R}$ there exists i such that (WLOG) $f'(a) < i < f'(b)$. Therefore f' always attains an irrational. If the interval is open, then we can find the midpoint, $m = \frac{a+b}{2}$, take the open ball around m, $V_\epsilon(m)$ guaranteed by it's entry in an open set, then take take the set $[m - \epsilon/2, m + \epsilon/2]$, which we know contain the end points as $m - \epsilon < m - \epsilon/2$ and $m + \epsilon/2 > m + \epsilon/2$, thus constructing a closed interval on which f' is defined.

(b) If f' exists on an open interval, and there is some point c where $f'(c) > 0$, then there exists a δ-neighborhood $V_\delta(c)$ around c in which $f'(x) > 0$ for all $x \in V_\delta(c)$).

False, consider the function

$$f(x) = \begin{cases} x + x^2 \sin(e^{1/|x|}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

at 0. If we evaluate the derivative manually we find $\lim_{x\to 0} 1 + x\sin(e^{1/|x|})$, which similar to evaluating $1 + x\sin(1/x)$ converges to 1 by the squeeze theorem. However with the actual evaluation of $f'(x) = 1 + 2x\sin(e^{1/|x|}) - e^{1/|x|}\cos(e^{1/|x|})$ clearly the $e^{1/|x|}\cos(e^{1/|x|})$ term will fluctuate wildly when approaching 0, contradicting that an open neighborhood around 0 will be purely positive.

(c) If f is differentiable on an interval containing zero and if $\lim_{x\to 0} f'(x) = L$, then it must be that $L = f'(0)$

True. Suppose for contradiction that $f'(0) \neq L$. Then there exists $\epsilon_0 > 0$ such that $|f'(0) - L| > \epsilon_0$. Since f' converges to L, then there exists $\delta_0 > 0$ such that $x \in V_{\delta_0}(0)$ implies $|f(x) - L| < \epsilon_0/2$. Since f is differentiable on $[-\delta_0/2, 0]$, then by darboux's theorem f' attains all values between $f'(-\delta_0/2)$ and $f'(0)$. This is a contradiction as for every x value in $(-\delta_0/2, 0)$, $|f'(x) - L| < |f'(0) - L|/2$, however all values between $f'(-\delta_0/2)$ and $f'(0)$ must be attained.

(d) The question above without the requirement of the limit existing

False, take $f(x) = \frac{x^2 - x}{x}$. Clearly $\lim_{x\to 0} \frac{x^2-x}{x} = \lim_{x\to 0} \frac{x(x-1)}{x} = -1$, however directly evaluating $f(0)$ is undefined.

5.3.8 Assume $g : (a,b) \to \mathbb{R}$ is differentiable at a point c. If $g'(c) \neq 0$ then there exists a δ-neighborhood $V_\delta(c) \subseteq (a,b)$ such that for every $x \in V_\delta(c) \backslash \{c\}$, $g(x) \neq g(c)$.
Proof: Assume $g'(c) \neq 0$. Suppose for contradiction that for every δ-neighborhood $V_\delta(c) \subseteq (a,b)$ there exists $x \in V_\delta(c) \backslash \{c\}$, $g(x) = g(c)$. If we consider $V_{\frac{1}{n}}(c) \backslash \{c\}$ then for each n we have an $x_n \in V_{\frac{1}{n}}(c)$ such that $g(x_n) = g(c)$. Therefore $\lim \frac{g(x_n) - g(c)}{x_n - c} = 0$, contradicting $g'(c) \neq 0$.

5.4.5 (a) Show that $g'(1)$ does not exists:
Consider the sequence $x_m = 1 + 2^{-m}$ where $m \in \mathbb{N}$. We will show that

$$\frac{g(x_m) - g(1)}{x_m - 1} = m + 1 - 2^{m+1}.$$

Note that

$$\begin{aligned}
g(x_m) &= \sum_{n=0}^{\infty} \frac{h(2^n(1+2^{-m}))}{2^n} \\
&= \sum_{n=0}^{\infty} \frac{h(2^n + 2^{n-m})}{2^n} \\
&= h(1+2^{-m}) + \sum_{n=1}^{\infty} \frac{h(2^n + 2^{n-m})}{2^n} \\
&= 1 - 2^m + \sum_{n=1}^{\infty} \frac{h(2^{n-m})}{2^n} && \text{periodicity of } h(x) \\
&= 1 - 2^m + \sum_{n=1}^{m} \frac{h(2^{n-m})}{2^n} && \text{if } n > m \text{ then } 2^{n-m} \text{ is a whole multiple of } 2 \\
&= 1 - 2^m + \sum_{n=1}^{m} \frac{2^{n-m}}{2^n} \\
&= 1 - 2^m + m2^{-m} \\
&= 1 + (m-1)2^{-m}.
\end{aligned}$$

Therefore,

$$\frac{g(x_m) - g(1)}{x_m - 1} = \frac{1 + (m-1)2^{-m} - 1}{1 + 2^{-m} - 1} = \frac{(m-1)2^{-m}}{2^{-m}} = m - 1.$$

Since the limit diverges to infinity, then $g'(1)$ does not exists.
Show that $g'(\frac{1}{2})$ does not exists.
Consider the sequence $x_m = 1 + 2^{-m}$ where $m \in \mathbb{N}$. We will show that

$$\frac{g(x_m) - g(1)}{x_m - 1} = m - 3$$

for a sufficently large m.

6.2.11 Assume (f_n) and (g_n) are uniformly convergent sequences of functions on the set A.

(a) Show that $(f_n + g_n)$ is a uniformly convergent sequence of functions
Proof: Let $\epsilon > 0$. Since $f_n \to f$ then there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1, |f(x) - f_n(x)| < \frac{\epsilon}{2}$. Similarly for g_n, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2, |f(x) - f_n(x)| < \frac{\epsilon}{2}$. Therefore if we take $N = \max\{N_1, N_2\}$, then consider $n \geq N$, we have

$$\begin{aligned} |(f(x) - g(x)) - (f_n(x) - g_n(x))| &= |(f(x) - f_n(x)) + (g(x) - g_n(x))| \\ &\leq |f(x) - f_n(x)| + |g(x) - g_n(x)| \qquad \text{triangle inequality} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

(b) Give an example to show that the product $(f_n g_n)$ may not converge uniformly.
Take $f_n = \frac{1}{x} + \frac{1}{n}, g_n = \frac{x^2+nx}{n}$. Since $f_n \to \frac{1}{x}$ and $g_n \to x$ then $f_n g_n \to 1$. Therefore if we compute their diference we find

$$\begin{aligned} |(\frac{1}{x} + \frac{1}{n})(\frac{x^2+nx}{n}) - 1| &= |\frac{x(x+n)(x+n)}{xn^2} - 1| \\ &= |\frac{x^2 + 2nx + n^2}{n^2} - 1| \\ &= |\frac{x^2 + 2nx}{n^2}| \\ &= |\frac{x^2}{n^2} + \frac{2x}{n}| \\ &\leq |\frac{x^2}{n} + \frac{2x}{n}| \\ &= \frac{1}{n}(x^2 + 2x) \end{aligned}$$

Therefore to have N be sufficiently large we must have $\frac{x^2+2x}{\epsilon} < N$. Since N depends on both x and ϵ then $f_n g_n$ is not uniformly continuous.

(c) Prove that if there exists an $M > 0$ such that $|f_n| \leq M$ and $|g_n| \leq M$ for all $n \in \mathbb{N}$, then $(f_n g_n)$ does converge uniformly.
Lemma: If $f_n \to f$ uniformly and $|f_n| \leq M$ for all $n \in \mathbb{N}$ then $|f| \leq M$. Since $f_n \to f$, then for every element in the sequence (k^{-1}) there exists $N_k \in \mathbb{N}$ such that for all $n \geq N_k$, $|f(x) - f_n)| < \frac{1}{k}$. Therefore by the triangle inequality and applying the bound we have that $|f(x)| < M + \frac{1}{k}$. Therefore by the algebraic order theorem $|f(x)| \leq M$
Proof: Let $\epsilon > 0$. Since $f_n \to f$ then there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1, |f(x) - f_n(x)| < \frac{\epsilon}{2M}$. Similarly for g_n, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2, |f(x) - f_n(x)| < \frac{\epsilon}{2M}$. Therefore if we take $N = \max\{N_1, N_2\}$, then

6.3.3 $f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}$. Since $\lim f_n = f = 0$, then $f' = 0$. Therefore we are solving $\frac{1-nx^2}{(1+nx^2)^2} = 0$. Since the denominator is strictly positive then we only have to solve for $1 - nx^2 = 0$. This is solved by $x = \pm\sqrt{\frac{1}{n}}$.

6.4.7 Let $h(x) = \sum_{n=1}^{\infty} \frac{1}{n^2+x^2}$

(a) Show that h is a continuous function defined on all of $\mathbb{R}$
Note that $\frac{1}{n^2+x^2}$ is a continuous function with the denominator never reaching 0. Additionally, observing that $\frac{1}{n^2+x^2} \leq \frac{1}{n^2}$, and as shown previously that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, then by the Weierstrauss M-test, then $h(x)$ is converges uniformly, and additionally since each $f_n(x)$ is continuous then $h(x)$ is continuous (theorem 6.4.2).

(b) Is h differentiable? If so, is the derivative function h' continuous?
Note that $f_n'(x) = \frac{-2x}{(x^2+n^2)^2}$, and since $f_n'(x)$ has a maximum value of $\frac{3\sqrt{3}}{8n^3}$, and that $\frac{3\sqrt{3}}{8n^3} \leq \frac{3\sqrt{3}}{8n^2}$, then by the very same reasoning as above, since $\frac{3\sqrt{3}}{8} \sum_{n=1} \frac{1}{n^2}$ converges by the algebraic limit theorem for series, then by the Weierstrauss M-test the sum $\sum_{n=1}^{\infty} f_n'(x)$ is uniformly convergent. Thus $h'(x)$ exists. Since both $-2x$ and $(x^2 + n^2)^2$ are continuous, and $(x^2 + n^2)^2 \neq 0$, then their quotient is continuous, therefore $h'(x)$ is continuous (theorem 6.4.2).

6.5.9 If $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ prove that $a_n = b_n$ for all $n \in \mathbb{N}$.
Proof. Consider $0 = h(x) = \sum_{n=0}^{\infty}(a_n - b_n)x^n$. Since both power series are continuous on the interval $(-R, R)$ then $h(x)$ is defined for 0. Therefore $0 = h(0) = (a_0 - b_0) + (a_1 - b_1)0 + \cdots = a_0 - b_0$. Therefore $a_0 = b_0$. Since each power series is differentiable, and the sum of differentiable functions is differentiable, then we have that $0 = h'(0) = (a_1 - b_1) + 2(a_2 - b_2)0 + \cdots$. Therefore by the principle of mathematical induction, for all $k \in \mathbb{N}$, if $k < n$, then $a_k = b_k$. Consider the nth derivative of $h(x)$, therefore $\frac{d^n}{dx^n}h(x) = \sum_{l=n}^{\infty} \frac{(l)!}{(l-n)!}(a_l - b_l)x^{l-n}$. We know by theorem 6.5.7 that convergent power series are infinitely differentiable, since $h(x)$ is defined on $(-R, R)$ then $\frac{d^n}{dx^n}h(0)$ is defined. Therefore $0 = \frac{d^n}{dx^n}h(0) = n!(a_n - b_n) + (n+1)!(a_n - b_n)0 + \cdots$. Thus $0 = a_n - b_n$, $a_n = b_n$. Therefore the power series are equivalent.