Recall: a set Y of real numbers has the no-gaps property provided that for all $x, y, z \in \mathbb{R}$, if $x < y < z$ and $x, z \in Y$ then $y \in Y$. In class we observed that every interval has the no gaps property. It is also true that every set of real numbers with the no-gaps property is an interval, but we have not proven this, and will not use it in this problem. Given a set X of real numbers define a relation N_X on the set X according to the following: for $x, y \in X, (x, y) \in pairs(N_X)$ provided that there is a set $I \subseteq X$ such that $x, y \in I$ and I has the no-gaps property.

1. Prove that this relation is an equivalence relation on X.

   - Reflexivity proof: Suppose $x \in X$. We want to show that for all $x \in X, (x, x) \in pairs(N_X)$. By definition of belonging to the relation we must show there exist $I \subseteq X$ such that $x, y \in I$, and I has the no-gaps property. Suppose $I = [x, x]$. We must show that I has the no-gaps property. Since the only real number in the set $[x, x]$, the no gaps property requirement of of $x < y < z, x, y \in \mathbb{R}$ can only be written as $x < y < x$, which is always false. Therefore I has the no gaps property.
   - Symmetry proof: Suppose $x, y \in X, x, y \in I \subseteq X$, I has the no gaps property. We must show that $(y, x) \in pairs(N_X)$. Therefore by definition we must show that there is a set $I \subseteq X$ such that $y, x \in I$, and I has the no gaps property. Since $x, y \in I$, then $x \in I$, and $y \in I$, therefore $y, x \in I$. Since $I \subseteq X$ exists and it has the no gaps property, then $(y, x) \in pairs(N_X)$.
   - Transitivity proof. Suppose $x, y, z \in X$, $(x, y), (y, z) \in pairs(N_X)$. We must show $(x, z) \in pairs(N_X)$. Therefore by definition we must show that there exist $I \subset X$ such that $x, z \in I$ and I has the no gaps property. By definition of being in the relation, there exists $I_1, I_2 \subseteq X$ such that $x, y \in I_1, y, z \in I_2$, and I_1, I_2 both have the no-gaps property. Let $I = I_1 \cup I_2$, since $I_1, I_2 \subseteq X$, then $I \subseteq X$. Therefore we must show that I has the no-gaps property. Since $x < y$ and $y < z$, then by composing the inequalities we get $x < y < z$, which since x, y, z are defined to be in I, demonstrates that I has the no-gaps property.

2. Let $\mathcal{N}(X)$ be the set of equivalence classes. Prove that every equivalence class has the no-gaps property.
   Suppose C is an equivalence class in $\mathcal{N}(X)$. We must show that all $C\mathcal{N}(X)$ has the no gaps property. By the definition of the no-gaps property we must show for all $x, y, z \in \mathbb{R}$ if $x, z \in C$ and $x < y < z$ then $y \in C$. Suppose $x, y, z \in \mathbb{R}$, $x, z \in C$, and $x < y < z$. We must show that $y \in C$. Since $x, z \in C$, then xN_Xz. By the definition of the relation there exists an interval $I \subseteq X$ such that $x, z \in I$, and I has the no-gaps property. Therefore since I has the no gaps property, $x < y < z$, and $x, z \in I$, then $y \in I$. Since $y \in I$, then by the definition of the relation, yN_Xz and yN_Xx, as they all exist in an interval with the no-gaps property. Therefore since y is related to elements in C, then $y \in C$.

3. Define a relationship $\ll$ on $\mathcal{N}(X)$ where for equivalence classes A and $B, A \ll B$ provided that for all $x \in A$ and $y \in B$, $x < y$. Prove that the relation $\ll$ is transitive, anti-reflexive, anti-symmetric, and full.

- Transitive proof:
  Suppose $A \ll B$ and $B \ll C$. We must show that $A \ll C$. By definition of the $\ll$, we must show for all $a \in A, c \in C$ that $a < c$. By definition, all $a \in A, b \in B, c \in C, a < b, b < c$. Therefore since $a < b$ and $b < c$ then by the transitivity of $<$, $a < c$.

- Anti-reflexive and Anti-symmetry proof:
  Suppose $A \ll B$. We must show that $B \not\ll A$. By definition of $\ll$, we must show that there exist $a \in A, b \in B$ such that $a \leq b$. By definition of $\ll$ we have for all $a \in A, b \in B, a < b$. Therefore all a, b satisfy the relationship $a \leq b$.

- Fullness proof: Suppose $A, B \in \mathcal{N}(X), A \neq B$. We must show that $A \ll B \vee B \ll A$. Since A, B are seperate equivalence classes, then by definition they are sets in a partition of X, therefore $A \cap B = \emptyset$. Assume $A \not\ll B$. We must show that $B \ll A$. Therefore by definition there exist $a \in A, b \in B$ such that $a \geq b$. Since A, B are disjoint then for all $a \in A, b \in B, a \neq b$. Therefore we satisfy the condition $a > b$. I'm now going to prove $B \ll A$ by contradiction. Suppose not. Let $a^* \in A, a^* < b$. Since $a, a^* \in A$, then there exists an inteval $a, a^* \in I$ with the no-gaps property. Since $a^* < b$, $b < a$, and $a, a^* \in I$, then by the definition of the no-gaps relation $b \in A$. This is a contradiction, as $A \cap B = \emptyset$. Therefore $B \ll A$