1. Note that if A is finite then trivially $\sum_{n\in A}\frac{1}{n}$ converges since it's the finite addition of rational numbers. Thus assume A is infinite. Since this is a sum over positive terms, then if $A\subseteq B$ we have that $\sum_{n\in A}\frac{1}{n}\leq\sum_{n\in B\setminus A}\frac{1}{n}+\sum_{n\in A}\frac{1}{n}=\sum_{n\in B}\frac{1}{n}$. Therefore we consider the largest A, the set of all natural numbers without 0 in their decimal expansion. Let S_k be the sums over all numbers in A with k digits, and let N_k be the subset of $\mathbb{N}$ with all of the k length numbers. Note that $|N_k|=9^k$, since for each decimal in the expansion we have 9 choices for possible digits. Additionally, each number $a\in N_k$ is bounded below by 10^{k-1}, since $10^{k-1}\leq 111\cdots 111$ k times $\leq a$, thus the reciprocals satisfy $\frac{1}{a}\leq\frac{1}{10^{k-1}}$. Therefore $S_k\leq|N_k|/10^{k-1}=9^k/10^{k-1}$. Thus $\sum_{n\in A}\leq\sum_{k=1}^{\infty}\frac{9^k}{10^{k-1}}=90$. Thus every sum as described above is bounded and increasing, thus it converges.

2. Suppose W is an open subset of X, our complete metric space (X,d), and $\{U_n : n\in\mathbb{N}\}$ is our countable set of open, dense subsets. Note for U_1, we can find $x_1\in U_1\cap W$, where for $0<r_1<1$, $\overline{B}(x_1,r_1)\subseteq W$, where $\overline{B}(x_1,r_1)$ is closed. Additionally, for U_2, we can find $x_2\in B(x_1,r_1)\cap U_2$, where for $0<r_2<\frac{1}{2}$, we have that $\overline{B}(x_2,r_2)\subsetneq B(x_1,r_1)$. For U_n, we can find $x_n\in B(x_{n-1},r_{x-1})\cap U_n$, where for $0<r_n<\frac{1}{n},\overline{B}(x_n,r_n)\subseteq B(x_{n-1},r_{n-1})$. Therefore for $\epsilon>0$, there exists $N\in\mathbb{N}$ where $\frac{1}{N}<\epsilon$, therefore by construction for all $2N\leq m<n, d(x_n,x_m)<\frac{1}{N}$ since by construction $x_n,x_m\in B(x_m,r_m)$, where $r_m<\frac{1}{2N}$, thus ensuring that all points within are at most a distance of $\frac{1}{N}$ from each other. Thus the sequence constructed is Cauchy. Additionally since the selection of each point is within a closed ball, then we can apply nested compact set property to get that $x\in\cap_{n\in\mathbb{N}}U_n$ Therefore it converges in $\cap_{n\in\mathbb{N}}U_n$, thus $W\cap\cap_{n\in\mathbb{N}}U_n$ contains a point.

3. Suppose for contradiction that X is of the first catagory. Let $(E_n)_{n\in\mathbb{N}}$ be a countable set of nowhere dense sets in our complete metric space X such that $\cup_{n\in\mathbb{N}}E_n=X$. Note that by definition $int(cl(E_n))=\emptyset$. Note that by taking the closure of the nowhere dense sets in our union we get that $X=\cup_{n\in\mathbb{N}}cl(E_n)$ Therefore let us consider $(\cup_{n\in\mathbb{N}}cl(E_n))^c=\cap_{n\in\mathbb{N}}cl(E_n)^c$. Note by the Baire Category theorem, since each E_n^c is an open, dense set then it's intersection is dense. However, this implies that $\emptyset=(\cup_{n\in\mathbb{N}}cl(E_n))^c$ is dense. This is a contradiction.

4. Suppose $\{F_n : n\in\mathbb{N}\}$ is a countable set of nowhere dense sets. Then $(\cup_{n\in\mathbb{N}}cl(F_n))^c=\cap_{n\in\mathbb{N}}cl(F_n)^c$ is by construction an intersection of open dense sets. Therefore it's intersection is dense in X. Suppose for contradiction that $\cup_{n\in\mathbb{N}}cl(F_n)$ contains an open interval, $B(x,r)\subset\cup_{n\in\mathbb{N}}cl(F_n)$, since the complement of $\cup_{n\in\mathbb{N}}cl(F_n)$ is dense, then $B(x,r)\cap(\cup_{n\in\mathbb{N}}cl(F_n))^c$ should be non-empty, however since $B(x,r)$ is contained entirely outside of $(\cup_{n\in\mathbb{N}}cl(F_n))^c$, then $B(x,r)\cap\cup_{n\in\mathbb{N}}F_n=\emptyset$. This is a contradiction, thus $\cup_{n\in\mathbb{N}}F_n$ does not contain an open interval, $int(\cup_{n\in\mathbb{N}}cl(F_n))=\emptyset$, which implies $int(\cup_{n\in\mathbb{N}}F_n)=\emptyset$.

5. Let E be a closed subset of a metric space (X,d)

   - $\Rightarrow$ Suppose E is nowhere dense, $x\in E,\epsilon>0$. Then E^c is dense, thus there exists $y\in E^c$ such that $y\in B(x,\epsilon)$. Therefore $d(x,y)\leq\epsilon$.

- $\Leftarrow$ We claim that E^c is dense. Note that since E^c is dense in E^c, consider a point in E, x. Then we know that for arbitrary $\epsilon > 0$ there exists $y \in E^c$ such that $d(x, y) \leq \epsilon$. Thus for all open sets in X we can place elements from E^c. If not, then there would exists an interval contained with E, which would ensure that E^c would not be dense in that interval, contradicting the fact that $E^c \cap E = \emptyset$.

6. Lemma: The set $E_k = \{f \in C([0,1]) : \exists x_0 \in [0,1], \forall x \in [0,1], |f(x) - f(x_0)| \leq n|x - x_0|\}$ is nowhere dense. Suppose $f \in E_k$. We will show that f is approximated by piecewise linear functions $g_n(x)$ with $|g'(x)| > 2k$ for all $x \in [0,1]$, where $g'(x)$ is defined. To show our claim, we note that since f is continuous on a compact interval then f is uniformly continuous. Therefore if we fix an $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in [0,1]$ if $|x - y| < \delta$ then $|f(x) - f(y)| < \frac{\epsilon}{2}$. Let $\delta' = \min(\{\frac{\epsilon}{4k}, \delta\})$. Then consider a partition $0 = x_0 \leq x_1 \leq \cdots \leq x_n = 1$ where $x_j - x_{j-1} < \delta'$. We construct $g_n(x)$ by assigning $g_n(x)$ on $[x_{j-1}, x_j]$ to be the line in between $g(x_{j-1})$ and $g(x_j)$. Note that if a slope is encountered which is less than $2n$, a point can be inserted such that the graph rises with rate $2n$ and descends with rate $-2n$ to hit the point. Note that this addition still satisfies both the slope and partition requirements. Therefore for all $x, y \in [x_{j-1}, x_j]$, our function satisfies

$$|g_n(x) - g_n(y)| < \max\{\epsilon/2, 2k|x - y|\} \leq \min\{\epsilon/2, 2k\delta'\} \leq \epsilon/2.$$

Therefore it remains to show that $\sup_{x\in[0,1]} |f(x) - g_n(x)| < \epsilon$. Note that for all $x \in [0,1]$, there exists a partition such that $x \in [x_{j-1}, x_j]$, thus letting us apply the inequalities

$$|f(x) - g_n(x)| \leq |f(x) - f(x_j) + f(x_j) - g_n(x_j) + g_n(x_j) - g_n(x)|$$

$$\leq |f(x) - f(x_j)| + |f(x_j) - g_n(x_j)| + |g_n(x_j) - g_n(x)| \leq \frac{\epsilon}{2} + 0 + \epsilon/2 = \epsilon$$

Therefore our piecewise linear function converges in the uniform metric. Note that $|g_n(x) - g_n(x_0)| \geq 2k|x - x_0| > 2k|x - x_0|$ by construction. Therefore $g_n \notin E_k$. Thus by the result proved above E_k is nowhere dense.

Let $\mathcal{C}$ denote the set of all nowhere differentiable functions. We will show that $\mathcal{C}^c$ is of the first category. Suppose $f \in \mathcal{C}^c$. Then there exists $x_0 \in [0,1]$ such that $f'(x_0)$ exists. If we consider the quotient function $\phi(x) = \frac{f(x)-f(x_0)}{x-x_0}$ on $x \in [0,1]\backslash\{x_0\}$ and $\phi(x_0) = f'(x_0)$ then we claim that $\phi(x)$ is continous. On $x \in [0,1]\backslash\{x_0\}$ then $\phi(x)$ is continuous since it is the quotient of continuous functions with a denominator which does not equal 0. At x_0, we have that $\lim_{x\to x_0} \phi(x) = f'(x_0) = \phi(x_0)$ by definition, thus $\phi(x)$ is continuous. Therefore $\phi(x)$ is uniformly continuous on $[0,1]$. Therefore it is bounded. Thus there exists $M \in \mathbb{N}$ such that $f \in E_M$. Therefore $\mathcal{C}^c \subseteq \cup_{n\in\mathbb{N}} E_n$. Furthermore, $\mathcal{C}^c = \cup_{n\in\mathbb{N}} \mathcal{C}^c \cap E_n$. Since E_n is nowhere dense, then trivially a subset, $\mathcal{C}^c \cap E_n$ is nowhere dense. Therefore by the corollary of the Baire Category theorem $\mathcal{C}^c$ is of the first category. Thus $\mathcal{C}$ is of the second category. Therefore there exists nowhere differentiable continuous functions on the unit interval.

7. Let $I(n) = \int_0^\pi \sin^n x dx$. Then

$$\begin{aligned}
I(n) &= \int_0^\pi \sin^n x dx \\
&= \sin^{n-1}(x)\cos(x)|_\pi^0 + (n-1)\int_0^\pi sin^{n-2}(x)\cos^2(x)dx \\
&= (n-1)\int_0^\pi sin^{n-2}(x)(1-\sin^2(x))dx \\
&= (n-1)I(n-2) - (n-1)I(n) \\
nI(n) &= (n-1)I(n-2) \\
I(n) &= \frac{n-1}{n}I(n-2)
\end{aligned}$$

Note that since $0 \leq \sin(x) \leq 1$ on $[0,\pi]$, then $\sin^n(x) \leq sin^{n-1}(x)$, and since the integral is being taken over where $\sin(x)$ is non-negative, then $I(n) \leq I(n-1)$, thus we have a decreasing sequence. Therefore, noting that $I(0) = \frac{\pi}{2}, I(1) = 1, I(0)/I(1) \geq 1$, we can apply the recurrence relation to solve the limit:

$$1 \leq \lim_{n\to\infty} \frac{I(2n)}{I(2n+1)} \leq \lim_{n\to\infty} \frac{I(2n-1)}{I(2n+1)} = \lim_{n\to\infty} \frac{n}{n-1}\frac{I(2n-1)}{I(2n-1)} = \lim_{n\to\infty} \frac{n}{n-1} = 1$$

Thus the limit converges to 1. Note that by induction we have that $I(2n) = \frac{\pi}{2}\prod_{i=1}^n \frac{2i-1}{2i}, I(2n+1) = \prod_{i=1}^n \frac{2i}{2i+1}$, therefore

$$\begin{aligned}
\lim_{n\to\infty} I(2n) &= \lim_{n\to\infty} I(2n+1) \\
\lim_{n\to\infty} \frac{\pi}{2}\prod_{i=1}^n \frac{2i-1}{2i} &= \prod_{i=1}^n \frac{2i}{2i+1} \\
\frac{\pi}{2} &= \lim_{n\to\infty} \prod_{i=1}^n \frac{2i}{2i+1}\frac{2i}{2i-1}
\end{aligned}$$

Thus proving the wallis product.
To get the wallis product in a form without explicit products, observe that

$$\begin{aligned}
\frac{\pi}{2} &= \lim_{n\to\infty} \prod_{i=1}^n \frac{2i}{2i+1}\frac{2i}{2i-1} \\
&= \lim_{n\to\infty} 2^{2n}n^2 \prod_{i=1}^n \frac{1}{(2i+1)(2i-1)} \\
&= \lim_{n\to\infty} 2^{2n}n^2 \frac{(2^n n!)^2(2n+1)}{((2n+1)!)^2} \\
&= \lim_{n\to\infty} \left(\frac{2^{2n}n!^2}{(2n)!}\right)^2 \left(\frac{1}{2n}\right)\left(\frac{2n}{2n+1}\right) \\
\pi &= \lim_{n\to\infty} \left(\frac{2^{2n}n!^2}{(2n)!\sqrt{n}}\right)^2
\end{aligned}$$

Therefore substituting in our definition for the factorial, $n! = Cn^{n+1/2}e^{-n}e^{r_n}$ where $1/(12n+1) < r_n < 1/(12n)$, we get

$$\begin{aligned}
\pi &= \lim_{n\to\infty}\left(\frac{2^{2n}C^2n^{2n+1}e^{-2n}e^{2r_n}}{C(2n)^{2n+1/2}e^{-(2n)}e^{r_{2n}}\sqrt{n}}\right)^2 \\
&= \lim_{n\to\infty}\left(\frac{Ce^{2r_n-r_{2n}}}{\sqrt{2}}\right)^2 \\
\sqrt{2\pi} &= C
\end{aligned}$$

8. Note that

$$\begin{aligned}
A(n) &= \int_0^{\frac{\pi}{2}} \cos^{2n}(x)dx = \sin(x)\cos^{2n-1}(x)|_0^{\frac{\pi}{2}} + (2n-1)\int_0^{\frac{\pi}{2}}\cos^{2n-2}(x)\sin^2(x)dx \\
&= (2n-1)\int_0^{\frac{\pi}{2}}\cos^{2n-2}(x)(1-\cos^2(x))dx \\
&= (2n-1)(A(n-1)-A(n)) \\
A(n) &= \frac{2n-1}{2n}A(n-1)
\end{aligned}$$

additionally

$$\begin{aligned}
A(n) &= x\cos^{2n}(x)|_0^{\frac{\pi}{2}} + 2n\int_0^{\frac{\pi}{2}} x\cos^{2n-1}(x)\sin(x)dx \\
&= n\left(x^2\cos^{2n-1}(x)\sin(x)|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} x^2((2n-1)\cos^{2n-2}(x)\sin^2(x) - \cos^{2n}(x))\right) \\
&= n\int_0^{\frac{\pi}{2}} x^2((2n-1)\cos^{2n-2}(x)(1-\cos^2(x)) - \cos^{2n}(x)) \\
&= n(2n-1)B(n-1) - 2n^2B(n)
\end{aligned}$$

Therefore by algebraic manipulations we get $\frac{1}{n^2} = 2\left(\frac{B(n-1)}{A(n-1)} - \frac{B(n)}{A(n)}\right)$. Thus by taking the sum

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{\infty} i^2 &= \sum_{i=1}^{\infty}\left(\frac{B(i-1)}{A(i-1)} - \frac{B(i)}{A(i)}\right) \\
&= \sum_{i=1}^{\infty}\frac{B(i-1)}{A(i-1)} - \sum_{i=1}^{\infty}\frac{B(i)}{A(i)} \\
&= \frac{B(0)}{A(0)} + \sum_{i=1}^{\infty}\frac{B(i)}{A(i)} - \sum_{i=1}^{\infty}\frac{B(i)}{A(i)} \\
&= \frac{B(0)}{A(0)}
\end{aligned}$$

Note that $B(0) = \frac{\pi^3}{3 \cdot 8}, A(0) = \frac{\pi}{2}$, therefore half of our sum is equal to $\frac{B(0)}{A(0)} = \frac{\pi^2}{12}$, therefore $\sum_{i=1}^{\infty} i^2 = \frac{\pi^2}{6}$.

Note that $\lim_{i \to \infty} \frac{B(i)}{A(i)} = 0$ since $\frac{1}{n^2} + \frac{2B(n)}{A(n)} = \frac{2B(n-1)}{A(n_1)}$, therefore the sequence $\frac{B(i)}{A(i)}$ is decreasing and positive since x^2 and $\cos^{2n}(x)$ is positive on $[0, \pi/2]$.