The purpose of this problem is to prove: (**) For any $n \geq 1$, if $|X| \geq 2^n$ and T is a tournament on ground-set X, then there is a subset Y of X of size $n+1$ such that the relation $T[Y]$ (T restricted to Y) is transitive.

1. Prove that a finite tournament on ground set X has exactly $|X|(|X|-1)/2$ pairs.

   Suppose X is a finite set and T is a tournament on the set X. We must show there are $|X|(|X|-1)/2$ pairs. By the principal of mathematical induction for all finite sets X' with a tournament defined on X' if $|X'| < |X|$ then the tournament has exactly $\frac{|X'|(|X'|-1)}{2}$ pairs. Note that since T is a tournament then we know that it is full and anti-reflexive, therefore for each $y \in X, x \in X \backslash y$ either xTy or yTx. Since $|X \backslash y| = |X| - 1$, then we have that each $x \in X$ occurs in $|X|-1$ pairs. We have two cases:

   - Assume $|X| = 1$. Since a tournament is anti-reflexive, there are 0 pairs. Therefore $0 = |pairs(T)| = \frac{|X|(|X|-1)}{2} = \frac{1(1-1)}{2} = 0$.
   - Assume $|X| > 1$. Suppose a is an element of X. Let X' be given by $X' = X \backslash a$. Since $|X'| = |X| - 1 < |X|$, then by the induction hypothesis the tournament T' defined on X' has exactly $\frac{|X'|(|X'|-1)}{2}$ number of pairs. Note that since $a \in X$, there are $|X|-1$ pairs which contain a in T. Note also that since $X' = X \backslash a$ that $|pairs(T')| = |pairs(T)| - (|X|-1)$. Therefore by algebraic manipulation:

$$\begin{aligned}
|pairs(T)| &= |pairs(T')| + |X| - 1 \\
&= \frac{|X'|(|X'|-1)}{2} + |X| - 1 \\
&= \frac{(|X|-1)(|X|-2)}{2} + |X| - 1 \\
&= (|X|-1)(\frac{|X|-2}{2} + 1) \\
&= (|X|-1)(\frac{|X|-2+2}{2}) \\
&= |X|(|X|-1)/2.
\end{aligned}$$

2. Prove: If T is a tournament on X then there is some $x \in X$ such that xTy for at least $(|X|-1)/2$ elements of X.

   Suppose for contradiction that for each $x \in X, xTy$ for strictly less than $(|X|-1)/2$ elements of X. Therefore, since $(|X|-1)/2$ is an upper bound on the potential number of connections per element, then $pairs(T) < |X|(|X|-1)/2$. This is a contradiction as by the previous lemma $pairs(T)$ is exactly $|X|(|X|-1)/2$.

3. Use induction and the previous part to prove (**).

   We must show for any $n \geq 1$, if $|X| \geq 2^n$ and T is a tournament on ground-set X, then there is a subset Y of X of size $n+1$ such that the relation $T[Y]$ is transitive.

Suppose $n \in \mathbb{N}, n \geq 1, |X| \geq 2^n$, and that T is a tournament on X. We must show there exists a subset $Y \subseteq X, |Y| = n+1$ such that $T[Y]$ is transitive. By the principal of mathematical induction for all sets X', tournaments T' defined on X', $k \in \mathbb{N}$, if $k < n, 2^n \geq |X'| \geq 2^k$ then there exists $Y' \subseteq X', |Y'| = k+1$ such that $T'[Y']$ is transitive. We have two cases:

- Assume $n = 1$. Then let $|X| = 2 = 2^1$. Since there are two elements in X, then there exists only one pair in T. Therefore T is vacuously transitive. Since the only subset of X of size 2 is X, and T, then the requirements have been satisifed.
- Assume $n > 1$. Since T is defined on a finite set X, then there exists an element $x_* \in X$ such that $x_* T y$ for at minimum $(|X| - 1)/2$ elements of X. Let X_* be the set of all elements s.t. $x_* T y$. Since there are $|X| - 1$ pairs which contain x_*, then the upper limit on X_* is $|X| - 1$. Since the lower limit on X_* is $\lceil (|X| - 1)/2 \rceil$, then $|X_*| \geq \lceil (|X| - 1)/2 \rceil \geq \lceil (2^n - 1)/2 \rceil = 2^n/2 = 2^{n-1}$. Since $|X| \geq 2^n \geq |X_*| \geq 2^{n-1}$, and $n - 1 < n$ then by the induction hypothesis there exists a set Y_* such that $Y_* \subseteq X_*, |Y_*| = n, T[Y_*]$ is transitive. We claim that $Y_* \cup \{x_*\}$ is transitive. Suppose $a, b \in Y_*$, therefore xTa, xTb since $a, b \in Y_*$. Note that since T is a tournament, then either aTb or bTa. We have two cases.
  - Suppose aTb. Therefore since xTa, aTb, we must show that xTb. Since xTb, then $T[Y_* \cap \{x_*\}]$ is transitive.
  - Suppose bTa. Therefore since xTa, bTa, we must show that xTa. Since xTa, then $T[Y_* \cap \{x_*\}]$ is transitive.

  Since $Y_* \cup \{x_*\} \subseteq X, |Y_* \cup \{x_*\}| = n + 1$, and that $T[Y_* \cup \{x_*\}]$ is transitive, then the requirements have been satisfied.