4.5.6 Let $f : [a, b] \to \mathbb{R}$ be continuous, $f(a) < 0 < f(b)$. We must show there exists $c \in [a, b]$ such that $f(c) = 0$. Let $I_0 = [a, b]$, consider the midpoint $z_1 = \frac{a+b}{2}$. If $f(z_1) > 0$, then set $b_1 = z_1, a_1 = a, I_1 = [a_1, b_1]$, and if $f(z_1) \leq 0$ then set $a_1 = z_1, b_1 = b$. This lends itself to a general formula for I_n, where for I_{n-1} with $z_{n-1} = \frac{a_{n-1}+b_{n-1}}{2}$, then if $f(z_{n-1}) > 0$, $I_n = [a_{n-1}, z_{n-1}]$, otherwise $I_n = [z_{n-1}, b_{n-1}]$. Since we have $I_0 \supseteq I_1 \supseteq \cdots$, then there exists $x \in \bigcap_{n=0}^{\infty} I_n$. We claim that (a_n) converges to x. Since each interval is half the length of the previous, then for an arbitrary $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - x| \leq 2^{-n}(b - a) < \epsilon$. A similar argument exists for $b_n \to x$. Since f is continuous then $\lim f(a_n) = \lim f(b_n) = f(x)$. We know by the algebraic limit theorem that since $f(a_n) \leq 0$ that $f(x) \leq 0$, and similarly $f(b_n) > 0$ implies that $f(x) \geq 0$. Therefore $f(x) = 0$.