3.5.3 (a) Show that a closed interval $[a, b]$ is a G_δ set.
Suppose $r \in (0, 1)$. We claim that $[a, b] = \bigcap_{n=1}^{\infty}(a - r^n, b + r^n)$.

- We want to show that $[a, b] \subseteq \bigcap_{n=1}^{\infty}(a - r^n, b + r^n)$.
Since for all $n \in \mathbb{N}, [a, b] \subseteq (a - r^n, b + r^n)$, then by the definition of set intersection, $[a, b] \subseteq \bigcap_{n=1}^{\infty}(a - r^n, b + r^n)$.
- We want to show that $\bigcap_{n=1}^{\infty}(a - r^n, b + r^n) \subseteq [a, b]$. Suppose $x \in \bigcap_{n=1}^{\infty}(a - r^n, b + r^n)$. We must show that $a \leq x \leq b$. First to show that $a \leq x$. We know that for all $n \in \mathbb{N}, a - r^n < x$, therefore x is an upper bound on $\{a - r^n : n \in \mathbb{N}\}$. Therefore all we need to prove is that $a \in \bigcap_{n=1}^{\infty}(a - r^n, b + r^n)$. Since a is the limit point of the sequence $(a - r^n)_{n=1}^{\infty}$, then there is no set $(a - r^n, b + r^n)$ which excludes it. Therefore $a \in \bigcap_{n=1}^{\infty}(a - r^n, b + r^n)$. Thus, since x is an upper bound on $(a - r^n)_{n=1}^{\infty}$, and our set contains a, then $a \leq x$. Next we must show that $x \leq b$. We know that for all $n \in \mathbb{N}, x < b + r^n$. Thus x is a lower bound on $(b + r^n)$. Therefore, similar to above, $x \leq b$. Therefore $a \leq x \leq b$. Thus $x \in [a, b]$

(b) Show that the half open interval $(a, b]$ is both G_δ and F_σ set

- Show that $(a, b]$ is a G_δ set
We claim that $\cap_{n=1}^{\infty}(a, b + r^n) = (a, b]$. As shown in the proof above, for the closed end point, the set above converges, and for the open right endpoint, it is trivial.
- Show that $(a, b]$ is a F_σ set
We claim that $\cup_{n=1}^{\infty}[a + \frac{1}{n}, b] = (a, b]$.

  - We must show that $\cup_{n=1}^{\infty}[a - \frac{1}{n}, b] \subseteq (a, b]$. Suppose $x \in \cup_{n=1}^{\infty}[a + \frac{1}{n}, b]$ We must show that $a < x \leq b$. Since $x \leq b$ is trivial, we must show that $a < x$. Since x is in the union, there must be a smallest $n_0 \in \mathbb{N}$ in which $x \in [a + \frac{1}{n_0}, b]$. Therefore $a < a + \frac{1}{n_0} \leq x$.
  - We must show that $(a, b] \subseteq \cup_{n=1}^{\infty}[a - \frac{1}{n}, b]$. Since $\{b\}$ is trivially in both, we must show that $(a, b) \subseteq \cup_{n=1}^{\infty}[a - \frac{1}{n}, b]$. Suppose $x \in (a, b)$. Therefore there exists $\epsilon > 0$ such that $V_\epsilon(x) \subset (a, b)$. Since the neighborhood is contained within (a, b) then $a < x - \epsilon$. Therefore $0 < x - \epsilon - a$. Thus by the archimedean principle there exists $n' \in \mathbb{N}$ such that $\frac{1}{n'} < x - \epsilon - a$. Therefore $a + \frac{1}{n'} < x - \epsilon$, and therefore $V_\epsilon(x) \subset [a + \frac{1}{n'}, b] \subseteq \cup_{n=1}^{\infty}[a + \frac{1}{n}, b]$. Thus $(a, b] \subseteq \cup_{n=1}^{\infty}[a + \frac{1}{n}, b]$

Making $(a, b]$ a F_σ set.

(c)
- Show that $\mathbb{Q}$ is a F_σ set.
Since $\mathbb{Q}$ is countable, then there exists a bijection between $\mathbb{N}$ and $\mathbb{Q}$. Therefore let $f : \mathbb{N} \to \mathbb{Q}$ be a bijection. Then we claim that $\bigcup_{n=1}^{\infty}[f(n), f(n)]$ is $\mathbb{Q}$. Since we're guaranteed to uniquely attain every rational number, then we have exactly $\mathbb{Q}$. Since we have a countable union of closed intervals, then we have satisifed the definition of F_σ
- Show that the set of irrationals forms a G_δ set.

Let $L_n = \displaystyle\bigcup_{x\in\mathbb{Z}} \left(\frac{x}{n}, \frac{x+1}{n}\right)$. We claim that $\mathbb{R}\backslash\mathbb{Q} = \displaystyle\bigcap_{n=1}^{\infty} L_n$.

- Suppose $x \in \mathbb{R}\backslash\mathbb{Q}$. We must show that $x \in \bigcap_{n=1}^{\infty} L_n$. Suppose for contradiction that $x \notin \bigcap_{n=1}^{\infty} L_n$. Therefore by the definition of set compliment and DeMorgan's law, $x \in \cup_{n=1}^{\infty} L_n^c$. Since x is in the union, there must exists $n_0 \in \mathbb{N}$ where $x \in L_{n_0}^c$. Therefore $x \in \{\frac{y}{n_0} : y \in \mathbb{Z}\}$. Thus there exists a $y_0 \in \mathbb{Z}$ such that $x = \frac{y_0}{n_0}$. This contradicts x being irrational. Therefore x is in the intersection.
- Suppose $x \in \bigcap_{n=1}^{\infty} L_n$. Then we must show that $x \in \mathbb{R}\backslash\mathbb{Q}$. Suppose for contradiction that $x \in \mathbb{Q}$. Then there exists $a, b \in \mathbb{Z}$ such that $x = \frac{a}{b}$. However, since x is in the intersection of L_n's, then $x \in L_b$. This is a contradiction as L_b excludes all rational numbers with a denominator of b. Thus x is irrational.

Therefore $\mathbb{R}\backslash\mathbb{Q} = \displaystyle\bigcap_{n=1}^{\infty} L_n$, thus making the irrationals G_δ.