5 (a) Let $Df(x_0) = 0, [D^2 f(x_0)]$ be a positive definite matrix. We want to show for a $\delta$ ball around $x_0$ that $f(x_0) \leq f(x)$. Note that for a positive definite matrix, we can do a coordinate transform $P$ to have that $h^T P[D^2 f(x_0)]P^{-1}h = \lambda_1 h_1^2 + \cdots + \lambda_n h_n^2$, therefore we can find a minimum $\lambda_i$ such that $h^T P^{-1}[D^2 f(x_0)]P^h \geq \lambda_i \|h\|^2$. For the rest we will assume that the basis allows for a diagonal $[D^2 f(x_0)]$. Let $\lambda_i > \epsilon > 0$, then there exists a $\delta$ such that for all $h \in B(\delta, 0), |f(x_0+h) - T_2(f; x_0)(h)| < \epsilon\|h\|^2$. Therefore $T_2(f; x_0)(h) - \epsilon\|h\|^2 < f(x_0 + h)$. This gets us that

$$\begin{aligned}
f(x_0 + h) - f(x_0) &> T_2(f; x_0)(h) - \epsilon\|h\|^2 - f(x_0) \\
&= f(x_0) - f(x_0) + h^T[D^2 f(x_0)]h - \epsilon\|h\|^2 \\
&= \lambda_1 h_1^2 + \cdots + \lambda_n h_n^2 - \epsilon(h_1^2 + \cdots + h_n^2) \\
&\geq \epsilon(h_1^2 + \cdots + h_n^2) - \epsilon(h_1^2 + \cdots + h_n^2) \\
&= 0
\end{aligned}$$

(b) Observe that $D(g(y)) = D(f(T(y)) = Df(T(y))DT(y)$. In order to compute $D^2 g(y)$, we must compute the intermediate terms $D(Df(T(y)))$ and $D(DT(y))$. Observe that we have $D(Df(T(y)))$ by another application of the chain rule, $D(Df(T(y))) = D^2 f(T(y))DT(y)$. To compute $D(DT(y))$, we must consider it multiplied via $Df(T(y))$ to make any sense. Therefore $Df(T(y))DDT(y) = \sum_{i=1}^n \frac{\partial f}{\partial y_i}(T(y))D(\frac{\partial T(y)}{\partial y_i}))$. Therefore $D^2(g(y)) = D^2 f(T(y))DT(y) + \sum_{i=1}^n \frac{\partial f}{\partial y_i}(T(y))D(\frac{\partial T(y)}{\partial y_i}))$. If $D^2 f(x_0)$ is a positive definite matrix there isn't a guarentee that $D^2 g(y_0)$ is positive definite. We observe in the formula that at points one would be computing $v^T D(\frac{\partial T(y)}{\partial y_i}))v$, which has no guarantees of being positive definite.

6 Consider $f$, and it's ith component $f_i$. Consider $x = (x_1, \cdots, x_n), \bar{x} = (\bar{x}_1, \cdots \bar{x}_n) \in U$. And let the vectors denoted $x_{\bar{i}}$ refer to vectors of the form $(x_1, \cdots, x_{i-1}, \bar{x}_i, \cdots, \bar{x}_n)$, where $x_{\bar{1}} = \bar{x}$, and $x_{\bar{n+1}} = x$. Then

$$\begin{aligned}
f_i(x) - f_i(\bar{x}) &= \sum_{j=1}^n f_i(x_{\bar{i+1}}) - f_i(x_{\bar{i}}) \\
&= \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(x_{\bar{j}} - (z_j + \bar{x}_j)e_j)(x_j - \bar{x}_j) \\
&= \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(\bar{x})(x_j - \bar{x}_j) + \sum_{j=1}^n [\frac{\partial f_i}{\partial x_j}(x_{\bar{j}} - (z_j + \bar{x}_j)e_j) - \frac{\partial f_i}{\partial x_j}(\bar{x})](x_j - \bar{x}_j)
\end{aligned}$$

where $z_i \in (x_i, \bar{x}_i)$ or $(\bar{x}_i, x_i)$ depending on size and $e_j$ is the standard $j$th basis vector. If we take our vectors to be within $\bar{U}$, then our partial derivatives are uniformly continuous, and if $\|x - \bar{x}\| < \delta$ then there exists an $\epsilon$ such that $|\frac{\partial f_i}{\partial x_j}(x) - \frac{\partial f_i}{\partial x_j}(\bar{x})| < \epsilon$.

Therefore

$$\sum_{j=1}^{n}[\frac{\partial f_i}{\partial x_j}(x_{\bar{j}} - (z_j + \bar{x}_j)e_j) - \frac{\partial f_i}{\partial x_j}(\bar{x})](x_j - \bar{x}_j)$$
$$\leq \epsilon \sum_{j=1}^{n} |(x_j - \bar{x}_j)| \leq \epsilon\sqrt{n}\|x - \bar{x}\|$$

Therefore $|f_i(x) - f_i(\bar{x}) - \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\bar{x})(x_j - \bar{x}_j)| < \sqrt{n}\epsilon\|x - \bar{x}\|$ has been shown for arbitrary $\epsilon$ for $x, \bar{x} \in U, \|x - \bar{x}\| < \delta$

7 Note that $S_N(f;x) = \frac{a_0}{2} + \sum_{n=1}^{N}(a_n\cos(nx) + b_n\sin(nx)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_N(x-t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)dt, 1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(t)$, $D_n(t)$ is even. Therefore,

$$= |S_N(f;x) - \frac{f(x+) + f(x-)}{2}|$$
$$= |\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)dt - \frac{f(x+) + f(x-)}{2}|$$

By applying the definition of $S_N(f;x)$

$$= |\frac{1}{2\pi}\int_{0}^{\pi} (f(x-t) + f(x+t))D_N(t)dt - \frac{f(x+) + f(x-)}{2}|$$

By the eveness of $D_N(t)$

$$= |\frac{1}{2\pi}\int_{0}^{\pi} (f(x-t) + f(x+t) - f(x+) - f(x-))D_N(t)dt|$$

By $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t) = 1, \frac{1}{2\pi}\int_{0}^{\pi} D_N(t) = \frac{1}{2}$

$$\leq |\frac{1}{2\pi}\int_{0}^{\pi} (f(x-t) - f(x-))D_N(t)dt| + |\frac{1}{2\pi}\int_{0}^{\pi} (f(x+t) - f(x+))D_N(t)dt|$$

Note that $\frac{f(x\pm t) - f(x\pm)}{\sin(\frac{t}{2})}$ are Riemann integrable on $(0,\pi)$, as when it isn't approaching 0 it's the quotient of two riemann integrable functions, and when at 0 we have that $\lim_{t\to 0}|\frac{f(x\pm t) - f(x\pm)}{\sin(\frac{t}{2})}| \leq \lim_{t\to 0}|\frac{t}{\sin(\frac{t}{2})}| = 2$, thus at the only possible discontinuity the quotient is bounded. Therefore, additionally, we can be extended to $(-\pi,\pi)$ by having them be zero on $(-\pi, 0]$, maintaining their Riemann integrability. Let these extended functions be noted as $f_-(t) = \begin{cases} \frac{f(x-t) - f(x-)}{\sin(\frac{t}{2})} & x \in (0,\pi) \\ 0 & x \in (-\pi, 0] \end{cases}$, $f_+(t) = \begin{cases} \frac{f(x+t) - f(x+)}{\sin(\frac{t}{2})} & x \in (0,\pi) \\ 0 & x \in (-\pi, 0] \end{cases}$. Therefore,

$$|\int_{0}^{\pi} (f(x-t) - f(x-))D_N(t)dt| + |\int_{0}^{\pi} (f(x+t) - f(x+))D_N(t)dt| =$$
$$|\int_{-\pi}^{\pi} f_-(t)\sin((N + \frac{1}{2})t)dt| + |\int_{-\pi}^{\pi} f_+(t)\sin((N + \frac{1}{2})t)dt|$$

thus we have the inequality

$$|S_N(f;x) - \frac{f(x+)+f(x-)}{2}| \leq |\int_{-\pi}^{\pi} f_-(t)\sin((N+\frac{1}{2})t)dt| + |\int_{-\pi}^{\pi} f_+(t)\sin((N+\frac{1}{2})t)dt|$$

Since both of the integrals on the right hand side are of the form $\int_{-\pi}^{\pi} h(t)\sin(\lambda t)$, where $h$ is Riemann integrable, then the Riemann-Lebesgue Lemma can be applied, thus $\lim_{N\to\infty} S_N(f;x) = \frac{f(x+)+f(x-)}{2}$