6 Let $\epsilon > 0$ be given. We will show that $\sum_{n=1}^{\infty}(-1)^n\frac{x^2+n}{n^2}$ converges uniformly on the interval (a,b). Let $B = \max\{|a|,|b|\}$ Note that the series $\sum_{n=1}^{\infty}\frac{B^2}{n^2} = \frac{\pi^2}{6}$ and $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n} = -\log(2)$. Since both of these separate series converges then they both satisfy the Cauchy criterion for series. Therefore there exists $N_1 \in \mathbb{N}$ such that for all $k \geq m \geq N_1, \sum_{n=m}^{k}\frac{B^2}{n^2} < \frac{\epsilon}{2}$ and there exists $N_2 \in \mathbb{N}$ such that $q \geq p \geq N_2, |\sum_{n=p}^{q}(-1)^n\frac{1}{n}| < \frac{\epsilon}{2}$. Therefore if we take $N = \max\{N_1, N_2\}$ then for all $r \geq s \geq N$,

$$|\sum_{n=s}^{r}(-1)^n\frac{x^2+n}{n^2}| \leq \sum_{n=s}^{r}\frac{B^2}{n^2} + |\sum_{n=s}^{r}(-1)^n\frac{1}{n}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since the series satisfies the Cauchy criterion, then it is uniformly convergent.
The function does not converge absolutely since

$$\sum_{n=1}^{\infty}\frac{x^2+n}{n^2} = \sum_{n=1}^{\infty}\frac{x^2}{n^2} + \frac{1}{n} > \sum_{n=1}^{\infty}\frac{1}{n} = \infty.$$

8 Note that for each function in the sum we have that $|c_n I(x-x_n)| \leq |c_n|$. Therefore by the Weierstrass M-test the series $\sum_{n=1}^{\infty} c_n I(x-x_n)$ converges uniformly. To show the continuity of the series when $x \neq x_n$ we must consider two cases. If x is not a limit point of the sequence $\{x_n\}$ then there must exists $\delta > 0$ such that $V(x,\delta) \cap \{x_n\} = \emptyset$. Therefore if we consider the subsequence $\{x_{n_k}\}$ that is to the left of x then the value function within $V(x,\delta)$ is simply the constant function with value $\sum_{n_k} c_{n_k}$, thus making it continuous.
If x is a limit point of $\{x_n\}$ then for all of the partial sums $\sum_{n=1}^{m} I(x-x_n)c_n$ is constant for some $\delta > 0$ around x. Therefore since it is true for the partial sums and since the series converges uniformly then we can apply the limit interchange theorem and get that $\lim_{t\to x}\lim_{m\to\infty}\sum_{n=0}^{m} c_n I(t-x_n)$ exists. Note that the value is exactly the sum of all the c_n to the left of x, which is exactly $\sum_{n=0}^{\infty} c_n I(x-x_n)$. Thus on all points such that $x \neq x_n$ the function is continuous.

11 Let M be the bound on all of the partial sums such that $|\sum_{n=1}^{k} f_n(x)| = |F_k(x)| \leq M$ for all $x \in E$ and let ϵ be given. Since $g_n \to 0$ uniformly then choose $N \in \mathbb{N}$ such that $|g_m(x)| < 2M\epsilon$ for all $m \geq N, x \in E$. Therefore, for all $N \leq p \leq q$, we have that

$$\begin{aligned}
|\sum_{n=p}^{q} f_n(x)g_n(x)| &= |\sum_{n=p}^{q-1} F_n(x)(g_n(x) - g_{n+1}(x)) + F_q(x)g_q(x) - F_{p-1}(x)g_p(x)| \\
&\text{abel summation} \\
&\leq M|\sum_{n=p}^{q-1}(g_n(x) - g_{n+1}(x)) + g_q(x) + g_p(x)| \\
&\text{uniform bound on } F_n(x) \\
&= 2Mg_p(x) \\
&< \epsilon
\end{aligned}$$

16 Let K be a compact set, $\{f_n\}$ be a set of equicontinuous functions which converge pointwise on K. We want to show that the functions converge uniformly. Let ϵ be given. Since $\{f_n\}$ is equicontinuous then there exists $\delta > 0$ such that for all $x, y \in K$, $|x - y| < \delta$ implies $|f_n(x) - f_n(y)| < \epsilon/3$ for all $n \in \mathbb{N}$. Since K is compact then there exists a finite set of points $\{x_1, \cdots, x_r\} \subset K$ such that $K \subseteq \cup_{i=1}^r V(x_i, \delta)$. Therefore for an arbitrary $x \in K$, there exists x_p such that $x \in V(x_p, \delta)$. Since $\{f_n\}$ converges pointwise for every point in K then there exists $N \in \mathbb{N}$ such that for all $n \geq m \geq N, |f_n(x_p) - f_m(x_p)| < \epsilon/3$. Now if take the same indicies and test whether $\{f_n\}$ converges at x we get that

$$\begin{aligned}
|f_n(x) - f_m(x)| &= |f_n(x) - f_n(x_p) + f_n(x_p) - f_m(x_p) + f_n(x_p) - f_m(x)| \\
&\leq |f_n(x) - f_n(x_p)| + |f_n(x_p) - f_m(x_p)| + |f_m(x_p) - f_m(x)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon
\end{aligned}$$

18 Let $\{f_n\}$ be a set of Riemann integrable functions on $[a, b]$ which are uniformly bounded by $M > 0$, and let F_n denote $F_n(x) \int_a^x f(t)dt$. We want to show that $\{F_n\}$ has a uniformly convergent subsequence. We claim that $\{F_n(x)\}$ is equicontinuous. Note that if $x, y \in [a, b]$ and $|x - y| < \frac{\epsilon}{M}$ then for all $n \in \mathbb{N}, |F_n(x) - F_n(y)| \leq M\frac{\epsilon}{M} = \epsilon$ (by theorem 6.12, Rudin). Therefore since $\{F_n\}$ is equicontinuous and uniformly bounded by $M(b - a)$ then we can apply theorem 7.25 of Rudin which gives us a uniformly convergent subsequence of $\{F_n\}$.