2 Let $z, w \in \mathbb{C}$ with $z = z_1 + iz_2, w = w_1 + iw_2, z_1, z_2, w_1, w_2 \in \mathbb{R}$. Additionally, let $\langle z, w\rangle = z_1w_1 + z_2w_2, (z, w) = z\bar{w}, Re(z, w) = Re(z\bar{w})$. First I must demonstrate that $\frac{1}{2}[(z, w) + (w, z)] = \langle z, w\rangle$:

$$\begin{aligned}
\frac{1}{2}[(z, w) + (w, z)] &= \frac{1}{2}(z\bar{w} + \bar{z}w) \\
&= \frac{1}{2}[(z_1 + iz_2)(w_1 - iw_2) + (z_1 - iz_2)(w_1 + iw_2)] \\
&= \frac{1}{2}(z_1w_1 - iz_1w_2 + iz_2w_1 + z_2w_2 + z_1w_1 + iz_1w_2 - iz_2w_1 + z_2w_2) \\
&= \frac{1}{2}(2z_1w_1 + 2z_2w_2) \\
&= z_1w_1 + z_2w_2 \\
&= \langle z, w\rangle.
\end{aligned}$$

Next I must demonstrate that $Re(z, w) = \langle z, w\rangle$:

$$\begin{aligned}
Re(z, w) &= Re(z\bar{w}) \\
&= Re((z_1 + iz_2)(w_1 - iw_2)) \\
&= Re(z_1w_1 - iz_1w_2 + iz_2w_1 + z_2w_2) \\
&= z_1w_1 + z_2w_2 \\
&= \langle z, w\rangle.
\end{aligned}$$

Thus $\langle z, w\rangle = \frac{1}{2}[(z, w) + (w, z)] = Re(z, w)$

3 Let $\omega = se^{i\phi}, s \geq 0, \phi \in [0, 2\pi)$. To solve $z^n = \omega$ where $z \in \mathbb{C}$ we must note that the polar form of $z = re^{i\theta}$, thus the equation is actually $r^ne^{in\theta} = se^{i\phi}$. Note that by the polar form of complex numbers $r \geq 0$. Therefore $r = \sqrt[n]{|s|}$. Now to divide out we're left with the equation $e^{i\phi} = e^{in\theta}$. Since $e^{i\theta} = e^{i\theta + 2\pi i}$ then to solve this equation we have to solve that $\phi \mod n\theta \mod 2\pi$. Note that if we add a multiple of $2\pi/n$ to θ then we have that $n(\theta + \frac{2\pi}{n}) = n\theta + 2\pi \equiv n\theta \equiv \phi \mod 2\pi$. Note that $\theta + \frac{2\pi}{n} \neq \theta \mod 2\pi$. Thus $\frac{\phi+2\pi a}{n}$ where $a \in \mathbb{Z}$ are all solutions. But which are unique? Note that if $a = n + b$ then $\frac{\phi+2\pi a}{n} = \frac{\phi+2\pi b}{n} + 2\pi \equiv \frac{\phi+2\pi b}{n} \mod 2\pi$. Therefore we have n solutions, where $a \in \{0, \cdots, n - 1\}$.

7 (a) Let $z, w \in \mathbb{C}$ such that $z = re^{i\theta}$, $\bar{z}w \neq 1$, $r = |z| < 1$, $|w| < 1$. Then we have that

$$\begin{aligned}
1 &\leq \frac{1}{r^2} \\
r^2 + |w|^2 &\leq \frac{1}{r^2}(r^2 + |w|^2) \\
r^2 + |w|^2 &\leq 1 + r^2|w|^2 \\
r^2 - rw - r\bar{w} + |w|^2 &\leq 1 + -rw - r\bar{w} + r^2|w|^2 \\
r^2 - rw - r\bar{w} + (-w)(-\bar{w}) &\leq 1 + -rw - r\bar{w} + (-rw)(-r\bar{w}) \\
(r - w)(r - \bar{w}) &\leq (1 - rw)(1 - r\bar{w})
\end{aligned}$$

Note the inequality is strict if one assumes that $r^2 < 1$ which implies that $r < 1$. Thus

$$\frac{r - w}{1 - r\bar{w}} \frac{r - \bar{w}}{1 - rw} < 1$$

$$\begin{aligned}
\frac{r - w}{1 - r\bar{w}} \frac{r - \bar{w}}{1 - rw} &< 1 \\
\|\frac{r - w}{1 - r\bar{w}}\|^2 &< 1 \\
\|\frac{r - w}{1 - r\bar{w}}\| &< 1 \\
\|\frac{r - w}{1 - r\bar{w}}\| &< 1
\end{aligned}$$

Note that our choice of z doesn't matter since if we define $w' = we^{i\theta}$ $|\frac{w - re^{i\theta}}{1 - \bar{w}re^{i\theta}}| = |e^{i\theta}||\frac{w - re^{i\theta}}{1 - \bar{w}re^{i\theta}}| = |\frac{we^{i\theta} - r}{1 - w'r} = |\frac{\bar{w} - r}{1 - \bar{w}'r}|$. Thus we have proven the desired inequality.

(b) For a fixed $w \in \mathbb{D}$ let $F(z) = \frac{w - z}{1 - \bar{w}z}$

i. We know that F maps from $\mathbb{D} \to \mathbb{D}$ by the proof above. F being holomorphic is equivalent to $\frac{\partial F}{\partial \bar{z}} = 0$ since

$$\frac{\partial F}{\partial \bar{z}} = \frac{0 \cdot (1 - \bar{w}z) - 0 \cdot (w - z)}{(1 - \bar{w}z)^2} = 0$$

then F is holomorphic.

ii. To show that F swaps 0 and w:

- $F(0) = \frac{w - 0}{1 - 0\bar{w}} = \frac{w}{1} = w$.
- $F(w) = \frac{w - w}{1 - \bar{w}w} = 0$. Note that $|w| < 1$ thus $1 - \bar{w}w \neq 0$.

iii. Note by the proof in a equality is attained when $r = 1$, which implies if $|z| = 1$ then $|F(z)| = 1$.

iv. Note that

$$\begin{aligned}
F \circ F(z) &= \frac{w - \frac{w - z}{1 - \bar{w}z}}{1 - \bar{w}\frac{w - z}{1 - \bar{w}z}} \\
&= \frac{w - |w|^2 z - w + z}{1 - |w|^2} \\
&= \frac{z - |w|^2 z}{1 - |w|^2} \\
&= z.
\end{aligned}$$

which holds if $|z| \leq 1$ since $|w| < 1$ and $|\bar{w}^{-1}| > 1$, thus ensuring the denominator is never 0. Furthermore this implies that F is bijective since we have found an inverse.

9 Let $f(z) = f(r,\theta) = u(r,\theta) + iv(r,\theta) = u(x(r,\theta), y(r,\theta)) + iv(x(r,\theta), y(r,\theta))$ where $z = re^{i\theta}$. with $-\pi < \theta < \pi$. Note if we treat f as a function from $\mathbb{R}^2 \to \mathbb{R}^2$ then by the multivariable chain rule we have that

$$\begin{aligned}\begin{bmatrix}\partial_r u & \partial_\theta u \\ \partial_r v & \partial_\theta v\end{bmatrix} &= \begin{bmatrix}\partial_x u & \partial_y u \\ \partial_x v & \partial_y v\end{bmatrix}\begin{bmatrix}\partial_r x & \partial_\theta x \\ \partial_r y & \partial_\theta y\end{bmatrix} \\ &= \begin{bmatrix}\partial_x u & \partial_y u \\ \partial_x v & \partial_y v\end{bmatrix}\begin{bmatrix}\cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta)\end{bmatrix} \\ &= \begin{bmatrix}\partial_x u\cos(\theta) + \partial_y u\sin(\theta) & -\partial_x ur\sin(\theta) + \partial_y ur\cos(\theta) \\ \partial_x v\cos(\theta) + \partial_y v\sin(\theta) & -\partial_x vr\sin(\theta) + \partial_y vr\cos(\theta)\end{bmatrix}\end{aligned}$$

Therefore

$$\begin{aligned}\partial_r u &= \partial_x u\cos(\theta) + \partial_y u\sin(\theta) \\ &= \partial_y v\cos(\theta) - \partial_x v\sin(\theta) \\ &= \frac{1}{r}(\partial_y vr\cos(\theta) - \partial_x vr\sin(\theta) \\ &= \frac{1}{r}\partial_\theta v\end{aligned}$$

and

$$\begin{aligned}\frac{1}{r}\partial_\theta u &= \frac{1}{r}(-\partial_x ur\sin(\theta) + \partial_y ur\cos(\theta)) \\ &= -\partial_x u\sin(\theta) + \partial_y u\cos(\theta) \\ &= -\partial_y v\sin(\theta) - \partial_x v\cos(\theta) \\ &= -\partial_r v\end{aligned}$$

Note that by the requirements above on z we ensure that z is uniquely determined. Using this fact we can observe that $\log(z) = \log(r) + i\theta = u + iv$, and we have that $\partial_r \log(r) = \frac{1}{r} = \frac{1}{r}1 = \frac{1}{r}\partial_\theta\theta$ and $\frac{1}{r}\partial_\theta u = \frac{1}{r}0 = 0 = -\partial_r\theta$. Thus $\log(z)$ is holomorphic on the spe

10 Note that

$$\begin{aligned}4\partial_z\partial_{\bar{z}} &= 4\frac{1}{2}(\partial_x - i\partial_y)\frac{1}{2}(\partial_x + i\partial_y) \\ &= \partial_x^2 + \partial_y^2 - i\partial_y\partial_x + i\partial_x\partial_y \\ &= \partial_x^2 + \partial_y^2 - i\partial_x\partial_y + i\partial_y\partial_x \\ &= (\partial_x + i\partial_y)(\partial_x - i\partial_y) \\ &= 4\frac{1}{2}(\partial_x + i\partial_y)\frac{1}{2}(\partial_x - i\partial_y) \\ &= 4\partial_{\bar{z}}\partial_z\end{aligned}$$

Thus $4\partial_z\partial_{\bar{z}} = 4\frac{1}{2}(\partial_x - i\partial_y)\frac{1}{2}(\partial_x + i\partial_y) = \partial_x^2 + \partial_y^2 - i\partial_y\partial_x + i\partial_x\partial_y = \partial_x^2 + \partial_y^2 - i\partial_y\partial_x + i\partial_y\partial_x = \partial_x^2 + \partial_y^2$

13 Assume $f : \Omega \to \mathbb{C}$ is holomorphic and Ω is an open set. Let $F : \mathbb{R}^2 \to \mathbb{C}$ where $f(x+iy) = F(x,y) = u(x,y) + iv(x,y)$.

(a) If $Re(f)$ is constant then u is constant. Thus $\partial_x u = \partial_y u = 0$. This implies by Cauchy-Riemann that $\partial_x v = \partial_y v = 0$. This implies that v is constant. Thus f is constant

(b) If $Im(f)$ is constant then v is constant. Thus $\partial_x v = \partial_y v = 0$. This implies by Cauchy-Riemann that $\partial_x u = \partial_y u = 0$. This implies that u is constant. Thus f is constant

(c) If $|f|$ is constant than $u^2 + v^2$ is constant. Thus $\partial_x(u^2+v^2) = \partial_y(u^2+v^2) = 0$ giving us the equations

$$\begin{aligned} 2u\partial_x u + 2v\partial_x v = 0 \\ 2u\partial_y u + 2v\partial_y v = 0 \end{aligned}$$

Note that this can be expressed in the form

$$\begin{bmatrix} \partial_x u & \partial_x v \\ \partial_y u & \partial_y v \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Observe that the matrix being used is the transpose of the jacobian, and we have found an element in it's null space. Thus $\det J_F = 0$. Therefore $|f'(z)| = 0$. Thus f is constant.

14

$$\begin{aligned} \sum_{n=M}^{N} a_n b_n &= \sum_{n=M}^{N} (B_n - B_{n-1}) a_n \\ &= \sum_{n=M}^{N} B_n a_n - \sum_{n=M}^{N} B_{n-1} a_n \\ &= \sum_{n=M}^{N} B_n a_n - \sum_{n=M-1}^{N-1} B_n a_{n+1} \\ &= B_N a_N - B_{M-1} a_M + \sum_{n=M}^{N-1} B_n (a_n - a_{n+1} \end{aligned}$$

16 (a) For $a_n = (\log(n))^2$, note that $1 \leq \log(n)^2$ for $n > 2$. Additionally since all polynomials grow faster than log, then $(\log(n))^2 \leq (\sqrt{n})^2 = n$. Thus $1 \leq (\log(n))^2 \leq n^{\frac{1}{n}} \to 1$ for sufficently large n. Thus the radius of convergence is 1.

(b) For $a_n = n!$, we know that for sufficently large n that $n! > (n/2)^{n/2}$. Therefore $\sqrt{n/2} = (n/2)^{\frac{n}{2n}} \leq \sqrt[n]{n!}$, which implies that $\sqrt[n]{n!} = \infty$. Therefore the radius of convergence is 0.

(c) If $a_n = \frac{n^2}{4^n+3n}$ then $\frac{n^2}{4^n} \geq a_n$ and $\frac{n^2}{2\cdot 4^n} \leq a_n$. Therefore $\frac{1}{4} \leq \sqrt[n]{\frac{n^2}{2\cdot 4^n}} \frac{n^{2/n}}{\sqrt[n]{2}4} \leq a_n \leq \frac{n^{2/n}}{4} \to \frac{1}{4}$. Thus by squeeze theorem $\lim a_n = \frac{1}{4}$ and the radius of convergence is 4.

17 Let $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ such that $L = \lim \frac{|a_{n+1}|}{|a_n|}$ and let $\epsilon > 0$. Note there exists some $N \in \mathbb{N}$ such that for all $n \geq N$ $|\frac{|a_{n+1}|}{|a_n|} - L| < \epsilon$. Therefore, $|a_n|(L-\epsilon) < |a_{n+1}| < |a_n|(L+\epsilon)$. Additionally, we have that $|a_n| = |\frac{a_n}{a_{n-1}}\frac{a_{n-1}}{a_{n-2}}\cdots\frac{a_{N+1}}{a_N}a_N|$. Thus, $(L-\epsilon)^{n-N}|a_N| < |a_{n+1}| < (L+\epsilon)^{n-N}|a_N|$. Taking the n-th root we find that $(L-\epsilon)^{1-\frac{N}{n}}|a_N|^{1/n} < |a_{n+1}|^{1/n} < (L+\epsilon)^{1-\frac{N}{n}}|a_N|^{1/n}$. Note that taking the n-th root of a constant sends it to 1, thus in the limit $L-\epsilon < |a_{n+1}|^{1/n} < L+\epsilon$. Note if we reindex a_n we get that $\lim \sqrt[n]{|a_n|} = L$.

19 (a) The power series $\sum nz^n$ has the radius of convergence 0 since $\lim n = \infty$. Thus it doesn't converge for any point on the unit circle

(b) The power series converges for every point on the unit circle since $|\sum_{n=1}^{N}\frac{z^n}{n^2}| \leq \sum_{n=1}^{N}\frac{|z^n|}{n^2} = \sum_{n=1}^{N}\frac{1}{n^2} \to \frac{\pi^2}{6}$

(c) Note if we fix z such that $|z| = 1$ and $z \neq 1$ that $\frac{1}{|1-z|}$ is bounded, and let it equal b. Additionally, $|1-z^n| \leq |1-(-1)| = 2$. Now let $\epsilon > 0$ be fixed. We know that there exists $N_1 \geq M_1 \in \mathbb{N}$ such that $\frac{8}{M_1 b} < \epsilon$. Furthermore there exists $N_2 \geq M_2 \in \mathbb{N}$ such that $|\sum_{n=M_2}^{N_2}\frac{1}{n(n+1)}| < \frac{b\epsilon}{4}$. Finally there exists $N_2 \geq M_3$ such that $|\sum_{n=M_3}^{N_3}\frac{z^2}{n^2}| < \frac{b\epsilon}{4}$. Now if we take $M = \max M_1, M_2.M_3$ and $N = \min N_1, N_2, N_3$ and that $M < N$, then

$$
\begin{aligned}
|\sum_{n=M}^{N}\frac{z^n}{n}| &= |\frac{1}{N}\frac{1-z^N}{1-z} - \frac{1}{M}\frac{1-z^{M-1}}{1-z} + \sum_{n=M}^{N-1}\left(\frac{1}{n} - \frac{1}{n+1}\right)\frac{1-z^n}{1-z}| \\
&\leq \frac{1}{N}\frac{2}{b} + \frac{1}{M}\frac{2}{b} + \frac{1}{b}\left(\sum_{n=M}^{N-1}\frac{1}{n(n+1)} + |\sum_{n=M}^{N-1}\frac{z^n}{n(n+1)}|\right) \\
&\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{1}{b}|\sum_{n=M}^{N-1}\frac{z^n}{n^2}| \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon
\end{aligned}
$$

Therefore the series converges when $z \neq 1$.

23 Let $f(x) = \begin{cases} 0 & x \leq 0 \\ e^{\frac{-1}{x^2}} & x > 0 \end{cases}$. Note that $\frac{d}{dx}f(x) = \begin{cases} 0 & x \leq 0 \\ \frac{2}{x^3}e^{\frac{-1}{x^2}} & x > 0 \end{cases}$ and $\frac{d^2}{dx^2}f(x) = \begin{cases} 0 & x \leq 0 \\ \frac{4-x^2}{x^6}e^{\frac{-1}{x^2}} & x > 0 \end{cases}$. I claim that $f^{(n)}(x) = \frac{P_n(x)}{x^{3n}}f(x)$ where $\deg(P_n(x)) = 2(n-1)$. Note that the base cases have been demonstrated. Now for my induction step, take $f^(n)(x) = \frac{P_n(x)}{x^{3n}}f(x)$.

If $x \leq 0$ then the function should be 0, otherwise if $x > 0$ then

$$\begin{aligned}
f^{(n+1)}(x) &= \frac{d}{dx}\frac{P_n(x)}{x^{3n}}e^{\frac{-1}{x^2}} \\
&= fracP_n(x)x^{3n}\frac{2}{x^3}e^{\frac{-1}{x^2}} + \frac{P_n'x^{3n} - 3nP_n(x)x^{3n-1}}{x^{6n}}e^{\frac{-1}{x^2}} \\
&= \frac{2P_n(x) + x^3P_n'(x) - 3nx^2P_n(x)}{x^{3n+3}}e^{\frac{-1}{x^2}}
\end{aligned}$$

which gives us the induction step in the degree of the bottom polynomial, and in the numerator the degree of the polynomial is $2n$ since $\deg(P_n') = 2(n-1) - 1$ and when multiplied by x^3 we get $2(n-1) - 1 + 3 = 2n$. Since the formula holds, then we claim that $\lim_{x\to 0} f^{(n)}(x) = 0$. Note that if we approach from the left then we trivially get 0. Therefore we must approch from the right. To do so I will make a change of variables with $x = \frac{1}{y}$. Now our limit becomes $\lim_{y\to\infty} f^{(n)}(\frac{1}{y}) = \frac{y^{3n}P_n(\frac{1}{y})}{e^{y^2}}$. Note that since P_n has degree $2(n-1)$ then $y^{3n}P_n(\frac{1}{y})$ is a polynomial in y with degree $n+2$. Thus we're taking the limit to infinity of a polynomial over an exponential. Thus the limit goes to 0. Since all derivatives of f vanish at the origin then the power series is just 0. However that implies that $f = 0$, however since $f \neq 0$ then f isn't represented by it's power series, thus implying that f is not analytic.