2.5.6 Note that $\sup S$ exists as (a_n) is bounded above by M. Therefore let $\sup S = s$ We claim for arbitrary $\epsilon > 0$ there are an infinite number of elements from the sequence (a_n) contained within $V_\epsilon(s)$. By definition of suprememum, for arbitrary $\epsilon > 0, s - \epsilon \in S$. Therefore by definition there are an infinite number of elements of (a_n) above $s - \epsilon$. Similiarly, by definition of supremum $s + \epsilon \notin S$. Therefore there are only finitely many elements of (a_n) above $s + \epsilon$. Since (a_n) is infinite then there must be infinitely many terms less than $s + \epsilon$. Since there are infinitely many terms greater than $s - \epsilon$ and above $s + \epsilon$ there are only finitely many terms then there must be an infinite number of terms between $s - \epsilon$ and $s + \epsilon$. Therefore $V_\epsilon(s)$ has an infinite number of terms from (a_n). We will now construct by induction a subsequence of (a_n) which converges to s, where each a_{n_k} sits within $V_{2^{-k}}(s)$. For $k = 1$, since $0 < 1$, we have an infinite number of terms in $V_1(s)$. Choose $n_1 \in \mathbb{N}$ satisfying $a_{n_1} \in V_1(s)$. By the principle of mathematical induction for all $j \in \mathbb{N}$ if $j < k$ then there exists $n_j \in \mathbb{N}$ satisfying $n_j > n_{j-1} > \cdots > n_1$ and $a_{n_j} \in V_{2^{-j}}(s)$. By the induction hypothesis $a_{n_{k-1}} \in V_{2^{1-k}}(s)$. As proved above there is an infinite number of terms in $V_{2^{-k}}(s)$, therefore we can go out far enough and select a $n_k \in \mathbb{N}$ such that $n_k > n_{k-1}, a_{n_k} \in V_{2^{-k}}(s)$. Now, to prove convergence, suppose $\epsilon > 0$. Since (2^{-n}) goes to 0 then we can find sufficient $k \in \mathbb{N}$ such that $2^{-k} < \epsilon$. Since (2^{-n}) is a decreasing sequence then for any $m > n, V_{2^{-n}}(s) \supseteq V_{2^{-m}}(s)$. Therefore for all $l \geq k, a_{n_l} \in V_{2^{-k}}(s)$. Therefore (a_{n_k}) converges to s.