10.4.3b

10.7.4 Let G be a group, ρ a representation, C a conjugacy class, and $T = \sum_{g \in C} \rho_g$. We want to show that T is G-invariant. Therefore for a group element $h \in G$, we have that $\rho_h(T) = \rho_h(\sum_{g \in C} \rho_g) = \sum_{g \in C} \rho_{hg}$. Note for each hg, there exists a unique $g' \in C$ such that $hg = g'h$, since C is a conjugacy class. Therefore the sum is equivalent to $\sum_{g' \in C} \rho_{g'h} = \sum_{g' \in C} \rho_{g'} \rho_h = T(\rho_h)$. Therefore T is G-invariant

11.1.8b What are the units in $\mathbb{Z}/8\mathbb{Z}$? If $n = 2k$, then $4 \cdot 2k \equiv 0 \mod 8$, thus all even elements are zero divisors. If $\gcd(n, 8) = 1$, then by Bezout's lemma there exists $x, y \in \mathbb{Z}$ such that $nx + 8y = 1$, therefore $nx \equiv 1 \mod 8$. Since $8 = 2^3$, then n must be odd. Thus the units of $\mathbb{Z}/8\mathbb{Z}$ is $\{1, 3, 5, 7\}$.

11.3.2 Let $a \subset \mathbb{Z}[i]$ be a non-zero ideal. Then there exists $x, y \in \mathbb{Z}$ with both not equal to 0 such that $x + iy \in a$. Therefore $(x - iy) \cdot (x + iy) = x^2 + y^2 \in a$. Since at least (WLOG) x is non-zero, x^2 is a non-zero integer. Thus a has a non-zero integer.

11.3.9 (a) Let x be nilpotent, therefore there exists $n \in \mathbb{N}$ such that $x^n = 0$. We want to find $a \in R$ such that $a(1 + x) = 1$. I claim that $a = 1 - x + x^2 - x^3 + x^4 + \cdots + (-1)^{n-1} x^{n-1} = \sum_{i=0}^{n-1} (-1)^i x^i$. Observe that

$$\begin{aligned}
a(1+x) &= (1+x) \sum_{i=0}^{n-1} (-1)^i x^i \\
&= \sum_{i=0}^{n-1} (-1)^i x^i + \sum_{i=0}^{n-1} (-1)^i x^{i+1} \\
&= 1 + \sum_{i=1}^{n-1} (-1)^i x^i + \sum_{i=1}^{n} (-1)^{i+1} x^i \\
&= 1 + \sum_{i=1}^{n-1} (-1)^i x^i + \sum_{i=1}^{n-1} (-1)^{i+1} x^i \\
&= 1
\end{aligned}$$

The final line works since $x^n = 0$. Thus $1 + x$ is a unit.

(b) Let R be a ring with prime characteristic p, and let $a \in R$ be a nilpotent element with $n \in \mathbb{N}$ such that $a^n = 0$. We want to show there exists $k \in \mathbb{N}$ such that $(1 + a)^k = 1$. We claim that $k = n \cdot p$. Observe that if $0 < l < pp$ then $\binom{p}{l} \mid p$ since both $l, p - l < p$, therefore $l!, (p - l)!$ do not contain the prime factor p. Thus $\frac{p!}{l!(p-l)!} \mid p$. Therefore $(1 + a)^p = \sum_{l=0}^{p} \binom{p}{l} a^l = 1 + a^p$