Lemma: Suppose $(p, q) \subset \mathbb{R}, |\alpha| < q - p$. Then there exists $r \in \alpha\mathbb{Z}$ such that $r \in (p, q)$. Suppose for contradiction that for all $r \in \mathbb{Z}, r\alpha \notin (p, q)$. Then there exists $n \in \mathbb{N}$ such that $n\alpha < p, q < (n+1)\alpha$. Therefore $q - p < (n + 1 - n)\alpha = \alpha$. This contradicts $\alpha < q - p$. Therefore there exists $r \in \mathbb{Z}$ such that $r\alpha \in (p, q)$.

1. *Show that the set of all dyadic rational numbers in* $[0, 1]$ *is dense*
   Suppose $r, s \in [0, 1], r > s$. Since $0 < r - s$, then by the Archimedean property there exists $l \in \mathbb{N}$ such that $\frac{1}{l} < r - s$. Since 2^m is unbounded then there exists $n \in \mathbb{N}$ such that $l < 2^n$. Therefore $2^{-n} < r - s$. Thus by the lemma above there exists $k \in \mathbb{N}$ such that $s < \frac{k}{2^n} < r$. Therefore the dyadics are dense on the unit interval.

2. *Let* $\alpha, Q \in \mathbb{R}, Q \geq 1$. *Show there exists* $a, q \in \mathbb{Z}, q < Q, gcd(a, q) = 1, |\alpha - \frac{a}{q}| < \frac{1}{qQ}$
   Let n be the smallest integer greater than Q. Consider the partition of the interval $[0, 1)$ by the set $E = \{[\frac{l}{n}, \frac{l+1}{n}) : l \in [n-1] \cup \{0\}\}$. Additionally, note that each $\{0, \{\alpha\}, \{2\alpha\}, \cdots, \{n\alpha\}\}$ can be assigned to one of the partitions. Since all elements are capable of fitting into the n partitions, and there are $n + 1$ elements then by the Pigeonhole principle there exists $k_1, k_2 \in \mathbb{N}_0$ such that $|\{k_1\alpha\} - \{k_2\alpha\}| < \frac{1}{n}$. Note that by the definition of $\{\}$, we can rewrite the expression inside of the absolute value signs as $|(k_1 - k_2)\alpha - ([k_1\alpha] - [k_2\alpha])| < \frac{1}{n}$. Therefore if we set $q = (k_1 - k_2)/gcd(k_1 - k_2, [k_1\alpha] - [k_2\alpha]), a = [k_1\alpha] - [k_2\alpha]/gcd(k_1 - k_2, [k_1\alpha] - [k_2\alpha])$ then we have the equations $|\alpha - \frac{a}{q}| < \frac{1}{nq} < \frac{1}{qQ}$. Note that by construction $gcd(a, q) = 1$. Therefore the theorem has been demonstrated.

3. *Show for every* $n \in \mathbb{N}$ *the closed form of the sum*

$$\sum_{k=1}^{n} \frac{1}{k(k+1)}$$

   Note that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. Therefore the sum expressed above can be rewritten as follows:

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k(k+1)} &= \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+1} \\ &= \sum_{k=1}^{n} \frac{1}{k} - \sum_{j=2}^{n+1} \frac{1}{j} \\ &= 1 - \frac{1}{n+1} + \sum_{k=2}^{n} \frac{1}{k} - \sum_{j=2}^{n} \frac{1}{j} \\ &= 1 - \frac{1}{n+1}.\end{aligned}$$

   Therefore since $\frac{1}{n} \to 0$ then the limit of S_n exists and is given by $\lim S_n = 1$.

4. *Assume that* $\alpha \notin \mathbb{Q}$ *show that the sequence* $\{\{n\alpha\} : n \in \mathbb{N}\}$ is dense in $[0, 1]$.
   Suppose $\alpha \in \mathbb{R}, (p, q) \subset [0, 1]$. Since $\alpha \notin \mathbb{Q}$ then there exists an infinite number

of rational approximations, $\frac{p_n}{q_n} \in \mathbb{Q}$ where $|\alpha - \frac{p_n}{q_n}| < \frac{1}{q_n^2}$ (consequence of problem 2). Note that when $n \to \infty$ then $q_n \to \infty$, therefore $\frac{1}{q_n} \to 0$. Thus there exists $n \in \mathbb{N}$ where $\frac{1}{q_n} < q - p$. Therefore $|q_n\alpha - p_n| < \frac{1}{q_n}$. Since p_n is an integer we can consider $|\{q_n\alpha\}| < \frac{1}{q_n}$. Therefore by the lemma by the lemma there exists $r \in \mathbb{Z}$ where $\pm\{rq_n\alpha\} \in (p, q)$. Thus the integer multiples of the fractional portion of α is dense in $[0, 1]$.

5. *Let $(\mathbb{F}, <)$ be an ordered field. Show that the axiom of completeness implies the Archimedean property.*
   Suppose $x \in \mathbb{R}$, and consider the set $S = \{n \in \mathbb{N} : n \leq x\}$. Note that if $x < 1$ then x is trivially bounded above by a natural number. Therefore we assume that $1 \leq x$. Therefore the set S is guarenteed to contain 1. Note that since S is non-empty and bounded above, by the axiom of completeness $supE$ exists. Let this number be denoted $s = \sup E$. By the definition of suprememum, $x < n + 1$. Since $n + 1 \in \mathbb{N}$, then $\mathbb{F}$ satisfies the archimedean property.

6. *Let $(\mathbb{F}, <)$ be an ordered field, which is Cauchy complete. Show that $\mathbb{F}$ satisfies the nested interval property*
   Suppose that $\mathbb{F}$ satisfies the nested interval property and we have a sequence of nested intervals $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots$ where $\lim |a_n - b_n| = 0$. Suppose $\epsilon > 0$, $(\epsilon_n)_{n\in\mathbb{N}} \to 0$. Note that since $\lim |a_n - b_n| = 0$ then there exists $N_1 \in \mathbb{N}$ where $|a_{N_1} - b_{N_1}| < \epsilon_1$, let $x_1 \in [a_{N_1}, b_{N_1}]$. Similarly for ϵ_2, we can find a N_2, x_2 such that $x_2 \in [a_{N_2}, b_{N_2}], |a_{N_2} - b_{N_2}| < \epsilon_2$. In general we can construct a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in [a_{N_n}, b_{N_n}], |a_{N_n} - b_{N_n}| < \epsilon_n$. Therefore for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\epsilon_N < \epsilon/2$. Thus for arbitrary $n > m \geq N$ we have that

$$\begin{aligned}
|x_n - x_m| &= |x_n - a_{N_N} + a_{N_N} - x_m| && \text{add 0} \\
&\leq |x_n - a_{N_N}| + |a_{N_N} - x_m| && \text{triangle inequality} \\
&\leq |b_{N_N} - a_{N_N}| + |a_{N_N} - b_{N_N}| && \text{the difference is bounded by the endpoints} \\
&< \epsilon_2 + \epsilon_2 \\
&= \epsilon
\end{aligned}$$

   Since (x_n) is Cauchy, then by Cauchy completeness $(x_n) \to x$. By a similar proof above we can show that $(a_n) \to x$ and $(b_n) \to x$. Therefore clearly $x \in \cap_{i=1}^{\infty}[a_i, b_i]$. Thus by the sqeeze theorem for any $y \in \cap_{i=1}^{\infty}[a_i, b_i]$ since $a_i \leq y \leq b_i$ for all $i \in \mathbb{N}$ and $(a_n), (b_n) \to x$ then $y = x$. Thus $\{x\} = \cap_{i=1}^{\infty}[a_i, b_i]$.

7. *Using the Archimedean property on $\mathbb{R}$ show that for every $x, y \in \mathbb{R}$ such that $x < y$ there exists $a \in \mathbb{Q}$ satisfying $x < a < y$.*
   Since $x < y$ then $0 < y - x$. Therefore by the Archimedean property there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < y - x$. Therefore by the lemma there exists $m \in \mathbb{Z}$ such that $\frac{m}{n} \in (x, y)$.

8. *Show that if $a_1, \ldots, a_n > 0$ and $a_1 \cdots a_n = 1$ then $a_1 + \cdots + a_n \geq n$*
   We will prove this statement by induction. Note that for the base case if $a_1 = 1$ then trivially $a_1 \geq 1$. Therefore by the principle of mathematical induction for all $k \in \mathbb{N}$

if $k < n$ then the proposition holds. WLOG assume a_1 is the minimal element and a_n is the maximal element. Then they satisfy the inequality $a_1 \leq 1 \leq a_n$. Therefore $(1-a_1)(a_n-1) \geq 0$. Therefore $a_1 + a_n - 1 \geq a_1 a_n$. Thus by the induction hypothesis

$$n - 1 \leq a_2 + \cdots + a_{n-1} + a_1 a_n \leq a_1 + \cdots + a_n - 1$$

$$n \leq a_1 + \cdots + a_n$$

9. *Using the previous problem show that for every positive numbers $x_1, x_2, \ldots, x_n$ we have*

$$(x_1 \cdot x_2 \cdot \ldots \cdot x_n)^{\frac{1}{n}} \leq \frac{x_1 + x_2 + \ldots + x_n}{n}$$

If we consider the product $x_1 \cdot x_2 \cdots x_n = p$, then if we set the sequence $a_k = \frac{x_k}{p^{\frac{1}{n}}}$, then $a_1 \cdots a_n = 1$. Therefore by the previous problem $a_1 + \cdots + a_n \geq n$. Therefore $\frac{x_1+\cdots+x_n}{p^{\frac{1}{n}}} \geq n$, giving us $\frac{x_1+\cdots+x_n}{n} \geq (x_1 \cdots x_n)^{\frac{1}{n}}$.

10. *Let $x_1 = 2$ and $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ for all $n \in \mathbb{N}$. Show that $(x_n)_{n\in\mathbb{N}}$ converges and find its limit.*
We will show (x_n) converges via monotone convergence theorem

- We will show that (x_n) is bounded below by $\sqrt{2}$. Since $x_n = \frac{1}{2}(x_{n-1} + \frac{2}{x_{n-1}})$, then it is an arithmetic mean. Therefore $x_n \geq \sqrt{\frac{2x_n}{x_n}} = \sqrt{2}$.
- We will show by induction that (x_n) is monotonically decreasing. For $n = 1$ we have that $x_1 = 2 > 1 + \frac{1}{2} = \frac{x_n}{2} + \frac{1}{x_n} = x_2$. Therefore by the principle of mathematical induction the theorem holds for all cases up to n. We want to show for $n+1$ that $x_{n+1} > x_n$. Observe that

$$\begin{aligned} x_{n+1} &= \frac{1}{2}(x_n + \frac{2}{x_n}) \\ &\leq \frac{1}{2}(x_n + \sqrt{2}) \\ &\leq \frac{1}{2}(x_n + x_n) \\ &= x_n. \end{aligned}$$

  Thus x_n is monotonically decreasing.

Since (x_n) is monotonically decreasing and bounded below then it converges. We claim that $\sqrt{2}$ is it's limit. Note that $\lim x_n = \lim x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$. Therefore the limit must satisfy it's own recurrence. Note that $x = \frac{x}{2} + \frac{1}{x}$ is solved by $x^2 = 2$. Since x_n is never negative then $x = \sqrt{2}$. Thus $x_n \to \sqrt{2}$