Alex Valentino

1.2.10 Let $y_1 = 1$ and for each $n \in \mathbb{N}$ define $y_{n+1} = \frac{3y_n+4}{4}$.

(a) Use induction to prove that the sequence satisfies $y_n < 4$ for all $n \in \mathbb{N}$.
Proof: For the base case, $y_1 = 1 < 4$. By the principle of mathematical induction for all $k \in \mathbb{N}$ if $k < n$ then $y_k < 4$. We must show that $y_n < 4$. Since $n - 1 < n$, then by the induction hypothesis $y_{n-1} < 4$. Therefore,

$$\begin{aligned} y_{n-1} &< 4 \\ 3y_{n-1} &< 12 \\ 3y_{n-1} + 4 &< 16 \\ (3y_{n-1} + 4)/4 &< 4 \\ y_n &< 4. \end{aligned}$$

(b) We must show that $(y_1, y_2, \cdots)$ is increasing. For the base case, $y_1 = 1, y_2 = \frac{3+4}{4} = \frac{7}{4}, 1 < \frac{7}{4}$. By PMI for all $k \in \mathbb{N}$ if $k < n$ then $y_k < y_{k+1}$. Since $n - 1 < n$, by the induction hypothesis $y_{n-1} < y_n$. Therefore,

$$\begin{aligned} y_{n-1} &< y_n \\ 3y_{n-1} + 4 &< 3y_n + 4 \\ \frac{3y_{n-1} + 4}{4} &< \frac{3y_n + 4}{4} \\ y_n &< y_{n+1} \end{aligned}$$