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Lemma 1: We claim that $C\backslash A \subseteq (B\backslash A)\cup(C\backslash B)$. Suppose $x \in C\backslash A, x \notin B\backslash A$. We must show $x \in C\backslash B$. Since $x \notin B\backslash A$ by the definition of set difference $x \notin B$ or $x \in A$. Since $x \in C\backslash A$ then $x \in C$ and $x \notin A$. Since $x \notin A$, then $x \notin B$. Therefore since $x \in C$ and $x \notin B$ then by the definition of set difference $x \in C\backslash B$.

Lemma 2: We claim that for all $M, N \in \mathcal{F}$, if $M \neq N, MRN$, then there exists an element $m_* \in M$ such that for all $x \in N\backslash M, m_* < x$. Suppose $M, N \in \mathcal{F}$, $M \neq N, MRN$. Since MRN, then $min(M\triangle N) \in M$. Let $m_* = min(M\triangle N)$. Therefore by definition of $M\triangle N$ and the minimum of a set, for all $y \in (M\backslash N)\cup(N\backslash M), m_* \leq y$. Since $N\backslash M \subset (M\backslash N)\cup(N\backslash M)$, then for all $z \in N\backslash M, m_* \leq z$. By definition of $N\backslash M$, for all $z \in N\backslash M$, $z \notin M$. Since $m_* \in M$, then $m_* \neq z$. Since $m_* \neq z, m_* \leq z$, then by definition of $<$, for all $z \in M\backslash N, m_* < z$.

Let $\mathcal{F}$ be the set of all finite subsets of $\mathbb{Z}$. We define a relation R on $\mathcal{F}$ as follows: For $A, B \in \mathcal{F}$ we say ARB if either $A = B$, or if the smallest member of $A\triangle B$ belongs to A. Prove that R is a total order on $\mathcal{F}$. (Recall $A\triangle B = (A\backslash B)\cup(B\backslash A)$.)

We must show that R is a total order on $\mathcal{F}$. Therefore we must show that R is anti-symmetric, full, transitive, and reflexive.

- We must show that R is anti-symmetric. There we must show for all $A, B \in \mathcal{F}$ if $A \neq B, ARB$ then $B\not{R}A$. Suppose $A, B \in \mathcal{F}, A \neq B, ARB$. We must show that $B\not{R}$. By definition of R, we must show that $\min(B\triangle A) \notin B$. Since $\triangle$ is symmetric then we must show $\min(A\triangle B) \notin B$. By definition of R we have $\min(A\triangle B) \in A$. Since $\min(A\triangle B) \notin B$ then it must be in the other set, thus $\min(A\triangle B) \in A$.

- We must show that R is full. Therefore we must show for all $A, B \in \mathcal{F}$, ARB or BRA. Suppose $B\not{R}A$. We must show ARB. By definition of R, $min(B\triangle A) \notin B$. Since no other set can contain the minimum except A, then $min(A\triangle B) \in A$. Therefore ARB.

- We must show that R is reflexive. We must show for all $A \in \mathcal{F}$ ARA. Since $A = A$, then ARA.

- We must show that R is transitive. We must show for all $A, B, C \in \mathcal{F}$ if ARB, BRC then ARC. Suppose $A, B, C \in \mathcal{F}$, ARB, BRC. Note that if $A = B$ or $B = C$, then it is vacuously transitive, therefore we assume $A \neq B, B \neq C$, which from here forward in the proof we assume lemma 2. Then by definition of R, $min(A\triangle B) \in A, min(B\triangle C) \in B$. let $a_* = min(A\triangle B), b_* = min(B\triangle C)$. Since $a_* \in A, b_* \in B$, then $a_* \in A\backslash B, b_* \in B\backslash C$. However we don't know if $a_* \in C, b_* \in A$. Therefore we have four cases.

  - Assume $a_* \notin C, b_* \notin A$. Since $b_* \notin A$, then $a_* < b_*$ as $a_*, b_* \in A\triangle B$. Since $a_* \notin C$, then by definition of set difference $a_* \in A\backslash C$. By lemma 2, $b_* < x$, for all $x \in C\backslash B$, and $a_* < y$ for all $y \in B\backslash A$. Since $a_* < b_*$ we may have for all $x \in C\backslash B, a_* < x$. By the definition of set union we may have for all $z \in C\backslash B \cup B\backslash A, a_* < z$. Since by the lemma 1 $C\backslash A \subset C\backslash B \cup B\backslash A$, then for all $w \in C\backslash A, a_* < w$. Therefore since $a_* \in A\backslash C$, and a_* is less than all of the elements in $C\backslash A$, then the minimum can't exists in $C\backslash A$. Therefore $C\not R A$. Since R is anti-symmetric, then ARC.

- Assume $a_* \notin C, b_* \in A$. Since $a_*, b_* \in A$, and $a_* \notin B$, then $a_* \neq b_*$. Since $a_*, b_* \in \mathbb{Z}$, then either $a_* < b_*$ or $b_* > a_*$. Therefore we have two cases:
  - Assume $a_* < b_*$. Since BRC, then $b_* < c$, for all $c \in C\backslash B$. Since ARB, then $a_* < b$ for all $b \in B\backslash A$. Since $a_* < b_* < c$, and for all $c \in C\backslash B, a_* < b$ for all $b \in B\backslash A$, then by the definition of union for all $x \in (B\backslash A) \cup (C\backslash B), a_* < x$. Since $C\backslash A \subseteq (B\backslash A) \cup (C\backslash B)$, then for all $y \in C\backslash A, a_* < y$. Therefore ARC.
  - Assume $b_* < a_*$. Since ARB, then $a_* < b$ for all $b \in B\backslash A$. Since BRC then for all $b_* < c$ for all $c \in C\backslash B$. Since $b_* < a_*$ then for all $b \in B\backslash A, b_* < b$. Therefore by the definition of set union for all $x \in (B\backslash A) \cup (C\backslash B), b_* < x$. Since $C\backslash A \subset (B\backslash A) \cup (C\backslash B)$, then for all $y \in C\backslash A, b_* < y$. Since $b_* \in A, ARC$.
- Assume $a_* \in C, b_* \notin A$. By definition of set difference, since $a_* \in C, a_* \notin B, b_* \in B, b_* \notin A$, then $a_* \in C\backslash B, b_* \in B\backslash A$. By lemma 2 and ARB, for all $x \in B\backslash A, a_* < b$. Since $b_* \in B\backslash A$, then $a_* < b_*$. By lemma 2 and BRC, for all $y \in C\backslash B, b_* < y$. Since $a_* \in C\backslash B$, then $b_* < a_*$. Therefore we have a contradiction, so this case can never occur.
- Assume $a_* \in C, b_* \in A$. Since $a_* \in C$, as shown before $a_* \in C\backslash B$. Since BRC and lemma 2, then as shown before $b_* < a_*$. By the definition of R and lemma 2 we have for all $x \in B\backslash A, a_* < x$, and for all $y \in C\backslash B, b_* < y$. Since $b_* < a_*$ then we have for all $x \in B\backslash A, b_* < x$. By the definition of set union, for all $z \in (B\backslash A) \cup (C\backslash B), b_* < z$. By lemma 1, for all $z \in C\backslash A, b_* < z$. As shown before $b_* \in A\backslash C$, since it is less than all the elements in $C\backslash A$, then the minimum can't exists in $C\backslash A$, therefore $C\not R A$. Since R is antisymmetric, then we have ARC.