6 a Let $y(t) = x'(t), \vec{x} = (x(t), y(t))$. Therefore $\vec{x}' = (x', y') = (y, -\frac{k}{m}x - \frac{a}{m}y) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{a}{m} \end{bmatrix} \vec{x}$. Thus $B = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{a}{m} \end{bmatrix}$.

b
- Suppose $(\frac{a}{m})^2 = \frac{4k}{m}$ (critically dampened). We must compute e^{tB}. Thus $0 = det(B - \lambda \mathbb{I}_2) = \lambda^2 + \frac{a}{m}\lambda + \frac{k}{m}$, therefore $\lambda = \frac{\frac{-a}{m} \pm \sqrt{(\frac{a}{m})^2 - \frac{4k}{m}}}{2} = \frac{\frac{-a}{m} \pm 0}{2} = \frac{-a}{2m}$. Thus $\lambda^2 + \frac{a}{m}\lambda + \frac{k}{m} = (\lambda + \frac{a}{2m})^2$. Therefore $\lambda = \frac{-a}{2m}$ is an eigenvalue with multiplicity 2. Since we are in $\mathbb{R}^2$, and that is the same as the multiplicity of the eigenvalue then we know that $(B + \frac{a}{2m})^2 \vec{x} = 0$ for all $\vec{x} \in \mathbb{R}^2$. Therefore $e^{tB} = e^{\frac{-a}{2m}t} \sum_{k=0}^{1} t^k (B + \frac{a}{2m}\mathbb{I}_2) = e^{\frac{-a}{2m}t}(\mathbb{I}_2 + t \begin{bmatrix} \frac{a}{2m} & 1 \\ \frac{-k}{m} & \frac{-a}{2m} \end{bmatrix}) = e^{\frac{-a}{2m}t} \begin{bmatrix} 1 + t\frac{a}{2m} & t \\ \frac{-k}{m}t & 1 - \frac{a}{2m}t \end{bmatrix}$.
- Suppose $(\frac{a}{m})^2 < \frac{4k}{m}$ (under dampened). We must compute e^{tB}. Since $(\frac{a}{m})^2 < \frac{4k}{m}$ then $(\frac{a}{m})^2 - \frac{4k}{m} = \frac{-\rho^2}{m^2}$. Therefore $\lambda = \frac{\frac{-a}{m} \pm \sqrt{(\frac{a}{m})^2 - \frac{4k}{m}}}{2} = \frac{\frac{-a}{m} \pm i\frac{\rho}{m}}{2} = \frac{1}{2m}(-a \pm i\rho)$. Let $\mu_1 = \frac{1}{2m}(-a + i\rho), \mu_2 = \frac{1}{2m}(-a - i\rho)$. Computing eigenvalues for μ_1 yields $\vec{v}_1 = (1, \frac{1}{2m}(-a + i\rho))$. Therefore $z_1 = e^{\mu_1 t}(1, \frac{1}{2m}(-a + i\rho)) = e^{\frac{-a}{2m}t}(\cos(\frac{\rho}{2m}t) + i\sin(\frac{\rho}{2m}t))(1, \frac{1}{2m}(-a + i\rho))$. Let $l = \frac{\rho}{2m}$.
Thus
$$\begin{aligned} z_1 &= e^{\frac{-a}{2m}t}(\cos(\frac{\rho}{2m}t) + i\sin(\frac{\rho}{2m}t))(1, \frac{1}{2m}(-a + i\rho)) \\ &= e^{\frac{-a}{2m}t}(\cos(lt) + i\sin(lt))(1, \frac{1}{2m}(-a + i\rho)) \\ &= e^{\frac{-a}{2m}t}(\cos(lt), \frac{1}{2m}(-a\cos(lt) - \rho\sin(lt))) + ie^{\frac{-a}{2m}t}(\sin(lt), \frac{1}{2m}(-a\sin(lt) + \rho\cos(lt))) \end{aligned}$$
Since we have a single complex solution, then we have the two real solutions forming the columns of $M(t)$:
$$e^{\frac{-a}{2m}t} \begin{bmatrix} \cos(lt) & \sin(lt) \\ \frac{1}{2m}(-a\cos(lt) - \rho\sin(lt) & \frac{1}{2m}(\rho\cos(lt) - a\sin(lt)) \end{bmatrix}$$
Since $M(0) = \begin{bmatrix} 1 & 0 \\ \frac{-a}{2m} & \frac{\rho}{2m} \end{bmatrix}$ then $M^{-1}(0) = \begin{bmatrix} 1 & 0 \\ \frac{a}{\rho} & \frac{2m}{\rho} \end{bmatrix}$. Therefore
$$\begin{aligned} e^{tB} = M(t)M^{-1}(0) &= e^{\frac{-a}{2m}t} \begin{bmatrix} \cos(lt) & \sin(lt) \\ \frac{1}{2m}(-a\cos(lt) - \rho\sin(lt) & \frac{1}{2m}(\rho\cos(lt) - a\sin(lt)) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{a}{\rho} & \frac{2m}{\rho} \end{bmatrix} \\ &= e^{\frac{-a}{2m}t} \begin{bmatrix} \frac{a}{\rho}\sin(lt) + \cos(lt) & \frac{2m}{\rho}\sin(lt) \\ -\frac{2k}{\rho}\sin(lt) & -\frac{a}{\rho}\sin(lt) + \cos(lt) \end{bmatrix}. \end{aligned}$$
- Suppose $(\frac{a}{m})^2 > \frac{4k}{m}$ (over dampened). We must compute e^{tB}. Since $(\frac{a}{m})^2 > \frac{4k}{m}$ then $\lambda = \frac{1}{2m}(-a \pm \rho)$. Let $\mu_1 = \frac{1}{2m}(-a + \rho), \mu_2 = \frac{1}{2m}(-a - \rho)$. These correspond to the eigenvectors $v_1 = (1, \frac{1}{2m}(-a + \rho)), v_2 = (1, \frac{1}{2m}(-a - \rho))$. Therefore $z_1 = e^{\frac{1}{2m}(-a+\rho)t}(1, \frac{1}{2m}(-a + \rho)), z_2 = e^{\frac{1}{2m}(-a-\rho)t}(1, \frac{1}{2m}(-a - \rho))$. Therefore
$$M(t) = e^{\frac{-a}{2m}t} \begin{bmatrix} e^{lt} & e^{-lt} \\ \frac{e^{l}t}{2m}(\rho - a) & -\frac{e^{-lt}}{2m}(\rho + a) \end{bmatrix}$$

. $M^{-1}(0) = \begin{bmatrix} 1 & 1 \\ \frac{\rho-a}{2m} & \frac{-\rho-a}{2m} \end{bmatrix}^{-1} = \frac{1}{\rho}\begin{bmatrix} \frac{\rho+a}{2} & -m \\ \frac{\rho-a}{2} & -m \end{bmatrix}$. Thus

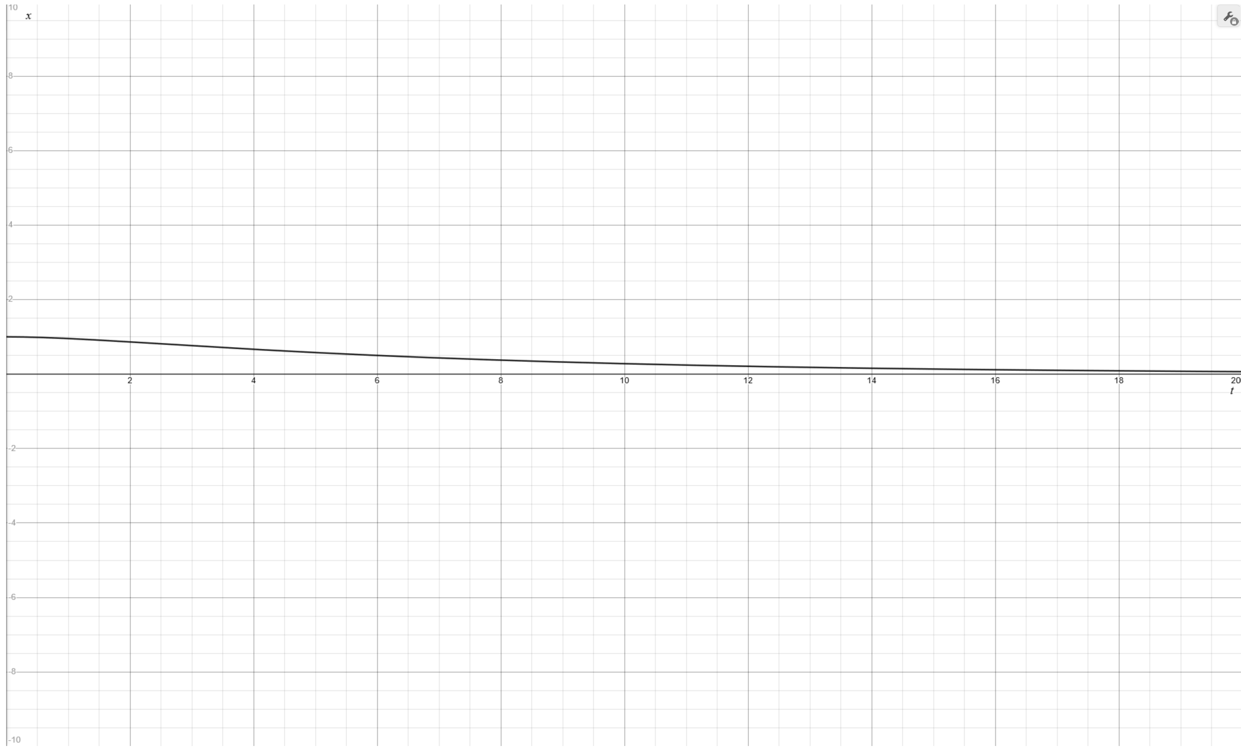
$$\begin{aligned}
e^{tB} &= M(t)M^{-1}(0) \\
&= \frac{e^{\frac{-a}{2m}t}}{\rho}\begin{bmatrix} e^{lt} & e^{-lt} \\ \frac{e^{l}t}{2m}(\rho-a) & -\frac{e^{-lt}}{2m}(\rho+a) \end{bmatrix}\begin{bmatrix} \frac{\rho+a}{2} & -m \\ \frac{\rho-a}{2} & -m \end{bmatrix} \\
&= \frac{e^{\frac{-a}{2m}t}}{\rho}\begin{bmatrix} \frac{\rho+a}{2}e^{lt} + \frac{\rho-a}{2}e^{-lt} & -m(e^{lt}+e^{-lt}) \\ \frac{1}{2m}(\rho^2+a^2)(e^{lt}-e^{-lt}) & \frac{\rho+a}{2}e^{lt} - \frac{\rho-a}{2}e^{-lt} \end{bmatrix} \\
&= e^{\frac{-a}{2m}t}\begin{bmatrix} \frac{a}{\rho}\sinh(lt) + \cosh(lt) & \frac{1}{l}\sinh(lt) \\ \frac{-2k}{\rho}\sinh(lt) & \frac{-a}{\rho}\sinh(lt) + \cosh(lt) \end{bmatrix}
\end{aligned}$$

b(2?) Since $\vec{x} = e^{tB}\vec{x}_0$, where $\vec{x}_0 = (x_0, v_0)$, then $x(t) = e^{\frac{-a}{2m}t}(x_0(1+t\frac{a}{2m}) + v_0 t)$. We want to find t such that $0 = x(t)$. Thus we have that $0 = e^{\frac{-a}{2m}t}(x_0(1+t\frac{a}{2m}) + v_0 t)$. Since $e^{\frac{-a}{2m}t}$ is strictly positive then $0 = x_0(1+t\frac{a}{2m}) + v_0 t$. This may be rewritten as $\frac{-x_0}{v_0+\frac{a}{2m}} = t$. If this value either diverges or is negative then the solution will not pass through the origin, and if it does we have found an explicit single value for t. Thus there is either one or zero times which $x(t) = 0$.

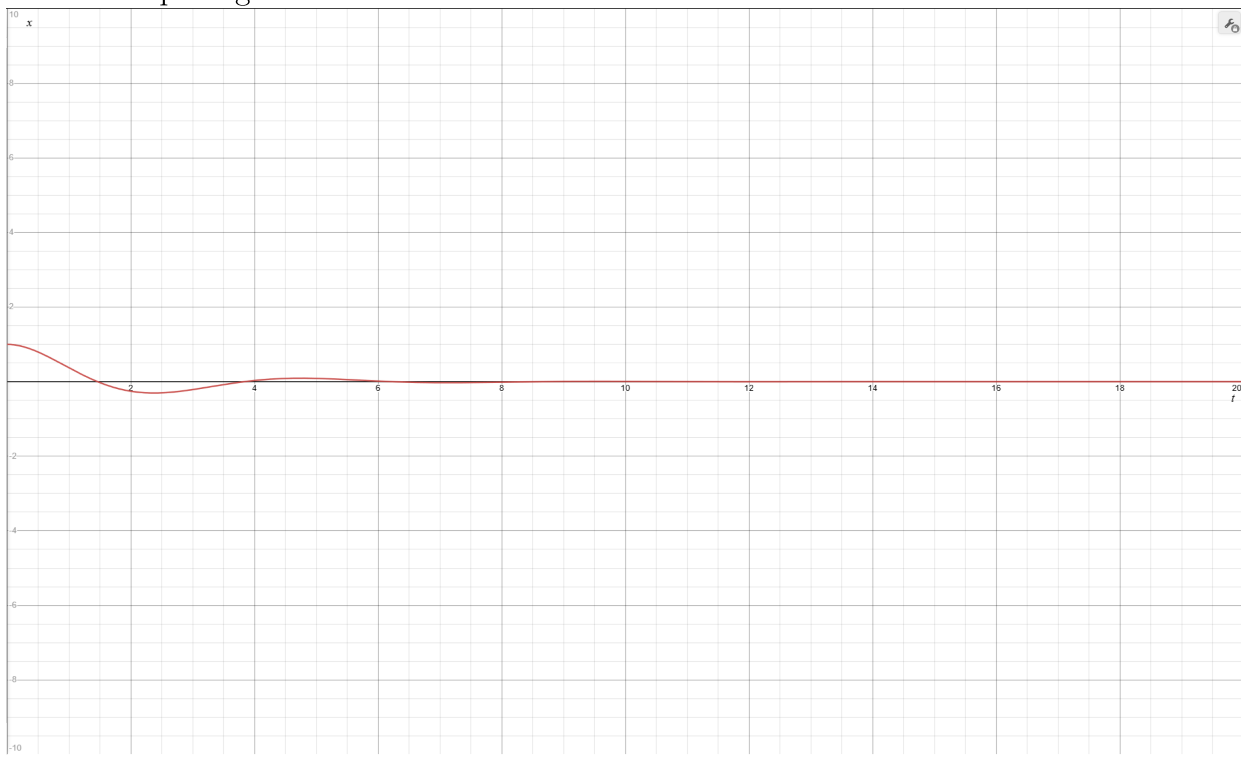
c Since $\vec{x} = e^{tB}\vec{x}_0$, where $\vec{x}_0 = (x_0, v_0)$, then $x(t) = e^{\frac{-a}{2m}t}(x_0(\frac{a}{\rho}\sinh(lt) + \cosh(lt)) + \frac{v_0}{l}\sinh(lt))$. We want to find when $x(t) = 0$ Since $e^{\frac{-a}{2m}t}$ is strictly positive then we have that $0 = x_0(\frac{a}{\rho}\sinh(lt) + \cosh(lt)) + \frac{v_0}{l}\sinh(lt)$. Rearrange and assuming that $\frac{x_0 a}{\rho} + \frac{v_0}{l} \neq 0$ we get that $-(\frac{\frac{x_0 a}{\rho}+\frac{v_0}{l}}{x_0})^{-1} = \tanh(lt)$. Since $tanh$ is 1-1 then we have a unique t if the value is in the range of tanh, if not then the function doesn't pass through the origin. If $\frac{x_0 a}{\rho} + \frac{v_0}{l} = 0$ then we have $0 = x_0\cosh(x)$, similarly, since arcosh is 1-1 then we have a unique t at which the function passes through the origin. Therefore $x(t) = 0$ occurs at most once for all parameters.

d Since $\vec{x} = e^{tB}\vec{x}_0$, where $\vec{x}_0 = (x_0, v_0)$, then $x(t) = e^{\frac{-a}{2m}t}(x_0(\frac{a}{\rho}\sin(lt) + \cos(lt)) + \frac{v_0}{l}\sin(lt))$. We want to find when $x(t) = 0$. Since $e^{\frac{-a}{2m}t}$ is strictly positive then we have that $0 = x_0(\frac{a}{\rho}\sin(lt) + \cos(lt)) + \frac{v_0}{l}\sin(lt)$. Rearrange and assuming that $\frac{x_0 a}{\rho} + \frac{v_0}{l} \neq 0$ we get that $-(\frac{\frac{x_0 a}{\rho}+\frac{v_0}{l}}{x_0})^{-1} = \tan(lt)$. Note that since tan is periodic and arctan is $1-1$ means that $t = \frac{\pi n + \arctan(-(\frac{\frac{x_0 a}{\rho}+\frac{v_0}{l}}{x_0})^{-1})}{l}$ where $n \in \mathbb{Z}$ are all valid solutions. If $\frac{x_0 a}{\rho} + \frac{v_0}{l} = 0$ then we have $0 = x_0\cos(x)$, similarly, since cos is periodic and not the zero function due to $x(t) \neq 0$ then we have $t = \frac{\pi n}{l}$ where $n \in \mathbb{Z}$. Since all cases have an infinite number of solutions, then $x(t)$ transits the origin an infinite number of times.

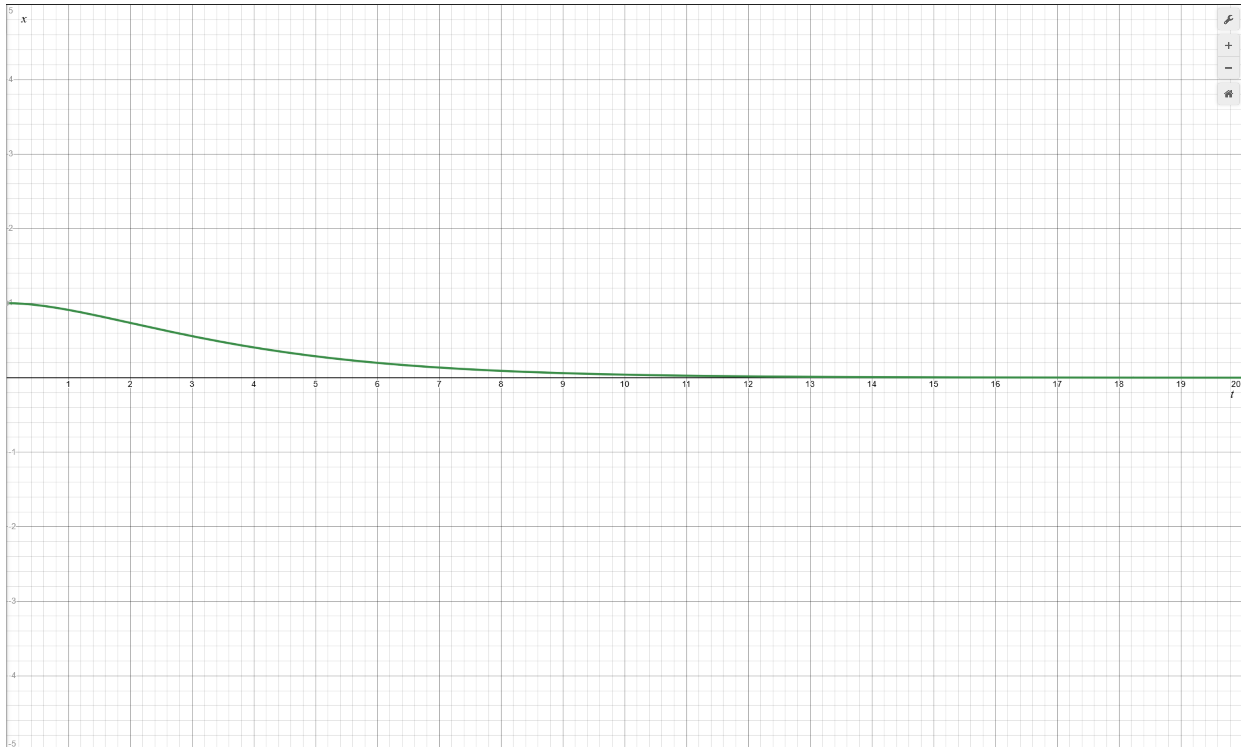
e
- Over dampening



- Under dampening



- Critical dampenening



7 By our previous expression for $\vec{x}'$, we can define our new $\vec{x}'(t) = B\vec{x}(t) + \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$.

Therefore by Duhamel's formula $\vec{x} = e^{tB}\vec{x}_0 + \int_{t_0}^{t} e^{(t-s)B} \begin{bmatrix} 0 \\ g(s) \end{bmatrix} ds$

a

- Suppose $(\frac{a}{m})^2 = \frac{4k}{m}$. Then our formula becomes

$$x(t) = e^{\frac{-a}{2m}t}(x_0(1 + t\frac{a}{2m}) + y_0 t) + \int_0^t e^{\frac{-a}{2m}(t-s)}(t-s)\cos(\omega s)ds$$

- Suppose $(\frac{a}{m})^2 < \frac{4k}{m}$. Then our formula becomes

$$x(t) = e^{\frac{-a}{2m}t}(x_0(\frac{a}{\rho}\sin(lt) + \cos(lt)) + \frac{v_0}{l}\sin(lt)) + \int_0^t e^{\frac{-a}{2m}(t-s)}\frac{\sin(l(t-s))}{l}\cos(\omega s)ds$$

- Suppose $(\frac{a}{m})^2 > \frac{4k}{m}$. Then our formula becomes

$$x(t) = e^{\frac{-a}{2m}t}(x_0(\frac{a}{\rho}\sinh(lt) + \cosh(lt)) + \frac{v_0}{l}\sinh(lt)) + \int_0^t e^{\frac{-a}{2m}(t-s)}\frac{\sinh(l(t-s))}{l}\cos(\omega s)ds$$

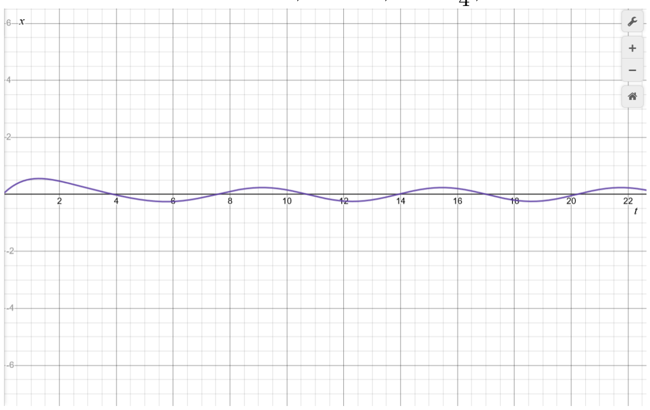
a'

- Suppose $(\frac{a}{m})^2 = \frac{4k}{m}$. Then our solution $x(t)$ becomes
$\frac{2me^{-\frac{at}{2m}}\left(-2me^{\frac{at}{2m}}\left(\left(a^2-4m^2\omega^2\right)\cos(t\omega)+4am\omega\sin(t\omega)\right)+a^2(at+2m)+4m^2\omega^2(at-2m)\right)}{\left(a^2+4m^2\omega^2\right)^2}$
- Suppose $(\frac{a}{m})^2 < \frac{4k}{m}$. Then our solution $x(t)$ becomes
$\frac{2m\left(2me^{-\frac{at}{2m}}\left(a\left(a^2+4m^2\omega^2+\rho^2\right)\sin\left(\frac{\rho t}{2m}\right)+\rho\left(a^2-4m^2\omega^2+\rho^2\right)\cos\left(\frac{\rho t}{2m}\right)\right)-2m\rho\left(\cos(t\omega)\left(a^2-4m^2\omega^2+\rho^2\right)+4am\omega\sin(t\omega)\right)\right)}{\rho(a^2+(\rho-2m\omega)^2)(a^2+(2m\omega+\rho)^2)}$

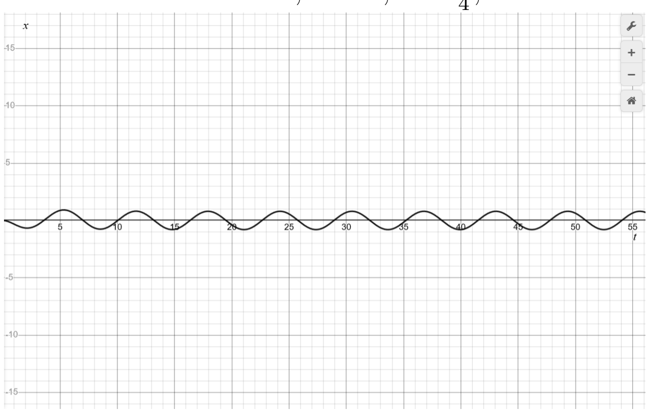
- Suppose $(\frac{a}{m})^2 > \frac{4k}{m}$. Then our solution $x(t)$ becomes

$$\frac{4m^2e^{-\frac{at}{2m}}\left(a\left(a^2+4m^2\omega^2-\rho^2\right)\sinh\left(\frac{\rho t}{2m}\right)+\rho\left(a^2-4m^2\omega^2-\rho^2\right)\cosh\left(\frac{\rho t}{2m}\right)+\rho e^{\frac{at}{2m}}\left(\cos(t\omega)\left(-a^2+4m^2\omega^2+\rho^2\right)-4am\omega\sin(t\omega)\right)\right)}{\rho(-a-2im\omega+\rho)(a-2im\omega+\rho)(-a+2im\omega+\rho)(a+2im\omega+\rho)}$$

b Solution when $m = 1, a = 1, k = \frac{5}{4}, \omega = 1$



c Solution when $m = 1, a = 1, k = \frac{1}{4}, \omega = 1$



d $m = 1, a = 1, k = \frac{1}{8}, \omega = 1$