2.4.1

2.4.6 (a) Suppose (a_n) is a bounded sequence. Prove that the sequence $y_n = \sup\{a_k : k \geq n\}$ converges.

Proof: We claim that (y_n) is a decreasing and bounded sequence.

- We claim that (y_n) is decreasing. Suppose $n \in \mathbb{N}$. We must show that $y_n \geq y_{n+1}$. We have two cases, $y_n = a_n, y_n \neq a_n$.
  - Suppose $a_n = y_n$. Therefore for all other elements in the sequence after a_n, $a_n \geq a_k$ where $k > n$. Therefore a_n is an upper bound on $\{a_k : k \geq n+1\}$. Since y_{n+1} is the supremum of the set mentioned before, and we have established that a_n is an upper bound, then by definition $y_n = a_n \geq y_{n+1}$.
  - Suppose $a_n \neq y_n$. Since a_n is already not a suprememum of the set $\{a_k : k \geq n\}$ then computing the supremum of the set excluding a_n, $\{a_k : k \geq n + 1\}$ should not change the supremum value. Therefore $y_n = y_{n+1}, y_n \geq y_{n+1}$.

  Therefore (y_n) is a decreasing sequence.
- We claim that (y_n) is bounded. Since (y_n) is decreasing we simply need to show that there is a quantity larger than y_1. Since $y_1 = \sup\{a_k : k \geq 1\}$ then y_1 is the supremum for (a_n). Since (a_n) is bounded, suppose for some quantity $M \in \mathbb{R}$, then by definition of supremum $y_1 \leq M$. Therefore (y_n) is bounded

Therefore by the monotone convergences theorem (y_n) converges.

(b) Let $z_n = \inf\{a_k : k \geq n\}$. Then $\lim z_n = \liminf a_n$. This should converge since (z_n) can easily be proved to be increasing and bounded.

(c)
- Prove that $\liminf a_n \leq \limsup a_n$
  Proof: Suppose $n \in \mathbb{N}$. For an arbitrary element $e \in \{a_k : k \geq n\}$, $e \leq y_n, e \geq z_n$ by the respective definitions of supremum and infimum. Therefore for all $n \in \mathbb{N}, z_n \leq y_n$. Since we know that $\liminf a_n, \limsup a_n$ exists, then by the algebraic order theorem $\liminf a_n \leq \limsup a_n$.
- An example of a strict inequality between $\liminf a_n$ and $\limsup a_n$ is the sequence $a_n = \frac{1}{n} + (-1)^{n+1}$. This is because $\liminf a_n = \lim \frac{1}{n} - 1 = -1 < 1 = \lim \frac{1}{n} + 1 = \limsup a_n$

(d) We must prove that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists

$\Rightarrow$

$\Leftarrow$ Suppose $\lim a_n$ exists and suppose for contradiction that $\liminf a_n \neq \limsup a_n$. Since $\liminf a_n \leq \limsup a_n$ then $\liminf a_n < \limsup a_n$. Therefore $0 < \limsup a_n - \liminf a_n$