2 Counterexample: Consider $f(x) = x$ from $\mathbb{R} \to \mathbb{R}$, however the domain is imbued with the topology $d(x, y) = \min |x - y|, 1$. (expand and solidfy proof) Note that this metric on $\mathbb{R}$ since

(a) $|x - y|, 1 \geq 0$ and $|x - y| = 0$ implies $x = y$
(b) if $|x - y| > 1$ then $|y - x| > 1$.
(c) $1 < 1 + 1 = 2$.

Which in conjuction with the standard l_1 distance on $\mathbb{R}$ being a metric implies d is a metric. Additionally, the topology generated by d is equivalent to the standard topology on $\mathbb{R}$ since for any (traditional) open set $U \subseteq \mathbb{R}$ and $x \in U$ there exists $\epsilon > 0$ such that $x \in B_\epsilon(x) \subseteq U$. If $\epsilon < 1$ then $B_\epsilon(x)$ is in our constructed topology. Otherwise if $\epsilon \geq 1$ then we can trivially find $\epsilon' < 1$ such that $B_{\epsilon'}(x) \subset B_\epsilon(x)$. Therefore any open ball in the standard topology on $\mathbb{R}$ can be constructed as a union of balls contained within our constructed topology. Therefore, our constructed topology works. Note that in our new topology for the domain that $\mathbb{R}$ is a bounded set, as for $x, y \in \mathbb{R}$, $d(x, y) \leq 1$. Therefore $\sup_{x,y\in\mathbb{R}} d(x, y) = 1$, thus making $\mathbb{R}$ have a diameter of 1. Additionally since $\mathbb{R}$ is complete then it satisfies the conditions. However $f(\mathbb{R}) = \mathbb{R}$, and in the image topology $\mathbb{R}$ is trivially unbounded. Therefore we have found a counterexample.

7 Let X be a bounded subset of ℓ_2, with bound D.

- Suppose $X \subset \ell_2$ is totally bounded, that $f \in X$, and $\epsilon > 0$. Then there exists $B_{\epsilon,d_{l_2}}(f_i)$ for $i \in [n]$ such that $X \subseteq \cup_{i=1}^n B_{\epsilon,d_{l_2}}(f_i)$. Therefore, there exists f_j such that $\sum_{i=1}^\infty |f_j(i) - f(i)|^2 < \epsilon^2$. Therefore
$$\sqrt{\sum_{n=N_\epsilon}^\infty |f(n)|^2} \leq \sqrt{\sum_{n=N_\epsilon}^\infty |f(n) - f_i(n)|^2} + \sqrt{\sum_{n=N_\epsilon}^\infty |f_i(n)|^2} < \epsilon + \epsilon$$
Therefore $\sum_{n=N_\epsilon}^\infty |f(n)|^2 \leq 4\epsilon^2$. Therefore we can choose $\epsilon/4 = \epsilon'$ and we're done.
- We want to show that X is totally bounded. Suppose for $\epsilon > 0$. Then there exists $N_\epsilon \in \mathbb{N}$ such that for all $f \in X$, $\sum_{n=N_\epsilon}^\infty |f(n)|^2 \leq \epsilon^2$. Note that $\|f\|_2 \leq \sqrt{\sum_{n=N_\epsilon}^\infty |f(n)|^2} + \sqrt{\sum_{n=1}^{N_\epsilon-1} |f(n)|^2} \leq D + \epsilon$. Note that if we take the closure of the ball B with radius $D + \epsilon$ in $\mathbb{C}^{N_\epsilon-1}$ and cover B with balls $x \in B, B_\epsilon(x) \subset \mathbb{C}^{N_\epsilon-1}$, then we can find an open subcover $B_\epsilon(x_i)$ where $i \in [n]$. Now for an arbitrary $y \in X$, for the first $N_\epsilon - 1$ coordinates, we can find a j such that $\{y_n\}_{n=1}^{N_\epsilon-1} \in B_\epsilon(x_j)$. Therefore $d_2(y, x_j) \leq d(x_j, \{y_n\}_{n=1}^{N_\epsilon-1}) + \sqrt{\sum_{n=N_\epsilon}^\infty |y_n|^2} < 2\epsilon$. Now note that we can find $N_{\epsilon/2}$ for $\epsilon/2$ and this completes the proof.

8 (a) Suppose for contradiction that $\{b_{n_k}\}$ is a sequence which converges pointwise for all $x \in [0, 1]$. Then one can construct the number $a \in [0, 1]$ with the binary representation being $a_n = \begin{cases} 0 & n = n_k, k \equiv 0 \mod 2 \\ 1 & n = n_k, k \equiv 1 \mod 2 \\ 0 & \text{otherwise} \end{cases}$. Then for the subsequence

of our subsequence $b_{n_{2k}}(a) = 0$ and $b_{n_{2k+1}}(a) = 1$. Since we have two subsequences which converge to different values, then our original subsequence of functions does not converge pointwise.

(b) Note that convergence in the product topology is convergence for each projection. However a projection is simply evaluating a function at a specific $x \in [0, 1]$. Since we can always construct a number which every subsequence of (b_n) fails on then necessarily no subsequence converges in the product topology.

9 Let (X, d) be a compact metric space, and define $d_s : X^{\mathbb{N}} \times X^{\mathbb{N}} \to [0, \infty)$ to be a function on the space of sequences of $X^{\mathbb{N}}$ by

$$(x_n), (y_n) \in X^{\mathbb{N}}, d_s((x_n), (y_n)) = \sum_{i=1}^{\infty} d(x_n, y_n)2^{-n}$$

(a) Show that d_s is a metric:

i. Since $d(x_n, y_n) \geq 0$ and $2^{-n} \geq 0$ then $d(x_n, y_n)2^{-n} \geq 0$ and $\sum_{i=1}^{\infty} d(x_n, y_n)2^{-n} \geq 0$. If $d_s((x_n), (y_n)) = 0$, then each $d(x_n, y_n) = 0$ since all terms are positive, therefore if all $d(x_n, y_n) = 0$ for all n then $x_n = y_n$ for all n, thus $(x_n)_{n\in\mathbb{N}} = (y_n)_{n\in\mathbb{N}}$.

ii. Note that since $d(x_n, y_n) = d(y_n, x_n)$ for all $n \in \mathbb{N}$ then

$$d_s((x_n), (y_n)) = \sum_{n=1}^{\infty} d(x_n, y_n)2^{-n} = \sum_{n=1}^{\infty} d(y_n, x_n)2^{-n} = d_s((y_n), (x_n))$$

iii. Note for $(x_n), (y_n), (z_n) \in X^{\mathbb{N}}$, $d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n)$ holds for all $n \in \mathbb{N}$. Additionally since $|d(x_n, y_n)| = d(x_n, y_n)$ then the series $\sum_{n=1}^{\infty}(d(x_n, y_n)+ d(y_n, z_n))2^{-n}$ converges absolutely if it doesn't diverge to ∞. Therefore if it doesn't diverge then via arbitrary rearrangement we have that $\sum_{i=1}^{n} d(x_n, z_n)2^{-n} \leq \sum_{i=1}^{n} d(x_n, y_n)2^{-n} + \sum_{i=1}^{n} d(y_n, z_n)2^{-n}$. If it does diverge then it goes to positive infinity. If $d_s((x_n), (y_n))$ is finite then the inequality holds. If $d_s((x_n), (y_n))$ is infinite then the inequality still holds. Therefore d_s obeys the triangle inequality

(b) Consider an open set $U \in X^{\mathbb{N}}$. Then U is a sequence of open sets in X of which finitely many of them aren't X. Therefore let $U_{k_1}, \cdots, U_{n_k}$ be the sets which aren't X in U. Then select $x_{k_i} \in U_{k_i}$, and find ϵ_{k_i} such that $x_{k_i} \in B_{\epsilon_{k_i}}(x_{k_i}) \subseteq U_{k_i}$. Then, one can find the smallest $\epsilon_{k_i}, \epsilon'$ from our set, and take the radius of our open ball in d_s to be of radius $\frac{\epsilon'}{2^{k_n+1}}$. Note for any x_i which aren't in the finite set of non-X open sets in the sequence U, then any radius works, as any other point chosen will be contained in X. If $x_i = x_{k_j}$, then the worst case scenario where the sequence y with $d(y_{k_j}, x_{k_j}) = \epsilon_{k_j} * 2^{k_j}$ and all other $y_i = x_i$ is avoided as:

$$\begin{aligned}\frac{d(y_{k_j}, x_{k_j})}{2^{k_J}} &< \frac{\epsilon'}{2^{k_n+1}} \\ d(y_{k_j}, x_{k_j}) &< \epsilon' 2^{k_j - k_n - 1} \\ &< \epsilon_{k_j}\end{aligned}$$

Note that $2^{k_j - k_n - 1}$ is at most 0.5 by construction. Therefore every element is guarenteed to be within the required open balls, thus our topology induced by d_s is atleast as strong as the product topology.

(c) Note that since (X, d) is compact then it is sequentially compact. Also let the sequence of sequences be denoted $(x_{ij})_{i\in\mathbb{N}, j\in\mathbb{N}}$, where x_i is a sequence in $X^{\mathbb{N}}$. Note that x_{0j} is a sequence in $X^{\mathbb{N}}$ as well. By the sequential compactness of X than x_{0j} has a convergent subsequence x_{0j_k}, which we shall say converges to l_1. Note that x_{1j_k} is a sequence in X, which has a convergent subsequence, say, $x_{1j_{k_l}}$. Note these subscripts are getting out of hand, therefore we will define $f_{m,n}$ to be n-th term in the subsequence starting at x_{0j_k} within x_{mj}. We will denote l_i by $\lim_{n\to\infty} f_{i,n}$. Now for $\epsilon > 0$, there exists some $r \in \mathbb{N}$ where by the diameter of X, D, is $\frac{D}{2^{2r}} < \frac{\epsilon}{2}$. Therefore we can choose the largest N such that for each $d(l_i, f_{i,N}) < \frac{\epsilon}{1-2^{1-2r}}$. Therefore

$$\sum_{i=1}^{\infty} \frac{d(f_{i,N}, l_i)}{2^i} = \sum_{i=1}^{2r-1} \frac{d(f_{i,N}, l_i)}{2^i} + \sum_{2r}^{\infty} \frac{d(f_{i,N}, l_i)}{2^i} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

(d) We will show that $X^{\mathbb{N}}$ is compact under the product topology. Let $\{U_\alpha\}_{\alpha\in I}$ be a cover where I is an index set, and all $U_\alpha \subset X^{\mathbb{N}}$ are open in the product topology. We know for each $x \in U_\alpha$, there exists $\epsilon > 0$ such that $B_{\epsilon, d_s}(x) \subset U$. Therefore we can write U_α as the union of d_s balls. Thus each U_α is open in the topology induced by d_s. Therefore since $(X^{\mathbb{N}}, d_s)$ is sequentially compact then $X^{\mathbb{N}}$ is compact with respect to d_s, and since our open cover is open with respect to d_s, there exists a subcover $U_{\alpha_1}, \cdots, U_{\alpha_n}$ such that $X \subseteq \cup_{i=1}^n U_{\alpha_i}$. Thus $X^{\mathbb{N}}$ is compact with respect to the product topology.

12 Let A, B be compact, non-empty, and disjoint in the topological space X, and for every $b \in B$, there exists $f_b : X \to [0, 1]$ continuous such that $f_b(b) = 1$, and vanishes on A. We want to show that there exists $U \subset A, V \subset B$ open such that $U \cap V = \emptyset$. Take the sets $V_b = \{f_b(x) > \frac{2}{3} : x \in X\}$ and $U_b = \{f(b) < \frac{1}{3} : x \in X\}$. Note that each U_b, V_b are open since f_b is open. Additionally by the compactness of B we can choose $V_{b_1}, \ldots, V_{b_n}$ such that $B \subseteq \cup_{i=1}^n V_{b_i}$. Let this subcover be denoted B. Let $U = \cap_{i=1}^n U_{b_i}$. Note that U is non-empty since each U_b is guarenteed to contain A as $0 < \frac{1}{3} < f_b(x), x \in U_b$ by construction. Therefore if $x \in U \cap V$, then there exists j such that $x \in V_{b_j}$. However, $U_{b_j} \cap V_{b_j}$ have an empty intersection since they're two different level sets of f_{b_j}, a continuous function. Thus $U \cap V = \emptyset$.