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6.4.7 Let $h(x) = \sum_{n=1}^{\infty} \frac{1}{n^2+x^2}$

(a) Show that h is a continuous function defined on all of $\mathbb{R}$
Note that $\frac{1}{n^2+x^2}$ is a continuous function with the denominator never reaching 0. Additionally, observing that $\frac{1}{n^2+x^2} \leq \frac{1}{n^2}$, and as shown previously that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, then by the Weierstrauss M-test, then $h(x)$ is converges uniformly, and additionally since each $f_n(x)$ is continuous then $h(x)$ is continuous (theorem 6.4.2).

(b) Is h differentiable? If so, is the derivative function h' continuous?
Note that $f_n'(x) = \frac{-2x}{(x^2+n^2)^2}$, and since $f_n'(x)$ has a maximum value of $\frac{3\sqrt{3}}{8n^3}$, and that $\frac{3\sqrt{3}}{8n^3} \leq \frac{3\sqrt{3}}{8n^2}$, then by the very same reasoning as above, since $\frac{3\sqrt{3}}{8} \sum_{n=1} \frac{1}{n^2}$ converges by the algebraic limit theorem for series, then by the Weierstrauss M-test the sum $\sum_{n=1}^{\infty} f_n'(x)$ is uniformly convergent. Thus $h'(x)$ exists. Since both $-2x$ and $(x^2+n^2)^2$ are continuous, and $(x^2+n^2)^2 \neq 0$, then their quotient is continuous, therefore $h'(x)$ is continuous (theorem 6.4.2).