Lemma: Suppose $f : (a,b) \to \mathbb{R}$ is differentiable. Then f is lipschitz if and only if $f'(x) < \infty$ for all $x \in (a,b)$

- Suppose f is lipschitz, differentiable. Since f is lipschitz, then there exists $c \in \mathbb{R}$ such that $|f(x) - f(y)| \leq L|x-y|$ for all $x, y \in (a,b)$. Then if we consider $x \neq y$ then we have a bound on the quotient function for f:

$$\left|\frac{f(x)-f(y)}{x-y}\right| \leq L$$

Therefore for a point $p \in (a,b)$ we can choose $x = p+h, y = p$, and take the limit as $h \to 0$.

$$\lim_{h\to 0}\left|\frac{f(p+h)-f(p)}{h}\right| \leq L.$$

Note that for every h, the quotient function satisfies the inequality, thus in the limit by the limit order theorem we have that $|f'(p)| \leq L$. Thus f' is bounded on (a,b).

- Suppose f' is bounded. Then for all $x \in (a,b), |f'(x)| \leq M$. We will show that f is Lipschitz. Suppose $l, r \in (a,b)$. Then we know by the mean value theorem that there exists $q \in (l,r)$ such that $\frac{f(r)-f(l)}{l-r} = f'(q)$. Thus $\left|\frac{f(r)-f(l)}{r-l}\right| = |f'(q)| \leq M$. Thus $|f(r) - f(l)| \leq M|l-r|$, giving us that f has lipschitz constant M.

1. (a)

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{1}{(3n-2)(3n+1)} &= \frac{1}{3}\left(\sum_{n=0}^{\infty}\frac{1}{3n-2} - \sum_{n=0}^{\infty}\frac{1}{3n+1}\right)\\
&= \lim_{s\to\infty}\frac{1}{3}\left(\sum_{n=0}^{s}\frac{1}{3n-2} - \sum_{n=0}^{s}\frac{1}{3n+1}\right)\\
&= \lim_{s\to\infty}\frac{1}{3}\left(\sum_{n=1}^{s}\frac{1}{3n-2} - \sum_{n=0}^{s-1}\frac{1}{3n+1} - \frac{1}{2} - \frac{1}{3s+1}\right)\\
&= \lim_{s\to\infty}\frac{1}{3}\left(\sum_{m=0}^{s-1}\frac{1}{3m+1} - \sum_{n=0}^{s-1}\frac{1}{3n+1} - \frac{1}{2} - \frac{1}{3s+1}\right)\\
&= \lim_{s\to\infty}\frac{-1}{3}\left(\frac{1}{2} + \frac{1}{3s+1}\right)\\
&= \frac{-1}{6}
\end{aligned}$$

Thus $\sum_{n=1}^{\infty}\frac{1}{(3n-2)(3n+1)} = \sum_{n=0}^{\infty}\frac{1}{(3n-2)(3n+1)} - \left(\frac{-1}{2}\right) = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

(b)

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} &= \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{(n+1)^2} \\
&= \lim_{s\to\infty} \sum_{n=1}^{s} \frac{1}{n^2} - \frac{1}{(n+1)^2} \\
&= \lim_{s\to\infty} \sum_{n=1}^{s} \frac{1}{n^2} - \sum_{n=1}^{s} \frac{1}{(n+1)^2} \\
&= \lim_{s\to\infty} 1 + \sum_{n=2}^{s} \frac{1}{n^2} - \frac{1}{(s+1)^2} - \sum_{n=1}^{s-1} \frac{1}{(n+1)^2} \\
&= \lim_{s\to\infty} 1 - \frac{1}{(s+1)^2} + \sum_{n=2}^{s} \frac{1}{n^2} - \sum_{m=2}^{s} \frac{1}{(m)^2} \\
&= \lim_{s\to\infty} 1 - \frac{1}{(s+1)^2} \\
&= 1
\end{aligned}$$

2. (a) Clearly $\lim \frac{\sqrt{n}}{2\sqrt{n}-1} = \frac{1}{2}$. Thus the series does not converge

(b) Note that for $n \geq 3, n^3 - 5n + 1 \leq n^3$. Thus $\frac{n^2}{n^3-5n+1} \geq \frac{1}{n}$. Since $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges then the sum diverges.

(c) Note that $\sqrt{n^2+1} - n = \frac{n^2+1-n^2}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2+1}+n}$. Additionally $(\sqrt{2}+1)n = \sqrt{n^2+n^2} + n \geq \sqrt{n^2+1} + n$. Therefore $\frac{1}{(\sqrt{2}+1)n} \leq \frac{1}{\sqrt{n^2+1}+n}$. Thus the sum diverges since $\sum \frac{1}{(\sqrt{2}+1)n}$ diverges.

(d) Note that by the ratio test $\lim \frac{2022 \cdot 2022^n}{(n+1)!} \frac{n!}{2022^n} = \lim \frac{2022}{n+1} = 0$, which being less than 1 ensures that the sum will converge.

(e) Note that $\sqrt{n} + n \leq n + n$. Thus $\frac{1}{2n} \leq \frac{1}{\sqrt{n}+n}$. Since $\frac{1}{2}\sum \frac{1}{n}$ is the harmonic series then it diverges. Thus the sum diverges.

(f) We know by the cauchy condensation test that $\sum \frac{1}{n\sqrt{n}}$ converges if $\sum 2^n \frac{1}{2^n 2^{n/2}}$ converges. After algebraic manipulation we have the sum $\sum \frac{1}{\sqrt{2}^n}$. Since $\sqrt{2} > 1$ then $\frac{1}{\sqrt{2}} < 1$. Thus that geometric series converges. Thus $\frac{1}{n\sqrt{n}}$ converges.

(g) Note that $\sqrt{n^6+n} - n^3 = \frac{n}{\sqrt{n^6+n}+n^3} = \frac{1}{\sqrt{n^4+\frac{1}{n}}+n^2}$. Note that since $n^2 \leq \sqrt{n^4+\frac{1}{n}} + n^2$ then $\frac{1}{\sqrt{n^4+\frac{1}{n}}+n^2} \leq \frac{1}{n^2}$. Thus since $\sum \frac{1}{n^2}$ converges then our desired sum converges.

(h) By the root test we have that $\lim \frac{2}{3}(\sqrt[n]{n})^2$. Note that we proved in the slides that $\lim \sqrt[n]{n} = 1$. Thus the ratio test has that the sum converges.

(i) Note that for $x \notin \mathbb{Z}$ we have for all $n \in \mathbb{N}, x + n \neq 0$. Thus the function $\frac{1}{x+n}$ is well defined for all $n \in \mathbb{N}$. Note that since $\lim \frac{1}{x+n} = 0$ then we have by the alternating series test that our desired sum converges

(j) By root test we have the limit $\lim \frac{(n!)^{\frac{2}{n}}}{2^n}$. Note that since we have a ratio of positive numbers that our limit is bounded below by 0. Additionally since $\sqrt[n]{n!} \leq n$, then our limit is bounded above by $\lim \frac{n^2}{2^n}$. However by the leacture slides we know that $\lim \frac{n^2}{2^n} = 0$ since exponentials are faster than any polynomial. Thus our limit is equal to 0. Thus our series converges

3. (a) Note that since $[1,2]$ and $[4,7]$ are both closed, and the finite union of closed sets is closed implies $[1,2] \cup [4,7]$ is closed. $cl(A) = [1,2] \cup [4,7], int(A) = (1,2) \cup (4,7)$
   (b) Note that $A = [0,1) \cup (4,5)$ is neither closed nor open since A contains a limit approaching 4 without having 4 and for any $r > 0$, $B(0,r) \not\subseteq [0,1)$.
   $cl(A) = [0,1] \cup [4,7], int(A) = (0,1) \cup (4,7)$.
   (c) $A = \mathbb{Q}$ is neither closed nor open since the set $\{q^2 < 2 : q \in \mathbb{Q}\} \subset \mathbb{Q}$, however $\sqrt{2} \notin \mathbb{Q}$. Additionally for any $q \in \mathbb{Q}, r > 0$ we know that $B(q,r) \not\subset \mathbb{Q}$ since the set of irrationals is dense in $\mathbb{R}$. Furthermore, $int(\mathbb{Q}) = \emptyset, cl(\mathbb{Q}) = \mathbb{R}$
   (d) $A = \mathbb{Q} \cap [0,1]$ is nether closed nor open. Note that A is not closed since $\{q \in \mathbb{Q} : 0 < q, q^2 < \frac{1}{2}\} \subset A$, however $\frac{1}{\sqrt{2}} \notin \mathbb{Q} \cap [0,1]$. Additionally, for every $r > 0, B(0,r) \not\subset A$, thus A is not open. $cl(A) = [0,1], int(A) = \emptyset$
   (e) Let $A = [0,3] \backslash \{\frac{2n+1}{3n} : n \in \mathbb{N}\}$. A is not open since for all $r > 0, B(3,r) \not\subseteq A$. A is not closed since we can find a sequence which converges to 1, however $1 \notin A$. Thus A does not contain one of it's limit points. $cl(A) = [0,3], int(A) = (0,3) \backslash \{\frac{2n+1}{3n} : n \in \mathbb{N}\}$.

4. (a) $B = [1,2)$ is not compact because it is not closed
   (b) $B = \{\frac{n+1}{n} : n \in \mathbb{N}\}$ is not compact because it is not closed because it does not contain it's limit point.
   (c) $B = \{\frac{n+1}{n} : n \in \mathbb{N}\} \cup \{1\}$ is compact since it is bounded by $M = 2$, and it contains the one limit point 1.
   (d) $B = \{\sqrt{n+1} - \sqrt{n} : n \in \mathbb{N}\} \cup$ is compact since the set is bounded by $M = \sqrt{2}$ and $\lim \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$ thus B is closed since it contains it's limit points.
   (e) $B = \bigcup_{n \in \mathbb{N}} [2n, 2n+1]$ is not compact since for any possible bound M, there exists $N \in \mathbb{N}$ such that $M < 2N$, and thus not contain the interval $[2N, 2N+1]$.

5. (a) $\partial([1,2) \cup (3,5]) = \{1,2,3,5\}$
   (b) $\partial \mathbb{Z} = \mathbb{Z}$
   (c) $\partial(\{1\} \cup (2,3)) = \{1,2,3\}$
   (d) $\partial \mathbb{Q} = cl(\mathbb{Q}) \backslash int(\mathbb{Q}) = \mathbb{R} \backslash \emptyset = \mathbb{R}$
   (e) Let C be the cantor set. We know that the triadic numbers are dense in the unit interval. Therefore for an arbitrary open interval in $[0,1]$ we can place an infinite number of triatic numbers within it. By the construction of the cantor set between any two triadyic number eventually an interval is removed. Thus for an arbitrary point $\alpha \in C$, there does not exists $\epsilon > 0$ such that $B(\alpha, \epsilon) \subset C$. Thus $int(C) = \emptyset$. Since C is closed then $cl(C) = C$. Thus $\partial C = C$.

6. (a) $[1,2] \cup [3,4]$ is compact since it is closed (finite union of closed sets is closed) and bounded (take $M = 4$), however 2.5 is not in the set.
   (b) $(1,2)$ is connected by definition, and is not closed since it is open and not $\mathbb{R}$ or the $\emptyset$.
   (c) $[1,2]$ is connected, and $\partial[1,2] = \{1,2\}$.
   (d) $[1,1]$ is compact since it is trivially closed and bounded, and since no open interval exists in the singleton set then $[1,1]$ is nowhere dense.
   (e) $\mathbb{Z}$ is nowhere dense since it's closure is $\mathbb{Z}$ and $\mathbb{Z}$ lacks any open intervals contained within. Similarly $\mathbb{Z}$ is not compact because $\mathbb{Z}$ is unbounded.
7. (a) $E = \bigcup_{n\in\mathbb{N}}[3n+1, 3n+2]$. Note that the first two intervals of the set are $[4,5]$ and $[7,8]$, and since the the left endpoints of each successive union'd interval is increasing then there are no intervals to be added between two "adjacent" intervals. Thus 6 is between 5 and 7, yet $6 \notin E$. E is not connected.
   (b) $E = \{1\} \cup [2,4]$. Note that $1.5 \notin E$, yet $1 < 1.5 < 2$. Thus E is not connected.
8. We know from class that $\sqrt{x}$, $\sqrt[3]{x}$ are continuous. Additionally both $\sqrt{x}$, $\sqrt[3]{x}$ are positive on $(0,1)$, thus their sum is never zero on $(0,1)$. Therefore the fraction $f(x) = \frac{1}{\sqrt[3]{x}+\sqrt{x}}$ is continuous. Suppose for contradiction that f is uniformly continuous on $(0,1)$. If we fix an $\epsilon > 0$, then there exists $\delta > 0$ such that for all $x, y \in (0,1)$ if $|x-y| < \delta$ then $|f(x) - f(y)| < \epsilon$. Note that if we take δ and $0 < x < \delta$, then we satisfy $|x - \delta| < \delta$. Thus $|f(x) - f(\delta)| < \epsilon$. Note that if we take $\lim_{x\to 0} f(x) = \infty$. Since $f(\delta)$ is finite then $|f(x)|$ exceeds ϵ at some point, otherwise contradicting the unbounded nature of f at 0. Therefore f is not uniformly continuous.
9. Suppose f is lipschitz on (a,b) with lipschitz constant L, $\epsilon > 0$, $x, y \in (a,b)$,
   If $|x-y| < \frac{\epsilon}{L}$, we have that

   $$|f(x) - f(y)| \leq L|x-y| < L\frac{\epsilon}{L} = \epsilon.$$

   Thus f is uniformly continuous.
10. We know that $f(x) = \frac{1}{x}$ is differentiable on $(0,1)$, however, if we consider that $f'(x) = \frac{-1}{x^2}$, then $\lim_{x\to 0} f'(x) = -\infty$. We know by the lemma that if f has an unbounded derivative then it is not lipschitz.