Alex Valentino Midterm 2 problem 2
300H

Suppose R is a relation on a set A that is transitive, symmetric, and reflexive. Prove that the set $\{R(x) : x \in A\}$ is a partition of A.

Proof: We must show that $\{R(x) : x \in A\}$ is a partition of A. Let $\Pi = \{R(x) : x \in A\}$. We must show that Π is a partition of A. Since the partition of $\emptyset$ behaves differently than any other set, then we have two cases:

- Assume $A = \emptyset$. Then the set $\{R(x) : x \in A\}$ will be empty because $x \in A$ is always false as $A = \emptyset$. Since there exists no sets satisfying condition to be an element of Π, then $\Pi = \emptyset$. Since the definition of partition requires that it be a collection of non-empty subsets of A, and there exists no set which is a subset of $\emptyset$, then the only partition of the empty set is $\emptyset$. Since $\Pi = \emptyset$, then Π is a partition of $\emptyset$.

- Assume $A \neq \emptyset$. By definition of partition we must show, (1) every $P \in \Pi$ is not empty, (2) for each $a \in A$, there exists a $P \in \Pi$ such that $a \in P$, and (3) for all $P_1, P_2 \in \Pi$ if $P_1 \neq P_2$ then $P_1 \cap P_2 = \emptyset$.

  1. We must show that every $P \in \Pi$ is non-empty. Suppose $P \in \Pi$. Since $P \in \Pi$ is defined by an element in $a \in A$ such that $P = R(a)$, and R is reflexive, then $a \in P$. Since $a \in P$, then P is non-empty.
  2. We must show that for all $a \in A$, there exists a $P \in \Pi$ such that $a \in P$. Suppose $a \in A$. We must show there exists a $P \in \Pi$ such that $a \in P$. Since $R(a) \in \Pi$, and $a \in R(a)$ by the reflexivity of R, then $P = R(a)$ satisfies the requirements.
  3. We must show for all $P_1, P_2 \in \Pi$ if $P_1 \neq P_2$ then $P_1 \cap P_2 = \emptyset$. We are going to prove this by contrapositive. Suppose $P_1 \cap P_2 \neq \emptyset$. We must show $P_1 = P_2$. Since $P_1 \cap P_2 \neq \emptyset$, then $P_1 \cap P_2 = S$, where there exists $x \in S$. Since $P_1, P_2 \in \Pi$, then there exists $a_1, a_2 \in A$ such that $P_1 = R(a_1), P_2 = R(a_2)$. Since $P_1 = R(a_1), P_2 = R(a_2)$, and $x \in P_1, P_2$, then by the definition of $R(a)$, $a_1 R x, a_2 R x$. By the symmetry of R, we have that $x R a_2$. By the transitivity of R we have that $a_1 R a_2$. Therefore by the transitivity of R $R(a_1) = R(a_2)$ as all elements that are related to a_1 are related to a_2 and all elements that are related to a_2 are related to a_1. Thus $P_1 = P_2$.

  Since Π satisfies the requirements, then $\{R(x) : x \in A\}$ is a Partition.