1.5.4 Let S be the set of all infinite binary strings. Suppose for contradiction there exists a bijection $f : \mathbb{N} \to S$. Let the binary string b be defined by $b_n = \begin{cases} f(n)_n + 1 & f(n)_n = 0 \\ f(n)_n - 1 & f(n)_n = 1 \end{cases}$ where $f(n)_n$ is the nth bit in the binary string given by $f(n)$. We claim that for all $n \in \mathbb{N}$ $f(n) \neq b$. For $n = 1$ we have that $f(1)_1 \neq b_1$, thus $f(1) \neq b$. By the principle of mathematical induction for all $k \in \mathbb{N}$ if $k < n$ then $b_k \neq f(k)_k$. Looking at $f(n)_n$, the bit at postition n is exact the inverted bit at b_n. Therefore $f(n) \neq b$. Therefore by induction, for all $m \in \mathbb{N}, f(m) \neq b$. This is a contradiction as f is a bijection from $f : \mathbb{N} \to S$. Therefore f is uncountable.

1.5.5 (a) Let $A = \{a, b, c\}$. Then $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$

(b) We must show that if A is a set and $|A| = n$ then $|\mathcal{P}(A)| = 2^n$. We will show this by induction. Assume WLOG that $A = \{1, 2, \ldots, n\}$. For $n = 1$ we have $A = \{1\}$, therefore $\mathcal{P}(A) = \{\emptyset, \{1\}\}$. There are two elements in $\mathcal{P}(A)$, thus $|\mathcal{P}(A)| = 2 = 2^1$. By PMI for all $k \in \mathbb{N}$ if $k < n$ then for a set B of size k we have that $|\mathcal{P}(B)| = 2^k$. Let $A' = A\backslash\{n\}$. Since $|A'| = n - 1 < n$ then by the induction hypothesis $|\mathcal{P}(A')| = 2^{n-1}$. We claim that $|\mathcal{P}(A)\backslash\mathcal{P}(A')| = 2^{n-1}$. Note that by definition if a subset $S \subseteq A$ is not in $\mathcal{P}(A')$ implies that $n \in S$. If we consider the set $S\backslash\{n\}$ since S is already a subset of A then $S\backslash\{n\} \subseteq A'$. Since all the sets in $\mathcal{P}(A)\backslash\mathcal{P}(A')$ are simply modified copies of sets from $\mathcal{P}(A')$ and $|\mathcal{P}(A')| = 2^{n-1}$ then $|\mathcal{P}(A)\backslash\mathcal{P}(A')| = 2^{n-1}$. Therefore

$$|\mathcal{P}(A)| = |\mathcal{P}(A') \cup \mathcal{P}(A)\backslash\mathcal{P}(A')| = |\mathcal{P}(A')| + |\mathcal{P}(A)\backslash\mathcal{P}(A')| = 2^{n-1} + 2^{n-1} = 2^n.$$

1.5.6 (a)

- Let $f : A \to \mathcal{P}(A)$ be given by:

$$f(a) = \{a\}, f(b) = \{b\}, f(c) = \{c\}.$$

- Let $h : A \to \mathcal{P}(A)$ be given by:

$$h(a) = \{a, b\}, h(b) = \{b, c\}, h(c) = \{a, c\}.$$

(b) Let $B = \{1, 2, 3, 4\}$. Let $g : B \to \mathcal{P}(B)$ be given by:

$$g(1) = \{1\}, g(2) = \{2\}, g(3) = \{3\}, g(4) = \{4\}.$$

(c) It is impossible to have an onto mapping for the previous two parts because for any arbitrary set A with size n, any arbitrary function between the two is mapping from n elements to 2^n elements. By definition a function must have a single output for every input