Alex Valentino | Midterm 2 problem 6 | 300H

Suppose that $(s_n : n \geq 1)$ and $(t_n : n \geq 1)$ are sequences satisfying $t_1 = 1, s_1 = 2, t_2 = 5$ and $s_2 = 6$ and for all $n \geq 3$:

$$s_n = 6s_{n-1} + 2s_{n-2}$$
$$t_n = 5t_{n-1} + 6s_{n-2}$$

Prove that for all $n \geq 2, t_n \geq 4t_{n-1}$.
Proof: We have two cases.

- Assume $n = 2$. Then we have $t_2 \geq 4t_1$. The inequality is equivalent to $5 \geq 4$.
- Assume $n > 2$. Therefore by algebraic manipulation we have

$$\begin{aligned} 4t_{n-1} &\geq 5t_{n-1} \\ &\geq 5t_{n-1} + 6t_{n-2} \\ &= t_n. \end{aligned}$$

Prove that for all $n \geq 1, t_n \leq s_n$. We have three cases.

- Assume $n = 1$. Then we have $t_1 \leq s_1$, which is equivalent to $1 \leq 2$.
- Assume $n = 2$. Then we have $t_2 \leq s_2$, which is equivalent to $5 \leq 6$.
- Assume $n > 2$. By the principal of mathematical induction for all $k \in \mathbb{N}$ if $k < n$ then we have $s_k \leq n_k$. Since $n-1, n-2 < n$, then by the induction hypothesis we have $t_{n-2} \leq s_{n-2}, t_{n-1} \leq s_{n-1}$. Note that by the first claim proved we have for all $n > 2, t_{n-1} \geq 4t_{n-2}$ Therefore by algebraic manipulation we have

$$\begin{aligned} t_n &= 5t_{n-1} + 6t_{n-2} \\ &= 5t_{n-1} + 4t_{n-2} + 2t_{n-2} \\ &\leq 6t_{n-1} + 2t_{n-2} \\ &\leq 6s_{n-1} + 2s_{n-2} \\ &= s_n. \end{aligned}$$

  Therefore $t_n \leq s_n$.