1 Let (x, d) be a seperable metric space, and $Y \subseteq X$ endowed with the metric d_Y. Let S be the seperable subset of X. Then we can define the topology of X, $\mathcal{O}_X$ as the arbitrary union of sets in $\mathcal{B} = \{B_r(x) : r \in \mathbb{Q}_{>0}, x \in S\}$. Note this definition holds because irrational from rational ball radiuses one can put together irrational ball radiuses, and then from there construct arbitrary open sets. Now to define a countable dense subset of Y, one just has to construct the set $\{Y \cap B : B \in \mathcal{B}, Y \cap B \neq \emptyset\}$. Note that this serves as an effective basis for the subset topology of Y, therefore for each B_Y in our basis if we pick one element, then we have found a countable dense subset for Y, thus making Y seperable.

3 Let (X, d) be a compact metric space

(a) Suppose not, then $f : X \to X$ is continuous and not onto, and has the property that for all possible $x_0 \in X$ and $r > 0$ there exists $x \in X$ such that $d(f(x), x_0) < r$. Since f is not onto then there exists a set $E \subset X$ such that for all $x \in X, f(x) \notin E$. Choose $e \in E$, then for all $n \in \mathbb{N}$, we can define x_n such that $d(f(x_n), x_0) < \frac{1}{n}$. Clearly $\lim f(x_n) = x_0$, and since X is a compact metric space then there should exists $x' \in X$ such that $\lim x_n = x'$ and $f(x') = x_0$. However this contradicts f being not onto, demonstrating the desired result.

(b) Let $f : X \to X$ be an isometry. We want to show that it is bijective, and thus need to show that is is injective and surjective.

- To show that f is injective, suppose for $x, y \in X$ that $f(x) = f(y)$. Then $d(f(x), f(y)) = 0$. Since f is an isometry then $d(x, y) = 0$. Note this is only true if $x = y$. Therefore f is injective.
- It is important to note that by the definition given above f must be continuous, as for all $\epsilon > 0$, if $x, y \in X, d(x, y) < \epsilon$ then $d(f(x), f(y)) < \epsilon$. If we then assume for contradiction that f is not onto then the theorem proved above holds and there exists $x_0 \in X, r > 0$ such that for all $x \in X, d(f(x), x_0) \geq r$. Let the sequence x_n be given by $x_{n+1} = f(x_n)$. Note that since compactness implies sequential compactness in a metric space then there exists x_{n_k} such that $\lim x_{n_k} = l$. Therefore there exists $p, q \in \mathbb{N}, q < p$ such that $d(x_{n_p}, L), d(x_{n_q}, L) < \frac{r}{2}$. Thus $d(x_0, x_{p-q}) = d(x_q, x_p) \leq d(x_{n_p}, L) + d(x_{n_q}, L) < r$. This contradicts the above statement since $x_{p-q} \in Im(f)$.

4 Let X be a compact metric space with metric d and $f : X \to \mathbb{C}$ be continuous. Let $\epsilon > 0$ be given. We know by the Heine-Cantor theorem that there exists $1 > \delta > 0$ such that if $x, y \in X, d(x, y) < \delta, |f(x) - f(y)| < \epsilon/2$. Therefore for each point $x \in X$ we can cover X with $\bigcup_{x \in X} B_\delta(x)$ and since X is compact we can choose $x_1, \cdots, x_n$ such that $X \subseteq \bigcup_{i=1}^n B_\delta(x_i)$. Furthermore we can arrange our found points $x_1, \cdots, x_n$ by which points are closest, where we can define to be $z_1 = x_1, z_i = \min_{x \in \{x_1, \cdots, x_n\} \setminus \{z_1, \cdots, z_{i-1}\}} d(x, z_{i-1})$. Furthermore, $d(z_i, z_{i-1}) < \delta$ since the balls are guarenteed to overlap to enclose X. Thus given an $x, y \in X$, we can choose the z_i, z_j balls to which x and y belong respectively, then define $|f(x) - f(y)| \leq |f(x) - f(z_i)| + \sum_{l=i+1}^{j} |f(z_l) - f(z_{l-1})| + |f(z_j) - f(y)| < \epsilon|i - j + 1| < \frac{n\epsilon}{\delta} d(x, y) + \epsilon$. Note that the path taken through the balls

might not be optimal, so our bets can be hedged via assuming the worse case scenario of having to pass through all the balls when the ideal path travels through one.

5 (a) Let (X, d) be a complete metric space where bounded sets are totally bounded and $A \subset X$ to be bounded and $B \subset X$ to be compact. Note that if one takes $A \cap B \neq \emptyset$ then the minimum is given by $x_1 \in A \cap B$ with $d(x_1, x_1) = 0 \leq d(x, y)$ for all $x \in A, y \in B$. Therefore one can assume that $A \cap B = \emptyset$. Additionally since X is a complete metric space then B is bounded and since B is bounded then B is totally bounded. Therefore B has sequential compactness. We want to show that there exists $x_1 \in A, x_2 \in B$ such that $d(x_1, x_2) \leq d(x, y)$ for all $x \in A, y \in B$. Assume not. Then there exists $(a_n) \subset A$ and $(b_n) \subset B$ such that $\lim d(a_n, b_n) = 0$. However, this implies that (b_n) has a convergent subsequence, (b_{n_k}) with $\lim b_{n_k} = b$, and since n_k can be chosen such that $d(a_{n_k}, b_{n_k}) < \frac{\epsilon}{2}$ and $d(b_{n_k}, b) < \frac{\epsilon}{2}$ then $d(a_{n_k}, b) \leq d(a_{n_k}, b_{n_k}) + d(b_{n_k}, b) < \epsilon$. This implies that A has a limit point contained within B, and since A is closed then that limit point is contained within A. Thus $A \cap B \neq \emptyset$ which is a contradiction.

(b) The sequences $(2^k + \frac{1}{k})_{k \in \mathbb{N}}$ and $(2^k)_{k \in \mathbb{N}}$ are examples, as the terms of the sequence will get arbitrarily close, but both diverge.