2.2.1 (a) Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ with $N > \sqrt{\frac{\frac{1}{\epsilon}-1}{6}}$. To verify that our choice of N is appropriate, let $n \in \mathbb{N}$ satisfy $n > N$. Therefore

$$\begin{aligned} n &> \sqrt{\frac{\frac{1}{\epsilon}-1}{6}} \\ n^2 &> \frac{\frac{1}{\epsilon}-1}{6} \\ 6n^2+1 &> \frac{1}{\epsilon} \\ \frac{1}{6n^2+1} &< \epsilon. \end{aligned}$$

Since $n^2 > 0$ for all $n \in \mathbb{N}$ then we have just proved $|\frac{1}{6n^2+1}| < \epsilon$.

2.2.7 (a) The convergence to infinity definition is thus: For all $\epsilon > 0$ there exists $B]in\mathbb{N}$ such that $n > N$ implies $|x_n| > \epsilon$.
Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > \epsilon^2$. To verify that our choice of N is correct, let $n \in \mathbb{N}$ satisfy $n > N$. Therefore $n > \epsilon^2 \Rightarrow \sqrt{n} > \epsilon$. Since $\sqrt{n} > 0$ for all $n \in \mathbb{N}$ then we have shown that $\lim_{n\to\infty} |\sqrt{n}| = \infty$

(b) The sequence $(1, 0, 2, 0, 3, 0, 4, \ldots)$ does not converge to infinity as choosing $\epsilon = 0.5$ does not satisfy the condition. As for all $n \in \mathbb{N}, x_{2n} = 0$ thus no matter how large N is choosen to be $x_{2n} < 0.5$.

2.2.8 (a) The sequence $(-1)^n$ is frequently in the set $\{1\}$ as given an arbitrary $N \in \mathbb{N}$ if N is even then the choice $n = N$ yields $(-1)^{2N} = 1$ and if N is odd where $N = 2l-1$ then $n = N+1$ yields $(-1)^n = (-1)^{2l+2} = ((-1)^2)^{l+1} = 1^{l+1} = 1$.
We claim that the sequence is not eventually in the set $\{1\}$ Supposes $N \in \mathbb{N}$. If N is odd then there exists $l \in \mathbb{Z}_{\leq 0}$ such that $N = 2l+1$. Since $N \leq N$ then $(-1)^N = (-1)^{2l+1} = -1 \notin \{1\}$. If N is even then we take $N+1 > N$, therefore $(-1)^{N+1} = (-1)^{2l+1} = -1 \notin \{1\}$. Since every choice of N results in the sequence not being 1 after a certain point then it is not eventually in $\{1\}$.

(b) Eventually is a stronger definition then frequently. As shown above we can have sequences frequently dance in and out of sets, but they are not guaranteed to stay inside after a certain point. We claim that eventually implies frequently and that the converse is not true:

- Suppose (a_n) is eventually in the set A. Therefore there exists $N_e \in \mathbb{N}$ such that for all $n \geq N_e, a_n \in A$. We must show that (a_n) is frequently in A. Suppose $N \in \mathbb{N}$. We must show there exists $n \geq N$ such that $a_n \in A$. We claim that all $n \geq N_e$ satisfy the definition. If $N \leq N_e$ then the definition is satisfied by $n = N_e$. If $N > N_e$ then any natural number k such that $k > N$ would suffice as $k > N > N_e$, therefore $a_k \in A$. Therefore (a_n) is frequently in A.
- Frequently does not imply eventually as part (a) of this problem serves as a counterexample.

(c) The sequence (a_n) converges to the real number r if for all $\epsilon > 0$ (a_n) is eventually in $V_\epsilon(r)$. We do not use the frequently definition here as we need the sequence to stay inside of $V_\epsilon(r)$ for all members of a_n for $n > N$ has been shown to not be satisfied by the frequently defintion.

(d) (x_n) is not necessarily eventually in the set $(1.9, 2.1)$. If (x_n) has an infinite number of terms that are not in the set $(1.9, 2.1)$, even if they are exceedingly rare, then for every $N \in \mathbb{N}$ there will always exists $n \geq N$ for which $x_n \notin (1.9, 2.1)$.
We claim that (x_n) is frequently in the set $(1.9, 2.1)$. Let $A \subseteq \mathbb{N}$ be the set containing all the indicies where $x_n = 2$. Note that since there is an infinite number of terms where $x_n = 2$ then A is a countable subset of $\mathbb{N}$. Suppose $N \in \mathbb{N}$. If $N \in A$ then $N \leq N$, $x_N = 2 \in (1.9, 2.1)$. If $N \notin A$ then since A is infinite there exists $a \in A$ such that $N < a$. By definition of being a member of A then $x_a = 2 \in (1.9, 2.1)$. Therefore (x_n) is frequently in $(1.9, 2.1)$.