1. Let the rows $A \in M_{n\times n}(F)$ be given by $a_1, a_2, \ldots, a_n$, and let B be the matrix where the rows are $a_n, a_{n-1}, \ldots, a_1$. Calculate $\det(B)$ in terms of $\det(A)$.
   Suppose $k \in \mathbb{N}$ if $k < j$, then the matrix $n \times n$ C having k row operations applied to the matrix A will have the determinate $\det(C) = (-1)^k \det(A)$. The base case is covered by part a of the corollary to theorem 4.6. For the induction step, let C be the matrix given by k row swap operations on A, and C' is the intermediary matrix after the first $k-1$ row operations. Therefore $\det(C') = (-1)^{k-1}\det(A)$. Since C' is lacking a row operation to all compared to C then $\det(C) = -\det(C')$. Therefore $\det(C) = -\det(C') = (-1)^k \det(A)$. Therefore to compute $\det(B)$ in terms of $\det(A)$ we need to know how many row swaps are between A and B. If n is even then only $n/2$ row swaps are required as we can get two interchanged rows $i, n-i+1$ for simply swapping i. If n is odd then $(n-1)/2$ swaps are required as the middle row is invariant under row reversal. The number of row swaps no matter the parity of n are equal to $\lfloor \frac{n}{2} \rfloor$. Therefore $\det(B) = (-1)^{\lfloor \frac{n}{2} \rfloor} \det(A)$

2. Let $M \in M_{n\times n}(F)$ be nilpotent where $M^k = 0_{n\times n}$. We must show that $\det(M) = 0$. Since $M^k = 0_{n\times n}$, then taking the determinant we get that $0 = \det(0_{n\times n}) = \det(M^k) = \det(M \cdots M) = \det(M)\cdots\det(M) = (\det(M))^k$. Therefore taking the kth root of both sides yields $0 = \det(M)$.

3. Suppose $\{u_1, \dots, u_n\} \subset F^n$ is a set of n distinct vectors, and let the matrix $U \in M_{n\times n}(F)$ be the matrix whose jth column is u_j. Then $\{u_1, \dots, u_n\}$ is a basis if and only if $det(U) \neq 0$.

   - ($\Rightarrow$) Since the columns of U are linearly independent and U is an $n \times n$ matrix, then U is invertible. Since U is invertible then by the corollary to theorem 4.7 $\det(U) \neq 0$.
   - ($\Leftarrow$) Suppose for contrapositive that $\{u_1, \dots, u_n\}$ is linearly dependent. Then $rank(U) < n$ as the range of U is simply the span of the columns. Therefore by corollary to theorem 4.6 $det(U) = 0$.

4. Suppose $A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & a_0 \\ -1 & 0 & 0 & \dots & 0 & a_1 \\ 0 & -1 & 0 & \dots & 0 & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & a_{n-1} \end{bmatrix}$. Compute $\det(A + t\mathbb{I}_n)$.

Proof: Suppose $k \in \mathbb{N}$, if $k < n$, then B taking the form of A above, then $\det(B+t\mathbb{I}_k) = t^k + \sum_{i=0}^{k-1} a_i t^i$.

Base case: Suppose $A = \begin{bmatrix} 0 & a_0 \\ -1 & a_1 \end{bmatrix}$. Then $\det(A + t\mathbb{I}_2) = \det\left(\begin{bmatrix} t & a_0 \\ -1 & t+a_1 \end{bmatrix}\right) = t(t+a_1) + a_0 = t^2 + a_1 t + a_0$. Note: If we apply our induction hypothesis to the k case then we get that $\det\left(\begin{bmatrix} t & 0 & 0 & \dots & 0 & a_0 \\ -1 & t & 0 & \dots & 0 & a_1 \\ 0 & -1 & t & \dots & 0 & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & t+a_{k-1} \end{bmatrix}\right) = t^k + \sum_{i=0}^{k-1} a_i t^i$. However if we evaluate the determinate itself we get

$$t * \det\left(\begin{bmatrix} t & 0 & \dots & 0 & a_1 \\ -1 & t & \dots & 0 & a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -1 & t+a_{k-1} \end{bmatrix}\right) + \det\left(\begin{bmatrix} 0 & 0 & \dots & 0 & a_0 \\ -1 & t & \dots & 0 & a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -1 & t+a_{k-1} \end{bmatrix}\right).$$

Since the term multiplied by t can be evaluated by the induction hypothesis, then we have the equation $t(t^{k-1} + \sum_{i=0}^{k-2} a_{i+1} t^i) + \det\left(\begin{bmatrix} 0 & 0 & \dots & 0 & a_0 \\ -1 & t & \dots & 0 & a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -1 & t+a_{k-1} \end{bmatrix}\right) = t^k + \sum_{i=0}^{k-1} a_i t^i$.

Therefore $\det\left(\begin{bmatrix} 0 & 0 & \dots & 0 & a_0 \\ -1 & t & \dots & 0 & a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -1 & t+a_{k-1} \end{bmatrix}\right) = a_0$.

Induction step: Let $A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & a_0 \\ -1 & 0 & 0 & \dots & 0 & a_1 \\ 0 & -1 & 0 & \dots & 0 & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & a_{n-1} \end{bmatrix}$. Using the definition of determinant we have that

$$\det(A+t\mathbb{I}_n) = t*\det\left(\begin{bmatrix} t & 0 & \dots & 0 & a_1 \\ -1 & t & \dots & 0 & a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -1 & t+a_{n-1} \end{bmatrix}\right) + \det\left(\begin{bmatrix} 0 & 0 & \dots & 0 & a_0 \\ -1 & t & \dots & 0 & a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -1 & t+a_{n-1} \end{bmatrix}\right).$$

Applying the induction hypothesis we have that

$\det(A+t\mathbb{I}_n) = t*(t^{n-1}+\sum_{i=0}^{n-2} a_{i+1}t^i)+\det\left(\begin{bmatrix} 0 & 0 & \dots & 0 & a_0 \\ -1 & t & \dots & 0 & a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -1 & t+a_{n-1} \end{bmatrix}\right)$. Since the only non-zero entry in the first column of the remaining matrix is at index $(2,1)$ and has a value of -1 means that the determinate can be evaluated giving us the equation $\det(A+t\mathbb{I}_n) = t*(t^{n-1}+\sum_{i=0}^{n-2} a_{i+1}t^i)+\det\left(\begin{bmatrix} 0 & 0 & \dots & 0 & a_0 \\ -1 & t & \dots & 0 & a_3 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -1 & t+a_{n-1} \end{bmatrix}\right)$. Applying the note about the induction hypothesis we have that the determinant of that matrix is 0. Therefore $\det(A+t\mathbb{I}_n) = t*(t^{n-1}+\sum_{i=0}^{n-2} a_{i+1}t^i) + a_0 = t^n + \sum_{i=0}^{n-1} a_i t^i$.