For a list $d = (d_1, \ldots, d_k)$ of nonnegative integers, define the *factorial sum* of d to be the sum $\sum_{j=1}^{k} d_j j!$. Say that the list $d = (d_1, \ldots, d_k)$ is *proper* provided that for each j, $d_j \leq j$ and $d_k \neq 0$. A factorial sum is *proper* if the associated list is proper. The purpose of this problem is to prove: Every natural number can be represented uniquely as a proper factorial sum.

1. Prove that every natural number can be represented as a proper factorial sum.
   Suppose $n \in \mathbb{N}$. We must show there exists a list $(d_1, \ldots, d_k) \in \mathbb{Z}_{\geq 0}, d_k \neq 0$ such that $n = \sum_{j=1}^{k} d_j j!$. By the principal of mathematical induction for all $m \in \mathbb{N}$ if $m < n$, then there exists a list $(d_1, \ldots, d_l) \in \mathbb{Z}_{\geq 0}, d_l \neq 0$ such that $n = \sum_{j=1}^{l} d_j j!$. We have two cases:

   - Assume $n = 1$. Then $1 = 1!$.
   - Assume $n > 1$. Then we have two cases, n is odd and n is even.
     - Assume n is odd. By definition, $n = 2p + 1, p \in \mathbb{N}$. Since $n - 1 < n$ then by the induction hypothesis there exists $(d_1, \ldots, d_l) \in \mathbb{Z}_{\geq 0}, d_l \neq 0$ such that $n - 1 = \sum_{j=1}^{l} d_j j!$. Therefore since $n = 2p + 1$, $2p = \sum_{j=1}^{l} d_j j!$. Since the only factorial that is not a multiple of 2 is $1!$, then $d_1 = 0$. Therefore the proper list representing n is $(1, \ldots, d_l)$.
     - Assume n is even. Since $n - 1 < n$ then by the induction hypothesis there exists $(d_1, \ldots, d_l) \in \mathbb{Z}_{\geq 0}, d_l \neq 0$ such that $n - 1 = \sum_{j=1}^{l} d_j j!$. To construct the proper list for n, append a 0 to the end of the list such that we have the list $(d_1, \ldots, d_k, 0)$, then find the first index i in the list such that $i \nmid (d_i + 1)$. Since by definition $n - 1$ is odd, and $2 \mid (1 + d_1) = (1 + 1)$, then we're guaranteed to have this process find an index. Then there are two cases, $i < k, i = k$. Assume $i < k$, then the list representing n would have all entries up to i be set to 0, then increment d_i by 1: $(0, \ldots, d_i + 1, \ldots, d_k)$. If $k = i$, then the list would be 0 for the first k terms, and the $k + 1$st term would be 1, thus satisfying the definition of a proper list.

2. Prove that if two proper lists have the same factorial sum then the lists are equal.
   Suppose there exists $n \in \mathbb{N}$, and list $(d_1, \ldots, d_k), (e_1, \ldots, e_l) \in \mathbb{Z}_{\geq 0}$ such that $\sum_{j=1}^{k} d_j j! = \sum_{i=1}^{l} e_i i! = n$. We must show that $(d_1, \ldots, d_k) = (e_1, \ldots, e_l)$. By the principal of mathematical induction for all $m \in \mathbb{N}$ if $m < n$ then there exists a unique proper list $(m_1, \ldots, m_s)$ such that $m = \sum_{r=1}^{s} m_r r!$. We have two cases:

   - Assume $n = 1$. Since $1 = 1 * 1!$, and the list representing that is (1), then the only proper lists of length 1 are (1), thus (1) is a unique representation of 1.
   - Assume $n > 1$. Then $n = \sum_{j=1}^{k} d_j j! = \sum_{j=1}^{k-1} d_j j! + d_k k!$. Let $m = n - d_k k!$. We now have two cases:
     - Assume $n - d_k k! = 0$. Then $n = d_k k!$, which means that the list $d = (0, \ldots, 0, d_k)$, which uniquely represents n as a proper factorial sum.
     - Assume $n - d_k k! > 0$. Then since $n - d_k k! < n$, by the induction hypothesis there exists a unique proper list $(m_1, \ldots, m_r)$ such that $n - d_k k! = \sum_{r=1}^{s} m_r r!$.

Therefore $\sum_{r=1}^{s} m_r r! = \sum_{j=1}^{k-1} d_j j!$. Therefore since the two sums are equal, and there's a unique representation for m, then $k!d_k + \sum_{r=1}^{s} m_r r!$ uniquely represents a natural number. Therefore $n = n - k!d_k + k!d_k = \sum_{r=1}^{s} m_r r! + k!d_k = \sum_{j=1}^{k-1} d_j j! + k!d_k = \sum_{j=1}^{k} d_j j!$.

A similar argument can be had for $\sum_{i=1}^{l} e_i i!$. Therefore both lists uniquely represent n. Thus they are equal.