4.8 (a) Solving for $v(x,y) = (0,0)$ yields two solutions of the form $(\frac{-1}{2}, \frac{1}{2})$ and $(\frac{3}{2}, \frac{1}{2})$. Putting these values into the jacobian of v corresponding to $\begin{bmatrix} 2x-1 & -2y-1 \\ 2x-1 & 3-2y \end{bmatrix}$ yields the matrices $\begin{bmatrix} -2 & -2 \\ -2 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$ respectively. These have eigenvalues of $\pm\sqrt{2}$ and $2 \pm 2i$ respectively, therefore since both have eigenvalues with a real part greater than 0 then they are both unstable.

(b) Solving for $v(x,y) = (0,0)$ yields two solutions of the form $(\frac{-1}{2}, \frac{1}{2})$ and $(\frac{3}{2}, \frac{1}{2})$. Putting these values into the jacobian of v corresponding to $\begin{bmatrix} 2x-1 & 3-2y \\ 2x-1 & -2y-1 \end{bmatrix}$ yields the matrices $\begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix}$. These have eigenvalues of $\pm 2\sqrt{2}$ and $-2 \pm 2i$. Therefore $(-\frac{1}{2}, \frac{1}{2})$ is stable and $(\frac{3}{2}, \frac{1}{2})$ is unstable.

4.10 (a) Solving for $v(x,y) = (0,0)$ yields three solutions of the form $(0,0), (1,-1), (1,-2)$. Putting these values into the jacobian of v corresponding to $\begin{bmatrix} -2(y+2) & -2(y+2)-2(x+y) \\ y & x-1 \end{bmatrix}$ yields $\begin{bmatrix} -4 & -4 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -2 & -2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ respectively. These correspond with the eigenvalues $\{-4,-1\}$, $-1 \pm \sqrt{3}$, and $\pm 2i$. Therefore $(0,0)$ is stable, $(1,-1)$ is unstable. However $(1,-2)$ has unknown behavior.

(b) Solving for $v(x,y) = (0,0)$ yields three solutions of the form $(0,0), (1,-1), (1,-2)$. Putting these values into the jacobian of v corresponding to $\begin{bmatrix} 2(y+2) & 2(y+2)+2(x+y) \\ y & x-1 \end{bmatrix}$ yields $\begin{bmatrix} 4 & 4 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$ respectively. These correspond with the eigenvalues $\{4,-1\}$, $1 \pm i$, and ± 2. Therefore all of the points are unstable.

5.1 1 $\mathbf{X}_1 = \Psi(\mathbf{X}_0) = x_0 + \int_0^t v(x_0, s)ds = \int_0^t 2sds = t^2$

2 $\mathbf{X}_2 = \Psi(\mathbf{X}_1) = x_0 + \int_0^t v(x_1, s)ds = \int_0^t 2s(1+s^2)ds = t^2 + \frac{1}{2}t^4$

3 $\mathbf{X}_3 = \Psi(\mathbf{X}_2) = x_0 + \int_0^t v(x_2, s)ds = \int_0^t 2s(1+s^2+\frac{1}{2}s^4)ds = t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6$

4 $\mathbf{X}_4 = \Psi(\mathbf{X}_3) = x_0 + \int_0^t v(x_3, s)ds = \int_0^t 2s(1+s^2+\frac{1}{2}s^4+\frac{1}{6}s^6)ds = t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + \frac{1}{24}t^8$

These terms correspond exactly with the solution of $e^{2t} - 1$ as

$$-1 + e^{2t} = -1 + \sum_{k=0}^{\infty} \frac{t^{2k}}{k!} = \sum_{k=1}^{\infty} \frac{t^{2k}}{k!} = t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + \frac{1}{24}t^8 + \cdots$$

which shows the first four terms of the series we've found manually via Picard iteration.

5.2 Credit to CT Lim for telling me about the series expansion trick.
Note that after performing the substitution $s = te^q$, the first three terms of our flow

transformation's power series in q are:

$$\begin{bmatrix} 1 & -qt \\ -\frac{q}{t} & q+1 \end{bmatrix}, \begin{bmatrix} \frac{q^2}{2}+1 & -q^2t-qt \\ -\frac{q}{t} & q^2+q+1 \end{bmatrix}, \begin{bmatrix} \frac{q^2}{2}+1 & -\frac{2q^3t}{3}-q^2t-qt \\ -\frac{q^3}{6t}-\frac{q}{t} & \frac{2q^3}{3}+q^2+q+1 \end{bmatrix}.$$

Also note that the first 3 terms in the power series expansions of e^q and e^{-q} around 0 are: $\frac{q^3}{6}+\frac{q^2}{2}+q+1$ and $-\frac{q^3}{6}+\frac{q^2}{2}-q+1$ respectively.

1 $\mathbf{X}_1 = \Psi(\mathbf{X}_0) = \begin{bmatrix} 1 & t-e^qt \\ \frac{e^{-q}}{t}-\frac{1}{t} & q+1 \end{bmatrix} \mathbf{x}_0$. Note that taking the first term of the power series of $\mathbf{X}_1$ we get $\begin{bmatrix} 1 & t-(1+q)t \\ \frac{1-q}{t}-\frac{1}{t} & q+1 \end{bmatrix} \mathbf{x}_0 = \begin{bmatrix} 1 & -qt \\ -\frac{q}{t} & q+1 \end{bmatrix} \mathbf{x}_0$, exactly the first term in the power series of $[\Phi_{t,s}]$.

2 $\mathbf{X}_2 = \Psi(\mathbf{X}_1) = \begin{bmatrix} q+e^{-q} & -2e^qt+qt+2t \\ \frac{e^{-q}q}{t}+\frac{2e^{-q}}{t}-\frac{2}{t} & \frac{q^2}{2}+e^q \end{bmatrix} \mathbf{x}_0$. Note that taking up to the second term of the power series of $\mathbf{X}_2$ we get

$$\begin{bmatrix} q+\frac{q^2}{2}-q+1 & -2(\frac{q^2}{2}+q+1)t+qt+2t \\ \frac{q-q^2}{t}+\frac{2(\frac{q^2}{2}-q+1)}{t}-\frac{2}{t} & \frac{q^2}{2}+\frac{q^2}{2}+q+1 \end{bmatrix} \mathbf{x}_0 = \begin{bmatrix} \frac{q^2}{2}+1 & -q^2t-qt \\ -\frac{q}{t} & q^2+q+1 \end{bmatrix} \mathbf{x}_0,$$

exactly the second term in the power series of $[\Phi_{t,s}]$.

3 $\mathbf{X}_3 = \Psi(\mathbf{X}_2) = \begin{bmatrix} e^{-q}q+2q+3e^{-q}-2 & \frac{tq^2}{2}-e^qtq+tq-e^qt+t \\ \frac{e^{-q}q^2}{2t}+\frac{e^{-q}q}{t}-\frac{q}{t}+\frac{e^{-q}}{t}-\frac{1}{t} & \frac{q^3}{6}-\frac{q^2}{2}-2q+3e^q-2 \end{bmatrix} \mathbf{x}_0$. Note that taking up to the third term of the power series of $\mathbf{X}_3$ we get

$$\begin{bmatrix} q-q^2+\frac{q^3}{2}+2q+3(-\frac{q^3}{6}+\frac{q^2}{2}-q+1)-2 & \frac{tq^2}{2}-(q+q^2+\frac{q^3}{2})t+tq-(\frac{q^3}{6}+\frac{q^2}{2}+q+1)t+t \\ \frac{q^2-q^3}{2t}+\frac{q-q^2+\frac{q^3}{2}}{t}-\frac{q}{t}+\frac{-\frac{q^3}{6}+\frac{q^2}{2}-q+1}{t}-\frac{1}{t} & \frac{q^3}{6}-\frac{q^2}{2}-2q+3(\frac{q^3}{6}+\frac{q^2}{2}+q+1)-2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{q^2}{2}+1 & -\frac{2q^3t}{3}-q^2t-qt \\ -\frac{q^3}{6t}-\frac{q}{t} & \frac{2q^3}{3}+q^2+q+1 \end{bmatrix} \mathbf{x}_0,$$

exactly the second term in the power series of $[\Phi_{t,s}]$.