1.4.2 Recall that $\mathbb{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbb{Q}$, then $a + b, ab \in \mathbb{Q}$.
Suppose $a, b \in \mathbb{Q}$. By definition of being members of $\mathbb{Q}$, there exists $c, d, e, f \in \mathbb{Z}$ such that $a = \frac{c}{d}, b = \frac{e}{f}$.

- We will show that $a + b \in \mathbb{Q}$

$$\begin{aligned} a + b &= \frac{c}{d} + \frac{e}{f} \\ &= \frac{cf + ed}{df} \end{aligned}$$

Since $\mathbb{Z}$ is closed under addition and multiplication, $cf + ed \in \mathbb{Z}, df \in \mathbb{Z}$. Therefore $a + b$ satisfies the definition of a rational number

- We will show that $ab \in \mathbb{Q}$. Since $ab = \frac{ce}{df}$ by definition, and $\mathbb{Z}$ is closed under multiplication then $ce \in \mathbb{Z}, df \in \mathbb{Z}$. Therefore $ab \in \mathbb{Q}$.

(b) Suppose $a \in \mathbb{Q}, t \in \mathbb{I}$.

- We must show that $a+t \in \mathbb{I}$. Suppose for contradiction that $a+t \in \mathbb{Q}$. Then there exists $r \in \mathbb{Q}$ such that $a+t = r$. Since $\mathbb{Q}$ is closed under addition then $t \in \mathbb{Q}$. This is a contradiction as $t \notin \mathbb{Q}$.
- We must show that if $a \neq 0$ then $at \in \mathbb{I}$. Suppose for contradiction that $at \in \mathbb{Q}$. Then there exists $r \in \mathbb{Q}$ such that $at = r$. Since $\mathbb{Q}$ is closed under non-zero division then $t \in \mathbb{Q}$. This is a contradiction as $t \notin \mathbb{Q}$.

(c) Given two irrational numbers $s, t \in \mathbb{I}$ we can say nothing about whether $st \in \mathbb{I}$ or $s + t \in \mathbb{I}$. As $\sqrt{3} * \sqrt{2} = \sqrt{6} \in \mathbb{I}$, however $\sqrt{2} * \sqrt{2} = 4 \in \mathbb{Q}$. Similarly $\frac{\sqrt{2}}{2} \in \mathbb{I}$, $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \in \mathbb{I}$, however $\sqrt{2}, -\sqrt{2} + 2 \in \mathbb{I}$, $\sqrt{2} - \sqrt{2} + 2 = 2 \in \mathbb{Q}$. Therefore you can't say anything conclusive about the product and sum of general irrational numbers.

1.4.6 (a) Let $T = \{x \in \mathbb{R} : x^2 < 2\}, \alpha = \sup T$. Suppose for contradiction that $\alpha^2 > 2$. Suppose $n \in \mathbb{N}$. Then we have that

$$\begin{aligned} \left(\alpha - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n} \end{aligned}$$

By the archimedean principle, we may choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \frac{\alpha^2-2}{2\alpha}$. Therefore if we set $n_0 = n$ we have that $\left(\alpha - \frac{1}{n}\right)^2 > \alpha^2 - \alpha^2 + 2 = 2$. This contradicts the fact that all upper bounds must be greater than or equal to α.

(b) Suppose $b \geq 0$. Let $T = \{x \in \mathbb{R} : x^2 < b\}, \alpha = \sup T$. We will show that $\alpha^2 = b$ by cases. Two notes before proceeding with the proof. First note that we already know that $0^2 = 0 \in \mathbb{R}$, therefore we will operate on the assumption that $b > 0$.

Second, we claim that $\alpha > 0$. Since $b > 0$ we can apply the archamedian principle to get $m \in \mathbb{N}$ such that $b > \frac{1}{m}$. Since $\frac{1}{m^2} < \frac{1}{m} < b$ then $\frac{1}{m} \in T$. By definition of $\alpha = \sup T$ then $\frac{1}{m} < \alpha$. Therefore $\alpha > 0$. We now begin evaluating cases.

- Suppose $\alpha^2 < b$. Let $n \in \mathbb{N}$. Therefore

$$\begin{aligned}\left(\alpha + \frac{1}{n}\right)^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2}\\ &< \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n}\\ &= \alpha^2 + \frac{2\alpha+1}{n}\end{aligned}$$

  Since $\alpha > 0$ then $\frac{b-\alpha^2}{2\alpha+1} > 0$. Therefore by the archamedian principle there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \frac{b-\alpha^2}{2\alpha+1}$. If we set $n = n_0$ then we have that $(\alpha + \frac{1}{n_0})^2 < \alpha^2 + b - \alpha^2 = b$. This contradicts the fact that $\alpha = \sup T$ as all elements of T must be less than α.

- Suppose $\alpha^2 > b$. Let $n \in \mathbb{N}$. Therefore

$$\begin{aligned}\left(\alpha - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2}\\ &> \alpha^2 - \frac{2\alpha}{n}\end{aligned}$$

  By the archimedean principle, we may choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \frac{\alpha^2-b}{2\alpha}$. Therefore if we set $n = n_0$ we have that $\left(\alpha - \frac{1}{n}\right)^2 > \alpha^2 - \alpha^2 + b = b$. This contradicts the fact that all upper bounds must be greater than or equal to α.

1.4.8 (a) We must show that for two countable sets A_1, A_2 that $A_1 \cup A_2$ is countable. For the proof we will be dealing with the set $B_2 = A_2 \backslash A_1$. We will assume that B_2 is countable. Therefore there exists $f_1 : \mathbb{N} \to A_1, f_2 : \mathbb{N} \to B_2$ such that both are bijections. We claim that $F : \mathbb{N} \to A_1 \cup A_2$ given by $F(x) = \begin{cases} f_1(\frac{x-1}{2}) & x \text{ even} \\ f_2(x/2) & x \text{ odd} \end{cases}$ is a bijection.

- Suppose $x_1, x_2 \in \mathbb{N}, F(x_1), F(x_2) \in A_1 \cup A_2, F(x_1) = F(x_2)$. We must show that $x_1 = x_2$. Since A_1, B_2 are disjoint then either $F(x_1) \in A_1$ or $F(x_1) \in B_2$. If $F(x_1) \in A_1$ then $f_1(\frac{x_1-1}{2}) = f_1(\frac{x_2-1}{2})$ where x_1, x_2 must be odd, if not F would output in B_2 by definition, contradicting the initial assumption. Since f_1 is a bijection then $x_1 = x_2$. Suppose $F(x_1) \in B_2$. Then our equation becomes $f_2(x_1/2) = f_2(x_2/2)$ where x_1, x_2 must be even by similar reasoning above. Since f_2 is a bijection then $x_1 = x_2$. Therefore F is an injective function.
- Suppose $y \in A_1 \cup A_2$. We must show there exists $x \in \mathbb{N}$ such that $y = f(x)$. Since $y \in A_1 \cup A_2$ then either $y \in A_1$ or $y \in B_2$. Suppose $y \in A_1$. Then

by the definition of countability there exists $n \in \mathbb{N}$ such that $f_1(n) = y$. We claim that $x = 2n + 1$. Observe that $F(2n + 1) = f_1(\frac{2n+1-1}{2}) = f_1(2n/2) = f_1(n) = y$. Suppose $y \in B_2$. Then by the definition of countability there exists $m \in \mathbb{N}$ such that $f_2(m) = y$. We claim that $x = 2m$. Observe that $F(2m) = f_2(\frac{2m}{2}) = f_2(m) = y$. Therefore F is surjective.

Since F has been shown to be a bijection between $\mathbb{N}$ and $A_1 \cup A_2$ then the union of any two countable sets is countable.

If B_2 was finite then we could have given an arbitrary indexing to B_2 by the bijection $\sigma : \{1, 2, \dots, n\} \to B_2$ then given F as $F(x) = \begin{cases} \sigma(x) & x \leq n \\ f_1(x - n) & x > n \end{cases}$

The greater proof of having $A_1, \dots, A_m$ countable sets having a countable union is by induction. Since any two countable sets can be unioned together to be a larger countable set, then we can apply that operation an arbitrary amount of times until we have $A_1 \cup \cdots \cup A_{m-1}$ as a countable set and A_m, then union then together and apply what has been proved above.

(b) Induction fails to prove part (ii) as ∞ is not a natural number. Part (i) is m sets, which is a finite number, and only requires a finite process to achieve.

(c) The arrangement as shown in the problem lends itself to a bijective function $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. If one is to take the sets $B_n = A_n \backslash (A_1 \cup A_2 \cup \cdots \cup A_{n-1} \cup A_{n+1} \cup \cdots)$ and assume that they remain countable after performing this process then we can take the respective bijection $f_n : \mathbb{N} \to B_n$ and arrange each function and it's output as so:

$$\begin{matrix} f_1(1) & f_1(2) & \cdots \\ f_2(1) & f_2(1) & \cdots \\ \vdots & \vdots & \ddots \end{matrix}$$

Since for each $(m, n) \in \mathbb{N}^2$ we have a unique $f_n(m)$ then we have another bijection $g : \mathbb{N}^2 \to \cup_{n=1}^{\infty} A_n$. Since the composition of bijections is a bijection, $g \circ f : \mathbb{N} \to \cup_{n=1}^{\infty} A_n$ is a bijection. Therefore $\cup_{n=1}^{\infty} A_n$ is countable.