3.4 (a) Let $A = \begin{bmatrix} -4 & 2 \\ 5 & -1 \end{bmatrix}$. To compute e^{tA} we need to know the eigenvectors and eigenvalues of A Characteristic polynomial of A: $(\lambda + 6)(\lambda - 1)$. Therefore the eigenvalues are $\lambda = -6, 1$, and let them be denoted μ_1, μ_2 respectively. These correspond to eigenvectors $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$. Since we know the eigenvectors, we now may compute the columns of $M(t)$, where $e^{tA} = M(t)M(0)^{-1}$:

i. $\vec{z}_1(t) := e^{tA}\vec{v}_1 = e^{-6t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

ii. $\vec{z}_2(t) := e^{tA}\vec{v}_2 = e^{t}\begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

Since we have $M(t)$, we now need $M(0)^{-1}$:

$$M(0)^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & 5 \end{bmatrix}^{-1} = \frac{-1}{7}\begin{bmatrix} 5 & -2 \\ -1 & -1 \end{bmatrix}.$$

Therefore $e^{tA} = \frac{-1}{7}e^{-6t}\begin{bmatrix} -5 - 2e^{7t} & 2 - 2e^{7t} \\ 5 - 5e^{7t} & -2 - 5e^{7t} \end{bmatrix}$

Let $\vec{x}_0 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Therefore the general solution to $\vec{x}' = A\vec{x}$ is the following:

$$e^{tA}\vec{x}_0 = \frac{-1}{7}e^{-6t}\begin{bmatrix} (-5 - 2e^{7t})x_1 + (2 - 2e^{7t})x_2 \\ (5 - 5e^{7t})x_1 + (-2 - 5e^{7t})x_2 \end{bmatrix}.$$

(b) To find when the entire solution goes to the zero vector, we simply need to find when one row of the vector goes to zero and test our solution on the entire vector. Therefore we must solve

$$\lim_{t\to\infty}(-5 - 2e^{7t})x_1 + (2 - 2e^{7t})x_2 = 0.$$

Rearranging we find $\lim_{t\to\infty}\frac{2 - 2e^{7t}}{5 + 2e^{7t}} = \frac{-2}{2} = -1 = \frac{x_1}{x_2}$. Therefore we have the relation $-x_2 = x_1$. Testing on the second row we find $\lim_{t\to\infty}(5 - 5e^{7t})x_1 + (-2 - 5e^{7t})x_2 = (5 - 5e^{7t})x_1 + -(-2 - 5e^{7t})x_1 = 7x_1$. Since $7x_1$ is constant and the vector is multiplied by e^{-6t} then we have $0 \cdot 7x_1 = 0$. Thus the vector $\vec{x}_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}c$, where $c \in \mathbb{R}$ solves the equation $\lim_{t\to\infty}\vec{x}(t) = \vec{0}$.

3.5 (a) Let $A = \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$. A has the characteristic polynomial $(\lambda - 3)^2$. Thus A has an eigenvalue of 3 with multiplicity 2. Note that $(A - 3\mathbb{I})^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, therefore any vector in $\mathbb{R}^2$ is a generalized eigenvector of A. Therefore we can go ahead and compute e^{tA} directly:

$$e^{tA} = e^{3t}e^{t(A-3\mathbb{I})} = e^{3t}\sum_{k=0}^{1} \frac{t^k}{k!}(A - 3\mathbb{I})^k = e^{3t}\left(\mathbb{I} + t\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\right) = \begin{bmatrix} 1+2t & -t \\ 4t & 1-2t \end{bmatrix}.$$

Therefore the general solution is $\vec{x}(t) = e^{tA}\vec{x}_0 = e^{3t}\begin{bmatrix} (1+2t)x_1 + (-t)x_2 \\ 4tx_1 + (1-2t)x_2 \end{bmatrix}$

(b) Note that

$$\lim_{t\to\infty} \vec{x}(t) = \lim_{t\to\infty} e^{3t}\left(\mathbb{I} + t\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\right)\vec{x}_0 = \lim_{t\to\infty} e^{3t}\vec{x}_0 + e^{3t}t\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\vec{x}_0$$

thus the $e^{3t}\vec{x}_0$ term will diverge for every non-zero vector. Therefore no matter the kernel of the matrix on the other term, the first term will diverge. Thus the only solution which converges to $\vec{0}$ is $\vec{0}$.

3.8 Let $A = \begin{bmatrix} -1 & 2 & 2 \\ -2 & -1 & 1 \\ -2 & -1 & -1 \end{bmatrix}$.

The characteristic polynomial of A is given by $det(A - \lambda\mathbb{I}) = -(\lambda + 1)(\lambda^2 + 2\lambda + 10)$. Therefore we have eigenvalues of $\mu_1 = -1, \mu_2 = -1+3i, \mu_3 = -1-3i$. These correspond to the eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2-6i \\ 4-3i \\ 5 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -2+6i \\ 4+3i \\ 5 \end{bmatrix}$. Columns of $M(t)$:

- $\vec{z}_1(t) := e^{-t}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$
- $\vec{z}_2(t) := e^{(-1+3i)t}\begin{bmatrix} -2-6i \\ 4-3i \\ 5 \end{bmatrix}$
- $\vec{z}_3(t) := e^{(-1-3i)t}\begin{bmatrix} -2+6i \\ 4+3i \\ 5 \end{bmatrix}$

Therefore we can now compute $M(0)^{-1}$:

$$M(0)^{-1} = \frac{1}{90}\begin{bmatrix} 10 & -20 & 20 \\ -2+6i & 4+3i & 5 \\ -2-6i & 4-3i & 5 \end{bmatrix}.$$

Therefore

$$e^{tA} = \frac{1}{90}e^{-t}\begin{bmatrix} 1 & e^{3it}(-2-6i) & e^{-3it}(-2+6i) \\ -2 & e^{3it}(4-3i) & e^{-3it}(4+3i) \\ 2 & 5e^{3it} & 5e^{-3it} \end{bmatrix}\begin{bmatrix} 10 & -20 & 20 \\ -2+6i & 4+3i & 5 \\ -2-6i & 4-3i & 5 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{e^{-t}}{9} & -\frac{2}{9}(-6i-2)e^{3\text{it}-t} & \frac{2}{9}(6i-2)e^{-3\text{it}-t} \\ \frac{(6i+18)e^{-t}}{135i} & -\frac{(9-12i)(4-3i)e^{3\text{it}-t}}{270i} & \frac{1}{18}(3i+4)e^{-3\text{it}-t} \\ -\frac{(6i-18)e^{-t}}{135i} & -\frac{(-12i-9)e^{3\text{it}-t}}{54i} & \frac{5}{18}e^{-3\text{it}-t} \end{bmatrix}$$

3.11 Let $A = \begin{bmatrix} 4 & -1 & 2 \\ 1 & 2 & 2 \\ -2 & 2 & 1 \end{bmatrix}$. The characteristic polynomial is given by $-(\lambda-1)(\lambda-3)^2$. Therefore the eigenvalues are $\lambda = 1, 3$. Since $\lambda = 3$ has a multiplicity of 2, then we will double count the eigenvalue of 3: let the eigenvalues be denoted $\mu_1 = 1, \mu_2 = 3, \mu_3 = 3$. The first two correspond to the eigenvectors are given by $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. The generalized eigenvector given by the kernel of $(A - 3\mathbb{I})$ denoted $\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Therefore we can compute the columns of $M(t)$:

- $\vec{z}_1(t) := e^t \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$
- $\vec{z}_2(t) := e^{3t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
- $\vec{z}_3(t) := e^{3t} \sum_{k=0}^{1} \frac{t^k}{k!}(A - 3\mathbb{I})^k \vec{v}_3 = e^{3t}\begin{bmatrix} -1+t \\ t \\ 1 \end{bmatrix}$

Since we have computed the columns of $M(t)$, we may now compute $M(0)^{-1}$:

$$M(0)^{-1} = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Therefore e^{tA} is given by:

$$e^{tA} = M(t)M(0)^{-1} = \begin{bmatrix} e^t & e^{3t} & e^{3t}(t-1) \\ e^t & e^{3t} & e^{3t}t \\ -e^t & 0 & e^{3t} \end{bmatrix}\begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -e^t\left(e^{2t}(t-2)+1\right) & e^{3t}(t-1)+e^t & e^t\left(e^{2t}-1\right) \\ -e^{3t}(t-1)-e^t & e^{3t}t+e^t & e^t\left(e^{2t}-1\right) \\ e^t - e^{3t} & e^t\left(e^{2t}-1\right) & e^t \end{bmatrix}$$