6.5.9 If $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ prove that $a_n = b_n$ for all $n \in \mathbb{N}$.
Proof. Consider $0 = h(x) = \sum_{n=0}^{\infty}(a_n - b_n)x^n$. Since both power series are continuous on the interval $(-R, R)$ then $h(x)$ is defined for 0. Therefore $0 = h(0) = (a_0 - b_0) + (a_1 - b_1)0 + \cdots = a_0 - b_0$. Therefore $a_0 = b_0$. Since each power series is differentiable, and the sum of differentiable functions is differentiable, then we have that $0 = h'(0) = (a_1 - b_1) + 2(a_2 - b_2)0 + \cdots$. Therefore by the principle of mathematical induction, for all $k \in \mathbb{N}$, if $k < n$, then $a_k = b_k$. Consider the nth derivative of $h(x)$, therefore $\frac{d^n}{dx^n} h(x) = \sum_{l=n}^{\infty} \frac{(l)!}{(l-n)!}(a_l - b_l)x^{l-n}$. We know by theorem 6.5.7 that convergent power series are infinitely differentiable, since $h(x)$ is defined on $(-R, R)$ then $\frac{d^n}{dx^n} h(0)$ is defined. Therefore $0 = \frac{d^n}{dx^n} h(0) = n!(a_n - b_n) + (n + 1)!(a_n - b_n)0 + \cdots$. Thus $0 = a_n - b_n$, $a_n = b_n$. Therefore the power series are equivalent.