2.5

2.6

2.10 (a) The inverse transform is given by:

$$(x, y) = (e^{-u}, \frac{v}{e^{2u}}).$$

(b) Time derivatives of u, v:

$$\frac{d}{dt}u = \frac{-x'}{x} = \frac{-x}{x} = 1$$

$$\begin{aligned}\frac{d}{dt}v &= 2xyx' + x^2y' \\ &= 2x^2y + x^2((xy - \frac{1}{x})^2 - \frac{2}{x^2}) \\ &= 2x^2y + (x^2y - 1)^2 - 1 \\ &= 2x^2y + x^4y^2 - 2x^2y + 1 - 2 \\ &= x^4y^2 - 1 \\ &= v^2 - 1\end{aligned}$$

Thus the vector field $\vec{w}$ for the system $\vec{u}' = \vec{w}(\vec{u})$ is given by: $\vec{w} = (-1, v^2 - 1)$. This is clearly decoupled as specified.

(c) Solving the decoupled system for $\vec{u}(0) = (u_0, v_0)$. Since $u' = -1$, then $u = u_0 - t$. For $v' = v^2 - 1$, by barrow's formula we get the equation

$$t = \int_{v_0}^{v} \frac{dz}{z^2 - 1}.$$

Splitting $\frac{1}{z^2-1}$ apart by partial fraction decomposition yields

$$\frac{1}{z^2 - 1} = -1(\frac{1}{2(1-v)} + \frac{1}{2(v+1)}).$$

This results in the integral being evaluated as

$$t = -ln(\sqrt{\frac{v+1}{1-v}}) + ln(\sqrt{\frac{v_0+1}{1-v_0}}).$$

Note that $ln\left(\sqrt{\frac{v+1}{1-v}}\right) = artanh(v)$, therefore we can directly express v now:

$$v = tanh(artanh(v_0) - t).$$

We must show that this solution for $\vec{u}$ with $\vec{u}(0) = (u_0, v_0)$ exists uniquely for all t if and only if $|v_0| \leq 1$.

- ($\Rightarrow$) Suppose the solution given above at $\vec{u} = (u_0, v_0)$ exists for all t and is unique. Then we must show $|v_0| \leq 1$. Suppose for contradiction that $|v_0| > 1$. Then we must evaluate $artanh(v_0)$, however for $v_0 > 1$, artanh is not defined. This contradicts u existing for all t. Therefore $|v_0| \leq 1$.
- ($\Leftarrow$) Suppose $|v_0| \leq 1$. We must show there exists a unique solution for all t at $\vec{u} = (u_0, v_0)$. Note that by definition we're operating inside of the maximal interval $(-1, 1)$ and the endpoints $\{-1, 1\}$. First for the cases where $v_0 \in (-1, 1)$. Since we need to show the existence and uniqueness of a solution, we simply need to show that $\vec{w}$ is Lipschitz on $(-1, 1)$. Note that since $w_1 = -1$, that for any value of v_0, w_1 is always bounded. For $v' = w_2 = v^2 - 1$, since $v \in (-1, 1)$, then $max(|w_2(v)|) = 1$. Then we have the inequality

  $$|x^2 - 1 - y^2 + 1| \leq |x^2 - y^2| \leq |x + y||x - y| \leq 2|x - y|$$

  Therefore on $(-1, 1)$ we have each component of $\vec{w}$ lipschitz continuous, thus $\|\vec{w}\|$ is lipschitz. For the case of $v_0 = 1$, we must show that the constant solution is the only one for $v' = v^2 - 1$. Since $\lim_{\delta \to 0} \int_{1-\delta}^{1} \frac{dz}{|z^2-1|} \geq \lim_{\delta \to 0} |artanh(1 - \delta) - artanh(1)| = \lim_{\delta \to 0} \infty = |artanh(1) - artanh(1 + \delta)| \leq \lim_{\delta \to 0} \int_{1}^{1+\delta} \frac{dz}{|z^2-1|}$, then the constant solution is the unique solution when $v_0 = 1$. Also note that $|artanh(x)| = |artanh(-x)|$, therefore this inequality also shows the uniqueness of the solution for $v = -1$.