1. Each of the subproblems apply AM-GM once.

   (a)
$$\begin{aligned}\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} &= 4\frac{1}{4}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)\\ &\le 4\left(\frac{abcd}{abcd}\right)^{\frac{1}{4}}\\ &= 4.\end{aligned}$$

   (b)
$$\begin{aligned}a^6+b^9+64 &= 3\frac{1}{3}(a^6+b^9+64)\\ &\le 3(a^6b^964)^{\frac{1}{3}}\\ &= 12a^2b^3\end{aligned}$$

2. By applying Cauchy-Schwarz we get that $1=(x_1+\cdots+x_n)^2 \le n\sum_i x_i^2$. Therefore the norm of the vector is bounded below via $\frac{1}{n}, \frac{1}{n} \le \sum_i x_i^2$. Thus it is minimized via having a norm of $\frac{1}{n}$. An example solution would be $x_i = \frac{1}{n}$ for all $i \in [n]$.

3. (a) Suppose $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\frac{1}{\sqrt{\epsilon}} < N$ by the archimedean principle. Therefore if $n \ge N$ then
$$\left|\frac{n^2}{n^4+n^2+1}\right| < \left|\frac{n^2}{n^4}\right| = \left|\frac{1}{n^2}\right| < \epsilon.$$

   (b) Suppose $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\frac{1}{\epsilon} < N$. Suppose $n \ge N$. Then we have that
$$\left|\frac{5n^2+n}{3n^2+1}-\frac{5}{3}\right| = \left|\frac{3n-5}{3(3n^2+1)}\right| < \left|\frac{1}{3n}\right| = \frac{1}{3}\left|\frac{1}{n}\right| < \frac{\epsilon}{3} < \epsilon.$$

4. Suppose $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n \ge N$ that $|a-a_n| < \epsilon$. Therefore by the reverse triangle inequality we have that $||a|-|a_n|| \le |a-a_n| < \epsilon$. Therefore $\lim |a_n| \to |a|$. The converse is not true. If we consider $a_n = (-1)^n(1-2^n)$, then clearly $\limsup a_n = 1, \liminf a_n = -1$. This obviously doesn't converge. However $|a_n| = 1-2^n$, which does converge to 1.

5. Let $a_n = 2^n, b_n = -2^n$. Clearly $a_n \to \infty, b_n \to -\infty$. However $a_n + b_n = 2^n - 2^n = 0$, which does converge since it's constant.

6. We will show that a_n is bounded below and decreasing.

   - We will show that a_n is bounded below by $\frac{1+\sqrt{13}}{2}$. Clearly $3 \ge \frac{1+\sqrt{13}}{2}$. By the principle of mathematical induction for all $k \in \mathbb{N}$ if $k < n$ then $a_k \ge \frac{1+\sqrt{13}}{2}$. Therefore $a_n = \sqrt{3+a_{n-1}} \ge \sqrt{3+\frac{1+\sqrt{13}}{2}} = \sqrt{\frac{7+\sqrt{13}}{2}} = \frac{\sqrt{1+\sqrt{13}}}{2}$. Thus a_n is bounded below.

- We will show that a_n is decreasing. Clearly $3 \geq \sqrt{6}$. By the principle of mathematical induction, for all $k \in \mathbb{N}$ if $k < n$ then $a_{k+1} \leq a_k$. Therefore $a_n = \sqrt{3 + a_{n-1}}$, by the induction hypothesis $\sqrt{3 + a_{n-1}} \leq \sqrt{3 + a_{n-2}} = a_{n-1}$. Thus $a_n \leq a_{n-1}$. Therefore (a_n) is decreasing.

Therefore by the monotone convergence theorem (a_n) converges, and it converges to the lower bound given above. The solution will satisfy $\alpha = \sqrt{3 + \alpha}, \alpha^2 - \alpha - 3 = 0$. This has a root of $\frac{1+\sqrt{13}}{2}$. Additionally $\left(\frac{1+\sqrt{13}}{2}\right)^2 = \frac{7+\sqrt{13}}{2} = 3 + \frac{1+\sqrt{13}}{2}$.

7. We will show that (a_n) is decreasing and bounded.

   - We will show that (a_n) is bounded. Since $a_n \in (0, 1)$ for all $n \in \mathbb{N}$, then we can say that for all $n \in \mathbb{N}$ $|a_n| \leq 1$. Thus (a_n) is bounded.
   - We will show that (a_n) is decreasing. Consider the given inequality $\frac{1}{4} < a_n(1 - a_{n+1})$. We can treat the product of elements from the sequence as a geometric mean, and apply AM-GM:

   $$\sqrt{a_n^2(1 - a_{n+1})^2} \leq \frac{a_n^2 + (1 - a_{n+1})^2}{2}.$$

   We know for $x \in (0, 1)$ that $x^2 \leq x$. Therefore we have from the previous inequality $\frac{1}{4} < \frac{a_n + 1 - a_{n+1}}{2}$ which yields $a_{n+1} < a_{n+1} + 1 < a_n$. Therefore (a_n) is decreasing.

   Thus (a_n) converges. But to what? Note that $\lim a_{n+1} = \lim a_n = \alpha$. Thus by the order limit theorem $\alpha(1 - \alpha) \geq \frac{1}{4}$. This is equivalent to the inequality $(\alpha - \frac{1}{2})^2 \leq 0$. Since $(\alpha - \frac{1}{2})^2 \geq 0$ then $(\alpha - \frac{1}{2})^2 = 0$. Thus $\alpha = \frac{1}{2}$. Therefore $\lim a_n = \frac{1}{2}$

8.