1. Let $Tri_{m \times n}(F)$ be the set of all upper triangular $m \times n$ matrices. Prove that $Tri_{m \times n}(F)$ is a subspace of $M_{m \times n}(F)$.

   - Let $\mathbb{0}$ denote the 0 $m \times n$ matrix. We must show that $\mathbb{0} \in Tri_{m \times n}(F)$. By definition of $\mathbb{0} \in M_{m \times n}(F)$, $\mathbb{0}_{ij} = 0$, for all $i \in [m], j \in [n]$. By definition of being an upper triangular matrix, for a matrix A, $A_{ij} = 0$ whenever $i > j$. Since all entries in $\mathbb{0}_{ij} = 0$, then it satisfies the requirements of $i > j$. Thus $\mathbb{0} \in Tri_{m \times n}(F)$
   - Suppose $X, Y \in Tri_{m \times n}(F)$. We must show $X + Y \in Tri_{m \times n}(F)$. Since $X, Y \in Tri_{m \times n}(F)$, then $X_{ij} = 0, Y_{ij} = 0$ whenever $i > j$. Therefore $(X + Y)_{ij} = X_{ij} + Y_{ij} = 0 + 0 = 0$ whenever $i > j$. Thus $X + Y \in Tri_{m \times n}(F)$.
   - Suppose $X \in Tri_{m \times n}(F), c \in F$. We must show $cX \in Tri_{m \times n}(F)$. Therefore by definition of being an upper triangular matrix, $c \cdot A_{ij} = c \cdot 0 = 0$ whenever $i > j$. Therefore $cA \in Tri_{m \times n}(F)$.

   Therefore $Tri_{m \times n}(F)$ is a subspace of $M_{m \times n}(F)$.

2. Let S be a non-empty set, and F a field. Let $C(S, F)$ denote the set of all functions $f \in \mathcal{F}(S, F)$ such that $f(s) = 0$ for all but a finite number of elements in S. Prove that $C(S, F)$ is a subspace of $\mathcal{F}(S, F)$.

- Let $\mathbb{0} : S \to S$ denote the function which is 0 for all elements of S. We must show $\mathbb{0} \in C(S, F)$. Since the set of elements for which $\mathbb{0} \neq 0$ is empty, and $|\emptyset| = 0$, then the non-zero element set is finite. Therefore $\mathbb{0} \in C(S, F)$.
- Let $f, g \in C(S, F)$, and let $A, B \subset S, |A| = n, |B| = m, n, m \in \mathbb{N}$ denote the sets for which f and g respectively are non-zero. We must show $f+g \in C(S, F)$. Since $f+g \in \mathcal{F}(S, F)$, then there exists a set D such that $S \subseteq D, \forall x \in D(f+g)(x) \neq 0$. We must show that D is finite. We claim that $D \subseteq A \cup B$. Suppose $x \in D$. We have three cases.

  (a) Suppose $f(x) \neq 0, g(x) = 0$. Therefore $(f+g)(x) = f(x)+g(x) = f(x)+0 = f(x) \neq 0$. Since $f(x) \neq 0, x \in A$

  (b) Suppose $f(x) = 0, g(x) \neq 0$. Therefore $(f+g)(x) = f(x)+g(x) = 0+g(x) = g(x) \neq 0$. Since $g(x) \neq 0, x \in B$.

  (c) Suppose $f(x) \neq 0, g(x) \neq 0$. Assume $f(x) \neq -g(x)$. Since both f, g are non-zero, and not inverses of each other, then $(f+g)(x) \neq 0$. Since both f, g are non-zero, then $x \in A$ and $x \in B$.
  Assume $f(x) = -g(x)$. Then $f(x) + g(x) = 0$. This is a contradiction, as $x \in D$.

  Since for all possible $x \in D, x \in A \cup B$, then we have shown the claim. Since $|D| \leq |A \cup B|$, $|A \cup B| \leq n + m$, and $n + m \in \mathbb{N}$, then $|D| \leq n + m$. Thus D is finite. Therefore $f + g \in C(S, F)$.
- Let $f \in C(S, F), c \in F$. We must show that $cf \in C(S, F)$. If $c = 0$, then $cf = \mathbb{0}$, which $\mathbb{0} \in C(S, F)$. Suppose $c \neq 0$. Since $f \in C(S, F)$, then there exists a set $A \subset S, |A| = n, n \in \mathbb{N}$ such that for all $s \in A, f(s) \neq 0$. We claim that A is the same set of non-zero points for cf. Suppose $x \in S$, we have two cases:
  - Suppose $x \in A$. Since $f(x) \neq 0, c \neq 0$, then $cf(x) \neq 0$.
  - Suppose $x \notin A$. Since $f(x) = 0$, then $cf(x) = c0 = 0$.

  Since A is the set of elements in S for which cf is non-zero, and A is finite, then $cf \in C(S, F)$.

Therefore $C(S, F)$ is a subspace of $\mathcal{F}(S, F)$.

Lemma 1: Suppose W is a subspace of the vector space V, $\vec{x}, \vec{y} \in V, \vec{x} \in W, \vec{y} \notin W$. We must show $\vec{x} + \vec{y} \notin W$. Suppose for contradiction that $\vec{x} + \vec{y} \in W$. Since $-\vec{x} \in W$, then $\vec{x} - \vec{x} + \vec{y} \in W$. Therefore $\vec{y} \in W$. This is a contradiction. Therefore $\vec{x} + \vec{y} \notin W$.

3. Let W_1, W_2 be subspaces of the vector space V. We must show that $W_1 \cup W_2$ is a subspace V if and only if $W_1 \subset W_2$ or $W_2 \subset W_1$.

    - ($\Rightarrow$) Suppose $W_1 \cup W_2$ is a subspace of V. We must show $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. Suppose for contradiction that $W_1 \nsubseteq W_2$ and $W_2 \nsubseteq W_1$. By definition of $\nsubseteq$, there exists vectors $\vec{x}, \vec{y}$ such that $\vec{x} \in W_1, \vec{x} \notin W_2, \vec{y} \in W_2, \vec{y} \notin W_1$. Since $\vec{x}, \vec{y} \in W_1 \cup W_2$, then $\vec{x} + \vec{y} \in W_1 \cup W_2$ by the closure property for subspaces. Since $\vec{x} \in W_1, \vec{y} \notin W_1$, then $\vec{x} + \vec{y} \notin W_1$ by lemma 1. Since $\vec{y} \in W_2, \vec{x} \notin W_2$, then $\vec{x} + \vec{y} \notin W_2$ by lemma 1. Therefore by definition of union $\vec{x} + \vec{y} \notin W_1 \cup W_2$. This is a contradiction. Therefore $W_1 \subseteq W_2$ or $W_1 \subseteq W_2$.
    - ($\Leftarrow$) Suppose $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. We must show $W_1 \cup W_2$ is a subspace of V. We have two cases. Suppose $W_1 \subseteq W_2$. Then by definition $W_1 \cup W_2 = W_2$. Since W_2 is a vector space of V, then the requirements have been satisfied. The proof is nearly identical for the $W_2 \subseteq W_1$ case.

4. Show that $P_n(F)$ is generated by $\{1, x, \cdots, x^n\}$. Suppose $f \in P_n(F)$. We must show that $f \in Span(\{1, x, \cdots, x^n\})$. By definition of being a member of $P_n(F)$, $f = a_0 \cdot 1 + a_1 \cdot x + \ldots + a_n \cdot x^n$, where $a_0, \cdots, a_n \in F$. Since $a_0, \cdots, a_n \in F$, and each of those is multiplied by an element in the generating set, then $f \in Span(\{1, x, \cdots, x^n\})$.

5. Let V be a vector space, $W \subseteq V$. We must show that $W \leq V$ if and only if $Span(W) = W$.

   - ($\Rightarrow$) Suppose $W \leq V$. We must show $Span(W) = W$. Therefore we must show $Span(W) \subseteq W, Span(W) \supseteq W$.
     - ($\subseteq$) Suppose $\vec{x} \in Span(W)$. We must show $\vec{x} \in W$. By definition of being a member of $Span(W)$, there exists vectors $\vec{w_1}, \ldots, \vec{w_n} \in W, c_1, \ldots c_n \in F$ such that $\vec{x} = \sum_{i=1}^{n} c_i \vec{w_i}$. By induction, W is closed under successive applications of closure under scalar multiplication and vector addition since W is a subspace. Therefore $\vec{x} \in W$.
     - ($\supseteq$) Suppose $\vec{x} \in W$. We must show $\vec{x} \in Span(W)$. By the identity property of vector spaces, $\vec{x} = 1 \cdot \vec{x}$. Since $\vec{x}$ is a scalar multiple of a vector in W, then $\vec{x} \in Span(W)$.
   - ($\Leftarrow$) Suppose $Span(W) = W$. We must show that $W \leq V$.
     - We must show $\vec{0} \in W$. Suppose $\vec{w} \in W$. Since $W = Span(W)$, and $0 \cdot \vec{w} \in Span(W)$, then $0 \cdot \vec{w} = \vec{0} \in W$.
     - Suppose $\vec{x}, \vec{y} \in W$. We must show $\vec{x} + \vec{y} \in W$. Since $\vec{x} + \vec{y} = 1 \cdot \vec{x} + 1 \cdot \vec{y}$, and these are scalar multiples of vectors we know to exists in W which are summed together, then $\vec{x} + \vec{y} \in Span(W)$. Therefore $\vec{x} + \vec{y} \in W$.
     - Suppose $\vec{x} \in W, c \in F$. We must show $c\vec{x} \in W$. Since $c\vec{x}$ is a scalar multiple of a vector we know to be within W, then $c\vec{x} \in Span(W)$. Thus $c\vec{x} \in W$.