Alex Valentino Assignment 9 problem 1

300H

Lemma: $\sum_{i=1}^n i = \frac{n(n+1)}{2}$.

We must show for all $n \in \mathbb{N}, \sum_{i=1}^n i = \frac{n(n+1)}{2}$. Suppose $n \in \mathbb{N}$. We must show $\sum_{i=1}^n i = \frac{n(n+1)}{2}$. Since we know that $\sum_{i=1}^n 2i - 1 = n^2$, and $\sum_{i=1}^n 1$, then

$$\sum_{i=1}^n 2i - 1 = n^2$$
$$\sum_{i=1}^n 2i - \sum_{i=1}^n 1 = n^2$$
$$\sum_{i=1}^n 2i - n = n^2$$
$$\sum_{i=1}^n 2i = n^2 + n$$
$$2\sum_{i=1}^n i = n^2 + n$$
$$\sum_{i=1}^n i = \frac{n(n+1)}{2}.$$

Prove that for all $n \in \mathbb{N}$, $\sum_{i=1}^n i^3 = (\sum_{i=1}^n i)^2$

We must show for all $n \in \mathbb{N}$, $\sum_{i=1}^n i^3 = (\sum_{i=1}^n i)^2$. Suppose $n \in \mathbb{N}$. We must show that $\sum_{i=1}^n i^3 = (\sum_{i=1}^n i)^2$. By the lemma above, we must show $\sum_{i=1}^n i^3 = (\frac{n(n+1)}{2})^2$. By the principal of induction, for all $k \in \mathbb{N}$, if $k < n$ then $\sum_{i=1}^k i^3 = (\frac{k(k+1)}{2})^2$. We now must show two cases:

- Assume $n = 1$. Then $1^3 = 1 = (\frac{1(1+1)}{2})^2 = (\frac{2}{2})^2 = 1^2 = 1$.
- Assume $n > 1$. Then $\sum_{i=1}^n i^3 = n^3 + \sum_{i=1}^{n-1} i^3$. Since $n - 1 < n$, by the induction hypothesis, $\sum_{i=1}^{n-1} i^3 = (\frac{n(n-1)}{2})^2$. Therefore $\sum_{i=1}^n i^3 = n^3 + \sum_{i=1}^{n-1} i^3 = n^3 + (\frac{n(n-1)}{2})^2 = n^2(n + \frac{(n-1)^2)}{4}) = n^2\frac{4n+n^2-2n+1}{4} = n^2\frac{n^2+2n+1}{4} = n^2\frac{(n+1)^2}{4} = (\frac{n(n+1)}{2})^2$.