1. Let $(W, \leq)$ be a linearly order set.

   - $\Rightarrow$ Suppose W is well ordered. We want to show that there does not exists a descending chain. Suppose for contradiction that there is a sequence $(w_n)_{n\in\mathbb{N}}$ where $w_n > w_{n+1}$. Since W is well order than $\min(w_n)$ exists. Since (w_n) has a minimum then there exists $n' \in \mathbb{N}$ such that for all $n \in \mathbb{N}, w_{n'} \leq w_n$. Note that $w_{n'+1} < w_{n'}$ by the definition of (w_n). Therefore $w_{n'+1} < w_{n'}$ and $w_{n'+1} \geq w_{n'}$. This is a contradiction. Therefore a descending chain does not exists.
   - $\Leftarrow$ Suppose W is not well ordered. We want to show that there exists a descending chain. Since W is not well ordered then there exists a set S where $S \neq \emptyset, S \subseteq W$ where $\min S$ does not exists. Since S is nonempty then there exists $x_1 \in S$. Note that $\min\{x_1\}$ exists, therefore there exists $x_2 \in S$ such that $x_2 < x_1$. Therefore by induction $\{x_1, \cdots, x_n\} \subset S$, since $\min\{x_1, \cdots, x_n\}$ exists. Therefore there exists $x_{n+1} \in S$ such that $x_{n+1} < x_n$. Therefore by induction we have constructed a descending chain.

2.

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7. - We will show that $\Phi : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ given by $\Phi(A) = \sum_{n\in A} \frac{2}{3^n}$ is an injection. Suppose $A, B \in \mathcal{P}(\mathbb{N}), A \neq B$. Since $A \neq B$ then the symmetric difference $A\Delta B$ is nonempty. Since $A\Delta B \subseteq \mathbb{N}$, then $A\Delta B$ has a minimal element. Let this element be denoted m, and WLOG assume $m \in A$. If we consider $\Phi(A) - \Phi(B) = \sum_{n\in A} \frac{2}{3^n} - \sum_{n\in B} \frac{2}{3^n} = \sum_{n\in A\setminus[m-1]} \frac{2}{3^n} - \sum_{n\in B\setminus[m-1]} \frac{2}{3^n}$ since A and B must share every element less than m, otherwise contradicting m is the smallest element in $A\Delta B$. Note that we can bound the difference below by considering $\sum_{n\in A\setminus[m-1]} \frac{2}{3^n} \geq \frac{2}{3^m}, -\sum_{n\in B\setminus[m-1]} \frac{2}{3^n} \geq -\sum_{n\in\mathbb{N}\setminus[m-1]} \frac{2}{3^n}$

8.

9.

10. (a) We claim that $card((\{0,1\}^{\mathbb{N}})^{\mathbb{N}}) = card(\{0,1\}^{\mathbb{N}\times\mathbb{N}})$. Note that $(\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$ is the set of of sequences of infinite binary sequences. Therefore for a given $f \in (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$ we have that $f(n) = (b_{nk})_{k\in\mathbb{N}}$ for all $n \in \mathbb{N}$. If we define $g : (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}) \to \{0,1\}^{\mathbb{N}\times\mathbb{N}}$ by $g(f) = (b_{nk})_{(n,k)\in\mathbb{N}^2}$, then this is clearly a bijection. Thus $card((\{0,1\}^{\mathbb{N}})^{\mathbb{N}}) = card(\{0,1\}^{\mathbb{N}\times\mathbb{N}})$. Therefore:

$$card(\mathbb{R}^{\mathbb{N}}) = card((\{0,1\}^{\mathbb{N}})^{\mathbb{N}}) = card(\{0,1\}^{\mathbb{N}\times\mathbb{N}}) = card(\{0,1\}^{\mathbb{N}}) = card(\mathbb{R})$$

.

(b) Let S be some countable set, and let $X = \{S^n : n \in \mathbb{N}\}$. We want to show that X is countable. Note that S is countable, and therefore S^n is countable by slide 22 of lecture 14. Since S^n and $\mathbb{N}$ is countable, then $\bigcup_{n\in\mathbb{N}}^{\infty} S^n$ is countable. Since $\bigcup_{n=1}^{\infty} S^n = X$, then we're done.

(c) Note that a polynomial is uniquely determined by it's coefficients. Therefore the set of polynomials over $\mathbb{Z}$ has the same cardinality as all of the finite integer sequences Thus $card(\mathbb{Z}[x]) = card(\{\mathbb{Z}^n : n \in \mathbb{N}\})$. Since $\{\mathbb{Z}^n : n \in \mathbb{N}\}$ is countable then $\mathbb{Z}[x]$ is countable

(d) Note that since $\mathbb{Z}[x]$ is countable and for $p \in \mathbb{Z}[x]$ the set $r(p) = \{p(x) = 0 : x \in \mathbb{R}\}$ is finite, then $\bigcup_{p\in\mathbb{Z}[x]} r(p)$ is countable. Note that this is exactly the set of algebraic numbers. Additionally, since we've found a countable subset of the real numbers, then there are an uncountable number of numbers not in our set. Thus real algebraic numbers exists

(e) Suppose for contradiction that there is a finite number of prime numbers. Let the set of primes be denoted $\{p_1, \cdots, p_n\}$. Consider the number $l = 1 + \prod_{i=1}^{n} p^i$. Note that for each prime $p_i, l \equiv 1 \mod p_i$. Thus l is divisible by none of the prime numbers. Since l can't be divide by primes, it can't be divide by the product of any of the primes. Thus l is only divisible by 1 and itself. Thus l is prime. This contradicts the fact that $\{p_1, \cdots, p_n\}$ is the set of all primes. Thus there is an infinite number of primes.