1. Let V be finite dimensional where $dim(V) = n$ and let $T : V \to V$ be linear.

   (a)
   - Suppose $rank(T) = rank(T^2)$, and let $rank(T) = m$. We must show that $R(T) \cap N(T) = \{0\}$. Suppose for contradiction that $R(T) \cap N(T) \neq \{0\}$. Therefore there exists $v_1 \in V$ such that $v_1 \in R(T) \cap N(T)$. Since $v_1 \in R(T)$, then we may extend v_1 to be a basis of $R(T)$, where $\{v_1, \dots, v_m\}$ is a basis for $R(T)$. Since $R(T^2) = \{T(v) : v \in R(T)\}$ by definition, then we may apply the book proof of rank nullity and have that $\{T(v_1), \dots, T(v_m)\}$ is a basis for $R(T^2)$. However since $T(v_1) = 0$ as $v_1 \in N(T)$ then $\{T(v_2), \dots, T(v_m)\}$ is a basis for $R(T^2)$, thus $rank(T^2) = m - 1$. This is a contradiction as $rank(T^2) = rank(T) = m$, therefore $R(T) \cap N(T) = \{0\}$.
   - We must show that $R(T) \oplus N(T) = V$. Since $R(T) \cap N(T) = \{0\}$ we must show that $R(T) + N(T) = V$. Since $dim(N(T) + R(T)) = dim(N(T)) + dim(R(T)) - dim(R(T) \cap N(T)) = n + n - m - 0 = n$. Since $N(T) + R(T)$ is a subspace of V and has the same dimension of V then $N(T) + R(T) = V$.

   (b) We must show that $R(T^k) \oplus N(T^k)$ for some $k \in \mathbb{N}$. Since $R(T) \oplus N(T)$, then the base case has been demonstrated. By the principle of mathematical induction for all $j \in \mathbb{N}$ if $j < k$ then $R(T^j) \oplus N(T^j)$. Since $k - 1 < k$ then by the induction hypothesis $R(T^{k-1}) \oplus N(T^{k-1}) = V$. Let $v_1, \dots v_m$ be a basis for $R(T^{k-1})$, and $v_{m+1}, \dots, v_n$ be a basis for $N(T^{k-1})$. Note that since $v_1, \dots, v_m$ is a set of linearly independent vectors not in the kernel, then by the book proof of the rank nullity theorem $\{T(v_1), \dots, T(v_m)\}$ is a basis for $R(T^k)$. Since $N(T^k) = N(T \circ T^{k-1})$ and $R(T) \cap N(T) = \{0\}$ then only vectors in the set $N(T^k)$ must belong to $N(T^{k-1})$ otherwise that implies there are vectors in the range of T that are also in it's kernel. Since we already have a basis for $N(T^{k-1})$, those same vectors will be the basis for $N(T^k)$. Assume for contradiction that $N(T^k) \cap R(T^k) \neq \{0\}$, and WLOG $T(v_1) \in N(T^k) \cap R(T^k)$. Since $T(v_1) \in N(T^k)$ then $T(v_1) \in T(^{k-1})$. Therefore $T(v_1) = a_{m+1}v_{m+1} + \dots a_n v_n$. Since $v_{m+1}, \dots, v_n$ are elements in the kernel then $T(T(v_1)) = 0$. This contradicts the fact that $R(T) \cap N(T) = \{0\}$. Therefore $N(T^k) \cap R(T^k) = \{0\}$. Since they have an empty intersection, then by the rank nullity theorem their sum has the same dimension as V. Thus their direct sum is V.

   Note that since $R(T) \cap N(T) = \{0\}$ then none of the vectors comprising the basis for $R(T^k)$ exists in the nullity, .

   Since we

2. Let A and B be $n \times n$ matrices such that AB is intvertible.

   (a) We must show that A and B are invertible. Since AB is invertible then L_{AB} is a bijection from $F^n \to F^n$ by defintion. By definition of matrix multiplication $L_{AB} = L_A \circ L_B$. Since L_{AB} is a composition and bijective then each function in the composition must be bijective. Since L_A and L_B are bijective then A and B must be invertible by definition.

   (b) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Since A, B are non-square then they cannot be inverted by definition. However $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Since $AB = \mathbb{I}_2$ then AB is invertible.

3. Let V and W be n-dimensional vector spaces, and let $T : V \to W$ be a linear transformation. Suppose that β is a basis for V. Prove that T is an isomorphism if and only if $T(\beta)$ is a basis for W.

   - ($\Rightarrow$) Suppose that T is an isomorphism. We must show that $T(\beta)$ is a basis for W. Since T is an isomorphism, then by definition $N(T) = \{0\}$. Since β is a set of n linearly independent vectors, then by the book proof of rank nullity $\{T(v) : v \in \beta\}$ is a basis for $R(T)$. Since $dim(W) = dim(V) = n$ and T is an isomorphism then $R(T) = W$. Since $T(\beta)$ is a basis for $R(T)$, then $T(\beta)$ is a basis for W.
   - ($\Leftarrow$) Suppose that $T(\beta)$ is a basis for W. We must show that T is an isomorphism. Since $T(\beta)$ is a basis for W and it is in the range of T then $R(T) = W$. Since $dim(W) = dim(V)$ then T is also one to one and onto. Therefore since T is also linear then T is an isomorphism.

4. Let B be an $n \times n$ invertible matrix. Define $\Phi : M_{n\times n}(F) \to M_{n\times n}(F)$ by $\Phi(A) = B^{-1}AB$. We must show that Φ is an isomorphism.

   - We must show that Φ is linear. Suppose $R, S \in M_{n\times n}(F), c \in F$, therefore:

$$\begin{aligned}\Phi(R + cS) &= B^{-1}(R + cS)B \\ &= (B^{-1}R + B^{-1}cS)B \\ &= B^{-1}RB + cB^{-1}SB \\ &= \Phi(R) + c\Phi(S).\end{aligned}$$

   - We must show for arbitrary $A \in M_{n\times n}(F)$ that there exists a unique $D \in M_{n\times n}(F)$ such that $\Phi(D) = A$. We claim that $D = BAB^{-1}$. Therefore we have that $\Phi(D) = B^{-1}DB^{-1} = B^{-1}BAB^{-1}B = A$. Note that since the inverse of B is unique then our choice for D is unique. Therefore Φ is a bijection.

   Therefore Φ is an isomorphism.