1.2.3 Suppose $G = \cup_l I_l, a_{k,l} = \int_{I_l} |c_k(x)|^p dx$. Let $\lim_{k\to\infty} a_{k,l} = b_l$. Then

$$|\sum_l a_{k,l} - \sum_l b_l| \leq |\sum_{i=1}^{N} a_{k,l} - b_l| + |\sum_{l=N+1}^{\infty} a_{k,l}| + |\sum_{l=N+1}^{\infty} b_l|.$$

Note since both $\sum_{l=1}^{\infty} a_{k,l}, \sum_{l=1}^{\infty} b_l$ converge then necessarily there must exist respective $N_1, N_2 \in \mathbb{N}$ such that for all $N \geq N_1, N_2, |\sum_{l=N+1}^{\infty} a_{k,l}| < \epsilon, |\sum_{l=N+1}^{\infty} b_l| < \epsilon$. Additionally since $|\sum_l a_{k,l} - \sum_l b_l| \leq \sum_{l=1}^{N} \int_{I_l} |c_k(x) - \phi(x)|^p dx$, and since $\lim_{k\to\infty} |c_k - \phi|_{L^p[a,b]} \to 0$ then there exists $K > 0$ such that for all $k \geq K, \sum_{l=1}^{N} \int_{I_l} |c_k(x) - \phi(x)|^p dx < \epsilon$. Therefore the entire sum is bounded above by 3ϵ. Since we have proved the limit interchange, we know that $\int_G |\phi(x)|^p dx = \lim_{k\to\infty} \int_G |c_k(x)|^p dx$. Therefore since it was already proven that the integrals of Cauchy sequences satisfy the UAC property, then for the ϵ already given, there exists $\delta > 0$ such that for arbitrary open $|G| < \delta, \int_G |c_k(x)|^p dx < \epsilon$. Therefore by taking the limit we find that $\int_G |\phi(x)|^p dx < \epsilon$

1.2.4 Consider $b_{k,l}$ where if $k \leq l$ then $b_{k,l} = 2^{k-l}$, otherwise if $k > l, b_{k,l} = 0$. Note for an arbitrary fixed k the sum $\sum_{l=k}^{\infty} 2^{k-l} = 2$. Note that since this is true for any k then $\lim_{k\to\infty} \sum_l b_{k,l} = 2$. However, if the limit is taken on the inside first then we get that $\lim_{k\to\infty} b_{k,l} = 0$. Therefore $\sum_l \lim_{k\to\infty} b_{k,l} = 0$

1.2.5 Consider the function

$$c_k(x) = \begin{cases} 12k^3x^2 & x \in [0, 1/2k] \\ 12k^3(x - \frac{1}{k})^2 & x \in [1/2k, 1/k] \\ 0 & \text{otws} \end{cases}$$

At $x = 0, c_k(0) = 12k^3 0^2 = 0$, and if $x \in (0, 1]$ then there exists $K \in \mathbb{N}$ such that $\frac{1}{k} < x$ which ensures for all $k \geq K$, $c_k(x) = 0$. Additionally, for every $k \in \mathbb{N}$ we get that

$$\int_0^1 c_k(x)dx = \int_0^{\frac{1}{2k}} 12k^3x^2 dx + \int_{\frac{1}{2k}}^{\frac{1}{2k}} 12k^3(x - \frac{1}{k})^2 = \frac{1}{2} + \frac{1}{2} = 1.$$

Therefore the integral limit swap does not work. Additionally by construction we have that $\int_0^{\frac{1}{k}} f(x)dx = 1$, therefore f is not UAC since we can take the arbitrarily small neighborhood of $(0, 1/k)$ and still have an integral of 1.

1.2.10 Let $f \in L^1[a, b]$, with $x \notin [a, b]$ taking on the value $f(x) = 0$. We want to show that $\lim_{h\to 0} \int_a^b |f(x + h) - f(x)|dx = 0$. Let $\epsilon > 0$ be given and $\int_a^b |f(x)|dx = M$. Since $f \in L^1[a, b]$ then $|f| \in R[a, b]$. Therefore there exists a partition P_1 such that $|U(|f|, P_1) - L(|f|, P_1)| < \epsilon$ Additionally note that

$$\int_a^b |f(x) - f(x + h)|dx \leq \int_a^b |f(x)|dx + \int_a^b |f(x + h)|dx \leq 2M.$$

Therefore $|f(x) - f(x+h)| \in R[a, b]$. Applying the previous fact, there exists a partition $P_2 = \{y_0, \cdots, y_m\}$ such that for arbitrary $s_i \in [y_{i-1}, y_i]$,

$$|\sum_{i=1}^{m} |f(s_i) - f(s_i + h)|\Delta y_i - \int_a^b |f(x) - f(x + h)|dx| < \epsilon.$$

Let $P = P_1 \cup P_2$. Since P is a refinement of both P_1 and P_2 then both properties hold for P. For $P = \{x_0, \cdots, x_n\}$, $s_i \in [x_{i-1}, x_i]$, and $\Delta x_i = x_i - x_{i-1}$ with a bit of rearrangement we get that

$$\int_a^b |f(x) - f(x+h)|dx < \epsilon + |\sum_{i=1}^n ||f(s_i)| - |f(s_i + h)||\Delta x_i|.$$

Since there is a finite number of Δx_i's, we can choose $d = \min\{\Delta x_1, \cdots \Delta x_n\}$. Therefore if we choose $c_i's$ such that $c_i = (x_i - x_{i-1})/2$ and $|h| < d/2$ then that guarantees that both c_i and $c_i + h$ will be within their given $[x_{i-1}, x_i]$. Therefore for every $||f(c_i)| - |f(c_i + h)| \leq M_i - m_i$, and if $c_i + h < a$ or $c_i + h > b$ then $|f(c_i)| \leq M_i$, which ensures that

$$|\sum_{i=1}^n ||f(s_i)| - |f(s_i + h)||\Delta x_i| \leq |U(|f|, P) - L(|f|, P)| < \epsilon$$

. Therefore $\int_a^b |f(x) - f(x+h)|dx < 2\epsilon$.

1.2.12 Consider $f \in L^1[a, b]$ with the extension that $x \notin [a, b]$ corresponding to $f(x) = 0$. Additionally let $\{c_k\} \subset C[a, b]$ be a Cauchy sequence for which $|c_k|_{L^1[a,b]} \to f$, $f_h(x) = h^{-1}\int_x^{x+h} f(y)dy, c_{k,h} = h^{-1}\int_x^{x+h} c_{k,h}(y)dy, C_k(x) = \int_0^x c_k(y)dy$. Then one can write

$$\int_a^b |f_h(x) - f(x)|dx \leq \int_a^b |f_h(x) - c_{k,h}(x)|dx + \int_a^b |c_{k,h}(x) - c_k(x)|dx + \int_a^b |c_k(x) - f(x)|.$$

Note that for both $\int_a^b |f_h(x) - c_{k,h}(x)|dx, \int_a^b |c_k(x) - f(x)|$ for sufficently large k, one has that $|c_k(x) - f(x)| < \epsilon$. Since $f_h(x) - c_{k,h}(x) \leq h^{-1}\int_x^{x+h} |f(x) - c_k(x)|dx < h^{-1}\epsilon(x + h - x) = \epsilon$, then both terms are bounded above by $\epsilon(b - a)$. For the term $c_{k,h}(x) - c_k(x)$, since $c_{k,h}(x) = h^{-1}\int_x^{x+h} c_k(y)dy = \frac{C(x+h) - C(x)}{h}$, and since $c_k \in C[a, b]$ implies that $C_k(x)$ is differentiable. Therefore there exists $x' \in (x, x+h)$ such that $\frac{C_k(x+h) - C_k(x)}{h} = c_k(x')$ by the mean value theorem. Therefore for sufficiently small h, by the continuity of c_k, $|c_k(x') - c_k(x)| < \epsilon$ bounding $\int_a^b |c_{k,h}(x) - c_k(x)|dx < \epsilon(b - a)$. Thus $\int_a^b |f_h(x) - f(x)|dx < 3(b - a)\epsilon$.