1. Suppose V, W are vector spaces over F and $T : V \to W$ is a linear transformation.

   (a) We must show that T is 1-1 if and only if T maps linearly independent subsets of V is linearly independent subsets of W.

   - ($\Rightarrow$). Suppose T is 1-1, and the set $S \subset V$ is linearly independent. We must show that the set $\{T(\vec{s}) : \vec{s} \in S\}$ is linearly independent. Suppose for contradiction $\{T(\vec{s}) : \vec{s} \in S\}$ is linearly dependent. Then by definition there exists $a_1, \ldots, a_n \in F$, and $\vec{s}_1, \ldots, \vec{s}_n \in S$ such that $\sum_{i=1}^n a_i T(\vec{s}_i) = 0$. Since T is linear we have that $T(\sum_{i=1}^n a_i \vec{s}_i) = 0$, and therefore since T is 1-1 and linear $\sum_{i=1}^n a_i \vec{s}_i = 0$. This is a contradiction as $\{\vec{s}_1, \ldots, \vec{s}_n\} \subseteq S$, and thus are linearly independent. Therefore $\{T(\vec{s}) : \vec{s} \in S\}$ is linearly independent.
   - ($\Leftarrow$) Suppose for all $S \subset V$ which are linearly independent $\{T(\vec{s}) : \vec{s} \in S\}$ is linearly independent. We must show that T is 1-1. Suppose for contradiction that T is not 1-1. Therefore $ker(T) \neq \{0\}$. Since $ker(T)$ is a subspace of V, then there exists a basis K for $ker(T)$. Since K is a basis, then it is a linearly independent subset of V. Therefore by definition of T, $\{T(\vec{v}) : \vec{v} \in K\}$ is linearly independent. This is a contradiction as every member of $\{T(\vec{v}) : \vec{v} \in K\}$ is $\vec{0}$. Therefore T is 1-1.

   (b) Suppose T is 1-1 and S is a subset of V We must show that S is linearly independent if and only if $T(S)$ is linearly independent.

   - ($\Rightarrow$) Since T is 1-1 and S is a linearly independent subset then $T(S)$ is linearly independent by the proof of (a) above.
   - ($\Leftarrow$) Suppose $T(S)$ is linearly independent. We must show that S is linearly dependent. Suppose for contradiction that S is linearly independent. Then there exists $\vec{s}^* \in S$ such that $\vec{s}^* = \sum_{i=1}^n a_i \vec{s}_i$ where $a_1, \ldots, a_n \in F, \vec{s}_1, \ldots, \vec{s}_n \in S$. Therefore $T(\vec{s}^*) = T(\sum_{i=1}^n a_i \vec{s}_i)$. Since T is $1-1$ then $\sum_{i=1}^n a_i T(\vec{s}_i) = T(\vec{s}^*)$, $\sum_{i=1}^n a_i T(\vec{s}_i) - T(\vec{s}^*) = 0$. Since we have found a linearly dependent subset of $T(S)$, then $T(S)$ is linearly dependent. This is a contradiction. Therefore S is linearly independent.

   (c) Suppose $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for V and T is 1-1 and onto. Let $dim(V) = n$. We must show that $T(\beta) = \{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ is a basis for W. Since T is 1-1 then $nullity(T) = 0$. Therefore by the rank nullity theorem $0 + rank(T) = dim(V)$. Therefore $rank(T) = dim(V)$. Since T is onto then $range(T) = W$, therefore $rank(T) = dim(W)$. Since β is a linearly independent subset of V, then $T(\beta)$ is linearly independent. Since $T(\beta)$ is a linearly independent subset of W with n vectors, then $T(\beta)$ is a basis for W.

2. Let V be the vector space of sequences. Define the function $T, U : V \to V$ by

$$T(a_1, a_2, \ldots) = (a_2, a_3, \ldots) \text{ and } U(a_1, a_2, \ldots) = (0, a_1, a_2, \ldots).$$

(a) Prove that T and U are linear. Suppose $a = (a_1, a_2, \ldots), b = (b_1, b_2, \ldots) \in V, c \in F$.

- We must show that T is linear, therefore by algebraic manipulation:

$$\begin{aligned} T(a + cb) &= T((a_1, \ldots) + c(b_1, \ldots)) \\ &= T(a_1 + cb_1, a_2 + cb_2, \ldots) \\ &= (a_2 + cb_2, \ldots) \\ &= (a_2, a_3, \ldots) + c(b_2, b_3, \ldots) \\ &= T(a) + cT(b). \end{aligned}$$

- We must show that U is linear, therefore by algebraic manipulation:

$$\begin{aligned} U(a + cb) &= T((a_1, \ldots) + c(b_1, \ldots)) \\ &= U(a_1 + cb_1, a_2 + cb_2, \ldots) \\ &= (0, a_1 + cb_1, a_2 + cb_2, \ldots) \\ &= (0, a_1, a_2, a_3, \ldots) + c(0, b_1, b_2, b_3, \ldots) \\ &= U(a) + cU(b). \end{aligned}$$

(b) We must show that T is onto and not 1-1.

- We must show that T is onto. Suppose $s = (s_1, s_2, \ldots) \in V$. We must show there exists $a = (a_1, a_2, \ldots) \in V$ such that $s = T(a)$. We claim that $(a_1, a_2, a_3, \ldots) = (0, s_1, s_2, \ldots)$. Therefore we have

$$\begin{aligned} T(a) &= T(0, s_1, s_2, \ldots) \\ &= (s_1, s_2, \ldots). \end{aligned}$$

  Therefore T is onto.
- We must show that T is not 1-1. Therefore we must show there exists $a = (a_1, \ldots), b = (b_1, b_2, \ldots) \in V$ such that $T(a) = T(b)$ and $b \neq a$. Suppose $a = (0, s_1, s_2, \ldots)$ and $b = (1, s_1, s_2, \ldots)$. Clearly $a \neq b$, and $T(a) = T(0, s_1, s_2, \ldots) = (s_1, s_2, \ldots) = T(1, s_1, s_2, \ldots) = T(b)$. Therefore T is not 1-1.

(c) We must show that U is 1-1 and not onto.

- We must show that U is 1-1. Suppose $a = (a_1, a_2, \ldots), b = (b_1, b_2, \ldots) \in V, U(a) = U(b)$. We must show that $a = b$. Since $U(a) = U(b)$, then by definition $(0, a_1, a_2, \ldots) = (0, b_1, b_2, \ldots)$. Therefore by definition of sequence $a_i = b_i$ for all $i \in \mathbb{N}$. Therefore $(a_1, a_2, \ldots) = (b_1, b_2, \ldots)$.
- We must show that U is not onto. We claim that $(1, 0, 0, \ldots)$ is not in the range of U. Since every sequence in the range of U takes the form $(0, s_1, s_2, \ldots)$, and the sequence $(1, 0, \ldots)$ has a 1 in the first position, then $(1, 0, 0, \ldots)$ is not in the range of U. Therefore U is not onto.

3. Prove that the subspaces $\{0\}, V, R(T), N(T)$ are T-invariant.

    (a) Suppose $\vec{x} \in \{0\}$. We must show that $T(x) \in \{0\}$. Since $\vec{x}, \{0\}$ then $x = 0$ Since T is linear then $T(0) = 0$. Therefore $T(x) \in \{0\}$.

    (b) Suppose $\vec{x} \in V$. We must show that $T(\vec{x}) \in V$. Since by definition $range(T) \subseteq V$, therefore V is T-invariant.

    (c) Suppose $\vec{x} \in range(T)$. We must show that $T(\vec{x}) \in range(T)$. Since by definition of T, $range(T) \subseteq V$. Therefore $\vec{x} \in V$. Therefore by definition of the range $T(\vec{x}) \in range(T)$. Therefore $range(T)$ is T-invariant

    (d) Suppose $\vec{x} \in ker(T)$. We must show that $T(\vec{x}) \in ker(T)$. Since $\vec{x} \in ker(T)$, then $T(\vec{x}) = \vec{0}$. Since T is linear then $T(\vec{0}) = \vec{0}$. Therefore $\vec{0} \in ker(T)$. Thus $T(\vec{x}) \in ker(T)$. Therefore $ker(T)$ is T-invariant.

4. Let V, W be vector spaces and let $T, U : V \to W$ be non-zero linear transformations. If $range(T) \cap range(U) = \{\vec{0}\}$ then show that T, U form a linearly independent subset of $\mathcal{L}(V, W)$. Suppose for contradiction that $\{T, U\}$ is linearly dependent. Therefore there exists non-zero constants $c_1, c_2 \in F$ such that $(c_1T + c_2U)(x) = 0$ for all $x \in V$. Therefore by definition $T(x) = \frac{-c_2}{c_1}U(x)$. Suppose $z = T(y)$, therefore $z = \frac{-c_2}{c_1}U(y) = U(\frac{-c_2}{c_1}y)$. Therefore $z \in range(U) \cap range(T)$. This is a contradiction, therefore $\{T, U\}$ is linearly independent.

5. Let V and W be vector spaces such that $dim(V) = dim(W)$, and let $T : V \to W$ be linear. Show there exists ordered basis β and γ for V and W, respectively, such that $[T]_\beta^\gamma$ is a diagonal matrix.

   Let $\beta = (\vec{v}_1, \dots, \vec{v}_n)$ be the ordered basis such that $\{\vec{v}_{i+m} : i \in [n-m]\}$ is a basis for $ker(T)$, and therefore $\{T(\vec{v}_i) : i \in [m]\}$ is a basis of $R(T)$ by the rank nullity theorem. Let the ordered basis γ be given by $T(\vec{v_i}) = \vec{w}_i$ for all $i \in [m]$, and the vectors $\vec{w}_{m+1}, \dots, \vec{w}_n$ be the extension to all of V. Now we must define the matrix $[T]_\beta^\gamma$, since T has a kernel, we will look at the vectors in β which do an don't get mapped into the kernel as seperate cases.

   - For all $j \in [m]$, we can define $a_{ij} = \delta_{i,j}$ such that $T(\vec{v}_j) = \sum_{i=1}^n a_{ij}\vec{w}_i = 0 \cdot \vec{w}_1 + \cdots + 0 \cdot \vec{w}_{j-1} + 1 \cdot \vec{w}_j + 0 \cdot \vec{w}_{j+1} + \cdots + 0 \cdot \vec{w}_n = \vec{w}_j$ to satisfy the definition of γ.
   - For all $j \in [n-m]$ since $T(v_{j+m}) = \vec{0}$, then $a_{ij} = 0$ is a unique representation as $T(v_{j+m}) = 0 = \sum_{i=1}^n a_{ij}\vec{w}_i$ being solved by anything but all scalars being 0 violates the linear independence of γ.

   Since the only non-zero scalars lie on the diagonal, then $[T]_\beta^\gamma$ is a diagonal matrix.