Let P be a partial order on the finite set X. A *chain* in P is a totally ordered subset. Let $c(P)$ be the size of the largest chain of P. An *antichain* A in P is a totally unordered subset, meaning that no two elements of A are comparable in the partial order. An *antichain partition* of P is a partition of X each of whose parts is an antichain. Let $\alpha(P)$ be the smallest number of parts in any antichain partition. The purpose of this problem is to prove the following interesting theorem: For any finite partially ordered set P, $c(P) = \alpha(P)$.

Hint Lemma: For each $x \in X$ define $h(x)$ to be the size of the largest chain that has x as its maximum element. Prove that for any integer j, $preim_h(j)$ is an antichain.

Proof: Let the function $h : X \to \mathbb{Z}_{\geq 0}$ be given by $h(x)$ is the size of the largest chain that has x as it's maximum element. We must show for all $j \in \mathbb{Z}_{\geq 0}$ that $preim_h(j)$ is an antichain. Suppose $j \in \mathbb{Z}_{\geq 0}$. We must show $preim_h(j)$ is an antichain. Assume for contradiction that $preim_h(j)$ is not an antichain. Then by definition of h we have a set of maximum elements of chains $M_j = \{m_{j1}, \ldots, m_{js}\}$. Since M_j is not an antichain then there exists $m_{jl}, m_{jk} \in M_j$ such that without loss of generality $m_{jl} \leq_P m_{js}$. Then by definition of chain m_{js} is the maximum element of the chain containing m_{jl}. This is a contradiction as $h(m_{js}) = j + 1$, but $h(m_{js})$ was defined to be j. Therefore $preim_h(j)$ is an antichain.

1. Prove $\alpha(P) \geq c(P)$.
   Suppose P is an arbitrary partial order on X. We must show that $\alpha(P) \geq c(P)$. Since $\alpha(P)$ is the minimum number for an anti chain, then by defininition of minimum, for all antichain partitions Π, $|\Pi| \geq \alpha(P)$. Since $c(P)$ is the maximum length of the chain then for all chains $C, c(P) \geq |C|$. Therefore we must show for all partitions Π and chains C that $|\Pi| \geq |C|$. Assume for contradiction that $|\Pi| < |C|$. Then since $|\Pi| < |C|$ there exists a part of the partition Π that has more than one element from a chain C. This is a contradiction as the parts of Π must be an antichain. Therefore $\alpha(P) \geq c(P)$.

2. Prove $\alpha(P) \leq c(P)$. For those who want a hint, see the footnote. (Acknowledge the hint if you use it.)[1]
   We must show for all posets P on X that $\alpha(P) \leq c(P)$. By the principal of mathematical induction for all posets Q of size k, if $k < n$, then $\alpha(Q) \leq c(Q)$. Let S denote the set of minimal elements of P, since P is finite S is guaranteed to be non-zero. We claim that $\alpha(P \backslash S) \leq c(P) - 1$. By the induction hypothesis we have that $\alpha(P \backslash S) \leq c(P \backslash S)$. Since $c(P)$ is the longest chain, and S contains the minimal elements of P, then the removal of just the minimal element of the longest chain is simply a decrement by one. Therefore $\alpha(P \backslash S) \leq c(P) - 1$. We claim that $\alpha(P) \leq \alpha(P \backslash S) + 1$. Since $S = preim_h(1)$, then S is an antichain. Therefore any antichain partition of $P \backslash S$ with S appended is a valid antichain partition of P, and therefore it's size of $\alpha(P \backslash S) + 1$ is bounded by $\alpha(P)$, thus $\alpha(P) \leq \alpha(P \backslash S) + 1$. Therefore since $\alpha(P) \leq \alpha(P \backslash S) + 1$ and

[1] For each $x \in X$ define $h(x)$ to be the size of the largest chain that has x as its maximum element. Prove that for any integer j, $preim_h(j)$ is an antichain. Use this to prove the theorem.

$\alpha(P\backslash S) \leq c(P) - 1$, then we can write:

$$\begin{aligned}
\alpha(P\backslash S) &\leq c(P) - 1 \\
\alpha(P\backslash S) + 1 &\leq c(P) \\
\alpha(P) &\leq c(P).
\end{aligned}$$