6.7 (a) We want to show for $f \in R[0,1]$ that $\int_0^1 f(x)dx = \lim_{c\to 0}\int_c^1 f(x)dx$. Note that $\int_0^1 f(x)dx - \lim_{c\to 0}\int_c^1 f(x)dx = \int_0^c f(x)dx$. Note that since as a function $F(x) = \int_0^x f(y)dy$ is continuous and that $\lim_{c\to 0}\int_0^c f(x)dx = \lim_{c\to 0} F(c) = F(\lim_{c\to 0} c) = F(0) = 0$ then equality holds.

(b) Consider the function $f(x)$ being defined piecewise on $x \in [\frac{1}{n+1}, \frac{1}{n}]$ such that $f(x) = (-1)^{n+1}(n+1)$. Note that since the function is simply a countable collection of constant functions that f is in fact Riemann integrable. Therefore if one takes the limit $\lim_{c\to 0}\int_c^1 f(x)dx = \sum_{i=1}^{\infty}\frac{(-1)^{n+1}}{n} = \log(2)$. However, if one considers $\int_0^1 |f(x)|dx$, all of the constant functions are made positive, and one is left with $\sum_{n=1}^{\infty}\frac{1}{n} \leq \int_0^1 |f(x)|dx$. Therefore the integral with the function $|f(x)|$ does not exists.

6.17 Suppose $P = \{x_0, \cdots, x_n\}$ is a partition of $[a,b]$, and choose $t_i \in [x_{i-1}, x_i]$ such that $g(t_i)\Delta x_i = G(x_i) - G(x_{i-1})$. Therefore one can apply the abel summation formula to the approximation to the integral $\int_a^b g(x)\alpha(x)dx, \sum_{i=1}^n g(t_i)\alpha(x_i)\Delta x_i$ as follows:

$$\begin{aligned}
\sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i &= \sum_{i=1}^n \alpha(x_i)(G(x_i) - G(x_{i-1})\\
&= \sum_{i=1}^n \alpha(x_i)G(x_i) - \sum_{i=1}^n \alpha(x_i)G(x_{i-1})\\
&= G(b)\alpha(b) + \sum_{i=1}^{n-1}\alpha(x_i)G(x_i) - \sum_{i=1}^n \alpha(x_i)G(x_{i-1})\\
&= G(b)\alpha(b) - G(a)\alpha(a) + \sum_{i=1}^n \alpha(x_{i-1})G(x_{i-1})\sum_{i=1}^n \alpha(x_i)G(x_{i-1})\\
&= G(b)\alpha(b) - G(a)\alpha(a) + \sum_{i=1}^n G(x_i)(\alpha(x_{i-1}) - \alpha(x_i))\\
&= G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^n G(x_i)\Delta\alpha_i
\end{aligned}$$

Note that in the limit the final sum on the right converges to $\int_a^b G(x)d\alpha(x)$. Additionally the original sum on the left converges to $\int_a^b g(x)\alpha(x)dx$. Thus $\int_a^b g(x)\alpha(x)dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G(x)d\alpha(x)$

7.12 Let $h_n(x) = f(x) - f_n(x)$. Note that since $f_n \leq g$ then $\int_0^\infty f_n(x)dx \leq \int_0^\infty g(x)dx < \infty$, and additionally the above inequality would imply that $\int_0^\infty f(x)dx \leq \int_0^\infty g(x)dx$. Therefore if we consider taking the integral and splitting it up we get $\lim_{n\to\infty}\int_0^\infty h_n(x)dx = \lim_{n\to\infty}\int_0^t h_n(x)dx + \int_t^T h_n(x)dx + \int_T^\infty h_n(x)dx$. First, independent of n if we take the limit as $t \to 0$ for $\int_0^t h_n(x)dx$ then as shown in exercise 6.7 (a) we can arbitrarily choose a t such that $\int_0^t h_n(x)dx < \epsilon/3$. Additionally for $\int_T^\infty h_n(x)dx$ since $\int_0^\infty f_n(x)dx = A_n$

and $\int_0^\infty f(x)dx = A$ then for each respective integral we can choose a T large enough such that $|A - A_n - \int_0^T h_n(x)| < \epsilon/3$ since they converge. Now, we know since $f_n \to f$ uniformly there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|f_n(x) - f(x)| < \frac{\epsilon}{3(T-t)}$ so that $\int_t^T h_n(x)dx \leq \int_t^T \frac{\epsilon}{3(T-t)}dx = \epsilon/3$. Therefore $\lim_{n\to\infty} \int_0^\infty h_n(x)dx = \epsilon$. Therefore $\lim_{n\to\infty} \int_0^\infty f_n(x) = \int_0^\infty f(x)dx$.

7.20 Note that by the Stone-Weierstrass theorem there exists a sequence of polynomials such that $\lim_{n\to\infty} P_n(x) = f(x)$ uniformly on $[a, b]$. Note that since the constituent parts of $P(n)$ are just mononomials, then $\int_0^1 P_n(x)f(x) = 0$. Therefore by theorem 7.16 in rudin, $0 = \lim_{n\to\infty} \int_0^1 P_n(x)f(x)dx = \int_0^1 f^2(x)$. Note that since $\int_0^1 f^2(x) = 0$ then $\bar{\int_0^1}|f(x)|^2 = 0$. This implies that the largest value on every interval of f is 0. Additionally since $f^2(x)$ is bounded below by zero implies that f is 0 at every point. Thus f is 0 on $[0, 1]$.

1.1.6

- $\overline{\int_{-1}^1} H(x)dH(x)$. Note for $y, x > 0$ that $H(x) - H(y) = 1 - 1 = 0$ and for $x, y < 0$ additionally $H(x) - H(y) = 0 - 0 = 0$. Therefore we only have to consider the partitions which either straddles the origin or contains the origin. So if $x_i < 0 < x_{i+1}$ then $sup_{[x_i,x_{i+1}]}H(x)(H(x_{i+1} - H(x_i)) = 1*1 = 1$. If we have $[x_i, x_{i+1}], x_{i+1} = 0$ then we have the same scenario of $sup_{[x_i,x_{i+1}]}H(x)(H(x_{i+1}) - H(x_i)) = 1*1 = 1$. Thus $\overline{\int_{-1}^1} H(x)dH(x) = 1$.
- $\underline{\int_{-1}^1} H(x)dH(x)$. Note by a similar construction above we have the scenario where the smallest value of the straddled interval is 0 and the smallest value of the interval on the origin is 0. Thus $\underline{\int_{-1}^1} H(x)dH(x) = 0$.
- $\overline{\int_{-1}^1} G(x)dH(x)$. Note that now if we choose to straddle the origin it matters in the case of the sum. If $x_i < 0 < x_{i+1}$ then the maximum value G attains on $[x_i, x_{i+1}]$ is 1, as opposed to having $x_{i-1} < x_i = 0 < x_{i+1}$, where on $x \in [x_{i-1}, x_i], G(x) = 0$. However since the upper sum chooses the smallest value here, then $\overline{\int_{-1}^1} G(x)dH(x) = 0$.
- $\underline{\int_{-1}^1} G(x)dH(x)$. Trivially for the lower sum, the only situation listed above in which the Stieltjets integral could be 1 is when taking the sup on the interval straddling the origin. Therefore taking the inf will result in 0. Thus all lower sums are 0, and the sup of the lower sums is 0. Therefore $\underline{\int_{-1}^1} G(x)dH(x) = 0$

Since both the upper and lower sums agree for $\int_{-1}^1 G(x)dH(x)$ then $G(x) \in R(dH)$.