6.2.1 Exercise 3. Let $\epsilon > 0$. Then there exists $k \in \mathbb{N}$ such that $\frac{1}{2^k} < 2\epsilon$. Let $\mathcal{P}_y = \{s_0, \cdots s_{2^k}\}$ be the partition of evenly spaced intervals of length $\frac{1}{2^k}$ for the y axis, where $I_{yi} = [s_{i-1}, s_i]$. Furthermore, for any $(x,y) \in [0,1]^2$, $0 \le f(x,y) \le \frac{1}{2}$. Therefore for any $U \subseteq [0,1]^2$, $osc(f,U) \le \frac{1}{2}$. Thus for an arbitrary $\mathcal{P}_x$ of the x axis with 2^k intervals denoted I_{xi}, $\sum_{i=1}^{2^k} osc(f, I_{xi} \times I_{yi})\frac{1}{2^k}|I_{xi}| \le \frac{1}{2^{k+1}}\sum_{i=1}^{2^k}|I_{xi}| = \frac{1}{2^{k+1}} < \frac{1}{2}2\epsilon = \epsilon$. Therefore the 2d thomae function $f(x,y)$ is Riemann integrable. Since the irrationals are dense in $[0,1]^2$, then for an arbitrary $I_x \times I_y \subseteq [0,1]^2$, $m(f, I_x \times I_y) = 0$, thus for an arbitrary partition $\mathcal{P}, L(f,\mathcal{P}) = 0$. Thus $\underline{\int}_{[0,1]^2} f = 0$, giving us that $\int_{[0,1]^2} f = 0$.

6.2.3 Exercise 1. Given that f, g are Riemann integrable then both of their sets of discontinuities are of measure 0. Then the union of those sets is also of measure 0. Since the discontinuities of $f \cdot g$ is at most the union of the previous sets, and that is measure zero implies the discontinuities of $f \cdot g$ is measure 0. Thus $f \cdot g$ is Riemann integrable.

6.2.3 Exercise 2. Note that for all x, $0 \le \sum_{i=2}^{\infty} \frac{x^i}{}i!$, therefore $x + 1 \le e^x$, and finally $\log(x+1) \le x$. Therefore since $\sum_{k=1}^{\infty} r_k < \infty$, then $\sum_{k=1}^{\infty}\log(1+r_k) < \infty$. This implies that $\prod_{k=1}^{\infty}(1+r_k) < \infty$. Therefore we get that $\prod_{k=1}^{\infty}(1-r_k) < \infty$. Note that by induction, if we remove ratio after ratio of the unit interval we get that $\mathcal{K}$ has length $\prod_{k=1}^{\infty}(1-r_k)$. If we consider the fact that $r_k \to 0$ as $k \to \infty$ implies that $1 - r_k$ will converge to 1. If $\prod_{k=1}^{\infty}(1-r_k) = 0$ then that would imply past some $K \in \mathbb{N}$, for all $k \ge K, 1 - r_k < r < 1$. However this is impossible for the afformentioned limit argumentation. Thus our given set isn't measure 0. Therefore, an upper sum can attain 1 for intervals of lengths which will sum to the length determined earlier by the product $\prod_{k=1}^{i} nfty(1-r_k)$. Additionally, the lower sums can trivially be made 0, as any open set can eventually have a multiple of $\frac{r_k}{2^k}$ put inside of it (end points of the k-th order cantor set process). Therefore the upper sums and the lower sums disagree, making $\chi_{\mathcal{K}}$ not Riemann integrable. Note that the character of $\mathcal{K}^c$ is also not Riemann integrable since it is equivalent to $1 - \chi_{\mathcal{K}}$, which is not Riemann integrable.

6.3.4 Note that the function defined as $f(x,y)$ in this problem is equivalent to $1 - f(y,x)$ as defined in Exercise 3 of 6.2.1 (first problem in the homework), and in this exercise f is shown to be Riemann integrable, therefore $1 - f(y,x)$ is Riemann integrable. Therefore the definition of f for this problem is Riemann integrable. Now to consider $\overline{\int}_0^1 f(x,y)dy$, for any possible partition of $\mathcal{P}$ for $[0,1]$ in y and any x, $U(f,\mathcal{P}) = 1$, as the irrationals are dense in $[0,1]$, therefore for any subinterval the max of 1 can always be attained. Therefore $\overline{\int}_0^1 f(x,y)dy = 1$. For $\underline{\int}_0^1 f(x,y)dy$, if $x = \frac{p}{q}, \gcd(p,q) = 1$ then in every possible interval of y for an arbitrary partition $\mathcal{P}$, $f(x,y) = 1 - \frac{1}{q}$, by the density of the rationals. Thus $L(f,\mathcal{P}) = 1 - \frac{1}{q}$. If x is irrational then $L(f,\mathcal{P}) = 1$ since every $f(x,y) = 1$. Thus $\underline{\int}_0^1 f(x,y)dy = \begin{cases} 1 - \frac{1}{q} & x = \frac{p}{q}, \gcd(p,q) = 1 \\ 1 & \text{otherwise} \end{cases}$. Therefore based off of this analysis it appears that $\int_0^1 f(x,y)dy$ is Riemann integrable for irrational x.

6.3.9 Suppose for contradiction that $g(x) \neq 0$ almost everywhere. Note that $0 = \overline{\int}_{R_1} g(\mathbf{x}) \ge \underline{\int}_{R_1} g(\mathbf{x}) \ge \min_{\mathbf{x} \in R_1} g(x)|R_1| \ge 0$, therefore $0 = \int_{R_1} g(\mathbf{x}) = \overline{\int}_{R_1} g(\mathbf{x}) = \underline{\int}_{R_1} g(\mathbf{x})$, making

g Riemann integrable on R_1. Since g is riemann integrable on R_1 then it is bounded, therefore there exists $m > 0$ such that $g(\mathbf{x}) \geq m$, as $g(\mathbf{x}) \geq 0, \neq 0$ by assumption. Thus, by definition of Riemann integrability, there exists $\delta > 0$ such that for any partition $\lambda(\mathcal{P}) < \delta$ we have that $\left|\sum_{S_\alpha \in \mathcal{P}} g(\mathbf{x}_\alpha)|S_\alpha|\right| < m|R_1|$. Since $g(x) \neq 0$ almost everywhere in R_1, then for each S_α we can choose an $\mathbf{x}_\alpha$ such that $g(\mathbf{x}_\alpha) \neq 0$. Therefore $m|R_1| \leq \left|\sum_{S_\alpha \in \mathcal{P}} g(\mathbf{x}_\alpha)|S_\alpha|\right| < m|R_1|$. This is a contradiction. Therefore $g(\mathbf{x}) = 0$ almost everywhere.