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350H

Let V, W be finite dimensional vector spaces, and $\mathcal{L}(V,W)$ be the set of all linear transforms $L : V \to W$. Note that cL where $c \in F$ and $L_1 + L_2$ where $L_1, L_2 \in \mathcal{L}(V,W)$ are linear transforms. Also note that when $L(x)$ is written for a linear transform, it is evaluating it's output for all elements in V, and represents an element in W. We must show that $\mathcal{L}$ is a vector space over F.

- Existence of the additive identity
  We claim that $0_{V,W} : V \to W$ given by $0_{V,W}(v) = 0$ for all $v \in V$ is the additive identity. Since $(0_{V,W} + L)(x) = 0_{V,W}(x) + L(x) = 0 + L(x) = L(x)$, then we have demonstrated that $0_{V,W}$ is the identity.

- Inverses for vectors
  We claim that $-L(x) := (-1)L(x)$ is the inverse of L. Since $(L - L)(x) = L(x) + (-1)L(x) = 0$ for all $x \in V$, then we found that $L - L = 0_{V,W}$. Therefore $-L$ is the inverse of L.

- Associativity
  Suppose $L_1, L_2, L_3 \in \mathcal{L}(V,W)$. Then we have $L_1(x) + (L_2 + L_3)(x) = L_1(x) + L_2(x) + L_3(x) = (L_1 + L_2)(x) + L_3(x)$. Thus we have associativity

- Commutativity
  Suppose $L_1, L_2 \in \mathcal{L}(V,W)$. Then we have $(L_1 + L_2)(x) = L_1(x) + L_2(x) = L_2(x) + L_1(x) = (L_2 + L_1)(x)$.

- Associativity of scalar and vector multiplication
  Suppose $a, b \in F, L \in \mathcal{L}(V,W)$. Then we have $a(bL)a(bL)(x) = abL(x) = (ab)L(x) = (ab)L$.

- Existance of the multiplicative field identity
  We claim that $1 \in F$ is the multiplicative scalar identity. Suppose $L \in \mathcal{L}(V,W)$. Since $1L = 1L(x) = L(x) = L$, then we have found the multiplicative identity.

- Distributivity of a scalar over vectors

  Suppose $L_1, L_2 \in \mathcal{L}(V,W), c \in F$. Then we have $c(L_1 + L_2)(x) = c(L_1(x) + L_2(x)) = cL_1(x) + cL_2(x)$.

- Distributivity of scalar multiplication with resepect to field addition
  Suppose $a, b \in F, L \in \mathcal{L}(V,W)$. Then we have $(a+b)L = (a+b)L(x) = aL(x)+bL(x) = aL + bL$.