1. (a) Suppose $\epsilon > 0$. Then there exists $N_\epsilon \in \mathbb{N}$ such that $\frac{1}{3}\left(\frac{7}{3\epsilon}+2\right) < N_\epsilon$. Therefore for $n \geq N_\epsilon$

$$\begin{aligned}\left|\frac{2n+1}{3n-2}-\frac{2}{3}\right| &= \left|\frac{6n+3-6n+4}{3(3n-2)}\right| \\ &= \frac{7}{3(3n-2)} \\ &< \epsilon\end{aligned}$$

(b) Suppose $\epsilon > 0$. Then there exists $N_\epsilon \in \mathbb{N}$ such that $\frac{2}{\epsilon} < N_\epsilon$. Therefore for $n \geq N_\epsilon$

$$\begin{aligned}\left|\frac{2n}{n^2+1}\right| &< \frac{2n}{n^2} \\ &= \frac{2}{n} \\ &< \epsilon\end{aligned}$$

(c) Suppose $\epsilon > 0$. Then there exists $N_{\epsilon,1,1} \in \mathbb{N}$ such that for all $n \geq N_{\epsilon,1,1}$, $\frac{n^1}{(1+1)^n} < \epsilon$ by the inequalities on slide 3 of the lecture 8 slides. Thus $|\frac{n}{2^n}| < \epsilon$.

(d) Suppose $\epsilon > 0$. Then there exists $N_\epsilon \in \mathbb{N}$ such that $\frac{2}{N_\epsilon} < \epsilon$. Therefore for $n \geq N_\epsilon$

$$\begin{aligned}\left|\frac{n^2-3n+1}{2n^2+n+1}-\frac{1}{2}\right| &= \left|\frac{-7n+1}{2(2n^2+n+1)}\right| \\ &< \left|\frac{1-7n}{4n^2}\right| \\ &= \frac{7n-1}{4n^2} \\ &= \frac{7}{4n}-\frac{1}{n^2} \\ &< \frac{7}{4n}+\frac{1}{4n} \\ &= \frac{2}{n} \\ &< \epsilon.\end{aligned}$$

(e) Suppose $\epsilon > 0$. Then there exists $N_{\epsilon,8,2010} \in \mathbb{N}$ such that for all $n \geq N_{\epsilon,8,2010}$,

$\frac{n^{2010}}{(1+8)^n} < 2\epsilon$ by the inequalities on slide 3 of the lecture 8 slides. Thus

$$\begin{aligned}
\left|\frac{3^n}{\sqrt{9^n+n^{2010}}}-1\right| &= \left|\frac{3^n-\sqrt{9^n+n^{2010}}}{\sqrt{9^n+n^{2010}}}\right| \\
&< \left|\frac{3^n-\sqrt{9^n+n^{2010}}}{3^n}\right| \\
&= \left|1-\sqrt{1+\frac{n^{2010}}{9^n}}\right| \\
&= \sqrt{1+\frac{n^{2010}}{9^n}}-1 \\
&\leq 1+\frac{n^{2010}}{2\cdot 9^n}-1 \text{ bernoulli inequality} \\
&< \epsilon
\end{aligned}$$

(a)

$$\begin{aligned}
\lim_{n\to\infty}\frac{5n^4+n^2-6}{3n^4+7} &= \lim_{n\to\infty}\frac{5+\frac{1}{n^2}-\frac{6}{n^4}}{3+\frac{7}{n^4}} \\
&= \frac{5+0-6\cdot 0}{3+7\cdot 0} \\
&= \frac{5}{3}
\end{aligned}$$

(b) Note that $0 < \frac{\sqrt[3]{n}}{1+\sqrt{n}} < n^{\frac{-1}{6}}$. Since $\lim_{n\to\infty} n^{\frac{-1}{6}} = 0$ (using the theorem on slide 3 of lecture slides 8) then by the squeeze theorem $\lim_{n\to\infty}\frac{\sqrt[3]{n}}{1+\sqrt{n}} = 0$

(c) Note that $\frac{2^7 n^{\frac{7}{2}}}{n^3(1+7\sqrt{n+2})} < \frac{(\sqrt{n+1}+\sqrt{n})^7}{n^3(1+7\sqrt{n+2})} < \frac{(\sqrt{n+1}+\sqrt{n})^7}{7n^{\frac{7}{2}}}$. Thus we will show be squeeze theorem that they both converge to $\frac{2^7}{7}$. First,

$$\begin{aligned}
\lim_{n\to\infty}\frac{2^7 n^{\frac{7}{2}}}{n^3(1+7\sqrt{n+2})} &= \lim_{n\to\infty}\frac{2^7}{n^{-\frac{1}{2}}+7\sqrt{1+\frac{2}{n}}} \\
&= \frac{2^7}{7}.
\end{aligned}$$

Second,

$$\begin{aligned}
\lim_{n\to\infty}\frac{(\sqrt{n+1}+\sqrt{n})^7}{7n^{\frac{7}{2}}} &= \lim_{n\to\infty}\frac{1}{7}\left(\sqrt{1+\frac{1}{n}}+1\right)^7 \\
&= \frac{2^7}{7}
\end{aligned}$$

Thus by squeeze theorem $\lim_{n\to\infty}\frac{(\sqrt{n+1}+\sqrt{n})^7}{n^3(1+7\sqrt{n+2})} = \frac{2^7}{7}$

(d)

$$\begin{aligned}\lim_{n\to\infty}\frac{\sqrt{3^n+2^n}}{\sqrt{3^n+1}} &= \lim_{n\to\infty}\frac{\sqrt{1+\left(\frac{2}{3}\right)}}{\sqrt{1+3^{-n}}}\\ &= \frac{\sqrt{1+0}}{\sqrt{1+0}}\\ &= 1\end{aligned}$$

(e)

$$\begin{aligned}\lim_{n\to\infty}\frac{7n+(\sqrt[3]{n}\sqrt[6]{n})\sqrt{9n+1}}{11n^3+7n+3} &= \lim_{n\to\infty}\frac{7n+\sqrt{9n^6+n^5}}{11n^3+7n+3}\\ &= \lim_{n\to\infty}\frac{\frac{7}{n^2}+\sqrt{9+\frac{1}{n}}}{11+7\frac{1}{n^2}+\frac{3}{n^3}}\\ &= \frac{3}{11}\end{aligned}$$

(a)

$$\begin{aligned}\lim_{n\to\infty}\frac{1-2+3-4+\cdots-2n}{\sqrt{n^2+2}} &= \lim_{n\to\infty}\frac{\sum_{i=1}^{2n}(-1)^{i+1}i}{\sqrt{n^2+2}}\\ &= \lim_{n\to\infty}\frac{\sum_{i=1}^{n}(2n-1)-2n}{\sqrt{n^2+2}}\\ &= \lim_{n\to\infty}\frac{-n}{\sqrt{n^2+2}}\\ &= \lim_{n\to\infty}\frac{-1}{1+\frac{2}{n^2}}\\ &= \frac{-1}{\sqrt{1+0}}\\ &= -1\end{aligned}$$

(b)

$$\begin{aligned}\lim_{n\to\infty}\frac{3^0+3^1+3^2+\cdots+3^n}{3^n} &= \lim_{n\to\infty}\frac{\sum_{i=0}^{n}3^i}{3^n}\\ &= \lim_{n\to\infty}\frac{1}{3^n}\frac{3^{n+1}-1}{2}\\ &= \lim_{n\to\infty}\frac{3}{2}-3^{-n}\\ &= \frac{3}{2}\end{aligned}$$

(c)

$$\begin{aligned}\lim_{n\to\infty}\frac{1+2+\cdots+n}{n} &= \lim_{n\to\infty}\frac{n(n+1)}{2n^2}\\ &= \lim_{n\to\infty}\frac{n+1}{2n}\\ &= \lim_{n\to\infty}\frac{1}{2}+\frac{1}{2n}\\ &= \frac{1}{2}\end{aligned}$$

(d) Note that $\frac{1}{n^2}+\frac{1}{n^2+1}+\cdots+\frac{1}{(n+1)^2}=\sum_{i=0}^{2n+1}\frac{1}{n^2+i}$ and $\sum_{i=0}^{2n+1}\frac{1}{n^2+2n+1}\le\sum_{i=0}^{2n+1}\frac{1}{n^2+i}\le\sum_{i=0}^{2n+1}\frac{1}{n^2}$. We will show that both of the limits go to 0:

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=0}^{2n+1}\frac{1}{n^2+2n+1} &= \lim_{n\to\infty}\frac{2n+2}{n^2+2n+1}\\ &= \lim_{n\to\infty}\frac{\frac{2}{n}+\frac{2}{n^2}}{1+\frac{2}{n}+\frac{1}{n^2}}\\ &= \frac{0+0}{1+0+0}\\ &= 0\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=0}^{2n+1}\frac{1}{n^2} &= \lim_{n\to\infty}\frac{2n+2}{n^2}\\ &= \lim_{n\to\infty}\frac{2}{n}+\frac{2}{n^2}\\ &= 0+0\\ &= 0\end{aligned}$$

(e) Note by part (d) on slide 3 of lecture slides 8 that for $\alpha\in\mathbb{R}, x\in\mathbb{N}$ that $\lim_{n\to\infty}\frac{n^\alpha}{(1+x)^\alpha}=0$. Therefore since $2\in\mathbb{N}$ and $100\in\mathbb{N}$ then $\lim_{n\to\infty}\frac{n^{100}}{(1+2)^n}=0$. Thus $\lim_{n\to\infty}\frac{n^{100}}{3^n}=0$.

2. (a) Note that for all $a,b\in\mathbb{R}$, $(a-b)^2\ge 0$, thus $a^2+b^2\ge 2ab$. Thus $(a+b)^2=a^2+2ab+b^2\le 2a^2+2b^2$.

(b) Note that $\frac{1}{2}(\frac{1}{a}+\frac{1}{b})\ge\frac{2}{a+b}$. Therefore $\frac{1}{a}+\frac{1}{b}\ge\frac{4}{a+b}$.

(c) Note that by AM-GM, we have that $\left(\frac{a+b}{2}\right)\ge\sqrt{ab}, \left(\frac{b+c}{2}\right)\ge\sqrt{bc}, \left(\frac{c+a}{2}\right)\ge\sqrt{ca}$. Therefore if we consider their product we have the inequality $\frac{1}{8}(a+b)(b+c)(c+a)\ge\sqrt{ab}\sqrt{bc}\sqrt{ca}=abc$. Thus $(a+b)(b+c)(c+a)\ge 8abc$.

(d) Note that $\frac{1}{3}\left(\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\right)\ge\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{abc}}$ by AM-GM. Furthermore by the previous problem we know that $(a+b)(b+c)(c+a)\ge 8abc$. Thus $\frac{1}{3}\left(\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\right)\ge 2$. Thus $\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\ge 6$.

(e) We know by AM-GM that $\sqrt[3]{abc} \leq \frac{1}{3}(a+b+c)$, $\sqrt[3]{bcd} \leq \frac{1}{3}(b+c+d)$, $\sqrt[3]{cda} \leq \frac{1}{3}(c+d+a)$, $\sqrt[3]{dab} \leq \frac{1}{3}(d+a+b)$. Thus their sum yields

$$\sqrt[3]{abc} + \sqrt[3]{bcd} + \sqrt[3]{cda} + \sqrt[3]{dab} \leq a+b+c+d.$$