1.

$$\begin{aligned}
\vec{x}'' &= (\vec{x}_0 + \sum_{j=1}^{4} \vec{x}_j)'' \\
&= \vec{x}_0'' + \sum_{j=1}^{4} \vec{x}_j'' \\
&= -K\vec{x}_0 + \sum_{j=1}^{4} (-K\vec{x}_j + \vec{f}_j) \\
&= -K(\vec{x}_0 + \sum_{j=1}^{4} \vec{x}_j) + \sum_{j=1}^{4} \vec{f}_j \\
&= -K\vec{x} + \sum_{j=1}^{4} \vec{f}_j
\end{aligned}$$

Therefore we have found a solution to the differential equation $\vec{x}'' = -K\vec{x} + \sum_{j=1}^{4} \vec{f}_j$.

2. The matrix $K = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$ has the characteristic polynomial $\lambda^2 - 9\lambda + 14$, thus K has eigenvalues $\mu_1 = 7, \mu_2 = 2$. These correspond with the normalized eigenvectors $\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Let $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$, and $D = \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}$.
Therefore $K = VDV^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix}$. Since we have shown that K diagonalizes, then we can show that $V^{-1}\vec{x}_0'' = -DV^{-1}\vec{x}_0$:

$$V^{-1}\vec{x}_0'' = -V^{-1}K\vec{x}_0 = -V^{-1}VDV^{-1}\vec{x}_0 = -DV^{-1}\vec{x}_0.$$

Since V^{-1} is a matrix then $V^{-1}\vec{x}_0(0) = V^{-1}(1, 2), V^{-1}\vec{x}_0'(0) = V^{-1}(1, 1)$ is just the definition of matrix multiplication.
Similarly for $V^{-1}\vec{x}_j'' = -DV^{-1}\vec{x}_j + V^{-1}\vec{f}_j$,

$$V^{-1}\vec{x}_j'' = -V^{-1}K\vec{x}_j + V^{-1}\vec{f}_j = -V^{-1}VDV^{-1}\vec{x}_j + V^{-1}\vec{f}_j = -DV^{-1}\vec{x}_j + V^{-1}\vec{f}_j.$$

And once again since the initial conditions are vectors you may simply multiply them.

3.

$$\begin{aligned}
y''(t) &= (y(0)\cos(\sqrt{\kappa}t) + \frac{y'(0)}{\sqrt{\kappa}}\sin(\sqrt{\kappa}t) + \frac{1}{\sqrt{\kappa}}\int_0^t \sin(\sqrt{\kappa}(t-s))g(s)ds)'' \\
&= y(0)\cos(\sqrt{\kappa}t)'' + \frac{y'(0)}{\sqrt{\kappa}}\sin(\sqrt{\kappa}t)'' + \frac{1}{\sqrt{\kappa}}(\int_0^t \sin(\sqrt{\kappa}(t-s))g(s)ds)'' \\
&= -\kappa y(0)\cos(\sqrt{\kappa}t) - \sqrt{\kappa}y'(0)\sin(\sqrt{\kappa}t) + (\int_0^t \cos(\sqrt{\kappa}(t-s))g(s)ds)' \\
&= -\kappa y(0)\cos(\sqrt{\kappa}t) - \sqrt{\kappa}y'(0)\sin(\sqrt{\kappa}t) + g(t) - \sqrt{\kappa}\int_0^t \sin(\sqrt{\kappa}(t-s))g(s)ds \\
&\text{(using the Leibniz integral rule)} \\
&= -\kappa(y(0)\cos(\sqrt{\kappa}t) + \frac{y'(0)}{\sqrt{\kappa}}\sin(\sqrt{\kappa}t) + \frac{1}{\sqrt{\kappa}}\int_0^t \sin(\sqrt{\kappa}(t-s))g(s)ds) + g(t) \\
&= -\kappa y(t) + g(t)
\end{aligned}$$

4. To solve for $\vec{x}_0$, since we know what the eigenvectors are and we have a formula for solving the equation the equation $w_j' = -\mu_j w_j$ where

$$w_j(t) = (\vec{x}_0(0)\cdot\vec{v}_j)\cos(\sqrt{\mu_j}t) + \frac{(\vec{x}_0'(0)\cdot\vec{v}_j)}{\sqrt{\kappa_j}}\sin(\sqrt{\mu_j}t)$$

with $\vec{x}_0 = \sum_{j=1}^2 w_j\vec{u}_j$. Therefore $w_1 = \frac{3\sin\left(\sqrt{7}t\right)}{\sqrt{35}} + \sqrt{5}\cos\left(\sqrt{7}t\right), w_2 = -\frac{\sin\left(\sqrt{2}t\right)}{\sqrt{10}}$. Thus

$$\vec{x}_0(t) = \left(\frac{1}{5}\sqrt{2}\sin(\sqrt{2}t) + \frac{3\sin(\sqrt{7}t)}{5\sqrt{7}} + \cos(\sqrt{7}t), -\frac{\sin(\sqrt{2}t)}{5\sqrt{2}} + \frac{6\sin(\sqrt{7}t)}{5\sqrt{7}} + 2\cos(\sqrt{7}t)\right)$$

To solve the general equation $\vec{x}_i = -K\vec{x}_i + \vec{f}_i$, since $\vec{f}_i = \cos(\phi_i + \omega_i t)\vec{u}_i$, then we have a solution for each component $w_{i,j} = (\vec{x}_i(0)\cdot\vec{v}_j)\cos(\sqrt{\mu_j}t) + \frac{(\vec{x}_i'(0)\cdot\vec{v}_j)}{\sqrt{\kappa_j}}\sin(\sqrt{\mu_j}t) + \frac{2(\vec{u}_i\cdot\vec{v}_j)}{\sqrt{\mu_j}}\left(\sin(\phi_i - \xi_{i,j}t)\frac{\sin(\eta_{i,j}t)}{\eta_{i,j}} + \sin(\phi_i + \eta_{i,j}t)\frac{\sin(\xi_{i,j}t)}{\xi_{i,j}}\right)$ where $\xi_{i,j} = \frac{\sqrt{\mu_j}+\omega_i}{2}, \eta_{i,j} = \frac{\sqrt{\mu_j}-\omega_i}{2}$. Since for all $i \in [4], \vec{x}_i(0) = \vec{x}_i' = (0,0)$, then our solution $\vec{x}_i$ for each $i \in [4]$ is:

$$\vec{x}_i = \sum_{j=1}^2 \frac{2(\vec{u}_i\cdot\vec{v}_j)}{\sqrt{\mu_j}}\left(\sin(\phi_i - \xi_{i,j}t)\frac{\sin(\eta_{i,j}t)}{\eta_{i,j}} + \sin(\phi_i + \eta_{i,j}t)\frac{\sin(\xi_{i,j}t)}{\xi_{i,j}}\right)\vec{v}_j.$$

5. $\vec{f}_1 = \cos(\omega_1)(1,3)$, thus $\phi_1 = 0, \vec{u}_1 = (1,3)$. Therefore by our formula:
$\vec{x}_1 = \left(\frac{4}{5}\omega_1\left(\frac{\sqrt{2}\left(\cos\left(\sqrt{2}t\right)-\cos(t\omega_1)\right)}{\omega_1^2-2} + \frac{\sqrt{7}\left(\cos(t\omega_1)-\cos\left(\sqrt{7}t\right)\right)}{\omega_1^2-7}\right), \frac{4}{5}\omega_1\left(\frac{\cos(t\omega_1)-\cos\left(\sqrt{2}t\right)}{\sqrt{2}\left(\omega_1^2-2\right)} + \frac{2\sqrt{7}\left(\cos(t\omega_1)-\cos\left(\sqrt{7}t\right)\right)}{\omega_1^2-7}\right)\right)$

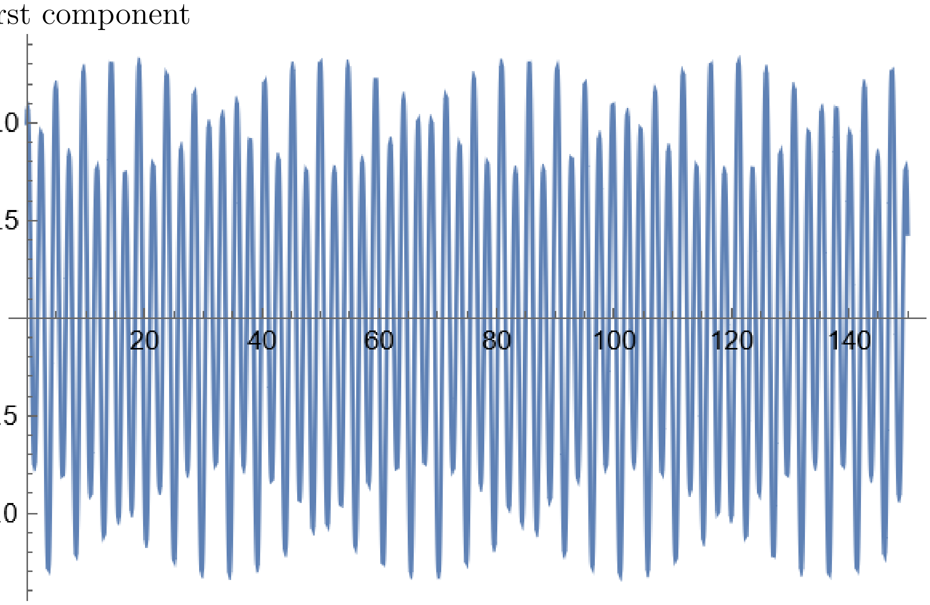
6. $\vec{f}_2 = \sin(\omega_2 t)(1,-1)$, thus $\phi_2 = -\frac{\pi}{2}, \vec{u}_2 = (1,-1)$. Therefore by our formula:
$\vec{x}_2 = (\frac{2}{35}\left(\frac{98\left(2\sin\left(\sqrt{2}t\right)-\sqrt{2}\omega_2\sin(t\omega_2)\right)}{\omega_2^2-2} + \frac{2\left(7\sin\left(\sqrt{7}t\right)-\sqrt{7}\omega_2\sin(t\omega_2)\right)}{\omega_2^2-7}\right),$
$\frac{14\left(\sqrt{2}\omega_2\sin(t\omega_2)-2\sin\left(\sqrt{2}t\right)\right)}{5\left(\omega_2^2-2\right)} + \frac{8\left(7\sin\left(\sqrt{7}t\right)-\sqrt{7}\omega_2\sin(t\omega_2)\right)}{35\left(\omega_2^2-7\right)})$

7. $\vec{f_3} = \sin(\omega_3 t)(3,-1)$, thus $\phi_3 = -\frac{\pi}{2}, \vec{u}_3 = (3,-1)$ Therefore by our formula:
$\vec{x}_3 = \frac{4}{35}\left(\frac{21\left(2\sin\left(\sqrt{2}t\right)-\sqrt{2}\omega_3\sin(t\omega_3)\right)}{\omega_3^2-2} + \frac{\sqrt{7}\omega_3\sin(t\omega_3)-7\sin\left(\sqrt{7}t\right)}{\omega_3^2-7}\right)$,
$\frac{6\left(\sqrt{2}\omega_3\sin(t\omega_3)-2\sin\left(\sqrt{2}t\right)\right)}{5\left(\omega_3^2-2\right)} + \frac{8\left(\sqrt{7}\omega_3\sin(t\omega_3)-7\sin\left(\sqrt{7}t\right)\right)}{35\left(\omega_3^2-7\right)}$

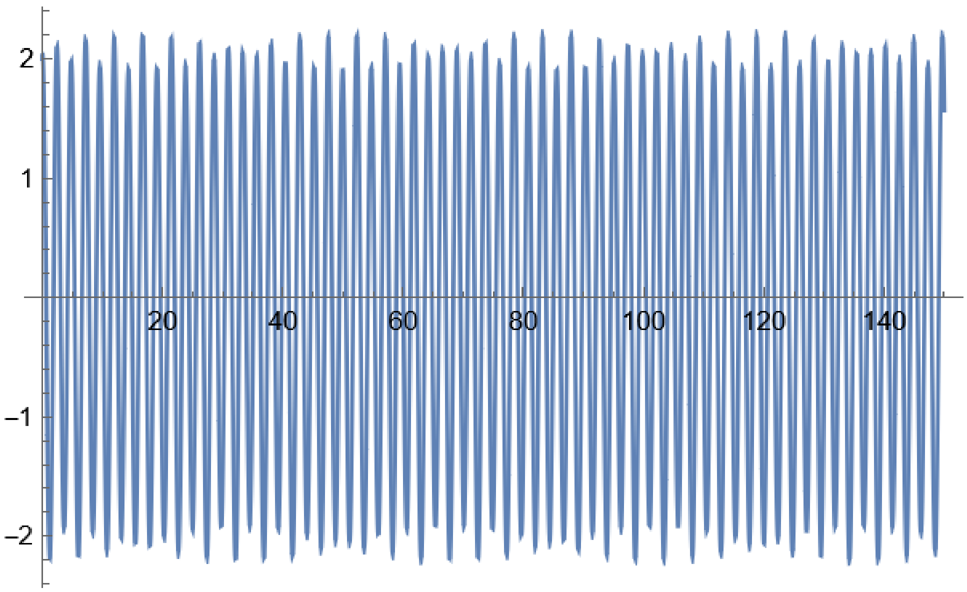
8. $\vec{f_4} = \cos(\omega_4)(1,0)$, thus $\phi_1 = 0, \vec{u}_1 = (1,0)$. Therefore by our formula:
$\vec{x}_4 = \frac{4}{35}\omega_4\left(\frac{14\sqrt{2}\left(\cos(t\omega_4)-\cos\left(\sqrt{2}t\right)\right)}{\omega_4^2-2} + \frac{\sqrt{7}\left(\cos(t\omega_4)-\cos\left(\sqrt{7}t\right)\right)}{\omega_4^2-7}\right)$,
$\frac{4}{35}\omega_4\left(\frac{7\sqrt{2}\left(\cos\left(\sqrt{2}t\right)-\cos(t\omega_4)\right)}{\omega_4^2-2} + \frac{2\sqrt{7}\left(\cos(t\omega_4)-\cos\left(\sqrt{7}t\right)\right)}{\omega_4^2-7}\right)$

9. $\vec{x}$ will experience resonance when any of the forcing function frequencies are either $\pm\sqrt{2}$ or $\pm\sqrt{7}$. If one takes ω_3 to approach $\sqrt{2}$ then the limit evaluates to:
$\lim_{t\to\sqrt{2}} \vec{x}_3(t) = (-\frac{2}{175}\left(\left(105+2\sqrt{14}\right)\sin\left(\sqrt{2}t\right) - 14\sin\left(\sqrt{7}t\right) + 105\sqrt{2}t\cos\left(\sqrt{2}t\right)\right)$,
$\frac{1}{175}\left(\left(105-8\sqrt{14}\right)\sin\left(\sqrt{2}t\right) + 56\sin\left(\sqrt{7}t\right) + 105\sqrt{2}t\cos\left(\sqrt{2}t\right)\right))$

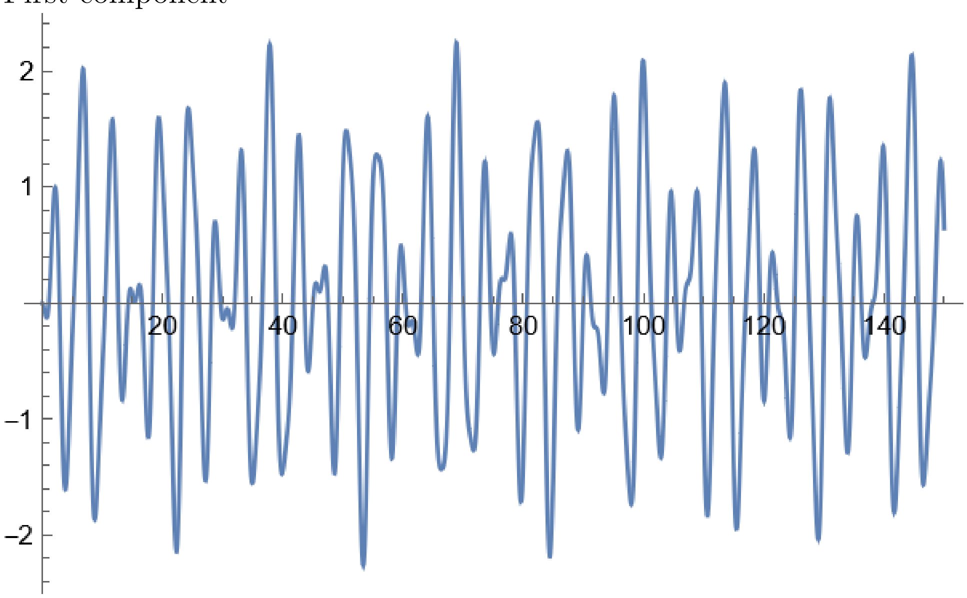
10. $\vec{x}_0$ First component



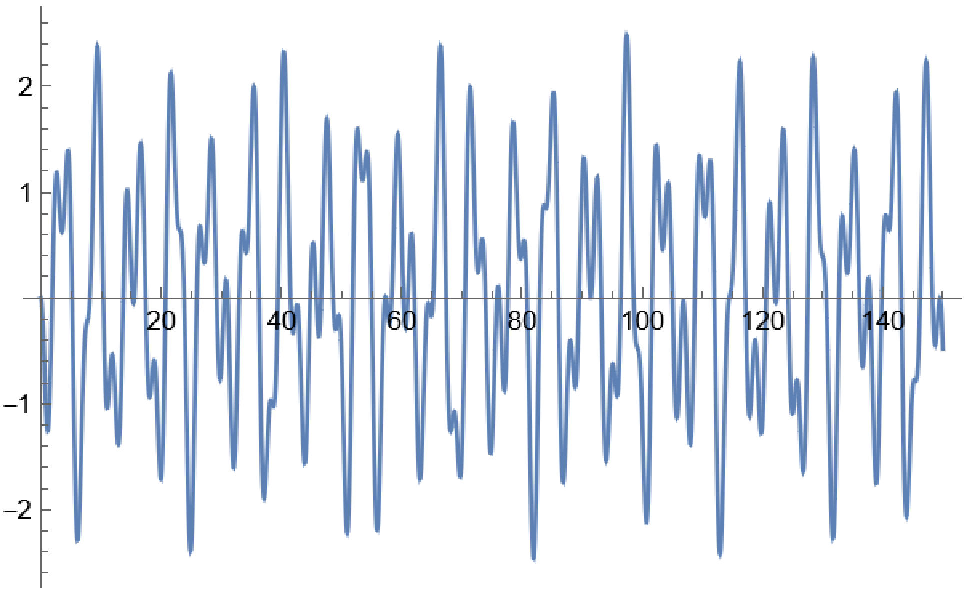
Second component



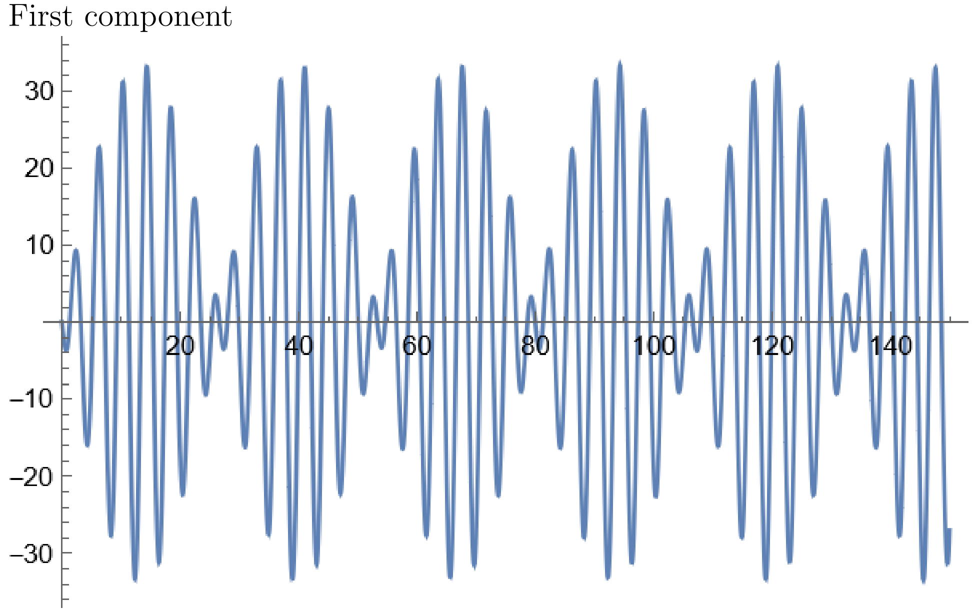
$\vec{x}_1$ First component



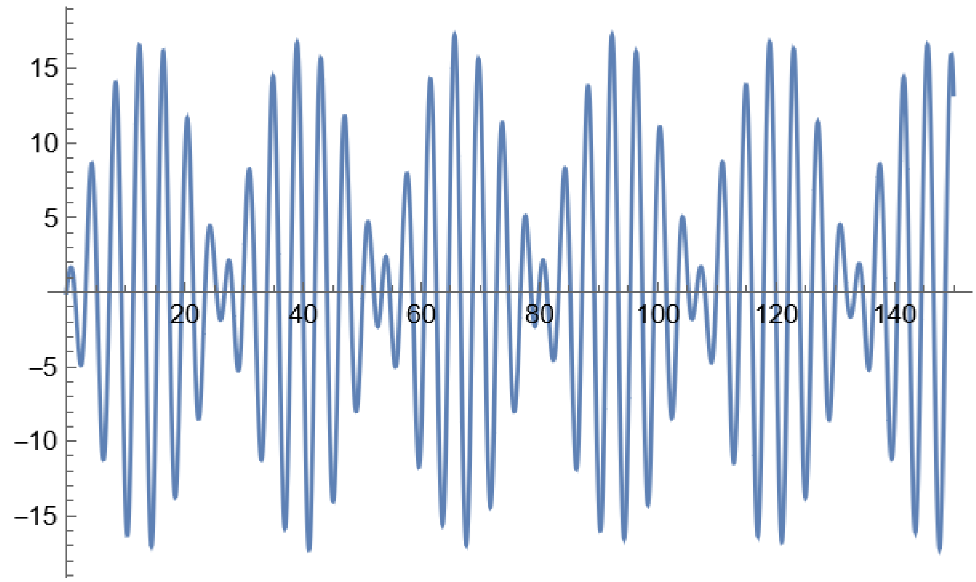
Second component



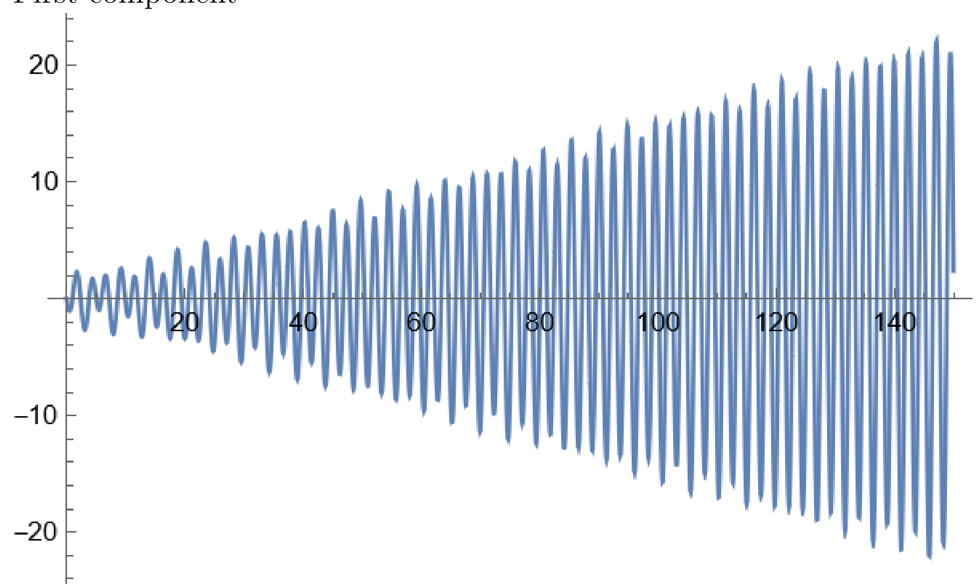
$\vec{x}_2$ First component



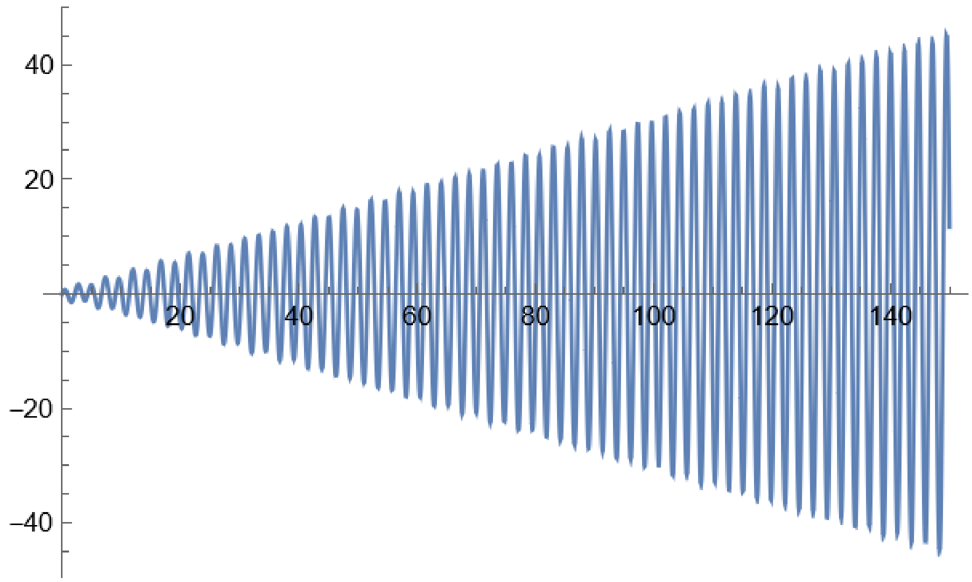
Second component



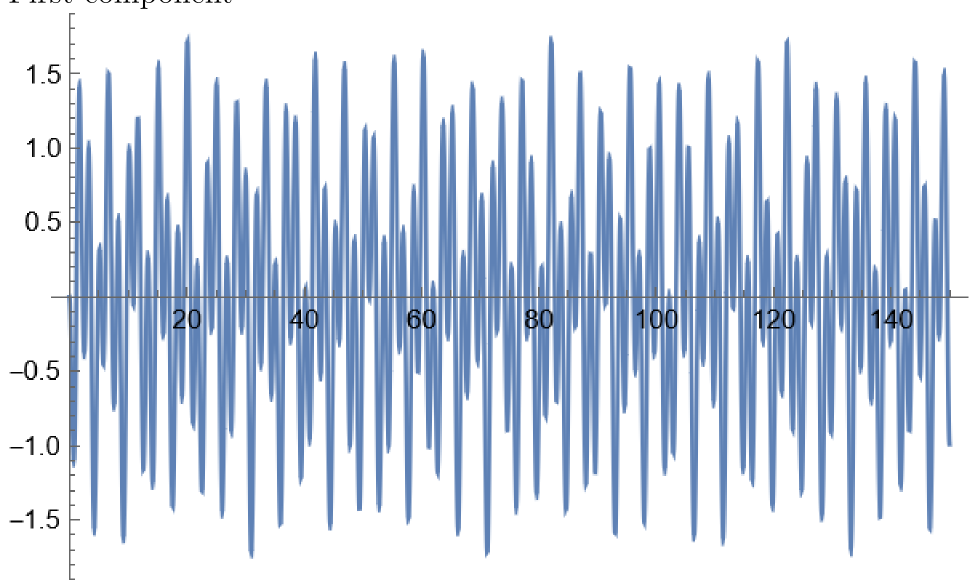
$\vec{x}_3$ First component



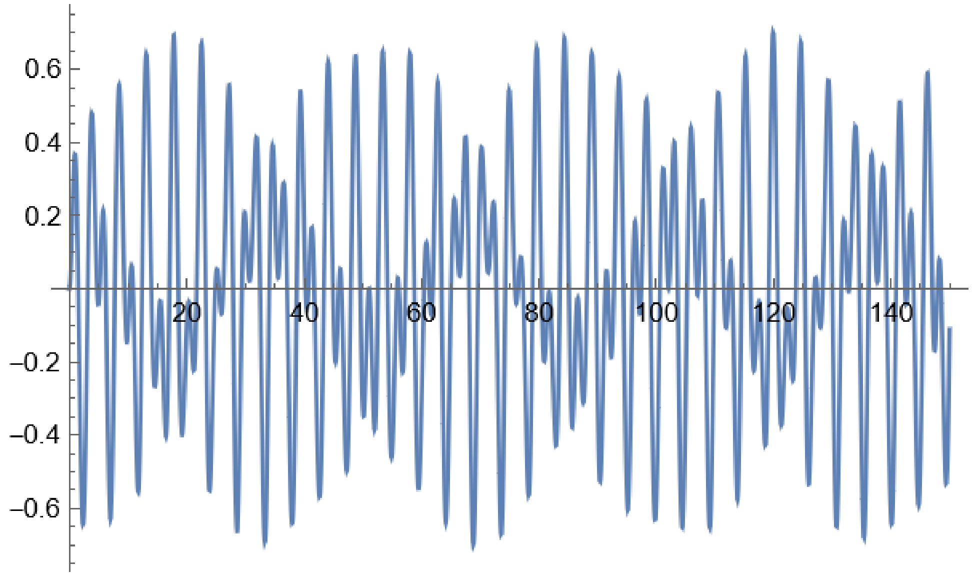
Second component



$\vec{x}_4$ First component



Second component



Clearly $\vec{x}_3$ is the dominating term, as $2.65 \approx \sqrt{7}$, therefore it's behavior is approaching resonant.