1. Each of the subproblems apply AM-GM once.

   (a)

$$\begin{aligned}\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} &= 4\frac{1}{4}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)\\ &\leq 4\left(\frac{abcd}{abcd}\right)^{\frac{1}{4}}\\ &= 4.\end{aligned}$$

   (b)

$$\begin{aligned}a^6+b^9+64 &= 3\frac{1}{3}(a^6+b^9+64)\\ &\leq 3(a^6b^964)^{\frac{1}{3}}\\ &= 12a^2b^3\end{aligned}$$

2. By applying Cauchy-Schwarz we get that $1=(x_1+\cdots+x_n)^2 \leq n\sum_i x_i^2$. Therefore the norm of the vector is bounded below via $\frac{1}{n}, \frac{1}{n} \leq \sum_i x_i^2$. Thus it is minimized via having a norm of $\frac{1}{n}$. An example solution would be $x_i = \frac{1}{n}$ for all $i \in [n]$.

3. (a) Suppose $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\frac{1}{\sqrt{\epsilon}} < N$ by the archimedean principle. Therefore if $n \geq N$ then

$$\left|\frac{n^2}{n^4+n^2+1}\right| < \left|\frac{n^2}{n^4}\right| = \left|\frac{1}{n^2}\right| < \epsilon.$$

   (b) Suppose $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\frac{1}{\epsilon} < N$. Suppose $n \geq N$. Then we have that

$$\left|\frac{5n^2+n}{3n^2+1}-\frac{5}{3}\right| = \left|\frac{3n-5}{3(3n^2+1)}\right| < \left|\frac{1}{3n}\right| = \frac{1}{3}\left|\frac{1}{n}\right| < \frac{\epsilon}{3} < \epsilon.$$

4. Suppose $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$ that $|a-a_n| < \epsilon$. Therefore by the reverse triangle inequality we have that $||a|-|a_n|| \leq |a-a_n| < \epsilon$. Therefore $\lim |a_n| \to |a|$. The converse is not true. If we consider $a_n = (-1)^n(1-2^n)$, then clearly $\limsup a_n = 1, \liminf a_n = -1$. This obviously doesn't converge. However $|a_n| = 1-2^n$, which does converge to 1.

5. Let $a_n = 2^n, b_n = -2^n$. Clearly $a_n \to \infty, b_n \to -\infty$. However $a_n + b_n = 2^n - 2^n = 0$, which does converge since it's constant.

6. We will show that $a_n$ is bounded below and decreasing.

   - We will show that $a_n$ is bounded below by $\frac{1+\sqrt{13}}{2}$. Clearly $3 \geq \frac{1+\sqrt{13}}{2}$. By the principle of mathematical induction for all $k \in \mathbb{N}$ if $k < n$ then $a_k \geq \frac{1+\sqrt{13}}{2}$. Therefore $a_n = \sqrt{3+a_{n-1}} \geq \sqrt{3+\frac{1+\sqrt{13}}{2}} = \sqrt{\frac{7+\sqrt{13}}{2}} = \frac{\sqrt{1+\sqrt{13}}}{2}$. Thus $a_n$ is bounded below.

- We will show that $a_n$ is decreasing. Clearly $3 \geq \sqrt{6}$. By the principle of mathematical induction, for all $k \in \mathbb{N}$ if $k < n$ then $a_{k+1} \leq a_k$. Therefore $a_n = \sqrt{3 + a_{n-1}}$, by the induction hypothesis $\sqrt{3 + a_{n-1}} \leq \sqrt{3 + a_{n-2}} = a_{n-1}$. Thus $a_n \leq a_{n-1}$. Therefore $(a_n)$ is decreasing.

Therefore by the monotone convergence theorem $(a_n)$ converges, and it converges to the lower bound given above. The solution will satisfy $\alpha = \sqrt{3+\alpha}, \alpha^2 - \alpha - 3 = 0$. This has a root of $\frac{1+\sqrt{13}}{2}$. Additionally $\left(\frac{1+\sqrt{13}}{2}\right)^2 = \frac{7+\sqrt{13}}{2} = 3 + \frac{1+\sqrt{13}}{2}$.

7. We will show that $(a_n)$ is decreasing and bounded.

   - We will show that $(a_n)$ is bounded. Since $a_n \in (0,1)$ for all $n \in \mathbb{N}$, then we can say that for all $n \in \mathbb{N}$ $|a_n| \leq 1$. Thus $(a_n)$ is bounded.
   - We will show that $(a_n)$ is decreasing. Consider the given inequality $\frac{1}{4} < a_n(1 - a_{n+1})$. We can treat the product of elements from the sequence as a geometric mean, and apply AM-GM:

   $$\sqrt{a_n^2(1-a_{n+1})^2} \leq \frac{a_n^2 + (1-a_{n+1})^2}{2}.$$

   We know for $x \in (0,1)$ that $x^2 \leq x$. Therefore we have from the previous inequality $\frac{1}{4} < \frac{a_n + 1 - a_{n+1}}{2}$ which yields $a_{n+1} < a_{n+1} + 1 < a_n$. Therefore $(a_n)$ is decreasing.

   Thus $(a_n)$ converges. But to what? Note that $\lim a_{n+1} = \lim a_n = \alpha$. Thus by the order limit theorem $\alpha(1-\alpha) \geq \frac{1}{4}$. This is equivalent to the inequality $(\alpha - \frac{1}{2})^2 \leq 0$. Since $(\alpha - \frac{1}{2})^2 \geq 0$ then $(\alpha - \frac{1}{2})^2 = 0$. Thus $\alpha = \frac{1}{2}$. Therefore $\lim a_n = \frac{1}{2}$

8. Lemma: $x^3$ is an increasing function. Suppose $x, y \in \mathbb{R}, x \leq y$. Consider $y^3 - x^3$. We want to show that $0 \leq y^3 - x^3$. We can write $y^3 - x^3$ as $(y-x)(y^2 + xy + y^2)$. Clearly $y - x \geq 0$. We must show that $y^2 + xy + y^2$ is greater than 0. It is trivially true if $x, y \geq 0$ or $x, y \leq 0$. Therefore we must consider the case $x \leq 0 \leq y$. Therefore $xy \leq 0$. Thus $2xy \leq xy$. Therefore

   $$x^2 + xy + y^2 \geq x^2 + 2xy + y^2 = (x+y)^2 \geq 0$$

   Therefore $0 \leq y^3 - x^3$, thus the cubic function on real numbers is increasing.
   We will show that the sequence converges by showing it is bounded and increasing:

   - We will show that $(a_n)$ is increasing by induction. Note that $a_1 = 0 \leq \frac{1}{2} = a_2$ for the base case. Therefore by the principle of mathematical induction for all $k \in \mathbb{N}$ if $k \leq n$ then $a_{k-1} \leq a_k$. Consider that $a_{n+1} = \frac{1}{3}(1 + a_n + a_{n-1}^3) \geq \frac{1}{3}(1 + a_{n-1} + a_{n-2}^3) = a_n$ as a consequence of the induction hypothesis and the fact that the cubic function is increasing. Thus $a_n \leq a_{n+1}$

- We will show that for all $n \in \mathbb{N}, a_n \leq \frac{\sqrt{5}-1}{2}$ by induction. Note that $0, \frac{1}{2} \leq \frac{\sqrt{5}-1}{2}$ thus the base case holds. By PMI for all $k \in \mathbb{N}$ if $k \leq n$ then $a_k \leq \frac{\sqrt{5}-1}{2}$. Note that
$$a_{n+1} = \frac{1}{3}(1+a_n+a_{n-1}^3) \leq \frac{1}{3}(1+\frac{\sqrt{5}-1}{2}+(\frac{\sqrt{5}-1}{2})^3) = \frac{1}{3}(\frac{12\sqrt{5}-12}{8}) = \frac{\sqrt{5}-1}{2}$$
. Thus $a_{n+1} \leq \frac{\sqrt{5}-1}{2}$.

Thus $(a_n)$ converges by MCT. Note that if $\alpha$ satisfies $\lim a_n = \alpha$ then $\alpha = \frac{1}{3}(1+\alpha+\alpha^3)$. Note that $\frac{\sqrt{5}-1}{2}$ is a solution of that equation. Thus $\lim a_n = \frac{\sqrt{5}-1}{2}$.

9.
$$\begin{aligned}
1 &= a+b+c+d \\
&\text{by the question definition} \\
&= \sqrt{a+b}\frac{a}{\sqrt{a+b}} + \sqrt{b+c}\frac{b}{\sqrt{b+c}} \\
&+ \sqrt{c+d}\frac{c}{\sqrt{c+d}} + \sqrt{d+a}\frac{d}{\sqrt{d+a}} \\
&\leq \sqrt{\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}}\sqrt{2a+2b+2c+2b} \\
&\text{by Cauchy-Schwarz} \\
\frac{1}{\sqrt{2}} &\leq \sqrt{\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}} \\
\frac{1}{2} &\leq \frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}
\end{aligned}$$
thus the inequality holds.

10. Note that
$$\begin{aligned}
\sum_{k=1}^n a_k b_k c_k &\leq \left(\sum_{k=1}^n a_k^3\right)^{\frac{1}{3}} \left(\sum_{k=1}^n (b_k c_k)^{\frac{3}{2}}\right)^{\frac{2}{3}} \\
&\text{by Hölders inequality} \\
&= \left(\sum_{k=1}^n a_k^3\right)^{\frac{1}{3}} \left(\sum_{k=1}^n b_k^{\frac{3}{2}} c_k^{\frac{3}{2}}\right)^{\frac{2}{3}} \\
&\leq \left(\sum_{k=1}^n a_k^3\right)^{\frac{1}{3}} \left(\sqrt{\sum_{k=1}^n b_k^3}\sqrt{\sum_{k=1}^n c_k^3}\right)^{\frac{2}{3}} \\
&\text{by Cauchy-Schwarz} \\
&= \left(\sum_{k=1}^n a_k^3\right)^{\frac{1}{3}} \left(\sum_{k=1}^n b_k^3\right)^{\frac{1}{3}} \left(\sum_{k=1}^n c_k^3\right)^{\frac{1}{3}}
\end{aligned}$$

thus the inequality holds.