- Suppose that Q is a partial order on B and that $f : A \longrightarrow B$. Define the relation R on A by xRy if $f(x)Qf(y)$. Prove that R is a TR relation.
  - Proof that $f(x) \leq_Q f(y) \Rightarrow xRy$. Suppose $x, y \in A, f(x) \leq_Q f(y)$. We must show that xRy. By definition of the relation xRy.
  - Proof that $xRy \Rightarrow f(x) \leq_Q f(y)$. Suppose $x, y \in A, xRy$. We want to show that $f(x) \leq_Q f(y)$. Since xRy is defined as only existing if $f(x) \leq_Q f(y)$, then we must have $f(x) \leq_Q f(y)$.
- For the rest of the problem, suppose R is an arbitrary TR relation on A. Define the relation W on A by xWy if and only if xRy and yRx. Prove that W is an equivalence relation.
  - Proof of the reflexivity of W. We must show for all $x \in A$ that xWx. Suppose $x \in A$. By definition of the relation xWx we must show xRx and xRx. Since R is a TR relation, then xRx. Therefore xWx.
  - Proof of the transitivity of W. We must show that for all $x, y, z \in A$ that if xWy and yWz then xWz. Suppose $x, y, z \in A, xWy, yWz$. We must show xWz. By definition of the relation we have xRy, yRx, yRz, zRy. By definition of the relation we must show xRz and zRx. Since R is transitive we have xRz and zRx.
  - Proof of the symmetry of W. We must show for all $x, y \in A$ if xWy then yWx. Suppose $x, y \in A, xWy$. We must show yWx. By definition of the relation we have xRy and yRx. Therefore we have yRx and xRy by the commutativity of and. Therefore by definition we have yWx.
- Let $\mathcal{C}$ denote the set of equivalence classes of W. Define a relation P on the set $\mathcal{C}$ where for $C, D \in \mathcal{C}$, CPD if there exists an $x \in C$ and a $y \in D$ such that xRy. Prove that this implies the stronger property that for all $C, D \in \mathcal{C}$ if CPD then for all $x \in C$ and $y \in D$, xRy.
  Proof: We must show for all $C, D \in \mathcal{C}$ that if CPD then for all $x \in C, y \in D, xRy$. Suppose $C, D \in \mathcal{C}, CPD$. We must show that for all $x \in C, y \in D$, xRy. Suppose $x \in C, y \in D$. We must show xRy. By definition of CPD we have the existence of $c \in C, d \in D$ such that cRd. Since c, x are members of the equivalence class C, then by transitivity xRd. Since d, y are members of the equivalence class D, then by transitivity xRy.
- Prove that P is a partial order on $\mathcal{C}$.
  - Proof of reflexivity. We must show for all $C \in \mathcal{C}$, CPC. Suppose $C \in \mathcal{C}$. We must show CPC. By the definition of P, we must show for all $x \in C, xRx$. Suppose $x \in C$. We must show xRx. Since R is reflexive, xRx.
  - Proof of transitivity. We must show for all $C, D, E \in \mathcal{C}$, if CPD and DPE then CPE. Suppose $C, D, E \in \mathcal{C}, CPD, DPE$. We must show CPE. By definition of P, for all $x \in C, y \in D, z \in E, xRy, yRz$. Since R is transitive, then for all $x \in C, z \in E, xRz$. Therefore by definition CPE.

- Proof of antisymmetry. We must show for all $C, D \in \mathcal{C}$ that if $C \neq D$ and CPD then $D\not{P}C$. Suppose $C, D \in \mathcal{C}, C \neq D, CPD$. We must show $D\not{P}C$. By definition of P we have for all $x \in C, y \in D, xRy$. By definition of P we must show for all $x \in C, y \in D, y\not{R}x$. Since $C \neq D$, and C, D are equivalence classes, then by definition for all $x \in C, y \in D$, we have $x\not{R}y$ or $y\not{R}x$. Since xRy, then for all $x \in C, y \in D$ we have $y\not{R}x$.

- Finish the proof of the $\Rightarrow$ direction.

  We must show that if R is a TR relation on the set A then there exists a set B with a partial order Q and a function $f : A \to B$ such that for all $x, y \in A, xRy$ if and only if $f(x) \leq_Q f(y)$. Suppose R is a TR relation on a set A. Let the class of representatives $Rep = \{a, b, \cdots\}$ for $\mathcal{C}$, let B be a set consisting of $R \backslash W$ and Rep, let $r : \mathcal{C} \to Rep$ be given by taking the equivalence class $\mathcal{C}$ and returning it's representative, let f be given by

  $$f(x) = \begin{cases} x & \text{if } x \in R\backslash W \\ r(C) & \text{if } \exists C \in \mathcal{C}, x \in C \end{cases}$$

  We claim Q is a poset defined as:

  - keeping all of the relations pre-existing between all of the elements in $R\backslash W$
  - mapping xRy, mRn with $x, n \in R\backslash W$,$m, y \in W$ to $x \leq_Q r(C), r(D) \leq_Q n$ where $C, D \in \mathcal{C}, m, y \in C$.
  - mapping all $C, D \in \mathcal{C}, CPD$ to $r(C) \leq_Q r(D)$.

  Proof that Q is a poset:

  - Proof of reflexivity: Suppose $x \in B$. We must show that $x \leq_Q x$. By definition of $x \in B$, either $x \in R\backslash W$ or $x \in Rep$.
    * Suppose $x \in R\backslash W$. We must show that $x \leq_Q x$. Since $x \in R$, and R is a TR relation, then $x \leq_Q x$.
    * Suppose $x \in Rep$. We must show that $x \leq_Q x$. Since x is a representative for an equivalence class X, and P is the relation from where it gets it's internal relations from, then we must show XPX. Since P is a poset, then XPX.
  - Proof of transitivity: Suppose $x, y, z \in B$. We must show that if $x \leq_Q y, y \leq_Q z$, then $x \leq_Q z$. Suppose $x \leq_Q y, y \leq_Q z$. We must show that $x \leq_Q z$. Since $x \leq_Q y$ and $y \leq_Q z$, then we know that there exists at least one element $x_a, y_a, z_a \in A$ such that x_aRy_a and y_aRz_a. Therefore by the transitivity of R, x_aRz_a. Therefore since x_aRz_a, then by the definition of Q that relationship maps onto the representative elements in B, therefore $x \leq_Q z$.
  - Proof of antisymmetry: Suppose $x, y \in B$. We must show that if $x \leq_Q y, x \neq y$, then $y \not\leq_Q x$. Suppose $x \leq_Q y, x \neq y$. Therefore we must work by cases of what portion of B they reside within:
    * Suppose $x, y \in Rep$. Then there exists $X, Y \in \mathcal{C}$ such that $x = r(X), y = r(Y)$. Therefore since Q maps all relations from XPY to $x \leq_Q y$, and P is a poset, then by definition $y \not\leq_Q x$.

    * Suppose $x, y \in R \backslash W$. Therefore since W has all elements with symmetric relationships in A, then by definition if $x, y \in R \backslash W$ and xRy then $y \not R x$. Therefore $y \not\leq_Q x$.
    * Suppose $x \in R \backslash W, y \in Rep$. Since $x \leq_Q y$, then there must exists $x_a, y_a \in A$ such that $x_a R y_a$. Since y_a is in an equivalence class and x_a isn't then by definition there isn't a symmetric relationship between x_a and y_a. Therefore $y_a \not R x_a$. Since Q only maps preexisting relationships, then $y \not\leq_Q x$.

Suppose $x, y \in A$. We must show that xRy implies $f(x) \leq_Q f(y)$ and $f(x) \leq_Q f(y)$ implies xRy.

- Suppose xRy. We must show that $f(x) \leq_Q f(y)$. We now have four cases, $x \in R \backslash W, y \notin R \backslash W$,$x \notin R \backslash W, y \in R \backslash W$, $x \in R \backslash W, y \in R \backslash W$, $x \notin R \backslash W, y \notin R \backslash W$.
    * Suppose $x \in R \backslash W, y \notin R \backslash W$. By definition there is a $C \in \mathcal{C}$ such that $y \in C$. Therefore $f(x) = x, f(y) = r(C)$. By definition of Q, $x \leq_Q r(C)$. Therefore $f(x) \leq_Q f(y)$.
    * Suppose $x \notin R \backslash W, y \in R \backslash W$. By definition there is a $C \in \mathcal{C}$ such that $x \in C$. Therefore $f(x) = r(C), f(y) = y$. By definition of Q, $r(C) \leq_Q x$. Therefore $f(x) \leq_Q f(y)$.
    * Suppose $x \in R \backslash W, y \in R \backslash W$. By definition of Q, xRy maps directly to $x \leq_Q y$. Since $f(x) = x, f(y) = y$, then $f(x) \leq_Q f(y)$.
    * Suppose $x \notin R \backslash W, y \notin R \backslash W$. By definition there exists $C, D \in \mathcal{C}$ such that $x \in C, y \in D$. Since xRy then CPD. Since Q maps CPD to $r(C) \leq_Q r(D)$, and $f(x) = r(C), f(y) = r(D)$, then $f(x) \leq_Q f(y)$.
- Suppose $f(x) \leq_Q f(y)$ We must show that xRy. We now have four cases, $x \in R \backslash W, y \notin R \backslash W$,$x \notin R \backslash W, y \in R \backslash W$, $x \in R \backslash W, y \in R \backslash W$, $x \notin R \backslash W, y \notin R \backslash W$.
    * Suppose $x \in R \backslash W, y \notin R \backslash W$. Since $y \notin R \backslash W$, then there exists $C \in \mathcal{C}$ such that $y \in C$. Therefore $f(x) \leq_Q f(y) = x \leq_Q r(C)$. By definition of Q, if $x \leq_Q r(C)$ then there exists an element e in C which xRe. Since $e, y \in C$, then xRy.
    * Suppose $x \notin R \backslash W, y \in R \backslash W$. The proof is nearly identical to the one above.
    * Suppose $x \in R \backslash W, y \in R \backslash W$. Then by definition of f, $f(x) \leq_Q f(y)$ is equivalent to $x \leq_Q y$. Therefore by definition of Q xRy.
    * Suppose $x \notin R \backslash W, y \notin R \backslash W$. Therefore there exists $C, D \in \mathcal{C}$ such that $x \in C, y \in D$. Therefore the inequality $f(x) \leq_Q f(y)$ becomes $r(C) \leq_Q r(D)$. Since $r(C) \leq_Q r(D)$ in Q corresponds to CPD, and $x \in C, y \in D$, then xRy.