1.1 Consider $y \in R\backslash F$, and the function $f : R \to R$ given by $x \mapsto yx$. Note that since $deg_F(R) = n$ then this implies that f is a linear operator. Therefore by rank nullity $rank(f) + 0 = rank(f) + nullity(f) = n$. Thus f is surjective. Therefore there exists $x' \in R$ such that $f(x') = 1$. Thus y has an inverse, making every element in R a unit, thus R is a field.

2.1 Let $f(x) = x^3 - 3x + 4$ and let $\alpha \in \mathbb{C}$ satisfy $f(\alpha) = 0$. Note that $a^3 = 3\alpha - 4$, $\alpha^4 = 3\alpha^2 - 4\alpha$, therefore to find an inverse $a\alpha^2 + b\alpha + c \in \mathbb{Q}(\alpha)$, we simply have to define a matrix which encodes $(a\alpha^2 + b\alpha + c)(\alpha^2 + \alpha + 1) = 1$. Note that since $\alpha(\alpha^2+\alpha+1) = \alpha^2+4\alpha-4$ and $\alpha^2(\alpha^2+\alpha+1) = 3\alpha^2-4\alpha+3\alpha-4+\alpha^2 = 4\alpha^2-\alpha-4$, then by the linearity of multiplying by $(\alpha^2 + \alpha + 1)$ we have the matrix equation to solve

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 1 \\ -1 & 4 & 1 \\ -4 & -4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

With the resulting solution being $a = \frac{-3}{49}, b = \frac{-5}{49}, c = \frac{17}{49}$

2.3 Note that the minimal polynomial for $\beta = \sqrt[3]{2}e^{\frac{2\pi i}{3}}$ is $x^3 - 2$. Note by theorem 15.2.8 there is an isomorphism between $\mathbb{Q}$ adjoined roots of an irreducible polynomial which fix $\mathbb{Q}$. Therefore since $\sqrt[3]{2}$ is another root of $x^3 - 2$ then there exists an isomorphism $\phi\mathbb{Q}(\beta) \to \mathbb{Q}(\sqrt[3]{2}$ in which $\phi(\mathbb{Q}) = \mathbb{Q}$. Therefore for the equation $x_1^2 + \cdots x_k^2 = -1$, by applying ϕ to it we get $\phi(x_1)^2 + \cdots + \phi(x_k)^2 = -1$. Note that $\phi(x_i) \in \mathbb{Q}(\sqrt[3]{2} \subset \mathbb{R}$, meaning that $\phi(x_1)^2 + \cdots + \phi(x_k)^2 > 0 > -1$. Therefore in $\mathbb{Q}(\beta)$ this equation is impossible.

3.1 Let F be a field and let α be an algebraic element over F such that $[F(\alpha) : F] = 5$. Since the degree of α over F is prime, and since $\alpha^2 \notin F$ then by corollary 15.3.7 $F(\alpha^2) = F(\alpha)$.

3.9 Let α be a complex root of $f(x)$, β be is a complex root of $g(x)$, $f(x), g(x)$ are both irreducible over $\mathbb{Q}$, and let $K = \mathbb{Q}(\alpha), L = \mathbb{Q}(\beta)$. ($\Rightarrow$) Suppose $f(x)$ is irreducible over L, then $deg(f) = [K : Q]$ must be equivalent to $[\mathbb{Q}(\alpha, \beta) : L]$ since f is irreducible over L by theorem 15.2.7. Therefore since $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}] = [\mathbb{Q}(\alpha, \beta) : K][K : \mathbb{Q}] = [\mathbb{Q}(\alpha, \beta) : L][L : \mathbb{Q}]$ then you can divide out by $[K : \mathbb{Q}]$ and get $[\mathbb{Q}(\alpha, \beta) : K] = [L : \mathbb{Q}]$. Thus g is irreducible over K. The converse is trivial.