1. Let $(W, \leq)$ be a linearly order set.

   - $\Rightarrow$ Suppose W is well ordered. We want to show that there does not exists a descending chain. Suppose for contradiction that there is a sequence $(w_n)_{n\in\mathbb{N}}$ where $w_n > w_{n+1}$. Since W is well order than $\min(w_n)$ exists. Since (w_n) has a minimum then there exists $n' \in \mathbb{N}$ such that for all $n \in \mathbb{N}, w_{n'} \leq w_n$. Note that $w_{n'+1} < w_{n'}$ by the definition of (w_n). Therefore $w_{n'+1} < w_{n'}$ and $w_{n'+1} \geq w_{n'}$. This is a contradiction. Therefore a descending chain does not exists.
   - $\Leftarrow$ Suppose W is not well ordered. We want to show that there exists a descending chain. Since W is not well ordered then there exists a set S where $S \neq \emptyset, S \subseteq W$ where $\min S$ does not exists. Since S is nonempty then there exists $x_1 \in S$. Note that $\min\{x_1\}$ exists, therefore there exists $x_2 \in S$ such that $x_2 < x_1$. Therefore by induction $\{x_1, \cdots, x_n\} \subset S$, since $\min\{x_1, \cdots, x_n\}$ exists. Therefore there exists $x_{n+1} \in S$ such that $x_{n+1} < x_n$. Therefore by induction we have constructed a descending chain.

2. - $(\Rightarrow)$ Suppose we have the hausdorff maximal principle, and we want to show zorn's lemma. Suppose $(X, \leq)$ is a poset, and all chains in X are bounded. Then by our original assumption there exists a chain $L \subseteq X$ which is maximal. We know since L is a chain then it is bounded above by assumption. Let $a \in X$ be an upper bound of L. We know that $a \in L$ since otherwise $L \cup \{a\}$ would be a chain larger than L, contradicting the maximality of L. Thus a must be a maximal element of X. Note that if we have $x \in X \backslash L, a \leq x$ then $a = x$ by maximality. Otherwise $L \cup \{x\}$ would be a chain, contradicting the maximality of L
   - $(\Leftarrow)$ Suppose we have zorn's lemma, and we want to prove the hausdorff maximal principle. Suppose $(X, \leq)$ is a poset, and we have the set of all chains, $\mathcal{C} \subseteq \mathcal{P}(X)$. We want to show that $\mathcal{C}$ has a maximal element. Therefore we must show all chains of chains (or 2chainz, if you will) of $\mathcal{C}$ are bounded, where we have the poset generated by $(\mathcal{C}, \subseteq)$, the set inclusion relation. Suppose $C \subset \mathcal{C}$ is a 2-chain. We claim that it is bounded above by $C^* = \bigcup_{c\in C} c$, and that $C^* \in \mathcal{C}$. Therefore we must show for all $x, y \in C^*$ that $x \leq y$ or $x \geq y$. Note that since $x, y \in C^*$, then there exists $C_1, C_2 \in C$ such that $x \in C_1, y \in C_2$. Since C is a 2-chain, then either $C_1 \subseteq C_2$ or $C_1 \supseteq C_2$. WLOG assume $C_1 \subseteq C_2$. Thus since $x, y \in C_2$, then since C_2 is a chain then $x \leq y$ or $x \geq y$. Therefore $C^* \in \mathcal{C}$. Thus every chain has an upper bound. Thus by Zorn's lemma $\mathcal{C}$ has a maximal element. Thus there is a maximal chain, hausdorff's principle holds.

3. Suppose $\{A_i : i \in I\}$ is a set of sets. Let P be a non-empty poset defined by $f \in P$ being a function with $dom(f) \subseteq I$ and $f : dom(f) \to \bigcup_{i\in I} A_i$ maintaing that for every $i \in dom(f)$ one has $f(i) \in A_i$. P is a poset with the relation $\leq$ given by for every $f, g \in P$ if $f \leq g$ then $dom(f) \subseteq dom(g)$ and for all $i \in dom(f)$ one has $f(i) = g(i)$. We want to show that every chain of functions is bounded in P. If one considers a chain in P, given by $(f_i)_{i\in I'}$, then one can consider the upper bound to trivially be the function $f : \bigcup_{i\in I'} dom(f_i) \to \bigcup_{i\in I} A_i$. Since for every $x \in \bigcup_{i\in I'} dom(f_i)$, every $g \in (f_i)_{i\in I'}$ with x in it's domain has the exact same value on x. Thus f has unique

values. Thus the chain is bounded. Therefore there exists a maximal h by zorn's lemma. Since h has a maximal domain, then it must be I itself. Otherwise, if there exists $j \in I$ such that $j \notin dom(h)$ then there exists a function defined on j, however one could make a new function which operates on both j and $dom(h)$, but this would contradict the maximality of h. Thus there exists an element in $\bigcup_{i \in I} A_i$.

4. Since A is finite then we can enumerate A with $\{a_1, \cdots, a_n\}$. Since we have the axiom of choice then we have the choice function $h : \mathcal{P}(X) \backslash \{\emptyset\} \to X$ where for every $S \in \mathcal{P}(X)$, $h(S) \in S$. Thus we can inductively define $a_{n+1} = h(X \backslash A)$, $a_{n+2} = h(X \backslash (A \cup \{a_{n+1}\})$, so on and so forth. Thus the function $f(x) = \begin{cases} a_{n+k} & \text{if} x = a_k \\ x & \text{otws} \end{cases}$ is necessarily a bijection from $X \to X \backslash A$ since all enumerated elements map 1-1 to enumerated elements in $X \backslash A$ and all unenumerated elements are mapped to themselves. Thus X and $X \backslash A$ have the same cardinality

5.

$$
\begin{aligned}
\Psi[\bigcup_{j\in J} A_j] &= A\backslash g[B\backslash f[\bigcup_{j\in J} A_j]] \\
&= A\backslash g[B\backslash \bigcup_{j\in J} f[A_j]] \\
&= A\backslash g[B\cap (\bigcup_{j\in J} f[A_j])^c) \\
&= A\backslash g[B\cap \bigcap_{j\in J} f[A_j]^c] \\
&= A\backslash (g[B]\cap \bigcap_{j\in J} g[f[A_j]^c]) \\
&= A\cap (g[B]\cap \bigcap_{j\in J} g[f[A_j]^c])^c \\
&= A\cap (g[B]^c\cup \bigcup_{j\in J} g[f[A_j]^c]^c) \\
&= (A\cap g[B]^c)\cup (\bigcup_{j\in J} A\cap g[f[A_j]^c]^c) \\
&= \bigcup_{j\in J}(A\cap g[B]^c)\cup (A\cap g[f[A_j]^c]^c) \\
&= \bigcup_{j\in J}(A\cap (g[B]^c\cup g[f[A_j]^c]^c)) \\
&= \bigcup_{j\in J}(A\cap (g[B\cap f[A_j]^c]^c)) \\
&= \bigcup_{j\in J}(A\backslash g[B\backslash f[A_j]])) \\
&= \bigcup_{j\in J} \Psi[A_j]
\end{aligned}
$$

6.

- We will show that $\Phi : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ given by $\Phi(A) = \sum_{n\in A} \frac{2}{3^n}$ is an injection. Suppose $A, B \in \mathcal{P}(\mathbb{N}), A \neq B$. Since $A \neq B$ then the symmetric difference $A\Delta B$ is nonempty. Since $A\Delta B \subseteq \mathbb{N}$, then $A\Delta B$ has a minimal element. Let this element be denoted m, and WLOG assume $m \in A$. If we consider $\Phi(A) - \Phi(B) = \sum_{n\in A} \frac{2}{3^n} - \sum_{n\in B} \frac{2}{3^n} = \sum_{n\in A\backslash[m-1]} \frac{2}{3^n} - \sum_{n\in B\backslash[m-1]} \frac{2}{3^n}$ since A and B must share every element less than m, otherwise contradicting m is the smallest element in $A\Delta B$. Note that we can bound the difference below by considering $\sum_{n\in A\backslash[m-1]} \frac{2}{3^n} \geq \frac{2}{3^m}, -\sum_{n\in B\backslash[m-1]} \frac{2}{3^n} \geq -\sum_{n\in\mathbb{N}\backslash[m]} \frac{2}{3^n}$.
  Since $-\sum_{n\in\mathbb{N}\backslash[m]} \frac{2}{3^n} = \frac{1}{3^m}$, then the difference $\Phi(A) - \Phi(B) \geq \frac{1}{3} > 0$. Thus $\Phi(A) > \Phi(B), \Phi(A) \neq \Phi(B)$. Thus Φ is injective
- Suppose $x, y \in \mathbb{R}, x \neq y$. Want to show that $\Psi(x) \neq \Psi(y)$. WLOG assume that $x < y$. Then by the density of $\mathbb{Q}$ in $\mathbb{R}$ there exists $q \in \mathbb{Q}$ such that $x < q < y$.

Since $\Psi(x) = \{r \in \mathbb{Q} : r < x\}$ then by definition $q \notin \Psi(x)$. However since $q < y$ and $q \in \mathbb{Q}$ then by definition $q \in \Psi(y)$. Thus $\Psi(x) \neq \Psi(y)$.

7. Since $\mathbb{N}$ has the same cardinality of $\mathbb{Q}$ we will consider subsets of $\mathbb{Q}$ in place of subsets of $\mathbb{N}$. The infinite family of subsets of $\mathbb{Q}$ we will consider is the set of arbitrarily chosen ration cauchy sequences of a given number. $\mathcal{A} = \{(x_n) \in \mathbb{Q}^n : \exists! r \in \mathbb{R}, x_n \to r\}$. Since each element uniquely corresponds to a single real number, then $\mathcal{A}$ has the same cardinality of $\mathbb{R}$. Note for any $(x_n), (y_n)$, since they converge to $x, y \in \mathbb{R}$, then by the topological definition of convergence we can choose $\epsilon = \frac{x-y}{2}$ in which infinitely many terms from each sequence would be in disjoint ϵ neighborhoods around x, y respectively. Thus any two elements have a finite intersection.

8. Note that the set of interval with rational endpoints corresponds to the set

$$\bigcup_{a \in \mathbb{Q}} [a, a] \cup \bigcup_{a,b \in \mathbb{Q}, a<b} \{(a, b), (a, b], [a, b), [a, b]\}.$$

Since each one of these has indexing over $\mathbb{Q}$ or $\mathbb{Q}^2$ then it's a countable union of countable sets. Thus it is countable.

9. Suppose $\{B(x, r)\}$ is a set of disjoint balls. Then uniquely for each $B(x, r)$, since $x - r < x + r$ then by the density of $\mathbb{Q}$ in $\mathbb{R}$ there exists $q \in \mathbb{Q}$ such that $q \in B(x, r)$. Since the balls are disjoint then there is a unique rational number within each ball. Since $\mathbb{Q}$ is countable then the set of balls is countable.

   (a) We claim that $card((\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}) = card(\{0, 1\}^{\mathbb{N}\times\mathbb{N}})$. Note that $(\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}$ is the set of of sequences of infinite binary sequences. Therefore for a given $f \in (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}$ we have that $f(n) = (b_{nk})_{k\in\mathbb{N}}$ for all $n \in \mathbb{N}$. If we define $g : (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}) \to \{0, 1\}^{\mathbb{N}\times\mathbb{N}}$ by $g(f) = (b_{nk})_{(n,k)\in\mathbb{N}^2}$, then this is clearly a bijection. Thus $card((\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}) = card(\{0, 1\}^{\mathbb{N}\times\mathbb{N}})$. Therefore:

   $$card(\mathbb{R}^{\mathbb{N}}) = card((\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}) = card(\{0, 1\}^{\mathbb{N}\times\mathbb{N}}) = card(\{0, 1\}^{\mathbb{N}}) = card(\mathbb{R})$$

   .

   (b) Let S be some countable set, and let $X = \{S^n : n \in \mathbb{N}\}$. We want to show that X is countable. Note that S is countable, and therefore S^n is countable by slide 22 of lecture 14. Since S^n and $\mathbb{N}$ is countable, then $\bigcup_{n\in\mathbb{N}}^{\infty} S^n$ is countable. Since $\bigcup_{n=1}^{\infty} S^n = X$, then we're done.

   (c) Note that a polynomial is uniquely determined by it's coefficients. Therefore the set of polynomials over $\mathbb{Z}$ has the same cardinality as all of the finite integer sequences Thus $card(\mathbb{Z}[x]) = card(\{\mathbb{Z}^n : n \in \mathbb{N}\})$. Since $\{\mathbb{Z}^n : n \in \mathbb{N}\}$ is countable then $\mathbb{Z}[x]$ is countable

   (d) Note that since $\mathbb{Z}[x]$ is countable and for $p \in \mathbb{Z}[x]$ the set $r(p) = \{p(x) = 0 : x \in \mathbb{R}\}$ is finite, then $\bigcup_{p\in\mathbb{Z}[x]} r(p)$ is countable. Note that this is exactly the set of algebraic numbers. Additionally, since we've found a countable subset of the real numbers, then there are an uncountable number of numbers not in our set. Thus real algebraic numbers exists

(e) Suppose for contradiction that there is a finite number of prime numbers. Let the set of primes be denoted $\{p_1, \cdots, p_n\}$. Consider the number $l = 1 + \prod_{i=1}^{n} p^i$. Note that for each prime $p_i, l \equiv 1 \mod p_i$. Thus l is divisible by none of the prime numbers. Since l can't be divide by primes, it can't be divide by the product of any of the primes. Thus l is only divisible by 1 and itself. Thus l is prime. This contradicts the fact that $\{p_1, \cdots, p_n\}$ is the set of all primes. Thus there is an infinite number of primes.