

1. Let $T : V \rightarrow W$ be a linear transformation from a finite-dimensional vector space V to a finite-dimensional vector space W . Let β and β' be ordered bases for V , and let γ' and γ be ordered bases for W . We must show that $[T]_{\beta'}^{\gamma'} = P^{-1}[T]_{\beta}^{\gamma}Q$ where Q is the β' to β change of coordinates matrix and P is the γ' to γ change of coordinates matrix. By definition $Q = [\mathbb{I}_V]_{\beta}^{\beta'}$, $P = [\mathbb{I}_W]_{\gamma'}^{\gamma}$. Note that the statement we're trying to prove is equivalent to $P[T]_{\beta'}^{\gamma'} = [T]_{\beta}^{\gamma}Q$. Therefore,

$$P[T]_{\beta'}^{\gamma'} = [\mathbb{I}_W]_{\gamma'}^{\gamma}[T]_{\beta'}^{\gamma'} = [\mathbb{I}_W T]_{\beta'}^{\gamma} = [T]_{\beta'}^{\gamma} = [T\mathbb{I}_V]_{\beta'}^{\gamma} = [T]_{\beta}^{\gamma}[\mathbb{I}_V]_{\beta'}^{\beta} = [T]_{\beta}^{\gamma}Q.$$

Therefore $[T]_{\beta'}^{\gamma'} = P^{-1}[T]_{\beta}^{\gamma}Q$.

2. Suppose $A, B \in M_{m \times n}(F)$, $P \in GL_m(F)$, $Q \in GL_n(F)$, $B = P^{-1}AQ$. We must show that there exists an n dimensional vector space V and an m dimensional vector space W over F , ordered bases β and β' for V and γ and γ' for W , and a linear transformation $T : V \rightarrow W$ such that

$$A = [T]_{\beta}^{\gamma} \text{ and } B = [T]_{\beta'}^{\gamma'}.$$

Let $V = F^n$, $W = F^m$, $T = L_A$, and β and γ be the standard ordered bases for F^n and F^m respectively. Therefore we trivially have $A = [L_A]_{\beta}^{\gamma} = [T]_{\beta}^{\gamma}$. We must find β', γ' such that $B = [T]_{\beta'}^{\gamma'}$.

3. Let V be a finite-dimensional vector space with the ordered basis β . Prove that $\psi(\beta) = \beta^{**}$. Let $\beta = \{x_1, \dots, x_n\}$. Therefore $\psi(\beta) = \{\hat{x}_1, \dots, \hat{x}_n\}$. By the corollary to theorem 2.26, there exists a basis $\{v_1, \dots, v_n\}$ of V^* such that $v_i^* = \hat{x}_i$. Applying the corollary again gives us $\{x_1, \dots, x_n\}$ as $\delta_{i,j} = \hat{x}_i(v_j) = v_j(x_i)$. Therefore $x_i^{**} = \hat{x}_i$, giving us $\psi(\beta) = \beta^{**}$.

4. Suppose that V, W are finite dimensional vector spaces over F and that $T : V \rightarrow W$ is linear. We must show that $N(T^t) = (R(T))^0$. Let $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$ be ordered bases β and γ of V and W respectively, and let $\{f_1, \dots, f_n\}$ and $\{g_1, \dots, g_m\}$ be ordered bases β^* and γ^* of V^* and W^* respectively.
- Suppose $v \in N(T^t)$. We must show that $v \in (R(T))^0$. By definition of being a member in $N(T^t)$, $v \in W^*, T^t(v) = v(T) = 0$. By theorem 2.24 of $T^t(v) = \sum_{s=1}^n (vT)(x_s)f_s$. Therefore $0 = \sum_{s=1}^n (vT)(x_s)f_s$. Since β^* is a basis for V^* , then for all $s \in [n]$, $(vT)(x_s) = 0$. Therefore for an arbitrary $x \in R(T)$, by the book proof of rank nullity $R(T)$ is spanned by $T(x_1), \dots, T(x_n)$, thus there exists a_1, \dots, a_n such that $x = a_1T(x_1) + \dots + a_nT(x_n)$. Therefore $v(x) = v(\sum_{i=1}^n a_iT(x_i)) = \sum_{i=1}^n a_iv(T(x_i)) = \sum_{i=1}^n a_i0 = 0$. Thus $v \in (R(T))^0$.
 - Suppose $v \in (R(T))^0$. We must show that $v \in N(T^t)$. By definition of $v \in N(T^t)$ we must show that $T^t(v) = v(T) = 0$. By definition of $v \in (R(T))^0$, for all $x \in R(T)$, $v(x) = 0$. By definition of being a member of W^* by theorem 2.24, $v = \sum_{l=1}^m v(y_l)g_l$.