2.4 (a) For equation (2), we can define x' explicitly by the following:

$$x' = \pm\sqrt{1+x^2}.$$

Therefore taking the absolute value of x' yields:

$$|x'| = \sqrt{1+x^2} \leq \sqrt{x^2+x^2} = \sqrt{2x^2} = \sqrt{2}|x|.$$

Since (2) has a lipschitz constant of $\sqrt{2}$, then by theorem 5 (2) has unique solutions for all $x_0 \in (-1,1)$.
Turning our attention to (1), then we have the equation

$$x' = \pm\sqrt{1-x^2}$$

Taking the absolute value of the derivative of x' yields

$$x'' = \frac{|x|}{\sqrt{1-x^2}}.$$

Taking the limit as x approaches -1 yields:

$$\lim_{x\to-1}|x''| = \frac{1}{\sqrt{1-1}} = \frac{1}{0} = \infty.$$

Since $|x'|$ is continuous on $(-1,1)$ and is not lipschitz then (1) has infinite solutions.

(b) Since (1) does not have a unique solutions we must show an infinite number of solutions to $(x')^2 + x^2 = 1, x(0) = x_0$. Since $x(t) = 1$ solves the equation as

$$(x_0')^2 + x_0 = 0^2 + 1^2 = 1$$

but not the initial value of $x_0 \in (-1,1)$, we must solve the differential equations by other means to give another solution to interpolate with.
Solving for non-steady state:

$$\begin{aligned}
x' &= \pm\sqrt{1-x^2} \\
1 &= \frac{\pm x'}{\sqrt{1-x^2}} \\
\int_{t_0}^{t} dt &= \pm\int_{x_0}^{x} \frac{dz}{\sqrt{1-z^2}} \\
t - t_0 &= \pm(\arcsin(x) - \arcsin(x_0)) \\
\pm(t - t_0 + \arcsin(x_0)) &= \arcsin(x) \\
x(t) &= \pm\sin(t - t_0 + arcsin(x_0))
\end{aligned}$$

Since $x(t) = 1, x(t) = \sin(t - t_0 + arcsin(x_0))$ both solve the differential equation, then we may create a new solution

$$x(t) = \begin{cases} 1 & t > t_0 - arcsin(x_0) + 2\pi a + \frac{\pi}{2} \\ \sin(t - t_0 + arcsin(x_0)) & t \leq t_0 - arcsin(x_0) + 2\pi a + \frac{\pi}{2} \end{cases}$$

where $a \in \mathbb{N} \cup \{0\}$. Note that at $t = t_0 - arcsin(x_0 + 2\pi a + \frac{\pi}{2})$, that for sin we have:

$$\sin(t - t_0 + arcsin(x_0)) = \sin(2\pi a + \frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$$

and for the derivative

$$cos(\frac{\pi}{2}) = 0$$

Which exactly aligns with the value and derivative of the constant function $x(t) = 1$. Therefore since our solutions are continuous, and there exist one for each natural number, then we have found an infinite number of solutions.

2.9 (a) Note that different classes of solutions are had for $\alpha = 1$ and $\alpha \neq 1$. Proceeding with the $\alpha \neq 1$ case:

$$x' = x|ln|x||^\alpha$$

Applying barrow's formula yields:

$$\int_{x_0}^{x} \frac{dx}{x|ln|x||^\alpha} = t - t_0$$

Note that x is within the maximal interval $(0, 1)$, therefore $|x| = x, |ln(x)| = -ln(x)$. Thus the substitutions of $u = |ln|x||$ and $du = \frac{-1}{x}dx$ may be made:

$$-\int_{x_0}^{x} u^{-\alpha} du = t - t_0.$$

Therefore after evaluation we have:

$$\frac{-1}{1-\alpha}(u(x)^{1-\alpha} - u(x_0)^{1-\alpha}) = t - t_0$$

Let $k = 1 - \alpha$, therefore by algebraic manipulation we have

$$\begin{aligned}
\frac{-1}{k}(u(x)^k - u(x_0)^k) &= t - t_0 \\
u(x)^k - u(x_0)^k &= k(t_0 - t) \\
u(x)^k &= k(t_0 - t) + u(x_0)^k \\
u(x) &= (k(t_0 - t) + u(x_0)^k)^{\frac{1}{k}} \\
-ln(x) &= (k(t_0 - t) + u(x_0)^k)^{\frac{1}{k}} \\
ln(x) &= -(k(t_0 - t) + u(x_0)^k)^{\frac{1}{k}} \\
x &= e^{-(k(t_0-t)+u(x_0)^k)^{\frac{1}{k}}}.
\end{aligned}$$

Some observations: if $\alpha > 1$ then $1 - \alpha = k < 1$. Since k is negative, then $u(0)^k = (-ln(0))^k = \infty^k = 0, u(1)^k = (-ln(1))^k = 0^k = \infty$. On the other hand if $\alpha < 1$ then $1 - \alpha = k > 1$. Since k is positive, then $u(1)^k = (-ln(1))^k = 0^k = 0, u(0)^k = (-ln(0))^k = \infty^k = \infty$. Therefore evaluating T_0, T_1 for the cases of $\alpha > 1, \alpha < 1$ yields:

i. $T_0, \alpha > 1$:

$$t_0 + -\int_{x_0}^{0} u^{-\alpha} = t_0 + \frac{-1}{k}(u(0)^k - u(x_0)^k) = t_0 + \frac{u(x_0)^k}{k} < \infty$$

ii. $T_1, \alpha > 1$:

$$t_0 + -\int_{x_0}^{1} u^{-\alpha} = t_0 + \frac{-1}{k}(u(1)^k - u(x_0)^k) = -\infty$$

iii. $T_0, \alpha < 1$:

$$t_0 + -\int_{x_0}^{0} u^{-\alpha} = t_0 + \frac{-1}{k}(u(0)^k - u(x_0)^k) = \infty$$

iv. $T_1, \alpha < 1$:

$$t_0 + -\int_{x_0}^{1} u^{-\alpha} = t_0 + \frac{-1}{k}(u(1)^k - u(x_0)^k) = t_0 + \frac{u(x_0)^k}{k} < \infty$$

Since T_0 is finite for $\alpha > 1$, then the solution does not hold for all t, however since T_1 is infinite, then we have found the solutions which are valid for $t > t_0$. Similarly for $\alpha < 1, T_1$ is finite, and T_0 is infinite, giving us another partial solution for $t < t_0$. Let us evaluate the $\alpha = 1$ case. The integral in terms of u from barrow's formula is still valid, but it's evaluation is different:

$$\begin{aligned}
-\int_{x_0}^{x} u^{-1}du &= t - t_0 \\
ln|u(x)| - ln|u(x_0)| &= t_0 - t \\
ln|u(x)| &= t_0 - t + ln|u(x_0)| \\
u(x) &= e^{t_0 - t + ln|u(x_0)|} \\
-ln(x) &= e^{t_0 - t + ln|u(x_0)|} \\
x &= e^{-e^{t_0 - t + ln|u(x_0)|}}
\end{aligned}$$

Evaluating on the endpoints:

i.

$$T_0 = t_0 + -\int_{x_0}^{0} u^{-1}du = t_0 - ln|u(0)| + ln|u(x_0)| = t_0 - ln(\infty) + ln|u(x_0)| = -\infty$$

ii.

$$T_1 = t_0 + -\int_{x_0}^{1} u^{-1}du = t_0 - ln|u(1)| + ln|u(x_0)| = t_0 - ln(0) + ln|u(x_0)| = \infty$$

Since the endpoints take an infinite amount of time to achieve, we have found the unique solution for all t.

(b) We claim that for $x(0) = 0$ unique steady state solution occurs when $\alpha \leq 1$. This gives us two cases to test:

i. Suppose $\alpha < 1$. Since we know from 2.9 that $k > 0$, and that $|T_0| = \infty$, which is equivalent to evaluating barrow's formula on the neighborhood $(-\delta, 0)$, where $x_0 = -\delta$, and that $u(x) = |ln|x||$, thus we only need to evaluate the following:

$$\lim_{\delta \to 0} \int_0^\delta |u^{-\alpha}| du = \lim_{\delta \to 0} \frac{-1}{k}(u(\delta)^k - u(0)^k) = \infty$$

Thus we have found the unique solution to be the steady state for $x(0) = 0$.

ii. Suppose $\alpha = 1$. Since we know that $|T_0| = \infty$, and that provides our cases for $(-\delta, x_0)$ as we simply can let $x_0 = -\delta$ and due to the absolute values we don't have to worry about sign. Thus we need to only evaluate the following:

$$\lim_{\delta \to 0} -\int_0^\delta |u^{-1}| du = \lim_{\delta \to 0} |-ln|u(0)| + ln|u(\delta)|| = \infty.$$

Thus the steady state solution is unique for $\alpha = 1, x(0) = 0$.

We claim that for $x(0) = 1$ unique steady state solution occurs when $\alpha \geq 1$. Thus gives us two cases to test:

i. Suppose $\alpha > 1$ Since we know that for $\alpha > 1$, $T_1 = \infty$, then we have evaluated on $(1 - \delta, 1)$. Thus we only need to evaluate the following:

$$\lim_{\delta \to 0} \int_1^{1+\delta} |u^{-\alpha}| du = |\frac{-1}{k}(u(1+\delta)^k - u(1)^k)| = \infty$$

ii. Suppose $\alpha = 1$. Since we know that for $\alpha = 1$, $T_1 = \infty$ then we have already evaluated on $(1 - \delta, 1)$. Thus we only need to evaluate the following:

$$\lim_{\delta \to 0} \int_1^{1+\delta} |u^{-1}| du = \lim_{\delta \to 0} |-ln|u(1)| + ln|u(\delta)|| = \infty.$$

(c) For which values of α is v Lipschitz? Note that since there does not exists solutions for all values of $\alpha \neq 1$, then $x|ln|x||^\alpha$ is not Lipschitz for those values. We must test for $\alpha = 1$. Note that since $x \in (0, 1)$ then $v(x) = x|ln|x|| = -xln(x)$. Now testing the boundedness of $|v'|$ at 0:

$$\lim_{x \to 0} |v'| = \lim_{x \to 0} |-1 - ln(x)| = |-1 - ln(0)| = \infty.$$

Since v' is unbounded then the $\alpha = 1$ case is not Lipschitz.

2.10 (a) To show that $H(x', x) = H(v_0, x_0)$, we simply have to show that $\frac{d}{dt}H(x', x) = 0$:

$$\begin{aligned}
\frac{d}{dt}H(x', x) &= \frac{d}{dt}(\frac{1}{2}(x')^2 + V(x)) \\
&= x'x'' + \frac{d}{dx}V(x)x' \\
&= x'x'' + -F(x)x' \\
&= x'x'' + -x''x' \\
&= 0.
\end{aligned}$$

Therefore $H(x', x)$ is an arbitrary constant, which can be set to $H(v_0, x_0)$. Therefore to solve $x' = \pm\sqrt{2(H(v_0, x_0) - V(x))}$ we simply apply barrows formula:

$$t(x) = t_0 + \int_{x_0}^{x} \frac{\pm dz}{\sqrt{2(H(v_0, x_0) - V(z))}}.$$

(b) Let $V(x) = \frac{1}{2}x^2$, and $x_0 = 1, v_0 = 0$. Then by the formula above we have:

$$t = t_0 + \int_1^x \frac{\pm dz}{\sqrt{2(\frac{1}{2} - \frac{1}{2}z^2)}} = t_0 + \int_1^x \frac{\pm dz}{\sqrt{1 - z^2}}.$$

This is solved by $x(t) = \pm\sin(t - t_0 + \frac{\pi}{2})$. Note that this is exactly the equation we found infinite solutions for in 2.4! Therefore the piecewise solution

$$x(t) = \begin{cases} 1 & t > t_0 + 2\pi a \\ \sin(t - t_0 + \frac{\pi}{2}) & t \leq t_0 + 2\pi a \end{cases}$$

provides solutions for all $a \in \mathbb{N}$ which have infinite rest periods at the equilibrium point $x = 1$.