2.3 (a)

$$S \circ T = (2x + 2y, 2x, 2x + y)$$

(b)

$$M_T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$M_S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$M_{S\circ T} = \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 2 & 1 \end{bmatrix}$$

2.4 (a) Note that $D(c) = 0$, implying that the first row of the matrix is all zeros. For $\sin(ix), i = 1 \ldots k$, $D(\sin(ix)) = i\cos(ix)$, this would imply that $D_{2i,2i+1} = i$ and the rest of the entries in the row of $2i+1$ would be zero. For $\cos(ix)$, $D(\cos(ix)) = -i\sin(ix)$, therefore $D_{2i+1,2i} = -i$. Furthermore, $D^2(\cos(it)) = -i^2\cos(it)$, $D^2(\sin(it)) = -i^2\sin(it)$, thus for $i \in [k]$, $D^2_{i,i} = -i^2$ and otherwise $D^2_{j,l} = 0$.

(b) Note that D over $\{\cos(t), \sin(t), \ldots, \cos(kt), \sin(kt)\}$ is guaranteed to have linearly independent columns since for columns with an index of $2i+1$ are of the form $[0, \cdots, i, \cdots, 0]$ in the $2i$ position, and columns with an index of $2i$ are of the form $[0, \cdots, -i, \cdots, 0]$ in the $2i+1$ position. Note this ensures that the rows of D have exactly 1 non-zero element, and as shown by the columns, there cannot be repeats. Therefore D is invertible.

2.10 (a) Let $a = a_1, \cdots, a_n \in \mathbb{R}$ be given. We want to find the operator norm of $S(x) = a \cdot x, x \in \mathbb{R}^n$ with respect to the l^1 norm. Note that $|\sum_{i=1}^n a_i x_i| \leq \sum_{i=1}^n |a_i||x_i|$ for all $x \in \mathbb{R}^n$. Thus we claim that the operator norm $\|S\|_1 = \max\{|a_1|, \ldots, |a_n|\}$. Let $a_k = \max\{|a_1|, \ldots, |a_n|\}$, and let $l^1(x) = 1$, therefore

$$|\sum_{i=1}^n a_i x_i| \leq \sum_{i=1}^n |a_i||x_i| \leq \sum_{i=1}^n |a_k||x_i| = |a_k| \sum_{i=1}^n |x_i| = |a_k|.$$

Note that S exactly attains this value when one chooses a vector of the form $(0, \cdots, 0, \pm 1, 0, \cdots, 0)$, the nonzero term is in position k, where the sign of the non-zero term is the opposite of a_k, then $|a_k| = a \cdot x \leq \sum_{i=1}^n |a_i||x_i| = |a_k|$.

(b) Note that for the l^∞ norm we have a very simular situtation as before. Since the only $x \in \mathbb{R}^n$ which satisfy $l^\infty(x) = 1$ are of the form $(0, \cdots, 0, 1, 0, \cdots, 0)$, then we have a finite number of x of the form e_k, so trivially $a \cdot e_k = a_k$, thus S is maximized via finding $\max\{a_1, \cdots, a_n\}$.

(c) For the l^p norm with $x \in \mathbb{R}^n, l^p(x) = 1$, we will have found the operator norm if we can find x such that $\sum_{i=1}^n |a_i x_i| \leq \left(\sum_{i=1}^n |a_i|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^n |a_i|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}$

becomes an equality. Note that if we let $x_i = \frac{|a_i|^{\frac{1}{p-1}}}{\left(\sum_{k=1}^n |a_i|^{\frac{p}{p-1}}\right)^{\frac{1}{p}}}$, then

$$\sum_{i=1}^n |x_i|^p = \frac{1}{\left(\sum_{k=1}^n |a_i|^{\frac{p}{p-1}}\right)} \sum_{i=1}^n |a_i|^{\frac{p}{p-1}} = 1$$

and furthermore we have that

$$\begin{aligned}
&= \sum_{i=1}^n |a_i x_i| \\
&= \sum_{i=1}^n \frac{|a_i|^{\frac{p-1+1}{p-1}}}{\left(\sum_{k=1}^n |a_i|^{\frac{p}{p-1}}\right)^{\frac{1}{p}}} \\
&= \frac{1}{\left(\sum_{k=1}^n a_i^{\frac{p}{p-1}}\right)^{\frac{1}{p}}} \left(\sum_{i=1}^n |a_i|^{\frac{p}{p-1}}\right)^{\frac{p}{p}} \\
&= \left(\sum_{i=1}^n |a_i|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}.
\end{aligned}$$

Thus in general if we choose our x_i such that the sign is always the same as a_i, then we can always ensure that $a_i x_i = |a_i x_i|$. Thus we have found the operator norm for the l^p norm.

3.11 Let $R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$\begin{aligned}
&= 0 \\
&= R_\theta A - A R_\theta \\
&= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \\
&= \begin{bmatrix} a\cos(\theta) - c\sin(\theta) & b\cos(\theta) - d\sin(\theta) \\ a\sin(\theta) + c\cos(\theta) & b\sin(\theta) + d\cos(\theta) \end{bmatrix} - \begin{bmatrix} a\cos(\theta) + b\sin(\theta) & -a\sin(\theta) + b\cos(\theta) \\ c\cos(\theta) + d\sin(\theta) & -c\sin(\theta) + d\cos(\theta) \end{bmatrix} \\
&= \begin{bmatrix} -(c+b)\sin(\theta) & (a-d)\sin(\theta) \\ (a-d)\sin(\theta) & -(b+c)\sin(\theta) \end{bmatrix},
\end{aligned}$$

implying that $-c = b, a = d$. Therefore $A = \begin{bmatrix} a & -c \\ c & a \end{bmatrix}$. If we consider the column vector of (a, c), then we can take it it polar form. Therefore there exists $r \geq 0$ and $\varphi \in [0, 2\pi)$ such that $(a, c) = (r\cos(\varphi), r\sin(\varphi))$. Therefore $A = \begin{bmatrix} r\cos(\varphi) & -r\sin(\varphi) \\ r\sin(\varphi) & r\cos(\varphi) \end{bmatrix}$