1. Let $T : V \to W$ be a linear transformation from a finite-dimensional vector space V to a finite-dimensional vector space W. Let β and β' be ordered bases for V, and let γ' and γ be ordered bases for W. We must show that $[T]_{\beta'}^{\gamma'} = P^{-1}[T]_\beta^\gamma Q$ where Q is the β' to β change of coordinates matrix and P is the γ' to γ change of coordinates matrix. By definition $Q = [\mathbb{I}_V]_{\beta'}^{\beta}$, $P = [\mathbb{I}_W]_{\gamma'}^{\gamma}$. Note that the statement we're trying to prove is equivalent to $P[T]_{\beta'}^{\gamma'} = [T]_\beta^\gamma Q$. Therefore,

$$P[T]_{\beta'}^{\gamma'} = [\mathbb{I}_W]_{\gamma'}^{\gamma}[T]_{\beta'}^{\gamma'} = [\mathbb{I}_W T]_{\beta'}^{\gamma} = [T]_{\beta'}^{\gamma} = [T\mathbb{I}_V]_{\beta'}^{\gamma} = [T]_\beta^\gamma[\mathbb{I}_V]_{\beta'}^{\beta} = [T]_\beta^\gamma Q.$$

Therefore $[T]_{\beta'}^{\gamma'} = P^{-1}[T]_\beta^\gamma Q$.

2. Suppose $A, B \in M_{m\times n}(F), P \in GL_m(F), Q \in GL_n(F), B = P^{-1}AQ$. We must show that there exists an n dimensional vector space V and an m dimensional vector space W over F, ordered bases β and β' for V and γ and γ' for W, and a linear transformation $T : V \to W$ such that
$$A = [T]_\beta^\gamma \text{ and } B = [T]_{\beta'}^{\gamma'}.$$
Let $V = F^n, W = F^m, T = L_A$, and β and γ denoted $\{e_V^1, \dots, e_V^n\}$ and $\{e_W^1, \dots, e_W^m\}$ be the standard ordered bases for F^n and F^m respectively. Therfore we trivially have $A = [L_A]_\beta^\gamma = [T]_\beta^\gamma$. We must find β', γ' such that $B = [T]_{\beta'}^{\gamma'}$. Note that since $Q \in GL_n(F)$, then L_Q^{-1} is a bijection from $V \to V$. By the book proof of rank nullity, $R(L_Q^{-1})$ has a basis $\{L_Q^{-1}(e_V^1), \dots, L_Q^{-1}(e_V^n)\}$, which since L_Q^{-1} is a bijection gives us $R(L_Q^{-1}) = V$, thus $\{L_Q^{-1}(e_V^1), \dots, L_Q^{-1}(e_V^n)\}$ is a basis for V. Let $\beta' = \{L_Q^{-1}(e_V^1), \dots, L_Q^{-1}(e_V^n)\}$. Suppose $x \in V, [x]_{\beta'} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$. Therefore $x = a_1 L_Q^{-1}(e_V^1) + \dots a_n L_Q^{-1}(e_V^n)$. Thus $Qx = Q(a_1 L_Q^{-1}(e_V^1) + \dots a_n L_Q^{-1}(e_V^n)) = Q(a_1 Q^{-1}(e_V^1) + \dots a_n Q^{-1}(e_V^n)) = QQ^{-1}\sum_{i=1}^n a_i e_V^i = \sum_{i=1}^n a_i e_V^i$. Thus $L_Q = [\mathbb{I}_V]_{\beta'}^\beta$. The proof for $P = [\mathbb{I}_W]_{\gamma'}^\gamma$ where $\gamma' = \{L_P^{-1}(e_W^1), \dots, L_P^{-1}(e_W^n)\}$ is identical. Therefore $P^{-1} = ([\mathbb{I}_W]_{\gamma'}^\gamma)^{-1} = [\mathbb{I}_W]_\gamma^{\gamma'}$. Thus $B = P^{-1}AQ = [\mathbb{I}_W]_\gamma^{\gamma'}[T]_\beta^\gamma[\mathbb{I}_V]_{\beta'}^\beta = [\mathbb{I}_W T \mathbb{I}_V]_{\beta'}^{\gamma'} = [T]_{\beta'}^{\gamma'}$.

3. Let V be a finite-dimensional vector space with the ordered basis β. Prove that $\psi(\beta) = \beta^{**}$. Let $\beta = \{x_1, \ldots, x_n\}$. Therefore $\psi(\beta) = \{\hat{x}_1, \ldots, \hat{x}_n\}$. By the corollary to theorem 2.26, there exists a basis $\{v_1, \ldots, v_n\}$ of V^* such that $v_i^* = \hat{x}_i$ and $\delta_{i,j} = \hat{x}_i(v_j) = v_j(x_i)$. Therefore, $x_i^* = \hat{x}$, thus $x_i^{**} = \hat{x}_i$. Therefore $\psi(\beta) = \beta^{**}$.

4. Suppose that V, W are finite dimensional vector spaces over F and that $T : V \to W$ is linear. We must show that $N(T^t) = (R(T))^0$. Let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$ be ordered bases β and γ of V and W respectively, and let $\{f_1, \ldots, f_n\}$ and $\{g_1, \ldots, g_m\}$ be ordered bases β^* and γ^* of V^* and W^* respectively.

   - Suppose $v \in N(T^t)$. We must show that $v \in (R(T))^0$. By definition of being a member in $N(T^t)$, $v \in W^*, T^t(v) = v(T) = 0$. By theorem 2.24 of $T^t(v) = \sum_{s=1}^n (vT)(x_s) f_s$. Therefore $0 = \sum_{s=1}^n (vT)(x_s) f_s$. Since β^* is a basis for V^*, then for all $s \in [n], (vT)(x_s) = 0$. Therefore for an arbitrary $x \in R(T)$, by the book proof of rank nullity $R(T)$ is spanned by $T(x_1), \ldots, T(x_n)$, thus there exists $a_1, \ldots, a_n$ such that $x = a_1 T(x_1) + \ldots + a_n T(x_n)$. Therefore $v(x) = v(\sum_{i=1}^n a_i T(x_i)) = \sum_{i=1}^n a_i v(T(x_i)) = \sum_{i=1}^n a_i 0 = 0$. Thus $v \in (R(T))^0$.
   - Suppose $v \in (R(T))^0$. We must show that $v \in N(T^t)$. By definition of $v \in N(T^t)$ we must show that $T^t(v) = v(T) = 0$. By definition of $v \in (R(T))^0$, for all $x \in R(T), v(x) = 0$. Since v acting on all the elements of $R(T)$ evaluate to 0, then $v(T) = 0$. Thus $v \in N(T^t)$.