4.4.6 (a) A continuous function $f : (0,1) \to \mathbb{R}$ and a cauchy sequence (x_n) such that $f(x_n)$ is not a cauchy sequence.
Consider $f(x) = \sin(\frac{1}{x})$ and the sequence

$$x_n = \begin{cases} \frac{1}{\frac{\pi}{2}+2\pi n} & \text{if } n \text{ odd} \\ \frac{1}{\frac{3\pi}{2}+2\pi n} & \text{if } n \text{ even} \end{cases}$$

Clearly $(x_n) \to 0$, however $f(x_n) = (-1)^{n+1}$ alternating between 1 and -1 forever, thus never converging, and therefore not cauchy.

(b) A continuous function $f : [0,1] \to \mathbb{R}$ and a cauchy sequence (x_n) such that $f(x_n)$ is not a cauchy sequence.
This is impossible since f is continuous then we know the sequence in the image converges, and as continuous functions map compact sets to compact sets (theorem 4.4.2), therefore guaranteed convergence within $f([0,1])$, thus making it cauchy

(c) A continuous function $f : [0,\infty) \to \mathbb{R}$ and a cauchy sequence (x_n) such that $f(x_n)$ is not a cauchy sequence.
This is impossible since a cauchy sequence is bounded by say a constant M, and since the cauchy sequence would be strictly non-negative then the closed interval $[0,M]$ would entirely contain the sequence. Therefore using the reasoning above on a larger compact set we arrive at the same conclusion.

(d) A continuous bounded function f on $(0,1)$ that attains a maximum value on this open interval but not a minimum value.
Consider $f(x) = \frac{1}{2} - |x - \frac{1}{2}|$. By repeated applications of the algebraic continuity theorem, f is continuous for all of $\mathbb{R}$. f attains its maximum at $x = \frac{1}{2}$. However it's minimum including limit points occur at $x = 0,1$ with a value of 0. However f with the restriction to $(0,1)$ cannot attain the minimum.