Let $p_0(x), p_1(x), p_2(x), \ldots,$ be an arbitrary infinite sequence of polynomials with real number coefficients, such that the degree of p_i is i. Prove that for any polynomial q if degree of q is n then there are real numbers $a_0, a_1, \ldots, a_n$ such that $q = \sum_{i=0}^{n} a_i p_i(x)$.

We must show that for any polynomial q of degree n that $q = \sum_{i=0}^{n} a_i p_i(x)$. Suppose q is a polynomial of degree n. We must show $q = \sum_{i=0}^{n} a_i p_i(x)$. By the principal of mathematical induction for all polynomials f where $deg(f) = k$, if $k < n$ then there exists a sequence of real numbers $b_0, \ldots, b_k$ such that $f = \sum_{j=0}^{k} b_j p_j(x)$. By definition of degree the largest term of q and p_n is n. Therefore if you divide q by p_n, since $n - n = 0$, then the largest possible quotient polynomial is constant, or a non-zero real number 0. Therefore $\frac{q(x)}{p_n(x)} = c + \frac{f(x)}{p_n(x)}$ where $c \in \mathbb{R}, c \neq 0$, and f is a polynomial whose degree is less than n. Since $deg(f) < n$, by the induction hypothesis there exists a list of real numbers $b_0, \ldots, b_{n-1}$ such that $f = \sum_{j=1}^{n-1} b_j p_j(x)$ (Note that if $deg(f) < n - 1$, then all $b_j, j > deg(f)$ can simply be set to 0 and still satisfy the requirements of a list of real numbers). Note also that we now have $q(x) = cp^n(x) + f(x)$. Therefore if we set $a_0 = b_0, \ldots a_{n-1} = b_{n-1}$, and $b_n = c$, then we have $\sum_{i=1}^{n} a_i p_i(x) = a_n p_n + \sum_{i=1}^{n} a_i p_i(x) = cp_n(x) + \sum_{j=1}^{n-1} b_j p_j(x) = cp_n + f(x) = q(x)$.