Math 292 Homework 1

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1 Exercises from section 1.3

1.2 Divide both sides of the differential equation by $1+t^2$ to yield

$$x' + \frac{2t}{1+t^2}x = \frac{\cot t}{1+t^2}$$

An integrating factor is $e^{\int \frac{2t}{1+t^2}\,dt} = 1+t^2$. Multiplying by the integrating factor results in

$$\frac{d}{dt}\left(\left(1+t^2\right)x\right) = \cot t$$

and integrating both sides in the t variable can be used to find the general solution

$$x(t) = \frac{\ln(\sin t) + C}{1+t^2}$$

To find the corresponding flow transformation, let $x(t_0) = x_0$ for some t_0 and solve for C to get $C = \left(1+t_0^2\right)x_0 - \ln(\sin t_0)$. By substitution, the flow transformation is

$$\Phi_{t_1,t_0}(x) = \frac{\ln(\sin t_1) + \left(1+t_0^2\right)x_0 - \ln(\sin t_0)}{1+t_1^2}$$

In the case $x\left(\frac{\pi}{2}\right) = 2$, $C = \frac{4+\pi^2}{2}$, so the solution is

$$x(t) = \frac{2\ln(\sin t) + 4 + \pi^2}{2\left(1+t^2\right)}$$

1.3 Replace x' with $\frac{1}{t'}$ and multiply both sides by t' to yield $e^x - 2tx = x^2t'$. Adding $2tx$ to both sides gives $e^x = x^2t' + 2tx$, and the right hand side is clearly the derivative of x^2t with respect to x. Integrating both sides in x gives $e^x + C = x^2t$. Solving for t gives

$$t(x) = \frac{e^x + C}{x^2}$$

1.6 $x' = \frac{1}{3}x + e^{-2t}x^{-2}$ is a Bernoulli equation, so make the substitution $V = x^{1-(-2)} = x^3$ to get a new differential equation

$$V' = V + 3e^{-2t}$$

which can be multiplied by e^{-t} on both sides to give

$$\left(e^{-t}V\right)' = 3e^{-3t}$$

and integrated to yield

$$e^{-t}V = C - e^{-3t}$$

so $x(t) = \sqrt[3]{Ce^t - e^{-2t}}$. To find the corresponding flow transformation, first solve for C using $x(t_0) = x_0$ to yield

$$C = x_0^3 e^{-t_0} + e^{-3t_0}$$

and replace t with t_1 to get

$$\Phi_{t_1,t_0}(x) = \sqrt[3]{\left(x_0^3 e^{-t_0} + e^{-3t_0}\right) e^{t_1} - e^{-2t_1}}$$

If $x(0) = 2$, then $C = 9$, so

$$x(t) = \sqrt[3]{9e^t - e^{-2t}}$$

1.8 The differential equation $x' = x(1-x) - c$ with initial condition $x(0) = x_0$ is separable:

$$\int_{x_0}^{x(t)} \frac{dx}{x(1-x) - c} = \int_0^t dt'$$

Performing partial fraction decomposition:

$$\frac{1}{x(1-x)-c} = \frac{1}{\sqrt{1-4c}} \left(\frac{1}{\sqrt{\frac{1}{4} - c} + \frac{1}{2} - x} + \frac{1}{\sqrt{\frac{1}{4} - c} - \frac{1}{2} + x} \right)$$

which when integrated yields

$$\frac{1}{\sqrt{1-4c}} \ln \left(\frac{\sqrt{\frac{1}{4} - c} - \frac{1}{2} + x}{\sqrt{\frac{1}{4} - c} + \frac{1}{2} - x} \right) \Bigg|_{x_0}^{x(t)} = t$$

so

$$e^{t\sqrt{1-4c}} = \frac{\left(\sqrt{\frac{1}{4} - c} - \frac{1}{2} + x(t)\right)\left(\sqrt{\frac{1}{4} - c} + \frac{1}{2} - x_0\right)}{\left(\sqrt{\frac{1}{4} - c} + \frac{1}{2} - x(t)\right)\left(\sqrt{\frac{1}{4} - c} - \frac{1}{2} + x_0\right)}$$

and solving for $x(t)$ yields

$$x(t) = \frac{\left(\frac{1}{2}x_0 - c + x_0\sqrt{\frac{1}{4} - c}\right) e^{t\sqrt{1-4c}} - \left(\frac{1}{2}x_0 - c - x_0\sqrt{\frac{1}{4} - c}\right)}{\sqrt{\frac{1}{4} - c} + \frac{1}{2} - x_0 - e^{t\sqrt{1-4c}}\left(\frac{1}{2} - x_0 - \sqrt{\frac{1}{4} - c}\right)}$$

Properties of the solution can be determined using Barrow's theorem. The solutions in x to the equation $x(1-x) - c = 0$ are $\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$, so the maximal intervals of x are $\left(-\infty, \frac{1}{2} - \sqrt{\frac{1}{4} - c}\right)$, $\left(\frac{1}{2} - \sqrt{\frac{1}{4} - c}, \frac{1}{2} + \sqrt{\frac{1}{4} - c}\right)$, and $\left(\frac{1}{2} + \sqrt{\frac{1}{4} - c}, \infty\right)$. So, for $\frac{1}{2} - \sqrt{\frac{1}{4} - c} < x_0 < \frac{1}{2} + \sqrt{\frac{1}{4} - c}$, $x(t)$ is defined over $t \in \mathbb{R}$, and $\lim_{t\to\infty} x(t) = \frac{1}{2} + \sqrt{\frac{1}{4} - c}$. When $x_0 > \frac{1}{2} + \sqrt{\frac{1}{4} - c}$, $x(t)$ is not defined everywhere, but it is defined for $t \geq 0$ with $\lim_{t\to\infty} x(t) = \frac{1}{2} + \sqrt{\frac{1}{4} - c}$. Meanwhile, for small x_0 in the range $0 < x_0 < \frac{1}{2} - \sqrt{\frac{1}{4} - c}$, $x(t)$ is defined only up to $t = \frac{1}{\sqrt{1-4c}} \ln \left(\frac{\frac{1}{2} - x_0 + \sqrt{\frac{1}{4} - c}}{\frac{1}{2} - x_0 - \sqrt{\frac{1}{4} - c}} \right)$. The limit as t approaches this time (from the left) is $-\infty$.

2 Exercises from section 2.5

2.1 The differential equation $x'(t) = \sin(x(t))$, $x(0) = x_0$ is separable, with

$$\int_0^t dt' = \int_{x_0}^{x(t)} \frac{1}{\sin x}\, dx = \int_{u_0}^{u(t)} \frac{1+u^2}{2u} \frac{2}{1+u^2} du$$

where $u = \tan\left(\frac{x}{2}\right)$, so $u(t) = \tan\left(\frac{x(t)}{2}\right)$ and $u_0 = \tan\left(\frac{x_0}{2}\right)$. Then it follows that

$$t = \ln\left(\frac{u(t)}{u_0}\right)$$

so $u(t) = u_0 e^t$, and hence

$$x(t) = 2\arctan\left(e^t \tan\left(\frac{x_0}{2}\right)\right) + 2\pi k$$

where $k = \lfloor \frac{x_0}{2\pi} + \frac{1}{2} \rfloor$. (This adjustment is necessary due to the range of the arctangent function. Since k is a constant that takes strictly integer values, this adjustment disappears when differentiating $x(t)$, and also disappears in $\sin(x(t))$ because the $2\pi k$ term simply adds k periods. This justifies its validity.) From the the formula for $x(t)$, it can be seen that $x(t)$ is defined for all $t \in \mathbb{R}$.

2.4 (a) Consider the second equation $(x')^2 - x^2 = 1$. Solving for x' yields $x' = \pm\sqrt{1+x^2}$. It can be shown that the expression on the right-hand side is Lipschitz continuous. Differentiating in terms of x on the right hand side gives $\pm\frac{x}{\sqrt{1+x^2}}$. Taking the absolute value gives $\left|\frac{x}{\sqrt{1+x^2}}\right| = \sqrt{\frac{x^2}{x^2+1}} < \sqrt{\frac{x^2+1}{x^2+1}} = 1$. So, absolute value of the derivative of this right-hand expression is bounded by 1. Letting $F(x) = \pm\sqrt{1+x^2}$, it follows that

$$|F(x_2) - F(x_1)| = |F'(c)\,(x_2 - x_1)| \leq |x_2 - x_1|$$

where $x_1 < c < x_2$, showing that F is Lipschitz with contant 1. Since it has been shown that equation (2) is Lipschitz continuous at all points, (2) is the equation that has a unique solution given the initial condition $x(0) = x_0$.

(b) Differentiate both sides of equation (1) to get

$$2x'x'' + 2x'x = 0$$

so either $x'(t) = 0$ or $x''(t) + x(t) = 0$. The first possibility leaves $x(t) = c$ for some constant c, and plugging it into the original equation gives $c^2 = 1$, so $x(t) = \pm 1$. However, since $-1 < x_0 < 1$, discard this solution. Instead, consider $x''(t) + x(t) = 0$. The function $x(t) = \sin(t+c)$ satisfies this relation, and plugging it into $(x')^2 + x^2$ gives $\cos(t+c) + \sin(t+c) = 1$, satisfying the original differential equation. Because of the initial condition $x(0) = x_0$, it is necessary to find some c such that $x_0 = \sin c$. There are infinitely many such c of the form $c = \arcsin x_0 + 2\pi k$ and $c = \pi - \arcsin x_0 + 2\pi k$, where k is an integer; $\arcsin x_0$ must be defined because $-1 < x_0 < 1$. For example, if $x_0 = \frac{1}{\sqrt{2}}$, then both $x(t) = \sin\left(t + \frac{\pi}{4}\right)$ and $x(t) = \sin\left(t + \frac{3\pi}{4}\right)$ are distinct solutions of $(x')^2 + x^2 = 1$ and $x(0) = \frac{1}{\sqrt{2}}$. These lead to two different solutions of the differential equation with the same x_0. However, it is possible to generate infinitely many solutions using piecewise functions. The sinusoidal solution around $t = 0$ can be used to satisfy the initial condition with $-1 < x(0) = x_0 < 1$, and outside of some region including $t = 0$, $x(t)$ can be constant (either 1 or -1), as this is also a solution to the differential equation. For example, the piecewise function

$$x(t) = \begin{cases} \sin\left(-\pi k_1 - \frac{\pi}{2}\right) & -\pi k_1 > t + \arcsin x_0 + \frac{\pi}{2} \\ \sin\left(t + \arcsin(x_0)\right) & -\pi k_1 < t + \arcsin(x_0) + \frac{\pi}{2} < \pi k_2 \\ \sin\left(\pi k_2 - \frac{\pi}{2}\right) & \pi k_2 < t + \arcsin(x_0) + \frac{\pi}{2} \end{cases}$$

where k_1 and k_2 are two positive integers is constructed this way, and it describes infinitely many solutions to $(x')^2 + x^2 = 1$ and $x(0) = x_0$ because there are infinitely many pairs of positive integers (note that this piecewise does not describe all of the infinite solutions for $x(t)$).