8.12(rudin) (a) Note that for $f(x) = 1$ on $x \in [-\delta, \delta]$ and is 2π periodic that $c_n = \frac{1}{2\pi}\int_{-\delta}^{\delta} e^{-inx}dx = \frac{\sin(n\delta)}{n\pi}$. Trivially $c_0 = \frac{1}{2\pi}\int_{-\delta}^{\delta} 1dx = \frac{\delta}{\pi}$.

(b) Therefore, since at $x = 0, f(x) = 1$, then we have that $1 = \sum_{n=-\infty}^{\infty} c_n = \frac{\delta}{\pi} + \sum_{i=1}^{\infty} \frac{\sin(n\delta)}{n\pi} + \sum_{i=1}^{\infty} \frac{\sin(-n\delta)}{-n\pi} = \frac{\delta}{\pi} + \sum_{i=1}^{\infty} \frac{2\sin(n\delta)}{n\pi}$. Therefore $\frac{\pi-\delta}{2} = \sum_{i=1}^{\infty} \frac{\sin(n\delta)}{n\pi}$

(c) Note by parsaval's theorem that since f is Riemann integrable then it satisfies $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2 = \sum_{n=-\infty}^{\infty} |c_n|^2$. Since f is a constant then we have $\frac{\delta}{\pi} = \sum_{n=-\infty}^{\infty} |c_n|^2$. Doing some rearranging we arrive at $\frac{\delta}{\pi} = \frac{\delta^2}{\pi^2} + 2\sum_{n=1}^{\infty} \frac{\sin^2(\delta n)}{n^2\pi^2}$. Therefore

$$\frac{\delta}{2\pi}\left(1 - \frac{\delta}{\pi}\right) = \sum_{n=1}^{\infty} \frac{\sin^2(\delta n)}{n^2\pi^2}$$

$$\frac{\pi - \delta}{2} = \sum_{n=1}^{\infty} \frac{\sin^2(\delta n)}{n^2\delta}$$

4.7 Verifying the orthogonality of $S = \{e^{i\frac{n\pi x}{l}} : n \in \mathbb{Z}\}$:

- Suppose $n \neq m$. Then

$$\begin{aligned}
\langle e^{i\frac{n\pi x}{l}}, e^{i\frac{m\pi x}{l}}\rangle &= \int_{-l}^{l} e^{i\frac{n\pi x}{l}} e^{-i\frac{m\pi x}{l}} dx \\
&= \int_{-l}^{l} e^{i\frac{(n-m)\pi x}{l}} dx \\
&= \frac{l}{i(n-m)\pi} e^{i\frac{(n-m)\pi x}{l}}]_{-l}^{l} \\
&= \frac{l}{i(n-m)\pi}\left(e^{i\frac{(n-m)\pi l}{l}} - e^{-i\frac{(n-m)\pi l}{l}}\right) \\
&= \frac{l}{i(n-m)\pi}(\cos((n-m)\pi) + \sin((n-m)\pi) - \cos((n-m)\pi) + \sin((n-m)\pi)) \\
&= \frac{l}{i(n-m)\pi} 2\sin((n-m)\pi) \\
&= \frac{l}{i(n-m)\pi} \cdot 0 \\
&= 0
\end{aligned}$$

- If $n = m$ then

$$\begin{aligned}
\|e^{i\frac{n\pi x}{l}}\| &= \sqrt{\int_{-l}^{l} e^{i\frac{n\pi x}{l}} e^{-i\frac{n\pi x}{l}} dx} \\
&= \sqrt{\int_{-l}^{l} e^{i\frac{(n-n)\pi x}{l}} dx} \\
&= \sqrt{\int_{-l}^{l} e^{0} dx} \\
&= \sqrt{2l}
\end{aligned}$$

Therefore S is an orthogonal set. Now for the second claim:

$$\begin{aligned}
\int_{-l}^{l} |\sum_{n=-N}^{N} c_n e^{i\frac{n\pi x}{l}}|^2 dx &= \int_{-l}^{l} \left(\sum_{n=-N}^{N} c_n e^{i\frac{n\pi x}{l}}\right) \overline{\left(\sum_{n=-N}^{N} c_n e^{i\frac{n\pi x}{l}}\right)} dx \\
&= \int_{-l}^{l} \sum_{n=-N}^{N} \sum_{m=-N}^{N} c_n \overline{c_m} e^{i\frac{(n-m)\pi x}{l}} dx \\
&= \sum_{n=-N}^{N} \int_{-l}^{l} \sum_{m=-N}^{N} c_n \overline{c_m} e^{i\frac{(n-m)\pi x}{l}} dx \\
&= \sum_{n=-N}^{N} |c_n|^2 2l \text{ by the orthogonality of } S
\end{aligned}$$

4.15 Let $a_n = \frac{1}{l}\int_{-l}^{l} g(x)\cos(\frac{n\pi x}{l})dx, b_n = \frac{1}{l}\int_{-l}^{l} g(x)\sin(\frac{n\pi x}{l})dx, c_n = \frac{1}{2l}\int_{-l}^{l} g(x)e^{-i\frac{n\pi x}{l}}dx$. Note that

$$\begin{aligned}
c_n &= \frac{1}{2l}\int_{-l}^{l} g(x)e^{-i\frac{n\pi x}{l}}dx \\
&= \frac{1}{2l}\int_{-l}^{l} g(x)\cos(\frac{n\pi x}{l})dx - i\frac{1}{2l}\int_{-l}^{l} g(x)\sin(\frac{n\pi x}{l})dx \\
&= \frac{1}{2}(a_n - ib_n) \\
c_{-n} &= \frac{1}{2}(a_n + ib_n).
\end{aligned}$$

Therefore

$$\begin{aligned}
c_n e^{i\frac{n\pi x}{l}} + c_{-n}e^{-i\frac{n\pi x}{l}} &= \frac{1}{2}(a_n - ib_n)e^{i\frac{n\pi x}{l}} + \frac{1}{2}(a_n + ib_n)e^{-i\frac{n\pi x}{l}} \\
&= a_n \frac{e^{i\frac{n\pi x}{l}} + e^{-i\frac{n\pi x}{l}}}{2} + b_n \frac{e^{i\frac{n\pi x}{l}} - e^{-i\frac{n\pi x}{l}}}{2i} \\
&= a_n\cos(\frac{n\pi x}{l}) + b_n\sin(\frac{n\pi x}{l}).
\end{aligned}$$

Thus for all $n \in \{-N, \cdots, N\} \backslash \{1\}$, $\sum_n c_n e^{i\frac{n\pi x}{l}} = \sum_{n=1}^{n} \left(a_n \cos(\frac{n\pi x}{l}) + b_n \sin(\frac{n\pi x}{l})\right)$. Note for the constant term we have trivially that, $a_0 = \int_{-l}^{l} g(x)\cos(0)dx = \int_{-l}^{l} g(x)e^0 dx = c_0$, Therefore $\sum_{-N}^{N} c_n e^{i\frac{n\pi x}{l}} = a_0 + \sum_{n=1}^{N}\left(a_n \cos(\frac{n\pi x}{l}) + b_n \sin(\frac{n\pi x}{l})\right)$. Furthermore, observe that $c_n e^{i\frac{n\pi x}{l}} = e^{i\frac{n\pi x}{l}} \frac{1}{2l}\int_{-l}^{l} g(t)e^{-i\frac{n\pi t}{l}}dt = \frac{1}{2l}\int_{-l}^{l} g(t)e^{-i\frac{n\pi(x-t)}{l}}dt$. Note that this is the n-th term of $D_N(x-t)$ if it was split into indvidual integrals, therefore $\sum_{-N}^{N} c_n e^{i\frac{n\pi x}{l}} = \int_{-l}^{l} g(t)D_N(x-t)dt$. Additionally,

$$\begin{aligned}
D_N(t) &= \sum_{n=-N}^{N} e^{-i\frac{n\pi t}{l}} \\
&= 1 + \sum_{n=1}^{N} e^{-i\frac{n\pi t}{l}} + \sum_{n=1}^{N} e^{i\frac{n\pi t}{l}} \\
&= 1 + \sum_{n=1}^{N}\left(\cos\left(\frac{n\pi t}{l}\right) - \sin\left(\frac{n\pi t}{l}\right) + \cos\left(\frac{n\pi t}{l}\right) + \sin\left(\frac{n\pi t}{l}\right)\right) \\
&= 1 + 2\sum_{n=1}^{N}\cos\left(\frac{n\pi t}{l}\right)
\end{aligned}$$

Finally, we are ready to show that computing a fourier series is equivalent to convolution by the dirchelet kernel:

$$\begin{aligned}
\sin\left(\frac{\pi t}{2l}\right)\left(1 + 2\sum_{n=1}^{N}\cos\left(\frac{n\pi t}{l}\right)\right) &= \sin\left(\frac{\pi t}{2l}\right) + \sum_{n=1}^{N}\sin\left(\frac{(n+\frac{1}{2})\pi t}{l}\right) - \sin\left(\frac{(n-\frac{1}{2})\pi t}{l}\right) \\
&= \sin\left(\frac{\pi t}{2l}\right) + \sum_{n=1}^{N}\sin\left(\frac{(n+\frac{1}{2})\pi t}{l}\right) - \sum_{j=0}^{N-1}\sin\left(\frac{(j+\frac{1}{2})\pi t}{l}\right) \\
&= \sin\left(\frac{\pi t}{2l}\right) - \sin\left(\frac{\pi t}{2l}\right) + \sin\left(\frac{(N+\frac{1}{2})\pi t}{l}\right) + \sum_{n=1}^{N-1}\sin\left(\frac{(n+\frac{1}{2})\pi t}{l}\right. \\
&= \sin\left(\frac{(N+\frac{1}{2})\pi t}{l}\right)
\end{aligned}$$

Therefore $1 + 2\sum_{n=1}^{N}\cos\left(\frac{n\pi t}{l}\right) = \frac{\sin\left(\frac{\left(N+\frac{1}{2}\right)\pi t}{l}\right)}{\sin\left(\frac{\pi t}{2l}\right)}$

4.18 We must first normalize what we're projecting by. This means we must compute $\int_0^{\pi}\sin(nx)dx$. This integral when evaluated yields $\frac{\pi}{2} - \frac{\sin(2\pi n)}{4n}$. Since $\sin(2\pi n) = 0$ for all $n \in \mathbb{N}$, then we are normalizing by $\frac{2}{\pi}$

- For $f(x) = 1$ we must compute the integral $\int_0^{\pi}\sin(nx)dx$, since we're on $[0, \pi]$. Note that

$$a_n = \frac{2}{\pi}\int_0^{\pi}\sin(nx)dx = \frac{2}{\pi}\frac{1}{n}\int_0^{n\pi}\sin(u)du = \frac{2}{\pi}\frac{1}{n}(1-\cos(n\pi))$$

Note that if n is even then $a_n = \frac{1-1}{n} = 0$. If n is odd then $a_n = \frac{4}{n\pi}$. Therefore $1 \approx \sum_{n=0}^{\infty} \frac{4}{\pi(2n+1)} \sin((2n+1)x)$ on $[0, \pi]$.

- For $g(x) = \cos(x)$ we must compute the integral $\int_0^\pi \cos(x)\sin(nx)dx$. Note that

$$\frac{2}{\pi}\int_0^\pi \cos(x)\sin(nx)dx = \frac{1}{\pi}\int_0^\pi \sin((n+1)x) - \sin((1-n)x)dx = \frac{1}{\pi}\left(\frac{1}{n+1} - \frac{1}{1-n}\right)$$

If n is odd then we have $a_n = 0$, and if n is even ($n = 2m$) we have that $a_n = \frac{4m}{\pi(4m^2-1)}$. Therefore $\cos(x) \approx \sum_{n=1}^{\infty} \frac{8n^2}{\pi(4n^2-1)} \sin(2nx)$