2.2 *A fair coin is flipped three times. What is the probability that the second flip is tails, given that there is at most one tails among the three flips?*
Let $A :=$ { the second flip is tails } and $B :=$ { there is at most one tails among the three flips }. We want to compute $\mathbb{P}(A|B)$. Therefore by the multiplication rule $\mathbb{P}(A|B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)}$. Note that $\mathbb{P}(B) = \frac{1}{2}$, as there are four three digit binary sequences with at max a single 1. For $\mathbb{P}(AB)$, the probability would be $\frac{1}{8}$ as $A \subset B$, and $\mathbb{P}(A) = \frac{1}{8}$. Therefore $\mathbb{P}(A|B) = \frac{1}{4}$.

2.8 *We shuffle a deck of cards and deal three cards (without replacement). Find the probability that the first card is a queen, the second is a king and the third is an ace.*
Let the events described above be given by $A =$ { first card is a queen }, $B =$ { second card is a king }, $C =$ { third card is an ace }. We want to compute $\mathbb{P}(ABC)$. Therefore we must apply the multiplication rule a few times:

$$\mathbb{P}(ABC) = \mathbb{P}(AB)\mathbb{P}(C \mid AB) = \mathbb{P}(A)\mathbb{P}(B \mid A)\mathbb{P}(C \mid AB)$$

Note that $\mathbb{P}(A) = \frac{4}{52}, \mathbb{P}(B \mid A) = \frac{4}{51}, \mathbb{P}(C \mid AB) = \frac{4}{50}$. Therefore $\mathbb{P}(ABC) = \frac{4^3}{50*51*52} \approx$ 0.05%

2.10 *I have a bag with 3 fair dice. One is 4-sided, one is 6-sided, and one is 12-sided. I reach into the bag, pick one die at random and roll it. The outcome of the roll is 4. What is the probability that I pulled out the 6-sided die?*
Let D_n where $n \in \{4, 6, 12\}$ denote the probability that a given die is drawn. The problem is asking to compute $\mathbb{P}(D_6 \mid 4)$. We don't have any easy way to compute this quantity, however if we apply Bayes' rule then we get $\mathbb{P}(D_6 \mid 4) = \frac{\mathbb{P}(4|D_6)\mathbb{P}(D_6)}{\mathbb{P}(4)}$. Note that $\mathbb{P}(D_6) = \frac{1}{3}$ since there are 3 dice, $\mathbb{P}(4 \mid D_6) = \frac{1}{6}$ since there is an equal chance for a 4 to be rolled among the 6 faces, and $\mathbb{P}(4)$ can be computed using the decomposition rule as $\mathbb{P}(4) = \sum_{n\in\{4,6,12\}} \mathbb{P}(D_n)\mathbb{P}(4 \mid D_n) = \frac{1}{4\cdot3} + \frac{1}{6\cdot3} + \frac{1}{12\cdot3} = \frac{1}{6}$. Therefore $\mathbb{P}(D_6 \mid 4) = \frac{\frac{1}{6}\frac{1}{3}}{\frac{1}{6}} = \frac{1}{3}$, which makes sense since every die can roll a four.

2.12 *We choose a number from the set* $\{1, 2, 3, \cdots, 100\}$ *uniformly at random and denote this number by* X. *For each of the following choices decide whether the two events in question are independent or not.*

(a) $A = \{X \textit{ is even}\}$, $B = \{X \textit{ is divisible by 5}\}$
$\mathbb{P}(A) = \frac{50}{100} = \frac{1}{2}, \mathbb{P}(B) = \frac{20}{100} = \frac{1}{5}$. Note that AB is the set of all numbers divisible by both 2 and 5, which would be 10. Therefore $Pro(AB) = \frac{10}{100} = \frac{1}{10}$. Since $\frac{1}{2}\frac{1}{5} = \frac{1}{10}$, then the probability of the sets of independent.

(b) $C = \{X \textit{ has two digits}\}$, $D = \{X \textit{ is divisible by 3}\}$.
Note that aside from the first 9 numbers and 100, every other element in the set has two digits. Therefore $\mathbb{P}(C) = \frac{90}{100} = \frac{9}{10}$. For $\mathbb{P}(D)$, there are 33 numbers under 100, thus $\mathbb{P}(D) = \frac{33}{100}$. Note that for $\mathbb{P}(CD)$ one has to take away three numbers, $\{3, 6, 9\}$, from D. Therefore $\mathbb{P}(CD) = \frac{3}{10}$. Note that this is not the same as $\frac{9}{10}\frac{33}{100}$. Therefore C, D do not have independent probabilities.

(c) $E = \{X \textit{ is a prime}\}$, $F = \{X \textit{ has a digit 5}\}$.. $\mathbb{P}(E) = \frac{25}{100} = \frac{1}{4}, \mathbb{P}(F) = \frac{19}{100}$. Note that the prime numbers under 100 w/ a 5 are $\{5, 53, 59\}$. Therefore $\mathbb{P}(EF) = \frac{3}{100}$. Clearly these sets are not independent since $\mathbb{P}(E)\mathbb{P}(F) = \frac{19}{400}$.

2.30 *Assume that $\frac{1}{3}$ of all twins are identical twins. You learn that Miranda is expecting twins, but you have no other information.*

(a) *Find the probability that Miranda will have two girls.*
Let GG denote the event that miranda has two girls, let F be the event that Miranda has fraternal twins, and let I be the event that Miranda has identical twins.
Since F and I partition the sample space, we can compute $\mathbb{P}(GG)$ via the decomposition rule as follows:

$$\mathbb{P}(GG) = \mathbb{P}(F)\mathbb{P}(GG \mid F) + \mathbb{P}(T)\mathbb{P}(GG \mid T) = \frac{2}{3}\frac{1}{4} + \frac{1}{3}\frac{1}{2} = \frac{1}{3}$$

(b) *You learn that Miranda gave birth to two girls. What is the probability that the girls are identical twins?*
We can compute this quantity in the following way:

$$\mathbb{P}(I \mid GG) = \frac{\mathbb{P}(GGI)}{\mathbb{P}(GG)} = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2}$$

2.38 *We choose one of the words in the following sentence uniformly at random and then choose one of the letters of that word, again uniformly at random:*

SOME DOGS ARE BROWN

(a) *Find the probability that the chosen letter is R*
$\mathbb{P}(R) = \mathbb{P}(SOME)\mathbb{P}(R \mid SOME) + \mathbb{P}(DOGS)\mathbb{P}(R \mid DOGS) + \mathbb{P}(ARE)\mathbb{P}(R \mid ARE) + \mathbb{P}(BROWN)\mathbb{P}(R \mid BROWN) = \frac{1}{4}(0 + 0 + \frac{1}{3} + \frac{1}{5}) = \frac{2}{15}$

(b) *Let X denote the length of the chosen word. Determine the probability mass function of X.*

$$p_X(L) = \begin{cases} L = 3 & \frac{1}{4} \\ L = 4 & \frac{1}{2} \\ L = 5 & \frac{1}{4} \\ 0 & \text{otherwise} \end{cases}$$

(c) *For each possible value k of X determine the conditional probability $\mathbb{P}(X = k|X > 3)$.*

- $\mathbb{P}(X = 3|X > 3) = \frac{\mathbb{P}((X=3)(X>3))}{\mathbb{P}(X>3)} = \frac{4*0}{3} = 0$
- $\mathbb{P}(X = 4|X > 3) = \frac{\mathbb{P}((X=4)(X>3))}{\mathbb{P}(X>3)} = \frac{4\frac{1}{4}}{3} = \frac{1}{3}$
- $\mathbb{P}(X = 5|X > 3) = \frac{\mathbb{P}((X=5)(X>3))}{\mathbb{P}(X>3)} = \frac{4\frac{1}{2}}{3} = \frac{2}{3}$

(d) *Determine the conditional probability* $\mathbb{P}(R \mid X > 3)$.

$$\begin{aligned}
Pro(R \mid X > 3) &= \sum_i \mathbb{P}(RWORDS(i) \mid X > 3) \\
&= \sum_i \frac{\mathbb{P}(WORDS(i)(X > 3))\mathbb{P}(R \mid WORDS(i)(X > 3))}{\mathbb{P}(X > 3)} \\
&= \sum_i \frac{\mathbb{P}(WORDS(i)|(X > 3))\mathbb{P}(X > 3)\mathbb{P}(R \mid WORDS(i)(X > 3))}{\mathbb{P}(X > 3)} \\
&= \sum_i \mathbb{P}(WORDS(i)|X > 3)\mathbb{P}(R \mid WORDS(i)X > 3) \\
&= \frac{1}{3}0 + \frac{1}{3} * 0 + 0 * 0 + \frac{1}{3}\frac{1}{5}
\end{aligned}$$

(e) *Given that the chosen letter is R, what is the probability that the chosen word was BROWN?*
$\mathbb{P}(BROWN|R) = \frac{\mathbb{P}(RBROWN)}{\mathbb{P}(BROWN)} = \frac{\mathbb{P}(R|BROWN)\mathbb{P}(BROWN)}{\mathbb{P}(R)} = \frac{\frac{1}{4}\frac{1}{5}}{\frac{2}{15}} = \frac{3}{8}$

2.58 *Suppose that a person's birthday is a uniformly random choice from the 365 days of a year (leap years are ignored), and one person's birthday is independent of the birthdays of other people. Alex, Betty and Conlin are comparing birthdays. Define these three events:*

$$\begin{aligned}
A &= \{\text{Alex and Betty have the same birthday}\} \\
B &= \{\text{Betty and Conlin have the same birthday}\} \\
C &= \{\text{Conlin and Alex have the same birthday}\}.
\end{aligned}$$

(a) *Are events A, B and C pairwise independent?*
Yes, observe that

$$\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C) = \frac{1}{365}$$

Since once one birthday is fixed the chance that the day is the same is $\frac{1}{365}$. Additionally,

$$\mathbb{P}(ABC) = \mathbb{P}(AB) = \mathbb{P}(BC) = \mathbb{P}(AC) = \frac{1}{365^2}$$

are all equivalent, and are the successive probabilities once one pair is fixed, you choose again. Therefore

$$\mathbb{P}(AB) = \frac{1}{365^2} = \mathbb{P}(A)\mathbb{P}(B)$$

$$\mathbb{P}(BC) = \frac{1}{365^2} = \mathbb{P}(B)\mathbb{P}(C)$$

$$\mathbb{P}(AC) = \frac{1}{365^2} = \mathbb{P}(A)\mathbb{P}(C)$$

thus the probabilities are pairwise independent.

(b) *Are events A, B and C independent?*
They are not independent, as

$$\frac{1}{365^3} = \mathbb{P}(ABC) \neq \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = \frac{1}{365^2}$$

2.60 *Assume that A, B and C are mutually independent events according to Definition 2.22. Verify the identities below, using the definition of independence, set operations and rules of probability*

(a) $\mathbb{P}(AB^c) = \mathbb{P}(A)\mathbb{P}(B^c)$

$$\begin{aligned}\mathbb{P}(AB^c) &= \mathbb{P}(A) - \mathbb{P}(AB)\\ &= \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B)\\ &= \mathbb{P}(A)(1 - \mathbb{P}(B))\\ &= \mathbb{P}(A)\mathbb{P}(B^c)\end{aligned}$$

(b) $\mathbb{P}(A^cC^c) = \mathbb{P}(A^c)\mathbb{P}(C^c)$

$$\begin{aligned}\mathbb{P}(A^cC^c) &= \mathbb{P}(C^c) - \mathbb{P}(AC^c)\\ &= \mathbb{P}(C^c)(1 - \mathbb{P}(A))\\ &= \mathbb{P}(C^c)\mathbb{P}(A^c)\end{aligned}$$

(c) $\mathbb{P}(AB^cC) = \mathbb{P}(A)\mathbb{P}(B^c)\mathbb{P}(C)$

$$\begin{aligned}\mathbb{P}(AB^cC) &= \mathbb{P}(AC) - \mathbb{P}(ABC)\\ &= \mathbb{P}(A)\mathbb{P}(C) - \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)\\ &= \mathbb{P}(A)\mathbb{P}(C)(1 - \mathbb{P}(B))\\ &= \mathbb{P}(A)(1 - \mathbb{P}(B))\mathbb{P}(C)\\ &= \mathbb{P}(A)\mathbb{P}(B^c)\mathbb{P}(C)\end{aligned}$$

(d) $\mathbb{P}(A^cB^cC^c) = \mathbb{P}(A^c)\mathbb{P}(B^c)\mathbb{P}(C^c)$

$$\begin{aligned}\mathbb{P}(A^cB^cC^c) &= \mathbb{P}(B^c) - \mathbb{P}(AB^cC)\\ &= \mathbb{P}(B^c) - \mathbb{P}(A)\mathbb{P}(B^c)\mathbb{P}(C)\\ &= \mathbb{P}(B^c)(1 - \mathbb{P}(A)\mathbb{P}(C))\\ &= \mathbb{P}(B^c)\mathbb{P}(A^cC^c)\\ &= \mathbb{P}(B^c)\mathbb{P}(A^c)\mathbb{P}(C^c)\\ &= \mathbb{P}(A^c)\mathbb{P}(B^c)\mathbb{P}(C^c)\end{aligned}$$