1. • Show that $x(t)$ satisifes $\|A^{-1}x(t)\|^2 = 1$.

$$\begin{aligned}\|A^{-1}x(t)\|^2 &= \|A^{-1}Au(t)\|^2\\ &= \|u(t)\|^2\\ &= \| \begin{bmatrix}\cos(t)\\ \sin(t)\end{bmatrix} \|^2\\ &= \cos^2(t)+\sin^2(t)\\ &= 1.\end{aligned}$$

• Show that the equation above may be written as $x \cdot Mx = 1$.

$$\begin{aligned}x\cdot Mx &= x^T Mx\\ &= x^T(A^{-1})^T A^{-1}x\\ &= (A^{-1}x)^T A^{-1}x\\ &= A^{-1}x\cdot A^{-1}x\\ &= \|A^{-1}x\|^2\\ &= 1.\end{aligned}$$

• Show that M is symmetric.

$$\begin{aligned}M^T &= ((A^{-1})^T A^{-1})^T\\ &= (A^{-1})^T((A^{-1})^T)^T\\ &= (A^{-1})^T A^{-1}\\ &= M\end{aligned}$$

• Suppose $M = \begin{bmatrix}a & c\\ c & b\end{bmatrix}$. We must show that $x\cdot Mx$ can be written as ax^2+by^2+2cxy.

$$\begin{aligned}1 &= x\cdot Mx\\ &= \begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}a & c\\ c & b\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}\\ &= \begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}ax+cy\\ cx+by\end{bmatrix}\\ &= ax^2+by^2+2cxy.\end{aligned}$$

2. • Show that both λ_1 and λ_2 are positive. Since $x\cdot Mx = \|A^{-1}x\|^2$, then all outputs of $x\cdot Mx$ are strictly positive. Suppose $x = u_1$, then $u_1\cdot Mu_1 = u_1\cdot\lambda_1 u_1 = \lambda_1\|u_1\|^2 = \lambda_1 > 0$. A similar proof exists for u_2. Therefore the eigenvalues are strictly positive.

• Suppose $\lambda_1 = \lambda_2$. We must show that $\|x(t)\| = \frac{1}{\sqrt{\lambda_1}}$. Let U be the matrix where u_1 and u_2 are columns. Since u_1, u_2 are orthonormal, then U is an orthogonal matrix. Therefore $U^{-1} = U^T$. Let $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, and let $q = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = U^T x$. Therefore we may rewrite $x \cdot Mx$ as follows:

$$1 = x \cdot Mx = q \cdot Dq = \lambda_1 q_1^2(t) + \lambda_2 q_2^2(t).$$

This equation defines an ellipse paramaterized by $q_1(t) = \pm\frac{1}{\lambda_1}\cos(t), q_2(t) = \pm\frac{1}{\lambda_2}\sin(t)$. Since $q = V^T x$, then we may explicitly solve for x via $x = Vq = \pm(\frac{1}{\lambda_1}\cos(t)u_1 + \frac{1}{\lambda_2}\sin(t)u_2)$. Therefore if $\lambda_1 = \lambda_2$, then $\|x\| = \sqrt{\frac{\cos^2(t)}{\lambda_1} + \frac{\sin^2(t)}{\lambda_2}} = \sqrt{\frac{\cos^2(t)}{\lambda_1} + \frac{\sin^2(t)}{\lambda_1}} = \frac{1}{\sqrt{\lambda_1}}$.

- Consider the case where $\lambda_1 > \lambda_2$. Note that $\|x\|^2 = \frac{\cos^2(t)}{\lambda_1} + \frac{\sin^2(t)}{\lambda_2} = u \cdot \begin{bmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{bmatrix} u$, let the matrix in the quadratic function above be denoted S^2. By lemma 19 in the previous carlen multivarible textbook, $\|x\|^2$ on the unit circle is maximized and minimized by the eigenvalues of S^2. For the matrix S^2 the eigenvalues are obviously $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}$. Since $\lambda_1 > \lambda_2$, then $\frac{1}{\lambda_1} < \frac{1}{\lambda_2}$, therefore making $\frac{1}{\lambda_2}$ the maximum of $\|x\|^2$, and therefore forcing $\frac{1}{\lambda_1}$ to be the minimum. Since $\sqrt{}$ is a monotonically increasing function on $\mathbb{R}_+$, then $\|x\|$ has a maximum of $\frac{1}{\sqrt{\lambda_2}}$ and a minimum of $\frac{1}{\sqrt{\lambda_1}}$. We must show that $\|x\|$ is maximal if and only if $x(t) = \pm\frac{1}{\lambda_2}u_2$.
  - Suppose $x(t) = \pm\frac{1}{\lambda_2}u_2$. We must show that $\|x(t)\|$ is maximal.

$$\|x(t)\| = \sqrt{\frac{\cos^2(t)}{\lambda_1} + \frac{\sin^2(t)}{\lambda_2}} \leq \sqrt{\frac{\cos^2(t)}{\lambda_2} + \frac{\sin^2(t)}{\lambda_2}} = \frac{1}{\sqrt{\lambda_2}} = \|\pm\frac{1}{\sqrt{\lambda_2}}u_2\|$$

  - Suppose $\|x(t)\|$ is maximal. We must show that $x(t) = \pm\frac{1}{\lambda_2}u_2$. Since x^2 is strictly increasing on $\mathbb{R}_+$ then $\|x(t)\|^2$ is maximal. This is equivalent to $\begin{bmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{bmatrix}$

3. Let

$$v_1 = \frac{1}{\sqrt{\lambda_1}}A^{-1}u_1, v_2 = \frac{1}{\sqrt{\lambda_2}}A^{-1}u_2.$$

- Show that $\{v_1, v_2\}$ is an orthonormal basis for $\mathbb{R}^2$. By definition of orthonormal basis we must show that $v_1 \cdot v_2 = 0, v_1 \cdot v_1 = v_2 \cdot v_2 = 1$.
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$$\begin{aligned} v_1 \cdot v_2 &= v_1^T v_2 \\ &= \frac{1}{\sqrt{\lambda_1}}(A^{-1}u_1)^T \frac{1}{\sqrt{\lambda_2}}A^{-1}u_2 \\ &= \frac{1}{\sqrt{\lambda_1\lambda_2}}u_1^T(A^{-1})^T A^{-1}u_2 \\ &= \frac{1}{\sqrt{\lambda_1\lambda_2}}\lambda_2 u_1^T u_2 \\ &= 0 \end{aligned}$$

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$$v_1 \cdot v_1 =$$