2.5 Note that the $x_1(t), x_2(t)$ are flow transformations, and can be given as $x_1 = \Psi_t(x_1), x_2 = \Psi_t(x_2)$. Thus the "catching up" condition can be interpreted as $\Psi_0(x_2) = \Psi_T(x_1)$. Therefore we have:

$$\Psi_{(k+1)T}(x_1) = \Psi_{kT} \circ \Psi_T(x_1) = \Psi_{kT} \circ \Psi_0(x_2) = \Psi_{kT}(x_2).$$

2.6 (a) showing that $v(x) = tanh(x)$ is Lipschitz:

$$tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} < \frac{e^x + e^{-x}}{e^x + e^{-x}} = 1$$

Therefore $|tanh(x) - tanh(y)| < 1 + 1 = 2$.
Finding the flow transformation:

$$\begin{aligned}
t - t_0 &= \int_{x_0}^{x} \frac{dx}{tanh(x)} \\
&= ln(sinh(x)) - ln(sinh(x_0)) \\
t - t_0 + ln(sinh(x_0)) &= ln(sinh(x)) \\
sinh(x) &= e^{t-t_0+ln(sinh(x_0))} \\
x &= arsinh(e^{t-t_0+ln(sinh(x_0))}).
\end{aligned}$$

Verifying that $\frac{d}{dx_0}\Psi_t(x_0) = \frac{v(\Psi_t(x_0))}{v(x_0)}$:

$$\begin{aligned}
\frac{d}{dx}\Psi_t &= \frac{d}{dx} arsinh(e^{t-t_0+ln(sinh(x_0))}) \\
&= \frac{e^{t-t_0+ln(sinh(x_0))}}{\sqrt{1 + e^{2(t-t_0+ln(sinh(x_0)))}}} cosh(x_0) \\
&= \frac{e^{t-t_0+ln(sinh(x_0))}}{\sqrt{1 + e^{2(t-t_0+ln(sinh(x_0)))}}} \frac{1}{tanh(x_0)} \\
&= \frac{tanh(arsinh(e^{t-t_0+ln(sinh(x_0))}))}{tanh(x_0)} \\
&= \frac{v(\Psi_t(x_0))}{v(x_0)}.
\end{aligned}$$

Verifying that $\frac{d}{dt}\Psi_t(x_0) = v(\Psi_t(x_0))$:

$$\begin{aligned}
\frac{d}{dt}\Psi_t &= \frac{d}{dt} arsinh(e^{t-t_0+ln(sinh(x_0))}) \\
&= \frac{e^{t-t_0+ln(sinh(x_0))}}{\sqrt{1 + e^{2(t-t_0+ln(sinh(x_0)))}}} \\
&= tanh(arsinh(e^{t-t_0+ln(sinh(x_0))})) \\
&= v(\Psi_t(x_0))
\end{aligned}$$

(b) Verifying that $\lim_{t\to\infty} x_2(t) - x_1(t) = \int_{x_1}^{x_2} \frac{1}{v(x)} dx$:

$$\begin{aligned}
\lim_{t\to\infty} x_2(t) - x_1(t) &= \lim_{t\to\infty} \int_{x_1}^{x_2} \frac{d}{dx}\Psi_t(x)dx \\
&= \int_{x_1}^{x_2} \frac{\lim_{t\to\infty} v(\Psi_t(x))}{v(x)} dx \\
&= \int_{x_1}^{x_2} \frac{\lim_{t\to\infty} \frac{e^{t-t_0+ln(sinh(x_0))}}{\sqrt{1+e^{2(t-t_0+ln(sinh(x_0)))}}}}{v(x)} dx \\
&= \int_{x_1}^{x_2} \frac{1}{v(x)} dx.
\end{aligned}$$

2.10 (a) The inverse transform is given by:

$$(x, y) = (e^{-u}, \frac{v}{e^{2u}}).$$

(b) Time derivatives of u, v:

$$\frac{d}{dt}u = \frac{-x'}{x} = \frac{-x}{x} = 1$$

$$\begin{aligned}
\frac{d}{dt}v &= 2xyx' + x^2y' \\
&= 2x^2y + x^2((xy - \frac{1}{x})^2 - \frac{2}{x^2}) \\
&= 2x^2y + (x^2y - 1)^2 - 1 \\
&= 2x^2y + x^4y^2 - 2x^2y + 1 - 2 \\
&= x^4y^2 - 1 \\
&= v^2 - 1
\end{aligned}$$

Thus the vector field $\vec{w}$ for the system $\vec{u}' = \vec{w}(\vec{u})$ is given by: $\vec{w} = (-1, v^2 - 1)$. This is clearly decoupled as specified.

(c) Solving the decoupled system for $\vec{u}(0) = (u_0, v_0)$. Since $u' = -1$, then $u = u_0 - t$. For $v' = v^2 - 1$, by barrow's formula we get the equation

$$t = \int_{v_0}^{v} \frac{dz}{z^2 - 1}.$$

Splitting $\frac{1}{z^2-1}$ apart by partial fraction decomposition yields

$$\frac{1}{z^2 - 1} = \frac{1}{2(v-1)} + \frac{-1}{2(v+1)}.$$

This results in the integral being evaluated as

$$t - t_0 = ln\left(\sqrt{\frac{v-1}{v+1}}\right) - ln\left(\sqrt{\frac{v_0 - 1}{v_0 + 1}}\right).$$

Inverting to get v yields:

$$v(t) = \frac{v_0 + 1 + (v_0 - 1)e^{2t}}{v_0 + 1 - (v_0 - 1)e^{2t}}$$

We must show that this solution for $\vec{u}$ with $\vec{u}(0) = (u_0, v_0)$ exists uniquely for all t if and only if $|v_0| \leq 1$.

- ($\Rightarrow$) Suppose the solution given above at $\vec{u} = (u_0, v_0)$ exists for all t and is unique. Then we must show $|v_0| \leq 1$. Suppose for contradiction that $|v_0| > 1$. Then let's see if we can get the denominator of v to be 0.

$$\begin{aligned} 0 &= v_0 + 1 - (v_0 - 1)e^{2t} \\ (v_0 - 1)e^{2t} &= v_0 + 1 \\ 2t &= ln\left(\frac{v_0 + 1}{v_0 - 1}\right) \end{aligned}$$

  Since $v_0 > 1$, then $v_0 - 1 > 0$, therefore the natural log is defined, and we can get a value for t, which means that v has a singularity, which contradicts the solution existing for all t. Therefore $|v_0| \leq 1$.

- ($\Leftarrow$) Suppose $|v_0| \leq 1$. We must show there exists a unique solution for all t at $\vec{u} = (u_0, v_0)$. Note that by definition we're operating inside of the maximal interval $(-1, 1)$ and the endpoints $\{-1, 1\}$. First for the cases where $v_0 \in (-1, 1)$. Since we need to show the existence and uniqueness of a solution, we simply need to show that $\vec{w}$ is Lipschitz on $(-1, 1)$. Note that since $w_1 = -1$, that for any value of v_0, w_1 is always bounded. For $v' = w_2 = v^2 - 1$, since $v \in (-1, 1)$, then $max(|w_2(v)|) = 1$. Then we have the inequality

$$|x^2 - 1 - y^2 + 1| \leq |x^2 - y^2| \leq |x + y||x - y| \leq 2|x - y|$$

  Therefore on $(-1, 1)$ we have each component of $\vec{w}$ lipschitz continuous, thus $\|\vec{w}\|$ is lipschitz. For the case of $v_0 = 1$, we must show that the constant solution is the only one for $v' = v^2 - 1$. Since $\lim_{\delta \to 0} \int_{1-\delta}^{1} \frac{dz}{|z^2-1|} \geq \lim_{\delta \to 0} |artanh(1 - \delta) - artanh(1)| = \lim_{\delta \to 0} \infty = |artanh(1) - artanh(1 + \delta)| \leq \lim_{\delta \to 0} \int_{1}^{1+\delta} \frac{dz}{|z^2-1|}$, then the times for which v leaves 1 is infinite, therefore the constant solution is the unique solution when $v_0 = 1$. Also note that $|artanh(x)| = |artanh(-x)|$, therefore these inequalities can be converted to also show the uniqueness of the steady state solution for $v = -1$.

(d)

$$\vec{u}(t) = \left(-1, \frac{x_0^2 y_0 + 1 + (x_0^2 y_0 - 1)e^{2t}}{x_0^2 y_0 + 1 - (x_0^2 y_0 - 1)e^{2t}}\right)$$

$$\vec{x}(t) = \left(x_0 e^t, \frac{1}{x_0^2 e^{2t}} \frac{x_0^2 y_0 + 1 + (x_0^2 y_0 - 1)e^{2t}}{x_0^2 y_0 + 1 - (x_0^2 y_0 - 1)e^{2t}}\right)$$

$$\begin{aligned}
\vec{x}' &= (x_0e^t, \frac{-2}{x_0^2e^{2t}}\frac{x_0^2y_0+1+(x_0^2y_0-1)e^{2t}}{x_0^2y_0+1-(x_0^2y_0-1)e^{2t}} \\
&+ \frac{2}{x_0^2}\frac{x_0^2y_0-1}{-e^{2t}(x_0^2y_0-1)+x_0^2y_0+1} + \frac{2(x_0^2y_0-1)(e^{2t}(x_0^2y_0-1)+x_0^2y_0+1)}{x_0^2(-e^{2t}(x_0^2y_0-1)+x_0^2y_0+1)^2}) \\
&= (x, -2y - \frac{1}{x^2} + x^2y^2)
\end{aligned}$$

(e) Skipped

(f) To solve the Ricatti equation, we assume there exists a solution for y, denoted y_1 given by $y_1 = ce^{\alpha t}$. Thus our equation becomes:

$$\alpha ce^{\alpha t} = c^2x_0^2e^{2(\alpha+1)t} - 2ce^{\alpha t} - x_0^2e^{2t}.$$

To remove the e terms, we must solve $\alpha = 2\alpha + 2, \alpha = -2$. Since $-2 = -4 + 2 = 2(-2) + 2$ then $\alpha = -2$. Thus our equation becomes:

$$-2c = c^2x_0^2 - 2c - x_0^2.$$

Thus $c = x_0^{-2}$. Let $g := y - y_1$. Solving for g' :

$$\begin{aligned}
g' &= g^2x_0^2e^{2t} \\
\int \frac{dg}{g^2} &= x_0^2 \int e^{2t}dt \\
\frac{-1}{g} &= \frac{x_0^2}{2}(e^{2t} + c_1) \\
g &= \frac{1}{x_0^2}\frac{-2}{e^{2t}+c_1} \\
y - x_0^{-2}e^{-2t} &= \\
y &= \frac{1}{x_0^2}\left(e^{-2t} - \frac{2}{e^{2t}+c_1}\right).
\end{aligned}$$

We can solve for c_1 by evaluating y at 0:

$$\begin{aligned}
y_0 &= \frac{1}{x_0^2}\left(1 - \frac{2}{1+c_1}\right) \\
x_0^2 y_0 &= 1 - \frac{2}{1+c_1} \\
v_0 &= 1 - \frac{2}{1+c_1} \\
v_0 - 1 &= -\frac{2}{1+c_1} \\
1 - v_0 &= \frac{2}{1+c_1} \\
\frac{2}{1-v_0} &= 1 + c_1 \\
\frac{v_0+1}{1-v_0} &= c_1.
\end{aligned}$$

Therefore y is given by:

$$\begin{aligned}
y &= \frac{1}{x_0^2}\left(e^{-2t} - \frac{2}{e^{2t} + \frac{v_0+1}{1-v_0}}\right) \\
&= \frac{1}{x_0^2}\left(\frac{-2 + 1 + \frac{v_0+1}{1-v_0}e^{-2t}}{e^{2t} + \frac{v_0+1}{1-v_0}}\right) \\
&= \frac{1}{x_0^2}\frac{\frac{v_0-1}{v_0+1} + e^{-2t}}{\frac{1-v_0}{v_0+1}e^{2t} + 1} \\
&= \frac{1}{x_0^2 e^{2t}}\frac{\frac{v_0-1}{v_0+1} + e^{-2t}}{\frac{1-v_0}{v_0+1} + e^{-2t}} \\
&= \frac{1}{x_0^2 e^{2t}}\frac{v_0 - 1 + (v_0+1)e^{-2t}}{v_0 - 1 + (v_0+1)e^{-2t}} \\
&= \frac{1}{x_0^2 e^{2t}}\frac{v_0 + 1 + (v_0-1)e^{2t}}{v_0 + 1 - (v_0-1)e^{2t}}
\end{aligned}$$

Since this is the same as the formula we found via the change of variables, this formula is correct.