16.4.1(b) Note that

$$x^2-2x-1=(x-(1+\sqrt{2}))(x-(1-\sqrt{2})), x^2-2x-1=(x-(1+2\sqrt{2}))(x-(1-2\sqrt{2})),$$

thus the roots are contained within $\mathbb{Q}(\sqrt{2})$, and since $\mathbb{Q}(\sqrt{2})$ is a galois extension for f implies that $[\mathbb{Q}(\sqrt{2}):\mathbb{Q}]=|Gal(\mathbb{Q}(\sqrt{2})/\mathbb{Q})|$.

16.6.1 For the equation x^3+x+1, $\Delta_f=-31$. Since $deg(\alpha)=3$, this implies that $[\mathbb{Q}(\alpha):\mathbb{Q}]=3$, which means that the order of the galois group $G(\mathbb{Q}(\alpha)/\mathbb{Q})$ is either 1 or 3. Since $\sqrt{-31}$ needs an extension of degree 2, then it is not contained in $\mathbb{Q}(\alpha)$. However, for the splitting field K, the square root of the discriminate is guaranteed to be within as $\sqrt{\Delta_f}=(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)(\alpha_1-\alpha_3)$, which is just the product and difference of the roots. Therefore $\sqrt{-31}=\sqrt{\Delta_f}\in K$.

16.6.2 Note that we have the inherited automorphisms from $\mathbb{Q}(\sqrt{2}),\mathbb{Q}(\sqrt{3}),\mathbb{Q}(\sqrt{5})$ of $\sqrt{p}\mapsto-\sqrt{p}$ where $p=2,3,5$. Furthermore, we have $8=[\mathbb{Q}(\sqrt{2},\sqrt{3},\sqrt{5}):\mathbb{Q}]$ as we have 3 items being adjoined to $\mathbb{Q}$, and there is $2^3=8$ distinct elements which can be created by multiplying them together, represented by $\sqrt{2}^{b_0}\sqrt{3}^{b_1}\sqrt{5}^{b_2}$, where $b_i=0,1, i=0,1,2$. Furthermore, we can generate other automorphisms by chaining the swapping of signs of different roots. Furthermore, the swapping of signs is commutative. Finally our 3 inherited automorphisms each generate a subgroup of order 2. Therefore the only possible galois group is $(\mathbb{Z}/2\mathbb{Z})^3$, or the field on 8 elements.

16.7.2

b $[F:L]=9$ cannot occur since $[K:L][L:F]=[K:F]=|G(K/f)|=24$, and $9\nmid 24$.

c Note that $C_2\times C_{12}\cong C_2\times C_3\times C_4$ by the chinese remainder theorem. Since the following is a direct product, this implies that C_4 is normal. Therefore there is exactly one copy of C_4, implying there is one intermediate field which has galois group C_4.

16.7.4 Subfields of $\mathbb{Q}(\sqrt{2},\sqrt{3},\sqrt{5})$:

(a) $\mathbb{Q}$
(b) $\mathbb{Q}(\sqrt{2})$
(c) $\mathbb{Q}(\sqrt{3})$
(d) $\mathbb{Q}(\sqrt{5})$
(e) $\mathbb{Q}(\sqrt{6})$
(f) $\mathbb{Q}(\sqrt{10})$
(g) $\mathbb{Q}(\sqrt{15})$
(h) $\mathbb{Q}(\sqrt{30})$
(i) $\mathbb{Q}(\sqrt{2},\sqrt{3})$
(j) $\mathbb{Q}(\sqrt{2},\sqrt{5})$
(k) $\mathbb{Q}(\sqrt{3},\sqrt{5})$

(l) $\mathbb{Q}(\sqrt{5}, \sqrt{6})$

(m) $\mathbb{Q}(\sqrt{10}, \sqrt{3})$

(n) $\mathbb{Q}(\sqrt{15}, \sqrt{2})$

(o) $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$