1.2 *What are the complex eigenvalues of the matrix A that represents a rotation of R^3 through the angle θ about a pole u?*
Note that if we consider the pole to be a unit vector, then one gets to that vector simply by two rotations. Therefore the rotation matrix $R_\theta(u)$ is equivalent to $PR_\theta(e_1)P^{-1}$. Therefore the characteristic polynomial $\det(R_\theta(u)-\lambda\mathbb{I}_3) = \det(R_\theta(e_1)-\lambda\mathbb{I}_3)$. Thus we have the characteristic polynomial $(1-\lambda)(\lambda^2-2\cos(\theta)\lambda+1) = (1-\lambda)(\lambda-e^{i\theta})(\lambda-e^{-i\theta})$. Thus the complex eigenvalues are $e^{i\theta}, e^{-i\theta}$

2.3 *Let A be an $n\times n$ complex matrix.*

(a) *Consider the linear operator T defined on the space $\mathbb{C}^{n\times n}$ of all complex $n\times n$ matrices by the rule $T(M) = AM - MA$. Prove that the rank of this operator is at most n^2-n.*
If we assume A is diagonalizable, where $A = P\Lambda_1P^{-1}$ with the diagonal entries of Λ_1 being $\lambda_1, \cdots, \lambda_n$, then if we consider any other diagonalizable matrix $M = P\lambda_2P^{-1}$ then we get that

$$\begin{aligned}
T(M) &= P\Lambda_1P^{-1}P\Lambda_2P^{-1} - P\Lambda_2P^{-1}P\Lambda_1P^{-1}\\
&= P\Lambda_1\Lambda_2P^{-1} - P\Lambda_2\Lambda_1P^{-1}\\
&= P(\Lambda_1\Lambda_2 - \Lambda_2\Lambda_1)P^{-1}\\
&= P(\Lambda_1\Lambda_2 - \Lambda_1\Lambda_2)P^{-1}\\
&= P0P^{-1}\\
&= 0.
\end{aligned}$$

Note that since Λ_1, Λ_2 are both diagonal, they commute, and since Λ_2 has n entries then $nullity(T) \geq n$. Therefore by the dimension formula $n^2 = rank(T) + nullity(T) \geq rank(T) + n, n^2+n \geq rank(T)$

(b) *Determine the eigenvalues of T in terms of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A.*
Consider $(E_{a,b})_{i,j} = \begin{cases} 1 & \text{if } i=a, j=b \\ 0 & \text{otherwise} \end{cases}$, If we note from before the eigenvectors of A are P, and evaluate $T(PE_{i,j}P^{-1}$ we attain the following

$$T(PE_{i,j}P^{-1}) = P(\Lambda_1E_{i,j}-E_{i,j}\Lambda_1)P^{-1} = P(\lambda_iE_{i,j}-\lambda_jE_{i,j})P^{-1} = (\lambda_i-\lambda_j)PE_{i,j}P^{-1}.$$

We have now found every eigenmatrix and eigenvalue! Namely, the set of eigenvectors of T are $\{\lambda_i - \lambda_j : i,j \in \{1,\ldots,n\}\}$.

3.2 (a)

$$\begin{aligned}
\frac{d}{dt}(A(t)^3) &= \frac{dA}{dt}A^2 + A\frac{dA^2}{dt}\\
&= \frac{dA}{dt}A^2 + A\frac{dA}{dt}A + A^2\frac{dA}{dt}\\
&= \frac{dA}{dt}A^2 + A(\frac{dA}{dt}A + A\frac{dA}{dt})\\
&= \frac{dA}{dt}A^2 + A\frac{dA}{dt}A + A^2\frac{dA}{dt}
\end{aligned}$$

(b)

$$0 = \frac{dI_n}{dt} = \frac{AA^{-1}}{dt}$$
$$0 = \frac{dA}{dt}A^{-1} + A\frac{dA^{-1}}{dt}$$
$$\frac{dA^{-1}}{dt} = -A^{-1}\frac{dA}{dt}A^{-1}$$

(c)
$$\frac{A^{-1}B}{dt} = \frac{dA^{-1}}{dt}B + A^{-1}\frac{dB}{dt} = -A^{-1}\frac{dA}{dt}A^{-1} + A^{-1}\frac{dB}{dt}$$

3.3 *Solve the equation* $\frac{dX}{dt} = AX$ *for the following matrices A:*

(a) Note that since $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, then the solution is given by $X = e^{tA} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{e^t}{2} + \frac{e^{3t}}{2} & \frac{e^{3t}}{2} - \frac{e^t}{2} \\ \frac{e^{3t}}{2} - \frac{e^t}{2} & \frac{e^t}{2} + \frac{e^{3t}}{2} \end{bmatrix}$

(b) Note that since $A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} = \begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -1 & -i \\ 1 & 1 \end{bmatrix}$ then the solution is given by

$$X = e^{tA} = \frac{-1}{2i}\begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{it} & 0 \\ 0 & e^t \end{bmatrix}\begin{bmatrix} -1 & -i \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} e^t\cos(it) & e^t\sin(it) \\ -e^t\sin(it) & e^t\cos(it) \end{bmatrix}.$$

3.6a *Let s be the rotation of the plane with angle* $\pi/2$ *about the point* $(1,1)^t$. *Write the formula for s as a product* $t_a\rho_\theta$
The transformation described in the problem can be trivially written as $t_{(1,1)}\rho_{\pi/2}t_{(-1,-1)}$, which shifts the point $(1,1)$ to the origin, has the rotation action, then restores the location of $(1,1)$. However we can reduce our number of isomorphism used on the plane by noting $\rho_{\pi/2}t_{(-1,-1)} = t_{(1,-1)}\rho_{\pi/2}$ from the transformation rules. Therefore $t_{(1,1)}\rho_{\pi/2}t_{(-1,-1)} = t_{(1,1)}t_{(1,-1)}\rho_{\pi/2} = t_{(2,0)}\rho_{\pi/2}$.