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Assignment 8 problem 1
300H

Recall that is k is a positive integer we define the relationship equiv$_k$ on $\mathbb{Z}$ by $m \equiv_k n$ provided that $k|(m-n)$. Prove that for any integers m_1, m_2, n_1, n_2 if $m_1 \equiv_k m_2$ and n_1 equiv$_k n_2$ then (1) $m_1 + n_1 \equiv_k m_2 + n_2$ and (2) $m_1 n_1 \equiv_k m_2 n_2$

1. Proof: Suppose $m_1, m_2, n_1, n_2 \in \mathbb{Z}$ and $m_1 \equiv_k m_2, n_1 \equiv_k n_2$. We must show that $m_1 + n_1 \equiv_k m_2 + n_2$. By definition of congruence mod k we must show $k \mid (m_1 + n_1 - (m_2 + n_2))$. By definition of congruence mod k we have $k \mid (m_1 - m_2)$ and $k \mid (n_1 - n_2)$. By the definition of divisibility we have $a, b \in \mathbb{Z}$ such that $ka = m_1 - m_2, kb = n_1 - n_2$. Adding both equations together yields $k(a + b) = m_1 + n_1 - m_2 - n_2$. Let $c = a + b$, due to the closure of $\mathbb{Z}$ under addition, $c \in \mathbb{Z}$. Therefore $kc = m_1 + n_1 - (m_2 + n_2)$. Therefore by the definition of divisibility $k \mid (m_1 + n_1 - (m_2 + n_2))$.

2. Proof: Suppose $m_1, m_2, n_1, n_2 \in \mathbb{Z}$ and $m_1 \equiv_k m_2, n_1 \equiv_k n_2$. We must show that $m_1 n_1 \equiv_k m_2 n_2$. By the definition of modular arithmetic, $0 \equiv_k m_1 - m_2, 0 \equiv_k n_1 - n_2$. Multiplying the two expressions together yields $0 \equiv_k (m_1 - m_2)(n_1 - n_2)$. Therefore,

$$\begin{aligned}
0 &\equiv_k (m_1 - m_2)(n_1 - n_2) \\
0 &\equiv_k m_1(n_1 - n_2) - m_2(n_1 - n_2) \\
m_2(n_1 - n_2) &\equiv_k m_1(n_1 - n_2) \\
m_2 n_1 - m_2 n_2 &\equiv_k m_1 n_1 - m_1 n_2 \\
m_2 n_2 - m_1 n_2 &\equiv_k m_1 n_1 - m_1 n_2 \\
m_2 n_2 &\equiv_k m_1 n_1 \\
m_1 n_1 &\equiv_k m_2 n_2.
\end{aligned}$$