Alex Valentino

Recall that if S is a set, a partition of S is a set $\mathcal{P}$ such that

1. Every member of $\mathcal{P}$ is a nonempty subset of S,
2. for each $s \in S$, there is an $M \in \mathcal{P}$ such that $s \in M$, and
3. For each pair $M_1, M_2 \in \mathcal{P}$ such that $M_1 \neq M_2$, we have $M_1 \cap M_2 = \emptyset$.

Now suppose X is a nonempty set. For $\mathcal{H}, \mathcal{G} \subseteq \mathcal{P}(X)$ define
$\mathcal{H} \sqcap \mathcal{G}$ to be the set $\{A \cap B : A \in \mathcal{H}, B \in \mathcal{G}, A \cap B \neq \emptyset\}$.
(a) Construct an example of two different partitions $\mathcal{H}$ and $\mathcal{G}$ of $\{1,2,3,4,5,6\}$ each having two parts, and construct $\mathcal{H} \sqcap \mathcal{G}$.
Let $\mathcal{H} = \{\{1,2,3\},\{4,5,6\}\}$, $\mathcal{G} = \{\{1,2\},\{3,4\},\{5,6\}\}$. Then $H \sqcap G = \{\{1,2,3\} \cap \{1,2\}, \{1,2,3\} \cap \{3,4\}, \{4,5,6\} \cap \{3,4\}, \{4,5,6\} \cap \{5,6\}\} = \{\{1,2\},\{3\},\{4\},\{5,6\}\}$
(b) Prove: for any two partitions $\mathcal{H}$ and $\mathcal{G}$ of $X, \mathcal{H} \sqcap \mathcal{G}$ is also a partition. (In your proof, carefully verify that $\mathcal{H} \sqcap \mathcal{G}$ satisfies the requirements of a partition.)
Suppose X is an arbitrary non-empty set, and $\mathcal{H}, \mathcal{G}$ are arbitrary partitions of X. We must show $\mathcal{H} \sqcap \mathcal{G}$ is a partition of X. Since there are three requirements for a set to be a partition of another, the proof will be split into three parts:

1. We must show that every member of $\mathcal{H} \sqcap \mathcal{G}$ is a non-empty subset of X. Suppose S is an arbitrary member of $\mathcal{H} \sqcap \mathcal{G}$. Then by definition $S = A \cap B$, where $A \in \mathcal{H}, B \in \mathcal{G}$ and $A \cap B \neq \emptyset$. By definition of partition $\forall A \in \mathcal{H}, A \subset X$. Suppose x is an arbitrary member of S. Then by definition of set intersection $x \in A$ and $x \in B$. Since $x \in A$ and $A \subset X$, then $x \in X$. Therefore we can say that S is a subset of X. It is non-empty since by defintion $S = A \cap B, A \cap B \neq \emptyset$. Therefore the first requirement is satisfied.

2. We must show that for each $e \in X$ that there is an $S \in \mathcal{H} \sqcap \mathcal{G}$ such that $e \in S$. Suppose e is an arbitrary element in X, and the set S is a member of $\mathcal{H} \sqcap \mathcal{G}$. By definition $S = A \cap B$, where $A \in \mathcal{H}, B \in \mathcal{G}$ and $A \cap B \neq \emptyset$. We can construct $\mathcal{H} = \{X \backslash e, \{e\}\}$ and $\mathcal{G} = \{X\}$, where $A = \{e\}$, and $B = X$. Then by definition $S = A \cap B = \{e\} \cap X = \{e\}$. Therefore $e \in S$.

3. We must show for each pair $S_1, S_2 \in \mathcal{H} \sqcap \mathcal{G}$ such that $S_1 \neq S_2$, we have $S_1 \cap S_2 = \emptyset$. Suppose S_1, S_2 are arbitrary unique members of $\mathcal{H} \sqcap \mathcal{G}$. By Definition $S_1 = A_1 \cap B_1, A_1 \in \mathcal{H}, B_1 \in \mathcal{G}, A_1 \cap B_1 \neq \emptyset$ and $S_2 = A_2 \cap B_2, A_2 \in \mathcal{H}, B_2 \in \mathcal{G}, A_2 \cap B_2 \neq \emptyset$. We must now show for each pair $S_1, S_2 \in \mathcal{H} \sqcap \mathcal{G}$ such that $S_1 \neq S_2$, we have $(A_1 \cap B_1) \cap (A_2 \cap B_2) = \emptyset$. Suppose x is an arbitrary element in $S_1 \cap S_2$. Therefore by the definition of set intersection (applied twice) $x \in A_1$ and $x \in B_1$ and $x \in A_2$ and $x \in B_2$). By implicitly applying the axiom of and commutativity to the definition of set intersection, $x \in (A_1 \cap A_2)$ and $x \in (B_1 \cap B_2)$. Since A_1, A_2 and B_1, B_2 are from the same partition, but are defined to be unique, then by the definition of a partition $A_1 \cap A_2 = \emptyset$ and $B_1 \cap B_2 = \emptyset$. Therefore $x \in \emptyset$ and $x \in \emptyset$. Therefore $S_1 \cap S_2 = \emptyset$.