

3.2.10 i A countable set contained in $[0, 1]$ with no limit points

This can't occur, as if we enumerate a countable set, then we have a sequence, and since it's contained within $[0, 1]$ then we have a bounded sequence. Therefore by the Bolzano-Weierstrauss theorem we have a convergent subsequence, i.e. a limit point of the set. Thus this description can't occur

ii A countable set contained in $[0, 1]$ with no isolated points

This can occur! Take $A = [0, 1] \cap \mathbb{Q}$. We know by the density of \mathbb{Q} in \mathbb{R} that we have a sequence of rational numbers to every real number. Since a rational number is a real number, then for every element in the set A we can generate a sequence with a sufficient ϵ contained within $[0, 1]$ for which an arbitrary rational number satisfies the definition of a limit point. And the countability of the rationals is a given.

iii A set with an uncountable number of isolated points

The issue arises with the fact that for each $x \in A$, there is an associated $\epsilon_x > 0$ in which $V_{\epsilon_x}(x) \cap A = \{x\}$. Note that since we have a unique ϵ_x enclosing x within $(x - \epsilon_x, x + \epsilon_x)$ then we know by the density of \mathbb{Q} in \mathbb{R} there must exist a rational number $r \in V_{\epsilon_x}(x)$. Therefore we can uniquely pair each $x \in A$ with $r \in \mathbb{Q}$ since $V_{\epsilon_x}(x)$ uniquely contains r and x . Since we have found a 1-1 mapping from A to \mathbb{Q} then clearly A must be countable.