Suppose $X$ is a set and $\mathcal{S}$ is a set of subsets of $X$. We say that a set $Z \subseteq X$ is a lower bound for $\mathcal{S}$ provided that $Z \subseteq S$ for all $S \in \mathcal{S}$. Prove that for any set $X$ and any collection $\mathcal{S}$ of subsets of $X$, there is a unique set $T$ with the following two properties:

(a) $T$ is a lower bound for $\mathcal{S}$.

Suppose $X$ is a set and $\mathcal{S}$ is a collection of subsets of $X$. We must show that there exist a set $T$ that is a lower bound for $\mathcal{S}$. By the definition of lower bound we must show that there exist a set $T$ such that for all $S \in \mathcal{S}$, $T \subseteq S$. Let $T$ be given by $T = \bigcap_{S\in\mathcal{S}} S$. Suppose $S^* \in \mathcal{S}$. We must show that $T \subseteq S^*$. Suppose $x \in T$. By the definition of subset we must show that $x \in S^*$. By definition of $x \in T, x \in \bigcap_{S\in\mathcal{S}} S$, therefore since $S^* \in \mathcal{S}$, by the definition of set intersection $x \in S^*$

(b) For any set $Z$ that is a lower bound for $\mathcal{S}$ we have $Z \subseteq T$.

Suppose $Z$ is a lower bound for $\mathcal{S}$. We must show that $Z \subseteq T$. By definition of lower bound for all $S \in \mathcal{S}$ $Z \subseteq S$. Suppose $\mathcal{S} = \{S_1, \cdots, S_n\}$ Therefore $Z \subseteq S_1$ and $Z \subseteq S_2 \cdots$ and $Z \subseteq S_n$. Therefore by definition of intersection $Z \subseteq \bigcap_{S\in\mathcal{S}} S$. Since by definition $\bigcap_{S\in\mathcal{S}} S = T$, then $Z \subseteq T$.