6.7.2.1 Note that the surface we care about is defined via $\begin{bmatrix}1 & 0 & -a & -c\\ 0 & 1 & -b & -d\end{bmatrix} x = \begin{bmatrix}0\\0\end{bmatrix}$. It is parameterized via $\{\begin{bmatrix}a & c\\ b & d\\ 1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_3\\x_4\end{bmatrix} : x_3, x_4 \in \mathbb{R}\}$. Let $A$ be the matrix of the parameterized kernel. Therefore to compute the area of the projected unit square in $x_3, x_4$ coordinates we need to compute $\sqrt{\det(A^TA)}$. This computation comes out to $\sqrt{\det(A^TA)} = \sqrt{a^2+b^2+c^2+d^2+(ad-bc)^2+1}$. Therefore we have to solve the integral $\int_0^1\int_0^1 \sqrt{\det(A^TA)}dx_3dx_4 = \sqrt{\det(A^TA)} = \sqrt{a^2+b^2+c^2+d^2+(ad-bc)^2+1}$

6.7.2.2 For the triangle defined by $\{(1,0,0), (\cos(\frac{\pi}{n}), \pm\sin(\frac{\pi}{n}), \frac{1}{2m})\}$, we can just recenter to the origin with the new coordinates $\{(0,0,0), (\cos(\frac{\pi}{n})-1, \pm\sin(\frac{\pi}{n}), \frac{1}{2m})\}$. Now we can compute the area of the parallelogram spanned by the two non-zero vectors, and then halve it to find the area of the triangle. If we let $a = (\cos(\frac{\pi}{n})-1, \sin(\frac{\pi}{n}), \frac{1}{2m}), b = (\cos(\frac{\pi}{n})-1, -\sin(\frac{\pi}{n}), \frac{1}{2m})$ the area computation becomes

$$\frac{1}{2}\sqrt{(a\cdot a)(b\cdot b)-(a\cdot b)^2} =$$

$$\frac{1}{2}\sqrt{(\frac{1+4m^2-8m^2\cos(\frac{\pi}{n})+4m^2\cos^2(\frac{\pi}{n})}{m^2})\sin^2(\frac{\pi}{n})}$$

$$= \frac{1}{2}\sin(\frac{\pi}{n})\sqrt{\frac{1}{m^2}+2^2(1-\cos(\frac{\pi}{n})))^2} = \frac{1}{2}\sin(\frac{\pi}{n})\sqrt{\frac{1}{m^2}+4^2\sin^4(\frac{\pi}{2n})}.$$

For the triangle $\{(1,0,0), (\cos(\frac{\pi}{n}), \sin(\frac{\pi}{n}), \pm\frac{1}{2m})\}$ we do the same process, where instead we have $a = (\cos(\frac{\pi}{n})-1, \sin(\frac{\pi}{n}), \frac{1}{2m}), b = (\cos(\frac{\pi}{n})-1, \sin(\frac{\pi}{n}), -\frac{1}{2m})$, yielding the computation

$$\begin{aligned}\frac{1}{2}\sqrt{(a\cdot a)(b\cdot b)-(a\cdot b)^2} &= \sqrt{\frac{1}{m^2}-\frac{2\cos(\frac{\pi}{n})}{m^2}+\frac{\cos^2(\frac{\pi}{n})}{m^2}+\frac{\sin^2(\frac{\pi}{n})}{m^2}}\\ &= \sqrt{\frac{1}{m^2}\sin^2(\frac{\pi}{2n})}\\ &= \frac{\sin(\frac{\pi}{2n})}{m}\end{aligned}$$

6.7.2.3 To compute the volume of the parallelpiped between 3 of the four verticies of $\{(1,0,0,0), (0,1,0,0), (0,$ one must take 3 of the vectors and form a matrix, take $A = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$, Observe that $\sqrt{\det(A^TA)} = 1$. Since one parallelpiped is $\frac{1}{6}$th of the total area of the tetrahedron then it's 6 times the area, thus the total area is 6.

6.7.3.3 (a) Since $f(x_1,x_2,x_3) = \sqrt{R^2-x_1^2-x_2^2-x_3^2}$ is a graph over $x_1, x_2, x_3$, we can find the surface area via computing $\sqrt{1+|Df(x)|^2}$ and computing over $x_1^2+x_2^+x_3^2 \leq$

$R^2$. Note that $Df(x) = \left(\frac{-x_1}{\sqrt{R^2-x_1^2-x_2^2-x_3^2}}, \frac{-x_2}{\sqrt{R^2-x_1^2-x_2^2-x_3^2}}, \frac{-x_3}{\sqrt{R^2-x_1^2-x_2^2-x_3^2}}\right)$, thus $Df(x)^T Df(x) = \frac{x_1^2+x_2^2+x_3^2}{R^2-(x_1^2+x_2^2+x_3^2)}$. Therefore to compute the general surface area we have to compute

$$\begin{aligned}
2\int\int\int_{x_1^2+x_2^+x_3^2\leq R^2}\sqrt{1+|Df(x)|^2} &= 2\int\int\int_{x_1^2+x_2^+x_3^2\leq R^2}\sqrt{\frac{R^2-x_1^2-x_2^2-x_3^2+x_1^2+x_2^2+x_3^2}{R^2-x_1^2-x_2^2-x_3^2}} \\
&= 2\int\int\int_{x_1^2+x_2^+x_3^2\leq R^2}\sqrt{\frac{R^2}{R^2-x_1^2-x_2^2-x_3^2}} \\
&= 2R\int_0^{2\pi}\int_0^{\pi}\int_0^R \frac{r^2\sin(\theta)}{\sqrt{R^2-r^2}}d\varphi d\theta dr \\
&= 8R^2\pi\int_0^R \frac{r^2}{\sqrt{R^2-r^2}}dr \\
&= 2\pi^2R^3
\end{aligned}$$

(b) Using the sphereical coordinates we have that the surface is parameterized by $S(\phi,\theta_1,\theta_2) = (R\sin(\theta_2)\sin(\theta_1)\cos(\phi), R\sin(\theta_2)\sin(\theta_1)\sin(\phi), R\sin(\theta_2)\cos(\theta_1), R\cos(\theta_2))$. Observe that
$D_\phi S = (-R\sin(\theta_2)\sin(\theta_1)\sin(\phi), R\sin(\theta_2)\sin(\theta_1)\cos(\phi), 0, 0)$
$D_{\theta_1} S = (R\sin(\theta_2)\cos(\theta_1)\cos(\phi), R\sin(\theta_2)\cos(\theta_1)\sin(\phi), -R\sin(\theta_2)\sin(\theta_1), 0)$
$D_{\theta_2} S = (R\cos(\theta_2)\sin(\theta_1)\cos(\phi), R\cos(\theta_2)\sin(\theta_1)\sin(\phi), R\cos(\theta_2)\cos(\theta_1), -R\sin(\theta_2))$
$\|D_\phi\|^2 = R^2\sin^2(\theta_2)\sin^2(\theta_1)$
$\|D_{\theta_1}\|^2 = R^2\sin^2(\theta_2)$
$\|D_{\theta_2}\|^2 = R^2$.
Since the angles have orthogonal tangents then the dot product between any two non-same terms will be zero. Thus $\sqrt{(\det((\nabla S)^T\nabla S))} = R^3\sin^2(\theta_2)\sin(\theta_1)$. Observe that $\int_0^\pi\int_0^\pi\int_0^{2\pi} R^3\sin^2(\theta_2)\sin(\theta_1)d\phi d\theta_1 d\theta_2 = R^3 2\pi^2$

6.7.3.4

- Let our parameterization be denoted $S_r(z,\theta) = (r(z)\cos(\theta), r(z)\sin(\theta), z)$. Observe that $D_zS_r = (r'(z)\cos(\theta), r'(z)\sin(\theta), 1)$ and that $D_\theta S_r = (-r(z)\sin(\theta), r(z)\cos(\theta), 0)$. Therefore $\|D_z\|^2 = |r'(z)|^2+1$, $\|D_\theta\|^2 = r^2(z)$, $D_z\cdot D_r = 0$. Thus the integral of the surface becomes $\int_a^b\int_0^{2\pi}\sqrt{r^2(z)(|r'(z)|^2+1)}d\theta dz = 2\pi\int_a^b r(z)\sqrt{|r'(z)|^2+1}dz$. Note that surface generated via $r(z) = cosh(1)$ is $4\pi\cosh(1)$. For $r(z) = cosh(z)$ we have the integral $2\pi\int_{-1}^1 cosh(z)^2dz = 2\pi(1+cosh(1)sinh(1))$. Note that $2\pi(1+cosh(1)sinh(1)$ is less then $4\pi\cosh(1)$, thus the radius of $cosh(z)$ is the smaller.

6.7.4.3 Since we're integrating with the condition that we're on $x^2+y^2 = 1$ implies that $x = \cos(\theta), y = \sin(\theta)$ is a valid parameterization. Furthermore, the condition on the planes gives us that $z = \pm 1 - \cos(\theta) - \sin(\theta)$. Therefore the integral can be translated into $\int_0^{2\pi}\int_{-1-\cos(\theta)-\sin(\theta)}^{1-\cos(\theta)-\sin(\theta)} z^2dzd\theta = \frac{8}{3}\int_0^{2\pi}d\theta = \frac{16\pi}{3}$