(Note: In this problem we will be defining a relation on the set $A \times A$. The elements of $A \times A$ are ordered pairs, so a pair of the relation we define is an *ordered pairs each of whose coordinates is an ordered pairs.* This can be confusing, so be sure to read and think about the problem carefully.) Suppose A is a set and R a partial order relation on A. Define a new relation Q on the set $A \times A$ as follows: For $a_1, a_2, b_1, b_2 \in A$ we have $(a_1, a_2)Q(b_1, b_2)$ provided that ($a_1 \neq b_1$ and $a_1 R b_1$) or ($a_1 = b_1$ and $a_2 R b_2$).

1. Prove that Q is a partial order on the set $A \times A$.

   - Proof of reflexivity: We must show that for all $(a_1, a_2) \in A^2, (a_1, a_2)Q(a_1, a_2)$. Suppose $(a_1, a_2) \in A^2$,. We must show $(a_1, a_2)Q(a_1, a_2)$ Since $a_1 = a_1$, and by the reflexivity of R, $a_2 R a_2$, then by the definition of Q, $(a_1, a_2)Q(a_1, a_2)$.
   - Proof of transitivity: We must show that for all $(a_1, a_2), (b_1, b_2), (c_1, c_2) \in A^2$ if $(a_1, a_2)Q(b_1, b_2)$ and $(b_1, b_2)Q(c_1, c_2)$ then $(a_1, a_2)Q(c_1, c_2)$. Suppose $(a_1, a_2), (b_1, b_2), (c_1, c_2) \in A^2$, $(a_1, a_2)Q(b_1, b_2)$ and $(b_1, b_2)Q(c_1, c_2)$. We must show that $(a_1, a_2)Q(c_1, c_2)$. By definition of Q we must show that ($a_1 \neq c_1$ and $a_1 R c_1$) or ($a_1 = c_1$ and $a_2 R c_2$). We now have four cases:
     - Suppose $a_1 \neq b_1, b_1 \neq c_1, a_1 R b_1, b_1 R c_1$. Then by transitivity of R we have $a_1 R c_1$. However we have the posibility that since $a_1 \neq b_1, b_1 \neq c_1$ that $a_1 = c_1$. This is not an issue because if $a_1 = c_1$ then we now have $a_1 R c_1$ and $c_1 R a_1$ by the reflexivity of R. However by the anti-symmetry of R we have since $a_1 \neq b_1$ and $a_1 R b_1$, then $b_1 \not{R} a_1$ and similarly $c_1 \not{R} b_1$. However since $c_1 \not{R} b_1 = a_1 \not{R} b_1$, we have a contradiction with original assumption of $a_1 R b_1$. Therefore $a_1 \neq c_1$. Thus we have $a_1 \neq c_1$ and $a_1 R c_1$.
     - Suppose $a_1 = b_1, a_2 R b_2, b_1 = c_1, b_2 R c_2$. Then by the transitivity $a_1 = c_1$ and $a_2 R c_2$.
     - Suppose $a_1 \neq b_1, a_1 R b_1, b_1 = c_1, b_2 R c_2$. Since $a_1 \neq b_1, b_1 = c_1$, then $a_1 \neq c_1$. Since $a_1 R b_1$, and $b_1 = c_1$, then $a_1 R c_1$.
     - Suppose $a_1 = b_1, a_2 R b_2, b_1 \neq c_1, b_1 R c_1$. Since $a_1 = b_1, b_1 \neq c_1$, then $a_1 \neq c_1$. Since $a_1 = b_1, b_1 R c_1$, then $a_1 R c_1$
   - Proof of anti-symmetry. Suppose $(a_1, a_2), (b_1, b_2) \in A^2, (a_1, a_2)Q(b_1, b_2), (a_1, a_2) \neq (b_1, b_2)$. We must show that $(b_1, b_2)\not{Q}(a_1, a_2)$ By definition of Q we must show $\neg((b_1 \neq a_1 \wedge b_1 R a_1) \vee (b_1 = a_1 \wedge b_2 R a_2))$. By first order logic we must show $b_1 = a_1$, or $b_1 \not{R} a_1$, and $b_1 \neq a_1$ or $b_2 \not{R} a_2$. Since $(a_1, a_2)Q(b_1, b_2)$ has an or statement, we have two cases:
     - Suppose $a_1 = b_1, a_2 R b_2$. By the anti-symmetry of R we have $b_2 \not{R} a_2$. Therefore since we have $a_1 = b_1$ and $b_2 \not{R} a_2$ then we have satisfied the requirement.
     - Suppose $a_1 \neq b_1, a_1 R b_1$. By the anti-symmetry of R we have $b_1 \not{R} a_1$. Therefore since we have $a_1 \neq b_1$ and $b_1 \not{R} a_1$ then we have satisfied the requirement.

2. Prove that if R is a total order (which means that for all $a, b \in A$ we have aRb or bRa) then so is Q.

Suppose R is a total order. We must show that Q is a total order. By definition of total order we have for all $a, b \in A, aRb$ or bRa. By definition of total order we must show for all $(a_1, a_2), (b_1, b_2) \in A^2$ that $(a_1, a_2)Q(b_1, b_2)$ or $(b_1, b_2)Q(a_1, a_2)$. Suppose $(a_1, a_2), (b_1, b_2) \in A^2$. By definition of R being a total order, and $a_1, a_2, b_1, b_2 \in A$, then we have a_1Rb_1 or b_1Ra_1 and a_2Rb_2 or b_2Ra_2. This provides four cases:

- Suppose a_1Rb_1 and a_2Rb_2. We now have two cases of either $a_1 = b_1$ and $a_1 \neq b_1$.
  - Suppose $a_1 = b_1$. Then since a_2Rb_2 and $a_1 = b_1$ by definition $(a_1, a_2)Q(b_1, b_2)$.
  - Suppose $a_1 \neq b_1$. Then since a_1Rb_1 and $a_1 \neq b_1$, then by definition $(a_1, a_2)Q(b_1, b_2)$.
- Suppose a_1Rb_1 and b_2Ra_2. We now have two cases of either $a_1 = b_1$ and $a_1 \neq b_1$.
  - Suppose $a_1 = b_1$. By reflexivity $b_1 = a_1$. Since b_2Ra_2, $(b_1, b_2)R(a_1, a_2)$.
  - Suppose $a_1 \neq b_1$.. Then since a_1Rb_1 then $(a_1, a_2)R(b_1, b_2)$.