

6 a Let $y(t) = x'(t)$, $\vec{x} = (x(t), y(t))$. Therefore $\vec{x}' = (x', y') = (y, -\frac{k}{m}x - \frac{a}{m}y) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{a}{m} \end{bmatrix} \vec{x}$. Thus $B = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{a}{m} \end{bmatrix}$.

- b • Suppose $(\frac{a}{m})^2 = \frac{4k}{m}$ (critically damped). We must compute e^{tB} . Thus $0 = \det(B - \lambda \mathbb{I}_2) = \lambda^2 + \frac{a}{m}\lambda + \frac{k}{m}$, therefore $\lambda = \frac{-a \pm \sqrt{(\frac{a}{m})^2 - \frac{4k}{m}}}{2} = \frac{-a \pm 0}{2} = \frac{-a}{2m}$. Thus $\lambda^2 + \frac{a}{m}\lambda + \frac{k}{m} = (\lambda + \frac{a}{2m})^2$. Therefore $\lambda = \frac{-a}{2m}$ is an eigenvalue with multiplicity 2. Since we are in \mathbb{R}^2 , and that is the same as the multiplicity of the eigenvalue then we know that $(B + \frac{a}{2m})^2 \vec{x} = 0$ for all $\vec{x} \in \mathbb{R}^2$. Therefore $e^{tB} = e^{\frac{-a}{2m}t} \sum_{k=0}^1 t^k (B + \frac{a}{2m} \mathbb{I}_2) = e^{\frac{-a}{2m}t} (\mathbb{I}_2 + t \begin{bmatrix} \frac{a}{2m} & 1 \\ \frac{k}{m} & -\frac{a}{2m} \end{bmatrix}) = e^{\frac{-a}{2m}t} \begin{bmatrix} 1 + t\frac{a}{2m} & t \\ \frac{k}{m}t & 1 - \frac{a}{2m}t \end{bmatrix}$.
- Suppose $(\frac{a}{m})^2 < \frac{4k}{m}$ (under damped). We must compute e^{tB} . Since $(\frac{a}{m})^2 < \frac{4k}{m}$ then $(\frac{a}{m})^2 - \frac{4k}{m} = \frac{-\rho^2}{m^2}$. Therefore $\lambda = \frac{-a \pm \sqrt{(\frac{a}{m})^2 - \frac{4k}{m}}}{2} = \frac{-a \pm i\frac{\rho}{m}}{2} = \frac{1}{2m}(-a \pm i\rho)$. Let $\mu_1 = \frac{1}{2m}(-a + i\rho)$, $\mu_2 = \frac{1}{2m}(-a - i\rho)$. Computing eigenvalues for μ_1 yields $\vec{v}_1 = (1, \frac{1}{2m}(-a + i\rho))$. Therefore $z_1 = e^{\mu_1 t} (1, \frac{1}{2m}(-a + i\rho)) = e^{\frac{-a}{2m}t} (\cos(\frac{\rho}{2m}t) + i \sin(\frac{\rho}{2m}t)) (1, \frac{1}{2m}(-a + i\rho))$. Let $l = \frac{\rho}{2m}$. Thus

$$\begin{aligned} z_1 &= e^{\frac{-a}{2m}t} (\cos(\frac{\rho}{2m}t) + i \sin(\frac{\rho}{2m}t)) (1, \frac{1}{2m}(-a + i\rho)) \\ &= e^{\frac{-a}{2m}t} (\cos(lt) + i \sin(lt)) (1, \frac{1}{2m}(-a + i\rho)) \\ &= e^{\frac{-a}{2m}t} (\cos(lt), \frac{1}{2m}(-a \cos(lt) - \rho \sin(lt))) + i e^{\frac{-a}{2m}t} (\sin(lt), \frac{1}{2m}(-a \sin(lt) + \rho \cos(lt))) \end{aligned}$$

Since we have a single complex solution, then we have the two real solutions forming the columns of $M(t)$:

$$e^{\frac{-a}{2m}t} \begin{bmatrix} \cos(lt) & \sin(lt) \\ \frac{1}{2m}(-a \cos(lt) - \rho \sin(lt)) & \frac{1}{2m}(\rho \cos(lt) - a \sin(lt)) \end{bmatrix}$$

Since $M(0) = \begin{bmatrix} 1 & 0 \\ \frac{-a}{2m} & \frac{\rho}{2m} \end{bmatrix}$ then $M^{-1}(0) = \begin{bmatrix} 1 & 0 \\ \frac{a}{\rho} & \frac{2m}{\rho} \end{bmatrix}$. Therefore

$$\begin{aligned} e^{tB} &= M(t)M^{-1}(0) = e^{\frac{-a}{2m}t} \begin{bmatrix} \cos(lt) & \sin(lt) \\ \frac{1}{2m}(-a \cos(lt) - \rho \sin(lt)) & \frac{1}{2m}(\rho \cos(lt) - a \sin(lt)) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{a}{\rho} & \frac{2m}{\rho} \end{bmatrix} \\ &= e^{\frac{-a}{2m}t} \begin{bmatrix} \frac{a}{\rho} \sin(lt) + \cos(lt) & \frac{2m}{\rho} \sin(lt) \\ -\frac{2k}{\rho} \sin(lt) & -\frac{a}{\rho} \sin(lt) + \cos(lt) \end{bmatrix}. \end{aligned}$$

- Suppose $(\frac{a}{m})^2 > \frac{4k}{m}$ (over damped). We must compute e^{tB} . Since $(\frac{a}{m})^2 > \frac{4k}{m}$ then $\lambda = \frac{1}{2m}(-a \pm \rho)$. Let $\mu_1 = \frac{1}{2m}(-a + \rho)$, $\mu_2 = \frac{1}{2m}(-a - \rho)$. These correspond to the eigenvectors $v_1 = (1, \frac{1}{2m}(-a + \rho))$, $v_2 = (1, \frac{1}{2m}(-a - \rho))$. Therefore $z_1 = e^{\frac{1}{2m}(-a+\rho)t} (1, \frac{1}{2m}(-a + \rho))$, $z_2 = e^{\frac{1}{2m}(-a-\rho)t} (1, \frac{1}{2m}(-a - \rho))$. Therefore

$$M(t) = e^{\frac{-a}{2m}t} \begin{bmatrix} e^{lt} & e^{-lt} \\ \frac{e^{lt}}{2m}(\rho - a) & -\frac{e^{-lt}}{2m}(\rho + a) \end{bmatrix}$$

$$M^{-1}(0) = \begin{bmatrix} 1 & 1 \\ \frac{\rho-a}{2m} & -\frac{\rho-a}{2m} \end{bmatrix}^{-1} = \frac{1}{\rho} \begin{bmatrix} \frac{\rho+a}{2} & -m \\ \frac{\rho-a}{2} & -m \end{bmatrix}. \text{ Thus}$$

$$\begin{aligned} e^{tB} &= M(t)M^{-1}(0) \\ &= \frac{e^{\frac{-a}{2m}t}}{\rho} \begin{bmatrix} e^{lt} & e^{-lt} \\ \frac{e^{lt}}{2m}(\rho-a) & -\frac{e^{-lt}}{2m}(\rho+a) \end{bmatrix} \begin{bmatrix} \frac{\rho+a}{2} & -m \\ \frac{\rho-a}{2} & -m \end{bmatrix} \\ &= \frac{e^{\frac{-a}{2m}t}}{\rho} \begin{bmatrix} \frac{\rho+a}{2}e^{lt} + \frac{\rho-a}{2}e^{-lt} & -m(e^{lt} + e^{-lt}) \\ \frac{1}{2m}(\rho^2 + a^2)(e^{lt} - e^{-lt}) & \frac{\rho+a}{2}e^{lt} - \frac{\rho-a}{2}e^{-lt} \end{bmatrix} \\ &= e^{\frac{-a}{2m}t} \begin{bmatrix} \frac{a}{\rho} \sinh(lt) + \cosh(lt) & \frac{1}{l} \sinh(lt) \\ -\frac{2k}{\rho} \sinh(lt) & -\frac{a}{\rho} \sinh(lt) + \cosh(lt) \end{bmatrix} \end{aligned}$$

b(2?) Since $\vec{x} = e^{tB}\vec{x}_0$, where $\vec{x}_0 = (x_0, v_0)$, then $x(t) = e^{\frac{-a}{2m}t}(x_0(1 + t\frac{a}{2m}) + v_0t)$. We want to find t such that $0 = x(t)$. Thus we have that $0 = e^{\frac{-a}{2m}t}(x_0(1 + t\frac{a}{2m}) + v_0t)$. Since $e^{\frac{-a}{2m}t}$ is strictly positive then $0 = x_0(1 + t\frac{a}{2m}) + v_0t$. This may be rewritten as $\frac{-x_0}{v_0 + \frac{a}{2m}} = t$. If this value either diverges or is negative then the solution will not pass through the origin, and if it does we have found an explicit single value for t . Thus there is either one or zero times which $x(t) = 0$.

c Since $\vec{x} = e^{tB}\vec{x}_0$, where $\vec{x}_0 = (x_0, v_0)$, then $x(t) = e^{\frac{-a}{2m}t}(x_0(\frac{a}{\rho} \sinh(lt) + \cosh(lt)) + \frac{v_0}{l} \sinh(lt))$. We want to find when $x(t) = 0$. Since $e^{\frac{-a}{2m}t}$ is strictly positive then we have that $0 = x_0(\frac{a}{\rho} \sinh(lt) + \cosh(lt)) + \frac{v_0}{l} \sinh(lt)$. Rearrange and assuming that $\frac{x_0a}{\rho} + \frac{v_0}{l} \neq 0$ we get that $-(\frac{\frac{x_0a}{\rho} + \frac{v_0}{l}}{x_0})^{-1} = \tanh(lt)$. Since \tanh is 1-1 then we have a unique t if the value is in the range of \tanh , if not then the function doesn't pass through the origin. If $\frac{x_0a}{\rho} + \frac{v_0}{l} = 0$ then we have $0 = x_0 \cosh(x)$, similarly, since arcosh is 1-1 then we have a unique t at which the function passes through the origin. Therefore $x(t) = 0$ occurs at most once for all parameters.

d Since $\vec{x} = e^{tB}\vec{x}_0$, where $\vec{x}_0 = (x_0, v_0)$, then $x(t) = e^{\frac{-a}{2m}t}(x_0(\frac{a}{\rho} \sin(lt) + \cos(lt)) + \frac{v_0}{l} \sin(lt))$. We want to find when $x(t) = 0$. Since $e^{\frac{-a}{2m}t}$ is strictly positive then we have that $0 = x_0(\frac{a}{\rho} \sin(lt) + \cos(lt)) + \frac{v_0}{l} \sin(lt)$. Rearrange and assuming that $\frac{x_0a}{\rho} + \frac{v_0}{l} \neq 0$ we get that $-(\frac{\frac{x_0a}{\rho} + \frac{v_0}{l}}{x_0})^{-1} = \tan(lt)$. Note that since \tan is periodic and \arctan is 1-1 means that $t = \frac{\pi n + \arctan(-(\frac{\frac{x_0a}{\rho} + \frac{v_0}{l}}{x_0})^{-1})}{l}$ where $n \in \mathbb{Z}$ are all valid solutions. If $\frac{x_0a}{\rho} + \frac{v_0}{l} = 0$ then we have $0 = x_0 \cos(x)$, similarly, since \cos is periodic and not the zero function due to $x(t) \neq 0$ then we have $t = \frac{\pi n}{l}$ where $n \in \mathbb{Z}$. Since all cases have an infinite number of solutions, then $x(t)$ transits the origin an infinite number of times.