If P is a partial order on X and Q is a partial order on Y, an *isomorphism* from P to Q is a bijection f between X and Y with the property that for all $x, x' \in X$, $x \leq_P x'$ if and only if $y \leq_P y'$. We say that P and Q are *isomorphic* provided that there is an isomorphism from P to Q. (Intuitively, two partial orders are isomorphic if they have identical structure and you can obtain Q from P by renaming the elements of P.) Suppose n is a natural number and consider the partially ordered set $(\mathcal{P}_n, \subseteq)$ (the power set of $\{1, \dots, n\}$ with the subset order) and the partially ordered set $(\{0,1\}^n, \leq^*)$ where for $y, z \in \{0,1\}^n$, $y \leq^* z$ means that $y_i \leq z_i$ for each $i \in \{1, \dots, n\}$. Prove that $(\mathcal{P}_n, \subseteq)$ is isomorphic to $(\{0,1\}^n, \leq^*)$.
We must show that there is an isomorphism between $(\mathcal{P}_n, \subseteq)$ and $(\{0,1\}^n, \leq^*)$. Therefore by definition we must show that there exists a bijection $f : \{0,1\}^n \to \mathcal{P}_n$ such that for all $a, b \in \{0,1\}^n$, $a \leq^* b$ if and only if $f(a) \subseteq f(b)$. We claim that f is given by $f(a) = \{i \in [n] : a_i = 1\}$. Suppose a, b are arbitrary functions in $\{0,1\}^n$. We must show that $a \leq^* b$ if and only if $f(a) \subseteq f(b)$.

- Suppose $a \leq^* b$. We must show that $f(a) \subseteq f(b)$. By definition of $\leq^*$ for all $i \in [n], a_i \leq b_i$. By definition of f, $f(a) = \{i \in [n] : a_i = 1\}, f(b) = \{i \in [n] : b_i = 1\}$. Since $a_i \leq b_i$, and $f(a)$ is the set of all elements where $a_i = 1$, then by the definition of $\leq$, b_i must equal 1 for the inequality to hold. Therefore by the definition of f, if $x \in f(a)$, then it must be in $f(b)$. Therefore by the definition of subset $f(a) \subseteq f(b)$.

- Suppose $f(a) \subseteq f(b)$. We must show that $a \leq^* b$. By definition of $\subseteq$ we have for all $i \in [n]$ if $i \in f(a)$, then $i \in f(b)$. Therefore if $i \in f(a)$, by the definition of f, $a_i = 1$, similarly $b_i = 1$. Therefore for all $i \in [n]$ such that $a_i = 1$, then $a_i = b_i = 1$, thus $a_i \leq b_i$ is equivalent to $1 \leq 1$. However, if we take $f(a)^c$, which would be the set where $a_i = 0$, then we have undefined behavior of b_i, as it could be either 1 or 0. Therefore this provides two cases:

  - $a_i = 0, b_i = 0, 0 \leq 0$.
  - $a_i = 0, b_i = 1, 0 \leq 1$.

  Thus for all possible $i \in [n], a_i \leq b_i$.