Note: all sets of the form $\{1, \cdots, n\}$ will be denoted by $[n]$

1.1 *We roll a fair die twice. Describe the sample space Ω and probability measure $\mathbb{P}$.*
The sample space is described by $\Omega = [6]^2$, with a probability measure of $\mathbb{P}(\{(i,j)\}) = \frac{1}{36}$ for all $i, j \in \{1, \cdots, 6\}$.
What is the probability that the second roll is larger than the first
If our first roll is a 1, then there are 5 numbers larger than 1 in the set $[6]$. For a roll of 2 we would have 1 less than if we rolled 1, or 4. For 3 to 6 it is a similar process. Therefore we have the sum $5+4+\cdots+1+0 = 15$ different rolls which would satisfy the condition. Therefore the set A containing all the rolls satisfying the condition would have a probability $\mathbb{P}(A) = \frac{5}{18} \approx 41\%$

1.4 *One of the 50 flags is put up at random 3 days of the week at a kindergarten*

(a) *What is the sample space and probability measure?*
- $\Omega = [50]^3$
- $\mathbb{P}(\{(i,j,k)\}) = \frac{1}{50^3}, i, j, k \in [50]$

(b) *What is the probability that the class hangs Wisconsin's flag on Monday, Michigan's flag on Tuesday, and California's flag on Wednesday?*
Since this is exactly one element in Ω, then the probability of this event is $\frac{1}{50^3}$

(c) *What is the probability that Wisconsin's flag will be hung at least two of the three days?*
There are 3 different ways in which the flag could occur exactly twice over the 3 days, and for the other day then there are 49 options which would fit the wisconsin flag on exactly 2 days. Then there is exactly one element in sample space where the wisconsin flag occurs three days in a row. Therefore there are $3*49+1 = 148$ valid flag combinations, thus the probability is $\frac{148}{50^3} \approx 0.1\%$

1.8 *Suppose that a bag of scrabble tiles contains 5 Es, 4 As, 3 Ns and 2 Bs. It is my turn and I draw 4 tiles from the bag without replacement. Assume that my draw is uniformly random. Let C be the event that I got two Es, one A and one N.*

(a) *Compute $\mathbb{P}(C)$ by imagining that the tiles are drawn one by one as an ordered sample.*
The initial factors are multiplying the probabilities of taking the two Es then an A and an N, then we multiple by the number of permutations of the drawing order and divide by 2 to avoid double counting the Es.
$$\frac{5}{14}\frac{4}{13}\frac{4}{12}\frac{3}{11}\frac{4!}{2} = \frac{120}{1001} \approx 1\%$$

(b) *Compute $\mathbb{P}(C)$ by imagining that the tiles are drawn all at once as an unordered sample.*
Here we multiply the number of valid ways to take Es then the N $\dfrac{\binom{5}{2}\binom{4}{1}\binom{3}{1}}{\binom{14}{4}} = \dfrac{120}{1001} \approx 1\%$

1a

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{cx-\frac{x^2}{2}}dx &= \int_{-\infty}^{\infty} e^{\frac{-1}{2}(x^2-2cx)}dx \\
&= \int_{-\infty}^{\infty} e^{\frac{-1}{2}(x^2-2cx+c^2)+\frac{c^2}{2}}dx \\
&= \int_{-\infty}^{\infty} e^{\frac{-1}{2}(x-c)^2+\frac{c^2}{2}}dx \\
&= e^{\frac{c^2}{2}}\int_{-\infty}^{\infty} e^{\frac{-1}{2}(x-c)^2}dx \\
&= e^{\frac{c^2}{2}}\int_{-\infty}^{\infty} e^{\frac{-1}{2}(x-c)^2}dx && \text{let } y = x-c, dy = dx \\
&= e^{\frac{c^2}{2}}\int_{-\infty}^{\infty} e^{\frac{-y^2}{2}}dy && \text{applying the fact } \int_{-\infty}^{\infty} e^{-x^2/2}dx = \sqrt{2\pi} \\
&= e^{\frac{c^2}{2}}\sqrt{2\pi}
\end{aligned}$$

2a

$$f(x,y) = \begin{cases} xe^{x^2-y} & \text{if } x \in (0,1), x^2 < y \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx &= \lim_{\gamma\to 0}\lim_{\delta\to 0}\int_{\gamma}^{1-\gamma}\int_{x^2+\delta}^{\infty} xe^{x^2-y}dydx \\
&\gamma \text{ and } \delta \text{ encode the non-inclusive nature of } f \\
&= \lim_{\gamma\to 0}\lim_{\delta\to 0}\int_{\gamma}^{1-\gamma} xe^{x^2}\int_{x^2+\delta}^{\infty} e^{-y}dydx \\
&= \lim_{\gamma\to 0}\lim_{\delta\to 0}\int_{\gamma}^{1-\gamma} xe^{x^2}[-e^{-y}]_{x^2+\delta}^{\infty}dx \\
&= \lim_{\gamma\to 0}\lim_{\delta\to 0}\int_{\gamma}^{1-\gamma} xe^{x^2}(0+e^{-x^2-\delta})dx \\
&= \lim_{\gamma\to 0}\lim_{\delta\to 0} e^{-\delta}\int_{\gamma}^{1-\gamma} xdx \\
&= \lim_{\gamma\to 0}\int_{\gamma}^{1-\gamma} xdx \\
&= \lim_{\gamma\to 0}\frac{(1-\gamma)^2-\gamma^2}{2} \\
&= \frac{1}{2}
\end{aligned}$$

b Let

$$A_1 = \{1, 2, 4, 8, 16\}$$
$$A_2 = \{2, 4, 6, 8, 10\}$$
$$A_3 = \{2, 10\}$$

Then

(a) $A_1 \cup A_3 = \{1, 2, 4, 8, 10, 16\}$

(b) $\bigcap_{i=1}^{3} A_i = \{2\}$

(c) $A_1 \backslash A_3 = \{1, 4, 8, 16\}$

(d) $A_1 \backslash A_2 = \{1, 16\}$

(e) $A_3 \cap A_1^c = \{10\}$

6c *In a lottery 5 different numbers are chosen from the first 90 positive integers.*

(a) *How many possible outcomes are there?*
There are $\binom{90}{5} = 43949268$ possible unordered combinations.

(b) *How many outcomes are there with the number 1 appearing among the five chosen numbers?*
Once 1 is chosen, then we have all of the combinations of 89 elements with a length of 4 elements, thus there are $\binom{89}{4} = 2441626$ combinations containing 1.

(c) *How many outcomes are there with two numbers below 50 and three numbers above 60?* There are 49 numbers below 50, and two are being chosen, so there are $\binom{49}{2}$ in that set, and for the 3 above 60 up to 90 would be $\binom{29}{3}$. Therefore there are $\binom{49}{2}\binom{29}{3} = 4297104$ valid combinations.

(d) *How many outcomes are there with the property that the last digits of all five numbers are different?* Since we're dealing with the first 90 numbers, then there are exactly 9 numbers with a given ones digit. Therefore for a particular drawing of digits we have $\binom{9}{1}\binom{9}{1}\binom{9}{1}\binom{9}{1}\binom{9}{1} = 59049$. However if we're to consider the number of ways in which distinct last digits could be chosen, we have a factor of $\binom{10}{5}$ to tack on, bringing our final total to 14880348.