Prove that for all $k, n \in \mathbb{Z}$ if $n > k \geq 0$, then $\binom{n}{k} = \sum_{i=1}^{k+1} \binom{n-i}{k-i+1}$

We must show for all $k, n \in \mathbb{Z}$ if $n > k \geq 0$, then $\binom{n}{k} = \sum_{i=1}^{k+1} \binom{n-i}{k-i+1}$. Suppose $n, k \in \mathbb{Z}, n > k \geq 0$. We must show $\binom{n}{k} = \sum_{i=1}^{k+1} \binom{n-i}{k-i+1}$. By the principal of mathematical induction we have for all $l \in \mathbb{Z}$, if $l < n, l > k \geq 0$, then $\binom{l}{k} = \sum_{i=1}^{k+1} \binom{l-i}{k-i+1}$. We now have two cases:

- Assume $n = 1$. Then $1 = \binom{1}{0} = \sum_{i=1}^{1} \binom{1-i}{1-i+1} = \binom{0}{0} = 1$.
- Assume $n > 1$. Then as proved on a previous assignment $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. Since $n - 1 < n$, then by the induction hypothesis we have $\binom{n-1}{k} = \sum_{i=1}^{k+1} \binom{n-1-i}{k-i+1}, \binom{n-1}{k-1} = \sum_{i=1}^{k} \binom{n-1-i}{k-i}$. Note that $\binom{n-k-2}{0} = \binom{n-k-2}{0} = 1$. Therefore by algebraic manipulation we have:

$$\begin{aligned}
\binom{n}{k} &= \binom{n-1}{k} + \binom{n-1}{k-1} \\
&= \sum_{i=1}^{k+1} \binom{n-1-i}{k-i+1} + \sum_{i=1}^{k} \binom{n-1-i}{k-i} \\
&= \binom{n-k-2}{0} + \sum_{i=1}^{k} \binom{n-1-i}{k-i+1} + \sum_{i=1}^{k} \binom{n-1-i}{k-i} \\
&= \binom{n-k-2}{0} + \sum_{i=1}^{k} \binom{n-i}{k-i+1} \\
&= \binom{n-k-1}{0} + \sum_{i=1}^{k} \binom{n-i}{k-i+1} \\
&= \sum_{i=1}^{k+1} \binom{n-i}{k-i+1}.
\end{aligned}$$