7.3 Since we want to find the number of elements of order 5, and since the largest power of 5 that divides 10 is 5 then we can apply the third sylow theorem. Let s be the number of 5-Sylow groups. Then by the third p-sylow theorem $s \mid 2$ and $s \equiv 1 \mod 5$. Therefore only 1 satisfies the equation. Since 5 is prime then our group is cyclic. Thus every element aside from the identity is of order 5. Thus our group has 4 elements of order 5.

7.5 (a)

(b)

(c)

(d)

7.6 We claim that $<(1234567),(124)(356)>= G$ is a non-abelian subgroup of order 21. Note that our generators are of orders 7 and 3 respectively. Additionally, in our non-abelian group by the third p-sylow theorem $<(1234567)>$ must be normal and $<(124)(356)>$ is not. This is demonstrated by the fact that $(124)(356)(1234567)(142)(365) = (1357246) = (1234567)^2$, which was shown in artin to be the requirement on being in the non-abelian isomorphism class of groups of order 21.

7.7 We know by orbit stabilizer that for the action of conjugation by elements of H on a given $s \in S$ satisfies $|H| = |C_H(s)||N_H(s)|$, where $C_H(s)$ is the conjugacy class of s by elements of H and $N_H(s)$ is the normalizer of s by elements in h. Recognizing that $|H| = p$ by construction gives us that either $|C_H(s)| = 1$ or $|C_H(s)| = 1$. What does this say? Either s is fixed by H or s after being conjugated by all elements of H forms a cycle since H is isomorphic to C_p then we can conjugate by successive powers of the generator of H until we return to the original element s.

7.8
- The order of $GL_n(F_p)$ is $\prod_{i=0}^{n-1}(p^n - p^i)$. This is obtained by considering the rows of a given matrix with F_p entries. Note that the first row can be any element in F_p^n except the 0 vector. Thus the first row has $p^n - 1$ possibilities. Going a row down one can note that every element in F_p^n satisfies except the 0 vector and the multiples of the first row. Thus there are $p^n - p$ options for the second. For the ith row one has to consider that any linear combination of the first $i-1$ rows. Since there are $i-1$ rows being multiplied by any element in F_p, then the ith row has $p^n - p^{i-1}$. Thus since there are n rows, $|GL_n(F_p)| = \prod_{i=1}^{n}(p^n - p^{i-1}) = \prod_{i=0}^{n-1}(p^n - p^i)$
- Note that $\prod_{i=0}^{n-1}(p^n - p^i) = p^{\frac{n(n-1)}{2}} \prod_{i=0}^{n-1}(p^{n-i} - 1)$. Thus the p-sylow subgroups of $GL_n(F_p)$ are of order $p^{\frac{n(n-1)}{2}}$. We claim that the subgroup of unitriangular matrices (the set of upper triangular matrices with 1s in the diagonal) is a p-sylow subgroup. Note that the determinate of all unitriangular matrices is 1, so it is within $GL_n(F_p)$. Additionally there are $\frac{n(n-1)}{2}$ entries with p possible values per, so the subgroup has order $p^{\frac{n(n-1)}{2}}$. Thus we have found a p-sylow subgroup.
- Since we want to find the number of p-sylow subgroups, and we know the subgroup of unitriangular matrices is a member, we need to find the number of conjugates.

Let U be the set of unitriangular matrices. We know by orbit stabilizer that $|GL_n(F_p)| = |N(U)||C(U)|$. We know from problem 7.6.2 that the normalizer of unitriangular matrices is the group of upper triangular matrices. Since upper triangular matrices have $\frac{n(n-1)}{2}$ entries above the diagonal that can take on p values and n entries along on the diagonal which can take on $p-1$ values (we exclude 0 to ensure the matrices remain inveritble). Thus $|N(U)| = (p-1)^n p^{\frac{n(n-1)}{2}}$. Therefore $|C(U)| = \frac{1}{(p-1)^n} \prod_{i=0}^{n-1} p^i - 1$

7.9 (a) By the third p-sylow theorem, if we consider s_p as the number of p-sylow groups, then $11 \mid s_3, s_3 \equiv 1 \mod 3$ and $3 \mid s_{11}, s_{11} \equiv 1 \mod 11$ is only solved by $s_3 = s_{11} = 1$. Thus they must be normal. Since they are of prime order then they are isomorphic to $\mathbb{Z}_3$ and $\mathbb{Z}_{11}$ respectively. Thus by the chinese remainder theorem $\mathbb{Z}_3 \times \mathbb{Z}_{11} \simeq \mathbb{Z}_{33}$. Thus all groups of order 33 are isomorphic to $\mathbb{Z}_{33}$.

(b)