Suppose $f : S \longrightarrow T$. Recall that for $X \subseteq S$, the image of X under f, denoted $im_f(X)$ is the set $\{f(x) : x \in X\}$ and for $Y \subseteq T$, the preimage of Y under f, $preim_f(Y)$ is the set $\{x \in S : f(x) \in Y\}$
For each of the following assertions determine whether it's true or false. If it's true prove it. If it's false, disprove it.
(a) For any two subsets X_1 and X_2 of $S, im_f(X_1 \cup X_2) = im_f(X_1) \cup im_f(X_2)$.
Proof: We must show that for all subsets X_1, X_2 of S that $im_f(X_1 \cup X_2) = im_f(X_1) \cup im_f(X_2)$. Suppose X_1, X_2 are arbitrary subsets of S. By the definition of set equality our goal becomes showing $im_f(X_1 \cup X_2) \subseteq im_f(X_1) \cup im_f(X_2)$ and $im_f(X_1) \cup im_f(X_2) \subseteq im_f(X_1 \cup X_2)$.

- We must show that $im_f(X_1 \cup X_2) \subseteq im_f(X_1) \cup im_f(X_2)$. Suppose p is an arbitrary member of $im_f(X_1 \cup X_2)$. Then by definition of image $p \in \{f(x) : x \in X_1 \cup X_2\}$. By definition of set union $p \in \{f(x) : x \in X_1 \text{ or } x \in X_2\}$. This provides two cases, when $x \in X_1$ and $x \in X_2$.
  - Assume $x \in X_1$. Then $\{f(x) : x \in X_1\} = im_f(X_1)$. Since $p \in im_f(X_1)$, then by definition of union $p \in im_f(X_1) \cup im_f(X_2)$.
  - Assume $x \in X_2$. Then $\{f(x) : x \in X_@\} = im_f(X_2)$. Since $p \in im_f(X_2)$, then by definition of union $p \in im_f(X_1) \cup im_f(X_2)$.
- We must show $im_f(X_1) \cup im_f(X_2) \subseteq im_f(X_1 \cup X_2)$. Suppose $p \in im_f(X_1) \cup im_f(X_2)$. By definition of set union $p \in im_f(X_1)$ or $p \in im_f(X_2)$. This provides two cases, when $p \in im_f(X_1)$ and when $p \in im_f(X_2)$.
  - Assume $p \in im_f(X_1)$. Then by definition of image $p \in \{f(x) : x \in X_1\}$. Therefore if $x \in X_1$ then it by definition $x \in X_1 \cup X_2$. Thus $p \in im_f(X_1 \cup X_2)$.
  - Assume $p \in im_f(X_2)$. Then by definition of image $p \in \{f(x) : x \in X_2\}$. Therefore if $x \in X_2$ then it by definition $x \in X_1 \cup X_2$. Thus $p \in im_f(X_1 \cup X_2)$.

Therefore $im_f(X_1 \cup X_2) \subseteq im_f(X_1) \cup im_f(X_2)$.
(b) For any two subsets X_1 and X_2 of $S, im_f\,(X_1 \cap X_2) = im_f\,(X_1) \cap im_f\,(X_2)$.
This is false. Take $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$, $X_1 = \{1, 2, 3\}$, and $X_2 = \{-1, -2, -3.\}$. Therefore $im_f(X_1 \cap X_2) = im_f(\{1, 2, 3\} \cap \{-1, -2, -3\}) = im_f(\emptyset) = \emptyset \neq \{1, 4, 9\} = \{1^2, 2^2, 3^2\} \cap \{(-1)^2, (-2)^2, (-3)^2\} = im_f(X_1) \cap im_f(X_2)$.

(c) For any two subsets Y_1 and Y_2 of T, $preim_f(Y_1 \cap Y_2) = preim_f(Y_1) \cap preim_f(Y_2)$.
We must show for any two subsets of T, that $preim_f(Y_1 \cap Y_2) = preim_f(Y_1) \cap preim_f(Y_2)$. By the definition of set equality, our goal is now to show that $preim_f(Y_1 \cap Y_2) \subseteq preim_f(Y_1) \cap preim_f(Y_2)$ and $preim_f(Y_1) \cap preim_f(Y_2) \subseteq preim_f(Y_1 \cap Y_2)$.

- We must show $preim_f(Y_1 \cap Y_2) \subseteq preim_f(Y_1) \cap preim_f(Y_2)$. Suppose p is an arbitrary element in $preim_f(Y_1 \cap Y_2)$. By the definition of set intersection we must show $p \in preim_f(Y_1 \cap Y_2) \subseteq preim_f(Y_1)$ and $p \in preim_f(Y_1 \cap Y_2) \subseteq preim_f(Y_2)$.
  - We must show $p \in preim_f(Y_1 \cap Y_2) \subseteq preim_f(Y_1)$. By definition $p \in \{x \in S : f(x) \in Y_1 \text{ and } f(x) \in Y_2\}$. If $f(x) \in Y_1$ and $f(x) \in Y_2$, then necessarily $f(x) \in Y_1$. Therefore by definition $p \in preim_f(Y_1)$.

  - We must show $p \in preim_f(Y_1 \cap Y_2) \subseteq preim_f(Y_2)$. By definition $p \in \{x \in S : f(x) \in Y_1$ and $f(x) \in Y_2\}$. If $f(x) \in Y_1$ and $f(x) \in Y_2$, then necessarily $f(x) \in Y_2$. Therefore by definition $p \in preim_f(Y_2)$.

- We must show $preim_f(Y_1) \cap preim_f(Y_2) \subseteq preim_f(Y_1 \cap Y_2)$. Suppose p is an arbitrary element of $preim_f(Y_1) \cap preim_f(Y_2)$. By the definition of set intersection, we must show $preim_f(Y_1) \cap preim_f(Y_2) \subseteq preim_f(Y_1$ and $Y_2)$. Therefore we now have two goals, $preim_f(Y_1) \cap preim_f(Y_2) \subseteq preim_f(Y_1)$ and $preim_f(Y_1) \cap preim_f(Y_2) \subseteq preim_f(Y_2)$.
  - We must show $preim_f(Y_1) \cap preim_f(Y_2) \subseteq preim_f(Y_1)$. By the definition of set intersection $p \in preim_f(Y_1)$ and $p \in preim_f(Y_2)$. Therefore $p \in preim_f(Y_1)$.
  - We must show $preim_f(Y_1) \cap preim_f(Y_2) \subseteq preim_f(Y_2)$. By the definition of set intersection $p \in preim_f(Y_1)$ and $p \in preim_f(Y_2)$. Therefore $p \in preim_f(Y_2)$.

(d) For any subset X of S, $preim_f(im_f(X)) \supseteq X$.
Proof: We must show for any subset X of S, $preim_f(im_f(X)) \supseteq X$. Suppose x is an arbitrary member of X. By the definition of image we must show $X \subseteq preim_f(\{f(x) : x \in X\})$ By definition of preimage we must show $X \subseteq \{s \in S : f(s) \in \{f(x) : x \in X\}\}$. Therefore since we have $x \in X$, we satisfy the internal definition of image, then the preimage of that simply takes all of the potential values in S which satisfy $f(s) = f(x)$, which we know include x.

(e) For any subset X of S, preim $_f$ $(\mathrm{im}_f(X)) = X$.
False: Let $f : \mathbb{R} \to \mathbb{R}$ be given as $f(x) = x^2$, and $X = \{1, 2\}$. then $X = \{1, 2\} \neq \{-2, -1, 1, 2\} = preim_f(\{1, 4\}) = preim_f(im_f(X))$.
(f) For any subset Y of T,im_f (preim $_f(Y)) \supseteq Y$.
False. Take $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sqrt{x}$, and $Y = \mathbb{R}$. Then $Y = \mathbb{R} \not\subseteq \mathbb{R}_{\geq 0} = im_f(\mathbb{R}_{\geq 0}) = im_f(preim_f((\mathbb{R})) = im_f(preim_f((Y))$.
(g) For any subset Y of T, $im_f(preim_f(Y)) = Y$.