1.1

1.8 (a) $\{1, 5, 7, 11\}$

(b) $\{1, 3, 5, 7\}$

(c) We claim that the set $\Phi(n) = \{k \in [n] : \gcd(n, k) = 1\}$ is the set of units of $\mathbb{Z}/n\mathbb{Z}$. Note that if $\gcd(a, n) = 1$ then there exists $x, y \in \mathbb{Z}$ such that $ax + ny = 1$. Therefore $1 = ax + ny \equiv ax \mod n$. Thus $x \mod n$ is the inverse of a. However consider for contradiction that $\Phi(n)$ does not contain all of the units. Thus there exists $u \in \mathbb{Z}/n\mathbb{Z}$ which $u \notin \Phi(n)$ and there exists $w \in \mathbb{Z}/n\mathbb{Z}$ such that $uw \equiv 1 \mod n$. Thus by the definition of modular arithmatic there exists $m \in \mathbb{Z}$ then $uw + my = 1$. Thus by definition of the gcd, $\gcd(u, n) = 1$. Thus $u \in \Phi(n)$. This is a contradiction. Thus $\Phi(n)$ contains all of the units of $\mathbb{Z}/n\mathbb{Z}$.

2.2 Proving that $F[[x]]$ is a ring

- Addition is an abelian group.
  - Commutativity: Suppose $a, b \in F[[x]]$ where $a = \sum_{i=0} a_i x^i, b = \sum_{i=0} b_i x^i$. Then

$$a+b = \sum_{i=0} a_i x^i + \sum_{i=0} b_i x^i = \sum_{i=0} (a_i + b_i) x^i = \sum_{i=0} (b_i + a_i) x^i = \sum_{i=0} b_i x^i + \sum_{i=0} a_i x^i = b+a$$

  - Identity: Suppose $a \in F[[x]]$. Then

$$0 + a = a + 0 = \sum_{i=0} a_i x^i + \sum_{i=0} 0x^i = \sum_{i=0} (a_i + 0) x^i = \sum_{i=0} a_i x^i = a.$$

    Thus 0 is the additive identity for $F[[x]]$.
  - Associativity: Suppose $a, b, c \in F[[x]]$. Then

$$\begin{aligned}
(a + b) + c &= \sum_{i=0} (a_i + b_i) x^i + \sum_{i=0} c_i x^i \\
&= \sum_{i=0} (a_i + b_i + c_i) x^i \\
&= \sum_{i=0} a_i x^i + (b_i + c_i) x^i \\
&= \sum_{i=0} a_i x^i + \sum_{i=0} (b_i + c_i) x^i \\
&= a + (b + c)
\end{aligned}$$

  - Additive inverses: Suppose $a \in F[[x]]$. Then by definition $a = \sum_{i=0} a_i x^i$. Since F is a field then the sequence $(-a_0, -a_1, \cdots) \subseteq F$. Therefore we can construct $b = \sum_{i=0} -a_i x^i$. Thus $a + b = \sum_{i=0} (a_i - a_i) x^i = \sum_{i=0} 0x^i = 0$. Thus b is the inverse of a.

- Multiplication is commutative: Suppose $a, b \in F[[x]]$ Then

$$\begin{aligned}(ab)_n &= \sum_{i+j=n} a_i b_j \\ &= \sum_{j+i=n} b_j a_i \text{ commutativity of } F \\ &= \sum_{l+k=n} b_l a_k \text{ let } l = j, k = i \\ &= (ba)_n\end{aligned}$$

  Since the nth coefficient is the same, then the power series is identical.
- Multiplication is associative
- Distributive rule.

The ideals of $F[[x]]$

3.2 Suppose $I \subset \mathbb{Z}[i]$ and consider $x \in I$. By definition of $\mathbb{Z}[i]$ there exists $a, b \in \mathbb{Z}$ such that $x = a + bi$ where at least one of the a, b is non-zero. Therefore the element $a - bi \in \mathbb{Z}[i]$ since $-b \in \mathbb{Z}$. Thus by the definition of an ideal $(a - bi)(a + bi) \in I$. Therefore $a^2 - b^2 \in I$. Since $a, b \in \mathbb{Z}$ then I contains an integer.

3.6

3.12

4.1

5.6

6.1