7.3 Since we want to find the number of elements of order 5, and since the largest power of 5 that divides 10 is 5 then we can apply the third sylow theorem. Let s be the number of 5-Sylow groups. Then by the third p-sylow theorem $s \mid 2$ and $s \equiv 1 \mod 5$. Therefore only 1 satisfies the equation. Since 5 is prime then our group is cyclic. Thus every element aside from the identity is of order 5. Thus our group has 4 elements of order 5.

7.5 (a) Since D_{10} has order 20, we must find a subgroup of order 4. $\{1, \theta^5, r, r\theta^5\}$ is a subgroup of order 4

(b) Note that we have shown before that the homomorphism from $S_4 \to S_3$ has a kernel of order 4, and additionally every element has even parity. Additionally each element of that kernel has order 2, therefore T contains an isomorphic copy of the Klein 4 group.

(c) Since O has order 24, we seek a subgroup of order 8. Note that if one fixes a pair of faces, one has a 90 degree rotation and a flip. Note that that is isomorphic to D_4, the set of symmetries of the square. Since $|D_4| = 8$, we are done

(d) Note that $I \cong A_5$, and we know that $|A_5|3 \cdot 4 \cdot 5$ therefore we need to find a subgroup of order 4. Since we found the klein 4 group inside of A_4 then we can find copies in A_5.

7.6 We claim that $<(1234567), (124)(356)>= G$ is a non-abelian subgroup of order 21. Note that our generators are of orders 7 and 3 respectively. Additionally, in our non-abelian group by the third p-sylow theorem $<(1234567)>$ must be normal and $<(124)(356)>$ is not. This is demonstrated by the fact that $(124)(356)(1234567)(142)(365) = (1357246) = (1234567)^2$, which was shown in artin to be the requirement on being in the non-abelian isomorphism class of groups of order 21.

7.7 We know by orbit stabilizer that for the action of conjugation by elements of H on a given $s \in S$ satisfies $|H| = |C_H(s)||N_H(s)|$, where $C_H(s)$ is the conjugacy class of s by elements of H and $N_H(s)$ is the normalizer of s by elements in h. Recognizing that $|H| = p$ by construction gives us that either $|C_H(s)| = 1$ or $|C_H(s)| = 1$. What does this say? Either s is fixed by H or s after being conjugated by all elements of H forms a cycle since H is isomorphic to C_p then we can conjugate by successive powers of the generator of H until we return to the original element s.

7.8

- The order of $GL_n(F_p)$ is $\prod_{i=0}^{n-1}(p^n - p^i)$. This is obtained by considering the rows of a given matrix with F_p entries. Note that the first row can be any element in F_p^n except the 0 vector. Thus the first row has $p^n - 1$ possibilities. Going a row down one can note that every element in F_p^n satisfies except the 0 vector and the multiples of the first row. Thus there are $p^n - p$ options for the second. For the ith row one has to consider that any linear combination of the first $i-1$ rows. Since there are $i-1$ rows being multiplied by any element in F_p, then the ith row has $p^n - p^{i-1}$. Thus since there are n rows, $|GL_n(F_p)| = \prod_{i=1}^{n}(p^n - p^{i-1}) = \prod_{i=0}^{n-1}(p^n - p^i)$

- Note that $\prod_{i=0}^{n-1}(p^n - p^i) = p^{\frac{n(n-1)}{2}} \prod_{i=0}^{n-1}(p^{n-i} - 1)$. Thus the p-sylow subgroups of $GL_n(F_p)$ are of order $p^{\frac{n(n-1)}{2}}$. We claim that the subgroup of unitriangular matrices (the set of upper triangular matrices with 1s in the diagonal) is a p-sylow subgroup. Note that the determinate of all unitriangular matrices is 1, so it is within $GL_n(F_p)$. Additionally there are $\frac{n(n-1)}{2}$ entries with p possible values per, so the subgroup has order $p^{\frac{n(n-1)}{2}}$. Thus we have found a p-sylow subgroup.
- Since we want to find the number of p-sylow subgroups, and we know the subgroup of unitriangular matrices is a member, we need to find the number of conjugates. Let U be the set of unitriangular matrices. We know by orbit stabilizer that $|GL_n(F_p)| = |N(U)||C(U)|$. We know from problem 7.6.2 that the normalizer of unitriangular matrices is the group of upper triangular matrices. Since upper triangular matrices have $\frac{n(n-1)}{2}$ entries above the diagonal that can take on p values and n entries along on the diagonal which can take on $p-1$ values (we exclude 0 to ensure the matrices remain inveritble). Thus $|N(U)| = (p-1)^n p^{\frac{n(n-1)}{2}}$. Therefore $|C(U)| = \frac{1}{(p-1)^n}\prod_{i=0}^{n-1} p^i - 1$

7.9 (a) By the third p-sylow theorem, if we consider s_p as the number of p-sylow groups, then $11 \mid s_3, s_3 \equiv 1 \mod 3$ and $3 \mid s_{11}, s_{11} \equiv 1 \mod 11$ is only solved by $s_3 = s_{11} = 1$. Thus they must be normal. Since they are of prime order then they are isomorphic to $\mathbb{Z}_3$ and $\mathbb{Z}_{11}$ respectively. Thus by the chinese remainder theorem $\mathbb{Z}_3 \times \mathbb{Z}_{11} \simeq \mathbb{Z}_{33}$. Thus all groups of order 33 are isomorphic to $\mathbb{Z}_{33}$.

(b) Note that by the third sylow theorem we know that the 3-sylow subgroup is normal and there are either 1, 3, or 9 2-sylow subgroups (represented by the number s_2). Note that there are two subgroups of order 9, $\mathbb{Z}_9$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$. Therefore we must work by cases

- Suppose $s_2 = 1$. If we have $\mathbb{Z}_9$ then by the normality of the sylow 2-subgroup we have that $\mathbb{Z}_2 \times \mathbb{Z}_9$. If we have $\mathbb{Z}_3 \times Z_3$ then we have the group $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$.
- Suppose that $s_2 = 3$. If our sylow subgroup is $\mathbb{Z}_9$ then we have a problem. We know that if we have $x \in \mathbb{Z}_9$ and our element of order 2, y that $x^i y = y x^{-i}$. This ensures that $\{y, x^3 y, x^6 y\}$ are the only elements which have order 2. Therefore xy should have order 6, since it's not in $\mathbb{Z}_9$, and doesn't have order 2 since it's not in the set of elements which have order 2, and can't be 18 since our group doesn't have a single generator. Thus $(xy)^2$ has order 3. However the only elements of order 3 are x^3, x^6. Thus $(xy)^3 = x^4 y$ or $(xy)^3 = x^7 y$. However $(xy)^3$ must have order 2, which contradicts the fact that it multiplies to something which is not in the set of three elements of order 2. Thus we must consider our 3-sylow subgroup to be $\mathbb{Z}_3 \times \mathbb{Z}_3$. Note that since we have found all other possible matrices which satisfy $B^2 = I$, the only other possible matrix with a diagonal containing 1 and 2. Note that if we consider the second entry in $\mathbb{Z}_3 \times \mathbb{Z}_3$ and multiplication by the matrix, we have a group of order 6 which is not commutative, yielding us S_3, and then the first coordinate is unchanged and of order 3, thus our group is isomorphic to $S_3 \times \mathbb{Z}_3$.
- Suppose that $s_2 = 9$. If we have $\mathbb{Z}_9$ as our 3-sylow subgroup then for our

element y of order 2, we know that for $x \in \mathbb{Z}_9$ we have that $yxy^{-1} = x^i$ since $\mathbb{Z}_9$ is normal. Thus since y is of order 2 and $xy \notin \mathbb{Z}_9$ then $1 = (xy)^2 = xyxy$ implies $yxy = x^{-1}$. Thus we have found D_9. If we have our sylow subgroup as $\mathbb{Z}_3 \times \mathbb{Z}_3$, then we must consider the homomorphism from $\mathbb{Z}_2 \to Aut(\mathbb{Z}_3 \times \mathbb{Z}_3)$. Since we need to have 9 order 2 subgroups which satisfies the relation $yxy = x^{-1}$. Since every element is it's own inverse in $\mathbb{Z}_3 \times \mathbb{Z}_3$, and the trivial automorphism implies commutativity ($\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$) then we must have the matrix $2I$, where I is the identity. This corresponds to the general dihedral group for $\mathbb{Z}_3 \times \mathbb{Z}_3$

(c) For all groups of order 20, the group of order 5 is normal via the sylow theorem, or $\mathbb{Z}_5$, whose generator is x, and there are 2 groups of order 4, $\mathbb{Z}_4$ and V_4. We will consider where the elements of $\mathbb{Z}_5$ are taken via conjugation by elements in our 2-sylow subgroup.

- Consider our group of order 4 to be $\mathbb{Z}_4$. Let y be a generator of $\mathbb{Z}_4$
  - If $yxy^{-1} = x$, then $\mathbb{Z}_4$ and $\mathbb{Z}_5$ commute, giving us $\mathbb{Z}_4 \times \mathbb{Z}_5$
  - If $yxy^{-1} = x^3$, then $y^3xy^{-3} = x^2$, thus WLOG assume $yxy^{-1} = x^2$. Since this homomorphism is well defined, we have found another group of order 20
  - Suppose $yxy^{-1} = x^4$. Then $y^2xy^{-2} = x$, thus x, y^2 form a subgroup of order 10. Note that since it has index 2, then it is normal. Let z be a generator of the subgroup. Then $z^5 = y^2$. Note that the only possible way to conjugate z with y is $yzy^{-1} = x^{-1}$, since conjugating by y twice would just be z, thus the power of z which gets mapped from conjugation must square to 1, this is only satisfied by -1. Note that this completely describes the multiplication table for our group.
- Consider our group of order 4 to be V_4 with generators y, z
  - If both generators commute with x, then we have $\mathbb{Z}_2 \times \mathbb{Z}_1 0$
  - If only (WLOG) $yxy^{-1} = x$, then x, y generates a subgroup of order 10, which is normal. Let x' be a generator of this subgroup. Since x, z don't commute, then x', z don't commute. Thus $zx'z^{-1} = x'^{-1}$, for the same reason that when our group of order 4 is $\mathbb{Z}_4$ for the automorphism $x \mapsto x^4$. Thus we have exactly described D_{10}
  - if neither y nor z commute with x then set $y = yz$, then repeat steps to get back D_{10}

(d) TODO order 30