5.5.3 Let $S(x,y) = (x^2 - y^2, 2xy)$. Note that $DS(x,y) = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$. Therefore the determinate of the jacobian is as follows: $|DS(x,y)| = 4(x^2 + y^2)$. The map can only be non-invertible when $|DS(x,y)| = 0$, however this only occurs when $x = y = 0$. Furthermore, the structure of the jacobian is all linear continuous terms, thus the jacobian is continuous everywhere. Therefore the inverse function theorem holds for S everywhere but the origin. Additionally, solving $(u,v) = S(x,y)$ for undetermined (x,y) generally, one finds that $x = \pm\sqrt{\frac{\pm\sqrt{u^2+v^2}+u}{2}}, y = \pm\sqrt{\frac{\pm\sqrt{u^2+v^2}-u}{2}}$. Since we're effectively doing complex exponentiation then swapping the sign should result in the same value, therefore if we consider approximations close to the origin, say within a ball $B((0,0),\delta)$ for $\delta > 0$, then the points $a = (\delta/2, \delta/2)$ and $b = (-\delta/2, -\delta/2)$ are in $B((0,0),\delta)$. Observe that $u(a) = u(b) = 0$ and $v(a) = v(b) = \frac{\delta^2}{2}$. Thus we fail to have a well defined inverse.

5.5.4 Let $(u,v) = f(x,y) = (e^x \cos(y), e^x \sin(y))$. Then $Df(x,y) = e^x \begin{bmatrix} \cos(y) & -\sin(y) \\ \sin(y) & \cos(y) \end{bmatrix}$. Note that the determinate $|Df(x,y)| = e^x(\cos^2(y) + \sin^2(y)) = e^x > 0$, therefore the Jacobian is always invertible. Furthermore, each term of the Jacobian is the product of continuous functions, thus making it continuous. Therefore the inverse function theorem holds everywhere. Note that the inverse is defined as $(\frac{1}{2}\log(u^2+v^2), \arctan(\frac{u}{v}))$. Note that issues don't arise for the inverse of the x coordinate, only the y, as $-\frac{\pi}{2} < \arctan(\frac{u}{v}) < \frac{\pi}{2}$.
In order to find the largest disc U around the origin in which f has an inverse, we only must consider the restrictions on y, as e^x is always invertible. Since we have $\cos(y)$ and $\sin(y)$, we claim that the largest ball around $(0,0)$ can be $B((0,0),\pi)$. Note if we have for a small δ the point $(x, \pi + \delta)$, then we have u, v such that $(u,v) = e^x(-\cos(\delta), -\sin(\delta))$, which is equivalent to $e^x(\cos(\delta - \pi), \sin(\delta - \pi))$, corresponding to the point $(x, \delta - \pi) \in B((0,0),\pi)$, for a small enough choice of x. Thus we have found the largest U.
To find the disk V For the point $(1,0)$, we have the arctan restriction which would make one think that $V = B((1,0), \frac{\pi}{2})$, however the point $(u,v) = (0,0)$ yields $x = \frac{1}{2}\log(0) = -\infty$, thus the largest disk we can have is $V = B((1,0),1)$. If we want to pick a larger open set for U, one could take $\mathbb{R} \times (-\pi, \pi)$. Since log is onto for all real numbers uniquely, then the inverse defined for x works everywhere. Note that the given set is open since for arbitrary $(a,b) \in \mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$, for any chosen r such that $(b-r, b+r) \subseteq (-\frac{\pi}{2}, \frac{\pi}{2})$ then any points $(c,d) \in B((a,b),r)$ have that $|d - b| \leq \sqrt{(c-a)^2 + (d-b)^2} < r$, thus $d \in (b-r, b+r)$, and trivially $c \in \mathbb{R}$, thus our given set is open. For a larger V, note that the image of f under U would be $\mathbb{R}^2 \backslash \{(x,0) : y \leq 0\}$, as the origin can't be mapped since $e^x > 0$, and the point $(-1,0) = e^0(\cos(\pi), \sin(\pi))$ can't be reached since $(x,\pi) \notin U$.

5.5.6 We want to find $\frac{\partial}{\partial y_j} g_k(y_1, \cdots, y_m) = \frac{\partial}{\partial y_j} f_k(\Phi(y_1, \cdots, y_m))$, where $k = m+1 \dots n$, Φ is the inverse of $(x_1, \cdots, x_m) \mapsto (f_1(x_1, \cdots, x_m), \dots, f_m(x_1, \cdots, x_m))$ around $(y_{10}, \cdots, y_{m0})$ which satisfies the inverse function theorem. Let $\vec{y} = (y_1, \cdots, y_m)$ Observe by the general chain rule that $Dg_k(\vec{y}) = Df_k(\Phi(\vec{y})) = [Df_k]_{\Phi(\vec{y})}[D\Phi]_{\vec{y}}$. Observe that $[Df_k]_{\Phi(\vec{y})} =$

$\left(\frac{\partial f_k(\Phi(\vec{y}))}{\partial x_1}, \cdots, \frac{\partial f_k(\Phi(\vec{y}))}{\partial x_m}\right)$ and since the partials of f are continuous then by the inverse function theorem we have that $[D\Phi]_{\vec{y}} = [D(f_1, \cdots, f_m)]^{-1}_{\Phi(\vec{y})}$. Therefore $\frac{\partial}{\partial y_j} g_k(y_1, \cdots, y_m) =$

$$Df_k(\Phi(\vec{y}))\hat{e}_j = \left(\frac{\partial f_k(\Phi(\vec{y}))}{\partial x_1}, \cdots, \frac{\partial f_k(\Phi(\vec{y}))}{\partial x_m}\right) \begin{bmatrix} \frac{\partial f_1(\Phi(\vec{y}))}{\partial x_1} & \cdots & \frac{\partial f_1(\Phi(\vec{y}))}{\partial x_m} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_m(\Phi(\vec{y}))}{\partial x_1} & \cdots & \frac{\partial f_m(\Phi(\vec{y}))}{\partial x_m} \end{bmatrix}^{-1} \hat{e}_j$$

5.5.7

- Inversion with respect to the x variable at $t = \pi$: $x'(\pi) = \cos(\pi) = -1$, thus the inversion formula can be applied. Since arcsin only has a domain of $[-\pi/2, \pi/2]$, we must consider the inversion at $x = \sin(t+\pi) = -\sin(t)$, thus $y = -x\sqrt{1-x^2}$. Since $y(0) = 0$ then $\frac{dy}{dx} y(0) = \frac{d}{dx}(-x\sqrt{1-x^2})|_{x=0} = -\sqrt{1-x^2}|_{x=0} + \frac{x^2}{\sqrt{1-x^2}}|_{x=0} = -1$.
- See if the inverse function theorem can be applied at y for the following x values:
  - $t = 0$: $y'(0) = \cos^2(0) - \sin^2(0) = 1$, thus it can be inverted, and $\frac{dx}{dy} = \frac{1}{1} = 1$
  - $t = \frac{\pi}{2}$: $y'(\frac{\pi}{2}) = \cos^2(\frac{\pi}{2}) - \sin^2(\frac{\pi}{2}) = -1$, thus it can be inverted and $\frac{dx}{dy} = \frac{0}{-1} = 0$
  - $t = \pi$: $y'(\pi) = \cos^2(\pi) - \sin^2(\pi) = 1$, thus it can be inverted and $\frac{dx}{dy} = \frac{-1}{1} = -1$
- For $t = \frac{\pi}{2}$, $x'(\frac{\pi}{2}) = \cos(\frac{\pi}{2}) = 0$, thus the inverse function theorem can't be applied.