5.4.5 (a) Show that $g'(1)$ does not exists:
Consider the sequence $x_m = 1 + 2^{-m}$ where $m \in \mathbb{N}$. We will show that

$$\frac{g(x_m) - g(1)}{x_m - 1} = m + 1 - 2^{m+1}.$$

Note that

$$\begin{aligned}
g(x_m) &= \sum_{n=0}^{\infty} \frac{h(2^n(1+2^{-m}))}{2^n} \\
&= \sum_{n=0}^{\infty} \frac{h(2^n + 2^{n-m})}{2^n} \\
&= h(1+2^{-m}) + \sum_{n=1}^{\infty} \frac{h(2^n + 2^{n-m})}{2^n} \\
&= 1 - 2^m + \sum_{n=1}^{\infty} \frac{h(2^{n-m})}{2^n} && \text{periodicity of } h(x) \\
&= 1 - 2^m + \sum_{n=1}^{m} \frac{h(2^{n-m})}{2^n} && \text{if } n > m \text{ then } 2^{n-m} \text{ is a whole multiple of 2} \\
&= 1 - 2^m + \sum_{n=1}^{m} \frac{2^{n-m}}{2^n} \\
&= 1 - 2^m + m2^{-m} \\
&= 1 + (m-1)2^{-m}.
\end{aligned}$$

Therefore,

$$\frac{g(x_m) - g(1)}{x_m - 1} = \frac{1 + (m-1)2^{-m} - 1}{1 + 2^{-m} - 1} = \frac{(m-1)2^{-m}}{2^{-m}} = m - 1.$$

Since the limit diverges to infinity, then $g'(1)$ does not exists.
Show that $g'(\frac{1}{2})$ does not exists.
Consider the sequence $x_m = 1 + 2^{-m}$ where $m \in \mathbb{N}$. We will show that

$$\frac{g(x_m) - g(1)}{x_m - 1} = m - 3$$

for a sufficently large m.

Note that

$$\begin{aligned}
g(x_m) &= \sum_{n=0}^{\infty} \frac{h(2^n(\frac{1}{2}+2^{-m}))}{2^n} \\
&= \sum_{n=0}^{\infty} \frac{h(2^{n-1}+2^{n-m})}{2^n} \\
&= \sum_{n=0}^{m} \frac{h(2^{n-1}+2^{n-m})}{2^n} \\
&= h(2^{-1}+2^{-m}) + h(1+2^{1-m}) + \sum_{n=2}^{m} \frac{h(2^{n-m})}{2^n} && n \text{ is large enough that } 2 \mid 2^{n-1} \\
&= h(2^{-1}+2^{-m}) + h(1+2^{1-m}) + (m-2)2^{-m} \\
&= 2^{-1}+2^{-m}+1-2^{1-m}+(m-2)2^{-m} && \text{take } m>2 \\
&= 2^{-1}+1+(m-3)2^{-m}
\end{aligned}$$

Noting that $h(\frac{1}{2}) = 2^{-1}+1$ we can compute:

$$\frac{g(x_m)-g(\frac{1}{2})}{x_m-\frac{1}{2}} = \frac{2^{-1}+1+(m-3)2^{-m}-2^{-1}-1}{\frac{1}{2}+2^{-m}-\frac{1}{2}} = m-3.$$

Therefore $g'(\frac{1}{2})$ diverges and is not well defined.

(b) For $p \in \mathbb{Z}, k \in \mathbb{N}_0, x = \frac{p}{2^k}$ show that $g'(x)$ does not exists.
Consider the sequence $x_m = \frac{p}{2^k} + \frac{1}{2^m}$ where $m \in \mathbb{N}$. We will show that

$$\frac{g(x_m)-g(\frac{p}{2^k})}{x_m-\frac{p}{2^k}} =$$

Note that

$$\begin{aligned}
g(\frac{p}{2^k}) &= \sum_{n=0}^{\infty} \frac{h(2^{n-k}p)}{2^n} \\
&= \sum_{n=0}^{m} \frac{h(2^{n-k}p)}{2^n} && \text{if } n>k \text{ then } h(2^{n-k}p)=0
\end{aligned}$$

Furthermore we can compute

$$\begin{aligned}
g(x_m) &= \sum_{n=0}^{\infty} \frac{h(2^{n-k}p + 2^{n-m})}{2^n} \\
&= \sum_{n=0}^{m} \frac{h(2^{n-k}p + 2^{n-m})}{2^n} && \text{Assume } m > k \\
&= \sum_{n=0}^{k} \frac{h(2^{n-k}p + 2^{n-m})}{2^n} + \sum_{n=k+1}^{m} \frac{h(2^{n-k}p + 2^{n-m})}{2^n} \\
&= \sum_{n=0}^{k} \frac{h(2^{n-k}p + 2^{n-m})}{2^n} + \sum_{n=k+1}^{m} \frac{h(2^{n-m})}{2^n} && \text{if } n > k \text{ then } 2 \mid 2^{n-k}p \\
&= \sum_{n=0}^{k} \frac{h(2^{n-k}p + 2^{n-m})}{2^n} + (m-k-1)2^{-m}.
\end{aligned}$$

Now we must estimate $\sum_{n=0}^{k} \frac{h(2^{n-k}p+2^{n-m})}{2^n}$. Since we have finite n then we can simply make m big enough such that for all $0 \leq n \leq k, h(2^{n-k}p) \notin \mathbb{Z}, |h(2^{n-k}p + 2^{n-m}) - h(2^{n-k}p)| < \min\{1 - h(2^{n-k}p), h(2^{n-k}p)\}$ (note that if every $h(2^{n-k}p) \in \mathbb{Z}$ then $g(x)$ is either $g(1)$ or $g(0)$, both of which have been shown to be nondifferentiable). Therefore $h(2^{n-k}p + 2^{n-m}) = h(2^{n-k}p) \pm 2^{n-m}$, as our specification on m put $h(2^{n-k}p + 2^{n-m})$ on the same line segment as $h(2^{n-k}p)$. Therefore,

$$\begin{aligned}
\sum_{n=0}^{k} \frac{h(2^{n-k}p + 2^{n-m})}{2^n} + (m-k-1)2^{-m} &= \sum_{n=0}^{k} \frac{h(2^{n-k}p) \pm 2^{n-m}}{2^n} + (m-k-1)2^{-m} \\
&= g(\frac{p}{2^k}) + (m-k-1)2^{-m} + \sum_{n=0}^{k} \pm 2^{m-n}.
\end{aligned}$$

Therefore we may compute

$$\begin{aligned}
\frac{g(x_m) - g(\frac{p}{2^k})}{x_m - \frac{p}{2^k}} &= \frac{g(\frac{p}{2^k}) + (m-k-1)2^{-m} + \sum_{n=0}^{k} \pm 2^{m-n} - g(\frac{p}{2^k})}{2^{-m}} \\
&= \frac{+(m-k-1)2^{-m} + \sum_{n=0}^{k} \pm 2^{m-n}}{2^{-m}} \\
&= (m-k-1) + \sum_{n=0}^{k} \pm 1 \\
&\geq m - k - 1 - k - 1 = m - 2k - 2.
\end{aligned}$$

Therefore $g'(p/2^k)$ diverges, and does not exists.