4.1 Let $A = \begin{bmatrix} 9 & 9 \\ -1 & 1 \end{bmatrix}$. We must find the solution to

$$\vec{x}' = A\vec{x} + e^{-4t}(1, t).$$

Note that since A is diagonalizeable, then $e^{tA} = Ve^{tD}V^{-1}$. Therefore

$$e^{tA} =$$

4.4 Let $y(t) = (y_1(t), y_2(t)) = (x(t), x'(t))$. Therefore our differential equation may be rewitten as

$$y'(t) = (y_1'(t), y_2'(t)) = (x'(t), x''(t)) = (y_2(t), -y_1(t) + f(t)) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} y(t) + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}.$$

Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. We know from previous sections in the textbook that $e^{tA} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$. Therefore we may use Duhamel's formula to solve for y:

$$y(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A} \begin{bmatrix} 0 \\ f(s) \end{bmatrix} ds.$$

Since $x(0) = x'(0) = 0$, then the first term on the right hand side goes to 0. Multiplying the vector through $e^{(t-s)A}$ yields $\begin{bmatrix} sin(t-s)f(s) \\ cos(t-s)f(s) \end{bmatrix}$. Therefore by the definition of y, $x = \int_0^t sin(t-s)f(s)ds$.