229352 Statistical Learning for Data Science 1 **Spring 2020**

Homework 1: due January 5

1. Find the unit vector in the same direction as $x = (1, 2, 3)$.
2. Find **a** two-dimensional unit vector that is orthogonal to $(1, 1)$.
3. For a certain pair of matrices A, B, the product AB has dimension 10×20. If A has 30 columns, what are the dimensions of A and B?
4. For $x = (1, 3, 5)$ compute $x^T x$ and xx^T.
5. Two d-dimensional vectors x, y both have length 2. If $x^T y = 2$, what is the angle between x and y?
6. A certain 3-dimensional random variable X has covariance as follows:
$$\mathrm{cov}(X) = \begin{pmatrix} 5 & -3 & 0 \\ -3 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$
   (a) Consider the direction $u = (1, 1, 1)/\sqrt{3}$. What is the variance of $X \cdot u$?
   (b) The eigenvectors of $\mathrm{cov}(X)$ can be found in the following list; which ones are they?
$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$
   (c) Find the eigenvalues corresponding to each of the eigenvectors in part (b). Make it clear which eigenvalue belongs to which eigenvector.
   (d) Suppose we used principal component analysis (PCA) to project points X into two dimensions. Which directions would it project onto?
   (e) Continuing from part (d), what would be the resulting two-dimensional projection of the point $x = (4, 0, 2)$?
   (f) Continuing from part (e), suppose that starting from the 2-dimensional projection, we tried to reconstruct the original x. What would the three-dimensional reconstruction be, exactly?
7. M is a 2×2 real-valued symmetric matrix with eigenvalues $\lambda_1 = 2$, $\lambda_2 = -1$ and corresponding eigenvectors
$$\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$
   (a) What is M?
   (b) What are the eigenvalues of $M^2 = MM$? Hint: we know that M can be written as $U\Lambda U^T$ where Λ has eigenvalues on the diagonal and $U^T = U^{-1}$.
8. We revisit the bias-variance decomposition.
   (a) Provide a sketch of typical (squared) bias, variance, training error and test error curves, on a single plot, as we go from less flexible statistical learning methods towards more flexible approaches. The x-axis should represent the amount of flexibility in the method, and they y-axis should represent the values for each curve. There should be four curves. Make sure to label each one.
   (b) Explain why each of the four curves has the shape displayed in part (a).