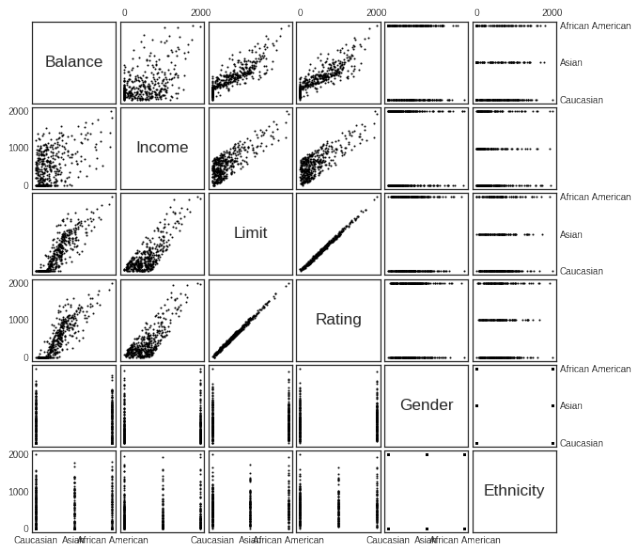


Linear Regression 3

Credit balance data



Predictor with two levels

Find the difference in credit card balance (y_i) between **male** and **female** (x_i).

$$x_i = \begin{cases} 0 & \text{if } i\text{th person is male.} \\ 1 & \text{if } i\text{th person is female.} \end{cases}$$

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

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Estimates of coefficients

	$\hat{\beta}_i$	$SE(\hat{\beta}_i)$	t -statistic	p -value
Intercept	509.80	33.13	15.389	<0.0001
gender(Female)	19.73	46.05	0.429	0.6690

$$\hat{y}_i = 509.80 + 19.73x_i.$$

Main takeaway:

1. Male has credit card debt of 509.80 **on average**.
2. Female has credit card debt of $509.80 + 19.73 = 529.53$ **on average**.
3. The difference in credit card debt is $\hat{\beta}_1 = 19.73$ **on average**.

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Main takeaway:

1. Male has credit card debt of 509.80 **on average**.
2. Female has credit card debt of $509.80 + 19.73 = 529.53$ **on average**.
3. The difference in credit card debt is $\hat{\beta}_1 = 19.73$ **on average**.

Question: Can we conclude that females have more credit debt on average than males?

Predictor with more than two levels

Find the difference in credit card balance (y_i) between **Asian**, **Caucasian** and **African American**.

$$y_i = \begin{cases} \beta_0 + \epsilon_i & \text{if } i\text{th person is African American.} \\ \beta_0 + \beta_1 + \epsilon_i & \text{if } i\text{th person is Asian.} \\ \beta_0 + \beta_2 + \epsilon_i & \text{if } i\text{th person is Caucasian.} \end{cases}$$

Predictor with more than two levels

Create two **dummy variables** x_{i1} and x_{i2} :

$$x_{i1} = \begin{cases} 1 & \text{if } i\text{th person is Asian.} \\ 0 & \text{if } i\text{th person is not Asian.} \end{cases}$$
$$x_{i2} = \begin{cases} 1 & \text{if } i\text{th person is Caucasian.} \\ 0 & \text{if } i\text{th person is not Caucasian.} \end{cases}$$

Using x_{i1} and x_{i2} , the regression can be written as

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i$$

Estimates of coefficients

	$\hat{\beta}_i$	$SE(\hat{\beta}_i)$	t -statistic	p -value
Intercept	531.00	46.32	11.464	<0.0001
ethnicity (Asian)	-18.69	65.02	-0.287	0.7740
ethnicity (Caucasian)	-12.50	56.68	-0.221	0.8260

Main takeaway: **On average,**

1. African American has credit debt of 531.00 .
2. Asian has 18.69 less debt than the African American.
3. Caucasian has 12.50 less debt than the African American.
4. Asian has _____ less debt than Caucasian.

Estimates of coefficients

	$\hat{\beta}_i$	$SE(\hat{\beta}_i)$	t -statistic	p -value
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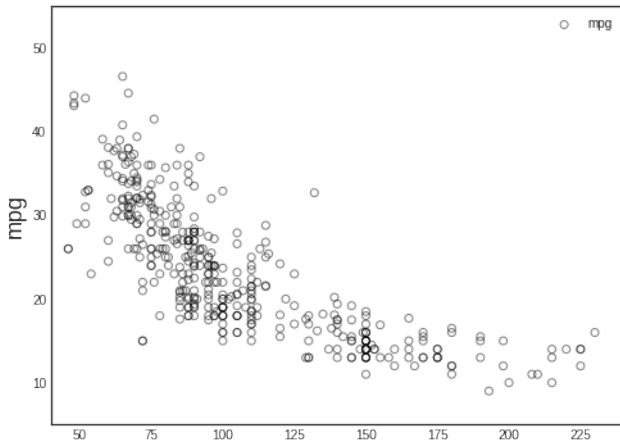
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Question: How can we decide if there is any difference in credit card balance between the ethnicities?

Linear model diagnosis

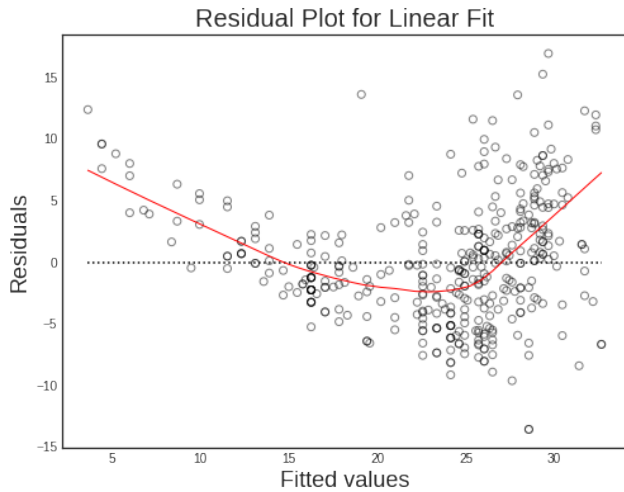
1. Non-linearity of the data

- Maybe the relationship between the predictors and the response is non-linear.



Residual plot

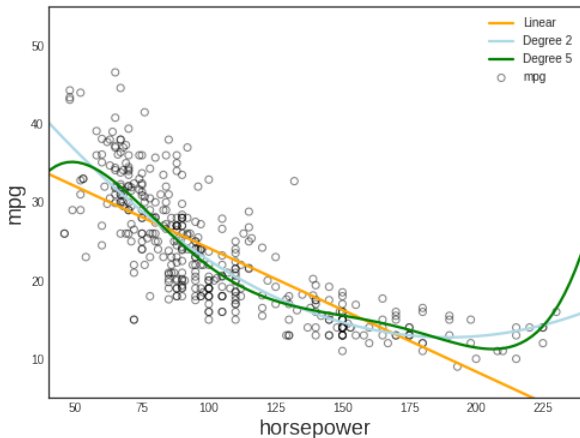
- Plot between the **fitted values** \hat{y}_i and the **residuals** $y_i - \hat{y}_i$.



Non-linear regression

Try a polynomial function of the horsepower:

$$\text{mpg} = \beta_0 + \beta_1 \times \text{horsepower} + \beta_2 \times \text{horsepower}^2 + \epsilon.$$



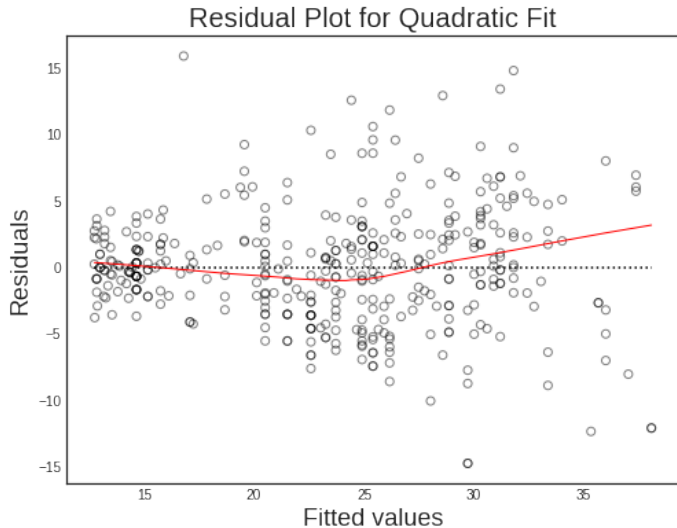
Estimates of coefficients

	$\hat{\beta}_i$	$SE(\hat{\beta}_i)$	t -statistic	p -value
Intercept	56.9001	1.8004	31.6	<0.0001
horsepower	-0.4662	0.0311	-15.0	<0.0001
horsepower ²	-0.0012	0.0001	10.1	<0.0001

Two things indicate that the quadratic fit is better:

- The p -value of **horsepower²** is significant.
- The R^2 of this model is 0.688 compared to 0.606 of the linear model.

Residual plot of non-linear regression



2. Correlation of error terms

- We assumed that the error terms

$$\epsilon_1, \epsilon_2, \dots, \epsilon_n$$

are independent to each other. This is an important assumption!

- What happens if this is not the case?

2. Correlation of error terms

Example: Suppose we accidentally doubled the data

$$(x_1, y_1), (x_1, y_1), (x_2, y_2), (x_2, y_2), \dots$$

and train the simple linear model

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i + \epsilon_i.$$

2. Correlation of error terms

Recall that the standard error of a coefficient is

$$\text{Model 1: } \text{SE}(\hat{\beta}_1)^2 = \frac{\sigma^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \quad (n \text{ points})$$

compared to

$$\text{Model 2: } \text{SE}(\hat{\beta}_1)^2 = \frac{\sigma^2}{\sum_{i=1}^{2n} (x_i - \bar{x})^2} \quad (2n \text{ points})$$

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- The standard error of Model 2 is smaller than that of Model 1.
- The confidence interval

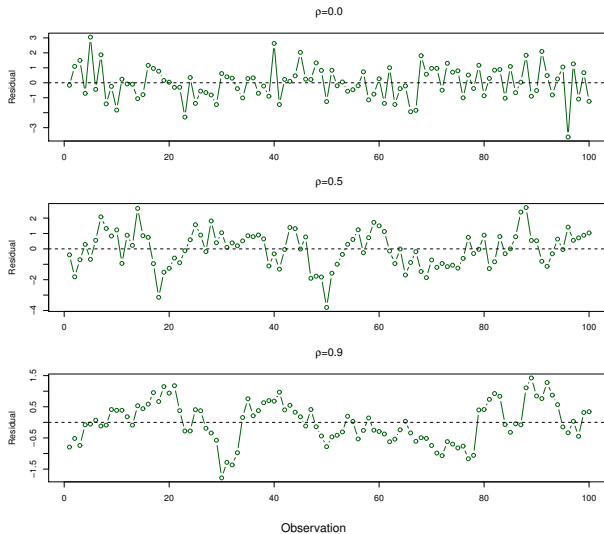
$$[\hat{\beta}_1 - 2 \cdot \text{SE}(\hat{\beta}_1), \hat{\beta}_1 + 2 \cdot \text{SE}(\hat{\beta}_1)]$$

is narrower.

2. Correlation of error terms

- From previous example, we learn that **correlated errors cause the confidence interval to be narrower.**
- As a result, we could mistakenly conclude that the coefficients are significant.
- **time series** is an example of data with correlated errors.

Time vs residual plot



3. Non-constant variance of error terms

- We also assumed that the variance of $\text{Var}(\epsilon_i) = \sigma^2$ for all i .
- The formula for standard error, hypothesis test and confidence interval are all derived **under this assumption**.

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- For example, the formula

$$\text{Cov}\hat{\beta} = \sigma^2(\mathbf{X}^T \mathbf{X})^{-1}$$

holds because we assumed that ϵ_i 's share the same variance σ^2 .

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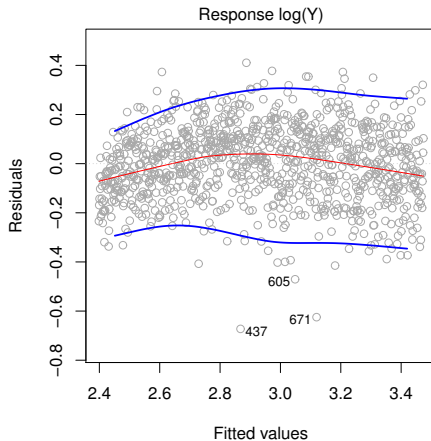
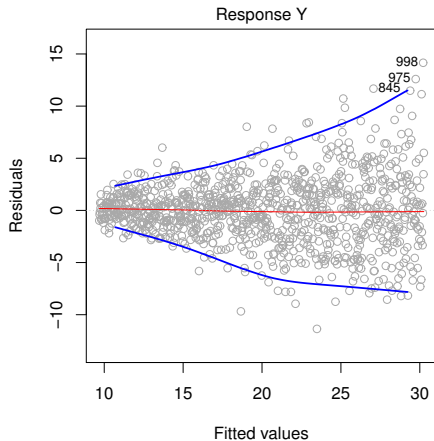
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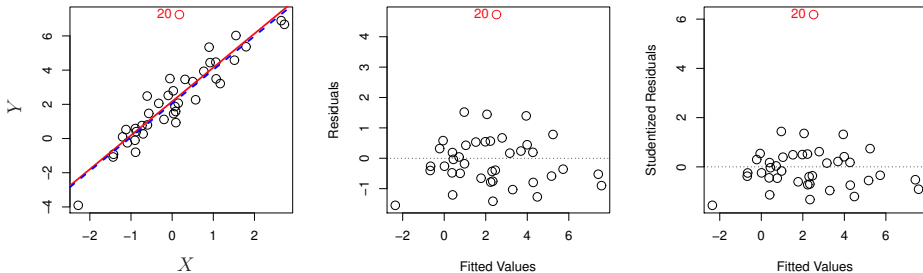
- Detect non-constant variance using **fitted value vs residual plot**.

Fitted value vs residual plot



4. Outliers

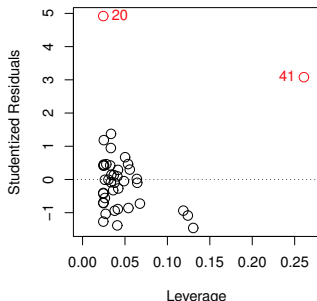
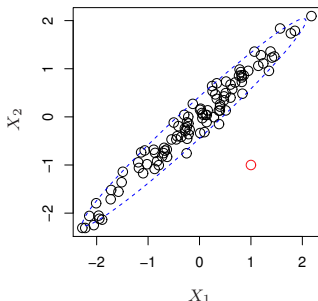
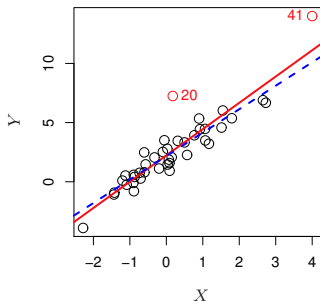
A single point can heavily influence the RSE and R^2 of the model.



	RSE	R^2
Model with outlier	1.09	0.805
Model without outlier	0.77	0.892
Improvement	29%	11%

5. High leverage points

- **High leverage point** is a point with an unusual value of x_i .
- Detect high leverage points using the **leverage statistic**.



6. Collinearity

- **collinearity problem** happens when two predictors are highly correlated to each other.
- Highly correlated variables cause problems when training the model.

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- Highly correlated variables cause problems when training the model.

Example: Suppose we have data with two predictors x and z .

$$(y_1, x_1, z_1), (y_2, x_2, z_2), \dots$$

where $z_i = 2x_i$.

6. Collinearity

Suppose that we have a solution $(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2) = (0, 1, 1)$

$$\hat{y}_i = x_i + z_i$$

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$$\begin{aligned}\hat{y}_i &= x_i + 2x_i \\ &= 3x_i\end{aligned}$$

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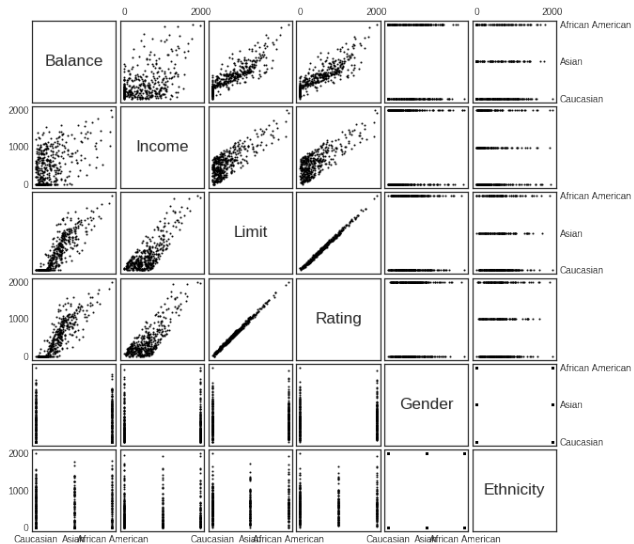
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Detect collinearity using **correlation matrix**. Remove a variable if the correlation is close to -1 or 1 .

Credit balance data



Multicollinearity

Multicollinearity happens when a predictor is a linear combination of other predictors.

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Example: Predictors x_i , z_i and w_i where $x_i = z_i + 2w_i$.

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Cannot be detected with correlation matrix. Instead, we use **variance inflation factor**

$$VIF(\hat{\beta}_i) = \frac{1}{1 - R_{X_i|X_{-i}}^2},$$

where $R_{X_i|X_{-i}}^2$ is the R^2 from a regression of X_i onto all other predictors.

Variance inflation factor

$$VIF(\hat{\beta}_i) = \frac{1}{1 - R_{X_i|X_{-i}}^2}.$$

[High multicol. in X_i] \rightarrow [$R_{X_i|X_{-i}}^2$ is close to 1] \rightarrow [high $VIF(\hat{\beta}_i)$]

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General rule: There is multicollinearity if VIF is higher than 5 or 10

Solution: Drop the variable (in this case, X_i).

Acknowledgement

Some of the figures in this presentation are taken from "An Introduction to Statistical Learning, with applications in R" (Springer, 2013) with permission from the authors: G. James, D. Witten, T. Hastie and R. Tibshirani