

Homework 1: December 25

- Find the unit vector in the same direction as $x = (1, 2, 3)$.
- Find a two-dimensional unit vector that is orthogonal to $(1, 1)$.
- For a certain pair of matrices A, B , the product AB has dimension 10×20 . If A has 30 columns, what are the dimensions of A and B ?
- For $x = (1, 3, 5)$ compute $x^T x$ and xx^T .
- Two d -dimensional vectors x, y both have length 2. If $x^T y = 2$, what is the angle between x and y ?
- A certain 3-dimensional random variable X has covariance as follows:

$$\text{cov}(X) = \begin{pmatrix} 5 & -3 & 0 \\ -3 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

- Consider the direction $u = (1, 1, 1)/\sqrt{3}$. What is the variance of $X \cdot u$?
 - The eigenvectors of $\text{cov}(X)$ can be found in the following list; which ones are they?

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$
 - Find the eigenvalues corresponding to each of the eigenvectors in part (b). Make it clear which eigenvalue belongs to which eigenvector.
 - Suppose we used principal component analysis (PCA) to project points X into two dimensions. Which directions would it project onto?
 - Continuing from part (d), what would be the resulting two-dimensional projection of the point $x = (4, 0, 2)$?
 - Continuing from part (e), suppose that starting from the 2-dimensional projection, we tried to reconstruct the original x . What would the three-dimensional reconstruction be, exactly?
7. M is a 2×2 real-valued symmetric matrix with eigenvalues $\lambda_1 = 2$, $\lambda_2 = -1$ and corresponding eigenvectors

$$\frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

- What is M ?
 - What are the eigenvalues of $M^2 = MM$? Hint: we know that M can be written as $U\Lambda U^T$ where Λ has eigenvalues on the diagonal and $U^T = U^{-1}$.
8. We revisit the bias-variance decomposition.
- Provide a sketch of typical (squared) bias, variance, training error and test error curves, on a single plot, as we go from less flexible statistical learning methods towards more flexible approaches. The x -axis should represent the amount of flexibility in the method, and the y -axis should represent the values for each curve. There should be four curves. Make sure to label each one.
 - Explain why each of the four curves has the shape displayed in part (a).