Linear algebra

229351

Points in $\mathbb{R}^n$

$\mathbb{R}^n$ is the set of n-dimensional **points**.

- $\mathbb{R}^1$ is a real line.

Points in $\mathbb{R}^n$

$\mathbb{R}^n$ is the set of n-dimensional **points**.

- $\mathbb{R}^2$: height & width

Points in $\mathbb{R}^n$

$\mathbb{R}^n$ is the set of n-dimensional **points**.

- $\mathbb{R}^3$: height, width & depth

Points in $\mathbb{R}^n$

$\mathbb{R}^n$ is the set of n-dimensional **points**.

- $\mathbb{R}^{3781}$

  Example: ratings of all movies I've watched

$$R = [\underbrace{4.0}_{\text{A-team}}, \underbrace{?}_{\text{ABBA}}, ?, 3.5, \ldots, \underbrace{5.0}_{\text{Zoolander}}]$$

Points in $\mathbb{R}^n$

$\mathbb{R}^n$ is the set of n-dimensional **points**.

- $\mathbb{R}^{171146}$

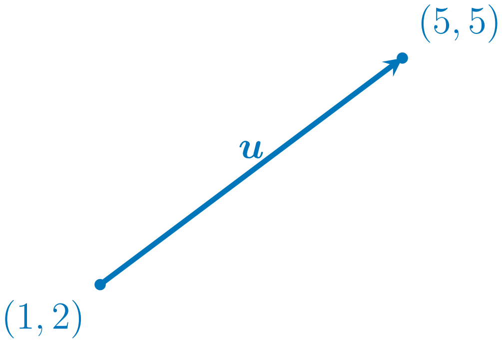
Example: ratings of all movies I've watched

$$W = \underset{\text{a}}{120}, \underset{\text{aardvard}}{0}, 0, 0, \ldots, \underset{\text{zyzzyva}}{0}]$$

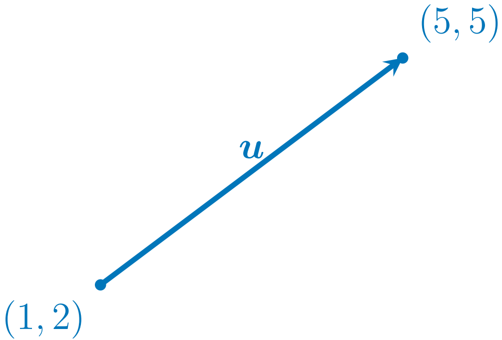
Is there a way to compare these high-dimensional vectors?

Vectors

- connects between two points.
- has **size** and **direction**.

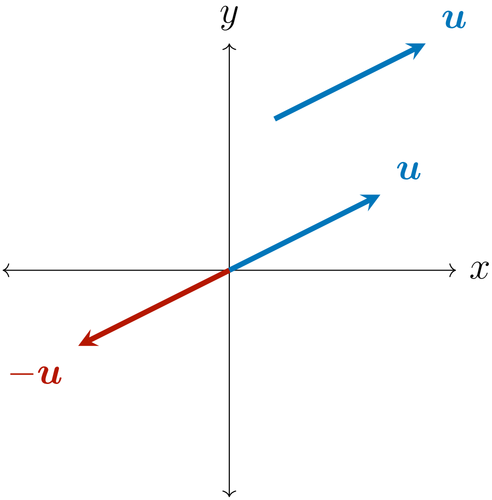


Two coordinate systems



- Rectangular coordinate
- Polar coordinate

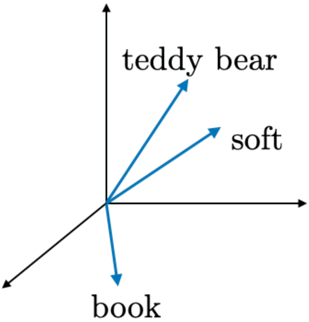
Vectors



From this point on, any element in $\mathbb{R}^n$ will be considered as a vector.

Word vectors

- Similar words
  $\rightarrow$ small angle
- Irrelevant words
  $\rightarrow$ right angle
- Opposite words
  $\rightarrow$ opposite directions



Dot product

Let $w_1, w_2 \in \mathbb{R}^n$ where

$$\begin{aligned} w_1 &= (a_1, a_2, \ldots, a_n) \\ w_2 &= (b_1, b_2, \ldots, b_n). \end{aligned}$$

Then the **dot product** between w_1 and w_2 is

$$w_1 \cdot w_2 = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$$

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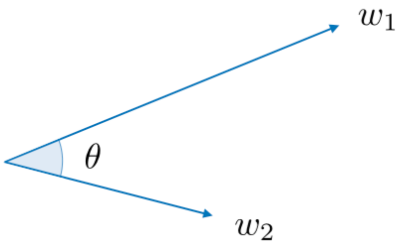
Then the **dot product** between w_1 and w_2 is

$$w_1 \cdot w_2 = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$$

and the size of w_1 is

$$\|w_1\| = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}.$$

Angle between vectors



Fact:

$$w_1 \cdot w_2 = \|w_1\| \|w_2\| \cos \theta.$$

Example

Find the angle between $w_1 = (1, 2)$ and $w_2 = (-2, 1)$

Another angle

$$w_1 \cdot w_2 = \|w_1\| \|w_2\| \cos\theta.$$

If u_1 and u_2 are **unit vector**, that is $\|u_1\| = \|u_2\| = 1$, then

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Thus, another way to use the formula is to <u>normalize</u> the vectors first

$$u_1 = \frac{w_1}{\|w_1\|}, \quad u_2 = \frac{w_2}{\|w_2\|}.$$

Matrix

Matrix is a collection of vectors

$$
n \text{ rows} \left\{ \begin{pmatrix} \underbrace{\begin{matrix} 2 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\ 0 & -2 & 1 & \cdot & \cdot & \cdot & 1 & 0 \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ 1 & 0 & 3 & \cdot & \cdot & \cdot & 1 & 1 \end{matrix}}_{m \text{ columns}} \end{pmatrix} \right.
$$

- Data is stored in a matrix.
- Cov/Corr of ≥ 2 variables is a matrix.

Matrix

Matrix is a collection of vectors

$$A = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ v_1 & v_2 & \dots & v_m \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$$

This is an $n \times m$ matrix (or $A \in \mathbb{R}^{n \times m}$).

Matrix & vectors

Matrix **transforms** vectors

$$\begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ v_1 & v_2 & \dots & v_m \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$$

$$= a_1 \begin{pmatrix} \uparrow \\ v_1 \\ \downarrow \end{pmatrix} + a_2 \begin{pmatrix} \uparrow \\ v_2 \\ \downarrow \end{pmatrix} + \dots + a_m \begin{pmatrix} \uparrow \\ v_m \\ \downarrow \end{pmatrix}.$$

In other words, a **span** of $v_1, v_2, \dots, v_m$.

Example

$$A = \begin{pmatrix} 2 & 0 & 3 \\ 1 & -1 & 0 \\ 0 & 3 & 10 \end{pmatrix}, u = (1, 2, -1).$$

Matrix & Matrix

Suppose that A and B are matrices.

$$
\begin{aligned}
AB &= A \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ u_1 & u_2 & \dots & u_p \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} \\
&= \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ Au_1 & Au_2 & \dots & Au_p \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}
\end{aligned}
$$

Example

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Matrix arithmetic

Sometimes we want to solve matrix equations. For example,

$$Y = X\boldsymbol{\beta}$$
$$\boldsymbol{\beta} = X^{-1}Y$$

but what is X^{-1} in the world of matrices?

Identity

In the world of numbers, the inverse x^{-1} of x is defined such that

$$x \cdot x^{-1} = 1.$$

What is 1 in the world of matrices?.

Identity matrix

Define I_n as

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Then for any $n \times m$ matrix A,

$$AI_m = I_n A = A.$$

Inverse matrices

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Inverse matrix

Let A be a square matrix (an $n \times n$ matrix) then the **inverse** of A is A^{-1} such that

$$AA^{-1} = A^{-1}A = I_n$$

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Why A has to be square? Try the following example:

$A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Transpose

The **tranpose** of a matrix flips its elements over the diagonal.
For example:

$$\text{If} \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$\text{then} \quad A^T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

Properties

Any matrices A, B and C satisfy:

$$A(B+C) = AB + AC \tag{1}$$

$$A(BC) = (AB)C \tag{2}$$

$$(A+B)^T = A^T + B^T \tag{3}$$

$$(AB)^T = B^T A^T \tag{4}$$

$$(AB)^{-1} = B^{-1} A^{-1} \tag{5}$$

However, $AB \neq BA$.

Symmetric matrices

A matrix A is **symmetric** if $A^T = A$.
For example:

$$\begin{pmatrix} 1 & 2 & 30 \\ 2 & 3 & 16 \\ 30 & 16 & 5 \end{pmatrix}$$

is symmetric.

Covariance and correlation matrices are symmetric.

Rotation matrices

In $\mathbb{R}^2$ the following matrix rotates any vector by an angle θ counterclockwise.

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

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$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =$$

Example

Pick $\theta = \frac{\pi}{2} = 90^\circ$