Time Series Analysis 2
DS351

Time series decomposition

Time series decomposition

Goal:

- Extract trend seasonality
- Visualize and improve understanding of time series

Moving averages

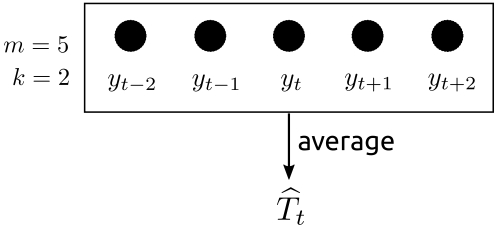
Moving average is a method to estimate the trend.

Time series: y_t

Moving average of order *m* of y_t is

$$\widehat{T}_t = \frac{1}{m} \sum_{i=-k}^{k} y_{t+i},$$

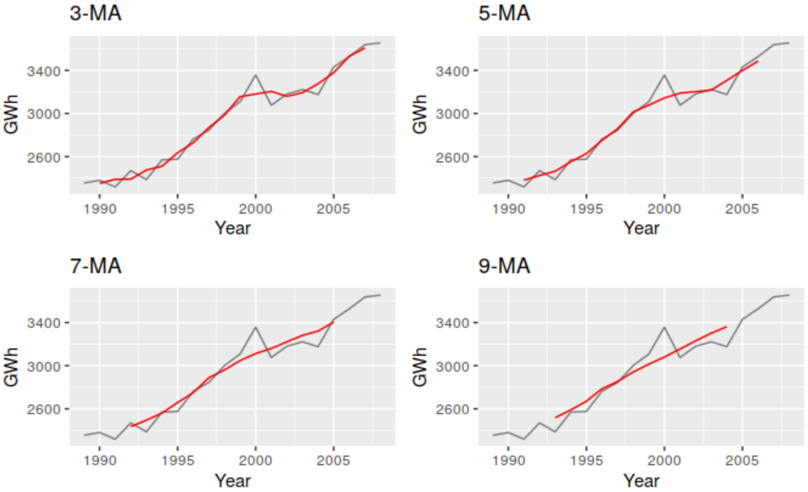
where $m = 2k + 1$.



Example: electricity sold to customers in South Australia

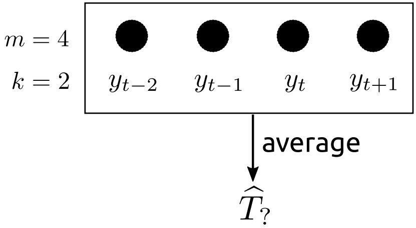
Year	Sales (GWh)	5-MA
1989	2354.34	
1990	2379.71	
1991	2318.52	2381.53
1992	2468.99	2424.56
1993	2386.09	2463.76
1994	2569.47	2552.60
1995	2575.72	2627.70
1996	2762.72	2750.62
1997	2844.50	2858.35
⋮	⋮	⋮
1997	2844.50	2858.35
2006	3527.48	3485.43
2007	3637.89	
2008	3655.00	

Example: moving average of different orders



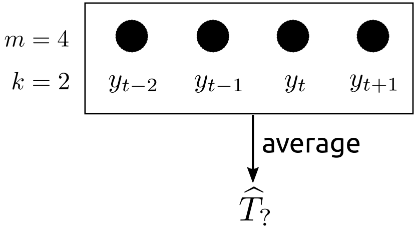
Moving average of even orders

For example, $m = 4$

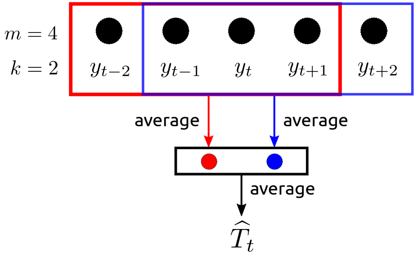


Moving average of even orders

For example, $m = 4$



Idea: use 2-MA **after** 4-MA



Australian quarterly beer production

Year	Quarter	Observation	4-MA	2x4-MA
1992	Q1	443		
1992	Q2	410	451.25	
1992	Q3	420	448.75	450
1992	Q4	532	451.5	450.12
1993	Q1	433	449	450.25
1993	Q2	421	444	446.5
1993	Q3	410	448	446
1993	Q4	512	438	443
1994	Q1	449	441.25	439.62
⋮	⋮	⋮	⋮	⋮
1996	Q3	398	433.75	430.88
1996	Q4	507	433.75	433.75

2×m-MA

The 2×4-MA of y_t is

$$\begin{aligned}\widehat{T}_t &= \frac{1}{2}\left[\frac{1}{4}(y_{t-2} + y_{t-1} + y_t + y_{t+1}) + \frac{1}{4}(y_{t-1} + y_t + y_{t+1} + y_{t+2})\right] \\ &= \frac{1}{8}y_{t-2} + \frac{1}{4}y_{t-1} + \frac{1}{4}y_t + \frac{1}{4}y_{t+1} + \frac{1}{8}y_{t+2}.\end{aligned}$$

2×m-MA

The 2×4-MA of y_t is

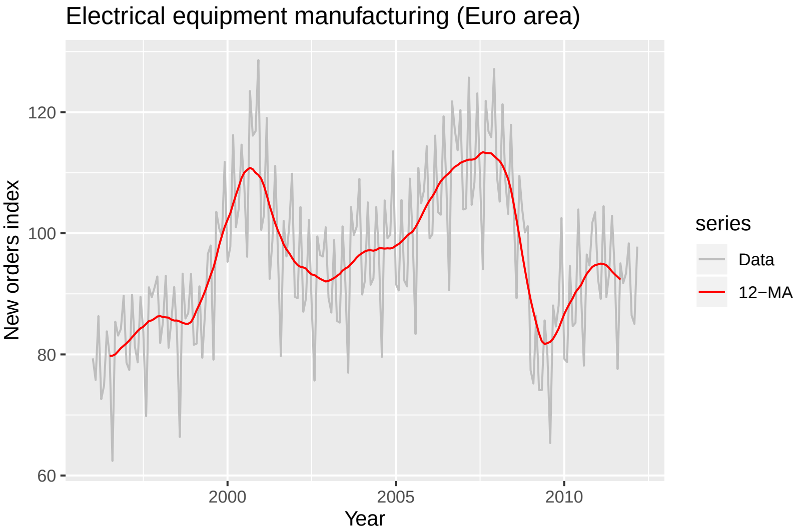
$$\begin{aligned}\widehat{T}_t &= \frac{1}{2}\left[\frac{1}{4}(y_{t-2}+y_{t-1}+y_t+y_{t+1})+\frac{1}{4}(y_{t-1}+y_t+y_{t+1}+y_{t+2})\right]\\ &= \frac{1}{8}y_{t-2}+\frac{1}{4}y_{t-1}+\frac{1}{4}y_t+\frac{1}{4}y_{t+1}+\frac{1}{8}y_{t+2}.\end{aligned}$$

In general, The 2×m-MA of y_t is

$$\widehat{T}_t = \frac{1}{2m}y_{t-k}+\ldots+\frac{1}{m}y_{t-1}+\frac{1}{m}y_t+\frac{1}{m}y_{t+1}+\ldots+\frac{1}{2m}y_{t+k},$$

where $m = 2k$.

Example: monthly data



2×4-MA for quarterly beer production, 7-MA for daily traffic data etc

Classical decomposition

Two types of decomposition:

1. Additive decomposition

$$y_t = S_t + T_t + R_t,$$

where

- S_t is the seasonal component.
- T_t is the trend component.
- R_t is the remainder component.

Classical decomposition

Two types of decomposition:

2. Multiplicative decomposition

$$y_t = S_t \times T_t \times R_t,$$

where

- S_t is the seasonal component.
- T_t is the trend component.
- R_t is the remainder component.

Additive decomposition

- *Step 1*: Pick m, usually the seasonal period.

$$\widehat{T}_t = \begin{cases} m\text{-MA} & \text{if } m \text{ is an odd number.} \\ 2 \times m\text{-MA} & \text{if } m \text{ is an even number.} \end{cases}$$

Additive decomposition

- *Step 1*: Pick m, usually the seasonal period.

$$\widehat{T}_t = \begin{cases} m\text{-MA} & \text{if } m \text{ is an odd number.} \\ 2 \times m\text{-MA} & \text{if } m \text{ is an even number.} \end{cases}$$

- *Step 2*: Calculate the detrended series

$$y_t - \widehat{T}_t$$

Additive decomposition

- *Step 3*: Compute the mean of $y_t - \widehat{T}_t$ for each seasonal unit. For example, for monthly data, we compute

$$\begin{aligned} S_1 &= \text{the mean of all values in January} \\ S_2 &= \text{the mean of all values in February} \\ &\text{and so on...} \end{aligned}$$

Additive decomposition

- *Step 3*: Compute the mean of $y_t - \widehat{T}_t$ for each seasonal unit. For example, for monthly data, we compute

$$
\begin{aligned}
S_1 &= \text{the mean of all values in January} \\
S_2 &= \text{the mean of all values in February} \\
&\text{and so on...}
\end{aligned}
$$

  Then, these seasonal values are adjusted to have zero mean.

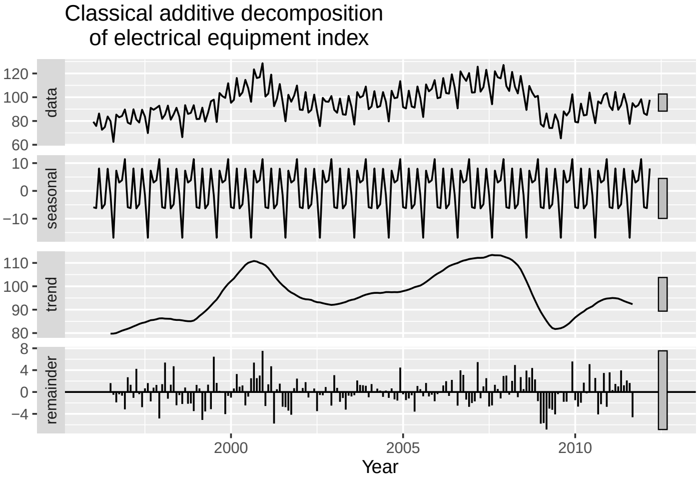
$$
\begin{aligned}
\widehat{S}_1 &= S_1 - \overline{S} \\
\widehat{S}_2 &= S_2 - \overline{S} \\
&\text{and so on...}
\end{aligned}
$$

  where $\overline{S} = \frac{1}{12}\sum_{i=1}^{12} S_i$

Additive decomposition

- *Step 4*: The remainder component is

$$\widehat{R}_t = y_t - \widehat{T}_t - \widehat{S}_t.$$



Multiplicative decomposition

- *Step 1*: Pick m, usually the seasonal period.

$$\widehat{T}_t = \begin{cases} m\text{-MA} & \text{if } m \text{ is an odd number.} \\ 2 \times m\text{-MA} & \text{if } m \text{ is an even number.} \end{cases}$$

Multiplicative decomposition

- *Step 1*: Pick m, usually the seasonal period.

$$\widehat{T}_t = \begin{cases} m\text{-MA} & \text{if } m \text{ is an odd number.} \\ 2 \times m\text{-MA} & \text{if } m \text{ is an even number.} \end{cases}$$

- *Step 2*: Calculate the detrended series

$$\frac{y_t}{\widehat{T}_t}.$$

Multiplicative decomposition

- *Step 3*: Compute the mean of $y_t / \widehat{T}_t$ for each seasonal unit. For example, for monthly data, we compute

$$
\begin{aligned}
S_1 &= \text{the mean of all values in January} \\
S_2 &= \text{the mean of all values in February} \\
&\quad \text{and so on...}
\end{aligned}
$$

Multiplicative decomposition

- *Step 3*: Compute the mean of $y_t / \widehat{T}_t$ for each seasonal unit. For example, for monthly data, we compute

$$S_1 = \text{the mean of all values in January}$$
$$S_2 = \text{the mean of all values in February}$$
$$\text{and so on...}$$

Then, these seasonal values are adjusted to have zero mean.

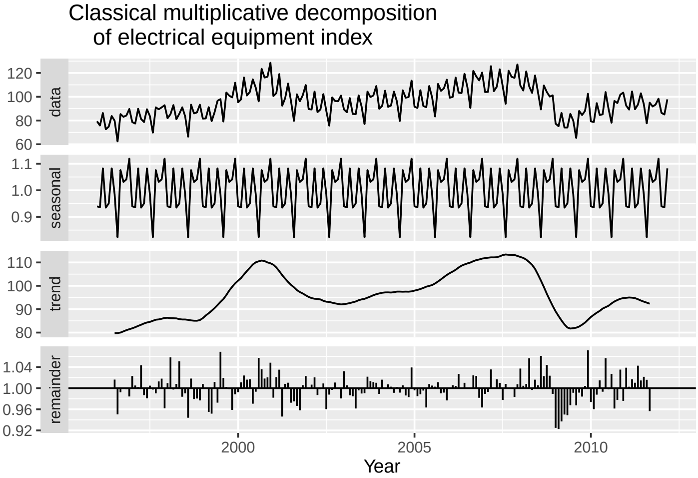
$$\widehat{S}_1 = S_1 / \sum_i S_i$$
$$\widehat{S}_2 = S_2 / \sum_i S_i$$
$$\text{and so on...}$$

where $\sum_i S_i = S_1 + S_2 + \ldots + S_{12}$.

Multiplicative decomposition

- *Step 4*: The remainder component is

$$\widehat{R}_t = \frac{y_t}{\widehat{T}_t \widehat{S}_t}.$$



Strength of trend (Wang, Smith & Hyndman, 2006)

Back to additive decomposition:

$$y_t = T_t + S_t + R_t.$$

Observation: for a time series with strong trend,

$$\frac{\mathrm{Var}(R_t)}{\mathrm{Var}(T_t + R_t)} \quad \text{should be small.}$$

Strength of trend (Wang, Smith & Hyndman, 2006)

Back to additive decomposition:

$$y_t = T_t + S_t + R_t.$$

Observation: for a time series with strong trend,

$$\frac{\mathrm{Var}(R_t)}{\mathrm{Var}(T_t + R_t)} \text{ should be small.}$$

for a time series with strong seasonality,

$$\frac{\mathrm{Var}(R_t)}{\mathrm{Var}(S_t + R_t)} \text{ should be small.}$$

Strength of trend (Wang, Smith & Hyndman, 2006)

Back to additive decomposition:

$$y_t = T_t + S_t + R_t.$$

Observation: for a time series with strong trend,

$$\frac{\text{Var}(R_t)}{\text{Var}(T_t + R_t)} \text{ should be small.}$$

for a time series with strong seasonality,

$$\frac{\text{Var}(R_t)}{\text{Var}(S_t + R_t)} \text{ should be small.}$$

Strength of trend (Wang, Smith & Hyndman, 2006)

So we define the strength of trend as

$$F_T = \max\left(0, 1 - \frac{\mathrm{Var}(R_t)}{\mathrm{Var}(T_t + R_t)}\right)$$

and the strength of seasonality as

$$F_S = \max\left(0, 1 - \frac{\mathrm{Var}(R_t)}{\mathrm{Var}(S_t + R_t)}\right).$$

Higher value = Stronger effect

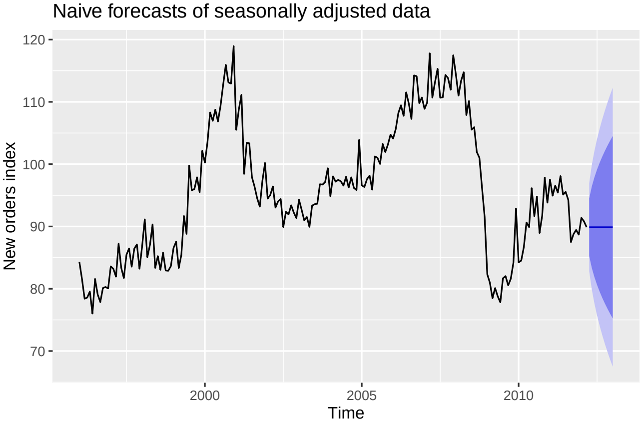
This is useful when we have a collection of time series and we want to find the one with the most trend or seasonality.

Forecasting with decomposition

We can make forecast from the decomposition

$$y_t = \widehat{S}_t + (\widehat{T}_t + \widehat{R}_t),$$

where we can use time series model to forecast the seasonally adjusted component $\widehat{A}_t = \widehat{T}_t + \widehat{R}_t$ and then add back the seasonal component S_t.

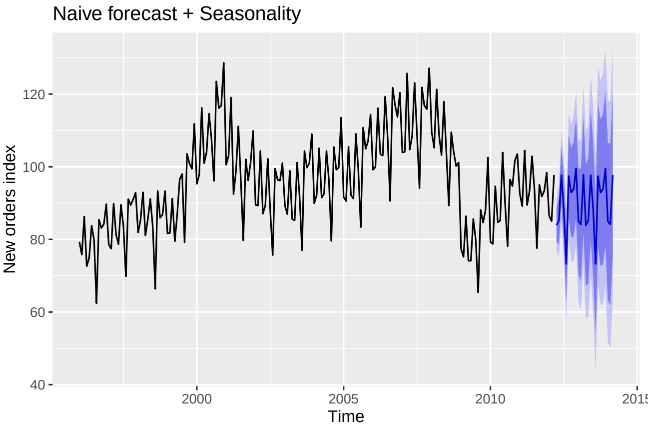


Forecasting with decomposition

We can make forecast from the decomposition

$$y_t = \widehat{S}_t + (\widehat{T}_t + \widehat{R}_t),$$

where we can use time series model to forecast the seasonally adjusted component $\widehat{A}_t = \widehat{T}_t + \widehat{R}_t$ and then add back the seasonal component S_t.



Exponential smoothing

Motivation

We can forecast with the **naïve method**:

$$\hat{y}_{T+h} = y_T \quad \text{for } h = 1, 2, \ldots$$

or a simple average

$$\hat{y}_{T+h} = \frac{1}{T} \sum_{i=1}^{T} y_t \quad \text{for } h = 1, 2, \ldots$$

- Notice that both forecasts use **weighted average** of previous observations.
- We want to make a forecasting model that lie between these two extremes.

Exponential smoothing

Idea: give the smallest weights to the oldest observations:

$$\hat{y}_{T+1} = \alpha y_T + \alpha(a - \alpha) y_{t_1} + \alpha(1 - \alpha)^2 y_{T_2} + \dots,$$

where $0 \leq \alpha \leq 1$ is the **smoothing parameter**.

Exponential smoothing

Idea: give the smallest weights to the oldest observations:

$$\hat{y}_{T+1} = \alpha y_T + \alpha(a-\alpha)y_{t_1} + \alpha(1-\alpha)^2 y_{T_2} + \dots,$$

where $0 \leq \alpha \leq 1$ is the **smoothing parameter**. This can be written as

$$\hat{y}_{T+1} = \alpha y_T + (1-\alpha)\hat{y}_T$$

where

$$\hat{y}_T = \alpha y_{T-1} + (1-\alpha)\hat{y}_{T-1}$$

where

$$\hat{y}_{T-1} = \alpha y_{T-2} + (1-\alpha)\hat{y}_{T-2} \qquad \text{and so on...}$$

Exponential smoothing

Two forms of ES:

$$\begin{aligned}\hat{y}_{T+1} &= \alpha y_T + (1-\alpha)\hat{y}_T \\ &= \sum_{j=0}^{T-1} \alpha(1-\alpha)^j y_{T-j} + (1-\alpha)^T l_0\end{aligned}$$

where l_0 is an **initial value**, a parameter to be learned from the data.

Exponential smoothing

Two forms of ES:

$$\hat{y}_{T+1} = \alpha y_T + (1-\alpha)\hat{y}_T$$
$$= \sum_{j=0}^{T-1} \alpha(1-\alpha)^j y_{T-j} + (1-\alpha)^T l_0$$

where l_0 is an **initial value**, a parameter to be learned from the data.

From input data $y_1, y_2, \ldots, y_T$, the model need to learn two parameters: the smoothing parameter α and initial value l_0.

Learning parameters from the data

ES model:

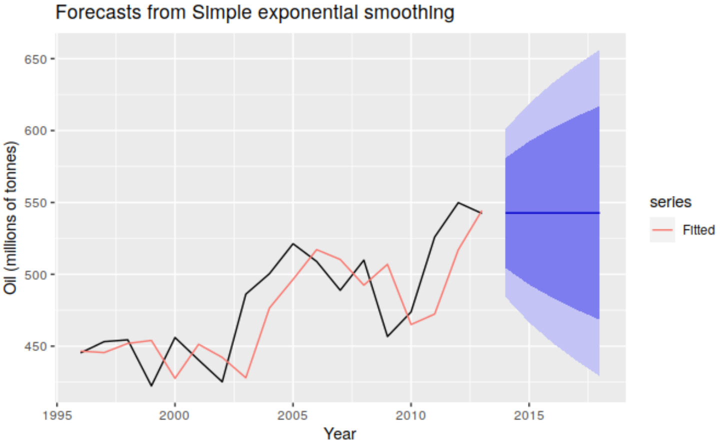
$$\hat{y}_{T+1} = \sum_{j=0}^{T-1} \alpha(1-\alpha)^j y_{T-j} + (1-\alpha)^T l_0$$

From input data $y_1, y_2, \ldots, y_T$, we need to find α and l_0 that minimize the SSE.

$$\text{SSE} = \sum_{t=1}^{T} (y_t - \hat{y}_t)^2$$

Note that this is not as easy as linear regression since there are α^k for high values of k in SSE.

Example: oil production in Saudi Arabia



Learned ES parameters: $\hat{\alpha} = 0.83$ and $\hat{l}_0 = 446.6$.