Advent of Code 2023 - Day 24 Solution
Linear Algebra Approach

Definitions

$i, j \in \mathbb{N}$	indices of specific hailstones
$\vec{r}_s = [r_{sx}, r_{sy}, r_{sz}]^T$	stone initial position (unknown)
$\vec{v}_s = [v_{sx}, v_{sy}, v_{sz}]^T$	stone velocity (unknown)
$\vec{r}_i, \vec{v}_i$	position and velocity of hailstone i
t_i	collision time with hailstone i

The Collinearity Constraint

For the stone to collide with hailstone i, they must occupy the same point in space at time t_i:

$$\vec{r}_s + t_i\vec{v}_s = \vec{r}_i + t_i\vec{v}_i \tag{1}$$

Rearranging to group terms involving time:

$$\vec{r}_s - \vec{r}_i = t_i(\vec{v}_i - \vec{v}_s) \tag{2}$$

This equation implies that the vector difference in positions $(\vec{r}_s - \vec{r}_i)$ is parallel to the vector difference in velocities $(\vec{v}_i - \vec{v}_s)$, as one is a scalar multiple (t_i) of the other.

Linearization via Cross Product

Since the two vectors are parallel, their cross product must be zero:

$$(\vec{r}_s - \vec{r}_i) \times (\vec{v}_i - \vec{v}_s) = \vec{0} \tag{3}$$

Expanding this cross product:

$$(\vec{r}_s \times \vec{v}_i) - (\vec{r}_s \times \vec{v}_s) - (\vec{r}_i \times \vec{v}_i) + (\vec{r}_i \times \vec{v}_s) = \vec{0} \tag{4}$$

Rearranging to isolate the non-linear term $(\vec{r}_s \times \vec{v}_s)$:

$$\vec{r}_s \times \vec{v}_s = (\vec{r}_s \times \vec{v}_i) + (\vec{r}_i \times \vec{v}_s) - (\vec{r}_i \times \vec{v}_i) \tag{5}$$

The term $\vec{r}_s \times \vec{v}_s$ contains the product of two unknowns. However, this term is identical for **every** hailstone. We can eliminate it by considering a second hailstone j and subtracting the two equations.

Using equation (5) for hailstone j:

$$\vec{r}_s \times \vec{v}_s = (\vec{r}_s \times \vec{v}_j) + (\vec{r}_j \times \vec{v}_s) - (\vec{r}_j \times \vec{v}_j) \tag{6}$$

Equating the RHS for i and j:

$$(\vec{r}_s \times \vec{v}_i) + (\vec{r}_i \times \vec{v}_s) - (\vec{r}_i \times \vec{v}_i) = (\vec{r}_s \times \vec{v}_j) + (\vec{r}_j \times \vec{v}_s) - (\vec{r}_j \times \vec{v}_j)$$

Grouping unknowns $(\vec{r}_s, \vec{v}_s)$ on the left and knowns on the right:

$$\vec{r}_s \times (\vec{v}_i - \vec{v}_j) + \vec{v}_s \times (\vec{r}_j - \vec{r}_i) = (\vec{r}_i \times \vec{v}_i) - (\vec{r}_j \times \vec{v}_j) \tag{7}$$

This is now a **linear vector equation**. Since it represents a cross product in 3D, it provides 3 scalar linear equations for every pair of hailstones.

System Construction

Let the unknowns be the state vector $\mathbf{x} = [r_{sx}, r_{sy}, r_{sz}, v_{sx}, v_{sy}, v_{sz}]^T$. We use two pairs of hailstones (e.g., $1 \to 2$ and $1 \to 3$) to generate 6 equations for our 6 unknowns.

Let $\Delta\vec{v}_{ij} = \vec{v}_i - \vec{v}_j$ and $\Delta\vec{r}_{ji} = \vec{r}_j - \vec{r}_i$. The equation is:

$$\vec{r}_s \times \Delta\vec{v}_{ij} + \vec{v}_s \times \Delta\vec{r}_{ji} = \vec{C}_{ij}$$

where $\vec{C}_{ij} = (\vec{r}_i \times \vec{v}_i) - (\vec{r}_j \times \vec{v}_j)$.

Matrix Form

The cross product $\vec{a} \times \vec{b}$ can be written as matrix multiplication $[\vec{a}]_\times \vec{b}$, where $[\vec{a}]_\times$ is the skew-symmetric matrix:

$$[\vec{a}]_\times = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

Note that $\vec{r}_s \times \Delta\vec{v}_{ij} = -\Delta\vec{v}_{ij} \times \vec{r}_s = -[\Delta\vec{v}_{ij}]_\times \vec{r}_s = [\Delta\vec{v}_{ji}]_\times \vec{r}_s$.

Therefore, for a pair of hailstones i, j, we obtain a 3×6 block of the matrix A:

$$\begin{bmatrix} [\vec{v}_j - \vec{v}_i]_\times & [\vec{r}_j - \vec{r}_i]_\times \end{bmatrix} \begin{bmatrix} \vec{r}_s \\ \vec{v}_s \end{bmatrix} = (\vec{r}_i \times \vec{v}_i) - (\vec{r}_j \times \vec{v}_j)$$

Explicit Matrix A

To solve for the 6 unknowns, we stack the blocks for pair $(1, 2)$ and pair $(1, 3)$.

Let $\Delta v_x = v_{2x} - v_{1x}$, $\Delta r_x = r_{2x} - r_{1x}$, etc. The first 3 rows (derived from pair 1, 2) are:

$$\begin{bmatrix} 0 & -\Delta v_z & \Delta v_y & 0 & -\Delta r_z & \Delta r_y \\ \Delta v_z & 0 & -\Delta v_x & \Delta r_z & 0 & -\Delta r_x \\ -\Delta v_y & \Delta v_x & 0 & -\Delta r_y & \Delta r_x & 0 \end{bmatrix} \begin{bmatrix} r_{sx} \\ r_{sy} \\ r_{sz} \\ v_{sx} \\ v_{sy} \\ v_{sz} \end{bmatrix} = \vec{b}_{12}$$

(Note the sign flip in the Δv block because $\vec{r}_s$ was the first term in the cross product).

Repeating this for pair $(1, 3)$ gives the bottom 3 rows. The resulting system $Ax = b$ can be solved via Gaussian elimination or standard linear solver libraries.