MPA Notes and Equations

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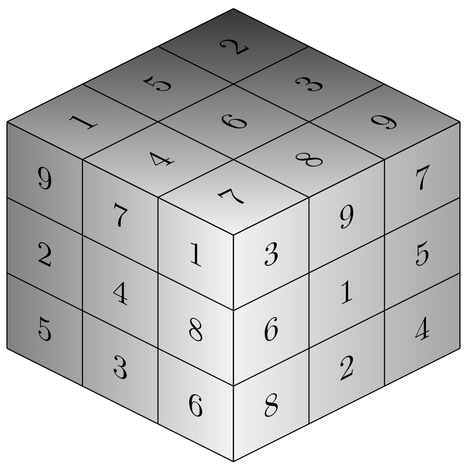


*Module Leader Mechanical Principles A

Basic Concepts and Equations for Mechanical Principles A written with LaTeX are Given in This Document!

1 Scalars and Vectors

A **scalar** has only **magnitude**, that is, it can be fully described by a single number.
Examples: Temperature, Distance, Speed, Mass.

A **vector** has both **magnitude and direction**, that is, as well as a number to quantify magnitude, we also need a way to indicate direction to describe a vector.
Examples: Displacement, Velocity, Force, Momentum. We can describe the

direction of a vector in several ways.
If the motion is only in one direction, for example, only along the x-axis and we take the usual convention of positive being to the right then we can say that a vector with a positive value (x +ve) has a direction to the right and a vector with a negative value (x -ve) has a direction to the left.
If the motion is in two directions we can; give the direction as an angle from the horizontal or indicate the direction by giving the vector's x and y components or by using unit vectors i and j.

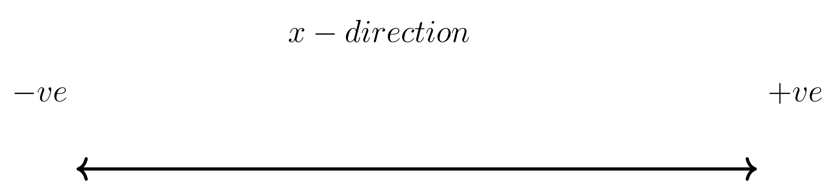


Figure 1 Indicating direction in one dimension, in this case the 'x-direction'.

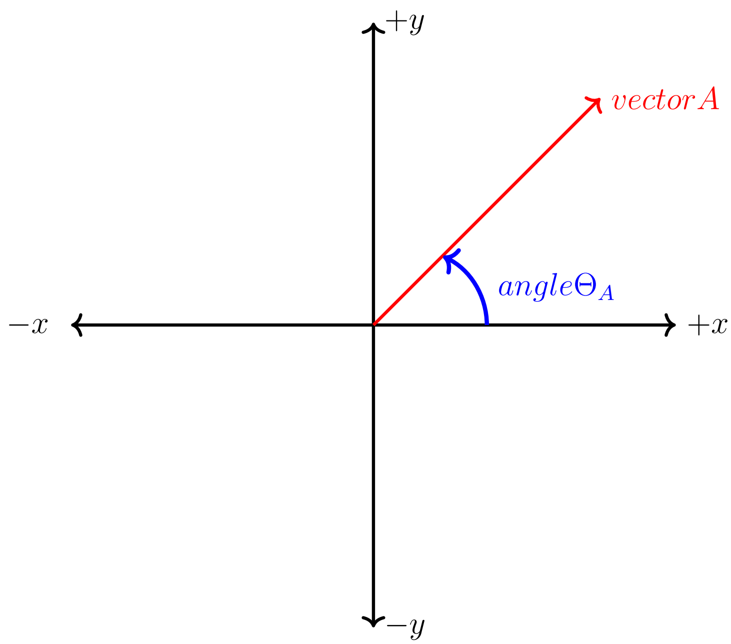


Figure 2 Indicating direction in two dimensions, the magnitude of the vector is shown by the length of the line in red and the direction of the vector is indicated by the angle in blue.

The component form of vector A in two dimensions can be given using unit vectors as:

$$\overrightarrow{A} = A_x\hat{i} + A_y\hat{j} \tag{1}$$

The Magnitude of vector A:

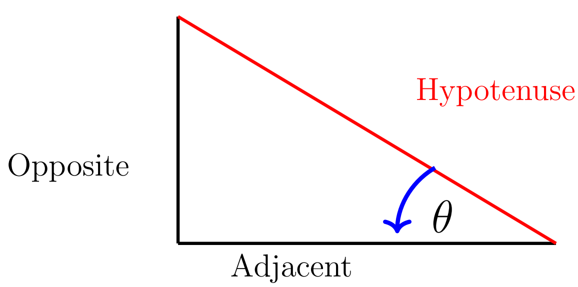
$$A = \sqrt{A_x^2 + A_y^2} \tag{2}$$

The direction angle of vector A:

$$\Theta_A = tan^{-1}\left(\frac{A_y}{A_x}\right) \tag{3}$$

It is helpful to remember the trigonometry functions; sine, cosine and tangent when dealing with vectors and one way to help remember is the mnemonic SohCahToa:

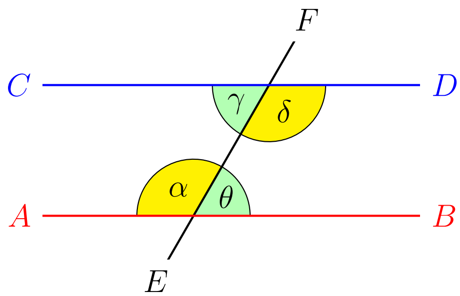
Mnemonic	Trig function	Formula	Ratio
SOH	Sine	$sin\theta = \frac{O}{H}$	Opposite over Hypotenuse
CAH	Cosine	$cos\theta = \frac{A}{H}$	Adjacent over Hypotenuse
TOA	Tangent	$tan\theta = \frac{O}{A}$	Opposite over Adjacent



In general when dealing with vectors [except when using the cross product] the problem can be tackled by:

1. break the vectors into their x and y components
2. separate and gather the x components and the y components
3. perform the operation or analysis separately for x and y components
4. write the separate results for x and y components
5. combine these x and y results together using vector addition

The following diagram might also help in selecting the correct angles.



If we assume that lines AB and CD are parallel, i. e., $AB \parallel CD$, then $\alpha = \delta$ and $\theta = \gamma$.

2 MOTION ALONG A STRAIGHT LINE

We can in many cases approximate an objects motion in space as being along a straight line. In which case we can determine aspects of its motion such as displacement, velocity and acceleration using relatively simple equations.

Displacement Δx is the change in position of an object from its initial position to its final position.

$$\Delta x = x_f - x_i \tag{4}$$

If an object moves in a number of individual 'steps' then we can calculate its *Total Displacement*

$$\Delta x_{Total} = \Sigma \Delta x_i \tag{5}$$

If an object moves from position x_1 at a time t_1 to a new position x_2 at a time t_2 we can determine its *Average Velocity*

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \tag{6}$$

Unless the motion is with constant velocity is will be different from the velocity at any particular instance in time, that is the *Instantaneous Velocity*

$$v(t) = \frac{dx(t)}{dt} \tag{7}$$

In a very similar way we can determine *Average Acceleration*

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1} \tag{8}$$

and *Instantaneous Acceleration*

$$a(t) = \frac{dv(t)}{dt} \tag{9}$$

If the motion is with ***constant acceleration*** then we can use a set of equations known as the kinematic equations.

Velocity from Acceleration

$$v = v_0 + at \tag{10}$$

Position from velocity and acceleration

$$x = x_0 + v_0 + \frac{1}{2}at^2 \tag{11}$$

Velocity from distance

$$v^2 = v_0^2 + 2a(x - x_0) \tag{12}$$

When considering object falling under the effect of gravity, that is with constant acceleration $-g$ where $g = 9.8m/s^2$, then a very similar set of equations can be used replacing a with $-g$, for example;

Height of free fall from starting position y_0 and initial velocity v_0

$$y = y_0 + v_0t - \frac{1}{2}gt^2 \tag{13}$$

3 MOTION IN TWO AND THREE DIMENSIONS

The position of a object in 3D space can be described by its *Position Vector*

$$\overrightarrow{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \tag{14}$$

Consider then a movement of an object from a position at t_1 to a new position at time t_2, the displacement of that object can be determined by the vector addition of the two position vectors, giving the *Displacement Vector*

$$\Delta\overrightarrow{r} = \overrightarrow{r}(t_2) - \overrightarrow{r}(t_1) \tag{15}$$

and the *Average Velocity* can be found from;

$$\overrightarrow{v_{avg}} = \frac{\overrightarrow{r}(t_2) - \overrightarrow{r}(t_1)}{t_2 - t_1} \tag{16}$$

As with motion in 1D the *Instantaneous Velocity Vector* in 3D is

$$\overrightarrow{v}(t) = \frac{d\overrightarrow{r}}{dt} \tag{17}$$

As with other vectors we can write the Velocity in terms of components;

$$\overrightarrow{v}(t) = v_x(t)\hat{i} + v_y(t)\hat{j} + v_z(t)\hat{k} \tag{18}$$

This approach is very useful in problem solving where each [*orthogonal*] component x, y and z can be treated independently and only re-combined at the end to give a final result. If acceleration is constant then we can use the kinematic equations independently in each component direction.

Where*Instantaneous Acceleration*

$$\overrightarrow{a}(t) = \frac{d\overrightarrow{v}(t)}{dt} \tag{19}$$

4 NEWTON'S LAWS OF MOTION

The study of how forces affect the motion of objects is known as *dynamics*, up to now we have restricted the study to equations the motion or *kinematic equations*. The most important relationships and equations in this section are based on **Newton's Laws of Motion** and to aid solve problems we will use *free body diagrams*. In many problems we can treat the mass of a

real object as acting at a point [centre of mass] and the forces acting on the object as force vectors acting on that point. This forms the basis of drawing free body diagrams.

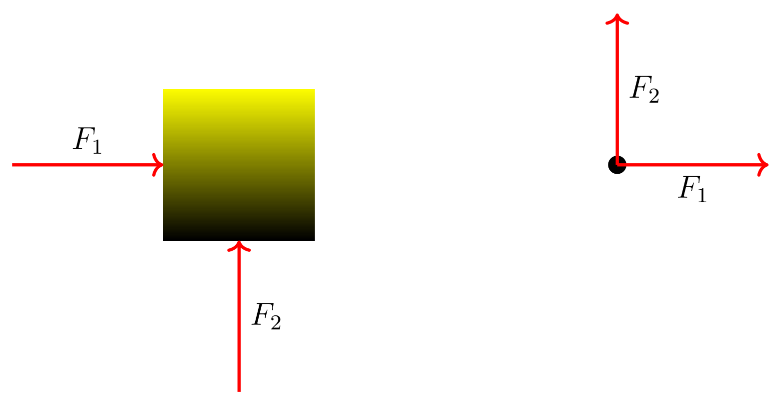


Figure 3 Two forces $\overrightarrow{F}_1$ and $\overrightarrow{F}_2$ acting on a box and the representation as a free body diagram.

This is a very simple case but we can use free body diagrams in more complex problems to help to visualise the magnitudes and directions of the forces acting on a body. Once this is clear we can calculate the *Net External Force*:

$$\overrightarrow{F}_{net} = \Sigma\overrightarrow{F} = \overrightarrow{F}_1 + \overrightarrow{F}_2 + ... \tag{20}$$

Problem-Solving Strategy: Drawing Free-Body Diagrams

1. Draw the object under consideration. If you are treating the object as a particle, represent the object as a point. Place this point at the origin of an xy-coordinate system.

2. Include all forces that act on the object, representing these forces as vectors. However, do not include the net force on the object or the forces that the object exerts on its environment.

3. Resolve all force vectors into x- and y-components.

4. Draw a separate free-body diagram for each object in the problem.

Newton's First Law of Motion

`A body at rest remains at rest or, if in motion, remains in motion at constant velocity unless acted on by a net external force.`

We can write Newton's First Law more mathematically

$$\overrightarrow{v} = \text{constant when} \overrightarrow{F}_{net} = \overrightarrow{0}N \tag{21}$$

Newton's Second Law of Motion

`The acceleration of a system is directly proportional to and in the same direction as the net external force acting on the system and is inversely proportional to its mass`

This can be written in equation form as:

$$\overrightarrow{a} = \frac{\overrightarrow{F}_{net}}{m} \tag{22}$$

It is more usual to see Newton's Second Law, written in *Vector Form* as:

$$\overrightarrow{F}_{net} = \Sigma\overrightarrow{F} = m\overrightarrow{a} \tag{23}$$

We can also write Newton's Second Law in *Component Form*, where we treat the x,y and z components separately:

$$\Sigma\overrightarrow{F}_{x} = m\overrightarrow{a_x},\ \ \Sigma\overrightarrow{F}_{y} = m\overrightarrow{a_y},\ \ \Sigma\overrightarrow{F}_{z} = m\overrightarrow{a_z} \tag{24}$$

By recognising that acceleration a can be written as $\frac{dv}{dt}$ and momentum $p = mv$, we can write Newton's Second Law in *Momentum Form*

$$\vec{F}_{net} = \frac{d\vec{p}}{dt} \tag{25}$$

That is, another way of expressing Newton's Second Law is that the net force acting on an object is equal to the rate of change of momentum of that object.

Newton's Second Law is used in our definition of **weight**, where in Vector Form, the *Definition of Weight* can be given as:

$$\vec{w} = m\vec{g} \tag{26}$$

Or simply in Scalar Form:

$$w = mg \tag{27}$$

An important thing to notice here is that weight is a force and will have SI units [N].

Newton's Third Law of Motion

`Whenever one body exerts a force on a second body, the first body experiences a force that is equal in magnitude and opposite in direction to the force that it exerts`

This can be written in equation form as:

$$\vec{F}_{AB} = -\vec{F}_{BA} \tag{28}$$

Newton's Third Law of Motion is used commonly in the following:

Normal force on an object resting on a horizontal surface, vector form.

$$\vec{N} = -m\vec{g} \tag{29}$$

Normal force on an object resting on a horizontal surface, scalar form.

$$N = -mg \tag{30}$$

Normal force on an object resting on an inclined surface θ from the horizontal, in scalar form.

$$N = -mgcos\theta \tag{31}$$

5 APPLICATION OF NEWTON'S LAWS

When applying Newton's Laws to solving problems it is useful to follow a consistent strategy or process.

Problem-Solving Strategy: Applying Newton's Laws of Motion

1. Identify the mechanical principles involved; list the information and values given and the unknowns to be calculated.
2. Sketch the problem, using arrows to represent all forces.
3. Determine the "system of interest". Represent this in a free-body diagram with component x, y and z forces.
4. Apply Newton's second law to solve the problem in component form. (If necessary, apply appropriate kinematic equations for motion along a straight line).
5. Treat as vectors to get a result.
6. Check that the solution is reasonable.

In these problems we may also have to deal with the forces associated with some form of resistance or friction which opposes the motion. The two most common are friction between two surfaces and the resistance due to motion through a fluid [drag].

Magnitude of static friction

$$f_s \leq \mu_s N \tag{32}$$

Magnitude of kinetic friction

$$f_k \leq \mu_k N \tag{33}$$

Drag Force

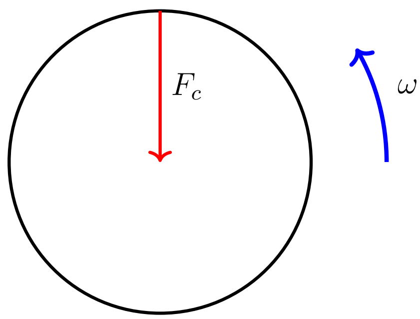
$$F_D = \frac{1}{2} C_D \rho A v^2 \tag{34}$$

Stokes' Law

$$F_s = 6\pi r \eta v \tag{35}$$

A body undergoing **Uniform Circular Motion** will experience *Centripetal Force*

$$F_c = m\frac{v^2}{r} = mr\omega^2 \tag{36}$$



where ω is the *angular velocity* and r is the *radius of motion*

$$\omega = \frac{\Delta\theta}{\delta t} \tag{37}$$

6 WORK AND KINETIC ENERGY

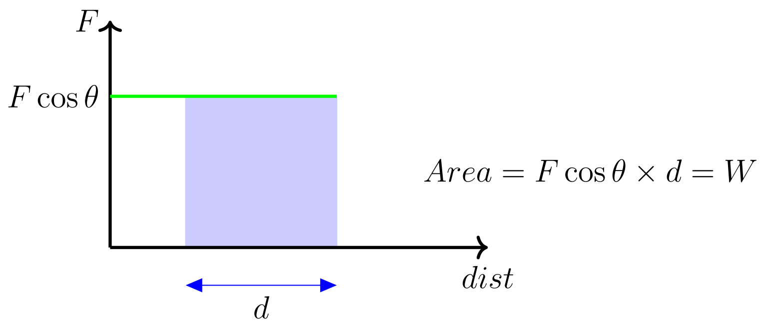
The work done on a system by a constant force is the product of the component of the force in the direction of motion times the distance through which the force acts.

Work Done by a constant force in line with displacement

$$W = F \cdot d \tag{38}$$

$$W = Fd\cos\theta \tag{39}$$

We can show this graphically.



If the force is not constant and or the path from A to B is not linear we can use the Work done by a force over an infinitesimal displacement;

$$dW = \overrightarrow{F} \cdot d\overrightarrow{r} = |\overrightarrow{F}||d\overrightarrow{r}|\cos\theta \tag{40}$$

and then calculate the Work done by a force acting along a path from A to B;

$$W_{AB} = \int_A^B \overrightarrow{F} \cdot d\overrightarrow{r} \tag{41}$$

If an object is lifted straight up from A to B against Earth's gravity near its surface at constant speed, then the force needed to lift it is equal to its weight mg . The Work done going from A to B by Earth's gravity near its surface is then;

$$W_{grav,AB} = -mg(y_B - y_A) \tag{42}$$

If we take the datum $y_A = 0$ and y_B is then some height h above the datum, we can make two related statements. One, the work done in raising the object of mass m to a height h is:

$$W = -mgh \tag{43}$$

and the gain in *Potential Energy* is:

$$PE = mgh \tag{44}$$

In a similar manner the work done going from A to B by one-dimensional spring force is:

$$W_{spring,AB} = -\frac{1}{2}k(x_B^2 - x_A^2) \tag{45}$$

and the gain in potential energy of the spring is:

$$PE_{spring,AB} = \frac{1}{2}k(x_B^2 - x_A^2) \tag{46}$$

The energy associated with a moving object is called *kinetic energy*. The translational kinetic energy (KE) of a mass m moving at a speed v . (Translational kinetic energy is distinct from rotational kinetic energy, which is considered later) can be written as:

$$KE = \frac{1}{2}mv^2 \tag{47}$$

The kinetic energy of an object can be related to its momentum

$$KE = \frac{1}{2}mv^2 = \frac{p^2}{2m} \tag{48}$$

The *Work-Energy Theorem* states that the net work on a system equals the change in its KE.

$$W_{net} = \frac{1}{2}mV^2 \tag{49}$$

We can also define here Power as a rate of doing work:

$$P = \frac{dW}{dt} \tag{50}$$

and Power as the dot product of force and velocity:

$$P = \overrightarrow{F} \cdot \overrightarrow{v} \tag{51}$$

7 LINEAR MOMENTUM AND COLLISIONS

Linear momentum is defined as the product of a system's mass multiplied by its velocity:

$$\overrightarrow{p} = m\overrightarrow{v} \tag{52}$$

Newton's second law of motion can also be written in terms of momentum, where the net external force F_{net} equals the change in momentum Δp of a system divided by the time taken for the change Δt.

$$F_{net} = \frac{\Delta p}{\Delta t} \tag{53}$$

We can rearrange this to give:

$$\Delta p = F_{net}\Delta t \tag{54}$$

The term $F_{net}\Delta t$ is know as *impulse* and is often given the symbol J.

In vector notation Impulse could be written as:

$$\overrightarrow{J} = \int_{t_i}^{t_f} \overrightarrow{F}(t)dt = \overrightarrow{J} = \overrightarrow{F}_{ave}\Delta t \tag{55}$$

and the Impulse-momentum theorem would then be written as:

$$\overrightarrow{J} = \Delta\overrightarrow{p} \tag{56}$$

we can then write the Average force from momentum as:

$$\overrightarrow{F}(t) = \frac{d\overrightarrow{p}}{dt} \tag{57}$$

An important property of momentum is that it is *conserved* in an isolated system. So we can write a Generalised conservation of momentum.

$$\sum_{j=1}^{N} \overrightarrow{p}_j = constant \tag{58}$$

An isolated system is defined to be one for which the net external force is zero ($F_{net.ext} = 0$), where:

$$\overrightarrow{F}_{net.ext} = \sum_{j=1}^{N} \frac{d\overrightarrow{p}_j}{dt} \tag{59}$$

This fact that momentum is conserved allows us to use this in the study of collisions. We will look at two basic types of collision

1. elastic collisions

2. non-elastic collisions

Beginning with elastic collisions we can define a elastic collision as one that conserves internal kinetic energy. If we consider two objects in an elastic collision with initial momentum p_1 and p_2 and final momentum p_1' and p_2' then the conservation of momentum means:

$$p_1 + p_2 = p_1' + p_2' \tag{60}$$

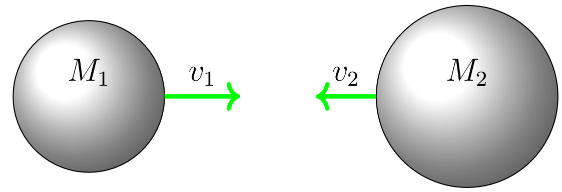
or

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \tag{61}$$

and because in an elastic collision internal kinetic energy is conserved

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \tag{62}$$

BEFORE

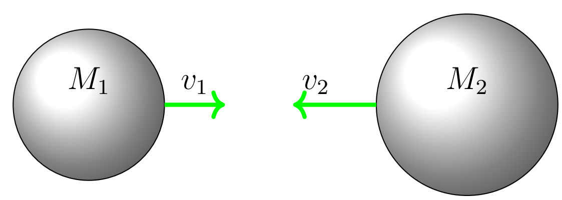


AFTER

An ***inelastic*** collision is one in which the internal kinetic energy changes (it is *not* conserved). A collision in which the objects stick together is sometimes called "perfectly inelastic."

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v_3 \tag{63}$$

BEFORE



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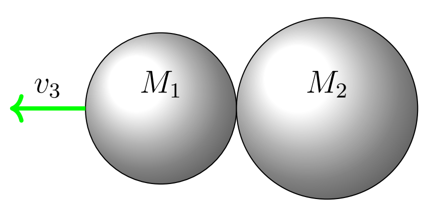


Collisions in 2D can be analysed using vectors and by treating the x-components and the y-components independently. For example for an ellastic collision.

$$p_{x1} + p_{x2} = p'_{x1} + p'_{x2} \tag{64}$$

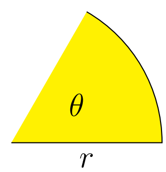
and

$$p_{y1} + p_{y2} = p'_{y1} + p'_{y2} \tag{65}$$

8 FIXED AXIS ROTATION

The *Angular Position* θ of an object rotating at a radius r which is displaced through an arc length s is given by:

$$\theta = \frac{s}{r} \tag{66}$$

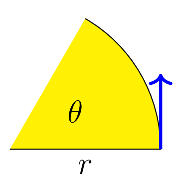


As shown earlier for uniform circular motion, [which is motion in a circle at constant speed and, hence, constant angular velocity] *angular velocity* ω was defined as the time rate of change of angle θ : Angular Velocity:

$$\omega = \frac{d\theta}{dt} \tag{67}$$

The *Tangential Velocity* v_t is then

$$v_t = r\omega \tag{68}$$



In a similar way we can write equations for *Angular Acceleration*

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}. \tag{69}$$

and *Tangential Acceleration*

$$a_t = r\alpha \tag{70}$$

The Kinematics for rotational motion is completely analogous to translational kinematics, where:

Angular Displacement

$$\theta_f = \theta_0 + \bar{\omega}t \tag{71}$$

Angular velocity with constant acceleration

$$\omega_f = \omega_0 + \alpha t \tag{72}$$

Angular displacement with constant acceleration

$$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \tag{73}$$

Just as there is analogous kinematic relations between translational and rotational there is also analogous kinetic relationships.
The first term we will look at is the *rotational* inertia or *moment of inertia* which is analogous to *mass*. If we have the simple case of an object we can treat as a point mass m rotating at a distance r from a fixed axis, we can calculate its moment of inertia I as:

$$I = mr^2$$

If we are dealing with a real object then we can define the moment of inertia I of an object to be the sum of mr^2 for all the point masses of which it is composed. That is,

Moment of Inertia:

$$I = \sum_j m_j r_j^2 \tag{74}$$

It is not easy to derive this for every shape of object in general so we will restrict our interest to four common situations:

1. A thin hoop of radius R and mass M rotating about its central axis

$$I = MR^2$$

2. A thick hoop of inner radius R_i and outer radius R_o and mass M rotating about its central axis

$$I = \frac{M}{2}(R_i^2 + R_o^2)$$

3. A solid cylinder or disk of radius R and mass M rotating about its central axis

$$I = \frac{MR^2}{2}$$

4. The rotation of a sphere of radius R and mass M about any axis

$$I = \frac{2MR^2}{5}$$

The next consideration is the moment of a force or the **torque**. If a force F is applied perpendicularly (at right angles, 90 deg) at a distance from the axis of r then the torque τ can be calculated from:

$$\tau = Fr$$

or at any angle, the *torque* in vector form:

$$\overrightarrow{\tau} = \overrightarrow{r} \times \overrightarrow{F} \tag{75}$$

We can also write an expression of Newton's second law [$F = ma$ in a translational context] for a rotational case:

Newton's 2nd Law of Rotation

$$\sum_i \tau_i = I\alpha \tag{76}$$

and define *Rotational Kinetic Energy* by the expression:

$$KE_{rot} = \frac{1}{2}I\omega^2 \tag{77}$$

and *Rotational Power*

$$P_{rot} = \tau\omega \tag{78}$$

As an analog to the definition of linear momentum $p = mv$ we can define define ***angular momentum*** L as:

$$L = I\omega \tag{79}$$

And analogous to linear momentum, it is conserved when the net external torque is zero. The relationship between torque and angular momentum is:

$$\tau_{net} = \frac{\Delta L}{\Delta t} \tag{80}$$

Expressing the conservation of angular momentum in terms of the moment of inertia, we can write:

$$L = L' \tag{81}$$

or

$$I\omega = I'\omega' \tag{82}$$

9 STATIC EQUILIBRIUM AND ELASTICITY

In mechanics the term "Statics" refers to the study of forces in equilibrium, so that the sum of all the forces and torques would be zero. So then we can describe two conditions which must be met for equilibrium.

1. First Equilibrium Condition: The first equilibrium condition for the static equilibrium of a rigid body expresses translational equilibrium:

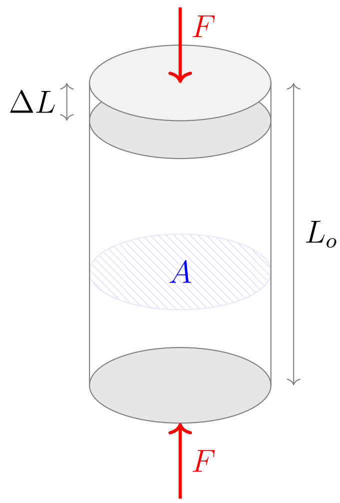
$$\sum_k \overrightarrow{F}_k = \overrightarrow{0}$$

2. Second Equilibrium Condition: The second equilibrium condition for the static equilibrium of a rigid body expresses rotational equilibrium:

$$\sum_k \overrightarrow{\tau}_k = \overrightarrow{0}$$

There are important concepts in mechanics which can be used in equilibrium conditions such as stress, strain, elastic modulus, etc.

The first situation we will consider is compressive stress and strain.



Compressive Stress and Strain

If we consider a cylinder of cross-section A and original length L_o is compressed by some force $F_\perp$ then the *stress* σ that the cylinder experiences is:

$$\sigma = \frac{F}{A} \tag{83}$$

and if the force changes the length of the cylinder by an amount ΔL, sometimes called a deformation of ΔL, then the compressive *strain* ϵ is given by:

$$\epsilon = \frac{\Delta L}{L_o} \tag{84}$$

As long as we stay in the elastic region, usually with small values of deformation or strain, then we can relate the stress and strain by the elastic modulus of the material, this is often known as the Young's modulus, where there is a Linear relation between stress and strain $stress = (elastic\ modulus) \times strain$ and Young's modulus E:

$$E = \frac{compressive\ stress}{compressive\ strain} = \frac{F_\perp}{A}\frac{L_0}{\Delta L} \tag{85}$$

If the forces were to pull rather than push on the faces of the cylinder putting it into *tension* and this time the cylinder extended by an amount ΔL then

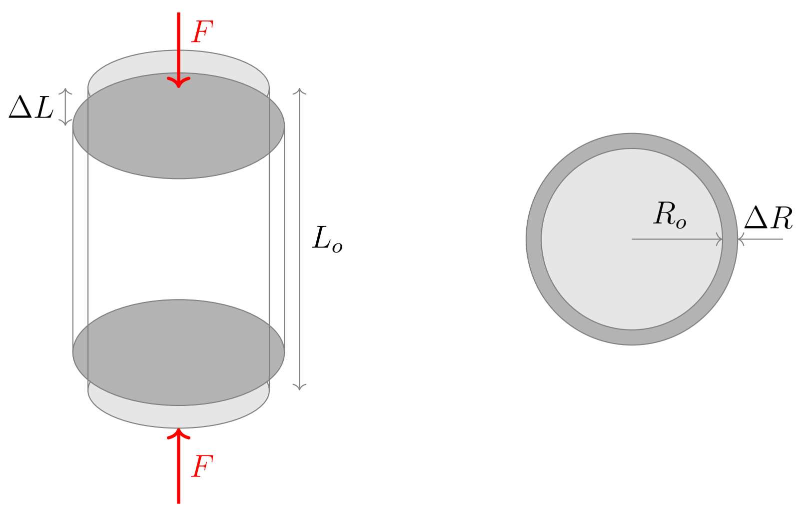
$$Y = \frac{tensile\ stress}{tensile\ strain} = \frac{F_\perp}{A}\frac{L_0}{\Delta L} \tag{86}$$

When we compress the cylinder it not only changes length, it gets shorter [a longitudinal change] it gets 'squashed' and so also changes radius, it gets wider [a lateral change]. We can define a lateral strain ϵ_{Lat} as:

$$\epsilon_{Lat} = \frac{\Delta R}{R} \tag{87}$$

The relationship between laterial strain and and longitudinal strain is given by **Poisson's Ratio** ν,

$$\nu = \frac{\epsilon_{Lat}}{\epsilon} \tag{88}$$



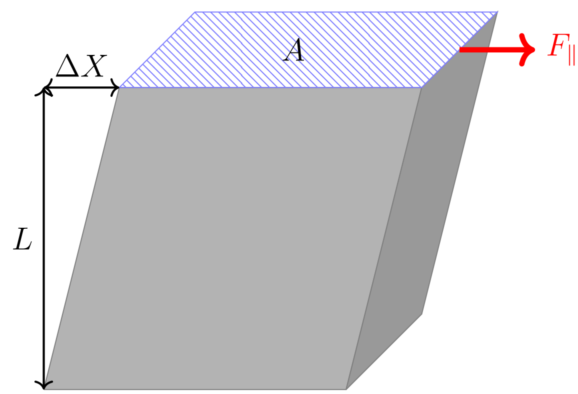
`Poisson's Ratio`

As well as deformation by compression or tension an object can deform when subjetc to a shear forces which causes a *Shear stress*, τ and gives a resultant shear strain γ, where:

$$\tau = \frac{F_{\parallel}}{A} \tag{89}$$

and

$$\gamma = \frac{\Delta X}{L} \tag{90}$$



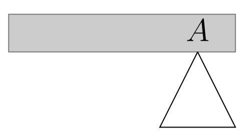
`Shear Stress and Strain`

This has an associated modulus, the *Shear modulus*, G, where:

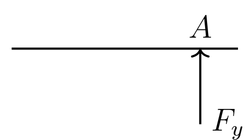
$$G = \frac{shear\ stress}{shear\ strain} = \frac{\tau}{\gamma} = \frac{F_{\parallel}}{A}\frac{L_0}{\Delta X} \tag{91}$$

10 BEAMS Part 1: Shear Force and Bending Moment

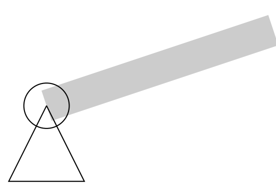
Structural Supports, Reactions and Free-Body Diagrams *Reaction forces* occur at supports or points of contact between bodies (Newton's 3rd law). The types of reaction that occur at a support depend on the type of support concerned. A support develops a reactive force on the supported member if the support prevents a translational movement of the beam (displacement), and it develops a reactive moment if it prevents rotation of the beam. Some common forms of supports and the reactions on free-body diagrams of the beams are shown below. A free-body diagram is used to show all external forces acting on a body, i.e. active and reactive forces, so that the body can be considered in isolation from its surroundings.



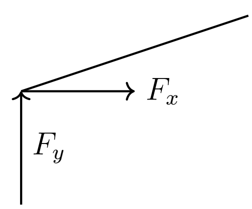
`Knife Edge Support`



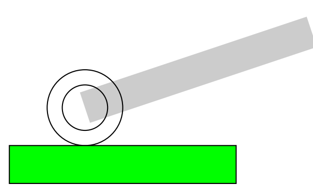
`Free Body Diagram`



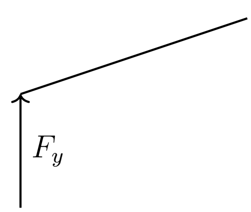
`Pinned Support`



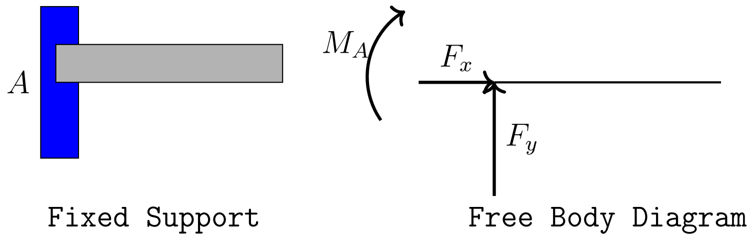
`Free Body Diagram`



`Roller Support`



`Free Body Diagram`



Beams, Shear Forces and Bending Moments A beam is a structure that is subjected to external forces inclined to its longitudinal axis. In most case, beams are horizontal and the loads they carry act vertically downwards, perpendicular to the longitudinal axis. In order to carry external forces, a beam has to be placed on supports which provide reactions to the external actions. The types of supports commonly used are:

1. knife edges, rollers or pins (simply-supported beam)

2. built-in at one end (cantilever beam)

3. built-in at one end, simply-supported at the other (propped cantilever)

4. built-in at both ends (encastre beam)

This course will focus on cases 1 and 2 only.

Equilibrium of Beams For a beam to be at rest (i.e. in equilibrium) when acted upon by external forces, the conditions of 2 dimensional equilibrium must be satisfied:

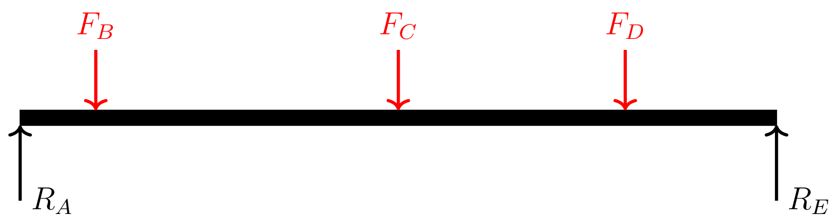
1. the algebraic sum of all the forces acting on the beam (including reactions) is zero [in the horizontal (x) and vertical (y) directions)], i.e.

$$\Sigma F_x = 0 \; and \; \Sigma F_y = 0$$

2. the algebraic sum of the moments of all the forces (including reactions) is zero about every point on the beam, i.e.

$$\Sigma M = 0$$

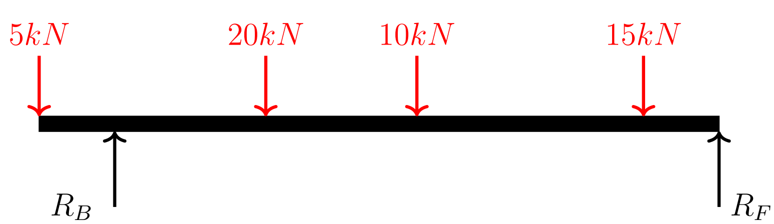
Types of Load There are two main types of loading which act on beams: concentrated or point loads that are applied to the beam through a 'knife-edge' and are shown on diagrams by arrows; uniformly distributed loads (UDI.) are spread over the whole or part of a beam so that the load carried by a given portion is proportional to the length of that portion. The magnitude of a UDL is measured in N/m and is usually denoted by ω . The figure below shows examples of beams with point loads and UDLs.



Reaction Supports and Point Loads

Reaction Supports and a UDL

Example of 2 Dimensional Equilibrium to Beams Consider the beam shown below which shows a simply-supported beam with 4 external point loads acting upon it.

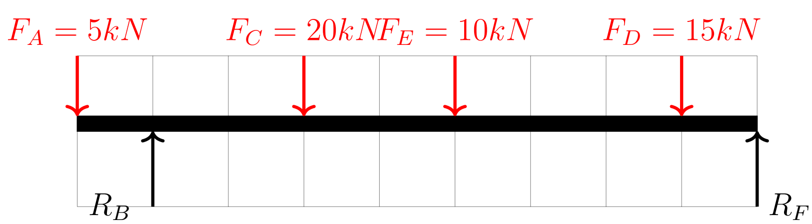


Reaction Supports and Point Loads

For equilibrium of forces, the sum of the four external vertical forces acting downwards must be equal to the sum of the two reaction forces acting vertically upwards, i.e.

$$\Sigma F_y = 0 = -5 + R_B - 20 - 10 - 15 + R_F \ therefore \ R_B + R_F = 50kN \quad (92)$$

Hence, this equation contains two unknown reaction forces and cannot be solved at this stage. If the distances between all the points of force application were known, then the moment caused by each force could be calculated about any point on the beam and the sum of these moments must be zero. However, if a moment equation was derived about points A, C, D and E, then an equation with the two unknown reaction forces R_B and R_F would result. Hence a moment equation should be derived about either of the reaction points B and F.
Consider the same beam with the distances between each of the points as shown on the grid spaced 1m apart.



Distances Between Reaction Supports and Point Loads

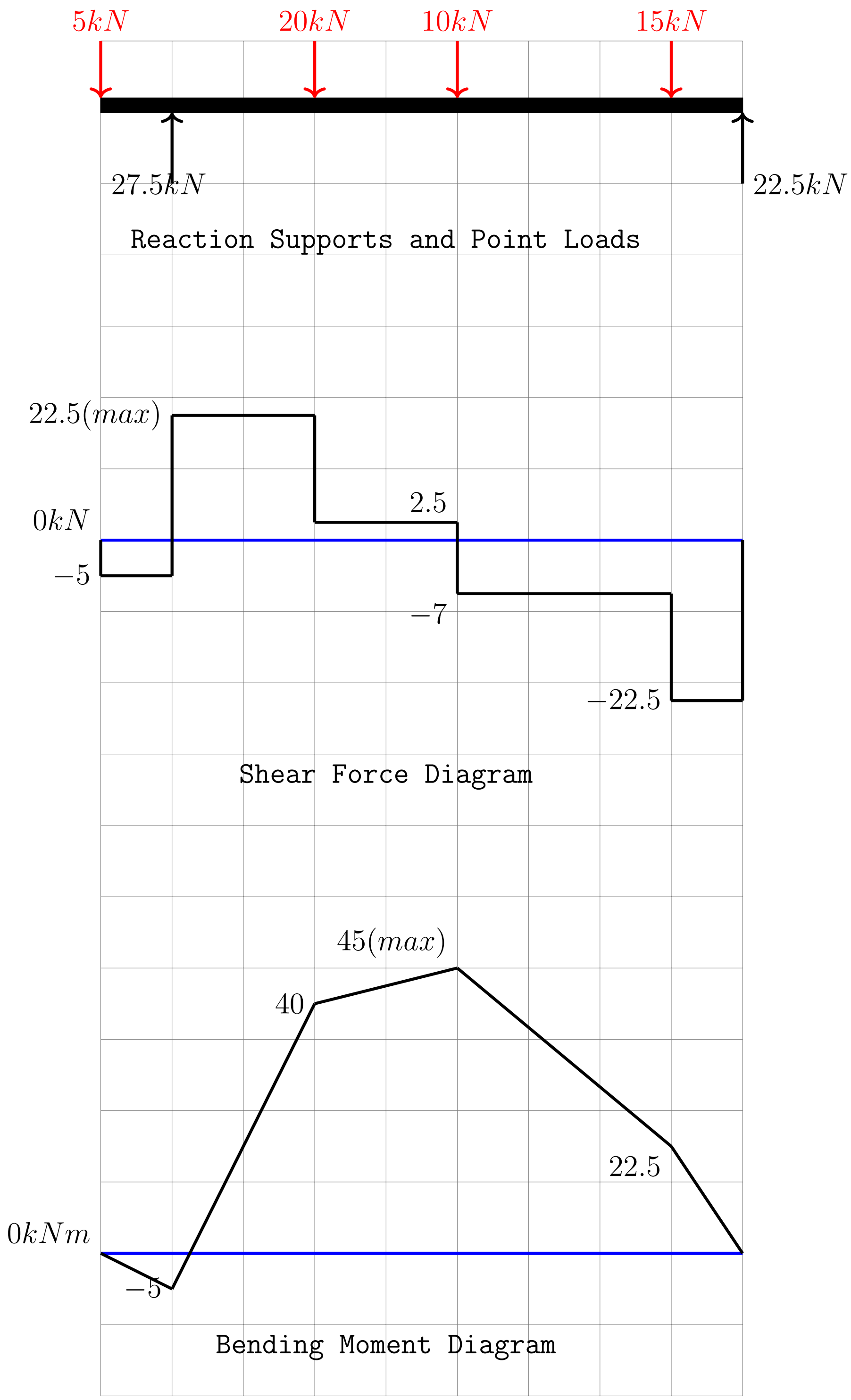
For equilibrium of moments, the sum of the moments caused by all forces must be equal to zero at any point along the beam. However, to have an equation containing just one unknown reaction force, moments are taken about B (noting that +M is clockwise and —M is anticlockwise), i.e.

$$\Sigma M_B = 0 - (5x1) + (20x2) + (10x4) + (15x7) - (8xR_F) = O, \implies R_F = 22.5kN \quad (93)$$

Hence substitute $R_F = 22.5kN$ in equation above gives: $RB + 22.5 = 50kN \implies R_B = 27.5kN$

Diagrams can now be drawn to show the variation of shear force and bending moment along the span of the beam to establish the location of maximum values.

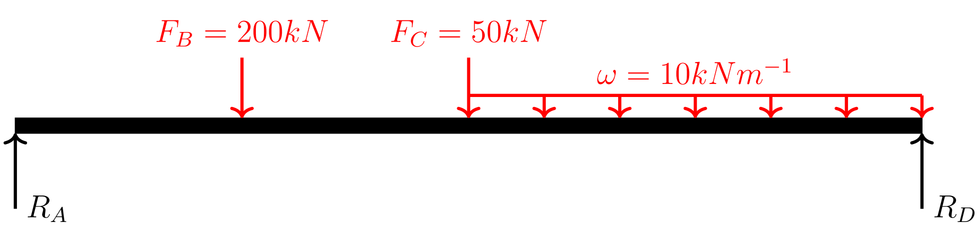
Shear Force and Bending Moment Diagrams Shear force (SF) and bending moment (BM) diagrams can be drawn that shows their variation along the span so that maximum values and their location can be established. The diagrams are usually drawn below the free-body diagram representing the beam.
Consider the figure below which shows the loaded beam and the calculated reactions at B and F. Lines are also drawn from each point on the beam to aid construction of the SF and M diagrams.



5kN
20kN
10kN
15kN
27.5kN
22.5kN
Reaction Supports and Point Loads
22.5(max)
0kN
2.5
−5
−7
−22.5
Shear Force Diagram
45(max)
40
22.5
0kNm
−5
Bending Moment Diagram

The maximum shear force is 22.5 kN along portions BC and EF, and the maximum bending moment is 45 kNm acting at D. To calculate the bending moments, use can be made of the shear force diagram: area AB — moment at B; area AB+BC = moment at C; area AB+BC+CD = moment at D (maximum in this case); area AB+BC+CD+DE = moment at E; area AB+BC+CD+DE+EF = moment at F e 0 in this case). Alternatively, moment equations can be derived to calculate the magnitude of the bending moment at each point, i.e.
$M_B = -(5x1) = -5kNm$,
$M_C = 40kNm$,
$M_D = 45kNm$,
$M_E = 22.5kNm$,
$M_F = OkNm$.
These values are the same as those calculated using the areas in the shear force diagrams.

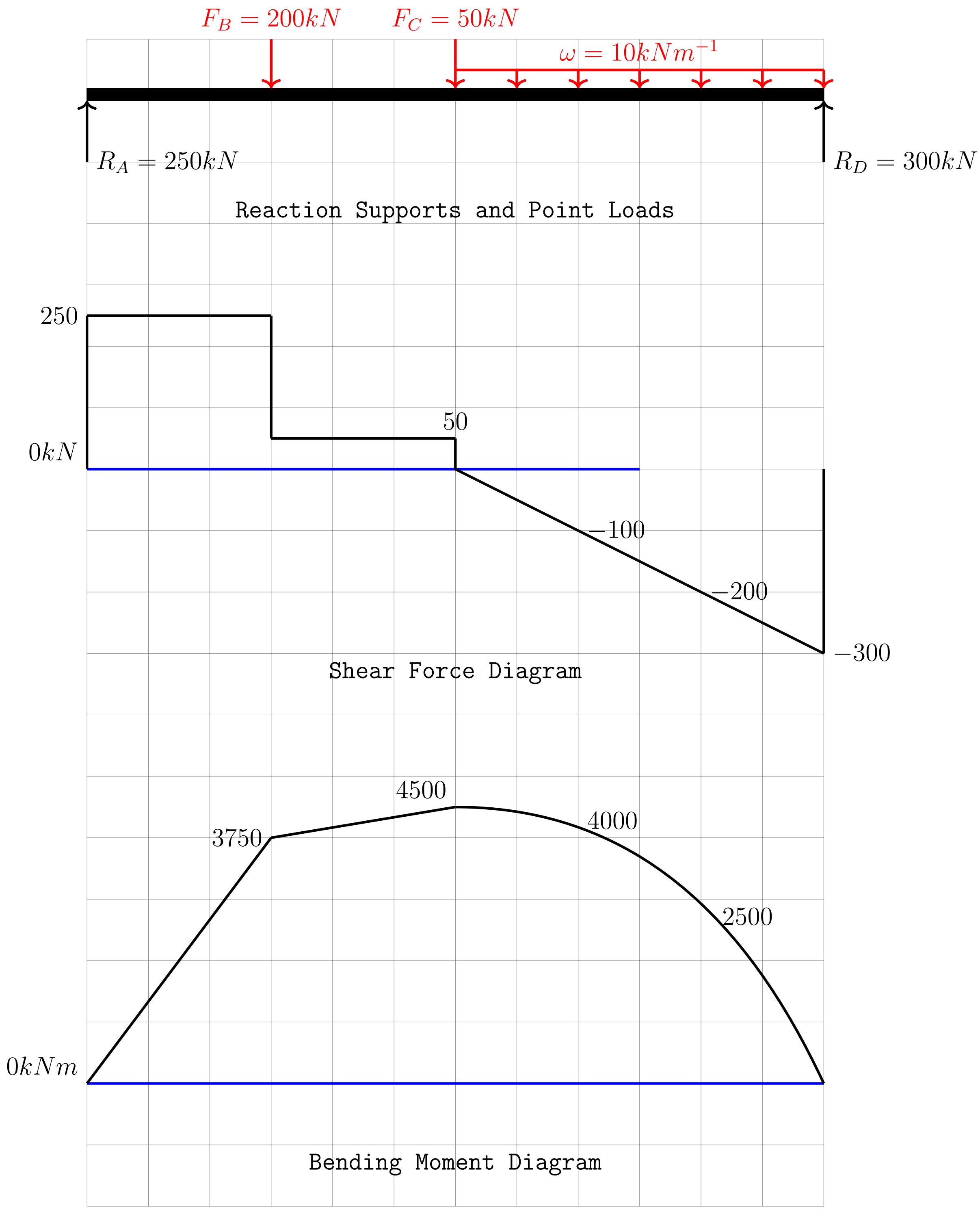
Consider now a beam which has both point loads and a uniform distributed load as shown below.



First calculate the Reactions at A and D using the grid spacing of 5m per unit:

$$\sum M_A = 200 \times 15 + 50 \times 30 + (10 \times 30) \times 45 + R_D \times 60 = 0 \rightarrow R_D = 300kN \tag{94}$$

$$\sum F_y = R_A - 200 - 50 - (10 \times 30) + 300 = 0 \rightarrow R_A = 250kN \tag{95}$$



$F_B = 200kN$
$F_C = 50kN$
$\omega = 10kNm^{-1}$
$R_A = 250kN$
$R_D = 300kN$
Reaction Supports and Point Loads
250
0kN
50
−100
−200
−300
Shear Force Diagram
4500
3750
4000
2500
0kNm
Bending Moment Diagram

The maximum shear force is 300 kN acting at D, and the maximum bending moment is 4500 kNm acting at C. The bending moments are calculated using the areas of the shear force diagrams or by moment calculation about each point along the beam.

When drawing SF and BM diagrams, some general rules apply:

1. In the absence of distributed loads (UDLs), the SF diagram consists of horizontal steps and the BM diagram is a series of straight lines.

2. For a beam (or part of a beam) carrying a UDL only, the SF diagram is a sloping straight line and the BM diagram is a parbola.

3. At a point where the SF diagram passes through zero (i.e. where the SF changes sign), the BM diagram has a maximum or minimum value.

4. Over a part of the span for which SF is zero, the BM has constant value.

5. At a point where the BM diagram passes through zero, this is called a point of contraflexure (or inflexion).

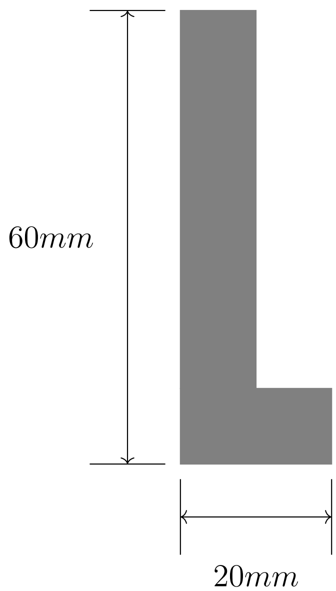
11 Beams Part 2: Bending Stresses

11.1 Centroids and 2nd Moment of Area

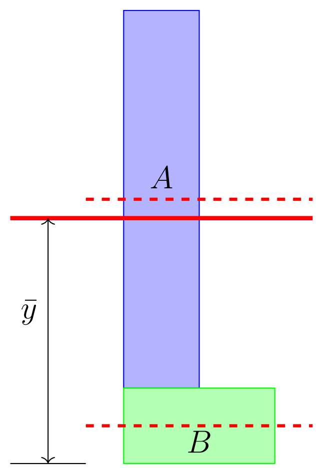
In statics, the 'centre of gravity' is the point through which the resultant weight or gravity effect passes regardless of the orientation of the body. Similar to the centre of gravity, a centre of area exists for every geometrical shape which is called the *centroid*. The position of the centroid of an area is usually defined by the coordinates $\bar{x}$ and $\bar{y}$ which are the distances from a reference set of axes.
For the common cross-sections; circle, rectangle, right angled triangled, the neutral axes coordinates $\bar{x}$ and $\bar{y}$ are found by inspection or from simple formulae. However, for more complex geometry the position of the centroid and hence the neutral axes is found by considering the *1st moments of area*. We will illustrate by example using geometric deconstruction. The centroid of a 2D figure X can be computed by dividing it into a finite number of simpler figures $X_1, X_2, \ldots, X_n$, computing the centroid C_i and area A_i of each part, and then computing

$$\bar{x} = \frac{\sum C_{i_x} A_i}{\sum A_i}, \bar{y} = \frac{\sum C_{i_y} A_i}{\sum A_i} \tag{96}$$

Consider the "L" section beam shown below left of web width 10mm and this broken down into two rectangles A and B shown below right. Taking the bottom left corner as (0,0) then the vertical distance to the central axis of rectangle A is 5mm and to the central axis of rectangle B 30mm.



"L" Cross-section



Two rectangles A and B

To calculate $\bar{y}$ we apply

$$\bar{y} = \frac{\sum C_{i_y} A_i}{\sum A_i} \tag{97}$$

$$\bar{y} = \frac{(5 \times (10 \times 20) + 30 \times (10 \times 50))}{(10 \times 20) + (10 \times 50)} = 22.86mm \tag{98}$$

The value for $\bar{x}$ can be found in a similar way, $\bar{x} = 6.43mm$.

Percentage Uncertainty

$$= \frac{\delta A}{A} \times 100\%$$

VECTORS

Multiplication by a scalar

$$\overrightarrow{B} = \alpha \overrightarrow{A}$$

Cummutative Law

$$\overrightarrow{A} + \overrightarrow{B} = \overrightarrow{B} + \overrightarrow{A}$$

Associative Law

$$(\overrightarrow{A} + \overrightarrow{B}) + \overrightarrow{C} = \overrightarrow{A} + (\overrightarrow{B} + \overrightarrow{C})$$

The component form of a vector in two dimensions

$$\overrightarrow{A} = A_x\hat{i} + A_y\hat{j}$$

The Magnitude of a vector in a plane

$$A = \sqrt{A_x^2 + A_y^2}$$

The direction angle of a vector in a plane

$$\Theta_A = tan^{-1}\left(\frac{A_y}{A_x}\right)$$

The component form of a vector in three dimensions

$$\overrightarrow{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

The Magnitude of a vector in three dimensions

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Definition of the **Scalar** product

$$\overrightarrow{A}.\overrightarrow{B} = AB\cos\theta$$

Scalar product in terms of the scalar components of vectors

$$\overrightarrow{A}.\overrightarrow{B} = A_xB_x + A_yB_y + A_zB_z$$

Dot products odf unit vectors

$$\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$$

Magnitude of the **vector** product

$$\mid \overrightarrow{A} \times \overrightarrow{B} \mid = AB\sin\theta$$

Cross products of unit vectors

$$\hat{i} \times \hat{j} = +\hat{k}, \hat{j} \times \hat{k} = +\hat{i}, \hat{k} \times \hat{i} = +\hat{j}$$

The Cross product in terms of scalar components of vectors

$$\overrightarrow{A} \times \overrightarrow{B} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$$

MOTION ALONG A STRAIGHT LINE

Displacement

$$\Delta x = x_f - x_i$$

Total Displacement

$$\Delta x_{Total} = \Sigma \Delta x_i$$

Average Velocity

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}$$

Instantaneous Velocity

$$v(t) = \frac{dx(t)}{dt}$$

Average Acceleration

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1}$$

Instantaneous Acceleration

$$a(t) = \frac{dv(t)}{dt}$$

Velocity from Acceleration [constant a]

$$v = v_0 + at$$

Position from velocity and acceleration [constant a]

$$x = x_0 + v_0 + \frac{1}{2}at^2$$

Velocity from distance [constant a]

$$v^2 = v_0^2 + 2a(x - x_0)$$

Height of free fall

$$y = y_0 + v_0 t - \frac{1}{2}gt^2$$

MOTION IN TWO AND THREE DIMENSIONS

Position Vector

$$\overrightarrow{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

Displacement Vector

$$\Delta\overrightarrow{r} = \overrightarrow{r}(t_2) - \overrightarrow{r}(t_1)$$

Velocity Vector

$$\overrightarrow{v}(t) = \frac{d\overrightarrow{r}}{dt}$$

Velocity in Terms of Components

$$\overrightarrow{v}(t) = v_x(t)\hat{i} + v_y(t)\hat{j} + v_z(t)\hat{k}$$

Average Velicity

$$\overrightarrow{v}_a vg = \frac{\overrightarrow{r}(t_2) - \overrightarrow{r}(t_1)}{t_2 - t_1}$$

Instantanious Acceleration

$$\overrightarrow{a}(t) = \frac{d\overrightarrow{v}(t)}{dt}$$

Centripetal Acceleration

$$a_c = \frac{v^2}{r}$$

NEWTON'S LAWS OF MOTION

Net External Force

$$\overrightarrow{F}_{net} = \Sigma\overrightarrow{F} = \overrightarrow{F}_1 + \overrightarrow{F}_2 + ...$$

Newton's First Law

$$\overrightarrow{v} = constantwhen\overrightarrow{F}_{net} = \overrightarrow{0}N$$

Newton's Second Law, Vector Form

$$\overrightarrow{F}_{net} = \Sigma\overrightarrow{F} = m\overrightarrow{a}$$

Newton's Second Law, Component Form

$$\Sigma\overrightarrow{F}_x = m\overrightarrow{a_x}, \Sigma\overrightarrow{F}_y = m\overrightarrow{a_y}, \Sigma\overrightarrow{F}_z = m\overrightarrow{a_z}$$

Newton's Second Law, Momentum Form

$$\overrightarrow{F}_{net} = \frac{d\overrightarrow{p}}{dt}$$

Defination of Weight, Vector Form

$$\overrightarrow{w} = m\overrightarrow{g}$$

Defination of Weight, Vector Form

$$w = mg$$

Newton's Thrid Law

$$\overrightarrow{F}_{AB} = -\overrightarrow{F}_{BA}$$

Normal force on an object resting on a horizontal surface, vector form.

$$\overrightarrow{N} = -m\overrightarrow{g}$$

Normal force on an object resting on a horizontal surface, scalar form.

$$N = -mg$$

Normal force on an object resting on an inclined surface, scalar form.

$$N = -mgcos\theta$$

APPLICATION OF NEWTON'S LAWS

Magnitude of static friction

$$f_s \leq \mu_s N$$

Magnitude of kinetic friction

$$f_k \leq \mu_k N$$

Centripetal Force

$$F_c = m\frac{v^2}{r} = mr\omega^2$$

Drag Force

$$F_D = \frac{1}{2}C_D\rho A v^2$$

Stokes' Law

$$F_s = 6\pi r\eta v$$

WORK AND KINETIC ENERGY

Work Done by a force in line with displacement

$$W = F \cdot d$$

Work done by a force over an infinitesimal displacement

$$dW = \overrightarrow{F} \cdot d\overrightarrow{r} = |\overrightarrow{F}||d\overrightarrow{r}| \cos\theta$$

Work done by a force acting along a path from A to B

$$W_{AB} = \int_A^B \overrightarrow{F} \cdot d\overrightarrow{r}$$

Work done going from A to B by Earth's gravity near its surface

$$W_{grav,AB} = -mg(y_B - y_A)$$

Work done going from A to B byone-dimensional spring force

$$W_{spring,AB} = -\frac{1}{2}k(x_B^2 - x_A^2)$$

Kinetic Energy of a particle

$$K = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

Power as a rate of doing work

$$P = \frac{dW}{dt}$$

Power as the dot product of force and velocity

$$P = \overrightarrow{F} \cdot \overrightarrow{v}$$

POTENTIAL ENERGY AND CONSERVATION OF ENERGY

Difference of potential energy

$$\Delta U_A B = U_B - U_A = -W_A B$$

Gravitational potential energy near Eath's surface

$$U(y) = mgy + const$$

Potential Energy of an ideal spring

$$U(x) = \frac{1}{2}kx^2 + const$$

Conservation of energy with no non-conservative forces

$$0 = W_{nc,AB} = \Delta(K + U)_{AB} = \Delta E_{AB}$$

LINEAR MOMENTUM AND COLLISIONS

Definition of momentum

$$\overrightarrow{P} = m\overrightarrow{v}$$

Impulse

$$\overrightarrow{J} = \int_{t_i}^{t_f} \overrightarrow{F}(t)dt = \overrightarrow{J} = \overrightarrow{F}_{ave}\Delta t$$

Impulse-momentum theorem

$$\overrightarrow{J} = \Delta\overrightarrow{p}$$

Average force from momentum

$$\overrightarrow{F}(t) = fracd\overrightarrow{p}dt$$

Conservation of momentum

$$\overrightarrow{p}_1 + \overrightarrow{p}_2 = constant$$

Generalised conservation of momentum

$$\sum_{j=1}^{N} \overrightarrow{p}_j = constant$$

External forces

$$\overrightarrow{F}_{ext} = \sum_{j=1}^{N} \frac{d\overrightarrow{p}_j}{dt}$$

FIXED AXIS ROTATION

Angular Position

$$\theta = \frac{s}{r}$$

Angular Velocity

$$\omega = \frac{d\theta}{dr}$$

Tangential Speed

$$v_t = r\omega$$

Angular Acceleration

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

Tangential Acceleration

$$a_t = r\alpha$$

Angular Displacement

$$\theta_f = \theta_0 + \bar{\omega}t$$

Angular velocity with constant acceleration

$$\omega_f = \omega_0 + \alpha t$$

Angular displacement with constant acceleration

$$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

Moment of Inertia

$$I = \sum_j m_j r_j^2$$

Rotational Kinetic Energy

$$K = \frac{1}{2}I\omega^2$$

Torque Vector

$$\overrightarrow{\tau} = \overrightarrow{r} \times \overrightarrow{F}$$

Newton's 2nd Law of Rotation

$$\sum_i \tau_i = I\alpha$$

Rotational Power

$$P = \tau\omega$$

ANGULAR MOMENTUM

Displacement of centre of mass of rolling object

$$d_{CM} = R\theta$$

Velocity of centre of mass of rolling object

$$v_{CM} = R\omega$$

Acceleration of centre of mass of rolling object

$$a_{CM} = R\alpha$$

Angular Momentum of a point mass

$$\overrightarrow{I} = \overrightarrow{r} \times \overrightarrow{p}$$

Angular Momentum

$$L = I\omega$$

Conservation of angular momentum

$$\frac{d\overrightarrow{L}}{dt} = 0$$

STATIC EQUILIBRIUM AND ELASTICITY

First Equilibrium Condition

$$\sum_k \overrightarrow{F}_k = \overrightarrow{0}$$

Second Equilibrium Condition

$$\sum_k \overrightarrow{\tau}_k = \overrightarrow{0}$$

Linear relation between stress and strain

$$stress = (elasticmodulus) \times strain$$

Young's modulus

$$E = \frac{tensilestress}{tensilestrain} = \frac{F_\perp}{A}\frac{L_0}{\Delta L}$$

Shear modulus

$$G = \frac{shearstress}{shearstrain} = \frac{F_\parallel}{A}\frac{L_0}{\Delta x}$$

OSCILLATIONS

Relationship between frequency and period

$$f = \frac{1}{T}$$

Position in Simple Harmonic Motion with $\phi = 0$

$$x(t) = A\cos(\omega t)$$

General Position in Simple Harmonic Motion

$$x(t) = A\cos(\omega t + \phi)$$

General Velocity in Simple Harmonic Motion

$$x(t) = -A\omega\sin(\omega t + \phi)$$

General Acceleration in Simple Harmonic Motion

$$x(t) = -A\omega^2\cos(\omega t + \phi)$$

Maximum displacement (amplitude) of SHM

$$x_{max} = A$$

Maximum velocity of SHM

$$|v_{max}| = A\omega$$

Maximum acceleration of SHM

$$|a_{max}| = A\omega^2$$

Angular frequency of a simple pendulum

$$\omega = \sqrt{\frac{g}{L}}$$

Natural angular frequency of a simple pendulum

$$\omega_0 = \sqrt{\frac{k}{m}}$$