

MODEL DESCRIPTION

1. MOTIVATION: SPARSITY + DIFFERENTIABLE

1. Most of the current graphical pooling method aims to learn an allocation matrix $[N * K]$. The computation for the embedding of the aggregated nodes requires $O(N^2)$. $([K * N] * [N * D])$, thus non-scalable.

2. None of the current sparse graphical pooling method are differentiable. Typically, an importance score is learned and then sparsity is achieved by top selection.

3. For top selection method, the embedding for coarsened graph only contain information for selected cluster/nodes, some information / structure in the original graph may be lost.

2. MODEL

2.1. Notations.

N : number of nodes,
 K : number of vectors in the codebook,
 D : dimension of embedding
 A : Adjacency matrix $[N * N]$
 X : Positional matrix $[N * D_x]$
 F : Feature matrix $[N * D_f]$
 C : codebook vectors $[K * D]$
 DS : distance matrix $[N * K]$

2.2. Model for learning the topological structure.

2.2.1. Algorithm.

1. Randomly initialize embedding X_0 , codebook C
2. Use two GCN layers to capture local structure: $X = \text{GCN}(\text{GCN}(X_0, A), A)$
3. Compute the distance matrix DS , where $DS[i, j] = \text{distances}(X_i, C_j)$
4. Replace the node embedding with closest vector in the notebook as Z_x

2.2.2. Loss function.

Consider the sum of vq loss and ELBO loss as the loss function, where vq Loss is the same as the original paper: $\|Z - \text{sg}(X)\|^2 + \beta \|X - \text{sg}(Z)\|^2$.

ELBO loss is adapted to encode Dirichlet process:

- Original ELBO is to maximize the lower bound

$$E_q[\log \frac{p(X, Z)}{q(Z)}] = E_q[\log p(X|Z)] + E_q[\log \frac{p(Z)}{q(Z)}],$$

or equivalently to minimize

$$E_q[\log p(A|Z)] + E_q[\log \frac{q(Z)}{p(Z)}],$$

where Z is the latent variable. The first term is also known as reconstruction term, and the second term is the KL-divergence between $q(Z)$ and $p(Z)$. In practice, the number of samples in MC simulation is set to 1.

- For the reconstruction term, we consider

$$-\frac{1}{|pos|} \sum_{(i,j) \in pos} \log p(Z_i^T Z_j) + \frac{1}{|neg|} \sum_{(i,j) \in neg} \log(1 - Z_i^T Z_j)$$

- Formally, the ELBO loss for vq-vae wouldn't contain the second term, since $q(Z)$ is a categorical distribution where the probability for the closest class equals to 1, i.e. $((\hat{\pi}_1, \dots, \hat{\pi}_i, \dots, \hat{\pi}_K) = (0, \dots, 1, \dots, 0))$. To encode Dirichlet process in the loss function, we consider a continuous relaxation of this distribution as $\hat{\pi}_k = \frac{1}{N} \sum_{n=1}^N \hat{\pi}_{nk}$, where $\hat{\pi}_{nk} \propto \exp(-DS[n, k])$ and $\sum_{k=1}^K \hat{\pi}_{nk} = 1$.
- Consider the prior distribution as stick-breaking prior, where π is generated as follows:

Sample v_i from $Beta(1, \alpha_0)$, $(i = 1, 2, \dots, K-1)$, $v_K = 1$.

Transform to $\pi_k = v_k \prod_{i=1}^{k-1} (1 - v_i)$, $(i = 1, \dots, K)$.

- Since it's hard to directly compute the marginal distribution $p(Z)$, we use a MC simulation to approximate the second term as

$$E_q[\log \frac{q(Z)}{p(Z)}] = \frac{1}{M} \sum_{m=1}^M E_q[\log \frac{q(Z)}{p(Z|v_m)}], \quad v_m \sim Beta(1, \alpha_0)$$

- Thus,

$$L = -\frac{1}{|pos|} \sum_{(i,j) \in pos} \log p(Z_i^T Z_j) + \frac{1}{|neg|} \sum_{(i,j) \in neg} \log(1 - Z_i^T Z_j) + \frac{1}{M} \sum_{m=1}^M E_q[\log \frac{q(Z)}{p(Z|v_m)}]$$

2.3. Model for learning the feature.

We assume that the feature matrix for each node is meaningful, and aim to learn a quantized vector to reflect the original feature vector. Similar algorithm and loss function is considered.

2.3.1. Algorithm.

1. Initialize codebook C
2. Use two GCN layers to absorb local structure: $F = GCN(GCN(F, A), A)$
3. Compute the distance matrix DS , where $DS[i, j] = distances(F_i, C_j)$
4. Quantize the embedding with closest vector in the notebook Z
5. Recover the initial embedding based on quantized latent embedding $F_d = GCN(GCN(Z, A), A)$

2.3.2. Loss function.

Similarly, the loss function is considered as the sum of reconstruction loss and KL loss, where KL loss is the same as last subsection. Reconstruction loss is computed based on the MSE between recovered feature embedding F_d and initial feature embedding F .

2.4. Supervised learning. For supervised learning, we consider the complete quantized embedding as a concatenation of quantized positional embedding and quantized feature embedding, and the label for each node is predicted using this quantized complete embedding.

3. PRACTICAL CONCERN

Several practical concern is observed and recorded as follows:

1. We use dirichlet prior with different parameter α_0 to control the number of clusters, and thus a smaller α_0 correspond with less number of clusters. In practice, however, the KL for a smaller α_0 may be larger than that for a larger α_0 , making the number of clusters for a smaller α_0 larger than that of a larger α_0 .
2. In practice, β -VAE (use β to balance reconstruction loss and KL loss) is observed to have better performance. Also, paying different attention for positional and feature embedding might have better effect. We need to balanced between too many loss functions, especially when considering supervised learning.
3. In Stochastic Model, the model doesn't perform well when the true number of clusters is large.
4. The model quantize the embedding for each node. Unique embedding might be more helpful for node prediction.
5. Since we consider positional embedding, it is not inductive.

Potentially, changing the structure of current method might solve these problems:

1. Other ways to update parameters without using loss function
2. Consider more efficient way of mixing the two embedding / distances
3. Consider the codebook as complete graph, update the codebook (similar to MP on coarsened graph)
4. Generate unique node embedding.