Separable Differential Equations and Homogeneous First-Order DE's

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1.4 Separable Differential Equations

Definition 1.4.1: A first-order differential equation is called separable if it can be written in the form

$$p(y) \cdot \frac{dy}{dx} = q(x)$$

1.4.2 Theorem:

If p(y) and q(x) are continuous, then Equation (1.4.1) has the general solution

$$\int q(y) \cdot dy = \int p(x) \cdot dx + C$$

where C is an arbitrary constant.

Example 1:

Find the general solution to $\frac{dy}{dx} = y^2 \cdot e^x dx$
Start by separating the variables to separate sides of the equation

$$\frac{1}{y^2} dy = e^x dx$$

Integrate both sides

$$\int \frac{1}{y^2} dy = \int e^x dx$$
$$\int y^{-2} dy = \int e^x dx$$
$$-y^{-1} = e^x + C$$
$$-\frac{1}{y} = e^x + C$$
$$y = -\frac{1}{e^x + C}$$

Example 2:

Find the solution to

$$\frac{dy}{dx} = \frac{4yx}{x^2+8}$$

Separate the variables to each side of the equation

$$\frac{1}{4y}dy = \frac{x}{x^2+8}dx$$

Integrate both sides

$$\int \frac{1}{4y}dy = \int \frac{x}{x^2+8}dx$$

Use U-substitution for the integration of the right hand side.

$$u = x^2 + 8 \qquad\qquad dx = \frac{du}{2x}$$

Obtaining:

$$\begin{aligned}
\frac{1}{4}\int \frac{1}{y}dy &= \int \frac{x}{u} \cdot \frac{du}{2x} \\
\frac{1}{4}\int \frac{1}{y}dy &= \int \frac{1}{u} \cdot \frac{du}{2} \\
\frac{1}{4}\int \frac{1}{y}dy &= \frac{1}{2}\int \frac{1}{u}du \\
\frac{1}{4}ln \mid y \mid &= \frac{1}{2}ln \mid u \mid +C
\end{aligned}$$

Replace u in terms of x back into the equation to get in original terms of x and y

$$\begin{aligned}
\frac{1}{4}ln \mid y \mid &= \frac{1}{2}ln \mid (x^2+8) \mid +C \\
ln \mid y \mid &= \frac{4}{2}ln \mid (x^2+8) \mid +C \\
ln \mid y \mid &= \frac{4}{2}ln(x^2+8) + C \\
e^{ln(y)} &= e^{2ln(x^2+8)+C} \\
y &= e^{ln((x^2+8)^2)} \cdot e^C \\
y &= (x^2+8)^2 \cdot e^C \\
\textbf{let:} \quad C_1 &= e^C \\
y &= C_1(x^2+8)^2
\end{aligned}$$

Example 3:

At 2PM on a cool (34°F) afternoon in March, Sherlock Holmes measured the temperature of a dead body to be at 38°F. One hour later, the temperature was at 38°F. After a quick calculation using Newton's law of cooling and taking the normal temperature of a living body to be 98°F, Holmes concluded that time of death was 10 AM. Was Holmes right?

To solve this problem we can use Newton's Law of Cooling.

$$\frac{dT}{dt} = -k(T - T_m)$$

$$T_m = 34$$

Setup:

$$\begin{aligned} \frac{dT}{dt} &= -k(T-34) && \text{[Plug in known variables]} \\ \int \frac{dT}{(T-34)} &= \int -kdt && \text{[Integrate both sides]} \end{aligned}$$

Solve for T:

$$\begin{aligned} ln(T-34) &= -kt + C \\ e^{ln(T-34)} &= e^{-kt+C} \\ T - 34 &= C_1 e^{-kt} && \text{[let: } e^C = C_1\text{]} \\ T &= C_1 e^{-kt} + 34 \end{aligned}$$

Solve for the given times to find k.
Note: t is found by taking time elapsed from t = 0 (10 AM).
At 2 PM the body's temperature was 38°F

$$\begin{aligned} T &= C_1 e^{-kt} + 34 && \text{[Using the equation]} \\ 38 &= C_1 e^{-k(4)} + 34 && \text{[t = 4, hours between 10 AM and 2 PM]} \\ 4 &= C_1 e^{-4k} \end{aligned}$$

At 3 PM the body's temperature was 36°F

$$\begin{aligned} 36 &= C_1 e^{-k(5)} + 34 && \text{[t = 5, hours between 10 AM and 3 PM]} \\ 2 &= C_1 e^{-5k} \end{aligned}$$

Solve for k.

$$\begin{aligned}\frac{4}{2} &= \frac{C_1e^{-4k}}{C_1e^{-5k}}\\ 2 &= e^k\\ ln(2) &= k\end{aligned}$$

Plug in k to solve for C_1 which is equivalent to e^C

$$\begin{aligned}4 &= C_1e^{-4ln(2)}\\ 4 &= \frac{1}{16}C_1\\ C_1 &= 64\end{aligned}$$

Substitute for $t = 0$ in the initial value equation to check if Holmes was correct about the crime occurring at 10 AM.

$$\begin{aligned}&T(0) = 34 + 64 \cdot e^{(0)(ln2)}\\ &T(0) = 34 + 64\\ &T(0) = 98\end{aligned}$$

Since the temperature at 10 AM was determined to be 98° Holmes was correct!

1.8 Homogeneous first-order DE's

1.8.1 Definition:

A function is homogeneous of degree zero if

$$f(tx, ty) = f(x, y)$$

for all positive values of t for which (tx, ty) is in the domain of f.

1.8.3 Theorem:

A function $f(x, y)$ is homogeneous of degree zero if and only if it depends on the ratio of $\frac{y}{x}$ or $\frac{x}{y}$ only.

1.8.4 Definition:

If $f(x, y)$ is homogeneous of degree zero, then the differential equation

$$\frac{dy}{dx} = f(x, y)$$

is called a homogeneous first-order differential equation.

1.8.5 Theorem:

The change of variables $y = x \cdot V(x)$ reduces a homogeneous first-order differential equation $\frac{dy}{dx} = f(x, y)$ to the separable equation

$$\frac{1}{F(V) - V} \cdot dV = \frac{1}{x} \cdot dx$$

Form of a First-Order Homogenous Differential Equation

$$\frac{dy}{dx} = F(\frac{y}{x})$$

Steps in solving a First-Order Homogenous DE

1. perform substitution: $v = \frac{y}{x} \rightarrow y = x \cdot v$ and $\frac{y}{x} = x \cdot \frac{dv}{dx} + v$
2. Solve the differential equation using separation of variables
3. Solve the original differential equation in terms of x and y

Example 1:

Determine if the following equation is homogeneous. If it is homogeneous, then solve.

$$\frac{dy}{dx} = \frac{3y^2 + xy}{x^2}$$

Multiplying both the numerator and the denominator by $\frac{1}{x^2}$ results in:

$$\frac{dy}{dx} = 3(\frac{y}{x})^2 + \frac{y}{x}$$

Which is in the form of:

$$\frac{dy}{dx} = F(\frac{y}{x})$$

Using our substitutions:

$$v = \frac{y}{x}$$
$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

We obtain:

$$x \cdot \frac{dv}{dx} + v = 3v^2 + v$$
$$x \cdot \frac{dv}{dx} = 3v^2$$

Rearrange to solve using separable DE method

$$\frac{1}{3v^2} \cdot dv = \frac{1}{x} \cdot dx$$
$$\frac{1}{3}v^{-2} \cdot dv = \frac{1}{x} \cdot dx$$
$$\int \frac{1}{3}v^{-2} \cdot dv = \int \frac{1}{x} \cdot dx$$
$$-\frac{1}{3}v^{-1} = ln \mid x \mid +C$$

Now acquire the equation in terms of x and y with our substitution ratios

$$-\frac{1}{3}v^{-1} = ln \mid x \mid +C$$
$$-\frac{1}{3}(\frac{y}{x})^{-1} = ln \mid x \mid +C$$
$$-\frac{1}{3}(\frac{x}{y}) = ln \mid x \mid +C$$
$$-\frac{1}{3}(\frac{x}{y}) = ln \mid x \mid +C$$
$$\frac{x}{y} = -3(ln \mid x \mid +C)$$
$$\frac{x}{y} = \frac{-3(ln \mid x \mid +C)}{1}$$
$$\frac{y}{x} = \frac{1}{-3(ln \mid x \mid +C)}$$
$$y = \frac{x}{-3(ln \mid x \mid +C)}$$

Example 2:

Determine if the following equation is homogeneous. If it is homogeneous, then solve.

$$\frac{dy}{dx} = \frac{y}{x + \sqrt{xy}}$$

Multiplying both the numerator and the denominator by $\frac{1}{x}$ results in:

$$\frac{dy}{dx} = \frac{(\frac{y}{x})}{\frac{x}{x} + \sqrt{\frac{xy}{x^2}}}$$

To get this in the form of:

$$\frac{dy}{dx} = F(\frac{y}{x})$$

We need to simplify, which will give us:

$$\frac{dy}{dx} = \frac{(\frac{y}{x})}{1 + \sqrt{(\frac{y}{x})}}$$

Once in the correct form, using our substitutions:

$$v = \frac{y}{x}$$
$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

We obtain:

$$x \cdot \frac{dv}{dx} + v = \frac{v}{1 + v^{\frac{1}{2}}}$$
$$x \cdot \frac{dv}{dx} = \frac{v}{1 + v^{\frac{1}{2}}} - \frac{v}{1}$$
$$x \cdot \frac{dv}{dx} = \frac{-v^{\frac{3}{2}}}{1 + v^{\frac{1}{2}}}$$

Rearrange to solve using separable DE method

$$\frac{1 + v^{\frac{1}{2}}}{-v^{\frac{3}{2}}} \cdot dv = \frac{1}{x} \cdot dx$$
$$(-v^{\frac{-3}{2}} - \frac{1}{v}) \cdot dv = \frac{1}{x} \cdot dx$$
$$\int (-v^{\frac{-3}{2}} - \frac{1}{v}) \cdot dv = \int \frac{1}{x} \cdot dx$$
$$2v^{\frac{-1}{2}} - ln \mid v \mid = ln \mid x \mid + C$$

Now acquire the equation in terms of x and y with our substitution ratios

$$2v^{\frac{-1}{2}} - ln \mid v \mid = ln \mid x \mid + C$$
$$2(\frac{x}{y})^{\frac{1}{2}} - ln \mid \frac{y}{x} \mid = ln \mid x \mid + C$$
$$2(\frac{x}{y})^{\frac{1}{2}} - (ln \mid y \mid - ln \mid x \mid) = ln \mid x \mid + C$$
$$2\sqrt{\frac{x}{y}} - ln \mid y \mid = C$$