

Separable Differential Equations and Homogeneous First-Order DE's

Donner HANSON Jack MCGRATH Sarah TOWNSEND
Anna GREENE Caitlin FELTS

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Instructor: Dr. Katherine Evans
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1.4 Separable Differential Equations

Definition 1.4.1: A first-order differential equation is called separable if it can be written in the form

$$p(y) \cdot \frac{dy}{dx} = q(x)$$

1.4.2 Theorem:

If $p(y)$ and $q(x)$ are continuous, then Equation (1.4.1) has the general solution

$$\int q(y) \cdot dy = \int p(x) \cdot dx + C$$

where C is an arbitrary constant.

Example 1:

Find the general solution to $\frac{dy}{dx} = y^2 \cdot e^x dx$

Start by separating the variables to separate sides of the equation

$$\frac{1}{y^2} dy = e^x dx$$

Integrate both sides

$$\begin{aligned}\int \frac{1}{y^2} dy &= \int e^x dx \\ \int y^{-2} dy &= \int e^x dx \\ -y^{-1} &= e^x + C \\ -\frac{1}{y} &= e^x + C \\ y &= -\frac{1}{e^x + C}\end{aligned}$$

Example 2:

Find the solution to

$$\frac{dy}{dx} = \frac{4yx}{x^2 + 8}$$

Separate the variables to each side of the equation

$$\frac{1}{4y} dy = \frac{x}{x^2 + 8} dx$$

Integrate both sides

$$\int \frac{1}{4y} dy = \int \frac{x}{x^2 + 8} dx$$

Use U-substitution for the integration of the right hand side.

$$u = x^2 + 8 \qquad dx = \frac{du}{2x}$$

Obtaining:

$$\begin{aligned} \frac{1}{4} \int \frac{1}{y} dy &= \int \frac{x}{u} \cdot \frac{du}{2x} \\ \frac{1}{4} \int \frac{1}{y} dy &= \int \frac{1}{u} \cdot \frac{du}{2} \\ \frac{1}{4} \int \frac{1}{y} dy &= \frac{1}{2} \int \frac{1}{u} du \\ \frac{1}{4} \ln | y | &= \frac{1}{2} \ln | u | + C \end{aligned}$$

Replace u in terms of x back into the equation to get in original terms of x and y

$$\begin{aligned} \frac{1}{4} \ln | y | &= \frac{1}{2} \ln | (x^2 + 8) | + C \\ \ln | y | &= \frac{4}{2} \ln | (x^2 + 8) | + C \\ \ln | y | &= \frac{4}{2} \ln(x^2 + 8) + C \\ e^{\ln(y)} &= e^{2\ln(x^2 + 8) + C} \\ y &= e^{\ln((x^2 + 8)^2)} \cdot e^C \\ y &= (x^2 + 8)^2 \cdot e^C \\ \text{let: } C_1 &= e^C \\ y &= C_1(x^2 + 8)^2 \end{aligned}$$

Example 3:

At 2PM on a cool (34°F) afternoon in March, Sherlock Holmes measured the temperature of a dead body to be at 38°F. One hour later, the temperature was at 36°F. After a quick calculation using Newton's law of cooling and taking the normal temperature of a living body to be 98°F, Holmes concluded that time of death was 10 AM. Was Holmes right?

To solve this problem we can use Newton's Law of Cooling.

$$\frac{dT}{dt} = -k(T - T_m)$$

$$T_m = 34$$

Setup:

$$\begin{aligned} \frac{dT}{dt} &= -k(T - 34) && \text{[Plug in known variables]} \\ \int \frac{dT}{(T - 34)} &= \int -k dt && \text{[Integrate both sides]} \end{aligned}$$

Solve for T:

$$\begin{aligned} \ln(T - 34) &= -kt + C \\ e^{\ln(T-34)} &= e^{-kt+C} \\ T - 34 &= C_1 e^{-kt} && \text{[let: } e^C = C_1\text{]} \\ T &= C_1 e^{-kt} + 34 \end{aligned}$$

Solve for the given times to find k.

Note: t is found by taking time elapsed from t = 0 (10 AM).

At 2 PM the body's temperature was 38°F

$$\begin{aligned} T &= C_1 e^{-kt} + 34 && \text{[Using the equation]} \\ 38 &= C_1 e^{-k(4)} + 34 && \text{[t = 4, hours between 10 AM and 2 PM]} \\ 4 &= C_1 e^{-4k} \end{aligned}$$

At 3 PM the body's temperature was 36°F

$$\begin{aligned} 36 &= C_1 e^{-k(5)} + 34 && \text{[t = 5, hours between 10 AM and 3 PM]} \\ 2 &= C_1 e^{-5k} \end{aligned}$$

Solve for k.

$$\begin{aligned}\frac{4}{2} &= \frac{C_1 e^{-4k}}{C_1 e^{-5k}} \\ 2 &= e^k \\ \ln(2) &= k\end{aligned}$$

Plug in k to solve for C_1 which is equivalent to e^C

$$\begin{aligned}4 &= C_1 e^{-4\ln(2)} \\ 4 &= \frac{1}{16} C_1 \\ C_1 &= 64\end{aligned}$$

Substitute for $t = 0$ in the initial value equation to check if Holmes was correct about the crime occurring at 10 AM.

$$\begin{aligned}T(0) &= 34 + 64 \cdot e^{(0)(\ln 2)} \\ T(0) &= 34 + 64 \\ T(0) &= 98\end{aligned}$$

Since the temperature at 10 AM was determined to be 98° Holmes was correct!

1.8 Homogeneous first-order DE's

1.8.1 Definition:

A function is homogeneous of degree zero if

$$f(tx, ty) = f(x, y)$$

for all positive values of t for which (tx, ty) is in the domain of f .

1.8.3 Theorem:

A function $f(x, y)$ is homogeneous of degree zero if and only if it depends on the ratio of $\frac{y}{x}$ or $\frac{x}{y}$ only.

1.8.4 Definition:

If $f(x, y)$ is homogeneous of degree zero, then the differential equation

$$\frac{dy}{dx} = f(x, y)$$

is called a homogeneous first-order differential equation.

1.8.5 Theorem:

The change of variables $y = x \cdot V(x)$ reduces a homogeneous first-order differential equation $\frac{dy}{dx} = f(x, y)$ to the separable equation

$$\frac{1}{F(V) - V} \cdot dV = \frac{1}{x} \cdot dx$$

Form of a First-Order Homogenous Differential Equation:

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Steps in solving a First-Order Homogenous DE

1. perform substitution: $v = \frac{y}{x} \rightarrow y = x \cdot v$ and $\frac{y}{x} = x \cdot \frac{dv}{dx} + v$
2. Solve the differential equation using separation of variables
3. Solve the original differential equation in terms of x and y

Example 1:

Determine if the following equation is homogeneous. If it is homogeneous, then solve.

$$\frac{dy}{dx} = \frac{3y^2 + xy}{x^2}$$

Multiplying both the numerator and the denominator by $\frac{1}{x^2}$ results in:

$$\frac{dy}{dx} = 3\left(\frac{y}{x}\right)^2 + \frac{y}{x}$$

Which is in the form of:

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Using our substitutions:

$$v = \frac{y}{x}$$
$$\frac{dy}{dx} = x \frac{dv}{dx} + v$$

We obtain:

$$x \cdot \frac{dv}{dx} + v = 3v^2 + v$$

$$x \cdot \frac{dv}{dx} = 3v^2$$

Rearrange to solve using separable DE method

$$\frac{1}{3v^2} \cdot dv = \frac{1}{x} \cdot dx$$

$$\frac{1}{3}v^{-2} \cdot dv = \frac{1}{x} \cdot dx$$

$$\int \frac{1}{3}v^{-2} \cdot dv = \int \frac{1}{x} \cdot dx$$

$$-\frac{1}{3}v^{-1} = \ln |x| + C$$

Now acquire the equation in terms of x and y with our substitution ratios:

$$-\frac{1}{3}v^{-1} = \ln |x| + C$$

$$-\frac{1}{3}\left(\frac{y}{x}\right)^{-1} = \ln |x| + C$$

$$-\frac{1}{3}\left(\frac{x}{y}\right) = \ln |x| + C$$

$$-\frac{1}{3}\left(\frac{x}{y}\right) = \ln |x| + C$$

$$\frac{x}{y} = -3(\ln |x| + C)$$

$$\frac{x}{y} = \frac{-3(\ln |x| + C)}{1}$$

$$\frac{y}{x} = \frac{1}{-3(\ln |x| + C)}$$

$$y = \frac{x}{-3(\ln |x| + C)}$$

Example 2:

Determine if the following equation is homogeneous. If it is homogeneous, then solve.

$$\frac{dy}{dx} = \frac{y}{x + \sqrt{xy}}$$

Multiplying both the numerator and the denominator by $\frac{1}{x}$ results in:

$$\frac{dy}{dx} = \frac{\left(\frac{y}{x}\right)}{\frac{x}{x} + \sqrt{\frac{xy}{x^2}}}$$

To get this in the form of:

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

We need to simplify, which will give us:

$$\frac{dy}{dx} = \frac{\left(\frac{y}{x}\right)}{1 + \sqrt{\left(\frac{y}{x}\right)}}$$

Once in the correct form, using our substitutions:

$$v = \frac{y}{x}$$

$$\frac{dy}{dx} = x \frac{dv}{dx} + v$$

We obtain:

$$x \cdot \frac{dv}{dx} + v = \frac{v}{1 + v^{\frac{1}{2}}}$$

$$x \cdot \frac{dv}{dx} = \frac{v}{1 + v^{\frac{1}{2}}} - \frac{v}{1}$$

$$x \cdot \frac{dv}{dx} = \frac{-v^{\frac{3}{2}}}{1 + v^{\frac{1}{2}}}$$

Rearrange to solve using separable DE method

$$\frac{1 + v^{\frac{1}{2}}}{-v^{\frac{3}{2}}} \cdot dv = \frac{1}{x} \cdot dx$$

$$\left(-v^{-\frac{3}{2}} - \frac{1}{v}\right) \cdot dv = \frac{1}{x} \cdot dx$$

$$\int \left(-v^{-\frac{3}{2}} - \frac{1}{v}\right) \cdot dv = \int \frac{1}{x} \cdot dx$$

$$2v^{-\frac{1}{2}} - \ln |v| = \ln |x| + C$$

Now acquire the equation in terms of x and y with our substitution ratios:

$$\begin{aligned}
 2v^{\frac{-1}{2}} - \ln |v| &= \ln |x| + C \\
 2\left(\frac{x}{y}\right)^{\frac{1}{2}} - \ln \left|\frac{y}{x}\right| &= \ln |x| + C \\
 2\left(\frac{x}{y}\right)^{\frac{1}{2}} - (\ln |y| - \ln |x|) &= \ln |x| + C \\
 2\sqrt{\frac{x}{y}} - \ln |y| &= C
 \end{aligned}$$

Example 3:

$$\frac{dy}{dx} = \frac{x-y}{x+y}$$

Get in the form of:

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Divide the numerator and denominator by x :

$$\frac{dy}{dx} = \frac{\frac{x}{x} - \frac{y}{x}}{\frac{x}{x} + \frac{y}{x}}$$

Simplify the equation to the correct form:

$$\frac{dy}{dx} = \frac{1 - \frac{y}{x}}{1 + \frac{y}{x}}$$

Using our substitutions:

$$\begin{aligned}
 y &= v \cdot x \\
 \frac{1-v}{1+v} &= v + x\left(\frac{dv}{dx}\right)
 \end{aligned}$$

Simplifying further to obtain:

$$x \frac{dv}{dx} = \frac{1-2v-v^2}{1+v}$$

Separate and integrate:

$$\begin{aligned}
 \int \frac{1+v}{1-2v-v^2} dv &= \int \frac{1}{x} dx \\
 -\frac{1}{2} \ln(1-2v-v^2) &= \ln(x) + C
 \end{aligned}$$

Substitute $\ln(k)$ for C:

$$-\frac{1}{2}\ln(1-2v-v^2) = \ln(x) + \ln(k)$$
$$1-2v-v^2 = \frac{1}{x^2k^2}$$

Since we have simplified the equation $1-2v-v^2 = \frac{1}{x^2k^2}$ replace $v = \frac{y}{x}$

$$1-2\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2 = \frac{1}{x^2k^2}$$
$$x^2 - 2xy - y^2 = \frac{1}{k^2}$$

Solve for y :

$$x^2 - 2xy - y^2 = \frac{1}{k^2}$$
$$y^2 + 2xy - x^2 = -\frac{1}{k^2}$$
$$y^2 + 2xy - x^2 = -\frac{1}{k^2}$$

Add $2x^2$ to both sides and consider $-\frac{1}{k^2}$ to be a constant C_1 so we can simplify:

$$y^2 + 2xy + x^2 = 2x^2 + C_1$$
$$(y+x)^2 = 2x^2 + C_1$$
$$y = \pm\sqrt{2x^2 + C_1} - x$$