Separable Differential Equations and Homogeneous First-Order DE's

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1.4 Separable Differential Equations

Definition 1.4.1: A first-order differential equation is called separable if it can be written in the form

$$p(y) \cdot \frac{dy}{dx} = q(x)$$

1.4.2 Theorem:

If p(y) and q(x) are continuous, then Equation (1.4.1) has the general solution

$$\int q(y) \cdot dy = \int p(x) \cdot dx + c$$

where c is an arbitrary constant.

Example 1:

Find the general solution to $\frac{dy}{dx} = y^2 \cdot e^x dx$
Start by separating the variables to separate sides of the equation

$$\frac{1}{y^2} dy = e^x dx$$

Integrate both sides

$$\begin{aligned} \int \frac{1}{y^2} dy &= \int e^x dx \\ \int y^{-2} dy &= \int e^x dx \\ -y^{-1} &= e^x + C \\ -\frac{1}{y} &= e^x + C \\ y &= -\frac{1}{e^x + C} \end{aligned}$$

Example 2:

Find the solution to

$$\frac{dy}{dx} = \frac{4yx}{x^2+8}$$

Separate the variables to each side of the equation

$$\frac{1}{4y}dy = \frac{x}{x^2+8}dx$$

Integrate both sides

$$\int \frac{1}{4y}dy = \int \frac{x}{x^2+8}dx$$

Use U-substitution for the integration of the right hand side.

$$u = x^2 + 8 \qquad\qquad dx = \frac{du}{2x}$$

Obtaining:

$$\begin{aligned}
\frac{1}{4}\int \frac{1}{y}dy &= \int \frac{x}{u} \cdot \frac{du}{2x} \\
\frac{1}{4}\int \frac{1}{y}dy &= \int \frac{1}{u} \cdot \frac{du}{2} \\
\frac{1}{4}\int \frac{1}{y}dy &= \frac{1}{2}\int \frac{1}{u}du \\
\frac{1}{4}ln \mid y \mid &= \frac{1}{2}ln \mid u \mid +C
\end{aligned}$$

Replace u in terms of x back into the equation to get in original terms of x and y

$$
\begin{aligned}
\frac{1}{4} ln \mid y \mid &= \frac{1}{2} ln \mid (x^2+8) \mid + C \\
ln \mid y \mid &= \frac{4}{2} ln \mid (x^2+8) \mid + C \\
ln \mid y \mid &= \frac{4}{2} ln(x^2+8) + C \\
e^{ln(y)} &= e^{2ln(x^2+8)+C} \\
y &= e^{ln((x^2+8)^2)} \cdot e^C \\
y &= (x^2+8)^2 \cdot e^C \\
\textbf{let:} \quad C_1 &= e^C \\
y &= C_1(x^2+8)^2
\end{aligned}
$$

1.8 Homogeneous first-order DE's

1.8.1 Definition:

A function is homogeneous of degree zero if

$$f(tx, ty) = f(x, y)$$

for all positive values of t for which (tx, ty) is in the domain of f.

1.8.3 Theorem:

A function $f(x, y)$ is homogeneous of degree zero if and only if it depends on the ratio of $\frac{y}{x}$ or $\frac{x}{y}$ only.

1.8.4 Definition:

If $f(x, y)$ is homogeneous of degree zero, then the differential equation

$$\frac{dy}{dx} = f(x, y)$$

is called a homogeneous first-order differential equation.

1.8.5 Theorem:

The change of variables $y = x \cdot V(x)$ reduces a homogeneous first-order differential equation $\frac{dy}{dx} = f(x, y)$ to the separable equation

$$\frac{1}{F(V)-V}\cdot dV = \frac{1}{x}\cdot dx$$

Form of a First-Order Homogenous Differential Equation

$$\frac{dy}{dx} = F(\frac{y}{x})$$

Steps in solving a First-Order Homogenous DE

1. perform substitution: $v = \frac{y}{x} \to y = x \cdot v$ and $\frac{y}{x} = x \cdot \frac{dv}{dx} + v$
2. Solve the differential equation using separation of variables
3. Solve the original differential equation in terms of x and y

[2-3 examples]
Determine if the following equation is homogeneous. If it is homogeneous, then solve.

$$\frac{dy}{dx} = \frac{3y^2 + xy}{x^2}$$

Multiplying both the numerator and the denominator by $\frac{1}{x^2}$ results in:

$$\frac{dy}{dx} = 3(\frac{y}{x})^2 + \frac{y}{x}$$

Which is in the form of:

$$\frac{dy}{dx} = F(\frac{y}{x})$$

Using our substitutions:

$$v = \frac{y}{x}$$
$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

We obtain:

$$x \cdot \frac{dv}{dx} + v = 3v^2 + v$$
$$x \cdot \frac{dv}{dx} = 3v^2$$

Rearrange to solve using separable DE method

$$\frac{1}{3v^2} \cdot dv = \frac{1}{x} \cdot dx$$
$$\frac{1}{3}v^{-2} \cdot dv = \frac{1}{x} \cdot dx$$
$$\int \frac{1}{3}v^{-2} \cdot dv = \int \frac{1}{x} \cdot dx$$
$$-\frac{1}{3}v^{-1} = ln \mid x \mid +C$$

Now aquire the equation in terms of x and y with our substitution ratios

$$-\frac{1}{3}v^{-1} = ln \mid x \mid +C$$
$$-\frac{1}{3}(\frac{y}{x})^{-1} = ln \mid x \mid +C$$
$$-\frac{1}{3}(\frac{x}{y}) = ln \mid x \mid +C$$
$$-\frac{1}{3}(\frac{x}{y}) = ln \mid x \mid +C$$
$$\frac{x}{y} = -3(ln \mid x \mid +C)$$
$$\frac{x}{y} = \frac{-3(ln \mid x \mid +C)}{1}$$
$$\frac{y}{x} = \frac{1}{-3(ln \mid x \mid +C)}$$
$$y = \frac{x}{-3(ln \mid x \mid +C)}$$