Normal Distributions

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January 2019

1 Introduction

Approximate a mixture of two normal distributions with a normal distribution and calculate the error of the approximation

We will use

$$\mathcal{N}(\mu, \sigma^2 | x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{1}$$

Let $A = \mathcal{N}(\mu_a, \sigma_a^2)$ and $B = \mathcal{N}(\mu_b, \sigma_b^2)$ then using a mixture factor $\alpha \in [0, 1]$ the mixture $A \oplus_\alpha B$ is

$$A \oplus_\alpha B = \alpha \mathcal{N}(\mu_a, \sigma_a^2) + (1-\alpha)\mathcal{N}(\mu_b, \sigma_b^2) \tag{2}$$

$$= \frac{\alpha}{\sqrt{2\pi}\sigma_a} e^{\frac{-(x-\mu_a)^2}{2\sigma_a^2}} + \frac{1-\alpha}{\sqrt{2\pi}\sigma_b} e^{\frac{-(x-\mu_b)^2}{2\sigma_b^2}} \tag{3}$$

We can calculate the expected value and standard deviation of this distribution to construct an approximate normal distribution C.

$$\mu = E\left[A \oplus_\alpha B\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(x \frac{\alpha}{\sqrt{2\pi}\sigma_a} e^{\frac{-(x-\mu_a)^2}{2\sigma_a^2}} + x \frac{1-\alpha}{\sqrt{2\pi}\sigma_b} e^{\frac{-(x-\mu_b)^2}{2\sigma_b^2}} \right) dx \tag{4}$$

$$\mu = \frac{\alpha}{\sqrt{2\pi}\sigma_a} \int_{-\infty}^{\infty} x e^{\frac{-(x-\mu_a)^2}{2\sigma_a^2}} dx + \frac{1-\alpha}{\sqrt{2\pi}\sigma_b} \int_{-\infty}^{\infty} x e^{\frac{-(x-\mu_b)^2}{2\sigma_b^2}} dx \tag{5}$$

$$\mu = \alpha E[A] + (1-\alpha) E[B] \tag{6}$$

$$\mu = \alpha \mu_a + (1-\alpha)\mu_b \tag{7}$$

The variance is more difficult to calculate:

$$\mathrm{Var}\left[A \oplus_\alpha B\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 \left(\frac{\alpha}{\sigma_a} e^{\frac{-(x-\mu_a)^2}{2\sigma_a^2}} + \frac{1-\alpha}{\sigma_b} e^{\frac{-(x-\mu_b)^2}{2\sigma_b^2}} \right) dx \tag{8}$$

$$= \frac{\alpha}{\sqrt{2\pi}\sigma_a}\int_{-\infty}^{\infty}(x-\mu)^2 e^{\frac{-(x-\mu_a)^2}{2\sigma_a^2}}dx + \frac{1-\alpha}{\sqrt{2\pi}\sigma_b}\int_{-\infty}^{\infty}(x-\mu)^2 e^{\frac{-(x-\mu_b)^2}{2\sigma_b^2}}dx \tag{9}$$

$$= \frac{\alpha}{\sqrt{2\pi}\sigma_a}\int_{-\infty}^{\infty}(x^2-2\mu x+\mu^2) e^{\frac{-(x-\mu_a)^2}{2\sigma_a^2}}dx + \frac{1-\alpha}{\sqrt{2\pi}\sigma_b}\int_{-\infty}^{\infty}(x^2-2\mu x+\mu^2) e^{\frac{-(x-\mu_b)^2}{2\sigma_b^2}}dx \tag{10}$$

Let's take just the left integral for now

$$= \frac{\alpha}{\sqrt{2\pi}\sigma_a}\left(\int_{-\infty}^{\infty}x^2 e^{\frac{-(x-\mu_a)^2}{2\sigma_a^2}}dx - 2\mu\int_{-\infty}^{\infty}x e^{\frac{-(x-\mu_a)^2}{2\sigma_a^2}}dx + \mu^2\int_{-\infty}^{\infty}e^{\frac{-(x-\mu_a)^2}{2\sigma_a^2}}dx\right)+\dots \tag{11}$$

$$= \alpha\left(\frac{1}{\sqrt{2\pi}\sigma_a}\int_{-\infty}^{\infty}x^2 e^{\frac{-(x-\mu_a)^2}{2\sigma_a^2}}dx - \frac{2\mu}{\sqrt{2\pi}\sigma_a}\int_{-\infty}^{\infty}x e^{\frac{-(x-\mu_a)^2}{2\sigma_a^2}}dx + \frac{\mu^2}{\sqrt{2\pi}\sigma_a}\int_{-\infty}^{\infty}e^{\frac{-(x-\mu_a)^2}{2\sigma_a^2}}dx\right)+\dots \tag{12}$$

The second term in the parenthesis is the formula for $2\mu E[\mathcal{N}(\mu_a, \sigma_a^2)] = 2\mu\mu_a$, and the third is simply equal to μ^2

$$= \alpha\left(-2\mu\mu_a + \mu^2 + \frac{1}{\sqrt{2\pi}\sigma_a}\int_{-\infty}^{\infty}x^2 e^{\frac{-(x-\mu_a)^2}{2\sigma_a^2}}dx\right)+\dots \tag{13}$$

add some terms we will need later

$$= \alpha\left(-2\mu\mu_a + \mu^2 + \frac{1}{\sqrt{2\pi}\sigma_a}\int_{-\infty}^{\infty}(x^2 + 2\mu_a x - 2\mu_a x - \mu_a^2 + \mu_a^2) e^{\frac{-(x-\mu_a)^2}{2\sigma_a^2}}dx\right)+\dots \tag{14}$$

$$= \alpha\left(-2\mu\mu_a + \mu^2 + \frac{1}{\sqrt{2\pi}\sigma_a}\int_{-\infty}^{\infty}((x-\mu_a)^2 + 2\mu_a x - \mu_a^2) e^{\frac{-(x-\mu_a)^2}{2\sigma_a^2}}dx\right)+\dots \tag{15}$$

$$= \alpha\left(-2\mu\mu_a + \mu^2 + \frac{1}{\sqrt{2\pi}\sigma_a}\int_{-\infty}^{\infty}(x-\mu_a)^2 e^{\frac{-(x-\mu_a)^2}{2\sigma_a^2}}dx + \frac{1}{\sqrt{2\pi}\sigma_a}\int_{-\infty}^{\infty}(2\mu_a x - \mu_a^2) e^{\frac{-(x-\mu_a)^2}{2\sigma_a^2}}dx\right)+\dots \tag{16}$$

The third term in parenthesis is $\mathrm{Var}[\mathcal{N}(\mu_a, \sigma_a^2)] = \sigma_a^2$

$$= \alpha\left(-2\mu\mu_a + \mu^2 + \sigma_a^2 + \frac{1}{\sqrt{2\pi}\sigma_a}\int_{-\infty}^{\infty}(2\mu_a x - \mu_a^2) e^{\frac{-(x-\mu_a)^2}{2\sigma_a^2}}dx\right)+\dots \tag{17}$$

$$= \alpha \left(-2\mu\mu_a + \mu^2 + \sigma_a^2 + \frac{2\mu_a}{\sqrt{2\pi}\sigma_a} \int_{-\infty}^{\infty} x e^{\frac{-(x-\mu_a)^2}{2\sigma_a^2}} dx - \frac{\mu_a^2}{\sqrt{2\pi}\sigma_a} \int_{-\infty}^{\infty} e^{\frac{-(x-\mu_a)^2}{2\sigma_a^2}} dx \right) + \dots \tag{18}$$

The fourth term in parenthesis is $2\mu_a E[\mathcal{N}(\mu_a, \sigma_a^2)] = 2\mu_a^2$ and the last term is equal to μ_a^2

$$= \alpha \left(-2\mu\mu_a + \mu^2 + \sigma_a^2 + 2\mu_a^2 - \mu_a^2 \right) + \dots \tag{19}$$

$$= \alpha \left(-2\mu\mu_a + \mu^2 + \sigma_a^2 + \mu_a^2 \right) + \dots \tag{20}$$

$$= \alpha \left(\sigma_a^2 + (\mu - \mu_a)^2 \right) + \dots \tag{21}$$

The right side is symmetric so we have:

$$\mathrm{Var}\left[A \oplus_\alpha B\right] = \alpha \left(\sigma_a^2 + (\mu - \mu_a)^2 \right) + (1 - \alpha) \left(\sigma_b^2 + (\mu - \mu_b)^2 \right) \tag{22}$$

This makes intuitive sense. The variance of the weighted sum of two Gaussians is the weighted sum of the variances plus the squared difference between each mean and the new mean.

We will substitute in for μ to get an alternative formula for $\mathrm{Var}\left[A \oplus_\alpha B\right]$

$$\begin{aligned} &\alpha \left(\sigma_a^2 + (\mu - \mu_a)^2 \right) + (1 - \alpha) \left(\sigma_b^2 + (\mu - \mu_b)^2 \right) \\ &= \alpha\sigma_a^2 + (1 - \alpha)\sigma_b^2 + \alpha(1 - \alpha)(\mu_a - \mu_b)^2 \end{aligned} \tag{23}$$

The error of this new distribution to approximate the mixture is:

$$\epsilon^2 = \int_{-\infty}^{\infty} \left(A \oplus_\alpha B - \mathcal{N}(\mu, \sigma^2 | x) \right)^2 dx \tag{24}$$

$$\epsilon^2 = \int_{-\infty}^{\infty} \left(\alpha\mathcal{N}(\mu_a, \sigma_a^2 | x) + (1 - \alpha)\mathcal{N}(\mu_b, \sigma_b^2 | x) - \mathcal{N}(\mu, \sigma^2 | x) \right)^2 dx \tag{25}$$

$$\begin{aligned}\epsilon^2 = \alpha^2 \int_{-\infty}^{\infty} \mathcal{N}^2(\mu_a, \sigma_a^2|x)dx \\ +(1-\alpha)^2 \int_{-\infty}^{\infty} \mathcal{N}^2(\mu_b, \sigma_b^2|x)dx \\ +\int_{-\infty}^{\infty} \mathcal{N}^2(\mu, \sigma^2|x)dx \\ +2\alpha(1-\alpha) \int_{-\infty}^{\infty} \mathcal{N}(\mu_a, \sigma_a^2|x)\mathcal{N}(\mu_b, \sigma_b^2|x)dx \\ -2\alpha \int_{-\infty}^{\infty} \mathcal{N}(\mu_a, \sigma_a^2|x)\mathcal{N}(\mu, \sigma^2|x)dx \\ -2(1-\alpha) \int_{-\infty}^{\infty} \mathcal{N}(\mu_b, \sigma_b^2|x)\mathcal{N}(\mu, \sigma^2|x)dx\end{aligned} \tag{26}$$

Each term is the product of two Gaussians so we will calculate the following and substitute:

$$P(\mu, \sigma^2; \nu, \tau^2) = \int_{-\infty}^{\infty} \mathcal{N}(\mu, \sigma^2|x)\mathcal{N}(\nu, \tau^2|x)dx \tag{27}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi}\tau} e^{-\frac{(x-\nu)^2}{2\tau^2}} dx \tag{28}$$

$$= \frac{1}{2\pi\sigma\tau} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} e^{-\frac{(x-\nu)^2}{2\tau^2}} dx \tag{29}$$

$$= \frac{1}{2\pi\sigma\tau} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2} - \frac{(x-\nu)^2}{2\tau^2}} dx \tag{30}$$

$$= \frac{1}{2\pi\sigma\tau} \int_{-\infty}^{\infty} e^{\frac{1}{2\sigma^2\tau^2}\left(-\tau^2(x-\mu)^2 - \sigma^2(x-\nu)^2\right)} dx \tag{31}$$

$$= \frac{1}{2\pi\sigma\tau} \int_{-\infty}^{\infty} e^{\frac{1}{2\sigma^2\tau^2}\left(-\tau^2(x^2-2\mu x+\mu^2) - \sigma^2(x^2-2\nu x+\nu^2)\right)} dx \tag{32}$$

$$= \frac{1}{2\pi\sigma\tau} \int_{-\infty}^{\infty} e^{\frac{1}{2\sigma^2\tau^2}\left(-\tau^2x^2+2\tau^2\mu x-\tau^2\mu^2-\sigma^2x^2+2\sigma^2\nu x-\sigma^2\nu^2\right)} dx \tag{33}$$

$$= \frac{1}{2\pi\sigma\tau} \int_{-\infty}^{\infty} e^{\frac{1}{2\sigma^2\tau^2}\left(-(\sigma^2+\tau^2)x^2+2(\sigma^2\nu+\tau^2\mu)x-(\sigma^2\nu^2+\tau^2\mu^2)\right)} dx \tag{34}$$

Now we complete the square in the exponent:

$$= \frac{1}{2\pi\sigma\tau} \int_{-\infty}^{\infty} e^{\frac{1}{2\sigma^2\tau^2}\left(-(\sigma^2+\tau^2)\left(x-\frac{\tau^2\mu+\sigma^2\nu}{\sigma^2+\tau^2}\right)^2-(\sigma^2\nu^2+\tau^2\mu^2)+\frac{(\tau^2\mu+\sigma^2\nu)^2}{\sigma^2+\tau^2}\right)} dx \tag{35}$$

$$= \frac{1}{2\pi\sigma\tau} \int_{-\infty}^{\infty} e^{\frac{\sigma^2+\tau^2}{2\sigma^2\tau^2}\left(-\left(x-\frac{\tau^2\mu+\sigma^2\nu}{\sigma^2+\tau^2}\right)^2-\frac{\sigma^2\nu^2+\tau^2\mu^2}{\sigma^2+\tau^2}+\left(\frac{\tau^2\mu+\sigma^2\nu}{\sigma^2+\tau^2}\right)^2\right)} dx \tag{36}$$

We will let $\rho^2 = \frac{\sigma^2\tau^2}{\sigma^2+\tau^2}$ and $\lambda = \frac{\tau^2\mu+\sigma^2\nu}{\sigma^2+\tau^2}$ which when substituted gives:

$$= \frac{1}{2\pi\sigma\tau} \int_{-\infty}^{\infty} e^{\frac{1}{2\rho^2}\left(-(x-\lambda)^2 - \frac{\sigma^2\nu^2+\tau^2\mu^2}{\sigma^2+\tau^2} + \lambda^2\right)} dx \tag{37}$$

$$= \frac{1}{2\pi\sigma\tau} \int_{-\infty}^{\infty} e^{-\frac{(x-\lambda)^2}{2\rho^2}} e^{-\frac{\sigma^2\nu^2+\tau^2\mu^2}{2\rho^2(\sigma^2+\tau^2)}} e^{\frac{\lambda^2}{2\rho^2}} dx \tag{38}$$

$$= \frac{1}{2\pi\sigma\tau} e^{-\frac{\sigma^2\nu^2+\tau^2\mu^2}{2\rho^2(\sigma^2+\tau^2)}} e^{\frac{\lambda^2}{2\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{(x-\lambda)^2}{2\rho^2}} dx \tag{39}$$

Substituting back in for ρ^2 in the first exponent:

$$= \frac{1}{2\pi\sigma\tau} e^{-\frac{\sigma^2\nu^2+\tau^2\mu^2}{2\frac{\sigma^2\tau^2}{\sigma^2+\tau^2}(\sigma^2+\tau^2)}} e^{\frac{\lambda^2}{2\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{(x-\lambda)^2}{2\rho^2}} dx \tag{40}$$

$$= \frac{1}{2\pi\sigma\tau} e^{-\frac{\sigma^2\nu^2+\tau^2\mu^2}{2\sigma^2\tau^2}} e^{\frac{\lambda^2}{2\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{(x-\lambda)^2}{2\rho^2}} dx \tag{41}$$

$$= \frac{1}{2\pi\sigma\tau} e^{-\frac{\nu^2}{2\tau^2}} e^{-\frac{\mu^2}{2\sigma^2}} e^{\frac{\lambda^2}{2\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{(x-\lambda)^2}{2\rho^2}} dx \tag{42}$$

The integral is a Gaussian with mean λ and variance ρ^2 and we know that $\int_{-\infty}^{\infty} \mathcal{N}(\lambda, \rho^2|x)dx = \sqrt{2\pi}\rho$

$$= \frac{1}{2\pi\sigma\tau} e^{-\frac{\nu^2}{2\tau^2}} e^{-\frac{\mu^2}{2\sigma^2}} e^{\frac{\lambda^2}{2\rho^2}} \sqrt{2\pi}\rho \tag{43}$$

$$P(\mu, \sigma^2; \nu, \tau^2) = \int_{-\infty}^{\infty} \mathcal{N}(\mu, \sigma^2)\mathcal{N}(\nu, \tau)dx = \frac{1}{\sqrt{2\pi}} \frac{\rho}{\sigma\tau} e^{-\frac{\nu^2}{2\tau^2}} e^{-\frac{\mu^2}{2\sigma^2}} e^{\frac{\lambda^2}{2\rho^2}} \tag{44}$$

Where $\rho^2 = \frac{\sigma^2\tau^2}{\sigma^2+\tau^2}$ and $\lambda = \frac{\tau^2\mu+\sigma^2\nu}{\sigma^2+\tau^2}$

As an aside we can calculate $P(\mu, \sigma^2; \mu, \sigma^2) = \int_{-\infty}^{\infty} \mathcal{N}^2(\mu, \sigma^2|x)dx$. In this special case we have the following values for λ and ρ^2

$$\lambda = \frac{\sigma^2\mu + \sigma^2\mu}{\sigma^2 + \sigma^2} = \frac{2\sigma^2\mu}{2\sigma^2} = \mu \tag{45}$$

$$\rho^2 = \frac{\sigma^4}{2\sigma^2} = \frac{1}{2}\sigma^2 \tag{46}$$

$$P(\mu, \sigma^2; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}} \frac{\rho}{\sigma^2} e^{-\frac{\mu^2}{2\sigma^2}} e^{-\frac{\mu^2}{2\sigma^2}} e^{\frac{\lambda^2}{2\rho^2}} \tag{47}$$

$$P(\mu, \sigma^2; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}} \frac{\rho}{\sigma^2} e^{-\frac{\mu^2}{2\sigma^2}} e^{-\frac{\mu^2}{2\sigma^2}} e^{\frac{\lambda^2}{2\rho^2}} \tag{48}$$

$$P(\mu, \sigma^2; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}} \frac{\rho}{\sigma^2} e^{-\frac{\mu^2}{\sigma^2}} e^{\frac{\lambda^2}{2\rho^2}} \tag{49}$$

Substituting the special-case values above for ρ^2 and λ gives

$$
\begin{aligned}
P(\mu,\sigma^2;\mu,\sigma^2) = \int_{-\infty}^{\infty} \mathcal{N}^2(\mu,\sigma^2)dx =& \frac{1}{\sqrt{2\pi}} \frac{\sigma}{\sqrt{2\sigma^2}} e^{-\frac{\mu^2}{\sigma^2}} e^{\frac{\mu^2}{\sigma^2}} \\
=& \frac{1}{2\sqrt{\pi}\sigma}
\end{aligned}
\tag{50}
$$

Next we will substitute ρ^2 and λ into 44:

$$
\begin{aligned}
P(\mu,\sigma^2;\nu,\tau^2) =& \frac{1}{\sqrt{2\pi}} \frac{\rho}{\sigma\tau} e^{-\frac{\nu^2}{2\tau^2}} e^{-\frac{\mu^2}{2\sigma^2}} e^{\frac{\lambda^2}{2\rho^2}} \\
=& \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\frac{\sigma^2\tau^2}{\sigma^2+\tau^2}}}{\sigma\tau} e^{-\frac{\nu^2}{2\tau^2}} e^{-\frac{\mu^2}{2\sigma^2}} e^{\frac{\left(\frac{\tau^2\mu+\sigma^2\nu}{\sigma^2+\tau^2}\right)^2}{2\frac{\sigma^2\tau^2}{\sigma^2+\tau^2}}} \\
=& \frac{1}{\sqrt{2\pi\left(\sigma^2+\tau^2\right)}} e^{-\frac{\nu^2}{2\tau^2}} e^{-\frac{\mu^2}{2\sigma^2}} e^{\frac{\left(\tau^2\mu+\sigma^2\nu\right)^2}{2\sigma^2\tau^2\left(\sigma^2+\tau^2\right)}} \\
=& \frac{1}{\sqrt{2\pi\left(\sigma^2+\tau^2\right)}} e^{-\frac{\nu^2}{2\tau^2}-\frac{\mu^2}{2\sigma^2}+\frac{\left(\tau^2\mu+\sigma^2\nu\right)^2}{2\sigma^2\tau^2\left(\sigma^2+\tau^2\right)}} \\
=& \frac{1}{\sqrt{2\pi\left(\sigma^2+\tau^2\right)}} e^{\frac{-\sigma^2\left(\sigma^2+\tau^2\right)\nu^2-\tau^2\left(\sigma^2+\tau^2\right)\mu^2+\left(\tau^2\mu+\sigma^2\nu\right)^2}{2\sigma^2\tau^2\left(\sigma^2+\tau^2\right)}} \\
=& \frac{1}{\sqrt{2\pi\left(\sigma^2+\tau^2\right)}} e^{\frac{-\sigma^4\nu^2-\sigma^2\tau^2\nu^2-\sigma^2\tau^2\mu^2-\tau^4\mu^2+\tau^4\mu^2+2\sigma^2\tau^2\mu\nu+\sigma^4\nu^2}{2\sigma^2\tau^2\left(\sigma^2+\tau^2\right)}} \\
=& \frac{1}{\sqrt{2\pi\left(\sigma^2+\tau^2\right)}} e^{\frac{-\sigma^2\tau^2\nu^2+2\sigma^2\tau^2\mu\nu-\sigma^2\tau^2\mu^2}{2\sigma^2\tau^2\left(\sigma^2+\tau^2\right)}} \\
=& \frac{1}{\sqrt{2\pi\left(\sigma^2+\tau^2\right)}} e^{\frac{-\nu^2+2\mu\nu-\mu^2}{2\left(\sigma^2+\tau^2\right)}} \\
=& \frac{1}{\sqrt{2\pi\left(\sigma^2+\tau^2\right)}} e^{\frac{-(\mu-\nu)^2}{2\left(\sigma^2+\tau^2\right)}} \\
=& \mathcal{N}(\mu,\sigma^2+\tau^2|\nu) = \mathcal{N}(\nu,\sigma^2+\tau^2|\mu)
\end{aligned}
\tag{51}
$$

That bears repeating:

$$
\int_{-\infty}^{\infty} \mathcal{N}(\mu,\sigma^2|x)\mathcal{N}(\nu,\tau^2|x)dx = \mathcal{N}(\mu,\sigma^2+\tau^2|\nu) = \mathcal{N}(\nu,\sigma^2+\tau^2|\mu) \tag{52}
$$

Now we substitute into 26:

$$
\begin{aligned}
\epsilon^2 =& \frac{\alpha^2}{2\sqrt{\pi}\sigma_a} + \frac{(1-\alpha)^2}{2\sqrt{\pi}\sigma_b} + \frac{1}{2\sqrt{\pi}\sigma} \\
&+ \frac{2\alpha(1-\alpha)}{\sqrt{2\pi\left(\sigma_a^2+\sigma_b^2\right)}} e^{-\frac{(\mu_a-\mu_b)^2}{2(\sigma_a^2+\sigma_b^2)}} \\
&- \frac{2\alpha}{\sqrt{2\pi\left(\sigma_a^2+\sigma^2\right)}} e^{-\frac{(\mu-\mu_a)^2}{2(\sigma^2+\sigma_a^2)}} \\
&- \frac{2(1-\alpha)}{\sqrt{2\pi\left(\sigma_b^2+\sigma^2\right)}} e^{-\frac{(\mu-\mu_b)^2}{2(\sigma^2+\sigma_b^2)}}
\end{aligned} \tag{53}
$$

Or written a different way

$$
\begin{aligned}
\epsilon^2 =& \frac{\alpha^2}{2\sqrt{\pi}\sigma_a} + \frac{(1-\alpha)^2}{2\sqrt{\pi}\sigma_b} + \frac{1}{2\sqrt{\pi}\sigma} \\
&+ 2\alpha(1-\alpha)\mathcal{N}(\mu_a, \sigma_a^2+\sigma_b^2|\mu_b) \\
&- 2\alpha\mathcal{N}(\mu, \sigma^2+\sigma_a^2|\mu_a) \\
&- 2(1-\alpha)\mathcal{N}(\mu, \sigma^2+\sigma_b^2|\mu_b)
\end{aligned} \tag{54}
$$

We will also need to estimate the difference between two Gaussian distributions which we will do as the following

$$
\begin{aligned}
\epsilon^2 &= \int_{-\infty}^{\infty} \left(\mathcal{N}(\mu,\sigma^2|x) - \mathcal{N}(\nu,\tau^2|x)\right)^2 dx \\
&= \int_{-\infty}^{\infty} \mathcal{N}^2(\mu,\sigma^2|x)dx - 2\int_{-\infty}^{\infty} \mathcal{N}(\mu,\sigma^2|x)\mathcal{N}(\nu,\tau^2|x)dx + \int_{-\infty}^{\infty} \mathcal{N}^2(\nu,\tau^2|x)dx \\
&= \mathcal{N}(\mu, 2\sigma^2|\mu) - 2\mathcal{N}(\mu, \sigma^2+\tau^2|\nu) + \mathcal{N}(\nu, 2\tau^2|\nu) \\
&= \frac{1}{2\sqrt{\pi}\sigma} + \frac{1}{2\sqrt{\pi}\tau} - \frac{2}{\sqrt{2\pi\left(\sigma^2+\tau^2\right)}} e^{-\frac{(\mu-\nu)^2}{2(\sigma^2+\tau^2)}}
\end{aligned} \tag{55}
$$

If we evaluate this when the distributions are equal ($\sigma = \tau$ and $\mu = \nu$) we get:

$$
\epsilon^2|_0 = \frac{1}{2\sqrt{\pi}\sigma} + \frac{1}{2\sqrt{\pi}\sigma} - \frac{2}{2\sqrt{\pi}\sigma} = 0 \tag{56}
$$

And as the difference between the means approaches ∞ we have

$$
\epsilon^2|_\infty = \frac{1}{2\sqrt{\pi}\sigma} + \frac{1}{2\sqrt{\pi}\tau} = \frac{1}{2\sqrt{\pi}} \left(\frac{\sigma+\tau}{\sigma\tau}\right) \tag{57}
$$

Since we want a value to go from zero to one we will divide by the

$$
\begin{aligned}
2\sqrt{\pi}\frac{\sigma\tau}{\sigma+\tau}\epsilon^2 &= \frac{\tau}{\sigma+\tau} + \frac{\sigma}{\sigma+\tau} - 2\left(\frac{\sigma\tau}{\sigma+\tau}\right)\sqrt{\frac{2}{\sigma^2+\tau^2}}e^{-\frac{(\mu-\nu)^2}{2(\sigma^2+\tau^2)}} \\
&= 1 - 2\left(\frac{\sigma\tau}{\sigma+\tau}\right)\sqrt{\frac{2}{\sigma^2+\tau^2}}e^{-\frac{(\mu-\nu)^2}{2(\sigma^2+\tau^2)}} \\
&= 1 - 2\left(\frac{\sigma\tau}{\sigma+\tau}\right)\sqrt{\frac{2}{\sigma^2+\tau^2}}\sqrt{2\pi\left(\sigma^2+\tau^2\right)}\mathcal{N}\left(\nu,\sigma^2+\tau^2|\mu\right) \\
&= 1 - 4\sqrt{\pi}\left(\frac{\sigma\tau}{\sigma+\tau}\right)\mathcal{N}\left(\nu,\sigma^2+\tau^2|\mu\right)
\end{aligned} \tag{58}
$$

We will also be interested to know when the mixture of two Gaussians has a variance less than one of the constituent Gaussians. We will assume without loss of generality that $\sigma > \tau$ so $\sigma - \tau > 0$

$$
\begin{aligned}
\rho^2 &= \mathrm{Var}\left[\alpha\mathcal{N}\left(\mu,\sigma^2|x\right) + (1-\alpha)\mathcal{N}(\nu,\tau^2|x)\right] \\
\rho^2 &= \alpha\sigma^2 + (1-\alpha)\tau^2 + \alpha(1-\alpha)(\mu-\nu)^2 \\
\rho^2 &= \alpha\sigma^2 + (1-\alpha)(\sigma^2 - (\sigma^2-\tau^2)) + \alpha(1-\alpha)(\mu-\nu)^2 \\
\rho^2 &= \alpha\sigma^2 + \sigma^2 - \alpha\sigma^2 - (1-\alpha)(\sigma^2-\tau^2) + \alpha(1-\alpha)(\mu-\nu)^2 \\
\frac{\rho^2-\sigma^2}{1-\alpha} &= \alpha(\mu-\nu)^2 - (\sigma^2-\tau^2)
\end{aligned} \tag{59}
$$

So when $\sigma^2 - \tau^2 > \alpha(\mu-\nu)$ then $\sigma^2 > \rho^2$. Let's find the zeros of this function:

$$
\begin{aligned}
0 &< \alpha(\mu-\nu)^2 - (\sigma^2-\tau^2) \\
\sigma^2-\tau^2 &< \alpha(\mu-\nu)^2 \\
\frac{\sigma^2-\tau^2}{(\mu-\nu)^2} &< \alpha \\
\frac{(\sigma+\tau)(\sigma-\tau)}{(\mu-\nu)^2} &< \alpha
\end{aligned} \tag{60}
$$