Theorem.

$$\int_0^1 x^{x^{1+x^{1+x^{1+x^{1+x^{\cdot^{\cdot^{\cdot}}}}}}}}\,dx = \log 2$$

Proof. $f(x) := x^{x^{1+x^{1+x^{1+x^{1+x^{\cdot^{\cdot^{\cdot}}}}}}}} = x^{xx^{x^{1+x^{1+x^{1+x^{\cdot^{\cdot^{\cdot}}}}}}}} = x^{xx^{xx^{x^{1+x^{1+x^{\cdot^{\cdot^{\cdot}}}}}}}} = \cdots = x^{xx^{xx^{xx^{xx^{xx^{\cdot^{\cdot^{\cdot}}}}}}}} = (x^x)^{x^{xx^{xx^{xx^{xx^{\cdot^{\cdot^{\cdot}}}}}}}} = (x^x)^{(x^x)^{x^{xx^{xx^{xx^{\cdot^{\cdot^{\cdot}}}}}}}} = \cdots = (x^x)^{(x^x)^{(x^x)^{(x^x)^{(x^x)^{\cdot^{\cdot^{\cdot}}}}}}} = (x^x)^{f(x)}$

We can take the natural logarithm of both sides to get:

$\log f(x) = e^{\log f(x)} x \log x \Rightarrow \log\left(f(x)^{-1}\right) e^{\log\left(f(x)^{-1}\right)} = -x \log x$

Now we will define a new function g using an "inverse definition":

$g^{-1}(x) := xe^x \Rightarrow \log\left(f(x)^{-1}\right) = g(-x\log x) \Rightarrow f(x)^{-1} = \exp(g(-x\log x)) \Rightarrow f(x) = \exp(g(-x\log x))^{-1}$

If we now use Lemma 1 we can get:

$f(x) = \frac{g(-x\log x)}{-x\log x}$

If we now use Lemma 2 we can get:

$f(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} (-x\log x)^{n-1}$

So Let's go back to the integral:

$\int_0^1 x^{x^{1+x^{1+x^{1+x^{1+x^{\cdot^{\cdot^{\cdot}}}}}}}}\,dx = \int_0^1 f(x)\,dx = \int_0^1 \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} (-x\log x)^{n-1}\,dx = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} \int_0^1 (-x\log x)^{n-1}\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-n)^{n-1}}{n!} \int_0^1 (x\log x)^{n-1} = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} \int_0^1 (x\log x)^{n-1}$

If we now use Lemma 3 we can get:

$\int_0^1 x^{x^{1+x^{1+x^{1+x^{1+x^{\cdot^{\cdot^{\cdot}}}}}}}}\,dx = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} \left(-\left(\frac{-1}{n}\right)^n \Gamma(n)\right) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

So using Lemma 4 we get:

$\int_0^1 x^{x^{1+x^{1+x^{1+x^{1+x^{\cdot^{\cdot^{\cdot}}}}}}}}\,dx = \log 2$

$\square$

Lemma 1.
$$g^{-1}(x) \coloneqq xe^{x} \Rightarrow \exp(g(x)) = \frac{x}{g(x)}$$

Proof. We know that: $g^{-1}(x) = xe^{x}$ so we can substitute x with $g(x)$ and get:

$g^{-1}(g(x)) = g(x)e^{g(x)}$ therefore, using the definition of an inverse we get:

$x = g(x)\exp(g(x))$ and then, we can devide both sides by $g(x)$ to get $\exp(g(x)) = \frac{x}{g(x)}$

□

Lemma 2.
$$g^{-1}(x) \coloneqq xe^{x} \Rightarrow g(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^{n}$$

Proof. We know by definition that: $x = g(x)e^{g(x)}$, let's derive both sides:

$1 = (g(x)+1)e^{g(x)}\frac{dg}{dx}$ if we multiply both sides by g and use the definition we get:

$g \stackrel{*}{=} (g+1)x\frac{dg}{dx} \Rightarrow g - x\frac{dg}{dx} = xg\frac{dg}{dx}$

So if we say that $a_n \coloneqq \frac{1}{\Gamma(n)}\frac{d^n g}{dx^n}\ |_{x=0}$ then:

$(1-n)a_n = \sum_{k=1}^{n-1} k a_k a_{n-k} = \frac{n}{2}\sum_{k=1}^{n-1} a_k a_{n-k}$

We can now divide both sides by $1-n$ and we will get:

$a_n = \frac{n}{2-2n}\sum_{k=1}^{n-1} a_k a_{n-k}$

And we know that $a_1 = \frac{dg}{dx}\ |_{x=0} \stackrel{*}{=} \lim_{x\to 0}\frac{g(x)}{xg(x)+x} = \lim_{x\to 0}\frac{g(x)}{x} = 1$ so using Lemma 5 and Taylor's theorem we get that:

$g(x) = \sum_{n=1}^{\infty}\frac{(-n)^{n-1}}{n!}x^n$

□

Lemma 3.
$$\int_0^1 (x\log x)^{n-1} = -\left(\frac{-1}{n}\right)^n \Gamma(n)$$

Proof. If we substitute x with $\exp\left(-\frac{x}{n}\right)$ in $\int_0^1 (x\log x)^{n-1}\ dx$ we will get:

$\int_{\infty}^{0}\left(-\frac{x}{n}\exp\left(-\frac{x}{n}\right)\right)^{n-1}\left(-\frac{1}{n}\exp\left(-\frac{x}{n}\right)\right)\ dx = -\frac{1}{n}\int_{\infty}^{0}\left(-\frac{x}{n}\right)^{n-1}\cdot\exp(-x)\ dx = -\left(-\frac{1}{n}\right)^n\int_0^{\infty} x^{n-1}\cdot\exp(-x)\ dx$

So using the definition of the Gamma function we get: $-\left(\frac{-1}{n}\right)^n\Gamma(n)$

□

Lemma 4.

$$-\sum_{n=1}^{\infty}\frac{(-x)^n}{n}=\log|x+1|$$

Proof. We know that $\frac{1}{1-q}=\sum_{n=0}^{\infty}q^n$ so if we substitute q with $-q$ we get $\frac{1}{1+q}=\sum_{n=0}^{\infty}(-q)^n$

Let's integrate over q from 0 to x on both side:

$\log|1+x|=\int_0^x\frac{1}{1+q}\,dq=\int_0^x\sum_{n=0}^{\infty}(-q)^n\,dq=\sum_{n=0}^{\infty}\int_0^x(-q)^n\,dq=\sum_{n=0}^{\infty}\frac{-(-x)^{n+1}}{n+1}=-\sum_{n=1}^{\infty}\frac{(-x)^n}{n}$ ∎

Lemma 5.

$$a_n=\begin{cases}1 & n=1\\ \frac{n}{2-2n}\sum_{k=1}^{n-1}a_ka_{n-k} & else\end{cases}\Leftrightarrow a_n=\frac{(-n)^{n-1}}{n!}$$

Proof. Let's prove that the right-hand side is the same as the left one:

We want to use Cayley's formula[1], which states that the number of distinct trees made by m vertices is m^{m-2}

Let's take n vertices (Let's call them A) and check in how many ways we can build a "double tree" from them:

- First method:

  Choose k vertices from A (there are $\binom{n}{k}$ way to choose those) and call them B, make a tree with the vertices in B (from Cayley's formula there are k^{k-2} trees like that) and another one from $A\setminus B$ (from Cayley's formula there are $(n-k)^{n-k-2}$ trees like that), then from each tree choose a point (there are k from the first and $n-k$ from the second) and connect them.

- Second method:

  Choose a tree with n vertices (from Cayley's formula there are n^{n-2} trees like that) then choose an edge and remove all its connection except one (there are $n-1$ edges) by doing this you divided the tree into two parts (separated by the edge), now each part can be either B or $A\setminus B$ so we can divide it in 2 ways

So the first method gives us $\binom{n}{k} \cdot k^{k-2} \cdot (n-k)^{n-k-2} \cdot k \cdot (n-k)$ and by summing for each k between 1 (So there could be a tree from B) and $n-1$ (So there could be a tree from $A \setminus B$) we get $\sum_{k=1}^{n-1} \binom{n}{k} k^{k-2}(n-k)^{n-k-2}k(n-k)$

And the second method gives us $n^{n-2} \cdot (n-1) \cdot 2$

Therefore:

$2(n-1)n^{n-2} = \sum_{k=1}^{n-1} \binom{n}{k} k^{k-2}(n-k)^{n-k-2}k(n-k) = \sum_{k=1}^{n-1} \binom{n}{k} k^{k-1}(n-k)^{n-k-1} = \sum_{k=1}^{n-1}(-1)^{n-1}n!\frac{(-k)^{k-1}}{k!}\frac{(k-n)^{n-k-1}}{(n-k)!}$

Let's devide both sides by $\frac{(2-2n)\cdot n! \cdot (-1)^{n-1}}{n}$:

$\frac{(-n)^{n-1}}{n!} = \frac{n}{2-2n}\sum_{k=1}^{n-1}\frac{(-k)^{k-1}}{k!}\frac{(k-n)^{n-k-1}}{(n-k)!}$

So the a_n from the right-hand side satisfies the recursive term from the left-hand side, all left to prove if that in the right-hand side $a_1 = 1$:

$\frac{(-1)^{1-1}}{1!} = \frac{1}{1} = 1$

$\square$

References

[1] Aigner M., Ziegler G.M. (2001) Cayley's formula for the number of trees. In: Proofs from THE BOOK. Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-662-04315-8_24