CHEN 2450
HOMEWORK 8
INITIAL VALUE PROBLEMS

Problem 1 (40 pts)

Consider a tank that is being filled at a volumetric flow rate of $\dot{V}_{\text{in}}$ (m^3/s) with a drain in the bottom that empties at $\dot{V}_{\text{out}}$ as shown in Figure 1. If the tank is gravity-drained, then the flow rate leaving the tank will be approximately

$$\dot{V}_{\text{out}} = \pi r^2 \sqrt{2gh} \tag{1}$$

where r is the radius of the drain (outlet) in the tank, g is the gravitational constant, and h is the height of the liquid in the tank[1]. The height of the water in the tank changes over time depending on the outflow and inflow. The equation governing the rate of change of height of the water in this tank is

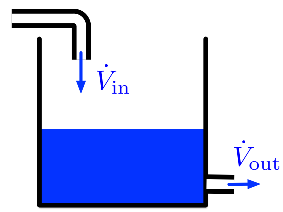

Figure 1: Schematic of the tank.

$$\begin{aligned} \frac{dh}{dt} &= \frac{1}{A_c}\left(\dot{V}_{\text{in}} - \dot{V}_{\text{out}}\right), \\ &= \frac{1}{A_c}\left(\dot{V}_{\text{in}} - \pi r^2 \sqrt{2gh}\right), \end{aligned} \tag{2}$$

where A_c is the cross-sectional area of the tank. Assume that the tank's diameter is 5 m and that the drain in the bottom of the tank is 10 cm in radius.

1. (10 pts) What is the steady-state liquid height in the tank for a constant $\dot{V}_{\text{in}}$? Express your answer in terms of quantities in (2). (HINT: @ steady state, $\frac{\text{dh}}{\text{dt}} = 0$)

2. (30 pts) For a steady-state liquid height of 1.0 m (note that this implies a particular value for $\dot{V}_{\text{in}}$ which you can find from your results in part 1) and initial liquid heights of $h_0 = \{0, 0.4, 0.8, 1.2, 1.6\}$, plot $h(t)$ for $t = [0, 60]$ minutes for all of these h_0 values. Plot all of them on the same figure and include an appropriate legend along with axes labels. Also provide a table that shows the value h at $t = 60$ minutes you find for each value of h_0.

   - You may use whatever time integrator you want to (forward Euler, backward Euler), but you should NOT use Python's built-in time integrators. Write your own like we did in class.
   - Your answer should be accurate. Be sure that your choice of Δt produces an accurate result.

[1] Note that h cannot be less than zero. Therefore, $\dot{V}_{\text{out}}$ cannot be less than zero.

Problem 2 (40 pts)

Consider the equations for first and second order reactions that we discussed in class:

$$\frac{dc}{dt} = -kc, \tag{3}$$

$$\frac{dc}{dt} = -kc^2, \tag{4}$$

with $c(0) = 1$ and $k = 1$ (here we won't be concerned with units, but note that the two k values have different units).

1. (10 pts) Derive the analytic solutions for $c(t)$ for each of these ODEs and plot $c(t)$ for $t = [0, 10]$. Show your work.

2. Show (by hand) the approximation to the solution to (3) and (4) for the first timestep using $\Delta t = 1$ for

   (a) (10 pts) Forward Euler

   (b) (10 pts) Backward Euler

   (c) (10 pts) Crank-Nicolson

   Also determine the absolute true error in your estimation of c at $t = 1$ using $\Delta t = 1$ for each of the methods above. Summarize the errors in a table like the following:

	Forward Euler	Backward Euler	Crank-Nicolson
$\frac{dc}{dt} = -kc$			
$\frac{dc}{dt} = -kc^2$			

Be sure to show all of your work. These solutions should all be "by hand" and then typeset in your Jupyter notebook.

Problem 3 (20 pts)

Predator-prey models were developed in the early part of the twentieth century by the Italian mathematician Vito Volterra and the American biologist Alfred J. Lotka. They describe how a population of predators responds to changes in its prey population and vice versa. These equations are commonly called Lotka-Volterra equations. The simplest example is the following pair of ODEs:

$$\frac{dx}{dt} = ax - bxy, \tag{5}$$

$$\frac{dy}{dt} = -cy + dxy, \tag{6}$$

where x is the number of prey and y is the number of predators, a is the prey growth rate, c is the predator death rate, and b and d are the rates characterizing the effect of the predator- prey interaction on prey death and predator growth, respectively.

1. (20 pts) Assuming the above model works for the rabbits (prey) and foxes (predator), starting with a population of five rabbits and two foxes, integrate the above system of equations using the Forward Euler method. Use $a = 1.2$, $b = 1.0$, $c = 1.0$, and $d = 0.5$. Use an appropriate timestep size and integrate the system until $t = 10$. Note that time is dimensionless in these equations. Plot the prey and predator populations on the same plot and discuss your results.