Karlsruhe Institute of Technology

Institute of Stochastics

Stochastic Geometry | Summer term 2020
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Work Sheet 6

Instructions for week 6 (May 25^{th} to May 29^{th}):

- Work through Sections 2.4 and 2.5 of the lecture notes.
- Answer the control questions 1) to 5), and solve Problems 1 to 3 of the exercises below.
- Please hand in your solutions to the exercises for correction until the morning of Monday, June 1^{st}. For the procedure, please have a look at the general information in Ilias. The submission of solutions is voluntary.

Control questions to monitor your progress:

1) Can you verify that K^* from Definition 2.33, that is,

$$K^*(\varphi, A) = \int \mathbb{1}\left\{\sum_{j=1}^{k} \delta_{(x_j,y_j)}\right\}\Big(\bigotimes_{i\in\mathbb{N}} K(x_i,\cdot)\Big)\big(\mathrm{d}(y_i)_{i\in\mathbb{N}}\big), \qquad \varphi = \sum_{j=1}^{k} \delta_{x_j} \in N(\mathbb{X}), \quad A \in \mathcal{N}(\mathbb{X}\times\mathbb{Y}),$$

   is a stochastic kernel from $N(\mathbb{X})$ to $N(\mathbb{X}\times\mathbb{Y})$?

   (**Hint:** Use that $\bigotimes_{i\in\mathbb{N}} K(x_i,\cdot)$ is a product of probability measures to conclude that $K^*(\varphi,\cdot)$ is a probability measure. To see the measurability of $\varphi \mapsto K^*(\varphi, A)$, convince yourself that

$$(\mathbb{N}\cup\{\infty\})\times\mathbb{X}^{\mathbb{N}} \ni \big(k,(x_i)_{i\in\mathbb{N}}\big) \mapsto \int f\big(k,(x_i)_{i\in\mathbb{N}},(y_i)_{i\in\mathbb{N}}\big)\Big(\bigotimes_{i\in\mathbb{N}} K(x_i,\cdot)\Big)\big(\mathrm{d}(y_i)_{i\in\mathbb{N}}\big)$$

   is measurable for every measurable $f : (\mathbb{N}\cup\{\infty\})\times\mathbb{X}^{\mathbb{N}}\times\mathbb{Y}^{\mathbb{N}} \to [0,\infty)$.)

2) Verify that the p-thinning Φ_p introduced in Definition 2.39 is indeed a point process in $\mathbb{X}$.

3) How does one have to choose the function p in Definition 2.39 such that the resulting K-marking Ψ of Φ (with K given by (2.7)) is an independent $\mathbb{Q}$-marking for some $\mathbb{Q}$? What are the possible distributions $\mathbb{Q}$?

4) Check that the map $M(\mathbb{R}^d) \ni \mu \mapsto \mu + x \in M(\mathbb{R}^d)$ is measurable for each $x \in \mathbb{R}^d$. Why is it important that this mapping is measurable?

5) Convince yourself that a Poisson process Φ in $\mathbb{R}^d$ with intensity measure Θ is stationary if, and only if, $\Theta = \gamma\cdot\lambda^d$ for some $\gamma > 0$ (where λ^d denotes the Lebesgue measure on $\mathbb{R}^d$).

   (**Hint:** To see the "if"-part, use Theorem 2.32 and the hint from Problem 2 below. For the "only if"-part use that every translation invariant and locally finite measure on $\mathbb{R}^d$ is a multiple of the Lebesgue measure.)

Exercises for week 6:

Problem 1 (Some advanced properties of the Poisson process)

Let $(\mathbb{X}, \rho)$ be a separable metric space, and let $(\Omega, \mathcal{A}, \mathbb{P})$ be the probability space which underlies the random quantities that appear in the following.

a) Let Φ be a Poisson process in $\mathbb{X}$ with intensity measure Θ. For $A \in \mathcal{X}$ such that $0 < \Theta(A) < \infty$, and $k \in \mathbb{N}$, we have

$$\mathbb{P}\Big(\Phi_A \in \cdot \;\Big|\; \Phi(A) = k\Big) = \mathbb{P}\Big(\sum_{j=1}^{k} \delta_{X_j} \in \cdot\Big)$$

with independent $X_1, \dots, X_k \sim \frac{\Theta_A}{\Theta(A)}$.

b) Let Ψ be a point process in $\mathbb{X}$. Then Ψ is a Poisson process with intensity measure $\lambda \in M(\mathbb{X})$ if, and only if,

$$\mathbb{E}\Big[F\big(\Psi_A\big)\Big] = e^{-\lambda(A)} \left(F(\mathbf{0}) + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{A^k} F\Big(\sum_{j=1}^{k} \delta_{y_j}\Big) \, \mathrm{d}\lambda^k(y_1, \dots, y_k)\right)$$

for all $A \in \mathcal{X}$ with $\lambda(A) < \infty$, and all measurable $F : N(\mathbb{X}) \to [0, \infty]$.

Problem 2

Let $m \in \mathbb{N}$ and $\gamma_1, \dots, \gamma_m > 0$ such that $\gamma_j \neq \gamma_k$ for all $j, k \in \{1, \dots, m\}$, $j \neq k$. Further, let $p_1, \dots, p_m \in (0, 1]$ with $\sum_{j=1}^{m} p_j = 1$. Let X be a discrete random variable taking values in $\{\gamma_1, \dots, \gamma_m\}$ such that

$$\mathbb{P}\big(X = \gamma_j\big) = p_j, \quad j = 1, \dots, m.$$

Consider a point process Φ in $\mathbb{R}^d$ whose distribution is specified by the conditional distributions

$$\mathbb{P}\Big(\Phi(B) = n \,\Big|\, X = \gamma_j\Big) = \frac{\big(\gamma_j \cdot \lambda^d(B)\big)^n}{n!} \cdot e^{-\gamma_j \cdot \lambda^d(B)},$$

where $B \in \mathcal{B}(\mathbb{R}^d)$, $n \in \mathbb{N}_0$, and $j = 1, \dots, m$. Prove the following assertions.

a) The process Φ is stationary.

b) The process Φ is not a Poisson process for $m \geqslant 2$.

Hint: Without proof you may use the fact that the sets $\big\{\mu \in N(\mathbb{R}^d) : \mu(B) = 0\big\}$, $B \in \mathcal{B}(\mathbb{R}^d)$, form a π-system which generates the σ-field $\mathcal{N}_s(\mathbb{R}^d) := \mathcal{N}(\mathbb{R}^d) \cap N_s(\mathbb{R}^d)$.

Problem 3 (Some properties of p-thinnings)

Let $(\mathbb{X}, \rho)$ be a separable metric space. Let $p : \mathbb{X} \to [0, 1]$ be measurable, and let Φ be a point process in $\mathbb{X}$ and Φ_p the p-thinning of Φ, both defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

a) Prove that, for any measurable $g : \mathbb{X} \to [0, \infty]$, the Laplace functional of Φ_p is given through

$$L_{\Phi_p}(g) := \mathbb{E}\left[\exp\left(-\int_{\mathbb{X}} g(x) \, \mathrm{d}\Phi_p(x)\right)\right] = \mathbb{E}\left[\exp\left(\int_{\mathbb{X}} \log\Big(1 - p(x)\big[1 - e^{-g(x)}\big]\Big) \, \mathrm{d}\Phi(x)\right)\right].$$

b) Let $B \in \mathcal{X}$. Can we interpret Φ_B as a p-thinning of Φ? If so, how do we have to choose p?

c) Prove that, for any $n \in \mathbb{N}$, Φ can be written as a superposition of identically distributed point processes $\Phi_1, \dots, \Phi_n$ such that $\mathbb{P}\big(\Phi(\mathbb{X}) \geqslant 1\big) > 0$ implies

$$\mathbb{P}\big(\Phi_k(\mathbb{X}) \geqslant 1\big) > 0, \quad k = 1, \dots, n.$$

d) Prove that c) is not necessarily true if we require $\Phi_1, \dots, \Phi_n$ to be independent.

e) Prove that if Φ is a Poisson process with intensity measure $\Theta \in M(\mathbb{X})$ then Φ is infinitely divisible, that is, for each $n \in \mathbb{N}$ we find i.i.d. point processes $\Phi_1, \dots, \Phi_n$ such that $\Phi \stackrel{d}{=} \Phi_1 + \ldots + \Phi_n$.

The solutions to these problems will be uploaded on June 1st.
Feel free to ask your questions about the exercises in the optional MS-Teams discussion on May 28th (09:15 h).