Institute of Stochastics

Stochastic Geometry | Summer term 2020
PD. Dr. Steffen Winter
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Work Sheet 9

Instructions for week 9 (June 15^{th} to June 19^{th}):

- Work through Sections 3.3 and 3.4 of the lecture notes.
- Answer the control questions 1) to 6), and solve Problems 1 to 4 of the exercises below.
- Please hand in your solutions to the exercises for correction until Monday, June 22^{nd}. For the procedure, please have a look at the general information in Ilias. The submission of solutions is voluntary.

Control questions to monitor your progress:

1) Provide examples of convex bodies $K, L \in \mathcal{K}^d$ such that their union $K \cup L$ is convex/is not convex.
2) Verify that, for any $K, L \in \mathcal{K}^d$ and any $\alpha, \beta > 0$, the relation
$$\alpha \cdot K + \beta \cdot L = \alpha \cdot \left(K + \frac{\beta}{\alpha} \cdot L\right)$$
holds.
3) Compute all intrinsic volumes of a singleton $\{x\}$, $x \in \mathbb{R}^d$, as well as those of a line segment $[x, y] := \big\{\alpha \cdot x + (1 - \alpha) \cdot y : \alpha \in [0, 1]\big\}$ for some $x, y \in \mathbb{R}^d$ with $x \neq y$.
4) Determine the intrinsic volumes of the unit cube in $\mathbb{R}^3$.
5) Convince yourself that, for $K \in \mathcal{K}^d \setminus \{\varnothing\}$, the metric projection $p(K, x)$ of a point $x \in \mathbb{R}^d$ onto K is uniquely determined.
6) Draw the normal cones of a triangle in $\mathbb{R}^2$ and a cube in $\mathbb{R}^3$ at different points of their boundary.

Exercises for week 9:

Problem 1 (The Euler characteristic)

Use the Steiner formula to prove that, for any $K \in \mathcal{K}^d \setminus \{\varnothing\}$, we have $V_0(K) = 1$.

Problem 2 (Additivity of intrinsic volumes)

a) Let $K, L \in \mathcal{K}^d \setminus \{\varnothing\}$ such that $K \cup L \in \mathcal{K}^d$. Prove that, for any $\varepsilon \geqslant 0$,

$$\mathbb{1}_{(K \cup L) + \varepsilon \cdot B^d} + \mathbb{1}_{(K \cap L) + \varepsilon \cdot B^d} = \mathbb{1}_{K + \varepsilon \cdot B^d} + \mathbb{1}_{L + \varepsilon \cdot B^d}.$$

b) Conclude from part a) that V_j is additive (for each $j \in \{0, \dots, d\}$).

Problem 3 (Intrinsic volumes of the unit ball)

Calculate the intrinsic volumes V_j ($j \in \{0, \dots, d\}$) of the d-dimensional unit ball B^d.

Problem 4 (A bound on the intrinsic volumes)

Consider a set $W \in \mathcal{K}^d$ which contains a ball of radius r, that is, there exists some $x \in \mathbb{R}^d$ and some $r > 0$ such that $x + r \cdot B^d \subset W$. Prove that, for any $j \in \{0, \dots, d\}$,

$$V_j(W) \leqslant \frac{(2^d - 1) \cdot V_d(W)}{\kappa_{d-j}\, r^{d-j}}.$$

The solutions to these problems will be uploaded on June 22nd.