Karlsruhe Institute of Technology

Institute of Stochastics

Stochastic Geometry | Summer term 2020
PD. Dr. Steffen Winter
Steffen Betsch, M.Sc.

Work Sheet 8

Instructions for week 8 (June 8^{th} to June 12^{th}):

- Work through Section 3.2 of the lecture notes.
- Answer the control questions 1) to 5), and solve Problems 1 to 4 of the exercises below.
- Please hand in your solutions to the exercises for correction until the morning of Monday, June 15^{th}. For the procedure, please have a look at the general information in Ilias. The submission of solutions is voluntary.

Control questions to monitor your progress:

1) Can you provide a non-isotropic shape distribution $\mathbb{Q}$ such that the stationary Boolean model with intensity $\gamma > 0$ and shape distribution $\mathbb{Q}$ is isotropic?

2) Let Φ be a stationary Poisson particle process in $\mathbb{R}^d$ with locally finite intensity measure Θ. Can you verify that the generated germ-grain (or Boolean) model $Z := \bigcup_{K \in \Phi} K$ is stationary?

   (**Hint:** You might want to have a look at the capacity functional.)

3) Can you verify that, for any locally finite measure μ on $\big(\mathcal{F}', \mathcal{B}(\mathcal{F}^d) \cap \mathcal{F}'\big)$, we have

$$\mu\big(\mathcal{F}^{C_0}_{C_1,\dots,C_k}\big) = \sum_{r=0}^{k} (-1)^{r-1} \sum_{1 \leqslant i_1 < \dots < i_r \leqslant k} \mu\big(\mathcal{F}_{C_0 \cup C_{i_1} \cup \dots \cup C_{i_r}}\big)$$

   for each $C_0, C_1, \dots, C_k \in \mathcal{C}^d$ and $k \in \mathbb{N}$?

   (**Hint:** Use induction and the fact that $\mathcal{F}^{C_0}_{C_1,\dots,C_k} = \mathcal{F}^{C_0}_{C_1,\dots,C_{k-1}} \setminus \mathcal{F}^{C_0 \cup C_k}_{C_1,\dots,C_{k-1}}$.)

4) Convince yourself that a locally finite measure μ on $\big(\mathcal{F}', \mathcal{B}(\mathcal{F}^d) \cap \mathcal{F}'\big)$ is determined uniquely by its values on the sets $\mathcal{F}_C$, $C \in \mathcal{C}^d$.

   (**Hint:** Use the Control question 3) together with the fact that $\big\{\mathcal{F}_{C_1,\dots,C_k} : C_1, \dots, C_k \in \mathcal{C}^d,\, k \in \mathbb{N}\big\}$ is a π-system which generates $\mathcal{B}(\mathcal{F}^d)$. Also use that for $C_n \nearrow \mathbb{R}^d$ we have $\mathcal{F}_{C_n} \nearrow \mathcal{F}'$ (as $n \to \infty$).)

5) Let $p \in [0, 1]$. Can you find a stationary germ-grain model Z (in $\mathbb{R}^d$) such that the volume fraction of Z is given through $p_Z = p$?

   (**Hint:** Consider a particle process $\Phi = \delta_C$ where you choose a stationary random set C such that 0 is contained in C with probability p.)

Exercises for week 8:

Problem 1 (Constructing a 'germ-grain model' which is not closed)

Find a dimension $d \in \mathbb{N}$, a probability measure $\mathbb{Q}$ on $\mathcal{C}^d$, and a point process Φ in $\mathbb{R}^d$ such that the germ-grain model corresponding to the independent $\mathbb{Q}$-marking of Φ is almost surely not a closed set.

Problem 2 (The mean covariogram of a Boolean model)

Let $Z = \bigcup_{k=1}^{\infty}(Z_k + \xi_k)$ be a stationary Boolean model in $\mathbb{R}^d$ with parameters γ and $\mathbb{Q}$. Let $Z_0 \sim \mathbb{Q}$ be the typical grain and $p_Z := \mathbb{E}\big[\lambda^d\big(Z \cap [0,1]^d\big)\big] = \mathbb{P}(0 \in Z)$ the volume fraction of Z (see Theorem 1.31). Further, let $\Phi = \sum_{j=1}^{\infty} \delta_{\xi_j}$ be the (stationary) Poisson process of the germs.

a) Prove that, for any $C \in \mathcal{C}^d$ and $x, y \in \mathbb{R}^d$,

$$(C + y) \cap \{x, 0\} \neq \varnothing \quad \Longleftrightarrow \quad y \in C^* \cup (x + C^*).$$

b) The mean covariogram C_0 of Z is given through

$$C_0(x) := \mathbb{E}\Big[\lambda^d\big(Z_0 \cap (Z_0 + x)\big)\Big], \qquad x \in \mathbb{R}^d.$$

Prove that, for each $x \in \mathbb{R}^d$,

$$\mathbb{E}\Big[\lambda^d\big(Z_0 \cup (Z_0 - x)\big)\Big] = 2 \cdot \mathbb{E}\big[\lambda^d(Z_0)\big] - C_0(x).$$

c) Use parts a) and b) to show that

$$\mathbb{P}\big(0 \in Z,\ x \in Z\big) = p_Z^2 + (1 - p_Z)^2\big(e^{\gamma \cdot C_0(x)} - 1\big).$$

Problem 3 (The concept of visibility)

Let $Z = \bigcup_{k=1}^{\infty}(Z_k + \xi_k)$ be a stationary Boolean model in $\mathbb{R}^d$ with intensity $\gamma > 0$ and shape distribution $\mathbb{Q}$ which is concentrated on those sets in $\mathcal{C}^d$ that contain $0 \in \mathbb{R}^d$. Further, let $v_d := \int_{\mathcal{C}^d} \lambda^d(C)\, \mathrm{d}\mathbb{Q}(C) < \infty$. A point $z \in Z$ is called visible if

$$\big|\big\{k \in \mathbb{N} : z \in Z_k + \xi_k\big\}\big| = 1,$$

that is, if z is contained in precisely one grain of the Boolean model. Define

$$\Phi := \sum_{j=1}^{\infty} \delta_{\xi_j} \cdot \mathbb{1}\big\{\xi_j \text{ is visible}\big\}.$$

Prove that Φ is a stationary point process in $\mathbb{R}^d$ with intensity $\gamma_{\mathbb{Q}} := \gamma \cdot e^{-\gamma \cdot \mathbb{E}[\lambda^d(Z_1)]}$.

Hint: For the calculation of the intensity you may use, without proof, Mecke's formula, which states that a Poisson process η with intensity measure Λ satisfies

$$\mathbb{E}\Big[\int_{\mathbb{R}^d} f(x, \eta)\, \mathrm{d}\eta(x)\Big] = \mathbb{E}\Big[\int_{\mathbb{R}^d} f(x, \eta + \delta_x)\, \mathrm{d}\Lambda(x)\Big]$$

for every measurable map $f : \mathbb{R}^d \times N(\mathbb{R}^d) \to [0, \infty)$.

Problem 4 (Spherical contact distribution functions)

Let Z be the stationary Boolean model in $\mathbb{R}^d$ with intensity $\gamma > 0$ and typical grain Z_0 given through a d-dimensional ball of random radius $R_0 \sim \mathrm{Exp}(\lambda)$ ($\lambda > 0$) around $0 \in \mathbb{R}^d$.

a) Calculate the spherical contact distribution function $H_{B(0,1)}(r)$, $r > 0$.

b) Now, let $d = 2$ and $r_1, r_2 > 0$ such that $r_1 \neq r_2$. Assume that $H_{B(0,1)}(r_1)$ and $H_{B(0,1)}(r_2)$ are known. Determine the intensity γ as well as the parameter λ in dependence of $H_{B(0,1)}(r_1)$ and $H_{B(0,1)}(r_2)$.

The solutions to these problems will be uploaded on June 15th.