Solutions for Work Sheet 10

Problem 1

Let $C \in \mathcal{C}^d$ and $W \in \mathcal{K}^d$ such that $V_d(W) > 0$. Prove the following assertions.

a) It holds that $\lim_{r\to\infty} V_d(W + r^{-1} \cdot C) = V_d(W)$.

b) It holds that $V_d(W - C) \leqslant c'_W \cdot V_d(C + B^d)$, where c'_W does not depend on C.

c) If $0 \in W$ and $r \geqslant 1$, then $V_d(W + r^{-1} \cdot C) \leqslant c_W \cdot V_d(W + C)$, where c_W does not depend on C or r.

Proposed solution:

a) Since C is compact, we find $\rho > 0$ such that $C \subset \rho \cdot B^d$. For each $r \geqslant 0$, we thus have

$$V_d(W) \leqslant V_d(W + r^{-1} \cdot C) \leqslant V_d\Big(W + \frac{\rho}{r} \cdot B^d\Big).$$

The Steiner formula yields

$$\lim_{r\to\infty} V_d\Big(W + \frac{\rho}{r} \cdot B^d\Big) = \lim_{r\to\infty} \sum_{j=0}^{d} \kappa_{d-j} \frac{\rho^{d-j}}{r^{d-j}} V_j(W) = V_d(W).$$

Therefore, we also have $\lim_{r\to\infty} V_d(W + r^{-1} \cdot C) = V_d(W)$.

b) Since W is compact, we find an index $n \in \mathbb{N}$ as well as $t_1, \dots, t_n \in \mathbb{R}^d$ such that $W \subset \bigcup_{k=1}^n (B^d + t_k)$. Hence,

$$W - C \subset \bigcup_{k=1}^{n} (B^d - C + t_k),$$

and we get $V_d(W - C) \leqslant n \cdot V_d(B^d - C) = n \cdot V_d(C + B^d)$.

c) Let $y_1, \dots, y_m \in C$ be such that $(W + y_i) \cap (W + y_k) = \varnothing$ for any $i, k \in \{1, \dots, m\}$ with $i \neq k$. We have $m \cdot V_d(W) \leqslant V_d(C + W)$, that is,

$$m \leqslant \frac{V_d(C + W)}{V_d(W)}. \tag{1}$$

We may thus choose m to be maximal. For each $x \in C$ we find $k \in \{1, \dots, m\}$ so that

$$(W + x) \cap (W + y_k) \neq \varnothing,$$

or, to put it differently, $x \in y_k + W - W$. Therefore,

$$C \subset \bigcup_{j=1}^{m} (y_j + W - W).$$

Using that $0 \in W$, $r \geqslant 1$, as well as that W is convex, we have $\frac{1}{r} \cdot W \subset W$ and $-\frac{1}{r} \cdot W \subset -W$. We conclude that

$$W + \frac{1}{r} \cdot C \subset \bigcup_{j=1}^{m} \Big(W + \frac{1}{r} \cdot (y_i + W - W) \Big) \subset \bigcup_{j=1}^{m} \Big(\frac{1}{r} \cdot y_i + 2 \cdot W - W \Big). \tag{2}$$

From (2) and (1) we obtain

$$V_d\Big(W + \frac{1}{r} \cdot C\Big) \leqslant m \cdot V_d(2 \cdot W - W) \leqslant \frac{V_d(K+W)}{V_d(W)} \cdot V_d(2 \cdot W - W) = \underbrace{\frac{V_d(2 \cdot W - W)}{V_d(W)}}_{=: c_W} \cdot V_d(W + C).$$

Problem 2 (The proof of part (iii) from Theorem 4.2)

Let Φ be a stationary particle process in $\mathbb{R}^d$ with shape distribution $\mathbb{Q}$ and intensity $\gamma > 0$, given some center function c. Let $f : \mathcal{C}' \to \mathbb{R}$ be a measurable and translation invariant map such that $f \geqslant 0$ or $\int_{\mathcal{C}'} |f| \,\mathrm{d}\mathbb{Q} < \infty$. Further, assume that

$$\int_{\mathcal{C}_0} |f(C)| \cdot \lambda^d(C + B^d) \,\mathrm{d}\mathbb{Q}(C) < \infty.$$

Prove that, for $W \in \mathcal{K}^d$ with $V_d(W) > 0$,

$$\gamma_f(\Phi) = \lim_{r \to \infty} \frac{1}{V_d(r \cdot W)} \mathbb{E}\left[\int_{\mathcal{C}'} \mathbb{1}\big\{ C \cap r \cdot W \neq \varnothing \big\} f(C) \,\mathrm{d}\Phi(C) \right].$$

Proposed solution: We have

$$\begin{aligned}
&\frac{1}{V_d(r \cdot W)} \mathbb{E}\left[\int_{\mathcal{C}'} \mathbb{1}\big\{ C \cap r \cdot W \neq \varnothing \big\} f(C) \,\mathrm{d}\Phi(C) \right] \\
&\qquad = \frac{\gamma}{V_d(r \cdot W)} \int_{\mathcal{C}_0} \int_{\mathbb{R}^d} \mathbb{1}\big\{ (C + x) \cap r \cdot W \neq \varnothing \big\} f(C) \,\mathrm{d}\lambda^d(x) \,\mathrm{d}\mathbb{Q}(C) \\
&\qquad = \frac{\gamma}{V_d(W)} \int_{\mathcal{C}_0} \int_{\mathbb{R}^d} \mathbb{1}\big\{ (r^{-1} \cdot C + y) \cap W \neq \varnothing \big\} f(C) \,\mathrm{d}\lambda^d(y) \,\mathrm{d}\mathbb{Q}(C) \\
&\qquad = \frac{\gamma}{V_d(W)} \int_{\mathcal{C}_0} V_d(W - r^{-1} \cdot C) \, f(C) \,\mathrm{d}\mathbb{Q}(C).
\end{aligned}$$

By part a) of Problem 1, we know that

$$\lim_{r \to \infty} V_d(W - r^{-1} \cdot C) = V_d(W).$$

Without loss of generality, we may assume that $0 \in W$. Parts c) and b) of Problem 1 yield

$$V_d(W - r^{-1} \cdot C) \leqslant c_W \cdot c_W' \cdot V_d(C + B^d), \qquad r \geqslant 1,$$

for some constants c_W and c_W' which do not depend on C or r. By assumption, we have

$$\int_{\mathcal{C}_0} V_d(C + B^d) \, |f(C)| \,\mathrm{d}\mathbb{Q}(C) = \int_{\mathcal{C}_0} |f(C)| \cdot \lambda^d(C + B^d) \,\mathrm{d}\mathbb{Q}(C) < \infty,$$

so dominated convergence gives

$$\begin{aligned}
\lim_{r \to \infty} \frac{1}{V_d(r \cdot W)} \mathbb{E}\left[\int_{\mathcal{C}'} \mathbb{1}\big\{ C \cap r \cdot W \neq \varnothing \big\} f(C) \,\mathrm{d}\Phi(C) \right] &= \frac{\gamma}{V_d(W)} \int_{\mathcal{C}_0} V_d(W) \, f(C) \,\mathrm{d}\mathbb{Q}(C) \\
&= \gamma \int_{\mathcal{C}_0} f(C) \,\mathrm{d}\mathbb{Q}(C) \\
&= \gamma_f(\Phi).
\end{aligned}$$

Problem 3 (An inclusion-exclusion principle – Lemma 4.4)

Let $\mathcal{R}^d$ be the collection of all finite unions of convex bodies.

a) Let $f : \mathcal{R}^d \to \mathbb{R}$ be an additive map in the sense of Definition 4.3, that is,

$$f(\varnothing) = 0 \qquad \text{as well as} \qquad f(K \cup L) + f(K \cap L) = f(K) + f(L), \quad K, L \in \mathcal{R}^d.$$

Prove that, for any $m \in \mathbb{N}$ and $K_1, \ldots, K_m \in \mathcal{R}^d$,

$$f(K_1 \cup \ldots \cup K_m) = \sum_{k=1}^{m} (-1)^{k-1} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant m} f(K_{i_1} \cap \ldots \cap K_{i_k}).$$

b) Prove that the map $\varphi_x : \mathcal{R}^d \to \mathbb{R}$, $B \mapsto \varphi_x(B) := \mathbb{1}_B(x)$, is additive for every $x \in \mathbb{R}^d$.

c) Prove that, for $m \in \mathbb{N}$, $K_1, \ldots, K_m \in \mathcal{K}^d$, and $K := \bigcup_{j=1}^m K_j$,

$$\mathbb{1}_K = \sum_{\ell=1}^{m} (-1)^{\ell-1} \sum_{1 \leqslant i_1 < \ldots < i_\ell \leqslant m} \mathbb{1}_{K_{i_1} \cap \ldots \cap K_{i_\ell}}$$

Proposed solution:

a) We prove the claim by induction. For $m = 1$, the statement is trivially true. Assume that the claim holds for some fixed $m \in \mathbb{N}$. Then, for $K_1, \ldots, K_{m+1} \in \mathcal{R}^d$, the additivity of f and the induction hypothesis yield

$$\begin{aligned}
f(K_1 \cup \ldots \cup K_{m+1}) &= f(K_1 \cup \ldots \cup K_m) + f(K_{m+1}) - f\big((K_1 \cap K_{m+1}) \cup \ldots \cup (K_m \cap K_{m+1})\big) \\
&= \sum_{k=1}^{m} (-1)^{k-1} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant m} f(K_{i_1} \cap \ldots \cap K_{i_k}) \ + \ f(K_{m+1}) \\
&\qquad - \sum_{k=1}^{m} (-1)^{k-1} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant m} f\big((K_{i_1} \cap K_{m+1}) \cap \ldots \cap (K_{i_k} \cap K_{m+1})\big) \\
&= \sum_{k=1}^{m} (-1)^{k-1} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant m} f(K_{i_1} \cap \ldots \cap K_{i_k}) \ + \ f(K_{m+1}) \\
&\qquad + \sum_{k=1}^{m} (-1)^{k} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant m} f(K_{i_1} \cap \ldots \cap K_{i_k} \cap K_{m+1}) \\
&= \sum_{k=1}^{m} (-1)^{k-1} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant m} f(K_{i_1} \cap \ldots \cap K_{i_k}) \\
&\qquad + \sum_{k=1}^{m+1} (-1)^{k-1} \sum_{1 \leqslant i_1 < \ldots < i_k = m+1} f(K_{i_1} \cap \ldots \cap K_{i_k}) \\
&= \sum_{k=1}^{m+1} (-1)^{k-1} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant m+1} f(K_{i_1} \cap \ldots \cap K_{i_k}).
\end{aligned}$$

b) Let $K, L \in \mathcal{R}^d$ and $x \in \mathbb{R}^d$ be arbitrary. We distinguish three cases in x:

In **case 1**, we consider $x \in K^c \cap L^c$. We clearly have

$$\varphi_x(K \cup L) + \varphi_x(K \cap L) = \mathbb{1}_{K \cup L}(x) + \mathbb{1}_{K \cap L}(x) = 0 + 0 = \mathbb{1}_K(x) + \mathbb{1}_L(x) = \varphi_x(K) + \varphi_x(L).$$

In **case 2**, we consider $x \in K \cap L^c$. It follows that

$$\varphi_x(K \cup L) + \varphi_x(K \cap L) = \mathbb{1}_{K \cup L}(x) + \mathbb{1}_{K \cap L}(x) = 1 + 0 = \mathbb{1}_K(x) + \mathbb{1}_L(x) = \varphi_x(K) + \varphi_x(L).$$

The case $x \in K^c \cap L$ follows by an identical argument.

In **case 3**, we consider $x \in K \cap L$. We have

$$\varphi_x(K \cup L) + \varphi_x(K \cap L) = \mathbb{1}_{K \cup L}(x) + \mathbb{1}_{K \cap L}(x) = 1 + 1 = \mathbb{1}_K(x) + \mathbb{1}_L(x) = \varphi_x(K) + \varphi_x(L).$$

c) The claim follows immediately from parts a) and b).

Problem 4 (From the proof of Lemma 4.7)

For $z \in \mathbb{Z}^d$ let $C_z := C^d + z$, where $C^d := [0,1]^d$. For $r > 0$ as well as $W \in \mathcal{K}^d$ with $V_d(W) > 0$ and $0 \in \operatorname{int}(W)$, define

$$Z_r^1 := \big\{ z \in \mathbb{Z}^d : C_z \cap r \cdot W \neq \varnothing,\, C_z \not\subset r \cdot W \big\} \qquad \text{and} \qquad Z_r^2 := \big\{ z \in \mathbb{Z}^d : C_z \subset r \cdot W \big\}.$$

Prove that

$$\lim_{r \to \infty} \frac{\operatorname{card}(Z_r^1)}{V_d(r \cdot W)} = 0 \qquad \text{and} \qquad \lim_{r \to \infty} \frac{\operatorname{card}(Z_r^2)}{V_d(r \cdot W)} = 1.$$

Proposed solution: First notice that there exist $s, t > 0$ as well as $r_0 \geqslant 0$ such that, for every $r \geqslant r_0$,

$$\bigcup_{z \in Z_r^1} C_z \subset (r+s) \cdot W \setminus (r-s) \cdot W \qquad \text{and} \qquad (r-t) \cdot W \subset \bigcup_{z \in Z_r^2} C_z \subset r \cdot W. \tag{3}$$

From the first inclusion of (3) it follows that, for $r \geqslant r_0$,

$$0 \leqslant \operatorname{card}(Z_r^1) \leqslant V_d\big((r+s) \cdot W\big) - V_d\big((r-s) \cdot W\big).$$

Since

$$\frac{V_d\big((r+s) \cdot W\big) - V_d\big((r-s) \cdot W\big)}{V_d(r \cdot W)} = \frac{(r+s)^d - (r-s)^d}{r^d} \to 0 \qquad (\text{as } r \to \infty),$$

we conclude that

$$\lim_{r \to \infty} \frac{\operatorname{card}(Z_r^1)}{V_d(r \cdot W)} = 0.$$

From the second inclusion of (3) it follows that, for $r \geqslant r_0$,

$$V_d\big((r-t) \cdot W\big) \leqslant \operatorname{card}(Z_r^2) \leqslant V_d(r \cdot W).$$

Since

$$\frac{V_d\big((r-t) \cdot W\big)}{V_d(r \cdot W)} = \frac{(r-t)^d}{r^d} \to 1 \qquad (\text{as } r \to \infty),$$

we conclude that

$$\lim_{r \to \infty} \frac{\operatorname{card}(Z_r^2)}{V_d(r \cdot W)} = 1.$$