Solutions for Work Sheet 7

Problem 1 (Estimation of the intensity of a Poisson process)

Let Φ be a Poisson process in $\mathbb{R}^d$ with intensity measure $\gamma \cdot \lambda^d$, where $\gamma > 0$.

a) Prove that $\widehat{\gamma}_B := \frac{\Phi(B)}{\lambda^d(B)}$, for bounded $B \in \mathcal{B}(\mathbb{R}^d)$ such that $\lambda^d(B) > 0$, is an unbiased estimator for γ.

b) Let $W_1, W_2, \ldots \in \mathcal{B}(\mathbb{R}^d)$ be Borel sets such that $0 < \lambda^d(W_n) < \infty$, $n \in \mathbb{N}$, and $\lambda^d(W_n) \overset{n\to\infty}{\longrightarrow} \infty$. Prove that the estimator $\widehat{\gamma}$ is weakly consistent, in the sense that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\Big(\big|\widehat{\gamma}_{W_n} - \gamma\big| > \varepsilon\Big) = 0.$$

c) Let $W_1, W_2, \ldots \in \mathcal{B}(\mathbb{R}^d)$ such that $0 < \lambda^d(W_n) < \infty$, $n \in \mathbb{N}$, and $\sup_{n\in\mathbb{N}} \frac{n^{1+\delta}}{\lambda^d(W_n)} < \infty$ for some $\delta > 0$. Prove that the estimator $\widehat{\gamma}$ is strongly consistent, in the sense that

$$\lim_{n\to\infty} \widehat{\gamma}_{W_n} = \gamma \quad \mathbb{P}\text{-almost surely.}$$

Hint: Recall from probability theory that if random variables $X, \{X_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ satisfy $\sum_{n=1}^{\infty} \mathbb{P}\big(|X_n - X| > \varepsilon\big) < \infty$, for every $\varepsilon > 0$, then $X_n \to X$ $\mathbb{P}$-almost surely.

d) Simulate a homogeneous Poisson process in $[-25, 25]^3$ with intensity $\gamma = 0.005$, that is, simulate a mixed binomial process with parameters $\tau \sim Po(50^3 \cdot 0.005)$ and $V = \mathcal{U}\big([-25, 25]^3\big)$. Proceed to estimate the intensity on the growing observation windows $W_n := B(0, n) \subset \mathbb{R}^3$, for $n = 1, \ldots, 25$, and see how the estimator behaves.

Proposed solution:

a) For bounded $B \in \mathcal{B}(\mathbb{R}^d)$, we have

$$\mathbb{E}\big[\widehat{\gamma}_B\big] = \mathbb{E}\left[\frac{\Phi(B)}{\lambda^d(B)}\right] = \frac{1}{\lambda^d(B)} \cdot \mathbb{E}\big[\Phi(B)\big] = \gamma.$$

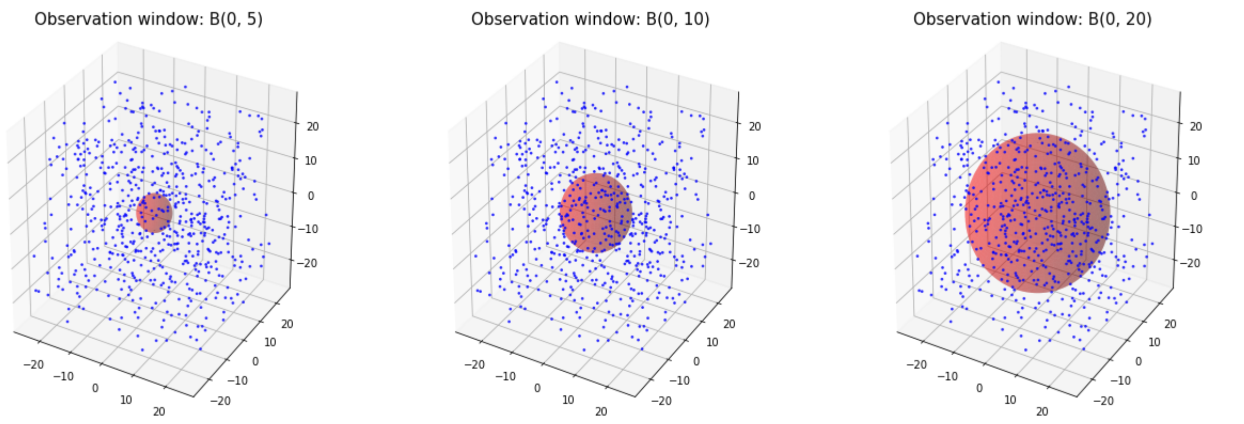
b) Using that $\mathbb{V}\mathrm{ar}\big(\Phi(B)\big) = \gamma \cdot \lambda^d(B)$, for $B \in \mathcal{B}(\mathbb{R}^d)$, Chebyshev's inequality yields

$$\mathbb{P}\Big(\big|\widehat{\gamma}_{W_n} - \gamma\big| > \varepsilon\Big) \leqslant \frac{\mathbb{V}\mathrm{ar}\big(\widehat{\gamma}_{W_n}\big)}{\varepsilon^2} = \frac{\gamma \cdot \lambda^d(W_n)}{\varepsilon^2 \big(\lambda^d(W_n)\big)^2} = \frac{\gamma}{\varepsilon^2\, \lambda^d(W_n)} \overset{n\to\infty}{\longrightarrow} 0.$$

c) We use the hint to prove almost sure convergence. Thus, let $\varepsilon > 0$ be arbitrary. Then Chebyshev's inequality and the assumption on $\{W_n\}_{n\in\mathbb{N}}$ imply

$$\sum_{n=1}^{\infty} \mathbb{P}\Big(\big|\widehat{\gamma}_{W_n} - \gamma\big| > \varepsilon\Big) \leqslant \sum_{n=1}^{\infty} \frac{\gamma}{\varepsilon^2\, \lambda^d(W_n)} \leqslant \frac{\gamma}{\varepsilon^2} \left(\sup_{n\in\mathbb{N}} \frac{n^{1+\delta}}{\lambda^d(W_n)}\right) \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} < \infty.$$

From the hint it follows that $\widehat{\gamma}_{W_n} \overset{n\to\infty}{\longrightarrow} \gamma$ $\mathbb{P}$-almost surely.



d) A suggestion for the simulation was implemented in Python and is available in the Ilias course. The following picture indicates what is going on: We have a Poisson process in the cube $[-25, 25]^3$, but we observe only the points in the balls of growing radius (suggested in red). The realization in the picture has a total of 605 points in the whole cube. The following table gives an idea of the behavior of the corresponding estimates for the intensity (which are rounded to four digits). The example seems to be perfectly in line with the consistency results in part c).

Observation window	$B(0,1)$	$B(0,3)$	$B(0,5)$	$B(0,10)$	$B(0,15)$	$B(0,20)$	$B(0,25)$
Number of points in the window	0	0	2	27	76	170	318
Value of the intensity estimator	0	0	0.0038	0.0064	0.0054	0.0051	0.0049

Problem 2 (Particle processes)

Defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, let Ψ be a stationary point process in $\mathbb{R}^d \times \mathcal{C}'$ such that $\mathbb{E}\big[\Psi(\cdot \times \mathcal{C}')\big]$ is locally finite. Let $\gamma \geqslant 0$ and the probability measure $\mathbb{Q}$ on $\mathcal{C}'$ be as in Theorem 2.42 (that is, $\mathbb{E}\Psi = \gamma \cdot \lambda^d \otimes \mathbb{Q}$) and assume that

$$\int_{\mathcal{C}'} \lambda^d(K + C)\,\mathrm{d}\mathbb{Q}(K) < \infty, \qquad C \in \mathcal{C}'.$$

Prove that $\Phi := \int_{\mathbb{R}^d \times \mathcal{C}'} \mathbb{1}\big\{K + x \in \cdot\,\big\}\,\mathrm{d}\Psi(x, K)$ is a particle process.

Proposed solution: As Ψ is a point process in $\mathbb{R}^d \times \mathcal{C}'$, the map $\Omega \to [0, \infty]$, $\omega \mapsto \Psi(\omega, \mathcal{H})$ is measurable for every $\mathcal{H} \in \mathcal{B}(\mathbb{R}^d) \otimes \big(\mathcal{B}(\mathcal{C}^d) \cap \mathcal{C}'\big)$. Therefore, simple monotone approximation shows that

$$\Omega \to [0, \infty], \quad \omega \mapsto \int_{\mathbb{R}^d \times \mathcal{C}'} f(x, K)\,\Psi\big(\omega, \mathrm{d}(x, K)\big)$$

is measurable for every measurable function $f : \mathbb{R}^d \times \mathcal{C}' \to [0, \infty]$. By Problem 2 a) of Work sheet 2 (and Theorem 1.17), the map $f(x, K) := \mathbb{1}\big\{K + x \in \mathcal{H}\big\}$ is measurable for every $\mathcal{H} \in \mathcal{B}(\mathcal{C}^d) \cap \mathcal{C}'$, and hence $\Omega \ni \omega \mapsto \Phi(\omega, \mathcal{H})$ is measurable. By Campbell's theorem, we have, for any set $C \in \mathcal{C}^d$,

$$\begin{aligned}
\mathbb{E}\big[\Phi(\mathcal{F}^d_C \cap \mathcal{C}')\big] = \gamma \int_{\mathbb{R}^d \times \mathcal{C}'} \mathbb{1}\big\{K + x \in \mathcal{F}^d_C\big\}\,\mathrm{d}(\lambda^d \otimes \mathbb{Q})(x, K) &= \gamma \int_{\mathcal{C}'} \int_{\mathbb{R}^d} \mathbb{1}\big\{(K + x) \cap C \neq \varnothing\big\}\,\mathrm{d}\lambda^d(x)\,\mathrm{d}\mathbb{Q}(K) \\
&= \gamma \int_{\mathcal{C}'} \lambda^d\big(K + (-C)\big)\,\mathrm{d}\mathbb{Q}(K) \\
&< \infty,
\end{aligned}$$

where we used that $(K + x) \cap C \neq \varnothing$ if, and only if, $x \in C + (-K)$, and that the Lebesgue measure is invariant under translations and rotations. By the hint for Control question 4, Φ is locally finite (almost surely). Thus, (technically after redefining Φ on the null set where it is not locally finite, to have a map $\Omega \to N(\mathcal{C}')$,) Problem 1 a) on Work sheet 4 implies that Φ is a point process in $\mathcal{C}'$, and hence a particle process as in Definition 3.1.

Problem 3 (Continuity of the circumball mapping)

a) Let $C \in \mathcal{C}^d$ be non-empty, that is, $C \in \mathcal{C}' = \mathcal{C}^d \setminus \{\varnothing\}$. Denote by

$$B(x,r) = \{y \in \mathbb{R}^d : \|y-x\| \leqslant r\}, \qquad x \in \mathbb{R}^d,\ r \geqslant 0,$$

the ball in $\mathbb{R}^d$ of radius r around x. Prove that among all balls that contain C there exists a unique ball $B(C)$ with smallest radius (the circumball of C).

b) Prove that the map $r : \mathcal{C}' \to [0,\infty)$, $C \mapsto r(C)$, where $r(C)$ denotes the radius of the circumball $B(C)$, is continuous with respect to the Hausdorff metric.

c) Prove that the map $c : \mathcal{C}' \to \mathbb{R}^d$, $C \mapsto c(C)$, where $c(C)$ denotes the center of the circumball $B(C)$, is continuous with respect to the Hausdorff metric.

d) Conclude that the map $\mathcal{C}' \to \mathcal{C}'$, $C \mapsto B(C)$ is continuous with respect to the Hausdorff metric.

Proposed solution:

a) Let $C \neq \varnothing$ be a compact set in $\mathbb{R}^d$.

(1) We first prove that a circumball exists. Consider

$$A := \{r \geqslant 0 : \exists x \in \mathbb{R}^d \text{ such that } B(x,r) \supset C\},$$

and put $\bar{r} := \inf A$. Choose a sequence $r_m \in A$ with $r_m \searrow \bar{r}$ and $x_m \in \mathbb{R}^d$, $m \in \mathbb{N}$, such that $B(x_m, r_m) \supset C$. Fix $y \in C$, and observe that, for any $m \in \mathbb{N}$,

$$\|x_m\| \leqslant \|x_m - y\| + \|y\| \leqslant r_m + \|y\| \leqslant r_1 + \|y\|.$$

Therefore, $\{x_m : m \in \mathbb{N}\} \subset B(0, r_1 + \|y\|)$, so there exists a convergent subsequence $(x_{m_k})_{k\in\mathbb{N}}$ with limit $x \in B(0, r_1 + \|y\|)$. In order to show that $C \subset B(x,\bar{r})$, let $z \in C$ and notice that, for every $k \in \mathbb{N}$,

$$\|x - z\| \leqslant \|x - x_{m_k}\| + \|x_{m_k} - z\| \leqslant \underbrace{\|x - x_{m_k}\|}_{\to 0} + \underbrace{r_{m_k}}_{\searrow \bar{r}}.$$

We conclude that $\|x - z\| \leqslant \bar{r}$, which implies $z \in B(x,\bar{r})$. Hence, the infimum $\bar{r}$ is attained and $B(x,\bar{r})$ is a circumball of C.

(2) We proceed to prove the uniqueness of the circumball. Suppose there exist two different circumballs $B(x_1,\bar{r})$ and $B(x_2,\bar{r})$, that is, $x_1 \neq x_2$ and $C \subset B(x_1,\bar{r})$ as well as $C \subset B(x_2,\bar{r})$. Then, apparently, $C \subset B(x_1,\bar{r}) \cap B(x_2,\bar{r}) =: D$. For $y \in D$, we have $\|y - x_1\| \leqslant \bar{r}$ and $\|y - x_2\| \leqslant \bar{r}$. Consequently, we get

$$\begin{aligned}
\Big\|y - \frac{x_1+x_2}{2}\Big\|^2 &= \Big\|\frac{y-x_1}{2} + \frac{y-x_2}{2}\Big\|^2 \\
&= \Big\|\frac{y-x_1}{2}\Big\|^2 + \Big\|\frac{y-x_2}{2}\Big\|^2 + 2\Big\langle \frac{y-x_1}{2}, \frac{y-x_2}{2}\Big\rangle \\
&\leqslant \frac{\bar{r}^2}{2} + \Big\langle \frac{y-x_2+x_2-x_1}{2}, \frac{y-x_2}{2}\Big\rangle + \Big\langle \frac{y-x_1}{2}, \frac{y-x_1+x_1-x_2}{2}\Big\rangle \\
&= \frac{\bar{r}^2}{2} + \Big\|\frac{y-x_2}{2}\Big\|^2 + \Big\langle \frac{x_2-x_1}{2}, \frac{y-x_2}{2}\Big\rangle + \Big\|\frac{y-x_1}{2}\Big\|^2 + \Big\langle \frac{y-x_1}{2}, \frac{x_1-x_2}{2}\Big\rangle \\
&\leqslant \bar{r}^2 - \Big\|\frac{x_1-x_2}{2}\Big\|^2,
\end{aligned}$$

so $D \subset B\Big(\frac{x_1+x_2}{2}, \sqrt{\bar{r}^2 - \big(\frac{\|x_1-x_2\|}{2}\big)^2}\Big)$. As

$$\sqrt{\bar{r}^2 - \Big(\frac{\|x_1-x_2\|}{2}\Big)^2} < \bar{r},$$

this a contradiction to the fact the $B(x_1,\bar{r})$ and $B(x_2,\bar{r})$ are circumballs. Hence, there cannot exist two different circumballs.

b) Notice that the space $\mathcal{C}'$ endowed with the topology induced by the Hausdorff metric δ is first countable (every metric space is first countable). Hence, continuity of a function defined on $\mathcal{C}'$ is equivalent to the sequential continuity of that function (see Control question 2 on Work sheet 2). Thus, let $C_1, C_2, \ldots \in \mathcal{C}'$ be a sequence which converges with respect to the Hausdorff metric, and denote its limit by $C \in \mathcal{C}'$. For each $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that, for all $n \geqslant n_0$,

$$C_n \subset C + \varepsilon B(0,1) \quad \text{and} \quad C \subset C_n + \varepsilon B(0,1).$$

On the one hand, this implies $C_n \subset B(C) + \varepsilon B(0,1)$ hence $r(C_n) - r(C) \leqslant \varepsilon$, and on the other hand, $C \subset B(C_n) + \varepsilon B(0,1)$ hence $r(C) - r(C_n) \leqslant \varepsilon$. We conclude that

$$\big| r(C) - r(C_n) \big| \leqslant \varepsilon.$$

In other words, $r(C_n) \to r(C)$ as $n \to \infty$.

c) Let $C_1, C_2, \ldots \in \mathcal{C}'$ be a convergent sequence in $(\mathcal{C}', \delta)$ with limit $C \in \mathcal{C}'$. Put $r := \sup_{n\in\mathbb{N}} r(C_n)$. We have $C_n \cap B(C_n) \neq \varnothing$, so $C_n \cap B\big(c(C_n), r\big) \neq \varnothing$ and thus $c(C_n) \in C_n + r\, B(0,1)$. Hence, the sequence $\big(c(C_n)\big)_{n\in\mathbb{N}}$ is contained in the set $\bigcup_{n\in\mathbb{N}} \big(C_n + r\, B(0,1)\big) = \big(\bigcup_{n\in\mathbb{N}} C_n\big) + r\, B(0,1)$ which is bounded as the sequence $(C_n)_{n\in\mathbb{N}}$ converges. To prove the continuity of c it suffices to show that every convergent subsequence of $\big(c(C_n)\big)_{n\in\mathbb{N}}$ converges to $c(C)$. Let $\big(c(C_{n_k})\big)_{k\in\mathbb{N}}$ be such a subsequence which converges to $\overline{c}$. Then, $B(C_n) = B\big(c(C_n), r(C_n)\big) \to B\big(\overline{c}, r(C)\big)$ as $n \to \infty$ by the same argument given in part d) below. For each $\varepsilon > 0$ we find an $n_0 \in \mathbb{N}$ such that, for any $n \geqslant n_0$,

$$C \subset C_n + \varepsilon B(0,1) \quad \text{and} \quad B(C_n) = B\big(c(C_n), r(C_n)\big) \subset B\big(\overline{c}, r(C)\big) + \varepsilon B(0,1).$$

It follows that, for each $n \geqslant n_0$, $C \subset B(C_n) + \varepsilon B(0,1)$ and therefore $C \subset B\big(\overline{c}, r(C)\big) + 2\varepsilon B(0,1)$. As $\varepsilon > 0$ was arbitrary, we conclude that

$$C \subset B\big(\overline{c}, r(C)\big),$$

and since circumballs are unique, we have $\overline{c} = c(C)$ which proves the claim.

d) Let $C_1, C_2, \ldots \in \mathcal{C}^d$ be a convergent sequence in $(\mathcal{C}', \delta)$ with limit $C \in \mathcal{C}'$. As b) and c) imply $r(C_n) \to r(C)$ and $c(C_n) \to c(C)$ (as $n \to \infty$), Problem 1 on Work sheet 2 implies

$$B(C_n) = B\big(c(C_n), r(C_n)\big) \to B\big(c(C), r(C)\big) = B(C)$$

as $n \to \infty$ with respect to the Fell-topology. However, the circumballs are contained in one sufficiently large compact set so Theorem 1.17 implies that $\delta\big(B(C_n), B(C)\big) \to 0$ (as $n \to \infty$). To see that the circumballs are contained in a compact set, use that, since $(C_n)_{n\in\mathbb{N}}$ converges in the Hausdorff metric, the set $\{C_n : n \in \mathbb{N}\}$ is bounded in $(\mathcal{C}', \delta)$, so, by definition of the Hausdorff metric, the sets C_n are all contained in the compact set $\widetilde{C} = C_1 + \rho\, B(0,1)$, where

$$\rho := \sup_{n,m\in\mathbb{N}} \delta(C_n, C_m) < \infty,$$

and taking the circumball of $\widetilde{C}$ gives a compact set which contains the circumballs of the sets C_n, $n \in \mathbb{N}$.

Problem 4 (On Theorem 3.6 and Remark 3.9)

Let Φ be a stationary particle process in $\mathbb{R}^d$ with locally finite intensity measure $\Theta \neq 0$. Let $c : \mathcal{C}' \to \mathbb{R}^d$ be some center function (in the sense of Definition 3.2). As in the lecture, we define $\mathcal{C}_0 := \big\{C \in \mathcal{C}' : c(C) = 0\big\}$. By Theorem 3.6 there exists a unique $\gamma > 0$ and a unique probability measure $\mathbb{Q}$ on $\mathcal{C}'$ such that $\mathbb{Q}(\mathcal{C}_0) = 1$ and

$$\Theta(\mathcal{H}) = \gamma \int_{\mathbb{R}^d} \int_{\mathcal{C}'} \mathbb{1}_{\mathcal{H}}(C + x)\, \mathrm{d}\mathbb{Q}(C)\, \mathrm{d}x, \qquad \mathcal{H} \in \mathcal{B}(\mathcal{C}').$$

a) Prove that γ does not depend on the choice of c.

b) Let $c' : \mathcal{C}' \to \mathbb{R}^d$ be another center function, and $\mathbb{Q}_{c'}$ the corresponding shape distribution. Also put $\mathcal{C}_{0,c'} := \big\{C \in \mathcal{C}' : c'(C) = 0\big\}$. Prove that $\mathbb{Q} = \mathbb{Q}_{c'} \circ T^{-1}$, where $T : \mathcal{C}_{0,c'} \to \mathcal{C}_0$, $C \mapsto C - c(C)$.

c) Find a center function c' such that $\mathbb{Q} \neq \mathbb{Q}_{c'}$.

Proposed solution:

a) Let c' be a second center function, and denote by $\gamma_{c'}$ the corresponding intensity. From the definition of the intensity (see the proof of Theorem 3.4), Campbell's theorem, Tonelli's theorem, and the translation covariance of c' it follows that

$$\begin{aligned}\gamma_{c'} = \mathbb{E}\Big[\int_{\mathcal{C}'} \mathbb{1}\{c'(C) \in [0,1)^d\}\,\mathrm{d}\Phi(C)\Big] &= \int_{\mathcal{C}'} \mathbb{1}\{c'(C) \in [0,1)^d\}\,\mathrm{d}\Theta(C) \\ &= \gamma \int_{\mathbb{R}^d}\int_{\mathcal{C}_0} \mathbb{1}\{c'(C+x) \in [0,1)^d\}\,\mathrm{d}\mathbb{Q}(C)\,\mathrm{d}x \\ &= \gamma \int_{\mathcal{C}_0}\int_{\mathbb{R}^d} \mathbb{1}\{c'(C)+x \in [0,1)^d\}\,\mathrm{d}x\,\mathrm{d}\mathbb{Q}(C) \\ &= \gamma \int_{\mathcal{C}_0} \lambda^d\Big([0,1)^d - c'(C)\Big)\,\mathrm{d}\mathbb{Q}(C) \\ &= \gamma \int_{\mathcal{C}_0} 1\,\mathrm{d}\mathbb{Q}(C) \\ &= \gamma.\end{aligned}$$

b) Define

$$\widetilde{T} : \mathcal{C}_0 \to \mathcal{C}_{0,c'},\, C \mapsto C - c'(C),$$

and observe that, for each $K \in \mathcal{C}_0$,

$$(T \circ \widetilde{T})(K) = T\big(K - c'(K)\big) = K - c'(K) - c\big(K - c'(K)\big) = K - c'(K) - c(K) + c'(K) = K.$$

Moreover, for any $K \in \mathcal{C}_{0,c'}$, we find

$$(\widetilde{T} \circ T)(K) = \widetilde{T}\big(K - c(K)\big) = K - c(K) - c'\big(K - c(K)\big) = K - c(K) - c'(K) + c(K) = K,$$

which implies $\widetilde{T} = T^{-1}$. For $\mathcal{H} \in \mathcal{B}(\mathcal{C}')$, the representation of Θ from Remark 3.5 (choose $B = [0,1]^d$) and Campbell's theorem yield

$$\begin{aligned}\mathbb{Q}(\mathcal{H}) &= \frac{1}{\gamma}\,\mathbb{E}\Big[\int_{\mathcal{C}'} \mathbb{1}_{[0,1]^d}\big(c(K)\big)\cdot \mathbb{1}_{\mathcal{H}}\big(C - c(C)\big)\,\mathrm{d}\Phi(C)\Big] \\ &= \frac{1}{\gamma}\int_{\mathcal{C}'} \mathbb{1}_{[0,1]^d}\big(c(K)\big)\cdot \mathbb{1}_{\mathcal{H}}\big(C - c(C)\big)\,\mathrm{d}\Theta(C) \\ &= \int_{\mathbb{R}^d}\int_{\mathcal{C}_{0,c'}} \mathbb{1}_{[0,1]^d}\big(c(C+x)\big)\cdot \mathbb{1}_{\mathcal{H}}\big(C + x - c(C+x)\big)\,\mathrm{d}\mathbb{Q}_{c'}(C)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d}\int_{\mathcal{C}_{0,c'}} \mathbb{1}_{[0,1]^d - c(C)}(x)\cdot \mathbb{1}_{T^{-1}(\mathcal{H})}\Big(T^{-1}\big(C - c(C)\big)\Big)\,\mathrm{d}\mathbb{Q}_{c'}(C)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d}\int_{\mathcal{C}_{0,c'}} \mathbb{1}_{[0,1]^d - c(C)}(x)\cdot \mathbb{1}_{T^{-1}(\mathcal{H})}\Big(C - c(C) - c'\big(C - c(C)\big)\Big)\,\mathrm{d}\mathbb{Q}_{c'}(C)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d}\int_{\mathcal{C}_{0,c'}} \mathbb{1}_{[0,1]^d - c(C)}(x)\cdot \mathbb{1}_{T^{-1}(\mathcal{H})}(C)\,\mathrm{d}\mathbb{Q}_{c'}(C)\,\mathrm{d}x \\ &= \int_{\mathcal{C}_{0,c'}} \underbrace{\Big(\int_{\mathbb{R}^d} \mathbb{1}_{[0,1]^d - c(C)}(x)\,\mathrm{d}x\Big)}_{=\lambda^d([0,1]^d)=1} \mathbb{1}_{T^{-1}(\mathcal{H})}(C)\,\mathrm{d}\mathbb{Q}_{c'}(C) \\ &= \mathbb{Q}_{c'}\big(T^{-1}(\mathcal{H})\big).\end{aligned}$$

c) Let $0 \neq y \in \mathbb{R}^d$ and $c'(C) := c(C) + y$. Then, $\mathbb{Q}_{c'}(\mathcal{C}_{0,c'}) = 1$ with

$$\mathcal{C}_{0,c'} = \big\{C \in \mathcal{C}' : c'(C) = 0\big\} = \big\{C \in \mathcal{C}' : c(C) = -y\big\}.$$

However, $\mathbb{Q}(\mathcal{C}_{0,c'}) = 0$ as $\mathcal{C}_0 \cap \mathcal{C}_{0,c'} = \varnothing$, and hence $\mathbb{Q} \neq \mathbb{Q}_{c'}$.