Solutions for Work Sheet 13

Problem 1

Let $K \in \mathcal{K}^3$ be such that $K \subset [0,1]^3$, and let X_1 be a random line (that is, a random 1-flat) through $[0,1]^3$, as defined in Problem 3 of Work sheet 11, on a given probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Suppose that, for a realization $X_1(\omega)$ ($\omega \in \Omega$), you can observe whether $X_1(\omega)$ intersects K, and, if so, that you can measure the length of $X_1(\omega) \cap K$. Construct an unbiased estimator for the surface area and the volume of K.

Proposed solution: According to Problem 3 on Work sheet 11, the distribution of X_1 is given by

$$\frac{1}{\mu_1\big(A_{[0,1]^3}\big)} \cdot \mu_1\big(\,\cdot\, \cap A_{[0,1]^3}\big),$$

where $A_{[0,1]^3} = \big\{E \in A(3,1) : E \cap [0,1]^3 \neq \varnothing\big\}$, and where μ_1 denotes the G_3-invariant measure on $A(3,1)$ from Theorem 4.26. Since $K \subset [0,1]^3$, the Crofton formula (Theorem 4.27) yields that, for each $j \in \{0,1\}$,

$$\mathbb{E}\Big[V_j(K \cap X_1)\Big] = \frac{\int_{A(3,1)} V_j(F \cap K)\,\mathrm{d}\mu_1(F)}{\int_{A(3,1)} V_0\big(F \cap [0,1]^3\big)\,\mathrm{d}\mu_1(F)} = \frac{c_{j,3}^{1,2+j} \cdot V_{2+j}(K)}{c_{0,3}^{1,2} \cdot V_2\big([0,1]^3\big)}.$$

Hence, for every $j \in \{0,1\}$,

$$V_{2+j}(K) = \frac{c_{0,3}^{1,2}}{c_{j,3}^{1,2+j}} \cdot V_2\big([0,1]^3\big) \cdot \mathbb{E}\Big[V_j(K \cap X_1)\Big] = \frac{2\pi \cdot j! \cdot \kappa_j}{(2+j)! \cdot \kappa_{2+j}} \cdot 3 \cdot \mathbb{E}\Big[V_j(K \cap X_1)\Big].$$

Choosing $j = 0$, we obtain

$$\widehat{V}_2(K) = 3 \cdot V_0(K \cap X_1)$$

as an unbiased estimator for $V_2(K)$, so $6 \cdot V_0(K \cap X_1)$ is an unbiased estimator of the surface area of K. Choosing $j = 1$ gives

$$\widehat{V}_3(K) = \frac{3}{2} \cdot V_1(K \cap X_1)$$

as an unbiased estimator for the volume of K.

Problem 2 (Lemma 5.2)

Let $\mathfrak{m}$ be a tessellation in $\mathbb{R}^d$ and $K \in \mathfrak{m}$.

a) Prove that one can find finitely many cells $K_1, \ldots, K_\ell \in \mathfrak{m} \setminus \{K\}$ such that $K_j \cap K \neq \varnothing$ $(j = 1, \ldots, \ell)$ and

$$\partial K = \bigcup_{j=1}^{\ell} (K_j \cap K).$$

b) Prove that K is a polytope.

Proposed solution:

a) By the local finiteness of $\mathfrak{m}$ there can only exist finitely many cell of $\mathfrak{m}$ that intersect K, and since $\bigcup_{C \in \mathfrak{m}} C = \mathbb{R}^d$ there exists at least one cell that intersects K. Hence, we denote by $K_1, \ldots, K_\ell \in \mathfrak{m} \setminus \{K\}$ all cells of $\mathfrak{m}$ such that $K_j \cap K \neq \varnothing$ for each $j \in \{1, \ldots, \ell\}$.

Now, let $x \in \partial K$. There exists a sequence $\{x_n : n \in \mathbb{N}\} \subset \mathbb{R}^d \setminus K$ such that $x_n \to x$ (as $n \to \infty$). The set $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact and hence intersects only finitely many cells of $\mathfrak{m}$, so we find a cell $K' \in \mathfrak{m} \setminus \{K\}$ which contains infinitely many of the points in $(x_n)_{n \in \mathbb{N}}$. As the sequence converges and K' is compact, we have $x \in K'$. However, since $x \in K$, and since the sets $K_1, \ldots, K_\ell$ are the only cells to intersect K, we must have $K' = K_j$ for some $j \in \{1, \ldots, \ell\}$. In particular, $x \in \bigcup_{j=1}^{\ell} (K_j \cap K)$.

For the reverse inclusion, assume that $x \in K_j \cap K$ for some $j \in \{1, \ldots, \ell\}$. As K_j and K are cells of the tessellation $\mathfrak{m}$, they satisfy $\operatorname{int}(K_j) \cap \operatorname{int}(K) = \varnothing$. Therefore, $x \notin \operatorname{int}(K_j) \cap \operatorname{int}(K)$. Suppose that $x \in \partial K_j \cap \operatorname{int}(K)$. As cells of a tessellation have non-empty interior, we find $y \in \operatorname{int}(K_j)$, and by convexity of K_j,

$$[y, x) := \big\{\lambda \cdot y + (1 - \lambda) \cdot x \, : \, \lambda \in (0, 1]\big\} \subset \operatorname{int}(K_j).$$

We also have $x \in \operatorname{int}(K)$ which implies $\operatorname{int}(K_j) \cap \operatorname{int}(K) \neq \varnothing$, a contradiction. We conclude that $x \notin \operatorname{int}(K)$, which leaves $x \in \partial K$.

b) Let $K_1, \ldots, K_\ell \in \mathfrak{m} \setminus \{K\}$ be the sets from part a). For each $j \in \{1, \ldots, \ell\}$ we have $\operatorname{int}(K_j) \cap \operatorname{int}(K) = \varnothing$, hence we find a hyperplane H_j which separates K_j and K, that is, $K_j \subset H_j^-$ and $K \subset H_j^+$, where H_j^- and H_j^+ denote the closed half-spaces whose boundary is H_j. We prove that

$$K = \bigcap_{j=1}^{\ell} H_j^+.$$

By construction, we have $K \subset \bigcap_{j=1}^{\ell} H_j^+$. Thus, let $x \in \bigcap_{j=1}^{\ell} H_j^+$. Suppose that $x \notin K$. As each cell of a tessellation has non-empty interior, we find an element

$$y \in \operatorname{int}(K) \subset \operatorname{int}\bigg(\bigcap_{j=1}^{\ell} H_j^+\bigg).$$

As the intersection of the half-spaces is convex, we have $(x, y) := \{\lambda \cdot x + (1 - \lambda) \cdot y : \lambda \in (0, 1)\} \subset \operatorname{int}\big(\bigcap_{j=1}^{\ell} H_j^+\big)$. Furthermore, there exists some

$$z \in (x, y) \cap \partial K \subset \operatorname{int}\bigg(\bigcap_{j=1}^{\ell} H_j^+\bigg).$$

By part a), we must have $z \in K_i$ for one $i \in \{1, \ldots, \ell\}$. The hyperplane H_i, however, was chosen such that $\operatorname{int}(H_i^+) \cap K_i = \varnothing$ which is a contradiction to the fact that $z \in \operatorname{int}\big(\bigcap_{j=1}^{\ell} H_j^+\big)$.

Problem 3 (Example 5.6)

Let $\varphi \in N_s(\mathbb{R}^d)$ such that $\varphi(\mathbb{R}^d) > 0$. For $x \in \varphi$ define the Voronoi cell of x as

$$C(\varphi, x) = \Big\{ z \in \mathbb{R}^d \ : \ \|z - x\| \leqslant \|z - y\| \text{ for each } y \in \varphi \Big\}.$$

Prove that all Voronoi cells are bounded if $\text{conv}(\varphi) = \mathbb{R}^d$, that is, if the convex hull of the points in φ is $\mathbb{R}^d$.

Proposed solution: Suppose there exists an unbounded Voronoi cell $C(\varphi, x)$ for some $x \in \varphi$. As $C(\varphi, x)$ is convex, we find $u \in \mathbb{R}^d$ with $\|u\| = 1$ such that

$$S := \big\{ x + \alpha \cdot u : \alpha \geqslant 0 \big\} \subset C(\varphi, x).$$

Let $\alpha > 0$. Then, trivially, $x \in \partial\big(B(x + \alpha \cdot u, \alpha)\big)$. If there exists some

$$y \in \varphi \cap \text{int}\big(B(x + \alpha \cdot u, \alpha)\big),$$

then

$$\big\| y - (x + \alpha \cdot u) \big\| < \alpha \qquad \text{as well as} \qquad \big\| x - (x + \alpha \cdot u) \big\| = \alpha$$

and therefore

$$x + \alpha \cdot u \in C(\varphi, y) \qquad \text{as well as} \qquad x + \alpha \cdot u \notin C(\varphi, x),$$

which is a contradiction to the fact that $S \subset C(\varphi, x)$. Hence, we must have

$$\varphi \cap \text{int}\big(B(x + \alpha \cdot u, \alpha)\big) = \varnothing$$

for all $\alpha > 0$. Thus, the open half-space

$$\bigcup_{n=1}^{\infty} \text{int}\big(B(x + n \cdot u, n)\big)$$

does not contain points of φ which is a contradiction to the assumption that $\text{conv}(\varphi) = \mathbb{R}^d$.