Solutions for Work Sheet 1

Problem 1 (Bertrand's paradox)

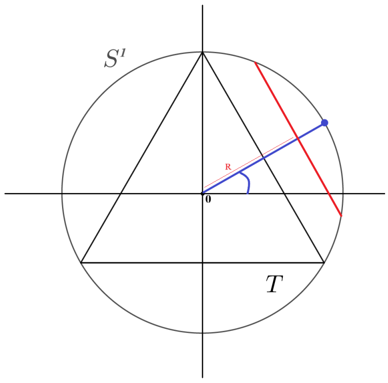
Consider the unit circle $S^1 \subset \mathbb{R}^2$ and an equilateral triangle T whose vertices lie on S^1. What is the probability that a randomly chosen circular chord is longer than the edges of the triangle? Does the probability depend on the choice of your model for this situation?

Proposed solution:

Notice that the chords given through the equilateral triangle have length $\sqrt{3}$ as we consider the unit circle. Thus, we can rephrase the question as follows: What is the probability that a random chord in the unit circle is longer than $\sqrt{3}$?

The following considerations serve to illustrate that the solution to this problem highly depends on our conception of a 'random chord', that is, the probability we obtain as a solution depends crucially on our choice of the model for this situation. In fact, we provide 3 reasonable models which yield different probabilities. We denote the random length of the chord by L.

- The first model for our random chord is as illustrated in the picture: The direction of the chord is determined by a random variable (a random angle) ϑ which is distributed uniformly over $(0, 2\pi)$, and its length is determined by its distance from the origin, that is, by another random variable R (independent of ϑ) which is distributed uniformly over $(0, 1)$. The length of the corresponding chord is (by the Pythagorean theorem) $L = 2\sqrt{1 - R^2}$, and we obtain



$$\mathbb{P}\big(L \geqslant \sqrt{3}\big) = \mathbb{P}\big(2\sqrt{1 - R^2} \geqslant \sqrt{3}\big) = \mathbb{P}\big(R \leqslant \tfrac{1}{2}\big) = \tfrac{1}{2}.$$

- The second model specifies the random chord through its random endpoints. Anticlockwise, we specify the starting point via some angle ϑ [uniformly distributed over $(0, 2\pi)$] and determine the endpoint by adding a second angle, namely ϑ' [also uniformly distributed over $(0, 2\pi)$ and independent of ϑ]. (In other words, ϑ' provides the circular distance between starting point and endpoint.) By the geometric definition of the sine in right-angled triangles, the length of the chord is $L = 2\sin(\vartheta'/2)$, and therefore

$$\mathbb{P}\big(L \geqslant \sqrt{3}\big) = \mathbb{P}\Big(\sin(\vartheta'/2) \geqslant \sqrt{3}/2\Big) = \mathbb{P}\Big(\tfrac{\vartheta'}{2} \in \big(\tfrac{\pi}{3}, \tfrac{2\pi}{3}\big)\Big) = \mathbb{P}\Big(\vartheta' \in \big(\tfrac{2\pi}{3}, \tfrac{4\pi}{3}\big)\Big) = \frac{\frac{4\pi}{3} - \frac{2\pi}{3}}{2\pi} = \frac{1}{3}.$$

- For the third model, notice that any chord is uniquely determined through its midpoint. Thus, the random chord is given through its random midpoint M which we take to be uniformly distributed over the unit disk $\{x \in \mathbb{R}^2 : \|x\|_2 < 1\}$. Similar to the first model, the length of the chord is $L = 2\sqrt{1 - \|M\|_2^2}$, and we obtain

$$\mathbb{P}\big(L \geqslant \sqrt{3}\big) = \mathbb{P}\Big(\|M\|_2 \leqslant \tfrac{1}{2}\Big) = \frac{Vol\big(B(0, 1/2)\big)}{Vol\big(B(0, 1)\big)} = \frac{1}{4}.$$

Problem 2 (Theorem 1.2)

Let $(E, \mathcal{O}_E)$ be a locally compact Hausdorff space with a countable base of the topology. Denote by $\mathcal{F}$ the collection of closed subsets of E and write $\mathcal{C}$ for the collection of compact subsets of E. For $A \subset E$, let $\mathcal{F}^A := \{F \in \mathcal{F} : F \cap A = \varnothing\}$ and $\mathcal{F}_A := \{F \in \mathcal{F} : F \cap A \neq \varnothing\}$. Prove the following assertions.

(T) The topology $\mathcal{O}_E$ has a countable base $\mathcal{D}$ which consists of open, relatively compact sets such that any set $O \in \mathcal{O}_E$ is the union of all sets $D \in \mathcal{D}$ that satisfy $\mathrm{cl}D \subset O$.

(1) The space $(\mathcal{F}, \mathcal{O}_\mathcal{F})$ is a compact Hausdorff space with a countable base of the topology (in particular, it is metrizable by Urysohn's metrization theorem).

(2) The subspace $\mathcal{F}' := \mathcal{F} \setminus \{\varnothing\}$ with the subspace topology is locally compact.

(3) The family $\{\mathcal{F}^C \mid C \in \mathcal{C}\}$ is a neighborhood base of $\varnothing$.

Proposed solution:

(T) Denote by $\widetilde{\mathcal{D}}$ a countable base of $\mathcal{O}_E$. Choose $\mathcal{D} \subset \widetilde{\mathcal{D}}$ as the collection of all open, relatively compact sets in $\widetilde{\mathcal{D}}$. Let $O \in \mathcal{O}_E$ be arbitrary. For $x \in O$ we find an open neighborhood U of x which is relatively compact [since E is locally compact]. As the set $U \cap O$ is open and contains x, there exists an open neighborhood V of x such that $\mathrm{cl}V \subset U \cap O$. *To verify this last claim, let K be a compact neighborhood of x. If $K \subset U \cap O$, choose $V := \mathrm{int}(K)$. Otherwise, $C := K \cap \big(E \setminus (U \cap O)\big)$ is non-empty and compact, and since E is Hausdorff, we find for every point $y \in C$ open neighborhoods $O_{x,(y)}, O_y \in \mathcal{O}_E$ of x and y, respectively, which separate the points. Then $\bigcup_{y \in C} O_y \supset C$ is an open cover of C for which we find a finite subcover $\{O_{y_1}, \dots, O_{y_m}\}$, say. Choosing for each O_{y_j} the corresponding $O_{x,(y_j)}$ as above, the set $V := \bigcap_{j=1}^m O_{x,(y_j)}$ satisfied the requirements.* Now, since $\widetilde{\mathcal{D}}$ is a base of the topology, we find $D = D_x \in \widetilde{\mathcal{D}}$ with $x \in D \subset V$, and from $\mathrm{cl}D \subset \mathrm{cl}V \subset U \cap O$ we conclude that D is relatively compact (as U is such) and $O = \bigcup_{x \in O} D_x$, which is the claim.

(1) *The Hausdorff property:* Let $F, F' \in \mathcal{F}$ be such that $F \neq F'$. Thus we find (without loss of generality) $x \in F \setminus F'$. As $E \setminus F'$ is open, there exists a set $D \in \mathcal{D}$ [where $\mathcal{D}$ is a in (T)] with $x \in D$ as well as $\mathrm{cl}D \cap F' = \varnothing$. Then, $F \in \mathcal{F}_D$, $F' \in \mathcal{F}^{\mathrm{cl}D}$, and $\mathcal{F}_D$, $\mathcal{F}^{\mathrm{cl}D}$ are open sets in $\mathcal{F}$ with $\mathcal{F}_D \cap \mathcal{F}^{\mathrm{cl}D} = \varnothing$.

A countable base of $\mathcal{O}_\mathcal{F}$ is given through

$$\tau' := \Big\{\mathcal{F}^{\mathrm{cl}D_1 \cup \dots \cup \mathrm{cl}D_m}_{D'_1, \dots, D'_k} \ : \ D_i, D'_j \in \mathcal{D},\ m \in \mathbb{N},\ k \in \mathbb{N}_0\Big\}.$$

Indeed, let $\mathcal{F}^C_{G_1,\dots,G_k}$ ($G_1, \dots, G_k \in \mathcal{O}_E$, $k \in \mathbb{N}_0$, $C \in \mathcal{C}$) be arbitrary, and let $F \in \mathcal{F}^C_{G_1,\dots,G_k}$. By definition, for each G_j we find a set $D'_j \in \mathcal{D}$ such that $\mathrm{cl}D'_j \subset G_j$ and $D'_j \cap F \neq \varnothing$. Moreover, $E \setminus F$ is open, so by (T) we find sets $D_1, D_2, \ldots \in \mathcal{D}$ such that $\mathrm{cl}D_i \subset E \setminus F$ (for each $i \in \mathbb{N}$) and $\bigcup_{i=1}^\infty D_i = E \setminus F$. As $C \subset E \setminus F$ is compact, we find $m \in \mathbb{N}$ such that $C \subset \bigcup_{i=1}^m D_i$. We conclude that $F \in \mathcal{F}^{\mathrm{cl}D_1 \cup \dots \cup \mathrm{cl}D_m}_{D'_1,\dots,D'_k} \subset \mathcal{F}^C_{G_1,\dots,G_k}$.

Compactness: A subbase of the topology $\mathcal{O}_\mathcal{F}$ is given by

$$\big\{\mathcal{F}^C \ : \ C \in \mathcal{C}\big\} \cup \big\{\mathcal{F}_G \ : \ G \in \mathcal{O}_E\big\}.$$

According to Alexander's subbase theorem, it suffices to prove that any cover of $\mathcal{F}$ with sets from this subbase has a finite subcover. Thus, let I, J be arbitrary index sets, and let $C_i \in \mathcal{C}$ for $i \in I$ and $G_j \in \mathcal{O}_E$ for $j \in J$ be such that

$$\mathcal{F} = \bigcup_{i \in I} \mathcal{F}^{C_i} \ \cup \ \bigcup_{j \in J} \mathcal{F}_{G_j}.$$

This fact is equivalent to

$$\bigcap_{i \in I} \mathcal{F}_{C_i} \ \cap \ \bigcap_{j \in J} \mathcal{F}^{G_j} = \varnothing$$

which, in turn, is equivalent to $\bigcap_{i \in I} \mathcal{F}^G_{C_i} = \varnothing$ when choosing $G := \bigcup_{j \in J} G_j \in \mathcal{O}_E$. Hence, we find $i_0 \in I$ such that $C_{i_0} \subset G$, since otherwise $(E \setminus G) \cap C_i \neq \varnothing$ for every $i \in I$ which would bring about $E \setminus G \in \bigcap_{i \in I} \mathcal{F}^G_{C_i}$, a contradiction. Therefore, we have $C_{i_0} \subset \bigcup_{j \in J} G_j$ and, as C_{i_0} is compact, we find a finite subset of indices $J_0 \subset J$ with $C_{i_0} \subset \bigcup_{j \in J_0} G_j$. We conclude that $\bigcap_{j \in J_0} \mathcal{F}^{G_j}_{C_{i_0}} = \varnothing$, so

$$\mathcal{F}^{C_{i_0}} \ \cup \ \bigcup_{j \in J_0} \mathcal{F}_{G_j} = \mathcal{F},$$

which proves the claim.

(2) For any $C \in \mathcal{C}$, the collection $\mathcal{F}_C = \mathcal{F} \setminus \mathcal{F}^C$ is closed (and hence compact) in $\mathcal{F}$. As $\varnothing \notin \mathcal{F}_C$, $\mathcal{F}_C$ is also compact in $\mathcal{F}'$. It remains to show that any point $F \in \mathcal{F}'$ admits a compact neighborhood. To this end, let $x \in F$ and let D be an open, relatively compact neighborhood of x. Then, $\mathcal{F}_{\mathrm{cl}D}$ is a compact subset of $\mathcal{F}$ which contains the open set $\mathcal{F}_D$, and $F \in \mathcal{F}_D$.

(3) We need to show that, for every neighborhood V of $\varnothing$ in $\mathcal{F}$, there exists a $C \in \mathcal{C}$ so that $\mathcal{F}^C \subset V$. Thus. take a neighborhood V of $\varnothing$, and a set $O \in \mathcal{O}_{\mathcal{F}}$ with $\varnothing \in O \subset V$. By definition of the Fell-topology, O is a union of sets

$$\mathcal{F}^C_{G_1,\dots,G_k}, \quad G_1,\dots,G_k \in \mathcal{O}_E, \quad C \in \mathcal{C}, \quad k \in \mathbb{N}_0.$$

As $\varnothing \in O$, there exist $k \in \mathbb{N}_0$, and $G_1,\dots,G_k \in \mathcal{O}_E$, $C \in \mathcal{C}$ such that

$$\varnothing \in \mathcal{F}^C_{G_1,\dots,G_k}.$$

However, $\varnothing \cap A = \varnothing$ for any $A \subset E$, so $k = 0$ and we are done.

Problem 3 (Theorem 1.3)

Let $(F_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{F}$, and $F \in \mathcal{F}$. Consider the following properties:

(1) $F_i \longrightarrow F$ in $(\mathcal{F}, \mathcal{O}_{\mathcal{F}})$, as $i \to \infty$.

(2) (a) $G \in \mathcal{G}$, $G \cap F \neq \varnothing \implies G \cap F_i \neq \varnothing$ for all $i \in \mathbb{N}$ except finitely many,
(b) $C \in \mathcal{C}$, $C \cap F = \varnothing \implies C \cap F_i = \varnothing$ for all $i \in \mathbb{N}$ except finitely many.

(3) (α) For each $x \in F$ and all but finitely many $i \in \mathbb{N}$ there exist some $x_i \in F_i$ such that $x_i \longrightarrow x$, as $i \to \infty$.
(β) For any subsequence $\left(F_{i_k}\right)_{k\in\mathbb{N}}$ and points $x_{i_k} \in F_{i_k}$ such that $x_{i_k} \overset{k\to\infty}{\longrightarrow} x$, we have $x \in F$.

Show that (2) and (3) are equivalent [the equivalence of (1) and (2) was discussed in the lecture].

Proposed solution:

We show that (a) and (α) as well as (b) and (β) are equivalent.

(a) $\Rightarrow$ (α) Let $x \in F$, and let $G_1 \supset G_2 \supset \cdots$ be a neighborhood base of x consisting of open sets. Apparently, $G_k \cap F \neq \varnothing$. By (a) we find for any $k \in \mathbb{N}$ some $i_k \in \mathbb{N}$ such that $G_k \cap F_i \neq \varnothing$ for $i \geqslant i_k$. Without loss of generality, assume $i_1 < i_2 < \cdots$. Choose a sequence $(x_\ell)_{\ell \geqslant i_1}$ with

$$x_\ell \in G_k \cap F_\ell \quad \text{for } \ell = i_k, \dots, i_{k+1} - 1,\ k \in \mathbb{N}.$$

Conclude that $x_\ell \to x$, as $\ell \to \infty$.

(α) $\Rightarrow$ (a) Let $G \in \mathcal{G}$ with $G \cap F \neq \varnothing$, and $x \in G \cap F$. In view of (α), we find an index $i_0 \in \mathbb{N}$ and elements $x_i \in F_i$, for $i \geqslant i_0$, such that $x_i \to x$. By the definition of convergence in topological spaces, we have $x_i \in G$ for i sufficiently large. Hence $F_i \cap G \neq \varnothing$ for all but finitely many $i \in \mathbb{N}$.

(b) $\Rightarrow$ (β) Let $x_{i_k} \in F_{i_k}$ with $x_{i_k} \overset{k\to\infty}{\longrightarrow} x$. If $x \notin F$ we find a compact neighborhood C of x with $C \cap F = \varnothing$ (apply Problem 1 to $E \setminus F$). Part (b) implies $C \cap F_i = \varnothing$ for $i \geqslant i_0$, where $i_0 \in \mathbb{N}$ is chosen large enough. This contradicts the convergence of the subsequence.

(β) $\Rightarrow$ (b) Let $C \in \mathcal{C}$ with $C \cap F = \varnothing$. Assume, for a contradiction, that there exists a sequence of indices with $C \cap F_{i_k} \neq \varnothing$ for $k \in \mathbb{N}$. Choose $x_{i_k} \in C \cap F_{i_k}$. Since C is compact and hence sequentially compact (as E is metrizable), we can find a further subsequence $\left(x_{i_{k_\ell}}\right)_{\ell\in\mathbb{N}}$ so that $x_{i_{k_\ell}} \overset{\ell\to\infty}{\longrightarrow} x \in C$. By (β), we get $x \in F$ and hence a contradiction to $C \cap F = \varnothing$.