Solutions for Work Sheet 8

Problem 1 (Constructing a 'germ-grain model' which is not closed)

Find a dimension $d \in \mathbb{N}$, a probability measure $\mathbb{Q}$ on $\mathcal{C}^d$, and a point process Φ in $\mathbb{R}^d$ such that the germ-grain model corresponding to the independent $\mathbb{Q}$-marking of Φ is almost surely not a closed set.

Proposed solution: We choose $d = 2$, and let $\Phi = \sum_{j=1}^{\infty} \delta_{(j, j^{-1})}$ be deterministic. Let X be a discrete random variable with

$$\mathbb{P}(X = k) = \frac{1}{k} - \frac{1}{k+1}, \qquad k \in \mathbb{N}.$$

We choose as $\mathbb{Q}$ the distribution of the random compact set $Z_0 := [-X, 0] \times \{0\}$. The corresponding germ-grain model Z almost surely has an accumulation point in the origin, but the origin is almost surely not contained in Z. Indeed, notice that the independent $\mathbb{Q}$-marking of Φ is given through

$$\Psi = \sum_{j=1}^{\infty} \delta_{\left((j, j^{-1}),\ [-X_j, 0] \times \{0\}\right)},$$

where $X_1, X_2, \ldots$ are i.i.d. copies of X, and that

$$\sum_{k=1}^{\infty} \mathbb{P}(X_k \geqslant k) = \sum_{k=1}^{\infty} \frac{1}{k} = \infty,$$

so the Borel-Cantelli lemma yields that (almost surely) infinitely many of the events

$$\left\{ \left(0, \tfrac{1}{k}\right) \in \left(k, \tfrac{1}{k}\right) + \left([-X_k, 0] \times \{0\}\right) \right\}$$

occur and thus Z has an accumulation point in the origin $(0, 0)$. However,

$$\mathbb{P}\big((0,0) \in Z\big) = \mathbb{P}\left((0,0) \in \bigcup_{k=1}^{\infty} \left(\left(k, \tfrac{1}{k}\right) + \left([-X_k, 0] \times \{0\}\right) \right) \right) = 0,$$

that is to say, Z is almost surely not closed.

Problem 2 (The mean covariogram of a Boolean model)

Let $Z = \bigcup_{k=1}^{\infty} (Z_k + \xi_k)$ be a stationary Boolean model in $\mathbb{R}^d$ with parameters γ and $\mathbb{Q}$. Let $Z_0 \sim \mathbb{Q}$ be the typical grain and $p_Z := \mathbb{E}\left[\lambda^d(Z \cap [0,1]^d)\right] = \mathbb{P}(0 \in Z)$ the volume fraction of Z (see Theorem 1.31). Further, let $\Phi = \sum_{j=1}^{\infty} \delta_{\xi_j}$ be the (stationary) Poisson process of the germs.

a) Prove that, for any $C \in \mathcal{C}^d$ and $x, y \in \mathbb{R}^d$,

$$(C + y) \cap \{x, 0\} \neq \varnothing \quad \Longleftrightarrow \quad y \in C^* \cup (x + C^*).$$

b) The mean covariogram C_0 of Z is given through

$$C_0(x) := \mathbb{E}\left[\lambda^d\big(Z_0 \cap (Z_0 + x)\big)\right], \qquad x \in \mathbb{R}^d.$$

Prove that, for each $x \in \mathbb{R}^d$,

$$\mathbb{E}\Big[\lambda^d\big(Z_0 \cup (Z_0 - x)\big)\Big] = 2 \cdot \mathbb{E}\big[\lambda^d(Z_0)\big] - C_0(x).$$

c) Use parts a) and b) to show that

$$\mathbb{P}\big(0 \in Z,\ x \in Z\big) = p_Z^2 + (1 - p_Z)^2\big(e^{\gamma \cdot C_0(x)} - 1\big).$$

Proposed solution:

a) For $C \in \mathcal{C}^d$ and $x, y \in \mathbb{R}^d$ we have

$$\begin{aligned}
(C + y) \cap \{x, 0\} \neq \varnothing \quad &\iff \quad \text{there exists } z \in C \text{ such that } z + y = x \text{ or } z + y = 0 \\
&\iff \quad \text{there exists } z \in C \text{ such that } y = x - z \text{ or } y = -z \\
&\iff \quad y \in (x + C^*) \cup C^*.
\end{aligned}$$

b) For $x \in \mathbb{R}^d$ we have

$$\begin{aligned}
\mathbb{E}\Big[\lambda^d\big(Z_0 \cup (Z_0 - x)\big)\Big] &= \mathbb{E}\Big[\lambda^d(Z_0) + \lambda^d(Z_0 - x) - \lambda^d\big(Z_0 \cap (Z_0 - x)\big)\Big] \\
&= \mathbb{E}\Big[2 \cdot \lambda^d(Z_0) - \lambda^d\big((Z_0 + x) \cap Z_0\big)\Big] \\
&= 2 \cdot \mathbb{E}\big[\lambda^d(Z_0)\big] - C_0(x).
\end{aligned}$$

c) From Equation (3.11) of the lecture notes we know that

$$p_Z = 1 - e^{-\gamma \cdot C_0(0)}.$$

Therefore,

$$\begin{aligned}
p_Z^2 + (1 - p_Z)^2\big(e^{\gamma \cdot C_0(0)} - 1\big) &= p_Z^2 + (1 - p_Z)^2\left(\frac{1}{-\big(1 - e^{-\gamma \cdot C_0(0)}\big) + 1} - 1\right) \\
&= p_Z^2 + (1 - p_Z)^2\left(\frac{1}{1 - p_Z} - 1\right) \\
&= p_Z^2 + 1 - p_Z - (1 - p_Z)^2 \\
&= p_Z \\
&= \mathbb{P}(0 \in Z),
\end{aligned}$$

which is the claim for $x = 0$. Now, let $x \neq 0$. Then, (using Theorem 3.16 a) and Theorem 3.6)

$$\begin{aligned}
\mathbb{P}\big(0 \in Z,\, x \in Z\big) &= 1 - \mathbb{P}\big(\{0 \notin Z\} \cup \{x \notin Z\}\big) \\
&= 1 - \mathbb{P}(0 \notin Z) - \mathbb{P}(x \notin Z) + \mathbb{P}\big(0 \notin Z,\ x \notin Z\big) \\
&= 1 - \mathbb{P}(0 \notin Z) - \mathbb{P}(x \notin Z) + \mathbb{P}\big(Z \cap \{0, x\} = \varnothing\big) \\
&= 1 - \mathbb{P}(0 \notin Z) - \mathbb{P}(x \notin Z) + \Big(1 - T_Z\big(\{0, x\}\big)\Big) \\
&= 2p_Z - 1 + \exp\left(-\gamma \int_{\mathbb{R}^d} \int_{\mathcal{C}^d} \mathbb{1}\big\{(K + y) \cap \{0, x\} \neq \varnothing\big\} \,\mathrm{d}\mathbb{Q}(K)\,\mathrm{d}\lambda^d(y)\right) \\
&= 2p_Z - 1 + \exp\left(-\gamma \int_{\mathbb{R}^d} \mathbb{E}\Big[\mathbb{1}\big\{(Z_0 + y) \cap \{0, x\} \neq \varnothing\big\}\Big] \mathrm{d}\lambda^d(y)\right) \qquad (1)
\end{aligned}$$

Part a) implies

$$\begin{aligned}
\int_{\mathbb{R}^d} \mathbb{1}\Big\{(Z_0 + y) \cap \{0, x\} \neq \varnothing\Big\} \,\mathrm{d}\lambda^d(y) &= \int_{\mathbb{R}^d} \mathbb{1}\Big\{y \in (x + Z_0^*) \cup Z_0^*\Big\} \,\mathrm{d}\lambda^d(y) \\
&= \lambda^d\big((x + Z_0^*) \cup Z_0^*\big) \\
&= \lambda^d\big((Z_0 - x) \cup Z_0\big).
\end{aligned}$$

Therefore, Equation (1) simplifies to

$$\mathbb{P}\big(0 \in Z,\, x \in Z\big) = 2p_Z - 1 + \exp\Big(-\gamma \cdot \mathbb{E}\Big[\lambda^d\big(Z_0 \cup (Z_0 - x)\big)\Big]\Big).$$

Part b) and Equation (3.11) of the lecture notes yield

$$\begin{aligned}\mathbb{P}\big(0 \in Z,\, x \in Z\big) &= 2p_Z - 1 + \exp\Big(-\gamma \cdot \big(2\,\mathbb{E}\big[\lambda^d(Z_0)\big] - C_0(x)\big)\Big) \\ &= 2p_Z - 1 + \exp\Big(-2\gamma \cdot \mathbb{E}\big[\lambda^d(Z_0)\big]\Big)\exp\big(\gamma \cdot C_0(x)\big) \\ &= 2p_Z - 1 + (1 - p_Z)^2 \cdot \exp\big(\gamma \cdot C_0(x)\big) \\ &= p_Z^2 + (1 - p_Z)^2\big(e^{\gamma \cdot C_0(x)} - 1\big).\end{aligned}$$

Problem 3 (The concept of visibility)

Let $Z = \bigcup_{k=1}^{\infty}(Z_k + \xi_k)$ be a stationary Boolean model in $\mathbb{R}^d$ with intensity $\gamma > 0$ and shape distribution $\mathbb{Q}$ which is concentrated on those sets in $\mathcal{C}^d$ that contain $0 \in \mathbb{R}^d$. Further, let $v_d := \int_{\mathcal{C}^d} \lambda^d(C)\, \mathrm{d}\mathbb{Q}(C) < \infty$. A point $z \in Z$ is called visible if

$$\big|\big\{k \in \mathbb{N} : z \in Z_k + \xi_k\big\}\big| = 1,$$

that is, if z is contained in precisely one grain of the Boolean model. Define

$$\Phi := \sum_{j=1}^{\infty} \delta_{\xi_j} \cdot \mathbb{1}\big\{\xi_j \text{ is visible}\big\}.$$

Prove that Φ is a stationary point process in $\mathbb{R}^d$ with intensity $\gamma_{\mathbb{Q}} := \gamma \cdot e^{-\gamma \cdot \mathbb{E}[\lambda^d(Z_1)]}$.

Hint: For the calculation of the intensity you may use, without proof, Mecke's formula, which states that a Poisson process η with intensity measure Λ satisfies

$$\mathbb{E}\Big[\int_{\mathbb{R}^d} f(x, \eta)\, \mathrm{d}\eta(x)\Big] = \mathbb{E}\Big[\int_{\mathbb{R}^d} f(x, \eta + \delta_x)\, \mathrm{d}\Lambda(x)\Big]$$

for every measurable map $f : \mathbb{R}^d \times N(\mathbb{R}^d) \to [0, \infty)$.

Proposed solution: Let $\Psi := \sum_{j=1}^{\infty} \delta_{(\xi_j, Z_j)}$ be the stationary, independently marked Poisson process which underlies the Boolean model at hand. Put

$$\mu_\Psi(x) := \sum_{j=1}^{\infty} \mathbb{1}_{Z_j + \xi_j}(x), \qquad x \in \mathbb{R}^d.$$

We obtain that a point $x \in \mathbb{R}^d$ is visible if, and only if, $\mu_\Psi(x) = 1$. We can thus write

$$\Phi(A) = \sum_{j=1}^{\infty} \delta_{\xi_j}(A) \cdot \mathbb{1}\big\{\mu_\Psi(\xi_j) = 1\big\} = \int_{\mathbb{R}^d \times \mathcal{C}^d} \mathbb{1}\{x \in A\} \cdot \mathbb{1}\big\{\mu_\Psi(x) = 1\big\}\, \mathrm{d}\Psi(x, K), \qquad A \in \mathcal{B}(\mathbb{R}^d),$$

which shows that Φ is a point process in $\mathbb{R}^d$. The stationarity of Φ follows from

$$\begin{aligned}(\Phi + t)(A) = \Phi(A - t) &= \int_{\mathbb{R}^d \times \mathcal{C}^d} \mathbb{1}\{x \in A - t\} \cdot \mathbb{1}\big\{\mu_\Psi(x) = 1\big\}\, \mathrm{d}\Psi(x, K) \\ &\overset{d}{=} \int_{\mathbb{R}^d \times \mathcal{C}^d} \mathbb{1}\{x \in A - t\} \cdot \mathbb{1}\big\{\mu_{T_{-t}(\Psi)}(x) = 1\big\}\, \mathrm{d}\Psi(x, K) \\ &\overset{d}{=} \int_{\mathbb{R}^d \times \mathcal{C}^d} \mathbb{1}\{x - t \in A - t\} \cdot \mathbb{1}\big\{\mu_{T_{-t}(\Psi)}(x - t) = 1\big\}\, \mathrm{d}\Psi(x, K) \\ &= \int_{\mathbb{R}^d \times \mathcal{C}^d} \mathbb{1}\{x \in A\} \cdot \mathbb{1}\big\{\mu_\Psi(x) = 1\big\}\, \mathrm{d}\Psi(x, K) \\ &= \Phi(A),\end{aligned}$$

where $A \in \mathcal{B}(\mathbb{R}^d)$ and $t \in \mathbb{R}^d$ are arbitrary. To calculate the intensity of Φ, note that Mecke's formula and the stationarity of Z imply

$$
\begin{aligned}
\gamma_{\mathbb{Q}} &= \mathbb{E}\left[\int_{\mathbb{R}^d} \mathbb{1}\{x \in [0,1]^d\}\, \mathrm{d}\Phi(x)\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}^d \times \mathcal{C}^d} \mathbb{1}\{x \in [0,1]^d\} \cdot \mathbb{1}\{\mu_\Psi(x) = 1\}\, \mathrm{d}\Psi(x,K)\right] \\
&= \gamma \int_{\mathbb{R}^d} \int_{\mathcal{C}^d} \mathbb{E}\Big[\mathbb{1}\{x \in [0,1]^d\} \cdot \mathbb{1}\{\mu_{\Psi+\delta_{(x,K)}}(x) = 1\}\Big]\, \mathrm{d}\mathbb{Q}(K)\, \mathrm{d}\lambda^d(x) \\
&= \gamma \int_{\mathbb{R}^d} \mathbb{E}\Big[\mathbb{1}\{x \in [0,1]^d\} \cdot \mathbb{1}\{\mu_\Psi(x) = 0\}\Big]\, \mathrm{d}\lambda^d(x) \\
&= \gamma \int_{\mathbb{R}^d} \mathbb{1}\{x \in [0,1]^d\} \cdot \mathbb{E}\Big[\mathbb{1}\{x \notin Z\}\Big]\, \mathrm{d}\lambda^d(x) \\
&= \gamma \cdot \mathbb{P}(0 \notin Z) \\
&= \gamma \cdot \big(1 - p_Z\big) \\
&= \gamma \cdot e^{-\gamma \cdot \mathbb{E}[\lambda^d(Z_1)]}.
\end{aligned}
$$

Note that in the fourth equality we have used that $\mathbb{Q}$ is concentrated on sets that contain $0 \in \mathbb{R}^d$, and in the last equality Equation (3.11) of the lecture notes comes into play.

Problem 4 (Spherical contact distribution functions)

Let Z be the stationary Boolean model in $\mathbb{R}^d$ with intensity $\gamma > 0$ and typical grain Z_0 given through a d-dimensional ball of random radius $R_0 \sim \mathrm{Exp}(\lambda)$ $(\lambda > 0)$ around $0 \in \mathbb{R}^d$.

a) Calculate the spherical contact distribution function $H_{B(0,1)}(r)$, $r > 0$.

b) Now, let $d = 2$ and $r_1, r_2 > 0$ such that $r_1 \neq r_2$. Assume that $H_{B(0,1)}(r_1)$ and $H_{B(0,1)}(r_2)$ are known. Determine the intensity γ as well as the parameter λ in dependence of $H_{B(0,1)}(r_1)$ and $H_{B(0,1)}(r_2)$.

Proposed solution:

a) Theorem 3.27 of the lecture notes, and the fact that $\mathbb{E}\big[R_0^k\big] = \frac{k!}{\lambda^k}$, imply

$$
\begin{aligned}
H_{B(0,1)}(r) &= 1 - \exp\Big(-\gamma \cdot \mathbb{E}\Big[\lambda^d\big((Z_0 + r \cdot B(0,1)^*) \setminus Z_0\big)\Big]\Big) \\
&= 1 - \exp\Big(-\gamma \cdot \mathbb{E}\Big[\lambda^d\big((R_0 + r) \cdot B(0,1) \setminus R_0 \cdot B(0,1)\big)\Big]\Big) \\
&= 1 - \exp\Big(-\gamma\kappa_d \cdot \mathbb{E}\Big[(R_0 + r)^d - R_0^d\Big]\Big) \\
&= 1 - \exp\left(-\gamma\kappa_d \cdot \mathbb{E}\left[\sum_{j=0}^{d-1} \binom{d}{j} r^{d-j} R_0^j\right]\right) \\
&= 1 - \exp\left(-\gamma\kappa_d \sum_{j=0}^{d-1} \binom{d}{j} r^{d-j}\, \mathbb{E}\big[R_0^j\big]\right) \\
&= 1 - \exp\left(-\gamma\kappa_d \sum_{j=0}^{d-1} \binom{d}{j} \frac{j!}{\lambda^j} r^{d-j}\right) \\
&= 1 - \exp\left(-\gamma\kappa_d \cdot \frac{d!}{\lambda^d} \sum_{j=0}^{d-1} \frac{(r \cdot \lambda)^{d-j}}{(d-j)!}\right) \\
&= 1 - \exp\left(-\gamma\kappa_d \cdot \frac{d!}{\lambda^d} \sum_{k=1}^{d} \frac{(r \cdot \lambda)^k}{k!}\right), \qquad r > 0.
\end{aligned}
$$

b) Define

$$
f(r) := -\log\big(1 - H_{B(0,1)}(r)\big), \qquad r > 0.
$$

Part a) yields

$$f(r) = \gamma\pi\left(r^2 + \frac{2}{\lambda}\,r\right), \qquad r > 0,$$

so, in particular,

$$\frac{f(r_1)}{r_1} = \gamma\pi\left(r_1 + \frac{2}{\lambda}\right) \quad \text{and} \quad \frac{f(r_2)}{r_2} = \gamma\pi\left(r_2 + \frac{2}{\lambda}\right),$$

which is equivalent to $Ax = b$ with

$$A := \pi\begin{pmatrix} r_1 & 2 \\ r_2 & 2 \end{pmatrix}, \quad x := \begin{pmatrix} \gamma \\ \gamma/\lambda \end{pmatrix}, \quad b := \begin{pmatrix} f(r_1)/r_1 \\ f(r_2)/r_2 \end{pmatrix}.$$

Notice that

$$A^{-1} = \frac{1}{2\pi(r_1 - r_2)}\begin{pmatrix} 2 & -2 \\ -r_2 & r_1 \end{pmatrix},$$

hence

$$\gamma = \frac{1}{\pi(r_1 - r_2)}\left(\frac{f(r_1)}{r_1} - \frac{f(r_2)}{r_2}\right),$$

and

$$\frac{\gamma}{\lambda} = \frac{1}{2\pi(r_1 - r_2)}\left(-\frac{r_2 \cdot f(r_1)}{r_1} + \frac{r_1 \cdot f(r_2)}{r_2}\right) = \frac{r_1 \cdot r_2}{2\pi(r_1 - r_2)}\left(-\frac{f(r_1)}{r_1^2} + \frac{f(r_2)}{r_2^2}\right).$$

As the function $r \mapsto \frac{f(r)}{r^2}$ is strictly decreasing, the right hand side of the previous equation is $\neq 0$ (recall that $r_1 \neq r_2$). We conclude that

$$\begin{aligned}
\lambda &= \frac{1}{\pi(r_1 - r_2)}\left(\frac{f(r_1)}{r_1} - \frac{f(r_2)}{r_2}\right)\Big/\frac{r_1 \cdot r_2}{2\pi(r_1 - r_2)}\left(-\frac{f(r_1)}{r_1^2} + \frac{f(r_2)}{r_2^2}\right) \\
&= \frac{2}{r_1\, r_2} \cdot \frac{r_1\, r_2^2\, f(r_1) - r_1^2\, r_2\, f(r_2)}{r_1^2\, r_2^2} \cdot \frac{r_1^2\, r_2^2}{-r_2^2\, f(r_1) + r_1^2\, f(r_2)} \\
&= 2 \cdot \frac{f(r_1) \cdot r_2 - f(r_2) \cdot r_1}{r_1^2 \cdot f(r_2) - r_2^2 \cdot f(r_1)}.
\end{aligned}$$