Masterthesis

External DLA

Tillmann Tristan Bosch

10. March 2020

Supervisor: PD. Dr. Steffen Winter

Faculty of Mathematics

Karlsruhe Institute of Technology

Contents

1 Introduction

2 Preliminaries

2.1 Symbols

Here we list all symbols which will be used in this paper. Let $d \in \mathbb{N}$ and $q \in \{0, \dots, d\}$.

$\mathbb{N} = \{1, 2, 3, \dots\}$, the set of natural numbers (without 0)

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$

$\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$

$\mathcal{B}^d$, d-dimensional Borel-σ-algebra of $\mathbb{R}^d$

$\mathcal{K}^d$, the set of convex and compact sets in $\mathbb{R}^d$

$B_d(x, r) = \{y \in \mathbb{R}^d \mid |x - y| \leq r\}$, the d-dimensional closed ball of radius r around x

$B_r := B_2(r, 0)$

$S_{d-1}(r, x) = \partial B_d(r, x)$, the $(d-1)$-dimensional surface of the d-dimensional ball

$A(d, q)$, the set of q-dimensional affine subspaces of $\mathbb{R}^d$

$\mathcal{A}(d, q)$, the σ-algebra of $A(d, q)$, as constructed later in the paper

$\mathcal{G} := A(2, 1)$, the set of lines in the real plane

$SO_d := \{\nu \in \mathbb{R}^{d \times d} \mid \nu\nu^\top = I_d \text{ and } \det \nu = 1\}$, identify $SO_2 = \{\nu_\beta := e^{i\beta} \in \mathbb{C} \mid \beta \in [0, 2\pi)\}$

$G_d := \{\varphi : \mathbb{R}^d \to \mathbb{R}^d, x \mapsto \nu x + b \mid \nu \in SO_d, b \in \mathbb{R}^d\}$, the set of euclidean motions

$\mathcal{P}_f$, the set of finite subsets of $\mathbb{Z}^d$ (or $\mathbb{Z}^2$ respective to the context)

2.2 Basic structures

2.2.1 Graphs

Let $d \in \mathbb{N}$. We will be interested in the undirected graph $(\mathbb{Z}^d, E)$ with its canonical graph structure, which is two vertices (or points) $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{Z}^d$ form an edge (e.q. $\{x, y\} \in E$) if and only if there exists exactly one $i \in \{1, \ldots, d\}$ such that $|x_i - y_i| = 1$ and $x_j = y_j$ for all $j \neq i$. For a point $x \in \mathbb{Z}^d$ its set of *neighbours* is defined as

$$N(x) := \{y \in \mathbb{Z}^d \mid \{x, y\} \in E\}.$$

For a set $A \subset \mathbb{Z}^d$ the *outer boundary* ∂A of A is defined as

$$\partial A := \{y \in \mathbb{Z}^d \setminus A \mid \exists x \in A : \ \{x, y\} \in E\}$$

and the closure $\bar{A}$ of A as

$$\bar{A} := A \cup \partial A.$$

Instead of $(\mathbb{Z}^d, E)$ we will write $\mathbb{Z}^d$ from now on.

2.2.2 Random Walks

Definition 2.2.1. Throughout the paper let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and for our space of interest $\mathbb{Z}^d$ we will always use the discrete σ-algebra which is the power set of $\mathbb{Z}^d$. A family $(S_n)_{n \in \mathbb{N}_0}$ of measurable functions $S_n : \Omega \to \mathbb{Z}^d$ is called a *random walk on $\mathbb{Z}^d$* (*starting at $x \in \mathbb{Z}^d$*) if and only if $S_0 = x$ a.s. and

$$\mathbb{P}(S_n = y \mid S_{n-1} = z) = \frac{1}{|N(z)|} = \frac{1}{2d}, \quad \text{for all } y \in N(z) \text{ and } z \in \mathbb{Z}^d.$$

Note that $|N(z)| = 2d$ for all $z \in \mathbb{Z}^d$ since every point has two neighbours in the direction of every dimensional component. We can therefore conclude easily that $\mathbb{P}(S_n = y \mid S_{n-1} = z) = 0$ for all $y \notin N(z)$ and $z \in \mathbb{Z}^d$. For $x, y \in \mathbb{Z}^d$ we introduce the short notation

$$\mathbb{P}_x(S_n = y) := \mathbb{P}(S_n = y \mid S_0 = x).$$

So a random walk can be understood as a particle starting from some point x and moving randomly on the grid choosing its next step uniformly from its neighbours. For the following let $(S_n)_{n \in \mathbb{N}}$ be a random walk on $\mathbb{Z}^d$ starting at $x \in \mathbb{Z}^d$.

Definition 2.2.2. Let $A \subset \mathbb{Z}^d$. We define the *hitting times* of A by

$$T_A := \min\{n \geq 0 \mid S_n \in A\} \text{ and } T_A^+ := \min\{n \geq 1 \mid S_n \in A\},$$

and $T_y := T_{\{y\}}$ and $T_y^+ := T_{\{y\}}^+$ for $y \in \mathbb{Z}^d$.

Definition 2.2.3. Define $T := T_x^+$ to be the first time for the random walk to come back to its origin x. The random walk is called *recurrent* if

$$\mathbb{P}_x(T < \infty) = 1,$$

and *transient* if

$$\mathbb{P}_x(T < \infty) < 1.$$

Lemma 2.2.1. A random walk $(S_n)_{n \in \mathbb{N}}$ on $\mathbb{Z}^d$ is recurrent if $d \leq 2$ and transient if $d \geq 3$.

Proof. A detailed proof of this result is presented in [4] Satz 5.1. $\square$

2.2.3 Expressions

If for $A \in \mathcal{F}$ we have $\mathbb{P}(A) = 1$ we will say that „A holds $\mathbb{P}$-a.s.", or short „A holds a.s." (A holds almost surely). Another short expression, if for a set of logical statements $(A_t)_{t \in I}$ with $I \in \{\mathbb{N}, \mathbb{R}\}$ we say „A_t holds for large t" it shall mean that there exists $T \in I$ such that A_t holds for all $t > T$. Here we mean that a logical statement holds if and only if the statement is true.

3 Incremental Aggregate

3.1 Definition

In this paper we will look at stochastic processes on the set of finite subsets of $\mathbb{Z}^d$, where we start with the one point set $\{0\}$ and incrementally add a point of the outer boundary of the current cluster according to some distribution. What we get is a randomly, point by point growing connected cluster which we will call *incremental aggregate*. Define

$$\mathcal{P}_f := \{A \subset \mathbb{Z}^d \mid A \text{ is finite}\}, \tag{3.1}$$

the set of finite subsets of $\mathbb{Z}^d$. Furthermore we will be interested in distributions on those sets, so for $A \in \mathcal{P}_f$ we define

$$\mathcal{D}_A := \{\mu : \mathbb{Z}^d \to [0,1] \mid \mu(y) = 0 \text{ for all } y \notin A \text{ and } \sum_{y \in A} \mu(y) = 1\}, \tag{3.2}$$

the set of distributions on A. Now we define an *incremental aggregate* as follows.

Definition 3.1.1. Let $\mu = (\mu_A)_{A \in \mathcal{P}_f}$ be a family of distributions with $\mu_A \in \mathcal{D}_A$ for all $A \in \mathcal{P}_f$. An *incremental aggregate* (*with distribution μ*) is a stochastic process $(\mathcal{E}_n)_{n \in \mathbb{N}}$ which evolves as follows. The process starts with one point $\mathcal{E}_1 = \{0\}$ (define $y_1 := 0$) at the origin of $\mathbb{Z}^d$. Knowing the process $\mathcal{E}_n$ at time n, let y_{n+1} be a random point in $\partial\mathcal{E}_n \in \mathcal{P}_f$ with distribution

$$\mathbb{P}(y_{n+1} = y \mid \mathcal{E}_n) := \mu_{\partial\mathcal{E}_n}(y), \quad y \in \mathbb{Z}^d. \tag{3.3}$$

We then define $\mathcal{E}_{n+1} := \mathcal{E}_n \cup \{y_{n+1}\}$ and the limit cluster as $\mathcal{E}_\infty := \bigcup_{n \in \mathbb{N}} \mathcal{E}_n$.

The above defines a Markov chain whose state space is the set of finite and connected subsets of $\mathbb{Z}^d$. For all incremental aggregates in this paper we will focus on the 2-dimensional case $\mathbb{Z}^2$ and will always identify $\mathbb{R}^2$ with $\mathbb{C}$ for a more convenient notation.

Remark 3.1.1. It is worth to mention that the distribution of a new added point y_n is not depending on the order of how previous points where added to the cluster since the harmonic measure only looks at the actual cluster as a set which doesn't contain any order information.

3.2 Notion of Fractal Dimension and Growth Rate

The notion of fractal dimension is usually used for sets with uncountable cardinality like continuous curves or surfaces. One way of defining a fractal dimension for curves like for example the cost line of New Zealand (see Figure 1) would be to look at the limit relation between the minimum number of boxes (squares) which we need to cover the cost line and the side length of these boxes. The following definition is motivated by [7] page 160. If for $\varepsilon > 0$ $N(\varepsilon)$ is the mininum number of boxes with side length ε which
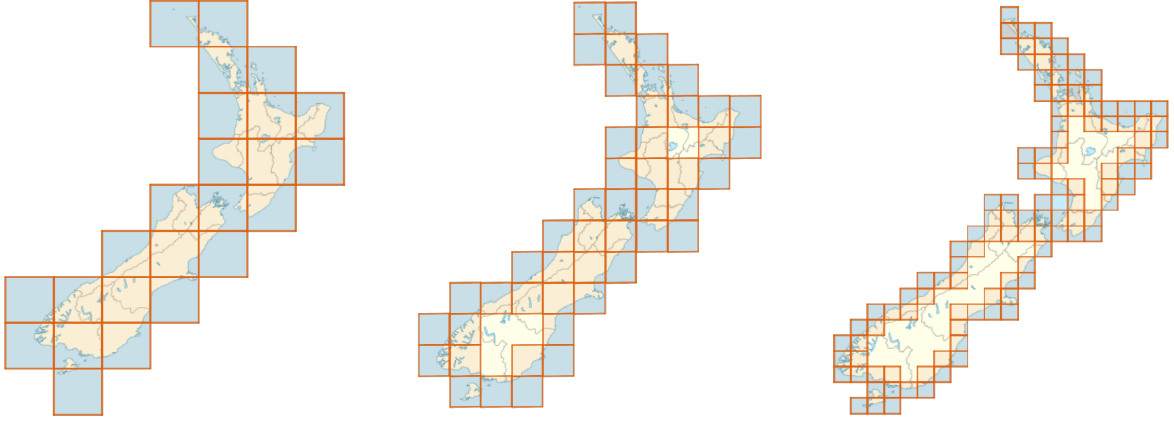
Figure 1: Box-covering of New Zealands outer cost with decreasing box sizes ε

we need to cover the cost line, then the so called *box-dimension* d_b is a constant such that $N(\varepsilon)$ grows as fast as ε^{-d_b} for letting ε tend to zero, so

$$d_b := -\lim_{\varepsilon \to 0} \frac{\ln(N(\varepsilon))}{\ln(\varepsilon)}. \tag{3.1}$$

This definition makes sense in many contexts, for example is the box-dimension of straight line segments 1 and of squares 2 and so on, so in those cases equal to the topological dimension. As with incremental aggregates we are dealing with finite point sets, this approach of defining a fractal dimension for our clusters is not senseful. It is not difficult to show, that the box-dimension of any finite set is 0. A helpful detail about the situation with finite sets as the ones we are looking at is that each point of the cluster can actually be interpreted and identified with a unique square since the cluster is living on the grid $\mathbb{Z}^2$. We will precise that in chapter 5. So instead of decreasing the sizes of the boxes with which we cover our set of interest, we leave the size of the boxes constant and increase the size of our set by adding points and looking at the limit cluster $\mathcal{E}_\infty$. Defining the radius of a finite set $A \in \mathcal{P}_f$ (with $|\cdot|$ the euclidean norm) as

$$\operatorname{rad}(A) := \max_{x \in A} |x|,$$

we therefore can identify the relation between the geometrical sizes of the boxes and the cluster as follows. For $n \in \mathbb{N}$ define $\varepsilon_n := \frac{1}{\mathbb{E}[\operatorname{rad}(\mathcal{E}_n)]}$. Before ε_n would have expressed the geometrical relation between one box and the whole cluster which we describe now by the fraction $\frac{1}{\mathbb{E}[\operatorname{rad}(\mathcal{E}_n)]}$ since a box size is now constantly 1 and the clusters geometrical size we can interpret by $\mathbb{E}[\operatorname{rad}(\mathcal{E}_n)]$ up to a constant. The minimum number of boxes with size 1 we need to cover $\mathcal{E}_n$ is always n, so $N(\varepsilon_n) = n$ for all $n \in \mathbb{N}$ and therefore we can rewrite the definition of the box-dimension (3.1) by replacing ε_n with $\frac{1}{\mathbb{E}[\operatorname{rad}(\mathcal{E}_n)]}$ and define the (*discrete*) *fractal dimension* of $\mathcal{E}_\infty$ as

$$d_f := \liminf_{n \to \infty} \frac{\ln(n)}{\ln(\mathbb{E}[\operatorname{rad}(\mathcal{E}_n)])} \tag{3.2}$$

Why we choose the limes inferior here we will see later. Another way of tackling the intuition for a fractal dimension of discrete sets is by considering subsets of the d-dimensional ball $B_d(0, m)$ of radius $m \geq 0$, which is motivated by [5] Part II page 98 and [3] 2.6 page 82. If we take a finite subset M of that ball, we could assign it the dimension k if its cardinality is of order m^k. If we apply the same argument to the clusters $\mathcal{E}_n$ and balls with radius $\mathbb{E}[\text{rad}(\mathcal{E}_n)]$ for all $n \in \mathbb{N}$, we get that the dimension of $\mathcal{E}_\infty$ could be interpreted as a constant k which fulfills that $n = |\mathcal{E}_n|$ grows as fast as $\mathbb{E}[\text{rad}(\mathcal{E}_n)]^k$ for letting n tend to infinity. We therefore get the same definition as in (3.2). This way of defining a fractal dimension for incremental aggregates strongly correlates with the growth rate of the aggregate which shall indicate how the radius of the cluster evolves while increasing the particle number. We can define the growth rate by looking for the smallest exponent α such that there exists a constant $c > 0$ with

$$\mathbb{E}[\text{rad}(\mathcal{E}_n)] \leq cn^\alpha$$

for large n. Rewriting this we come to the equivalent inequality

$$\frac{\ln(\mathbb{E}[\text{rad}(\mathcal{E}_n)])}{\ln(n)} - \frac{\ln(c)}{\ln(n)} \leq \alpha$$

for large n and we could finally define the growth rate α_f of an incremental aggregate as the smallest value satisfying this inequality, so

$$\alpha_f := \limsup_{n \to \infty} \frac{\ln(\mathbb{E}[\text{rad}(\mathcal{E}_n)])}{ln(n)}. \tag{3.3}$$

Since $\text{rad}(\mathcal{E}_n) \leq n$ for all $n \in \mathbb{N}$ a.s., we get

$$\alpha_f \leq 1.$$

We further chose the limes inferior in the definition (3.2) to simply have

$$d_f = \frac{1}{\alpha_f} \tag{3.4}$$

which includes the cases $d_f = \infty$ or $\alpha_f = \infty$ ($\frac{1}{0} := \infty$). We therefore get

$$d_f \geq 1.$$

In $\mathbb{Z}^d$ for some $d \in \mathbb{N}$ we can fill the cube with center 0 and side length n with an order of $n^{\frac{1}{d}}$ points. Therefore we can fill the ball of radius n with an order of $n^{\frac{1}{d}}$ points aswell, so there is a constant $c > 0$ such that $cn^{\frac{1}{d}} \leq \text{rad}(\mathcal{E}_n)$ for all $n \in \mathbb{N}$ a.s.. Hence

$$d_f \leq \liminf_{n \to \infty} \frac{\ln(n)}{\ln(cn^{\frac{1}{d}})} = d.$$

So in total we get the trivial boundaries

$$1 \leq d_f \leq d$$

and

$$\frac{1}{d} \leq \alpha_f \leq 1$$

for any incremental aggregate in $\mathbb{Z}^d$.

4 External Diffusion Limited Aggregate

4.1 Definition

External DLA is a model of an incremental aggregate as defined above using a very natural family of distributions, called the *harmonic measures*.

Definition 4.1.1. (*harmonic measure*) Let $A \subset \mathbb{Z}^d$. The hitting probability of A is the function

$$H_A : \mathbb{Z}^d \times A \to [0,1], \quad (x,y) \mapsto H_A(x,y) := \mathbb{P}_x(S_{T_A^+} = y).$$

In literature you can find the same definition where T_A is used instead of T_A^+. Since in the following for finite sets $A \in \mathcal{P}_f$ the limit $|x| \to \infty$ of $H_A(x,y)$ is of interest, T_A^+ is chosen for convenience. In fact, for a fixed element $x \in \mathbb{Z}^d$ the function $H_A(x, \cdot)$ defines a measure on A with total mass $\mathbb{P}_x(T_A^+ < \infty)$ and it can be adapted to a probability measure by conditioning the random walk to hit A in finite time. Define

$$\bar{H}_A : \mathbb{Z}^d \times A \to [0,1], \quad (x,y) \mapsto \bar{H}_A(x,y) := \mathbb{P}_x(S_{T_A^+} = y \mid T_A^+ < \infty),$$

so for fixed $x \in \mathbb{Z}^d$ the function $\bar{H}_A(x, \cdot)$ defines a probability measure on A. Indeed this definition is motivated by [3] (Chapter 2, Definition 2.1) and in the same chapter it is proved, that for finite sets $A \in \mathcal{P}_f$ the limit

$$\lim_{|x| \to \infty} \bar{H}_A(x,y) =: h_A(y)$$

exists for each $y \in A$. The function $h_A : A \to [0,1]$ is called the *harmonic measure of A*. For an element $y \in A$, h_A can be interpreted as the probability that a random walk starting at „infinity" hits A the first time at y. If we look at the 2-dimensional case, by Lemma 2.2.1, we have that a random walk is recurrent on $\mathbb{Z}^2$ and therefore we get $\bar{H}_A = H_A$ since $\mathbb{P}_x(T_A^+ < \infty) = 1$ by recurrence.

Definition 4.1.2. (*external diffusion limited aggregate*) *External diffusion limited aggregate*, short *external DLA* or just *DLA*, is an incremental aggregate with the family of harmonic measures $(h_A)_{A \in \mathcal{P}_f}$ as distribution.

4.2 Fractal Dimension and Growth Rate of External DLA

There is not yet a rigorous proof on the exact fractal dimension of external DLA. There are rather few rigorously proved results and it seems that it is very hard to prove such results on DLA. Looking at computer simulations of DLA in $\mathbb{Z}^2$ it seems that the clusters are relatively sparse and they appear to have a noninteger fractal dimension. In [6] DLA is observed in a magnetic aggregation context, and empirically they find a fractal dimension of around 1.8. Other simulations seem to suggest a value a little less than 1.7 for d_f in two dimensions ([3] page 83). There is also a theory that predicts

$$d_f = \frac{d^2 + 1}{d + 1}$$

in $\mathbb{Z}^d$ which seems to agree fairly well with simulations ([3] page 83). There are only few rigorously proved results and we will see one of them in the following. First we will proof some lemmas. For that define two random functions

$$r : \mathbb{N} \to [0, \infty), \quad n \mapsto \mathrm{rad}(\mathcal{E}_n)$$

and

$$T : [0, \infty) \to \mathbb{N}, \quad s \mapsto \min\{j \in \mathbb{N} \mid r(j) \geq s\}.$$

Then it is easy to show that for all $n \in \mathbb{N}$, $s \in [0, \infty)$ and $\omega \in \Omega$

(i) $r(\omega)$ and $T(\omega)$ grow monotonously

(ii) $T(\omega)(r(\omega)(n)) \leq n$

(iii) $r(\omega)(T(\omega)(s)) \geq s$

hold.

Lemma 4.2.1. For both random functions r and T we have that

$$r(\omega)(n) \to \infty \text{ for } n \to \infty$$

and

$$T(\omega)(s) \to \infty \text{ for } s \to \infty$$

for all $\omega \in \Omega$.

Proof. Since we are moving on the grid $\mathbb{Z}^d$ we have that for a ball $B_d(0, n)$ with radius $n \geq 0$ we have a finite number $N := |B_d(0, n) \cap \mathbb{Z}^d|$ and for any $\omega \in \Omega$ get

$$r(\omega)(2N) \geq n.$$

Therefore for all $\omega \in \Omega$ and any $n \in \mathbb{N}$ we can find a $M \in \mathbb{N}$ such that $r(\omega)(M) \geq n$, and since $r(\omega)$ grows monotonously we get $r(\omega)(n) \to \infty$ for $n \to \infty$ for all $\omega \in \Omega$. Very similarly we can argument for T. $\qquad\square$

Lemma 4.2.2. Let $a > 0$ and $h : [0, \infty) \to [0, \infty)$ be a bijective, multiplicative and monotonously growing function. Then the following are equivalent:

(i) $\exists c > 0 : \mathbb{P}(r(n) \leq ch(n) \text{ for large } n) = 1$

(ii) $\exists c > 0 : \mathbb{P}(T(as) \geq ch^{-1}(s) \text{ for large } s) = 1$

Proof. $\Rightarrow$: For $c > 0$ define

$$A_c := \{r(n) \leq ch(n) \text{ for large } n\}$$

and

$$B_c := \{T(as) \geq ch^{-1}(s) \text{ for large } s\}.$$

Choose $c > 0$ such that $\mathbb{P}(A_c) = 1$. Take $\omega \in A_c$ and choose $N \in \mathbb{N}$ such that $r(\omega)(n) \leq ch(n)$ for all $n > N$. Hence there exists $\tilde{c} > 0$ such that $\tilde{c}h^{-1}(r(\omega)(n)) \leq n$ for all $n > N$. By Lemma 4.2.1 we can choose $M \in \mathbb{N}$ big enough such that $T(\omega)(aM) > N$, hence $T(\omega)(as) \geq T(\omega)(aM) > N$ for all $s > M$ since $T(\omega)$ grows monotonously. Hence we can write $\tilde{c}h^{-1}(r(\omega)(T(\omega)(as))) \leq T(\omega)(as)$ for all $s > M$ and since $r(T(\omega)(as)) \leq as$ we finally get $\tilde{c}h^{-1}(a)h^{-1}(s) = \tilde{c}h^{-1}(as) \leq T(\omega)(as)$ for all $s > M$, hence $\omega \in B_{\tilde{c}h^{-1}(a)}$, where we used the multiplicativity of h and that h^{-1} falls monotonously. We therefore get $A_c \subset B_{\tilde{c}h^{-1}(a)}$, hence $\mathbb{P}(B_{\tilde{c}h^{-1}(a)}) = 1$.

$\Leftarrow$: The proof for this direction works analogously and needs Lemma 4.2.1 aswell. $\square$

Lemma 4.2.3. Let $n \in \mathbb{N}$ and $T_1, \dots, T_n$ be independent geometrically distributed random variables with parameter $p < \frac{1}{2}$. Let $Y := T_1 + \cdots + T_n$, then for every $a \in [2p, 1)$ we have

$$\mathbb{P}(Y \leq \frac{an}{p}) \leq (ae^2)^n.$$

Proof. The moment generating function of Y is

$$\mathbb{E}[e^{tY}] = (pe^t)^n(1 - e^t(1 - p))^{-n} = p^n(e^{-t} - (1 - p))^{-n}.$$

By Chebyshev for any random variable X we know the inequality

$$\mathbb{P}(X \geq x) \leq \inf_{t > 0} \frac{\mathbb{E}[e^{tX}]}{e^{tx}}$$

and can therefore follow that for any $t > 0$

$$\mathbb{P}(Y \leq \frac{an}{p}) = \mathbb{P}(-Y \geq -\frac{an}{p}) \leq \exp(\frac{ant}{p})\mathbb{E}[e^{-tY}] = \exp(\frac{ant}{p})p^n(e^t - (1 - p))^{-n}.$$

Choose $t = \ln(\frac{a(1-p)}{a-p})$ (note that $t > 0$), then

$$\begin{aligned}
\mathbb{P}(Y \leq \frac{an}{p}) &\leq (\frac{a(1-p)}{a-p})^{\frac{an}{p}} p^n(1-p)^{-n}(\frac{p}{a-p})^{-n} \\
&= (\frac{a}{a-p})^{\frac{an}{p}}(1-p)^{\frac{an}{p}}(1-p)^{-n}(\frac{1}{a-p})^{-n} \\
&\leq (1 + \frac{p}{a-p})^{\frac{an}{p}}(1-p)^n(1-p)^{-n}(a-p)^n \\
&\leq (1 + \frac{p}{a-p})^{\frac{2(a-p)n}{p}} a^n \\
&\leq (ae^2)^n.
\end{aligned}$$

$\square$

Definition 4.2.1. For $x \in \mathbb{Z}^2$ and $n \in \mathbb{N}$ define

$$\mathcal{P}_n^x := \{A \in \mathcal{P}_f \mid x \in A, n = \max_{y \in A} |x - y| \text{ and } A \text{ is connected}\}.$$

The following theorem gives a bound on the harmonic measure which will be necessary in the proof we present here for the growth rate of DLA. The theorem is proved by [3] in Theorem 2.5.2 and we state it slightly different here to generalize it easier afterwards.

Theorem 4.2.1. There exists a constant $c > 0$ such that for all $n \in \mathbb{N}$

$$h_A(0) \leq cn^{-\frac{1}{2}} \quad \text{for all } A \in \mathcal{P}_n^0.$$

Proof. If $A \in \mathcal{P}_n^0$ then $0 \in A$ and $n = \mathrm{rad}(A)$ and for that case the proof is presented for all dimensions $d \in \mathbb{N}$ in [3] Theorem 2.5.2. The proof is very technical and requires various results of other theorems and lemmas which are also to find in [3]. $\qquad \square$

Proposition 4.2.1. Theorem 4.2.1 can be generalized to the following. Let $y \in \mathbb{Z}^2$, then there exists $c > 0$ such that for all $n \in \mathbb{N}$

$$h_A(y) \leq cn^{-\frac{1}{2}} \quad \text{for all } A \in \mathcal{P}_n^y.$$

Proof. For $y \in \mathbb{Z}^2$ define the translation function

$$\Phi_y : \mathbb{Z}^2 \to \mathbb{Z}^2, x \mapsto x + y.$$

Recall that in $\mathbb{Z}^2$ the harmonic measure can be written as $h_A(z) = \lim_{|x| \to \infty} \mathbb{P}_x(S_{T_A^+} = z)$ for each $z \in A$ for a set $A \subset \mathbb{Z}^2$ (see Definition 4.1.1). Since the distribution of random walks in $\mathbb{Z}^2$ is invariant under translation we can conlude that

$$\begin{aligned} h_A(z) &= \lim_{|x| \to \infty} \mathbb{P}_x(S_{T_A^+} = z) \\ &= \lim_{|x| \to \infty} \mathbb{P}_{\Phi_y(x)}(S_{T_{\Phi_y(A)}^+} = \Phi_y(z)) \\ &= h_{\Phi_y(A)}(\Phi_y(z)). \end{aligned}$$

By Theorem 4.2.1 we know that there exists a $c > 0$ such that for all $n \in \mathbb{N}$

$$h_A(0) \leq cn^{-\frac{1}{2}} \quad \text{for all } A \in \mathcal{P}_n^0.$$

Let $n \in \mathbb{N}$. We get

$$h_A(0) = h_{\Phi_y(A)}(\Phi_y(0)) = h_{\Phi_y(A)}(y) \leq cn^{-\frac{1}{2}} \quad \text{for all } A \in \mathcal{P}_n^0$$

and since $A \in \mathcal{P}_n^0 \iff \Phi_y(A) \in \mathcal{P}_n^{\Phi_y(0)}$ we get

$$h_A(y) \leq cn^{-\frac{1}{2}} \quad \text{for all } A \in \mathcal{P}_n^y.$$

$\qquad \square$

Theorem 4.2.2. For the growth rate of external DLA in $\mathbb{Z}^2$ as defined in 3.2 (3.3) we have

$$\alpha_f \leq \frac{2}{3}. \tag{4.1}$$

Proof. For $c > 0$ define

$$A_c := \{\omega \in \Omega \mid \mathrm{rad}(\mathcal{E}_n(\omega)) \leq cn^{\frac{3}{2}} \text{ for large } n\}$$

and

$$D_c := \{\omega \in \Omega \mid T(\omega)(2n) \geq cn^{\frac{2}{3}} \text{ for large } n\}.$$

If we have a $c > 0$ such that

$$\mathbb{P}(A_c) = 1, \tag{4.2}$$

then we have

$$
\begin{aligned}
\alpha_f &= \limsup_{n \to \infty} \frac{\ln(\mathbb{E}[\mathrm{rad}(\mathcal{E}_n)])}{\ln(n)} \\
&= \limsup_{n \to \infty} \frac{\ln(\int_\Omega \mathrm{rad}(\mathcal{E}_n)d\mathbb{P})}{\ln(n)} \\
&= \limsup_{n \to \infty} \frac{\ln(\int_{A_c} \mathrm{rad}(\mathcal{E}_n)d\mathbb{P})}{\ln(n)} \\
&\overset{(+)}{\leq} \limsup_{n \to \infty} \frac{\ln(\int_{A_c} cn^{\frac{3}{2}}d\mathbb{P})}{\ln(n)} \\
&= \limsup_{n \to \infty} \frac{\ln(cn^{\frac{3}{2}})}{\ln(n)} \\
&= \frac{3}{2}\limsup_{n \to \infty} \frac{\ln(c^{\frac{2}{3}}) + \ln(n)}{\ln(n)} = \frac{3}{2} \cdot 1 = \frac{3}{2}
\end{aligned}
$$

If we choose

$$h : [0, \infty) \to [0, \infty), x \mapsto x^{\frac{2}{3}},$$

and $a = 2$, then by Lemma 4.2.2 we can show (4.2) if we find a constant $c > 0$ such that

$$\mathbb{P}(D_c) = 1. \tag{4.3}$$

Note that h is bijective, multiplicative and monotonously growing. Lets first argument why the inequality at $(+)$ indeed holds. We define ...

So now we will try to prove (4.3). For $n \in \mathbb{N}$ write $\mathcal{E}_n = \{y_1, \ldots, y_n\}$ according to the definition in 3.1.1, where y_j is the j-th point added to the cluster. Let $\beta > 0$ which will be determined lateron. For $n \in \mathbb{N}$ let $\tilde{m}_n := \beta n^{\frac{3}{2}}$ and define

$$V_n := \{\omega \in \Omega \mid T(\omega)(2n) < \tilde{m}_n\}.$$

Further define the set of realised random walk paths of length n with starting point in $\tilde{B}_n := \{x \in \mathbb{Z}^2 \mid d(x, \partial B_n) \leq 1 \text{ and } |x| \geq n\}$

$$Z_n := \{[z] := \{z_1, \ldots, z_n\} \subset \mathbb{Z}^2 \mid z_1 \in \tilde{B}_n \text{ and } z_i \in N(z_{i-1}) \text{ for all } i \in \{2, \ldots, n\}\},$$

for $[z] \in Z_n$ define events

$$W_n([z]) := \{\omega \in \Omega \mid \exists j_1 < \cdots < j_n \leq \tilde{m}_n \text{ such that } y_{j_i}(\omega) = z_i \text{ for all } i \in \{1, \ldots, n\}\}$$

and the union of these events

$$W_n := \bigcup_{[z] \in Z_n} W_n([z]).$$

With $\tilde{B}_n$ we mean that the random walks in Z_n start „on the boundary" of B_n. We will quickly prove that

$$V_n \subset W_n \text{ for all } n \in \mathbb{N}.$$

Let $n \in \mathbb{N}$ and $\omega \in V_n$, then $T(\omega)(2n) < \tilde{m}_n$. For $m_n := \max\{j \in \mathbb{N} \mid j \leq \tilde{m}_n\}$ we therefore have $\text{rad}(\mathcal{E}_{m_n}(\omega)) \geq 2n$ (Note that $T(\omega)(2n) \in \mathbb{N}$). Therefore since $\mathcal{E}_{m_n}(\omega)$ is connected there must exist indices $j_1 < \cdots < j_n$ s.t. $y_{j_1}(\omega) \in \tilde{B}_n$ and $[z_0] := [y_{j_1}(\omega), \ldots, y_{j_n}(\omega)] \subset \mathcal{E}_{m_n}(\omega)$, since $\max_{x \in [z_0]} |x| \leq 2n$. Therefore $j_n \leq m_n \leq \tilde{m}_n$ and therefore $\omega \in W_n([z_0]) \subset W_n$ which proofs the inclusion.

For a sequence of events $(A_n)_{n \in \mathbb{N}}$ recall that

$$\limsup_{n \to \infty} A_n := \bigcap_{n \in \mathbb{N}} \bigcup_{i \geq n} A_i = \{\omega \in \Omega \mid \omega \in A_n \text{ for infinitely many } n \in \mathbb{N}\}.$$

We will use the Lemma of Borel-Cantelli on the sequence of events $(W_n)_{n \in \mathbb{N}}$. If we can show that

$$\sum_{n \in \mathbb{N}} \mathbb{P}(W_n) < \infty, \tag{4.4}$$

then with $V_n \subset W_n$ for all $n \in \mathbb{N}$ and Borel-Cantelli we get

$$\mathbb{P}(\limsup_{n \to \infty} V_n) \leq \mathbb{P}(\limsup_{n \to \infty} W_n) = 0.$$

Since

$$(\limsup_{n \to \infty} V_n)^C = \{\omega \in \Omega \mid \exists N \in \mathbb{N} \text{ s.t. } \omega \in V_n^C \text{ for all } n > N\} = D_\beta$$

we can then conclude that $\mathbb{P}(D_\beta) = 1$ and have finished the proof. So we want to show (4.4).

For $n \in \mathbb{N}$, $[z] \in Z_n$ and $i \in \{1, \ldots, n\}$ we define random variables

$$\tau_i : \Omega \to \mathbb{N}^\infty, \tau_i(\omega) = j :\Leftrightarrow \begin{cases} y_j(\omega) = z_i, & j < \infty, \\ z_i \notin \mathcal{E}_\infty(\omega), & j = \infty, \end{cases}$$

so τ_i is either the index j such that the j-th added point to the cluster is equal to z_i, or infinity if the final cluster $\mathcal{E}_\infty$ doesn't contain z_i. τ_i is measurable because for $j \in \mathbb{N}$ we have

$$\tau_i^{-1}(j) = \{y_j = z_i\} = y_j^{-1}(z_i) \in \mathcal{F}$$

and for $j = \infty$ we have

$$\begin{aligned} \tau_i^{-1}(\infty) &= \{z_i \notin \mathcal{E}_\infty = \bigcup_{k \in \mathbb{N}} \mathcal{E}_k\} \\ &= \{z_i \notin \mathcal{E}_k \text{ for all } k \in \mathbb{N}\} \\ &= \bigcap_{k \in \mathbb{N}} \{z_i \notin \mathcal{E}_k\} \\ &= \bigcap_{k \in \mathbb{N}} \bigcap_{l=1}^{k} \{y_l \neq z_i\} \\ &= \bigcap_{k \in \mathbb{N}} \bigcap_{l=1}^{k} y_l^{-1}(\mathbb{Z}^2 \setminus \{z_i\}) \in \mathcal{F}. \end{aligned}$$

Further for $i \in \{1, \ldots, n-1\}$ we define waiting times

$$\sigma_i : \Omega \to \mathbb{N}^\infty, \omega \to \begin{cases} \tau_{i+1}(\omega) - \tau_i(\omega), & \text{if } \tau_{i+1}(\omega) < \infty \text{ and } \tau_i(\omega) < \infty, \\ \infty, & \text{else}, \end{cases}$$

so σ_i is the waiting time between adding z_i and z_{i+1} to the cluster. We quickly argument that σ_i is measurable aswell. For $j \in \mathbb{N}$ we have

$$\sigma_i^{-1}(j) = \{\tau_{i+1} - \tau_i = j\} = \bigcup_{k \in \mathbb{N}} \{\tau_{i+1} = k\} \cap \{\tau_i = k - j\} \in \mathcal{F}$$

and

$$\sigma_i^{-1}(\infty) = \{\tau_{i+1} = \infty\} \cap \{\tau_i = \infty\} \in \mathcal{F},$$

since τ_{i+1} and τ_i are measurable.

Consider the event $W_n([z])$ and take $i \in \{1, \ldots, n-1\}$. Note that in this case $\tau_j(\omega)$ is finite for all $j \in \{1, \ldots, n\}$ and $\omega \in W_n([z])$. We will now argue that the distribution of σ_i is bounded by that of a geometrically distributed random variable with paramter

$$p_n := c_1 n^{-\frac{1}{2}}$$

15

for some constat $c_1 > 0$ which is the same constant for all $n \in \mathbb{N}$.

TEST

For $j \geq 2$ define $D_j := \{A \in \mathcal{P}_f \mid \mathbb{P}(\mathcal{E}_{j-1} = A) > 0 \text{ and } z_i \in \partial A\}$. Then

$$\mathbb{P}(\sigma_i = l) = \mathbb{P}(\exists j \geq 2 \text{ such that } y_j = z_i, y_{j+1} \neq z_{i+1}, \ldots, y_{j+l-1} \neq z_{i+1}, y_{j+l} = z_{i+1})$$

$$= \mathbb{P}(\bigcup_{j \geq 2} \{y_j = z_i, y_{j+1} \neq z_{i+1}, \ldots, y_{j+l-1} \neq z_{i+1}, y_{j+l} = z_{i+1}\})$$

$$\leq \sum_{j \geq 2} \mathbb{P}(y_j = z_i, y_{j+1} \neq z_{i+1}, \ldots, y_{j+l-1} \neq z_{i+1}, y_{j+l} = z_{i+1})$$

$$\overset{(\cdot)}{=} \sum_{j \geq 2} \sum_{A \in D_j} \mathbb{P}(\mathcal{E}_{j-1} = A) \mathbb{P}(y_j = z_i, \ldots, y_{j+l-1} \neq z_{i+1}, y_{j+l} = z_{i+1} \mid \mathcal{E}_{j-1} = A)$$

$$\overset{(\cdot\cdot)}{=} \sum_{j \geq 2} \sum_{A \in D_j} \mathbb{P}(\mathcal{E}_{j-1} = A) \mathbb{P}(y_j = z_i \mid \mathcal{E}_{j-1} = A)$$
$$\cdot \mathbb{P}(y_{j+1} \neq z_{i+1}, \ldots, y_{j+l} = z_{i+1} \mid \mathcal{E}_j = A \cup \{z_i\})$$

$$\leq \sum_{j \geq 2} \sum_{A \in D_j} \mathbb{P}(\mathcal{E}_{j-1} = A) \mathbb{P}(y_{j+1} \neq z_{i+1}, \ldots, y_{j+l} = z_{i+1} \mid \mathcal{E}_j = A \cup \{z_i\})$$

$$\overset{(\cdot\cdot)}{=} \sum_{j \geq 2} \sum_{A \in D_j} \mathbb{P}(\mathcal{E}_{j-1} = A) \mathbb{P}(y_{j+1} \neq z_{i+1} \mid \mathcal{E}_j = A \cup \{z_i\})$$
$$\cdot \mathbb{P}(y_{j+2} \neq z_{i+1}, \ldots, y_{j+l} = z_{i+1} \mid \mathcal{E}_j = A \cup \{z_i\} \text{ and } y_{j+1} \neq z_{i+1})$$

$$= \sum_{j \geq 2} \sum_{A \in D_j} \mathbb{P}(\mathcal{E}_{j-1} = A)(1 - h_{A \cup \{z_i\}}(z_{i+1}))$$
$$\cdot \mathbb{P}(y_{j+2} \neq z_{i+1}, \ldots, y_{j+l} = z_{i+1} \mid \mathcal{E}_j = A \cup \{z_i\} \text{ and } y_{j+1} \neq z_{i+1})$$

$$\overset{(\cdots)}{\leq} \sum_{j \geq 2} \sum_{A \in D_j} \mathbb{P}(\mathcal{E}_{j-1} = A)(1 - h_{A \cup \{z_i\}}(z_{i+1}))$$
$$\cdot \sum_{y \in \partial(A \cup \{z_i\}), y \neq z_{i+1}} \mathbb{P}(y_{j+2} \neq z_{i+1}, \ldots, y_{j+l} = z_{i+1} \mid \mathcal{E}_j = A \cup \{z_i\} \text{ and } y_{j+1} = y)$$

$$\overset{(\cdots\cdot)}{\leq} \sum_{j \geq 2} \sum_{A \in D_j} \mathbb{P}(\mathcal{E}_{j-1} = A)(1 - h_{A \cup \{z_i\}}(z_{i+1}))$$
$$\cdot 4j \; \mathbb{P}(y_{j+2} \neq z_{i+1}, \ldots, y_{j+l} = z_{i+1} \mid \mathcal{E}_j = A \cup \{z_i\} \text{ and } y_{j+1} = \tilde{y})$$

$$= \ldots$$

TEST Now choose β such that $4e^2 \beta c_1 < 1$ und choose $N \in \mathbb{N}$ such that $\beta c_1 \geq 2 p_n$ for all $n > N$. If we then define $a := \beta c_1$ and $Y := \sum_{i=1}^{n-1} \sigma_i = \tau_n - \tau_1$ we can use Lemma 4.2.3 and get

$$PROBLEM\mathbb{P}(\tau_n \leq \tilde{m}_n) \leq \mathbb{P}(\tau_n - \tau_1 \leq \tilde{m}_n) = \mathbb{P}(Y \leq \frac{an}{p_n}) \leq (e^2 a)^{n-1} \quad \text{for all } n > N$$

and since $W_n([z]) \subset \{\tau_n \leq \tilde{m}_n\}$ we get

$$\mathbb{P}(W_n([z])) \leq (e^2 a)^n \quad \text{for all } n > N.$$

Counting the elements in Z_n we have less or equal $c_2 n$ points in $\tilde{B}_n$ for some constant $c_2 > 0$ as starting points and 4^{n-1} possibilities for the next $n-1$ steps of a random walk of length n. So $|Z_n| \leq c_2 n 4^{n-1}$ and therefore

$$\sum_{[z] \in Z_n} \mathbb{P}(W_n([z])) \leq c_2 n 4^{n-1} (e^2 a)^{n-1} = c_2 n (4e^2 a)^{n-1} \quad \text{for all } n > N$$

and finally

$$\sum_{n > N} \mathbb{P}(W_n) \leq \sum_{n > N} c_2 n (4e^2 \beta c_1)^{n-1} =: r_\beta.$$

Since β was chosen such that $4e^2 \beta c_1 < 1$, r_β is finite and therefore in total we get

$$\sum_{n \in \mathbb{N}} \mathbb{P}(W_n) < \infty,$$

which completes the proof. $\qquad\square$

Remark 4.2.1. With this theorem and 3.4 we conclude that the fractal dimension of external DLA is at least $\frac{3}{2}$. In the next chapter we will look at another incremental aggregate which tries to approximate DLA and compare simulations of both to get a empirical comparision of their growth rates.

5 Line Hitting Aggregate

5.1 Motivation

In the following we will look at a process which is the approach of a simple approximation of external DLA on $\mathbb{Z}^2$. The idea is to let particles move on straight lines coming from infinity and add them to the cluster where they hit it. Obviously in most cases particles cannot move completely straight on $\mathbb{Z}^2$. Therefore we will consider points in $\mathbb{Z}^2$ as the centers of unit squares and let the particles move on straight lines in the full plane $\mathbb{R}^2$. We consider a line hitting a point in $\mathbb{Z}^2$ if and only if it intersects with its unit square as defined in the following.

Definition 5.1.1. Define

$$\mathbb{C}_{sq} := \{[k - \frac{1}{2}, k + \frac{1}{2}] + [l - \frac{1}{2}, l + \frac{1}{2}]i \subset \mathbb{C} \mid k, l \in \mathbb{Z}\}, \tag{5.1}$$

note that $\mathbb{C} = \bigcup_{s \in \mathbb{C}_{sq}} s$. The canonical function

$$sq : \mathbb{Z}^2 \to \mathbb{C}_{sq}, \quad (k, l) \to [k - \frac{1}{2}, k + \frac{1}{2}] + [l - \frac{1}{2}, l + \frac{1}{2}]i \tag{5.2}$$

is bijective and intuitively identifies points in $\mathbb{Z}^2$ with squares in $\mathbb{C}$ which is p is the center of the square $sq(p)$ for all $p \in \mathbb{Z}^2$. In the following when using a point $p \in \mathbb{Z}^2$ it will reference the point in $\mathbb{Z}^2$ or the corresponding square in $\mathbb{C}$ respecting the context. This bijection also naturally defines a graph structure on $\mathbb{C}_{sq}$, which is two squares $s_1, s_2 \in \mathbb{C}_{sq}$ form an edge if and only if $sq^{-1}(s_1)$ and $sq^{-1}(s_2)$ form an edge in $\mathbb{Z}^2$. For the following we say a line g *hits* a point $p \in \mathbb{Z}^2$ if and only if $g \cap sq(p) \neq \emptyset$ (see in *Figure* 2).

Definition 5.1.2. Let $g = g_{\alpha,p} \in \mathcal{G}$ and $A \in \mathcal{P}_f$. We define

$$g \cap A := \{p \in A \mid g \text{ hits } p\}$$

which is the subset of all points in A which are hit by g (see *Figure* 3). For the following we suppose $g \cap A \neq \emptyset$. We will define a total ordered relation $\lhd$ on $g \cap A$ which shall be defined equivalently for all $g \in \mathcal{G}$ and $A \in \mathcal{P}_f$ with $g \cap A \neq \emptyset$. We choose two points $x, y \in g \cap A$ and split the definition of the relation $\lhd$ into four cases, depending on whether the line g goes from left-bottom to right-top, left-top to right-bottom, parallel to the x-axis and parallel to the y-axis. Denote the real part of x with $Re(x)$ and the imaginary one with $Im(x)$.

Case 1 : g is parallel to the x-axis $(\Leftrightarrow \quad \alpha = \frac{\pi}{2})$

$$x \lhd y \quad :\Leftrightarrow \quad Re(x) < Re(y)$$
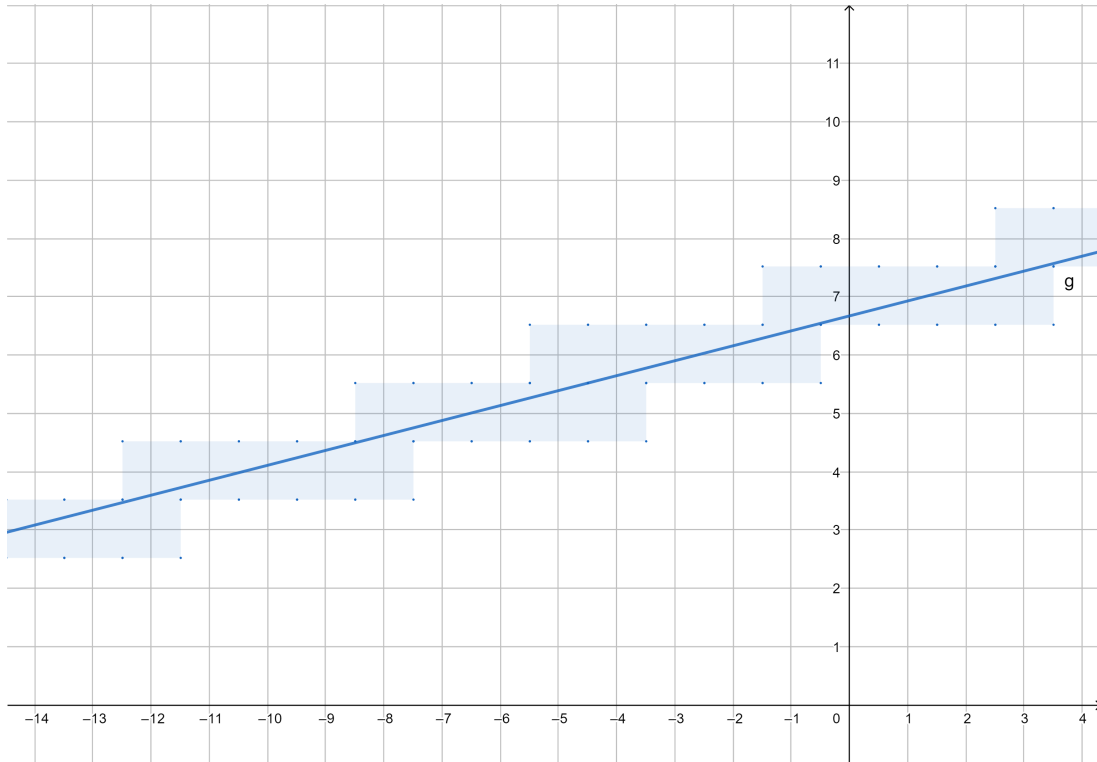
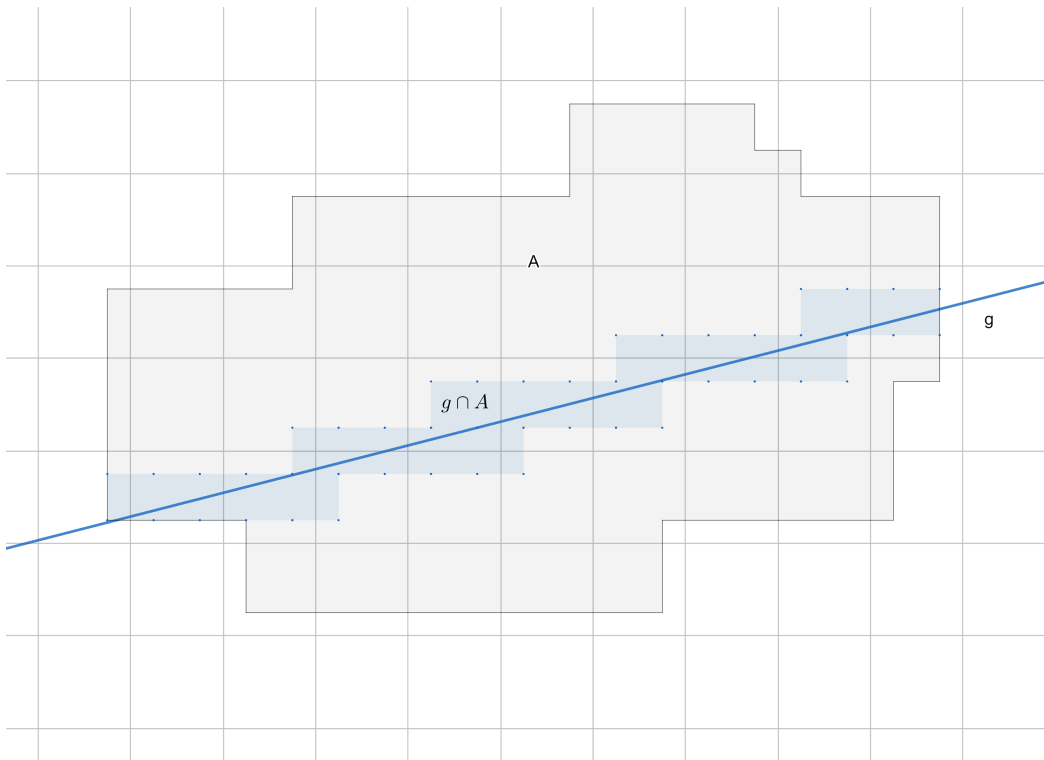Figure 2: g hits squares around points

Figure 3: g hits squares in A

Case 2 : g is parallel to the y-axis $(\Leftrightarrow \quad \alpha = 0)$

$$x \triangleleft y \quad :\Leftrightarrow \quad Im(x) < Im(y)$$

Case 3 : g is going from left-bottom to right-top $(\Leftrightarrow \quad \alpha \in (\frac{\pi}{2}, \pi))$

$$x \triangleleft y \quad :\Leftrightarrow \quad \begin{cases} Re(x) < Re(y), & \text{if } Re(x) \neq Re(y), \\ Im(x) < Im(y), & \text{if } Re(x) = Re(y). \end{cases}$$

Case 4 : g is going from left-top to right-bottom $(\Leftrightarrow \quad \alpha \in (0, \frac{\pi}{2}))$

$$x \triangleleft y \quad :\Leftrightarrow \quad \begin{cases} Re(x) < Re(y), & \text{if } Re(x) \neq Re(y), \\ Im(x) > Im(y), & \text{if } Re(x) = Re(y). \end{cases}$$

It is easy to see that this relation on $g \cap A$ is well-defined. In the following we will prove that this relation is strictly and totally ordered.

Lemma 5.1.1. For a line $g = g_{\alpha,p} \in \mathcal{G}$ and $A \in \mathcal{P}_f$ with $g \cap A \neq \emptyset$ the relation $\triangleleft$ on $g \cap A$ is totally ordered.

Proof. We will only proove the case where g is going from left-bottom to right-top, which is *Case* 3 of the definition. In this case we have $\alpha \in (\frac{\pi}{2}, \pi)$. Note, that the proof for *Case* 4 will work very similar and in the case of g being parallel to one of the axes (*Case* 1 or 2), all properties for a totally ordered relation follow directly from the totally ordered relation $<$ on $\mathbb{R}$. So let $\alpha \in (\frac{\pi}{2}, \pi)$.

Antisymmetry : For antisymmetry let $x \triangleleft y$ and $y \triangleleft x$. Suppose $Re(x) \neq Re(y)$, then $Re(x) < Re(y)$ and $Re(y) < Re(x)$, a contradiction because of the total order $<$ in $\mathbb{R}$. So $Re(x) = Re(y)$. But then we have $Im(x) < Im(y)$ and $Im(y) < Im(x)$ and therefore also $Im(x) = Im(y)$, hence $x = y$.

Transitivity : For transitivity let $x \triangleleft y$ and $y \triangleleft z$. We find four cases. In case $Re(x) \neq Re(y)$ and $Re(y) \neq Re(z)$ we get $Re(x) < Re(z)$ by transitivity of $<$, hence $x \triangleleft z$. In case $Re(x) \neq Re(y)$ and $Re(y) = Re(z)$ we get $Re(x) < Re(y) = Re(z)$, therefore $x \triangleleft z$. In case $Re(x) = Re(y)$ and $Re(y) \neq Re(z)$ we get $Re(x) = Re(y) < Re(z)$, similar as the last case. In the last case $Re(x) = Re(y) = Re(z)$ we get $Im(x) < Im(y)$ and $Im(y) < Im(z)$ and again by transitivity of $<$ we get $Im(x) < Im(z)$, hence $x \triangleleft z$ again.

Connexity : Connexity is given since for any two points $x, y \in g \cap A$ we have either $Re(x) \neq Re(y)$ or $Re(x) = Re(y)$ and therefore either $x \triangleleft y$ or $y \triangleleft x$. $\qquad \square$

Remark 5.1.1. The relation $\lhd$ on $g \cap A$ basically orders the hitting points of g with A from left to right (or bottom to top in case of a line parallel to the y-axis). This order allows us to identify the outermost hitting points which are the minimum and maximum of $g \cap A$ with respect to $\lhd$. To clarify, we define $\min(g \cap A) := x_0$ if and only if $x_0 \lhd x$ for all $x \in g \cap A, x \neq x_0$, analogously $\max(g \cap A)$. This means when moving on g facing A coming from infinity this order allows to know where in A the line g hits first when „entering" A and where it hits last when „leaving" A. What we want to do next is to choose a line randomly out of all lines which hit the current cluster. This is isn't an obvious task and we will have to develop a most fair underlying distribution on lines in the next section where we will touch basic aspects of Integral Geometry.

5.2 Integral Geometry

In the next section we want to define an approximation for External DLA. This approximation will be an incremental aggregate for which distribution definition we need some concepts and results from Integral Geometry which we will discuss and develop in this section. In the process we want to define we will want to choose a random line out of all lines which intersect with the current cluster of the aggregate. This random choosing is not obvious since most of the time the cluster will be strongly non symmetric and it is even less obvious how to actually get a realisation of a random line when simulating with Python. In our case we are looking for a parametrisation of lines in the plane and a reasonable way of choosing parameters randomly.

We will introduce a possible solution for this problem first through the abstract and general concepts of integral geometry and later through a simple parametrisation for the case of lines in the plane which goes hand in hand with the general result.

5.2.1 General results

In the general context we are in $\mathbb{R}^d$ for $d \in \mathbb{N}$ and consider q-dimensional affine subspaces where $q \in \{0, \ldots, d\}$, short q-flats in $\mathbb{R}^d$. The set of q-flats in $\mathbb{R}^d$ is denoted by $A(d,q)$. Later we will be interested in choosing random lines in the real plane (i.e. 1-flats in $\mathbb{R}^2$). In order to get a probability measure on some set of q-flats, we first need a measure and a σ-algebra on $A(d,q)$ in total.

Definition 5.2.1. For $B \in \mathcal{B}^d$ define

$$[B]_{d,q} := \{F \in A(d,q) \mid F \cap B \neq \emptyset\}.$$

If the context is clear, we will only write $[B]$ instead of $[B]_{d,q}$.

Definition 5.2.2. The σ-algebra $\mathcal{A}(d,q)$ on $A(d,q)$ is defined by

$$\mathcal{A}(d,q) := \sigma(\{[K] \mid K \in \mathcal{K}^d\}).$$

Theorem 5.2.1. On $A(d, q)$ there exists a unique G_d-invariant Radon measure μ_q such that

$$\mu_q(A_{B_d(1,0)}) = \kappa_{d-q}, \tag{5.1}$$

where $\kappa_n := \lambda_n(B_n(1, 0))$ is the n-dimensional Lebesque meausure of the n-dimensional unit ball for $n \in \mathbb{N}$, and $\kappa_0 := 1$.

Proof. [1] Theorem 4.26

MAKE PROOF CLEARER $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

5.2.2 Construction in the plane: Isotropic lines

For our special case we choose $d = 2$ and $q = 1$, thus lines in the plane. We denote this set of lines by $\mathcal{G}$. The following construction in this chapter is completely motivated by [2] 2.1.1. Firstly we propose a parametrisation of lines which works as follows. Every line can be uniquely determined by an angle $\alpha \in [0, \pi)$ and a real number $p \in \mathbb{R}$. Let $\langle \cdot, \cdot \rangle$ be the standard scalar product on $\mathbb{R}^2$, respectively used for values in $\mathbb{C}$ as we identify $\mathbb{R}^2$ with $\mathbb{C}$ as $\mathbb{R}$-vectorspaces. Let $e_\alpha := e^{\alpha i} = \cos(\alpha) + \sin(\alpha)i$ and $s_\alpha := -\sin(\alpha) + \cos(\alpha)i$ be the unit vectors 1 and i rotated by α counterclockwise. Lets consider the representation $x = \langle x, e_\alpha \rangle e_\alpha + \langle x, s_\alpha \rangle s_\alpha$ for $x \in \mathbb{C}$. Since e_α and s_α form a base of $\mathbb{C}$ as a $\mathbb{R}$-vectorspace, the parameters $\langle x, e_\alpha \rangle$ and $\langle x, s_\alpha \rangle$ are unique for each x. It thus is easy to realize that $g_{\alpha,p} := \{x \in \mathbb{C} \mid \langle x, e_\alpha \rangle = p\}$ defines a line (compare with *Figure* 4) and that every line has a unique pair of α and p for such a representation. In words, $g_{\alpha,p}$ contains all points which have length p in direction of e_α. With $\Phi := [0, \pi) \times \mathbb{R}$ this naturally defines a bijection

$$\chi : \Phi \to \mathcal{G}, \quad (\alpha, p) \mapsto g_{\alpha,p}.$$

We take the subspace Borel-σ-algebra $\mathcal{B}_\Phi := \mathcal{B}^2 \cap \Phi$ on Φ and define the σ-algebra $\mathfrak{G}$ on $\mathcal{G}$ by $\mathfrak{G} := \chi(\mathcal{B}_\Phi)$. This works well since χ is a bijection. We want to show in the following that this way of defining a σ-algebra on $\mathcal{G}$ makes sense as it is indeed equivalent to the general context as defined above. To do that it is convenient to use a special generator set for the Borel-σ-algebra $\mathcal{B}_\Phi$.

Lemma 5.2.1. Define

$$\mathcal{R}_+ := \{[\alpha, \beta] \times (0, b] \mid 0 \leq \alpha < \beta < \pi, b \geq 0\},$$

$$\mathcal{R}_- := \{[\alpha, \beta] \times [b, 0) \mid 0 \leq \alpha < \beta < \pi, b \leq 0\},$$

$$\mathcal{R}_0 := \{[\alpha, \beta] \times \{0\} \mid 0 \leq \alpha < \beta < \pi\},$$
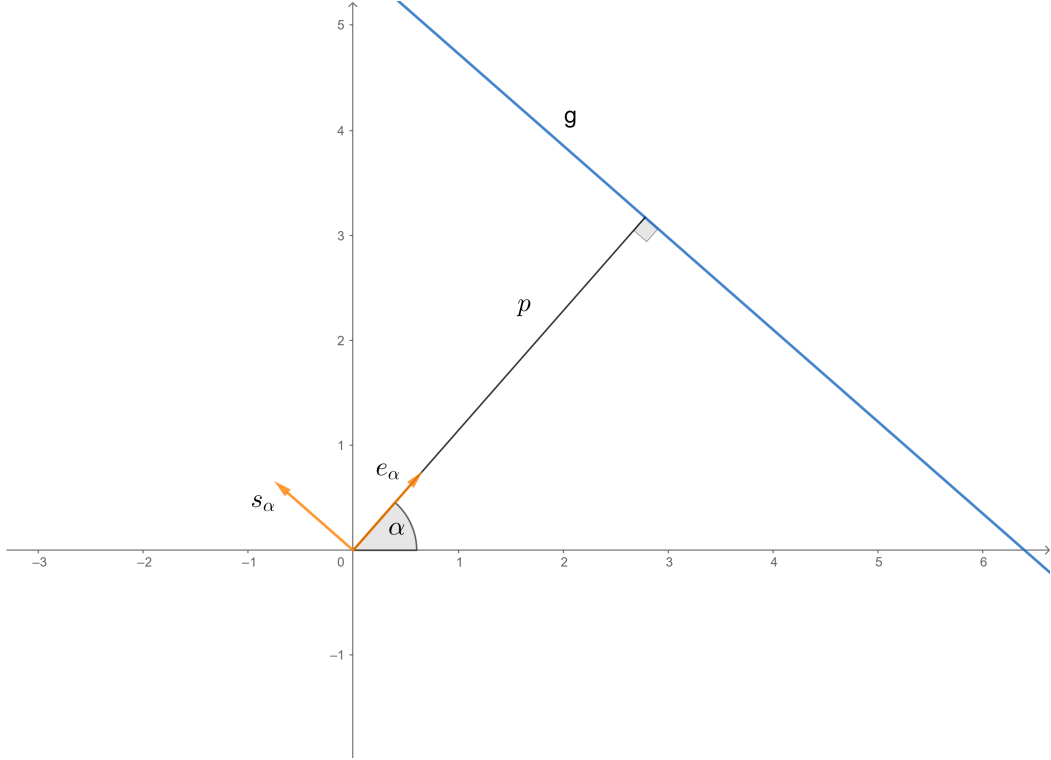
22

Figure 4: Line parameters α and p

and

$$\mathcal{R} := \mathcal{R}_+ \cup \mathcal{R}_- \cup \mathcal{R}_0.$$

Then $\sigma(\mathcal{R}) = \mathcal{B}_\Phi$.

Proof. We show that $\sigma(\mathcal{R})$ contains all rectangles in Φ of the form $[\alpha, \beta] \times (a, b]$ with $a, b > 0$, $[\alpha, \beta] \times [a, b)$ with $a, b < 0$ and $[\alpha, \beta] \times [a, b]$ with $0 \in [a, b]$. First let $a, b > 0$ and $R = [\alpha, \beta] \times (a, b]$, then

$$R = ([\alpha, \beta] \times (0, b]) \setminus ([\alpha, \beta] \times (0, a])$$

and therefore $R \in \sigma(\mathcal{R}_+) \subset \sigma(\mathcal{R})$. Similarly it works if $a, b < 0$. If $0 \in [a, b]$ then we can write $R = [\alpha, \beta] \times [a, b]$ with three components $R = [\alpha, \beta] \times [a, 0) \cup [\alpha, \beta] \times \{0\} \cup [\alpha, \beta] \times (0, b]$ which lie in $\mathcal{R}_-, \mathcal{R}_0$ and $\mathcal{R}_+$ respectively. Therefore $R \in \sigma(\mathcal{R})$ aswell. By measure theory the above described rectangles form a generator set of $\mathcal{B}_\Phi$, which completes the proof. $\qquad\square$

Lemma 5.2.2. We have $\mathcal{A}(2, 1) = \mathfrak{G}$.

Proof. We will consider generators of these σ-algebras. By Lemma 5.2.1 we know that $\sigma(\mathcal{R}) = \mathcal{B}_\Phi$ and since χ is a bijection, we have $\chi(\sigma(\mathcal{R})) = \sigma(\chi(\mathcal{R}))$ and finally $\mathfrak{G} = \sigma(\chi(\mathcal{R}))$. For $\tilde{A} := \{[K] \mid K \in \mathcal{K}^2\}$ we have by definition $\mathcal{A}(2, 1) = \sigma(\tilde{A})$.
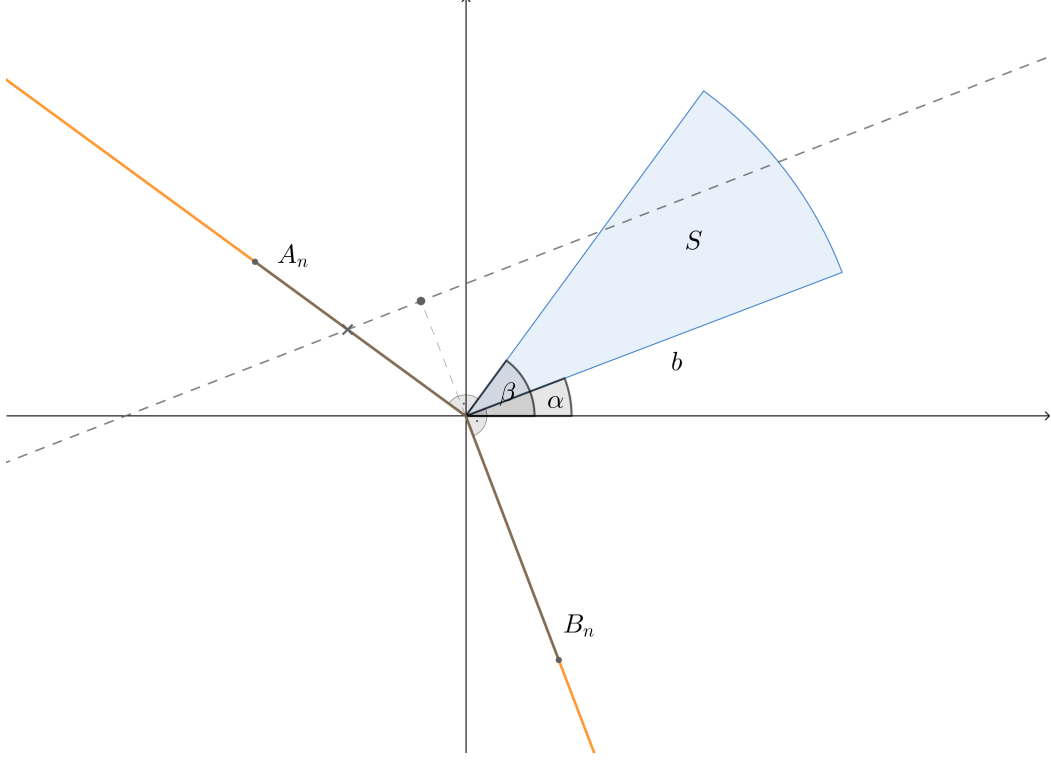
Figure 5: S and the sets A_n and B_n

$\subset$: Let $K \in \mathcal{K}^2$. We will show that $\chi^{-1}([K])$ is a closed set in Φ. If that is the case we have $\chi^{-1}([K]) \in \mathcal{B}_\Phi$, therefore $[K] \in \chi(\mathcal{B}_\Phi) = \mathfrak{G}$ and finally $\mathcal{A}(2,1) = \sigma(\tilde{A}) \subset \mathfrak{G}$. To show that $\chi^{-1}([K])$ is closed let $(\alpha_0, p_0) \in \Phi \setminus \chi^{-1}([K])$. Then $\chi(\alpha_0, p_0) \notin [K]$ and therefore $\chi(\alpha_0, p_0) \cap K = \emptyset$. Since K is closed we can find small values $\tilde{\alpha}, \tilde{p} > 0$ such that $\chi(\alpha, p) \cap K = \emptyset$ for all $(\alpha, p) \in [\alpha_0, \alpha_0 + \tilde{\alpha}] \times [p_0, p_0 + \tilde{p}] =: R$. Hence we have $R \subset \Phi \setminus \chi^{-1}([K])$, so $\Phi \setminus \chi^{-1}([K])$ is open. Hence $\chi^{-1}([K])$ is closed.

$\supset$: For this inclusion we will show that $\chi(R) \in \mathcal{A}(2,1)$ for all $R \in \mathcal{R}$. First let $R = [\alpha, \beta] \times (0, b] \in \mathcal{R}_+$ for some $b > 0$. Define

$$S := \{pe_\gamma \in \mathbb{C} \mid (\gamma, p) \in R\}$$

Furthermore for $n \in \mathbb{N}$ define

$$A_n := \{tns_\beta \mid t \in [0,1]\} \text{ and } B_n := \{-tns_\alpha \mid t \in [0,1]\},$$

the segments from 0 to ns_β and 0 to $-ns_\alpha$ (*Figure* 5). We will show now that

$$\chi(R) = [\bar{S}] \setminus \left(\bigcup_{n \in \mathbb{N}} [A_n] \cup \bigcup_{n \in \mathbb{N}} [B_n] \right) =: \tilde{S},$$

where $\bar{S}$ is the closure of S (note that $\bar{S} = S \cup \{0\}$). Let $(\gamma, p) \in R$. Then $pe_\gamma \in \chi(\gamma, p) \cap S$ and therefore $\chi(\gamma, p) \in [\bar{S}]$. Assume that there exits an $n \in \mathbb{N}$ such

that $\chi(\gamma, p) \cap A_n \neq \emptyset$. Then $\beta + \frac{\pi}{2} - \gamma < \frac{\pi}{2}$, hence $\beta < \gamma$, a contradiction. Similarly argument for any B_n, so we finally have $\chi(\gamma, p) \notin [A_n]$ and $\chi(\gamma, p) \notin [B_n]$ for any $n \in \mathbb{N}$. Hence $\chi(\gamma, p) \in \tilde{S}$ and therefore $\chi(R) \subset \tilde{S}$.

Now let $(\gamma, p) \in \Phi$ such that $\chi(\gamma, p) \in \tilde{S}$. Assume that $\gamma \notin [\alpha, \beta]$ then with a similar argument as in the first inclusion it is easy to see that there must be an $n \in \mathbb{N}$ such that $\chi(\gamma, p) \cap A_n \neq \emptyset$ or $\chi(\gamma, p) \cap B_n \neq \emptyset$, a contradiction. Therefore $\gamma \in [\alpha, \beta]$. Now assume $p \notin (0, b]$. If $p > b$ then $\chi(\gamma, b) \cap B_b = \emptyset$ and since $\bar{S} \subset B_b$ it is $\chi(\gamma, p) \notin [\bar{S}]$, a contradiction. If $p < 0$ then, since the angle between the segments A_n and B_n opposite of S is strictly smaller than π, $\chi(\gamma, p)$ must intersect with A_n or B_n for some $n \in \mathbb{N}$, again a contradiction. Note that $p \neq 0$ since $[\{0\}] \subset [A_1]$. Thus we have $p \in (0, b]$. Therefore we have $(\gamma, p) \in R$ and finally $\tilde{S} \subset \chi(R)$.

It is left to show that $\tilde{S} \in \mathcal{A}(2, 1)$. All the segments A_n and B_n are compact and convex for all $n \in \mathbb{N}$, and since S is bounded and convex as a circle segment with angle smaller than π, $\bar{S}$ is compact and convex. Finally $\tilde{S} \in \mathcal{A}(2, 1)$.

In total we get $\mathcal{R}_+ \subset \mathcal{A}(2, 1)$. $\mathcal{R}_- \subset \mathcal{A}(2, 1)$ can be shown analogously. So for the last case let $R = [\alpha, \beta] \times \{0\} \in \mathcal{R}_0$. Define the line segment

$$T := \{(1 - t)s_\alpha + ts_\beta \mid t \in [0, 1]\}.$$

Then $[\{0\}] \cap [T] \in \mathcal{A}(2, 1)$. We show that $\chi(R) = [\{0\}] \cap [T]$. Let $(\gamma, 0) \in R$. Then $\chi(\gamma, 0)$ contains 0 and since γ is in between α and β, $\chi(\gamma, 0)$ must intersect with T (*Figure* 6). For the other inclusion let $(\gamma, p) \in \Phi$ such that $\chi(\gamma, p) \in [\{0\}] \cap [T]$. Since $0 \in \chi(\gamma, p)$ it must be $p = 0$ and since it intersects with T its angle must lay in between α and β. All in all this completes the proof. $\qquad\square$

Definition 5.2.3. A $\mathcal{F}$-$\mathfrak{G}$-measurable function $g : \Omega \to \mathcal{G}$ is called a *random line*.

Definition 5.2.4. We define the measure $\mu := \lambda_{2|\Phi} \circ \chi^{-1}$ on $(\mathcal{G}, \mathfrak{G})$ where $\lambda_{2|\Phi}$ is the 2-dimensional Lebesgue measure restricted to Φ. We say a measure ν on $(\mathcal{G}, \mathfrak{G})$ is locally finite if for any $K \in \mathcal{K}^2$ we have $\nu([K]) < \infty$.

Lemma 5.2.3. μ is locally finite and G_2-invariant.

Proof. Let $K \in \mathcal{K}^2$ and since K is compact choose $r > 0$ such that $K \subset B_r$. Then we have $[K] \subset A_{B_r}$ and

$$\mu([K]) \leq \mu(A_{B_r}) = \lambda_{2|\Phi}(\chi^{-1}(A_{B_r})) = \lambda_{2|\Phi}([0, \pi) \times [-r, r]) = 2\pi r < \infty,$$

hence μ is locally finite. To show that μ is G_2-invariant, that means euclidean motion invariant, we must show it is translation and rotation invariant. First we clarify what exactly translation and rotation mean for lines. We denote x *modulo* r as $(x)_r$. For $b \in \mathbb{C}$ and $\beta \in [0, 2\pi)$ we define

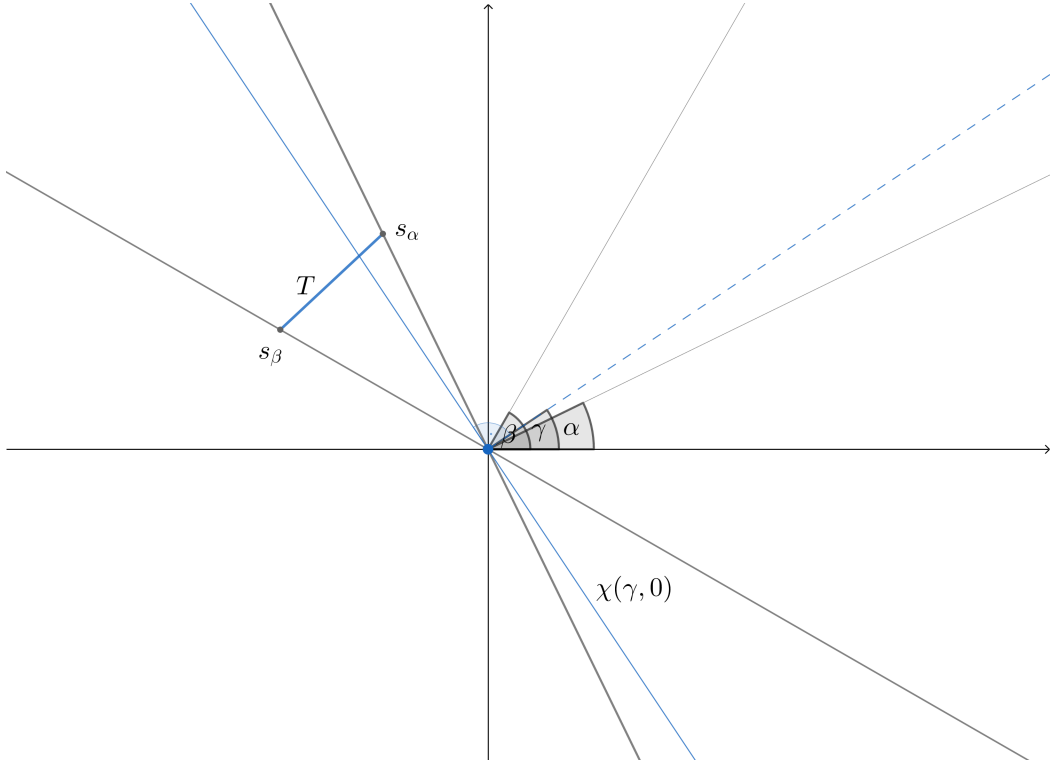$$T_b : \Phi \to \Phi, \quad (\alpha, p) \mapsto (\alpha, p + \langle e_\alpha, b \rangle) \tag{5.2}$$

Figure 6: Line in $[\{0\}] \cap [T]$

and

$$D_\beta : \Phi \to \Phi, \quad (\alpha, p) \mapsto ((\alpha + \beta)_\pi, \delta((\alpha + \beta)_{2\pi})p), \tag{5.3}$$

where

$$\delta : [0, 2\pi) \to \{-1, 1\}, \quad \gamma \to \begin{cases} 1, & \gamma \in [0, \pi) \\ -1, & \gamma \in [\pi, 2\pi) \end{cases}.$$

It is easy to see that both functions all well-defined. T_b defines a translation by b and D_β a rotation by β. Lets proof that first. Let $(\alpha, p) \in \Phi$, then

$$\begin{aligned}
x \in \chi(\alpha, p) + b &\Leftrightarrow x - b \in \chi(\alpha, p) \\
&\Leftrightarrow \langle e_\alpha, x - b \rangle = p \\
&\Leftrightarrow \langle e_\alpha, x \rangle = p + \langle e_\alpha, b \rangle \\
&\Leftrightarrow x \in \chi(\alpha, p + \langle e_\alpha, b \rangle) \\
&\Leftrightarrow x \in \chi(T_b(\alpha, p))
\end{aligned}$$

and therefore $\chi(\alpha, p) + b = \chi(T_b(\alpha, p))$. Hence $T_b(\alpha, p)$ are indeed the parameters for the by b translated line. For the rotation lets devide it into two cases. First let $(\alpha + \beta)_{2\pi} \in [0, \pi)$, then $\delta((\alpha + \beta)_{2\pi}) = 1$ and $(\alpha + \beta)_\pi = \alpha + \beta$ and therefore

$$D_\beta(\alpha, p) = (\alpha + \beta, p).$$

26

In the second case with $(\alpha + \beta)_{2\pi} \in [\pi, 2\pi)$ we have $\delta((\alpha + \beta)_{2\pi}) = -1$ and $(\alpha + \beta)_\pi = \alpha + \beta - \pi$ and therefore

$$D_\beta(\alpha, p) = (\alpha + \beta - \pi, -p).$$

In the second case we have to carefully understand the parametrisation of $\mathcal{G}$, but finally we can see that $D_\beta(\alpha, p)$ are indeed the parameters of the by β rotated line.

We will further show now, that μ is invariant in respect to both these functions. Let $A \in \mathfrak{G}$, $b \in \mathbb{C}$ and $\nu_\beta \in SO_2$ for some $\beta \in [0, 2\pi)$. We will understand $A + b = \{g + b \in \mathcal{G} \mid g \in A\}$ and $\nu_\beta A = \{\nu_\beta g \in \mathcal{G} \mid g \in A\}$ pointwise, and $g + b = \{x + b \mid x \in g\}$ and $\nu_\beta g = \{\nu_\beta x \mid x \in g\}$ pointwise aswell. We furthermore define $A_p := \{\alpha \in [0, \pi) \mid (\alpha, p) \in \chi^{-1}(A)\}$ and $A_\alpha := \{p \in \mathbb{R} \mid (\alpha, p) \in \chi^{-1}(A)\}$ for $(\alpha, p) \in \Phi$. For a translation we get

$$
\begin{aligned}
\mu(A + b) &= \int_\mathfrak{G} \mathbb{1}_{A+b}(g)\mu(dg) \\
&= \int_\mathfrak{G} \mathbb{1}_A(g - b)\mu(dg) \\
&= \int_\mathfrak{G} \mathbb{1}_A(g - b)\lambda_{2|\Phi}(\chi^{-1}(dg)) \\
&= \int_{\chi^{-1}(\mathfrak{G})} \mathbb{1}_{\chi^{-1}(A)}(\chi^{-1}(g - b))\lambda_{2|\Phi}(d(\chi^{-1}(g))) \\
&\overset{(5.2)}{=} \int_\Phi \mathbb{1}_{\chi^{-1}(A)}(\alpha, p - \langle e_\alpha, b\rangle)\lambda_{2|\Phi}(d(\alpha, p)) \\
&= \int_0^\pi \int_\mathbb{R} \mathbb{1}_{A_\alpha}(p - \langle e_\alpha, b\rangle)\lambda_1(dp)\lambda_{1|[0,\pi)}(d\alpha) \\
&= \int_0^\pi \int_\mathbb{R} \mathbb{1}_{A_\alpha + \langle e_\alpha, b\rangle}(p)\lambda_1(dp)\lambda_{1|[0,\pi)}(d\alpha) \\
&= \int_0^\pi \lambda_1(A_\alpha + \langle e_\alpha, b\rangle)\lambda_{1|[0,\pi)}(d\alpha) \\
&\overset{(+)}{=} \int_0^\pi \lambda_1(A_\alpha)\lambda_{1|[0,\pi)}(d\alpha) \\
&= \dots \\
&= \mu(A),
\end{aligned}
$$

and for a rotation we get

$$
\begin{aligned}
\mu(\nu_\beta A) &= \int_\mathcal{G} \mathbb{1}_{\nu_\beta A}(g)\mu(dg) \\
&= \int_\mathcal{G} \mathbb{1}_A(\nu_{-\beta}g)\lambda_{2|\Phi}(\chi^{-1}(dg)) \\
&= \int_{\chi^{-1}(\mathcal{G})} \mathbb{1}_{\chi^{-1}(A)}(\chi^{-1}(\nu_{-\beta}g))\lambda_{2|\Phi}(d(\chi^{-1}(g))) \\
&\overset{(5.3)}{=} \int_\Phi \mathbb{1}_{\chi^{-1}(A)}((\alpha - \beta)_\pi, \delta((\alpha - \beta)_{2\pi})p)\lambda_{2|\Phi}(\alpha, p) \\
&= \int_\Phi \mathbb{1}_{\chi^{-1}(A)}((\alpha - \beta)_\pi, \delta((\alpha - \beta)_{2\pi})p)\lambda_{2|\Phi}(\alpha, p)
\end{aligned}
$$

$$= \int_0^{2\pi} \int_{\mathbb{R}} \mathbb{1}_{\chi^{-1}(A)}((\alpha - \beta)_\pi, \delta((\alpha - \beta)_{2\pi})p) \lambda_1(dp) \lambda_{1|[0,\pi)}(d\alpha)$$

$$= \int_0^{2\pi} \int_{\mathbb{R}} \mathbb{1}_{A_{(\alpha-\beta)_\pi}}(\delta((\alpha - \beta)_{2\pi})p) \lambda_1(dp) \lambda_{1|[0,\pi)}(d\alpha)$$

$$\overset{(+)}{=} \int_0^{2\pi} \int_{\mathbb{R}} \mathbb{1}_{A_{(\alpha-\beta)_\pi}}(p) \lambda_1(dp) \lambda_{1|[0,\pi)}(d\alpha)$$

$$= \int_{\mathbb{R}} \int_0^{2\pi} \mathbb{1}_{\chi^{-1}(A)}((\alpha - \beta)_\pi, p) \lambda_{1|[0,\pi)}(d\alpha) \lambda_1(dp)$$

$$= \int_{\mathbb{R}} \int_0^{2\pi} \mathbb{1}_{A_p}((\alpha - \beta)_\pi) \lambda_{1|[0,\pi)}(d\alpha) \lambda_1(dp)$$

$$\overset{(+)}{=} \int_{\mathbb{R}} \int_0^{2\pi} \mathbb{1}_{A_p}(\alpha) \lambda_{1|[0,\pi)}(d\alpha) \lambda_1(dp)$$

$$= \int_{\mathbb{R}} \int_0^{2\pi} \mathbb{1}_{\chi^{-1}(A)}(\alpha, p) \lambda_{1|[0,\pi)}(d\alpha) \lambda_1(dp)$$

$$= \ldots$$

$$= \mu(A).$$

where in $(+)$ we used the translation and rotation invariance of the Lebesgue measure. This completes the proof. $\qquad\square$

By 5.2.1 we know that μ is, up to a factor, the only euclidean motion invariant measure on $\mathcal{G}$. Since it is locally finite, for $K \in \mathcal{K}^2$ we can define a probability measure on $\mathcal{G}$ by

$$\mathbb{P}_\mu^K(A) := \frac{\mu(A \cap [K])}{\mu([K])}, \quad A \in \mathfrak{G}.$$

Definition 5.2.5. Let $K \in \mathcal{K}^2$. A random line $g : \Omega \to \mathcal{G}$ is called *K-isotropic* if

$$\mathbb{P}(g \in A) = \mathbb{P}_\mu^K(A), \quad A \in \mathfrak{G}.$$

Lemma 5.2.4. Let $M, K \in \mathcal{K}^2$ with $M \subset K$. Let f be a random K-isotropic and g be a random M-isotropic line. Then for all $A \in \mathfrak{G}$ we have

$$\mathbb{P}(f \in A \mid f \in [M]) = \mathbb{P}(g \in A).$$

Proof. Note that since $M \subset K$ it is $[M] \subset [K]$. For $A \in \mathfrak{G}$ we therefore directly get

$$\mathbb{P}(f \in A \mid f \in [M]) = \frac{\mathbb{P}(f \in A \cap [M])}{\mathbb{P}(f \in [M])}$$

$$= \frac{\mu(A \cap [M] \cap [K])}{\mu([K])} \frac{\mu([K])}{\mu([M] \cap [K])}$$

$$= \frac{\mu(A \cap [M])}{\mu([M])}$$

$$= \mathbb{P}(g \in A).$$

$$\qquad\square$$

If we choose a simple convex set such as $K = B_r$ the ball around the origin with radius r then choosing random K-isotropic lines becomes a very intuitive and easy realizable task as the following lemma shows.

Lemma 5.2.5. Let $K = B_r \in \mathcal{K}^2$ and let (α, p) be uniformly distributed in $\tilde{\Phi} := [0, \pi) \times [-r, r] = \chi^{-1}([K]) \subset \Phi$. Then $\chi(\alpha, p)$ is a random K-isotropic line.

Proof. For $A \in \mathfrak{G}$ we get

$$\mathbb{P}(\chi(\alpha, p) \in A) = \mathbb{P}((\alpha, p) \in \chi^{-1}(A)) = \frac{\lambda_{2|\tilde{\Phi}}(\chi^{-1}(A) \cap \chi^{-1}([K])))}{\lambda_{2|\tilde{\Phi}}(\chi^{-1}([K]))}$$

$$= \frac{\lambda_{2|\Phi}(\chi^{-1}(A \cap [K]))}{\lambda_{2|\Phi}(\chi^{-1}([K]))} = \frac{\mu(A \cap [K])}{\mu([K])}.$$

$\square$

Both lemmas 5.2.4 and 5.2.5 give a help for realizing K-isotropic lines for complicated sets K. Lemma 5.2.4 tells us that we if we are looking for a K-isotropic line, we can actually take a convex, compact set B which contains K and realize B-isotropic lines. If we realize such a line and it happens that it intersects K, we know that its distribution is equal to trying to realize K-isotropic lines directly. And how to realize B-isotropic lines? Lemma 5.2.5 tells us that if we choose $B = B_r$ a ball with a big enough radius such that it contains K, then realizing B-isotropic lines comes by choosing the line parameters α and p uniformly in $[0, \pi)$ and $[-r, r]$. Finally we have a practicable process of choosing random K-isotropic lines, even if K happens to be very asymmetric and complicated. This gives the base to define a new Incremental Aggregate in the next section which tries to approximize External DLA.

5.3 Definition

We are now able to choose a line randomly out of all lines hitting a cluster in a fair and agreeable way, which are K-isotropic lines as presented in Definition 5.2.5. All in all we have realized the mathematical structure to define the incremental aggregate as we have planned it in the motivation in the beginning of this chapter. Recall that for a bounded subset $A \subset \mathbb{C}$ the convex hull $conv(A)$ of A is defined to be the smallest convex set containing A, formally

$$conv(A) := \bigcap_{A \subset K \in \mathcal{K}^2} K \in \mathcal{K}^2.$$

For a set $A \in \mathcal{P}_f$ we define

$$conv(A) := conv(\bigcup_{p \in A} sq(p))$$

and since $\bigcup_{p \in A} sq(p)$ is a bounded set we have $conv(A) \in \mathcal{K}^2$.

Definition 5.3.1. (*Random Line Hitting Distribution*) Let $A \in \mathcal{P}_f$ and $K := conv(A) \in \mathcal{K}^2$. For $x \in \mathbb{Z}^2$ and $g \in \mathcal{G}$ define

$$\gamma_A(g) := \frac{1}{2}\mathbb{1}\{|g \cap A| \geq 2\} + \mathbb{1}\{|g \cap A| = 1\},$$

$$\tilde{\mu}_A(x, g) := \gamma_A(g) \begin{cases} \mathbb{1}\{x \in \{\min(g \cap A), \max(g \cap A)\}\}, & \text{if } g \cap A \neq \emptyset, \\ 0, & \text{if } g \cap A = \emptyset. \end{cases} \tag{5.1}$$

and

$$\mu_A(x) := \frac{1}{\mathbb{P}_\mu^K([A])} \int_{\mathcal{G}} \tilde{\mu}_A(x, g) \, \mathbb{P}_\mu^K(dg). \tag{5.2}$$

We quickly show that $\mu_A \in \mathcal{D}_A$. For all $x \in \mathbb{Z}^2 \setminus A$ and $g \in \mathcal{G}$ we have $\tilde{\mu}_A(x, g) = 0$ and therefore $\mu_A(x) = 0$. Furthermore for all $g \in \mathcal{G}$ we have

$$\sum_{x \in A} \tilde{\mu}_A(x, g) = \begin{cases} \frac{1}{2}2, & |g \cap A| \geq 2, \\ 1, & |g \cap A| = 1, \\ 0, & |g \cap A| = 0. \end{cases} = \mathbb{1}\{g \cap A \neq \emptyset\}$$

and therefore

$$\sum_{x \in A} \mu_A(x) = \frac{1}{\mathbb{P}_\mu^K([A])} \int_{\mathcal{G}} \mathbb{1}\{g \cap A \neq \emptyset\} \, \mathbb{P}_\mu^K(dg) = \frac{\mathbb{P}_\mu^K([A])}{\mathbb{P}_\mu^K([A])} = 1.$$

Hence, the family of distributions $(\mu_A)_{A \in \mathcal{P}_f}$ defines an incremental aggregate.

Definition 5.3.2. (*line hitting aggregate*) An incremental aggregate with the random line hitting distribution we call *line hitting aggregate*, short *LHA*.

Remark 5.3.1. Note, that the indicator function in (5.1) is well-defined, since a line g chosen by the distribution $\mathbb{P}_\mu^K$ certainly hits $K = conv(A)$ and it is easy to understand that g hits $conv(A)$ if and only if g hits A.

5.4 Fractal Dimension and Growth Rate of LHA

6 Python Simulation of the Line Hitting Aggregate

In this section we will present a simulation for the line hitting aggregate and approximate its fractal dimension empirically. The simulation is written in *Python* and images are created with *pygame*. If you want to check and try the code yourself, feel welcomed to checkout this repository on github:

LINK

Each vertex in $\mathbb{Z}^2$ can be naturally identified with values in $\mathbb{C}$ and is represented in this way in python code. Every value in $\mathbb{C}$ with integer components can be represented by squares as presented in Definition 5.1.1, and each square will be represented by exactly one pixel on a monitor screen, so every picture created by the simulation consists of pixels each naturally representing exactly one vertex in $\mathbb{Z}^2$. The aim of this simulation is to create realisations of LHA and guess its fractal dimension as close as possible to their mathematical definitions, so the only source of error are external python packages and unexact machine calculation.

6.1 Images and Empirical Fractal Dimension

6.2 Code

In the simulation we are using imports for systemic purposes (pygame, sys, json, time) and for mathematical purposes (random, math.sin, math.cos). Only the imports for mathematical purposes are used for the calculation of a realisation of LHA. So on the side of imports only these packages will have an effect on the mathematical preciseness of the realised processes. Other than that every calculations preciseness is bounded to some extend by the preciseness of machine calculation. We will ignore a study on that in this simulation.

When using the software you will have the possibility to enter some parameters which include colorings of background and particle pixels, how many iterations you want to run the process and how big the image shall be at the end (more on that you can find in the README file of the above mentioned github repository). If correct parameters are set, you can run a shell script which in turn will run a single python file which reads the entered parameters and executes two steps. First is the execution of the mathematical process which needs the mathematical imports as described above. Second is the creation of an image of the calculated process which needs the systemic imports as described above.

References

[1] Daniel Hug, Günter Last, Steffen Winter. *Stochastic Geometry, Lecture Notes (summer term 2020).* Institute of Technology, Karlsruhe, 2020

[2] Franz Sackmann. *Zufällige Geraden, Staatsexamensarbeit.* University Karlsruhe (TH), Karlsruhe, 2007

[3] Gregory F. Lawler *Intersections of Random Walks.* University of Chicago, 1996

[4] Norbert Henze *Irrfahrten - Faszination der Random Walks* Institute of Technology, Karlsruhe, 2018

[5] Uta Freiberg, Ben Hambly, Michael Hinz and Steffen Winter *Fractal Geometry and Stochastics VI* 2020

[6] J. Gonzalez-Gutierrez, J. L. Carrillo-Estrada and J. C. Ruiz-Suarez *Aggregation and dendritic growth in a magnetic granular system* 2013

[7] Egbert Brieskorn *Felix Hausdorff zum Gedächtnis, Band I, vieweg Verlag* 1996 page 160

Erklärung

Hiermit versichere ich, dass ich diese Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt, die wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht und die Satzung des Karlsruher Instituts für Technologie zur Sicherung guter wissenschaftlicher Praxis in der jeweils gültigen Fassung beachtet habe.

Karlsruhe, den 10. März 2020