Version vom 5.2.
S. 10–19

rnal Diffusion Limited Aggregate

.1 Definition

External DLA is a model of an incremental aggregate as defined above using a very natural family of distributions, called the *harmonic measures.*

Definition 4.1.1. (*harmonic measure*) Let $A \subset \mathbb{Z}^d$. The hitting probability of A is the function

$$H_A : \mathbb{Z}^d \times A \to [0,1], \quad (x,y) \mapsto H_A(x,y) := \mathbb{P}_x(S_{T_A^+} = y).$$

In literature you can find the same definition where T_A is used instead of T_A^+. Since in the following for finite sets $A \in \mathcal{P}_f^d$ the limit $|x| \to \infty$ of $H_A(x,y)$ is of interest, T_A^+ is chosen for convenience. In fact, for a fixed element $x \in \mathbb{Z}^d$ the function $H_A(x,\cdot)$ defines a measure on A with total mass $\mathbb{P}_x(T_A^+ < \infty)$ and it can be adapted to a probability measure by conditioning the random walk to hit A in finite time. Define

$$\bar{H}_A : \mathbb{Z}^d \times A \to [0,1], \quad (x,y) \mapsto \bar{H}_A(x,y) := \mathbb{P}_x(S_{T_A^+} = y \mid T_A^+ < \infty),$$

so for fixed $x \in \mathbb{Z}^d$ the function $\bar{H}_A(x,\cdot)$ defines a probability measure on A. Indeed this definition is motivated by [3] (Chapter 2, Definition 2.1) and in the same chapter it is proved, that for finite sets $A \in \mathcal{P}_f^d$ the limit

$$\lim_{|x|\to\infty} \bar{H}_A(x,y) =: h_A(y)$$

exists for each $y \in A$. The function $h_A : A \to [0,1]$ is called the *harmonic measure of A*. For an element $y \in A$, h_A can be interpreted as the probability that a random walk starting at „infinity“ hits A the first time at y.

Definition 4.1.2. (*external diffusion limited aggregate*) *External diffusion limited aggregate* (on $\mathbb{Z}^d$), short *external DLA* or just *DLA*, is an incremental aggregate with the family of harmonic measures $(h_A)_{A \in \mathcal{P}_f^d}$ as distribution.

Remark 4.1.1. If we look at the 2-dimensional case, by Lemma 2.2.2, we have that $\mathbb{P}_x(T_A^+ < \infty) = 1$ for any $x \in \mathbb{Z}^2$, and therefore get $\bar{H}_A = H_A$. So in two dimensions the harmonic measure of $A \subset \mathbb{Z}^2$ can be written as

$$h_A(y) = \lim_{|x|\to\infty} \mathbb{P}_x(S_{T_A^+} = y), \quad y \in A.$$

4.2 Fractal Dimension and Growth Rate of External DLA in $\mathbb{Z}^2$

There is not yet a rigorous proof on the exact fractal dimension of external DLA. There are rather few rigorously proved results and it seems that it is very hard to prove such results on DLA. Looking at computer simulations of DLA in $\mathbb{Z}^2$ it seems that the clusters are relatively sparse and they appear to have a noninteger fractal dimension. In [6] DLA is observed in a magnetic aggregation context, and empirically they find a fractal dimension of around 1.8. Other simulations seem to suggest a value a little less than 1.7 for d_f in two dimensions ([3] page 83). There is also a theory that predicts

$$d_f = \frac{d^2+1}{d+1}$$

in $\mathbb{Z}^d$ which seems to agree fairly well with simulations ([3] page 83). There are only few rigorously proved results and we will see one of them in the following. First we will proof some lemmas. For that define two random functions

was bedeutet rad? und was war nochmal $\mathcal{E}_n$? (Leser erinnern!)

$$r : \mathbb{N} \to [0,\infty), \quad n \mapsto \mathrm{rad}(\mathcal{E}_n)$$

and

$$T : [0,\infty) \to \mathbb{N}, \quad s \mapsto \min\{j \in \mathbb{N} \mid r(j) \geq s\}.$$

Then it is easy to show that for all $n \in \mathbb{N}$, $s \in [0,\infty)$ and $\omega \in \Omega$

(i) $r(\omega)$ and $T(\omega)$ grow monotonously   in welcher Variable? präziser!

(ii) $T(\omega)(r(\omega)(n)) \leq n$

(iii) $r(\omega)(T(\omega)(s)) \geq s$

hold.

Lemma 4.2.1. For both random functions r and T we have that

$$r(\omega)(n) \to \infty \text{ for } n \to \infty$$

and

$$T(\omega)(s) \to \infty \text{ for } s \to \infty$$

for all $\omega \in \Omega$.

A *Proof.* Since we are moving on the grid $\mathbb{Z}^d$ we have that for a ball $B_d(0,n)$ with radius $n \geq 0$ ~~we have a finite~~ the number $N := |B_d(0,n) \cap \mathbb{Z}^d|$ is finite and for any $\omega \in \Omega$ we get

$$r(\omega)(2N) \geq n.$$

Therefore for all $\omega \in \Omega$ and any $n \in \mathbb{N}$ we can find a(n) $M \in \mathbb{N}$ such that $r(\omega)(M) \geq n$, and since $r(\omega)$ grows monotonously we get $r(\omega)(n) \to \infty$ for $n \to \infty$ for all $\omega \in \Omega$. Very similarly we can argument for T. □

Lemma 4.2.2. Let $d > 0$ and $h : [0,\infty) \to [0,\infty)$ be a bijective, multiplicative and monotonously growing function. Then the following are equivalent:

Kein Verb! (argue)

increasing

$0 : \mathbb{P}(r(n) \leq ch(n) \text{ for large } n) = 1$

$, \exists c > 0 : \mathbb{P}(T(as) \geq ch^{-1}(s) \text{ for large } s) = 1$ nicht wohldef.! wie ist das genau gemeint?

$\{\omega : \exists N = N(\omega) : r(\omega)(n) \leq c\, h(n) \; \forall n \geq N\}$??

Proof. $\Rightarrow$: For $c > 0$ define

$$A_c := \{r(n) \leq ch(n) \text{ for large } n\}$$

and

$$B_c := \{T(as) \geq ch^{-1}(s) \text{ for large } s\}.$$

Choose $c > 0$ such that $\mathbb{P}(A_c) = 1$. Take $\omega \in A_c$ and choose $N \in \mathbb{N}$ such that $r(\omega)(n) \leq ch(n)$ for all $n > N$. Hence there exists $\tilde{c} > 0$ such that $\tilde{c}h^{-1}(r(\omega)(n)) \leq n$ for all $n > N$. By Lemma 4.2.1 we can choose $M \in \mathbb{N}$ big enough such that $T(\omega)(aM) > N$, hence $T(\omega)(as) \geq T(\omega)(aM) > N$ for all $s \geq M$ since $T(\omega)$ grows monotonously. Hence we can write $\tilde{c}h^{-1}(r(\omega)(T(\omega)(as))) \leq T(\omega)(as)$ for all $s > M$ and since $r(T(\omega)(as)) \leq as$ we finally get $\tilde{c}h^{-1}(a)h^{-1}(s) = \tilde{c}h^{-1}(as) \leq T(\omega)(as)$ for all $s > M$, hence $\omega \in B_{\tilde{c}h^{-1}(a)}$, where we used the multiplicativity of h and that h^{-1} falls monotonously. We therefore get $A_c \subset B_{\tilde{c}h^{-1}(a)}$, hence $\mathbb{P}(B_{\tilde{c}h^{-1}(a)}) = 1$.
decreases

$\Leftarrow$: The proof for this direction works analogously and needs Lemma 4.2.1 aswell. $\square$

Lemma 4.2.3. Let $n \in \mathbb{N}$ and $T_1, \ldots, T_n$ be independent geometrically distributed random variables with parameter $p < \frac{1}{2}$. Let $Y := T_1 + \cdots + T_n$, then for every $a \in [2p, 1)$ we have

$$\mathbb{P}(Y \leq \frac{an}{p}) \leq (ae^2)^n.$$

Proof. The moment generating function of Y is

$$\mathbb{E}[e^{tY}] = (pe^t)^n(1 - e^t(1-p))^{-n} = p^n(e^{-t} - (1-p))^{-n}.$$

By Chebyshev for any random variable X we know the inequality

$$\mathbb{P}(X \geq x) \leq \inf_{t>0} \frac{\mathbb{E}[e^{tX}]}{e^{tx}}$$

and can therefore follow that for any $t > 0$

$$\mathbb{P}(Y \leq \frac{an}{p}) = \mathbb{P}(-Y \geq -\frac{an}{p}) \leq \exp(\frac{ant}{p})\mathbb{E}[e^{-tY}] = \exp(\frac{ant}{p})p^n(e^t - (1-p))^{-n}.$$

Choose $t = \ln(\frac{a(1-p)}{a-p})$ (note that $t > 0$), then

$$\begin{aligned}\mathbb{P}(Y \leq \frac{an}{p}) &\leq (\frac{a(1-p)}{a-p})^{\frac{an}{p}} p^n (1-p)^{-n} (\frac{p}{a-p})^{-n} \\ &= (\frac{a}{a-p})^{\frac{an}{p}} (1-p)^{\frac{an}{p}} (1-p)^{-n} (\frac{1}{a-p})^{-n}\end{aligned}$$

$$
\begin{aligned}
&\le (1+\frac{p}{a-p})^{\frac{an}{p}}(1-p)^n(1-p)^{-n}(a-p)^n \\
&\le (1+\frac{p}{a-p})^{\frac{2(a-p)n}{p}}a^n \\
&\le (ae^2)^n.
\end{aligned}
$$

□

Definition 4.2.1. For $x \in \mathbb{Z}^2$ and $r \in [1,\infty)$ define

$$\mathcal{P}_r^x := \{A \in \mathcal{P}_f \mid x \in A, r = \max_{y\in A}|x-y| \text{ and } A \text{ is connected}\}.$$

needed/required/used

The following theorem gives a bound on the harmonic measure which will be necessary in the proof we present here for the growth rate of DLA. The theorem is proved by [3] in Theorem 2.5.2 and we state it slightly different here to generalize it easier afterwards.

-ly in

Theorem 4.2.1. There exists a constant $c > 0$ such that for all $r \in [1,\infty)$

$$h_A(0) \le cr^{-\frac{1}{2}} \quad \text{for all } A \in \mathcal{P}_r^0.$$

Proof. If $A \in \mathcal{P}_r^0$ then $0 \in A$ and $r = \mathrm{rad}(A)$ and for that case the proof is presented for all dimensions $d \in \mathbb{N}$ in [3] Theorem 2.5.2. The proof is very technical and requires various results of other theorems and lemmas which are also to find in [3]. □

vor die Prop.!

as follows constant

Proposition 4.2.1. Theorem 4.2.1 can be generalized to the following. There exists a $c > 0$ such that for all $y \in \mathbb{Z}^2$ and $s, r \in [1,\infty)$ with $s \ge r$ we have

$$h_A(y) \le cr^{-\frac{1}{2}} \quad \text{for all } A \in \mathcal{P}_s^y.$$

Proof. For $y \in \mathbb{Z}^2$ define the translation function

$$\Phi_y : \mathbb{Z}^2 \to \mathbb{Z}^2, x \mapsto x + y.$$

Recall that in $\mathbb{Z}^2$ the harmonic measure of $A \subset \mathbb{Z}^2$ can be written as

$$h_A(z) = \lim_{|x|\to\infty} \mathbb{P}_x(S_{T_A^+} = z), \quad z \in A,$$

(see Remark 4.1.1). Since the distribution of random walks in $\mathbb{Z}^2$ is invariant under translation we can conlude that

$$
\begin{aligned}
h_A(z) &= \lim_{|x|\to\infty} \mathbb{P}_x(S_{T_A^+} = z) \\
&= \lim_{|x|\to\infty} \mathbb{P}_{\Phi_y(x)}(S_{T^+_{\Phi_y(A)}} = \Phi_y(z)) \\
&= h_{\Phi_y(A)}(\Phi_y(z))
\end{aligned}
$$

for all $z \in A$. By Theorem 4.2.1 we know that there exists a $c > 0$ such that for all $s \in [1,\infty)$

$$h_A(0) \le cs^{-\frac{1}{2}} \quad \text{for all } A \in \mathcal{P}_s^0.$$

or $s, r \in \mathbb{R}$ with $s \geq r$ we get

$$h_A(0) = h_{\Phi_y(A)}(\Phi_y(0)) = h_{\Phi_y(A)}(y) \leq cs^{-\frac{1}{2}} \leq cr^{-\frac{1}{2}} \quad \text{for all } A \in \mathcal{P}_s^0,$$

and since $A \in \mathcal{P}_s^0 \Leftrightarrow \Phi_y(A) \in \mathcal{P}_s^{\Phi_y(0)}$ we get

$$h_A(y) \leq cr^{-\frac{1}{2}} \quad \text{for all } A \in \mathcal{P}_s^y.$$

Since c was chosen independently of y, s and r, this completes the proof. Section? □ ok

Theorem 4.2.2. For the growth rate of external DLA in $\mathbb{Z}^2$ as defined in 3.2 (3.3) we have
in (equation)

$$\alpha_f \leq \frac{2}{3}. \tag{4.1}$$

Proof. For $c > 0$ define

$$A_c := \{\omega \in \Omega \mid \mathrm{rad}(\mathcal{E}_n(\omega)) \leq cn^{\frac{2}{3}} \text{ for large } n\}$$

and

$$D_c := \{\omega \in \Omega \mid T(\omega)(2n) \geq cn^{\frac{3}{2}} \text{ for large } n\}.$$

there is a constant
If we have a $c > 0$ such that

$$\mathbb{P}(A_c) = 1, \tag{4.2}$$

then we have

$$\begin{aligned}
\alpha_f &= \limsup_{n\to\infty} \frac{\ln(\mathbb{E}[\mathrm{rad}(\mathcal{E}_n)])}{\ln(n)} \\
&= \limsup_{n\to\infty} \frac{\ln(\int_\Omega \mathrm{rad}(\mathcal{E}_n) d\mathbb{P})}{\ln(n)} \\
&= \limsup_{n\to\infty} \frac{\ln(\int_{A_c} \mathrm{rad}(\mathcal{E}_n) d\mathbb{P})}{\ln(n)} \\
&\overset{(+)}{\leq} \limsup_{n\to\infty} \frac{\ln(\int_{A_c} cn^{\frac{2}{3}} d\mathbb{P})}{\ln(n)} \\
&= \limsup_{n\to\infty} \frac{\ln(cn^{\frac{2}{3}})}{\ln(n)} \\
&= \frac{2}{3} \limsup_{n\to\infty} \frac{\ln(c^{\frac{3}{2}}) + \ln(n)}{\ln(n)} = \frac{2}{3} \cdot 1 = \frac{2}{3}
\end{aligned}$$

? (✓)

If we choose

$$h : [0, \infty) \to [0, \infty), x \mapsto x^{\frac{2}{3}},$$

and $a = 2$, then by Lemma 4.2.2 we can show (4.2) if we find a constant $c > 0$ such that

$$\mathbb{P}(D_c) = 1. \tag{4.3}$$

Note that h is bijective, multiplicative and increasing ~~monotonously growing~~. Lets first argue ~~argument~~ why the inequality at (+) indeed holds. We define CONTINUE

So now we will try to prove (4.3). For $n \in \mathbb{N}$ write $\mathcal{E}_n = \{y_1, \dots, y_n\}$ according to ~~the~~ definition ~~in~~ 3.1.1, where y_j is the j-th point added to the cluster. Let $\beta > 0$ which will be determined later on. For $n \in \mathbb{N}$ let $\tilde{m}_n := \beta n^{\frac{3}{2}}$ and define

$$V_n := \{\omega \in \Omega \mid T(\omega)(2n) < \tilde{m}_n\}.$$

Further define the set of realised random walk paths of length n with starting point in $\tilde{B}_n := \{x \in \mathbb{Z}^2 \mid d(x, \partial B_n) \le 1 \text{ and } |x| \ge n\}$ ← warum nicht $\tilde{B}_n = \partial B_n$?

$$Z_n := \{[z] := \{z_1, \dots, z_n\} \subset \mathbb{Z}^2 \mid z_1 \in \tilde{B}_n \text{ and } z_i \in N(z_{i-1}) \text{ for all } i \in \{2, \dots, n\}\},$$

↳ besser als Folge schreiben: $(z_1, \dots, z_n) \in (\mathbb{Z}^2)^n$

for $[z] \in Z_n$ define events

$$W_n([z]) := \{\omega \in \Omega \mid \exists j_1 < \dots < j_n \le \tilde{m}_n \text{ such that } y_{j_i}(\omega) = z_i \text{ for all } i \in \{1, \dots, n\}\}$$

and the union of these events

$$W_n := \bigcup_{[z] \in Z_n} W_n([z]).$$

With $\tilde{B}_n$ we mean that the ~~(random)~~ walks (or paths) in Z_n start „on the boundary" of B_n. We will quickly prove that

$$V_n \subset W_n \text{ for all } n \in \mathbb{N}. \qquad (W)$$

Let $n \in \mathbb{N}$ and $\omega \in V_n$, then $T(\omega)(2n) < \tilde{m}_n$. For $m_n := \max\{j \in \mathbb{N} \mid j \le \tilde{m}_n\}$?= $\tilde{m}_n$!? we therefore have $\mathrm{rad}(\mathcal{E}_{m_n}(\omega)) \ge 2n$ (Note that $T(\omega)(2n) \in \mathbb{N}$). Therefore, since $\mathcal{E}_{m_n}(\omega)$ is connected, there must exist indices $j_1 < \dots < j_n$ s.t. $y_{j_1}(\omega) \in \tilde{B}_n$ and $[z_0] := [y_{j_1}(\omega), \dots, y_{j_n}(\omega)] \subset \mathcal{E}_{m_n}(\omega)$, since $\max_{x \in [z_0]} |x| \le 2n$. Therefore $j_n \le m_n \le \tilde{m}_n$ and therefore $\omega \in W_n([z_0]) \subset W_n$ which proofs the inclusion. (T(2n) statt $\tilde{m}_n$ hier?!)

For a sequence of events $(A_n)_{n \in \mathbb{N}}$ recall that

$$\limsup_{n \to \infty} A_n := \bigcap_{n \in \mathbb{N}} \bigcup_{i \ge n} A_i = \{\omega \in \Omega \mid \omega \in A_n \text{ for infinitely many } n \in \mathbb{N}\}.$$

We will use the Lemma of Borel-Cantelli on the sequence of events $(W_n)_{n \in \mathbb{N}}$. If we can show that

$$\sum_{n \in \mathbb{N}} \mathbb{P}(W_n) < \infty, \tag{4.4}$$

then with $V_n \subset W_n$ for all $n \in \mathbb{N}$ and Borel-Cantelli we get

$$\mathbb{P}(\limsup_{n\to\infty} V_n) \leq \mathbb{P}(\limsup_{n\to\infty} W_n) = 0.$$

Since

$$(\limsup_{n\to\infty} V_n)^C = \{\omega \in \Omega \mid \exists N \in \mathbb{N} \text{ s.t. } \omega \in V_n^C \text{ for all } n > N\} = D_\beta$$

we can then conclude that $\mathbb{P}(D_\beta) = 1$ and have finished the proof. So we want to show (4.4).

For $n \in \mathbb{N}$, $[z] \in Z_n$ and $i \in \{1, \dots, n\}$ we define random variables

$$\tau_i : \Omega \to \mathbb{N}^\infty, \tau_i(\omega) = j :\Leftrightarrow \begin{cases} y_j(\omega) = z_i, & j < \infty, \\ z_i \notin \mathcal{E}_\infty(\omega), & j = \infty, \end{cases}$$

kann man missverstehen!
$\mathbb{N}\cup\{\infty\}$?!

z_i

so τ_i is either the index j such that the j-th added point to the cluster is equal to z_i, or infinity if the final cluster $\mathcal{E}_\infty$ doesn't contain z_i. τ_i is measurable because for $j \in \mathbb{N}$ we have

$$\tau_i^{-1}(j) = \{y_j = z_i\} = y_j^{-1}(z_i) \in \mathcal{F}$$

and for $j = \infty$ we have

$$\begin{aligned}
\tau_i^{-1}(\infty) &= \{z_i \notin \mathcal{E}_\infty = \bigcup_{k\in\mathbb{N}} \mathcal{E}_k\} \\
&= \{z_i \notin \mathcal{E}_k \text{ for all } k \in \mathbb{N}\} \\
&= \bigcap_{k\in\mathbb{N}} \{z_i \notin \mathcal{E}_k\} \\
&= \bigcap_{k\in\mathbb{N}} \bigcap_{l=1}^{k} \{y_l \neq z_i\} \\
&= \bigcap_{k\in\mathbb{N}} \bigcap_{l=1}^{k} y_l^{-1}(\mathbb{Z}^2 \setminus \{z_i\}) \in \mathcal{F}.
\end{aligned}$$

Further for $i \in \{1, \dots, n-1\}$ we define waiting times

$$\sigma_i : \Omega \to \mathbb{N}^\infty, \omega \to \begin{cases} \tau_{i+1}(\omega) - \tau_i(\omega), & \text{if } \tau_{i+1}(\omega) < \infty \text{ and } \tau_i(\omega) < \infty, \\ \infty, & \text{else,} \end{cases}$$

-e

so σ_i is the waiting time between adding z_i and z_{i+1} to the cluster. We quickly argu~~ment~~ that σ_i is measurable aswell. For $j \in \mathbb{N}$ we have

$$\sigma_i^{-1}(j) = \{\tau_{i+1} - \tau_i = j\} = \bigcup_{k\in\mathbb{N}} \{\tau_{i+1} = k\} \cap \{\tau_i = k - j\} \in \mathcal{F}$$

and

$$\sigma_i^{-1}(\infty) = \{\tau_{i+1} = \infty\} \cup \{\tau_i = \infty\} \in \mathcal{F},$$

since τ_{i+1} and τ_i are measurable.

Let $i \in \{1, \ldots, n-1\}$ and define the event

$< \infty$?

$$U^i_{[z]} := \{\tau_1 \leq \cdots \leq \tau_i\}.$$

Note that $\mathbb{P}(U^i_{[z]}) > 0$ and also note again, that all τ_i are defined based on one $[z]$. We will now prove that the distribution of σ_i conditioned on $U^i_{[z]}$ is bounded by that of a geometrically distributed random variable with parameter

$$p_n := c_1 n^{-\frac{1}{2}} \tag{4.5}$$

for some constant $c_1 > 0$, which is the same constant for all $i \in \{1, \ldots, n-1\}$ and for all $n \in \mathbb{N}$. We will want to use Proposition 4.2.1 here, for which we need to create terms where the harmonic measure appears. For that we need probabilities which condition on a given cluster which we will develop now. We first define some helpful sets. For $m \in \mathbb{N}$ define ?

$$C_m := \{\mathcal{E}_m(\omega) \mid \omega \in \Omega\}$$

the set of realized clusters of size m and the disjoint union of them as $C := \bigcup_{m \in \mathbb{N}} C_m$. Further define another disjoint partition of C with sets einmal sollte "<" stehen!

$$E_m := \{\mathcal{E} \in C \mid m - 1 \leq \mathrm{rad}(\mathcal{E}) \leq m\}\text{, } m \in \mathbb{N}$$

such that $C = \bigcup_{m \in \mathbb{N}} E_m$ aswell. Since $|z_1| \geq n$ and considering $U^i_{[z]}$, we need a minimal amount of time steps such that it is possible for the cluster to contain z_i. So there exists a $n_0 \in \mathbb{N}$ such that

$$\mathbb{P}(\tau_i = j, U^i_{[z]}) \begin{cases} > 0, & \text{for } j \geq n_0, \\ = 0, & \text{for } j < n_0. \end{cases}$$

Further if we choose $n_0 \leq j < \infty$ and considering $U^i_{[z]}$ again, then $\mathcal{E}_{j-1}$ must have some minimal radius if we want it to be possible that the next added point is z_i, similarly argumented as above. So there exists a $m_0 \in \mathbb{N}$ such that

$$\mathbb{P}(\mathcal{E}_{j-1} = \mathcal{E}, \tau_i = j, U^i_{[z]}) \begin{cases} > 0, & \text{for } \mathcal{E} \in E_m \text{ and } m \geq m_0, \\ = 0, & \text{for } \mathcal{E} \in E_m \text{ and } m < m_0. \end{cases}$$

Having made that clear we can use the law of total probability twice in the following. For $k \in \mathbb{N}_0$ we get

$$\mathbb{P}(\sigma_i > k \mid U^i_{[z]}) = \sum_{n_0 \leq j \leq \infty} \mathbb{P}(\tau_i = j \mid U^i_{[z]}) \mathbb{P}(\sigma_i > k \mid \tau_i = j, U^i_{[z]})$$

$$\geq \sum_{n_0 \leq j < \infty} \mathbb{P}(\tau_i = j \mid U^i_{[z]}) \mathbb{P}(\sigma_i > k \mid \tau_i = j, U^i_{[z]}).$$

For shorter expressions write $\Gamma_{ij} := \{\mathcal{E}_{j-1} = \mathcal{E}\} \cap \{\tau_i = j\} \cap U^i_{[z]}$. Further for $n_0 \leq j < \infty$ we get

$$\mathbb{P}(\sigma_i > k \mid \tau_i = j, U^i_{[z]}) = \sum_{m_0 \leq m} \sum_{\mathcal{E} \in E_m} \mathbb{P}(\mathcal{E}_{j-1} = \mathcal{E} \mid \tau_i = j, U^i_{[z]}) \mathbb{P}(\sigma_i > k \mid \Gamma_{ij}).$$

We now state that there exists a constant $c_1 > 0$ such that for all $k \in \mathbb{N}$, $n_0 \leq j < \infty$, $m_0 \leq m$ and $\mathcal{E} \in E_m$ we have

$$\mathbb{P}(\sigma_i > k \mid \Gamma_{ij}) \geq (1 - c_1 n^{-\frac{1}{2}})^k \tag{4.6}$$

which we prove by induction. For $k = 1$ by Proposition 4.2.1 there exists a $c > 0$ such that

$$\begin{aligned}
\mathbb{P}(\sigma_i > 1 \mid \Gamma_{ij}) &\geq \mathbb{P}(\tau_{j+1} \neq z_{i+1} \mid \Gamma_{ij}) \\
&= 1 - h_{\mathcal{E} \cup \{z_i\}}(z_{i+1}) \\
&\geq 1 - cn^{-\frac{1}{2}} \\
&= (1 - cn^{-\frac{1}{2}})^1.
\end{aligned}$$

Note that we can use Proposition 4.2.1 because $\mathrm{rad}(\mathcal{E} \cup \{z_i\}) \geq n$ since $z_1 \in \mathcal{E} \cup \{z_i\}$ as we conditioned on $U^i_{[z]}$. Note that the harmonic measure doesn't care about the order of how points where added to form the current cluster, as mentioned in Remark 3.1.1. Note that by Proposition 4.2.1 the chosen c has a strong universal property such that it doesn't depend on $n_0 \leq j < \infty$, $m_0 \leq m$ or $\mathcal{E} \in E_m$. Now let the statement be true for some $k = l$. Then again by Proposition 4.2.1 there exists a $c > 0$ such that

$$\begin{aligned}
\mathbb{P}(\sigma_i > l + 1 \mid \Gamma_{ij}) &= \mathbb{P}(\sigma_i > l, y_{l+1} \neq z_{i+1} \mid \Gamma_{ij}) \\
&= \mathbb{P}(\sigma_i > l \mid \Gamma_{ij}) \mathbb{P}(y_{l+1} \neq z_{i+1} \mid \Gamma_{ij}, \sigma_i > l) \\
&\overset{(+)}{\geq} (1 - cn^{-\frac{1}{2}})^l \mathbb{P}(y_{l+1} \neq z_{i+1} \mid \Gamma_{ij}, \sigma_i > l) \\
&\overset{(++)}{\geq} (1 - cn^{-\frac{1}{2}})^l (1 - cn^{-\frac{1}{2}}) \\
&= (1 - cn^{-\frac{1}{2}})^{l+1},
\end{aligned}$$

where in (+) we used the induction assumption. In order to show (++) we would need to split the probability with the law of total probability by conditioning on all clusters such that Γ_{ij} is fulfilled and the next l added points are not equal to z_{i+1}. We would then have the same inequality as in the induction beginning using Proposition 4.2.1 and reput together the contioned probabilites to get this inequality here. Same as in the induction beginning we have now found a constant $c_1 > 0$ such that (4.6) holds for all $k \in \mathbb{N}$ and this c_1 is independent of $n_0 \leq j < \infty$, $m_0 \leq m$ or $\mathcal{E} \in E_m$. Therefore we can go backwards the two splittings where we used the law of total probability and finally get

$$\mathbb{P}(\sigma_i > k \mid U^i_{[z]}) \geq (1 - c_1 n^{-\frac{1}{2}})^k$$

with a constant $c_1 > 0$ which doesn't depend neither on $i \in \{1, \dots, n-1\}$ or $n \in \mathbb{N}$ by the strong universal property given by Proposition 4.2.1. We therefore have shown what we stated at (4.5).

We now continue to show (4.4). Choose β such that $4e^2\beta c_1 < 1$ und choose $N \in \mathbb{N}$ such that $\beta c_1 \geq 2p_n$ for all $n > N$. If we then define $a := \beta c_1$ and $Y := \sum_{i=1}^{n-1} \sigma_i = \tau_n - \tau_1$ we can use Lemma 4.2.3 and get PROBLEM

$$\mathbb{P}(\tau_n \leq \tilde{m}_n \mid U^n_{[z]}) \leq \mathbb{P}(\tau_n - \tau_1 \leq \tilde{m}_n \mid U^n_{[z]}) = \mathbb{P}(Y \leq \frac{an}{p_n} \mid U^n_{[z]}) \leq (e^2 a)^{n-1}$$

for all $n > N$. Since $W_n([z]) \subset \{\tau_n \leq \tilde{m}_n\} \cap U^n_{[z]}$ we get

$$\mathbb{P}(W_n([z])) \leq \mathbb{P}(W_n([z]) \mid U^n_{[z]}) \leq \mathbb{P}(\tau_n \leq \tilde{m}_n \mid U^n_{[z]}) \leq (e^2 a)^n$$

for all $n > N$. Counting the elements in Z_n we have less or equal $c_2 n$ points in $\tilde{B}_n$ for some constant $c_2 > 0$ (which doesn't depend on n) as starting points and 4^{n-1} possibilities for the next $n-1$ steps of a random walk of length n. So $|Z_n| \leq c_2 n 4^{n-1}$ and therefore

$$\sum_{[z] \in Z_n} \mathbb{P}(W_n([z])) \leq c_2 n 4^{n-1} (e^2 a)^{n-1} = c_2 n (4e^2 a)^{n-1} \quad \text{for all } n > N$$

and finally

$$\sum_{n > N} \mathbb{P}(W_n) \leq \sum_{n > N} c_2 n (4e^2 \beta c_1)^{n-1} =: r_\beta.$$

Since β was chosen such that $4e^2\beta c_1 < 1$, r_β is finite and therefore in total we get

$$\sum_{n \in \mathbb{N}} \mathbb{P}(W_n) < \infty,$$

which completes the proof. □

Remark 4.2.1. With this theorem and 3.4 we conclude that the fractal dimension of external DLA is at least $\frac{3}{2}$. In the next chapter we will look at another incremental aggregate which tries to approximate DLA and compare simulations of both to get an empirical comparision of their growth rates.