Karlsruhe Institute of Technology

Institute of Stochastics

Stochastic Geometry | Summer term 2020
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Solutions for Work Sheet 5

Problem 1 (Simulating point processes)

Use your favorite programming language to do the following simulations (we suggest the use of R or Python).

a) Simulate a binomial process with parameters $m = 50$ and $\mathbb{V} = \mathcal{U}\big([0,1]^2\big)$.

b) Simulate a homogeneous Poisson process in $[0,1]^2$ with intensity $\gamma = 50$, or equivalently, simulate a mixed binomial process with parameters $\tau \sim Po(50)$ and $\mathbb{V} = \mathcal{U}\big([0,1]^2\big)$.

c) Simulate a Poisson process in $[0,1]^3$ with intensity measure $\Theta = 100 \cdot \big(B(0.5, 0.5) \otimes B(1, 1.5) \otimes B(2, 1)\big)$, where $B(\alpha, \beta)$ denotes the beta distribution with parameters $\alpha, \beta > 0$.

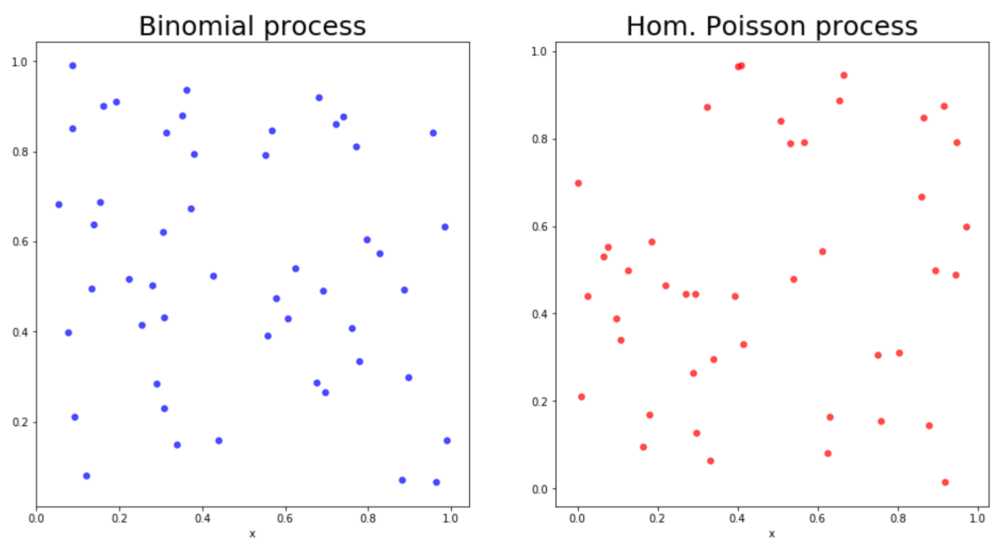
d) Visit the website Morphometry.org and try to replicate the results in parts a) and b) using the 'morphometer'.

Note: Python-code for the simulations will be made available in Ilias together with the solutions to this problem sheet. The website morphometry.org can be accessed at any time. It also includes more complicated processes and visualizations of other objects from stochastic geometry. Those who are interested in the simulation of point patterns or in the analysis of spatial data should take a look at the R-package `spatstat` in connection with the book

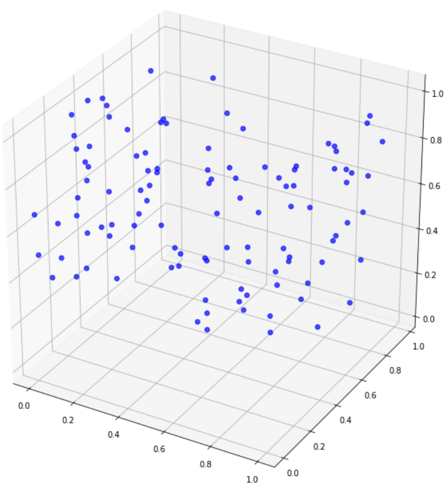
A. Baddeley, E. Rubak, and R. Turner (2016); *Spatial Point Patterns: Methodology and Applications with R;* CRC Press, Taylor & Francis Group; Boca Raton.

Those who use Python might want to have a look at the `PySAL`-package and its components.

Possible realizations:
One realization of the processes in a) and b) looks like this:



Notice that the number of points in the binomial process is fixed to 50. While the expected number of points in the Poisson process is also 50, the given realization only has 44 points. A realization of the Poisson process in c) might look like this:



Problem 2 (Some properties of the Poisson process)

Let $(\mathbb{X}, \rho)$ be a separable metric space, and let Φ be a Poisson process in $\mathbb{X}$, defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, with intensity measure $\Theta \in M(\mathbb{X})$.

a) Prove that, for $A, B \in \mathcal{X}$,

$$\operatorname{Cov}\Big(\Phi(A), \Phi(B)\Big) = \Theta(A \cap B).$$

b) Let $B \in \mathcal{X}$. Show that the map $M(\mathbb{X}) \to M(\mathbb{X})$, $\mu \mapsto \mu_B := \mu \llcorner B := \mu_{|_B}$ is measurable. Further, prove that the restriction Φ_B of Φ onto B is a Poisson process in $\mathbb{X}$ with intensity measure Θ_B.

c) Let $B_1, B_2, \ldots \in \mathcal{X}$ be pairwise disjoint. Prove that $\Phi_{B_1}, \Phi_{B_2}, \ldots$ are independent Poisson processes with intensity measures $\Theta_{B_1}, \Theta_{B_2}, \ldots$.

Proposed solution:

a) For $A, B \in \mathcal{X}$, we get

$$\begin{aligned}
\operatorname{Cov}\Big(\Phi(A), \Phi(B)\Big) &= \operatorname{Cov}\Big(\Phi(A \backslash B) + \Phi(A \cap B), \Phi(B \backslash A) + \Phi(A \cap B)\Big) \\
&= \operatorname{Cov}\Big(\Phi(A \backslash B), \Phi(B \backslash A)\Big) + \operatorname{Cov}\Big(\Phi(A \backslash B), \Phi(A \cap B)\Big) \\
&\quad + \operatorname{Cov}\Big(\Phi(A \cap B), \Phi(B \backslash A)\Big) + \operatorname{Cov}\Big(\Phi(A \cap B), \Phi(A \cap B)\Big) \\
&= 0 + 0 + 0 + \mathbb{V}\mathrm{ar}\big(\Phi(A \cap B)\big)
\end{aligned}$$

Next, notice that, for $Y \sim Po(\gamma)$, $\gamma \geqslant 0$, we have $\mathbb{E}Y = \mathbb{V}\mathrm{ar}(Y) = \gamma$. From $\Phi(A \cap B) \sim Po\big(\Theta(A \cap B)\big)$, conclude that

$$\operatorname{Cov}\Big(\Phi(A), \Phi(B)\Big) = \mathbb{V}\mathrm{ar}\big(\Phi(A \cap B)\big) = \Theta(A \cap B).$$

b) For $A \in \mathcal{X}$ and $r \geqslant 0$, we have

$$\begin{aligned}
\big\{\mu \in M(\mathbb{X}) : \mu_B \in M_{A,r}\big\} &= \big\{\mu \in M(\mathbb{X}) : \mu_B(A) \leqslant r\big\} \\
&= \big\{\mu \in M(\mathbb{X}) : \mu(A \cap B) \leqslant r\big\} \\
&\in \mathcal{M}(\mathbb{X}),
\end{aligned}$$

as $\mu \mapsto \mu(A \cap B)$ is measurable. From this measurability property and the fact that $\Phi_B \leqslant \Phi$, it follows immediately that Φ_B is a point process. Now, take $B_1, \ldots, B_n \in \mathcal{X}$ pairwise disjoint. Then the sets

$B \cap B_1, \ldots, B \cap B_n \in \mathcal{X}$ are pairwise disjoint and, since Φ is Poisson process, the random variables $\Phi_B(B_1) = \Phi(B \cap B_1), \ldots, \Phi_B(B_n) = \Phi(B \cap B_n)$ are independent. Furthermore,

$$\Phi_B(A) = \Phi(B \cap A) \sim Po\Big(\Theta(B \cap A)\Big),$$

so Φ_B is a Poisson process with intensity measure Θ_B.

c) By part b) we only have to show that $\Phi_{B_1}, \Phi_{B_2}, \ldots$ are independent [as random elements of $N(\mathbb{X})$]. It is no loss of generality to assume that $\bigcup_{j=1}^{\infty} B_j = \mathbb{X}$ and that $\Theta(B_j) < \infty$, $j \in \mathbb{N}$ (if not, just add the complement of the union of the B_js to the sequence and cut each set into bounded parts). As in the proof of Theorem 2.31, we find independent Poisson processes $\widetilde{\Phi}_1, \widetilde{\Phi}_2, \ldots$ restricted to the sets $B_1, B_2, \ldots$ (respectively) such that $\widetilde{\Phi}_j \overset{d}{=} \Phi_{B_j}$ for each $j \in \mathbb{N}$ as well as $\widetilde{\Phi} := \sum_{j=1}^{\infty} \widetilde{\Phi}_j \overset{d}{=} \Phi$. For any $k \in \mathbb{N}$ and measurable maps $f_1, \ldots, f_k : N(\mathbb{X}) \to [0, \infty)$, we have

$$\mathbb{E}\Big[\prod_{j=1}^{k} f_j(\Phi_{B_j})\Big] = \mathbb{E}\Big[\prod_{j=1}^{k} f_j(\widetilde{\Phi}_{B_j})\Big] = \mathbb{E}\Big[\prod_{j=1}^{k} f_j(\widetilde{\Phi}_j)\Big] = \prod_{j=1}^{k} \mathbb{E}\Big[f_j(\widetilde{\Phi}_j)\Big] = \prod_{j=1}^{k} \mathbb{E}\Big[f_j(\Phi_{B_j})\Big].$$

Choosing $f_j(\mu) = \mathbb{1}\{\mu \in C_j\}$, $C_j \in \mathcal{N}(\mathbb{X})$, for each $j = 1, \ldots, k$, yields

$$\mathbb{P}\Big(\Phi_{B_1} \in C_1, \ldots, \Phi_{B_k} \in C_k\Big) = \mathbb{P}(\Phi_{B_1} \in C_1) \cdot \ldots \cdot \mathbb{P}(\Phi_{B_k} \in C_k)$$

and hence the claim.

Problem 3 (A mapping theorem for point processes)

Let $(\mathbb{X}, \mathcal{X})$ and $(\mathbb{Y}, \mathcal{Y})$ be separable metric spaces with their corresponding Borel σ-fields. Let Φ be a point process in $\mathbb{X}$, defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and consider a measurable map $g : \mathbb{X} \to \mathbb{Y}$ for which the preimage of any bounded subset of $\mathbb{Y}$ is bounded in $\mathbb{X}$.

a) Prove that $\widetilde{\Phi} := \Phi \circ g^{-1}$ is a point process in $\mathbb{Y}$.

b) Now, assume that Ψ is a Poisson process in $\mathbb{X}$ with intensity measure $\Theta \in M(\mathbb{X})$. Show that $\widetilde{\Psi} = \Psi \circ g^{-1}$ is a Poisson process in $\mathbb{Y}$ with intensity measure $\widetilde{\Theta} := \Theta \circ g^{-1} \in M(\mathbb{Y})$.

Proposed solution:

a) First, we have to show that $\widetilde{\Phi}$ is a measurable map $\Omega \to N(\mathbb{Y})$. For fixed $\omega \in \Omega$, $\widetilde{\Phi}(\omega)$ is a measure on $\mathbb{Y}$, namely the push-forward measure of $\Phi(\omega)$ with respect to g. To see that $\widetilde{\Phi}(\omega)$ is locally finite, let $B \in \mathcal{Y}_b$. By assumption, $g^{-1}(B) \in \mathcal{X}_b$, so the local finiteness of $\Phi(\omega)$ (as it is a point process in $\mathbb{X}$) implies

$$\widetilde{\Phi}(\omega, B) = \Phi(\omega, g^{-1}(B)) < \infty,$$

so $\widetilde{\Phi}$ maps into $N(\mathbb{Y})$. Moreover, for any $B \in \mathcal{Y}$, we have $g^{-1}(B) \in \mathcal{X}$ and Problem 1 a) from Sheet 4 implies that

$$\widetilde{\Phi}(B) = \Phi(g^{-1}(B))$$

is a random variable. Hence $\widetilde{\Phi}$ is a point process, again by Problem 1 a) from Sheet 4.

b) By part a), $\widetilde{\Psi}$ is a point process in $\mathbb{Y}$, and a similar argument as above (concerning local finiteness) confirms that $\widetilde{\Theta} \in M(\mathbb{Y})$. It remains to prove that $\widetilde{\Psi}$ satisfies the defining properties of a Poisson process. First, let $A \in \mathcal{Y}$. Then $g^{-1}(A) \in \mathcal{X}$, and we have, for $k \in \mathbb{N}_0$,

$$\mathbb{P}\Big(\widetilde{\Psi}(A) = k\Big) = \mathbb{P}\Big(\Psi(g^{-1}(A)) = k\Big) = e^{-\Theta\left(g^{-1}(A)\right)} \frac{\Theta(g^{-1}(A))^k}{k!} = e^{-\widetilde{\Theta}(A)} \frac{\widetilde{\Theta}(A)^k}{k!}.$$

Now, let $B_1, \ldots, B_n \in \mathcal{Y}$ be pairwise disjoint. Then $g^{-1}(B_1), \ldots, g^{-1}(B_n) \in \mathcal{X}$ are also pairwise disjoint, and the random variables

$$\widetilde{\Psi}(B_1) = \Psi(g^{-1}(B_1)), \ldots, \widetilde{\Psi}(B_n) = \Psi(g^{-1}(B_n))$$

are independent. We conclude that $\widetilde{\Psi}$ is a Poisson process in $\mathbb{Y}$.

Problem 4 (Superposition)

Let $\Phi_1, \ldots, \Phi_n$ be independent point processes in a separable metric space $(\mathbb{X}, \rho)$ with intensity measures $\Theta_1, \ldots, \Theta_n$ and Laplace functionals $L_1, \ldots, L_n$.

a) Denote by Θ and L the intensity measure and Laplace functional (respectively) of the superposition $\Phi := \Phi_1 + \ldots + \Phi_n$. Express Θ and L in terms of $\Theta_1, \ldots, \Theta_n$ and $L_1, \ldots, L_n$, respectively.

b) Use the Laplace functional to prove that the superposition of independent Poisson processes $\Psi_1, \ldots, \Psi_n$ (with intensity measures $\Theta_1, \ldots, \Theta_n$) is itself a Poisson process.

Proposed solution:

a) For $B \in \mathcal{X}$ we have

$$\Theta(B) = \mathbb{E}\big[\Phi(B)\big] = \sum_{j=1}^{n} \mathbb{E}\big[\Phi_j(B)\big] = \sum_{j=1}^{n} \Theta_j(B),$$

that is, $\Theta = \sum_{j=1}^{n} \Theta_j$. Moreover, we obtain (for any measurable $f : \mathbb{X} \to [0, \infty])$

$$L(f) = \mathbb{E}\Big[\exp\Big(-\int_{\mathbb{X}} f \, \mathrm{d}\Phi\Big)\Big] = \mathbb{E}\Big[\prod_{j=1}^{n} \exp\Big(-\int_{\mathbb{X}} f \, \mathrm{d}\Phi_j\Big)\Big] = \prod_{j=1}^{n} L_j(f),$$

where, in the last step, we used the independence of $\Phi_1, \ldots, \Phi_n$.

b) We use Theorem 2.29 of the lecture notes to show that $\Psi := \Psi_1 + \ldots + \Psi_n$ is a Poisson process. Applying part a) and Theorem 2.29 to $\Psi_1, \ldots, \Psi_n$, we obtain

$$\begin{aligned} L_\Psi(f) = \prod_{j=1}^{n} L_{\Psi_j}(f) &= \prod_{j=1}^{n} \exp\Big(-\int_{\mathbb{X}} \big(1 - e^{-f(x)}\big) \, \mathrm{d}\Theta_j(x)\Big) \\ &= \exp\Big(-\int_{\mathbb{X}} \big(1 - e^{-f(x)}\big) \, \mathrm{d}\Big(\textstyle\sum_{j=1}^{n} \Theta_j\Big)(x)\Big) \\ &= \exp\Big(-\int_{\mathbb{X}} \big(1 - e^{-f(x)}\big) \, \mathrm{d}\Theta(x)\Big), \end{aligned}$$

for all measurable $f : \mathbb{X} \to [0, \infty]$. Theorem 2.29 implies that Ψ is a Poisson process with intensity measure $\Theta := \Theta_1 + \ldots + \Theta_n$.