Karlsruhe Institute of Technology

**Institute of Stochastics**

**Stochastic Geometry | Summer term 2020**
PD. Dr. Steffen Winter
Steffen Betsch, M.Sc.

# Work Sheet 10

## Instructions for week 10 (June $22^{nd}$ to June $26^{th}$):

- The course is completed by an oral examination at the end, which can be taken either directly after the end of the semester (July, $27^{th} - 31^{st}$) or in September. In order to plan the exams, please write an email to Steffen Winter (steffen.winter@kit.edu) in case you intend to take the exam and indicate your preferred week(s). At the moment we collect this information for planning purposes only. This is not the official registration and it does not impose any obligations on you to take the exam.
- Work through Sections 4.1 and 4.2 of the lecture notes.
- Answer the control questions 1) to 5), and solve Problems 1 to 4 of the exercises below.
- Please hand in your solutions to the exercises for correction until the morning of Monday, June $29^{th}$. For the procedure, please have a look at the general information in Ilias. The submission of solutions is voluntary.

## Control questions to monitor your progress:

1) Let $K \in \mathcal{K} \setminus \{\varnothing\}$ be a convex body in $\mathbb{R}^d$. Can you construct a sequence of polytopes $K_n$ in $\mathbb{R}^d$ such that $\delta(K_n, K) \to 0$ (as $n \to \infty$), where $\delta$ denotes the Hausdorff metric?
2) Verify that the union and the intersection of any two polyconvex sets is itself polyconvex.
3) Write out the formula from Lemma 4.4 (inclusion-exclusion principle) for the cases $m = 2$ and $m = 3$.
4) Use Lemma 4.4 (inclusion-exclusion principle) to verify that, for a square $W$ in $\mathbb{R}^2$, the intrinsic volumes satisfy $V_j(2 \cdot W) = 2^j \cdot V_j(W)$ for $j = 0, 1, 2$.
5) Convince yourself that the formula in Lemma 4.5 is true for the volume functional $V_d$.

## Exercises for week 10:

### Problem 1

Let $C \in \mathcal{C}^d$ and $W \in \mathcal{K}^d$ such that $V_d(W) > 0$. Prove the following assertions.

a) It holds that $\lim_{r\to\infty} V_d(W + r^{-1} \cdot C) = V_d(W)$.

b) It holds that $V_d(W - C) \leqslant c'_W \cdot V_d(C + B^d)$, where $c'_W$ does not depend on $C$.

c) If $0 \in W$ and $r \geqslant 1$, then $V_d(W + r^{-1} \cdot C) \leqslant c_W \cdot V_d(W + C)$, where $c_W$ does not depend on $C$ or $r$.

### Problem 2 (The proof of part (iii) from Theorem 4.2)

Let $\Phi$ be a stationary particle process in $\mathbb{R}^d$ with shape distribution $\mathbb{Q}$ and intensity $\gamma > 0$, given some center function $c$. Let $f : \mathcal{C}' \to \mathbb{R}$ be a measurable and translation invariant map such that $f \geqslant 0$ or $\int_{\mathcal{C}'} |f| \,\mathrm{d}\mathbb{Q} < \infty$. Further, assume that

$$\int_{\mathcal{C}_0} |f(C)| \cdot \lambda^d(C + B^d) \,\mathrm{d}\mathbb{Q}(C) < \infty.$$

Prove that, for $W \in \mathcal{K}^d$ with $V_d(W) > 0$,

$$\gamma_f(\Phi) = \lim_{r\to\infty} \frac{1}{V_d(r \cdot W)} \mathbb{E}\left[\int_{\mathcal{C}'} \mathbb{1}\{C \cap r \cdot W \neq \varnothing\} f(C) \,\mathrm{d}\Phi(C)\right].$$

### Problem 3 (An inclusion-exclusion principle – Lemma 4.4)

Let $\mathcal{R}^d$ be the collection of all finite unions of convex bodies.

a) Let $f : \mathcal{R}^d \to \mathbb{R}$ be an additive map in the sense of Definition 4.3, that is,

$$f(\varnothing) = 0 \qquad \text{as well as} \qquad f(K \cup L) + f(K \cap L) = f(K) + f(L), \quad K, L \in \mathcal{R}^d.$$

Prove that, for any $m \in \mathbb{N}$ and $K_1, \dots, K_m \in \mathcal{R}^d$,

$$f(K_1 \cup \ldots \cup K_m) = \sum_{k=1}^{m} (-1)^{k-1} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant m} f(K_{i_1} \cap \ldots \cap K_{i_k}).$$

b) Prove that the map $\varphi_x : \mathcal{R}^d \to \mathbb{R}$, $B \mapsto \varphi_x(B) := \mathbb{1}_B(x)$, is additive for every $x \in \mathbb{R}^d$.

c) Prove that, for $m \in \mathbb{N}$, $K_1, \dots, K_m \in \mathcal{K}^d$, and $K := \bigcup_{j=1}^m K_j$,

$$\mathbb{1}_K = \sum_{\ell=1}^{m} (-1)^{\ell-1} \sum_{1 \leqslant i_1 < \ldots < i_\ell \leqslant m} \mathbb{1}_{K_{i_1} \cap \ldots \cap K_{i_\ell}}.$$

### Problem 4 (From the proof of Lemma 4.7)

For $z \in \mathbb{Z}^d$ let $C_z := C^d + z$, where $C^d := [0,1]^d$. For $r > 0$ as well as $W \in \mathcal{K}^d$ with $V_d(W) > 0$ and $0 \in \mathrm{int}(W)$, define

$$Z_r^1 := \{z \in \mathbb{Z}^d : C_z \cap r \cdot W \neq \varnothing,\ C_z \not\subset r \cdot W\} \qquad \text{and} \qquad Z_r^2 := \{z \in \mathbb{Z}^d : C_z \subset r \cdot W\}.$$

Prove that

$$\lim_{r\to\infty} \frac{\mathrm{card}(Z_r^1)}{V_d(r \cdot W)} = 0 \qquad \text{and} \qquad \lim_{r\to\infty} \frac{\mathrm{card}(Z_r^2)}{V_d(r \cdot W)} = 1.$$

**The solutions to these problems will be uploaded on June 29th.**