For $c > 0$ define

$$A_c := \{\omega \in \Omega \mid \mathrm{rad}(\mathcal{E}_n(\omega)) \leq cn^{\frac{2}{3}} \text{ for large } n\}$$

and

$$D_c := \{\omega \in \Omega \mid T(\omega)(2n) \geq cn^{\frac{3}{2}} \text{ for large } n\}.$$

If there is a constant $c > 0$ such that

$$\mathbb{P}(A_c) = 1, \tag{4.2}$$

then we have

$$\begin{aligned}
\alpha_f &= \limsup_{n\to\infty} \frac{\ln(\mathbb{E}[\mathrm{rad}(\mathcal{E}_n)])}{\ln(n)} \\
&= \limsup_{n\to\infty} \frac{\ln(\int_\Omega \mathrm{rad}(\mathcal{E}_n) d\mathbb{P})}{\ln(n)} \\
&= \limsup_{n\to\infty} \frac{\ln(\int_{A_c} \mathrm{rad}(\mathcal{E}_n) d\mathbb{P})}{\ln(n)} \\
&\overset{(+)}{\leq} \limsup_{n\to\infty} \frac{\ln(\int_{A_c} cn^{\frac{2}{3}} d\mathbb{P})}{\ln(n)} \\
&= \limsup_{n\to\infty} \frac{\ln(cn^{\frac{2}{3}})}{\ln(n)} \\
&= \frac{2}{3} \limsup_{n\to\infty} \frac{\ln(c^{\frac{3}{2}}) + \ln(n)}{\ln(n)} = \frac{2}{3} \cdot 1 = \frac{2}{3}
\end{aligned}$$

Hier muss man anders argumentieren!

If we choose

$$h : [0, \infty) \to [0, \infty), x \mapsto x^{\frac{2}{3}},$$

and $a = 2$, then by Lemma 4.2.2 we can show (4.2) if we find a constant $c > 0$ such that

$$\mathbb{P}(D_c) = 1. \tag{4.3}$$

Note that h is bijective, multiplicative and increasing. Lets first argue why the inequality at (+) indeed holds. We define CONTINUE

So now we will try to prove (4.3). For $n \in \mathbb{N}$ write $\mathcal{E}_n = \{y_1, \ldots, y_n\}$ according to Definition 3.1.1, where y_j is the j-th point added to the cluster. Let $\beta > 0$ which will be determined later on. For $n \in \mathbb{N}$ let $\tilde{m}_n := \beta n^{\frac{3}{2}}$ and define

$$V_n := \{\omega \in \Omega \mid T(\omega)(2n) < \tilde{m}_n\}.$$

? Further define the set of ~~realised~~ all possible random walk paths of length n with starting point in the set $\tilde{B}_n := \{x \in \mathbb{Z}^2 \mid n \leq |x| < n + 1\}$ by

$$Z_n := \{[z] := (z_1, \ldots, z_n) \in (\mathbb{Z}^2)^n \mid z_1 \in \tilde{B}_n, z_i \in N(z_{i-1}) \text{ for } i \in \{2, \ldots, n\}\},$$

for $[z] \in Z_n$ define the events

$$W_n([z]) := \{\omega \in \Omega \mid \exists j_1 < \cdots < j_n \leq \tilde{m}_n \text{ such that } y_{j_i}(\omega) = z_i \text{ for all } i \in \{1, \ldots, n\}\}$$

and the union of these events by

$$W_n := \bigcup_{[z] \in Z_n} W_n([z]).$$

With $\tilde{B}_n$ we mean ~~that the paths in Z_n start „on~~ the boundary "of B_n, the ball with radius n. We will quickly prove that

$$V_n \subset W_n \text{ for all } n \in \mathbb{N}.$$

Let $n \in \mathbb{N}$ and $\omega \in V_n$, then $T(\omega)(2n) < \tilde{m}_n$. For $m_n := \max\{j \in \mathbb{N} \mid j \leq \tilde{m}_n\}$ we therefore have $T(\omega)(2n) \leq m_n$, hence $\operatorname{rad}(\mathcal{E}_{m_n}(\omega)) \geq 2n$. Therefore, since $\mathcal{E}_{m_n}(\omega)$ is connected, there must exist indices $j_1 < \cdots < j_n$ such that $[z_0] := (y_{j_1}(\omega), \ldots, y_{j_n}(\omega)) \in Z_n$ and $[z_0] \subset \mathcal{E}_{m_n}(\omega)$, since $\max_{x \in [z_0]} |x| \leq 2n$. Therefore $j_n \leq m_n \leq \tilde{m}_n$ and therefore $\omega \in W_n([z_0]) \subset W_n$ which proofs the inclusion. ok

For a sequence of events $(A_n)_{n \in \mathbb{N}}$ recall that

$$\limsup_{n \to \infty} A_n := \bigcap_{n \in \mathbb{N}} \bigcup_{i \geq n} A_i = \{\omega \in \Omega \mid \omega \in A_n \text{ for infinitely many } n \in \mathbb{N}\}.$$

We will use the Lemma of Borel-Cantelli on the sequence of events $(W_n)_{n \in \mathbb{N}}$. If we can show that

$$\sum_{n \in \mathbb{N}} \mathbb{P}(W_n) < \infty, \tag{4.4}$$

then with $V_n \subset W_n$ for all $n \in \mathbb{N}$ and Borel-Cantelli we get

$$\mathbb{P}(\limsup_{n \to \infty} V_n) \leq \mathbb{P}(\limsup_{n \to \infty} W_n) = 0.$$

Since

$$(\limsup_{n \to \infty} V_n)^C = \{\omega \in \Omega \mid \exists N \in \mathbb{N} \text{ s.t. } \omega \in V_n^C \text{ for all } n > N\} = D_\beta$$

we can then conclude that $\mathbb{P}(D_\beta) = 1$ and have finished the proof. So we want to show (4.4). ok

For $n \in \mathbb{N}$, $[z] \in Z_n$ and $i \in \{1, \ldots, n\}$ we define random variables

$$\tau_i : \Omega \to \mathbb{N}^\infty, \tau_i(\omega) = j :\Leftrightarrow \begin{cases} y_j(\omega) = z_i, & j < \infty, \\ z_i \notin \mathcal{E}_\infty(\omega), & j = \infty, \end{cases}$$

so τ_i is either the index j such that the j-th added point is z_i, or infinity if the final cluster $\mathcal{E}_\infty$ doesn't contain z_i. τ_i is measurable because for $j \in \mathbb{N}$ we have

$$\tau_i^{-1}(j) = \{y_j = z_i\} = y_j^{-1}(z_i) \in \mathcal{F}$$

and for $j = \infty$ we have

$$\begin{aligned}\tau_i^{-1}(\infty) &= \{z_i \notin \mathcal{E}_\infty = \bigcup_{k\in\mathbb{N}} \mathcal{E}_k\} \\ &= \{z_i \notin \mathcal{E}_k \text{ for all } k \in \mathbb{N}\} \\ &= \bigcap_{k\in\mathbb{N}} \{z_i \notin \mathcal{E}_k\} \\ &= \bigcap_{k\in\mathbb{N}} \bigcap_{l=1}^{k} \{y_l \neq z_i\} \\ &= \bigcap_{k\in\mathbb{N}} \bigcap_{l=1}^{k} y_l^{-1}(\mathbb{Z}^2 \setminus \{z_i\}) \in \mathcal{F}.\end{aligned}$$

Further for $i \in \{1, \dots, n-1\}$ we define waiting times

$$\sigma_i : \Omega \to \mathbb{N}^\infty, \omega \to \begin{cases} \tau_{i+1}(\omega) - \tau_i(\omega), & \text{if } \tau_{i+1}(\omega) < \infty \text{ and } \tau_i(\omega) < \infty, \\ \infty, & \text{else,} \end{cases}$$

so σ_i is the waiting time between adding z_i and z_{i+1} to the cluster, if both are added. We quickly argue that σ_i is measurable as well. For $j \in \mathbb{N}$ we have

$$\sigma_i^{-1}(j) = \{\tau_{i+1} - \tau_i = j\} = \bigcup_{k\in\mathbb{N}} \{\tau_{i+1} = k\} \cap \{\tau_i = k - j\} \in \mathcal{F}$$

and

$$\sigma_i^{-1}(\infty) = \{\tau_{i+1} = \infty\} \cup \{\tau_i = \infty\} \in \mathcal{F},$$

since τ_{i+1} and τ_i are measurable.

der Gleichheitsfall ist nicht möglich, oder?

Let $i \in \{1, \dots, n-1\}$ and define the event

$$U_{[z]}^i := \{\tau_1 \leq \dots \leq \tau_i\}.$$

dominated !

Note that $\mathbb{P}(U_{[z]}^i) > 0$ and also note again, that all τ_i are defined based on one $[z]$. We will now prove that the distribution of σ_i conditioned on $U_{[z]}^i$ is bounded by that of a geometrically distributed random variable with parameter

$$p_n := c_1 n^{-\frac{1}{2}} \tag{4.5}$$

for some constant $c_1 > 0$, which is the same constant for all $i \in \{1, \dots, n-1\}$ and for all $n \in \mathbb{N}$. We want to use Proposition 4.2.1 here, for which we need to create terms where the harmonic measure appears. For that we need probabilities where we condition on a given cluster which we will develop now. We first define some helpful sets. For $m \in \mathbb{N}$ define

possible/realisable

$$C_m := \{\mathcal{E}_m(\omega) \mid \omega \in \Omega\}$$

the set of realized clusters of size m and the disjoint union of them as $C := \bigcup_{m \in \mathbb{N}} C_m$. Further define another disjoint partition of C ~~with~~ into sets

$$E_m := \{\mathcal{E} \in C \mid m - 1 \le \mathrm{rad}(\mathcal{E}) < m\}$$

for $m \in \mathbb{N}$ such that $C = \bigcup_{m \in \mathbb{N}} E_m$ as well. Since $|z_1| \ge n$ and considering $U^i_{[z]}$, we need a minimal amount of time steps such that it is possible for the cluster to contain z_i. So there exists a $n_0 \in \mathbb{N}$ such that

so sofort gewählt hängt n_0 von i, und z_i ab (vielleicht sollte man n_i schreiben statt n_0 ?!)

$$\mathbb{P}(\tau_i = j, U^i_{[z]}) \begin{cases} > 0, & \text{for } n_0 \le j \le \infty, \\ = 0, & \text{for } j < n_0. \end{cases}$$

Further if we choose $n_0 \le j < \infty$ and considering $U^i_{[z]}$ again, then $\mathcal{E}_{j-1}$ must have some minimal radius if we want it to be possible that the next added point is z_i, similarly argumented as above. So there exists a $m_0 \in \mathbb{N}$ such that

$$\mathbb{P}(\mathcal{E}_{j-1} = \mathcal{E}, \tau_i = j, U^i_{[z]}) \begin{cases} > 0, & \mathcal{E} \in E^i_m, \text{ for some } m \ge m_0, \\ = 0, & \mathcal{E} \notin E^i_m, m \ge m_0 \text{ or } \mathcal{E} \in E_m, m < m_0, \end{cases}$$

unklar $\mathcal{E} \in E_m \setminus E^i_m$ for some $m \ge m_0$ or .... ?

where $E^i_m := \{\mathcal{E} \in E_m \mid \{z_1, \ldots, z_{i-1}\} \subset \mathcal{E} \text{ and } z_i \in \partial\mathcal{E}\}$ with $m \in \mathbb{N}$. Having made that clear we can use the law of total probability twice in the following. For $k \in \mathbb{N}$ we get

$$\begin{aligned} \mathbb{P}(\sigma_i > k \mid U^i_{[z]}) &= \sum_{n_0 \le j \le \infty} \mathbb{P}(\tau_i = j \mid U^i_{[z]}) \mathbb{P}(\sigma_i > k \mid \tau_i = j, U^i_{[z]}) \\ &= \mathbb{P}(\tau_i = \infty \mid U^i_{[z]}) \mathbb{P}(\sigma_i > k \mid \tau_i = \infty, U^i_{[z]}) \\ &\qquad + \sum_{n_0 \le j < \infty} \mathbb{P}(\tau_i = j \mid U^i_{[z]}) \mathbb{P}(\sigma_i > k \mid \tau_i = j, U^i_{[z]}) \\ &= \mathbb{P}(\tau_i = \infty \mid U^i_{[z]}) + \sum_{n_0 \le j < \infty} \mathbb{P}(\tau_i = j \mid U^i_{[z]}) \mathbb{P}(\sigma_i > k \mid \tau_i = j, U^i_{[z]}), \end{aligned}$$

note that $\mathbb{P}(\sigma_i > k \mid \tau_i = \infty, U^i_{[z]}) = 1$ since $\{\tau_i = \infty\} \subset \{\sigma_i = \infty\} \subset \{\sigma_i > k\}$ for all $k \in \mathbb{N}$. For shorter expressions, write $\Gamma_{ij} := \{\mathcal{E}_{j-1} = \mathcal{E}\} \cap \{\tau_i = j\} \cap U^i_{[z]}$. Further for $n_0 \le j < \infty$ we get

$$\mathbb{P}(\sigma_i > k \mid \tau_i = j, U^i_{[z]}) = \sum_{m_0 \le m} \sum_{\mathcal{E} \in E^i_m} \mathbb{P}(\mathcal{E}_{j-1} = \mathcal{E} \mid \tau_i = j, U^i_{[z]}) \mathbb{P}(\sigma_i > k \mid \Gamma_{ij}).$$

ok

We now state that there exists a constant $c_1 > 0$ such that for any $n_0 \le j < \infty$, $m_0 \le m$ and $\mathcal{E} \in E^i_m$ we have

$$\mathbb{P}(\sigma_i > k \mid \Gamma_{ij}) \ge (1 - c_1 n^{-\frac{1}{2}})^k \tag{4.6}$$

for all $k \in \mathbb{N}$, which we prove by induction. For $k=1$ by Proposition 4.2.1 there exists a $c>0$ such that

$$\begin{aligned}\mathbb{P}(\sigma_i > 1 \mid \Gamma_{ij}) &\geq \mathbb{P}(\tau_{j+1} \neq z_{i+1} \mid \Gamma_{ij}) \\ &= 1 - h_{\partial(\mathcal{E}\cup\{z_i\})}(z_{i+1}) \\ &\geq 1 - cn^{-\frac{1}{2}} \\ &= (1 - cn^{-\frac{1}{2}})^1.\end{aligned}$$

Note that we can use Proposition 4.2.1 because $\mathrm{rad}(\mathcal{E}\cup\{z_i\}) \geq n$ since $z_1 \in \mathcal{E}\cup\{z_i\}$ as we conditioned on $U^i_{[z]}$. Note that the harmonic measure doesn't care about the order of how points where added to form the current cluster, as mentioned in Remark 3.1.1. Note that by Proposition 4.2.1 the chosen c has a strong universal property such that it doesn't depend on $n_0 \leq j < \infty$, $m_0 \leq m$ or $\mathcal{E} \in E_m$. Now let the statement be true for some $k = l$. Then again by Proposition 4.2.1 there exists a $c > 0$ such that

$$\begin{aligned}\mathbb{P}(\sigma_i > l+1 \mid \Gamma_{ij}) &= \mathbb{P}(\sigma_i > l, y_{l+1} \neq z_{i+1} \mid \Gamma_{ij}) \\ &= \mathbb{P}(\sigma_i > l \mid \Gamma_{ij})\mathbb{P}(y_{l+1} \neq z_{i+1} \mid \Gamma_{ij}, \sigma_i > l) \\ &\overset{(+)}{\geq} (1 - cn^{-\frac{1}{2}})^l\mathbb{P}(y_{l+1} \neq z_{i+1} \mid \Gamma_{ij}, \sigma_i > l) \\ &\overset{(++)}{\geq} (1 - cn^{-\frac{1}{2}})^l(1 - cn^{-\frac{1}{2}}) \\ &= (1 - cn^{-\frac{1}{2}})^{l+1},\end{aligned}$$

where in (+) we used the induction assumption. In order to show (++) we would need to split the probability with the law of total probability by conditioning on all clusters such that Γ_{ij} is fulfilled and the next l added points are not equal to z_{i+1}. We would then have the same inequality as in the induction beginning using Proposition 4.2.1 and reput together the conditioned probabilites to get this inequality here. Same as in the induction beginning we have now found a constant $c_1 > 0$ such that (4.6) holds for all $k \in \mathbb{N}$ and this c_1 is independent of $n_0 \leq j < \infty$, $m_0 \leq m$ or $\mathcal{E} \in E_m$.

Since $\{\tau_i = j\} \cap U^i_{[z]} \subset \bigcup_{m_0 \leq m} \bigcup_{\mathcal{E}\in E^i_m} \{\mathcal{E}_{j-1} = \mathcal{E}\}$ by construction of E^i_m and m_0 we can use Lemma 4.2.4 and for $n_0 \leq j \leq \infty$ get

$$\mathbb{P}(\sigma_i > k \mid \tau_i = j, U^i_{[z]}) \geq (1 - c_1 n^{-\frac{1}{2}})^k$$

for all $k \in \mathbb{N}$. Because $U^i_{[z]} \subset \bigcup_{n_0 \leq j \leq \infty}\{\tau_i = j\}$ by construction of n_0 we can use Lemma 4.2.4 again and finally get

$$\mathbb{P}(\sigma_i > k \mid U^i_{[z]}) \geq (1 - c_1 n^{-\frac{1}{2}})^k \quad ✓$$

for all $k \in \mathbb{N}$. We therefore have shown what we stated ~~at~~ in (4.5).

Note again that we needed the constant c_1 to not depend on any j, m or $\mathcal{E}$ which it does not by its universal property given by Proposition 4.2.1. The universality is so strong that it does not even depend on $i \in \{1, \dots, n-1\}$ or $n \in \mathbb{N}$.

We now continue to show (4.4). Choose β such that $4e^2\beta c_1 < 1$ und choose $N \in \mathbb{N}$ such that $\beta c_1 \geq 2p_n$ for all $n > N$. If we then define $a := \beta c_1$ and $Y := \sum_{i=1}^{n-1} \sigma_i$ we can use Lemma 4.2.3 and get

$\leq \tilde{m} = \beta n^{\frac{3}{2}}$

$$\mathbb{P}(\tau_n \leq \tilde{m}_n \mid U_{[z]}^n) \leq \mathbb{P}(\tau_n - \tau_1 \leq \tilde{m}_n \mid U_{[z]}^n) = \mathbb{P}(Y \leq \frac{an}{p_n} \mid U_{[z]}^n) \leq \frac{1}{a}(e^2 a)^n$$

for all $n > N$. Since $W_n([z]) \subset \{\tau_n \leq \tilde{m}_n\} \cap U_{[z]}^n$ we get

$$\mathbb{P}(W_n([z])) \leq \mathbb{P}(W_n([z]) \mid U_{[z]}^n) \leq \mathbb{P}(\tau_n \leq \tilde{m}_n \mid U_{[z]}^n) \leq \frac{1}{a}(e^2 a)^n$$

to

than

for all $n > N$. Counting the elements in Z_n we have less or equal $c_2 n$ points in $\tilde{B}_n$ for some constant $c_2 > 0$ as starting points and 4^{n-1} possibilities for the next $n-1$ steps of a random walk of length n. So $|Z_n| \leq c_2 n 4^{n-1}$ and therefore

Sind es nicht nur 3 Möglichkeiten jeweils, weil man nicht zurück gehen kann auf dem Pfad ?!

$$\sum_{[z] \in Z_n} \mathbb{P}(W_n([z])) \leq c_2 n 4^{n-1} \frac{1}{a}(e^2 a)^n = \frac{c_2}{4a} n (4e^2 a)^n \quad \text{for all } n > N$$

and finally

$$\sum_{n>N} \mathbb{P}(W_n) \leq \sum_{n>N} \frac{c_2}{4a} n (4e^2 \beta c_1)^n =: r_\beta.$$

Now note again, that c_1, c_2 and β are not depending on n. Since β was chosen such that $4e^2\beta c_1 < 1$, r_β is finite and therefore in total we get

$$\sum_{n \in \mathbb{N}} \mathbb{P}(W_n) < \infty,$$

which completes the proof. □

In the next chapter we will look at another incremental aggregation which tries to approximate DLA and compare simulations of both to get an empirical comparision of their growth rates.

Der Beweis ist so korrekt, denke ich, allerdings noch recht schwer zu lesen. Kann man sicher noch besser strukturieren und ein bisschen glätten, evtl. die Notation noch eleganter wählen