Solutions for Work Sheet 14

Problem 1 (Simulating Boolean models)

Use your favorite programming language to do the following simulations (we suggest the use of R or Python). You might want to use some of the code from Problem 1 b) of Work sheet 5 as a starting point. Assume that the underlying center function is the center of the circumball of a compact set.

a) Simulate a stationary Boolean model in $[0, 1]^2$ with intensity $\gamma = 50$ and deterministic grains that are disks with radius 0.05.

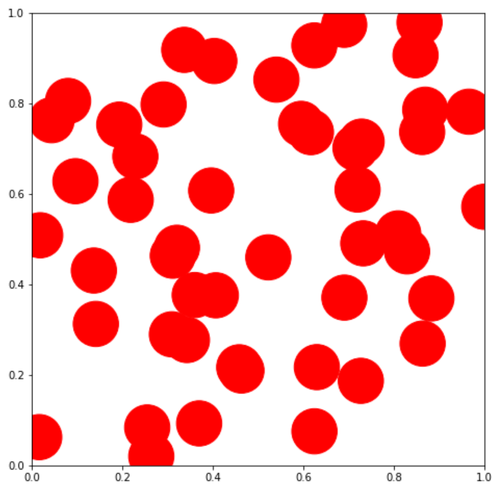
b) Vary the intensity parameter in part a). For which intensity do you get a connected component which connects the top and the bottom (or the left with the right side) of the unit square?

c) Simulate a stationary Boolean model in the square $[0, 10] \times [0, 5]$ with intensity $\gamma = 3$ and typical grain $Z_0 = [-S, S]^2$, where S is uniformly distributed on the interval $\left(0, \frac{1}{2}\right)$.

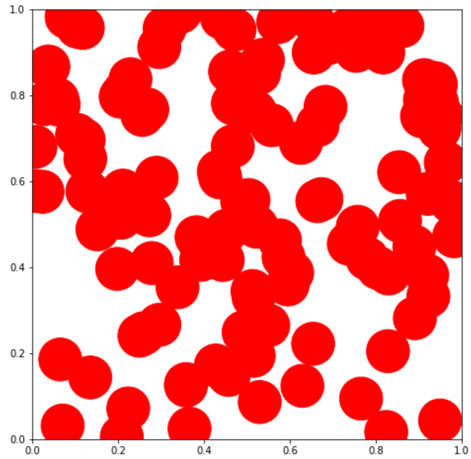
d) Simulate a stationary Boolean model in the cube $[0, 5]^3$ with intensity $\gamma = 0.25$ and typical grain $Z_0 = B(0, R) \subset \mathbb{R}^3$, where R has an exponential distribution with parameter $\lambda = \frac{10}{3}$.

Proposed solution:

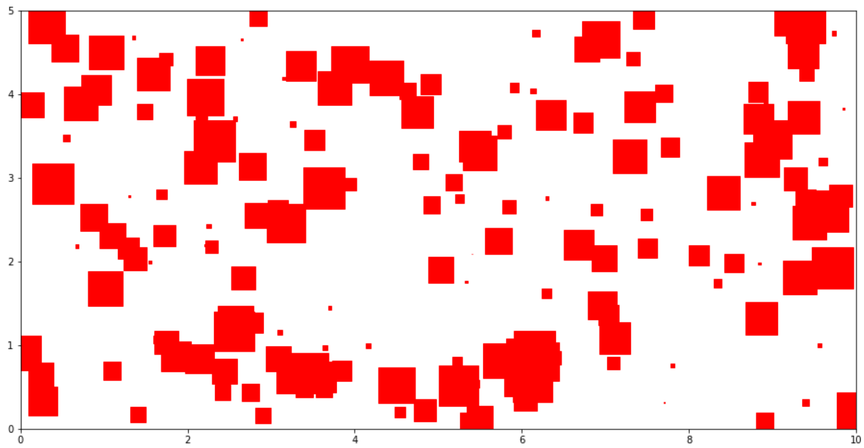
a) The following realization of the Boolean model consists of a total of 49 disks.



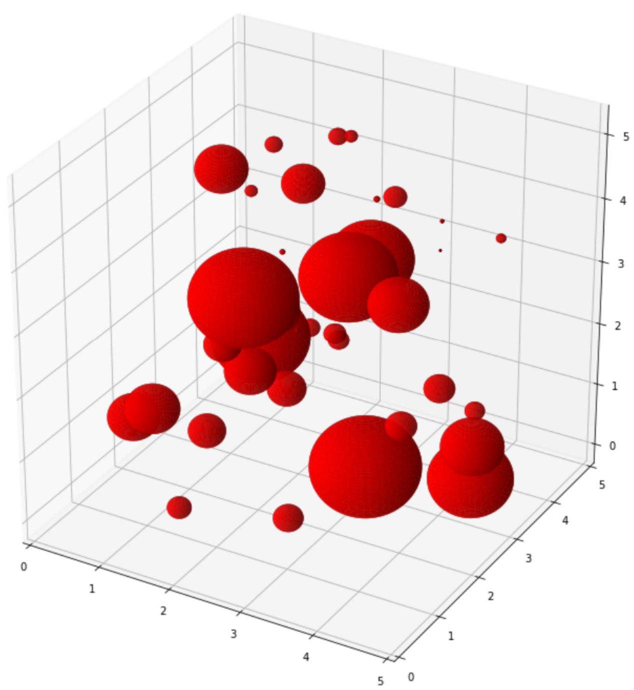
b) For a solution of this part of the exercise, use the simulation of part a), but increase the intensity by 1 each time you run the simulation until you hit a realization where the top and bottom (or the left and right side) of the unit square are connected via the Boolean model. The following picture shows what this might look like. In the corresponding simulation, the given realization arose for an intensity of 116. Recall that we consider realizations of a random object, so no worries if the phenomenon of interest occurred much earlier or later in your own simulations.



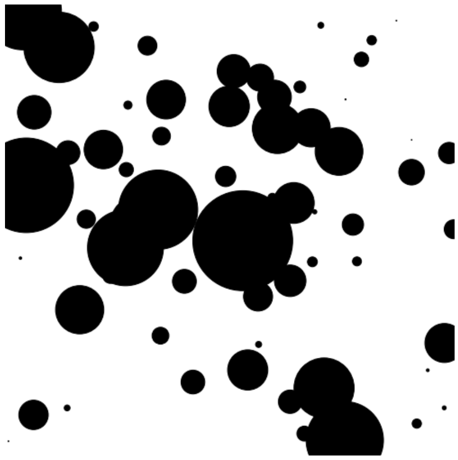
c) The following realization of the Boolean model consists of a total of 170 squares.



d) The following realization of the Boolean model consists of a total of 39 balls.



Problem 2 (Estimation in the Boolean model)



On the right hand side you find the realization of a stationary Boolean model Z observed in the unit square. You are given no information other than the fact that the picture shows the Boolean model in the unit square, $Z \cap [0,1]^2$, and that the picture, provided under the name 'Sheet14_2' in 'png' format in Ilias, has a size of 500×500 pixels.

It is your task to find estimates for the intensity parameter γ of the Boolean model and for the expected radius of the typical grain, which apparently is a ball with some random radius.

Proceed in the following steps:

a) Visit the 'morphometer' software tool on the website Morphometry.org and make sure that the settings are put to 'Image analysis' in 'Expert' mode and that you 'analyze black phase'. The correct settings are indicated in the picture below (in red). Proceed to upload the picture of the Boolean model (the option is indicated in blue below).

b) After the analysis of the 'basic parameters' you can read off the area and perimeter of the Boolean model in the picture you have uploaded (indicted in orange in the picture below). The result is given in pixels. Use these values to calculate the intrinsic volumes of the model, $V_j\big(Z \cap [0,1]^2\big)$, for $j = 0, 1, 2$.

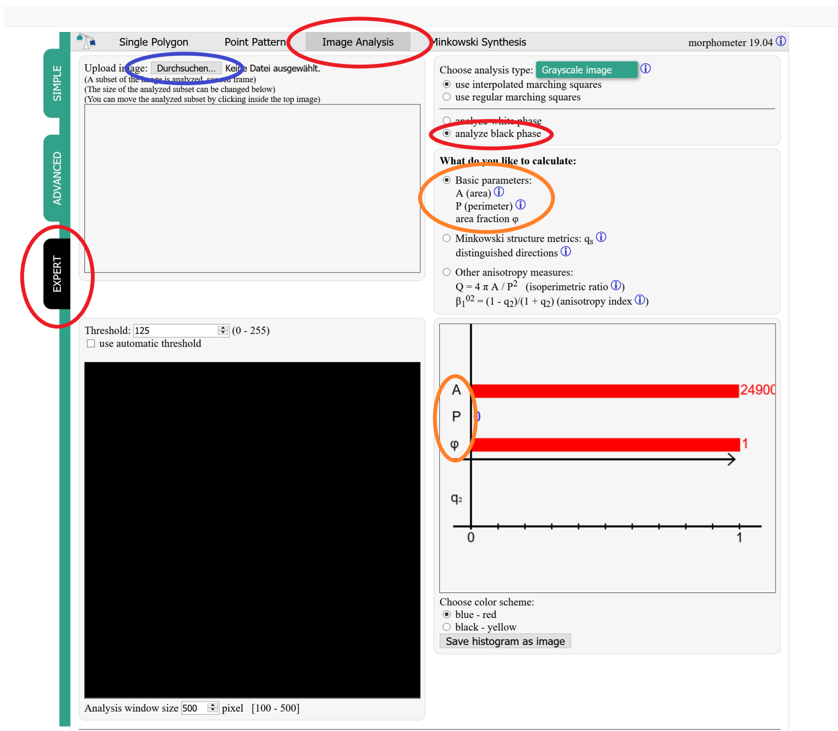
   Hint: Recall that the picture has $500 \times 500 = 250000$ pixels and represents the unit square. To calculate the Euler characteristic, recall that it is just the number of connected components minus the number of holes in the set.

c) Use your results from part b) to estimate the intrinsic volume densities of the Boolean model.

   Hint: The definition of $\delta_j = \delta_{V_j}$ suggests that $\widehat{\delta}_j = V_2\big([0,1]^2\big)^{-1} \cdot V_j\big(Z \cap [0,1]^2\big)$ is a reasonable estimator.

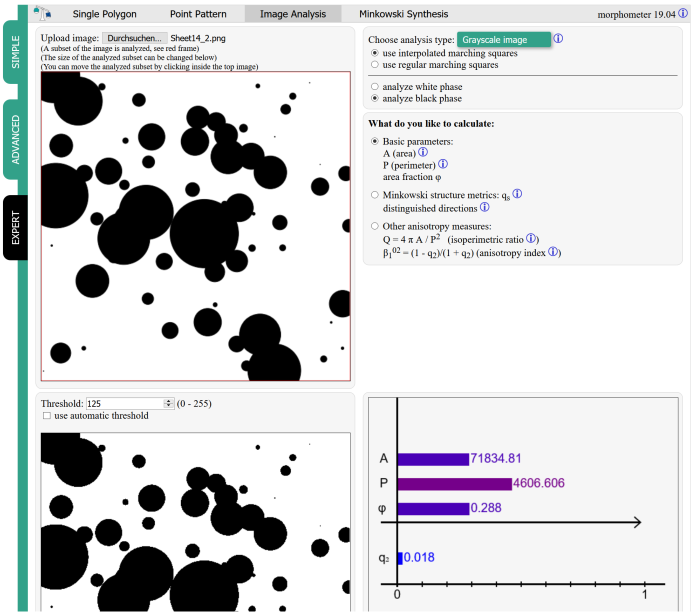
d) Use the relations from Remark 4.38 to estimate the intensity γ of the given Boolean model. Also use the information you have acquired so far to estimate the expected radius of the typical grain.

e) You are given the additional information that the radius of the typical grain follows an exponential distribution with some parameter $\lambda > 0$. Can you find an estimate for λ based on your previous observations?



Proposed solution:

a) After uploading the picture, your screen should look somewhat like the following graphic.



b) From the graphic in part a) we immediately infer that the area covered by the balls is 71834.81 pixels. As the full picture consists of 250000 pixels and represents the unit square, the area covered by the balls is $A = \frac{71834.81}{250000} = 0.28733924$. The perimeter of the given realization is 4606.606 pixel lengths, and as 500 pixels stand for a length of 1, the perimeter of the realization is $P = \frac{4606.606}{500} = 9.213212$. Thus, we immediately know what the intrinsic volumes V_1 and V_2 are:

$$V_1\left(Z \cap [0,1]^2\right) = \frac{P}{2} = 4.606606 \qquad \text{and} \qquad V_2\left(Z \cap [0,1]^2\right) = A = 0.28733924.$$

In order to calculate V_0 recall that this is nothing but the Euler characteristic. Simply by counting the connected components (there are 41) and subtracting the number of holes (there are none) in the realization of the Boolean model, we obtain $V_0\left(Z \cap [0,1]^2\right) = 41$.

c) From the hint and part b) it is immediately obvious that $\widehat{\delta}_0 = 41$,

$$\widehat{\delta}_1 = 4.606606 \qquad \text{and} \qquad \widehat{\delta}_2 = 0.28733924.$$

d) The estimate for the intensity of the Boolean model suggested by Remark 4.38 is

$$\widehat{\gamma} = \frac{\widehat{\delta}_0}{1-\widehat{\delta}_2} + \frac{1}{\pi} \cdot \left(\frac{\widehat{\delta}_1}{1-\widehat{\delta}_2}\right)^2 \approx 70,83072.$$

We know that $\gamma_1 = \gamma \cdot \mathbb{E}\left[V_1(Z_0)\right]$, where Z_0 is the typical grain which is a ball of random radius R. As $\gamma_1 = \frac{\delta_1}{1-\delta_2}$, we have $\mathbb{E}\left[V_1(Z_0)\right] = \frac{1}{\gamma} \cdot \frac{\delta_1}{1-\delta_2}$. Since V_1 measures half the perimeter of a compact set, we know that $\mathbb{E}\left[V_1(Z_0)\right] = \pi \cdot \mathbb{E}[R]$. As an estimate of $\mathbb{E}[R]$ we thus have

$$\widehat{R} = \frac{1}{\pi \cdot \widehat{\gamma}} \cdot \frac{\widehat{\delta}_1}{1-\widehat{\delta}_2} \approx 0,02905.$$

e) If the random variable R in part d) follows an exponential distribution with parameter λ, then $\mathbb{E}[R] = \frac{1}{\lambda}$. Hence, we obtain as an estimate

$$\widehat{\lambda} = \frac{1}{\widehat{R}} \approx 34,42495.$$

Usually estimators based on the volume functional are more reliable than estimators based on the perimeter. Thus, let us construct a second estimator for λ based on the volume of the realization of the

Boolean model. As R is exponentially distributed with parameter λ, we have $\mathbb{E}\big[V_2(Z_0)\big] = \pi \cdot \mathbb{E}[R^2] = \frac{2\pi}{\lambda^2}$. Moreover, we have $\gamma_2 = \gamma \cdot \mathbb{E}\big[V_2(Z_0)\big]$, so $\lambda^2 = \frac{2\pi \cdot \gamma}{\gamma_2}$. Since $\gamma_2 = -\log(1-\delta_2)$, we obtain

$$\lambda = \sqrt{-\frac{2\pi \cdot \gamma}{\log(1-\delta_2)}}$$

and, as an estimate for λ,

$$\widehat{\lambda} = \sqrt{-\frac{2\pi \cdot \widehat{\gamma}}{\log(1-\widehat{\delta}_2)}} \approx 36.2461.$$

Note: The given picture stems from the simulation of a Boolean model with intensity $\gamma = 65$, and with the radius distribution being exponential with parameter $\lambda = \frac{100}{3}$. We see that the perimeter-based estimator in part e) fares better than the volume-based estimator for the given realization. Repeating the experiment a large number of times would give more reliable insight on the performance of the estimation methods.