Karlsruher Institut für Technologie

Lecture notes
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Stochastic Geometry

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Contents

0 Introduction

0.1 Random Geometric Structures

Some typical structures studied in stochastic geometry (not all of them will be addressed in the course; some motivation is given on the introductory slides):

1. Random closed sets – the basic theoretical model of random geometric objects;
2. Germ-grain models (systems of convex (or more general) particles and their union sets): The germs are given by a point process. To each germ a grain (a convex set) is attached.
3. Convex hulls of random points: e.g. in a given bounded convex set n uniform i.i.d. random points are chosen and their convex hulls are studied as $n \to \infty$.
4. Random connection model: The points of a point process are randomly connected according to certain rules which may depend on their positions (and the position of other points) and form a graph or more generally a cell complex.
5. Tessellations: A tessellations is a partition of a space into bounded convex regions. Examples are Voronoi and hyperplane tessellations.
6. Excursion sets of real-valued random fields $(X_t)_{t\in I}$, $I \subset \mathbb{R}^d$:

$$Z_c := \{t \in I \colon X_t \geq c\}, \qquad c \in \mathbb{R}.$$

0.2 Some typical questions

In $\mathbb{R}^d$ let Z be a union of particles or of their parts (e.g. faces). Let $f_1, \dots, f_m$ be real-valued geometric functionals (e.g. volume, surface area, Euler characteristic etc.).

1. Do the limits $\bar{f}_i := \lim_{W \uparrow \mathbb{R}^d} V_d(W)^{-1} f_i(Z \cap W)$ exist (e.g. $\mathbb{P}$-a.s.)?
2. Do the limits $\sigma_{ij} := \lim_{W \uparrow \mathbb{R}^d} V_d(W)^{-1} \operatorname{Cov}(f_i(Z \cap W), f_j(Z \cap W))$ exist and if yes, how do they depend on model parameters like the intensity of underlying process? Is there a multivariate central limit theorem?
3. What are the connectivity properties of Z? Does Z percolate, i.e., does Z have an unbounded connected component? This question ist studied in percolation theory.
4. What are the probabilities $\mathbb{P}(x_1 \in Z, \dots, x_k \in Z)$ for $x_1, \dots, x_k \in \mathbb{R}^d$? These are the so-called *k-point correlation functions*.
5. Let X be a random point in $\mathbb{R}^d \setminus Z$. What is the distribution of the distance $d(X, Z)$? This question leads to the notion of *contact distributions*.

1 Random closed sets

Let $(E, \mathcal{O}_E)$ be a topological space and $\mathcal{F}(E)$ the family of closed subsets of E. If $(\Omega, \mathcal{A}, \mathbb{P})$ is some probability space and $Z : \Omega \to \mathcal{F}(E)$ a measurable mapping, then we call Z a *random closed set*. Measurability is meant here with respect to the σ-algebra $\mathcal{A}$ on Ω and the Borel σ-algebra on $\mathcal{F}(E)$. This requires to specify a topology $\mathcal{O}_{\mathcal{F}(E)}$ on $\mathcal{F}(E)$. We will do this in a way such that natural operations between closed sets will at least be measurable and in many cases even continuous or semi-continuous.

1.1 The space of closed sets

In this section we introduce a topology on the space of closed subsets of a topological space E. The definition will only require sections of closed sets with open and compact sets, respectively, such that the construction works indeed in a general topological space. Examples are metric spaces and in particular the spaces $\mathbb{R}^d$ and $\mathbb{S}^{d-1}$ (with the Eucidean metric for instance).

Definition 1.1 For a topological space $(E, \mathcal{O}_E)$, we define

$$\mathcal{F}(E) := \{F \subset E \mid F \text{ closed}\},$$
$$\mathcal{C}(E) := \{F \subset E \mid F \text{ compact}\},$$
$$\mathcal{G}(E) := \{F \subset E \mid F \text{ open}\} = \mathcal{O}_E.$$

Moreover, we introduce a topology on $\mathcal{F} = \mathcal{F}(E)$ by specifying a base. For this let $A, A_1, \ldots, A_k \subset E$ for $k \in \mathbb{N}_0$ and set

$$\mathcal{F}^A := \{F \in \mathcal{F} \mid F \cap A = \emptyset\},$$
$$\mathcal{F}_A := \{F \in \mathcal{F} \mid F \cap A \neq \emptyset\},$$
$$\mathcal{F}^A_{A_1,\ldots,A_k} := \mathcal{F}^A \cap \mathcal{F}_{A_1} \cap \ldots \cap \mathcal{F}_{A_k}.$$

Notice that we have $\emptyset \in \mathcal{F}^A$, $\emptyset \notin \mathcal{F}_A$ and $(\mathcal{F}^A)^c = \mathcal{F}_A$. Moreover, the system

$$\tau := \{\mathcal{F}^C_{G_1,\ldots,G_k} \mid C \in \mathcal{C}, G_1, \ldots, G_k \in \mathcal{G}, k \in \mathbb{N}_0\}$$

is $\cap$-stable with $\mathcal{F} = \mathcal{F}^\emptyset \in \tau$ and $\emptyset = \mathcal{F}^\emptyset_\emptyset \in \tau$. Therefore, τ is the base of a uniquely determined topology $\mathcal{O}_{\mathcal{F}(E)}$ on $\mathcal{F}(E)$. The elements of $\mathcal{O}_{\mathcal{F}(E)}$ (i.e. the open sets) are the unions of sets of τ. This topology will be used in the sequel. It is called *hit-or-miss topology*, *Fell topology* (after James M.G. Fell, who introduced this topology in 1962) or *topology of closed convergence*.

In the sequel $(E, \mathcal{O}_E)$ will always denote a locally compact Hausdorff space with countable base. Therefore, $(E, \mathcal{O}_E)$ is in particular a Polish space (i.e. a complete metrizable space with countable base). The following statement shows that $(\mathcal{F}(E), \mathcal{O}_{\mathcal{F}(E)})$ has at least as good properties as $(E, \mathcal{O}_E)$. If E is clear from the context, we will omit it in the notation and simply write $\mathcal{F}$ instead of $\mathcal{F}(E)$, $\mathcal{C}$ instead of $\mathcal{C}(E)$ and $\mathcal{G}$ instead of $\mathcal{G}(E)$. In large parts of the manuscript, E will be the Euclidean space $\mathbb{R}^d$, $d \geq 1$, but in some places it is convenient to have the subsequent notions available more generally.

Theorem 1.2 *(Properties of the Fell topology)*
Let $(E, \mathcal{O}_E)$ be a locally compact Hausdorff space with countable base. Then the following assertions hold:

(1) *$(\mathcal{F}, \mathcal{O}_\mathcal{F})$ is a compact Hausdorff space with countable base and therefore metrizable. In particular, $(\mathcal{F}, \mathcal{O}_\mathcal{F})$ is also sequentially compact.*

(2) *If $C \subset E$ is compact, then $\mathcal{F}_C$ is compact in $\mathcal{F}' := \mathcal{F} \setminus \{\emptyset\}$. Here $\mathcal{F}'$ equipped with the trace topology is a locally compact Hausdorff space with countable base.*

(3) *The family $\{\mathcal{F}^C \mid C \in \mathcal{C}\}$ is a neighbourhood basis of $\emptyset$.*

Proof. The proof is discussed on Work sheet 1. □

The notion of convergence with respect to a given topology is essential. Let us recall it for $\mathcal{F}$ and the Fell topology $\mathcal{O}_\mathcal{F}$: A sequence $(F_j)_{j\in\mathbb{N}}$ in $\mathcal{F}$ is said to *converge* to a set $F \in \mathcal{F}$, as $j \to \infty$, if each neighbourhood $\mathcal{U}$ of F (that is, each $\mathcal{U} \in \mathcal{O}_\mathcal{F}$ such that $F \in \mathcal{U}$) contains almost all of the F_j. (Recall that in this context *almost all* means all except finitely many, which we will abbreviate by a.a. in the sequel.) For showing the convergence of sequences of closed sets the following criteria are often helpful.

Theorem 1.3 *(Convergence of closed sets)*
Let $(F_j)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{F}$ and let $F \in \mathcal{F}$. Then the following three assertions are equivalent:

(1) *$F_j \to F$, as $j \to \infty$.*

(2) *For each $G \in \mathcal{G}$, $G \cap F \neq \emptyset$ implies $G \cap F_j \neq \emptyset$ for a.a. $j \in \mathbb{N}$ and, for each $C \in \mathcal{C}$, $C \cap F = \emptyset$ implies $C \cap F_j = \emptyset$ for a.a. $j \in \mathbb{N}$.*

(3) *For each $x \in F$ there exists a sequence (x_j) in E such that $x_j \in F_j$ for a.a. $j \in \mathbb{N}$ and $x_j \to x$ as $j \to \infty$.*
If $(i_j)_{j\in\mathbb{N}}$ is some increasing sequence in $\mathbb{N}$ and $(x_{i_j})_{j\in\mathbb{N}}$ a convergent sequence with $x_{i_j} \in F_{i_j}$ for each $j \in \mathbb{N}$, then the limit $\lim_{j\to\infty} x_{i_j}$ is in F.

Proof. Assume (1) holds. Let $G \in \mathcal{G}$ with $G \cap F \neq \emptyset$. Then $F \in \mathcal{F}_G$ and so $\mathcal{F}_G$ is a neighbourhood of F. Therefore $F_j \in \mathcal{F}_G$ for a.a. $j \in \mathbb{N}$, i.e. $F_j \cap G \neq \emptyset$. Furthermore, let $C \in \mathcal{C}$ with $C \cap F = \emptyset$. Then $F \in \mathcal{F}^C$ and so $\mathcal{F}^C$ is a neighbourhood of F. Thus $F_j \in \mathcal{F}^C$ for a.a. $j \in \mathbb{N}$, i.e. $F_j \cap C = \emptyset$. Hence (2) is satisfied.

Assume now that (2) holds. Let $\mathcal{U}$ be a neighbourhood of F. Then there are sets $C \in \mathcal{C}$ and $G_1, \dots, G_k \in \mathcal{G}$ such that $F \in \mathcal{F}^C_{G_1,\dots,G_k} \subset \mathcal{U}$. This means $F \cap C = \emptyset$ and $F \cap G_j \neq \emptyset$ for $j = 1, \dots, k$ which, by (2), implies $F_j \in \mathcal{F}^C_{G_1,\dots,G_k} \subset \mathcal{U}$ for a.a. $j \in \mathbb{N}$.

The equivalence of (2) and (3) will be discussed on Work sheet 1. See also Theorem 12.2.2 in [6, S. 656]. □

We give some examples of sequences of closed sets in $\mathbb{R}^d$ converging to some limit set. Denote by $B(x, r)$ the closed ball in $\mathbb{R}^d$ with centre $x \in \mathbb{R}^d$ and radius $r \geq 0$.

Examples 1.4 Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^d$ and $x \in \mathbb{R}^d$. Let $(r_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers and let $r > 0$.

(i) If $x_n \to x$, then $\{x_n\} \to \{x\}$, as $n \to \infty$. (This follows directly from criterion (3) in Theorem 1.3.)

(ii) If $x_n \to x$ and $r_n \to r$, then $B(x_n, r_n) \to B(x, r)$.

(iii) If $||x_n|| \to \infty$, then $\{x_n\} \to \emptyset$.

For a discussion of (ii) and (iii) see Work sheet 2.

Recall that a mapping $f : E \to E'$ between topological spaces $(E, \mathcal{O})$, $(E', \mathcal{O}')$ is called *continuous* if and only if for every open set $V \subset E'$, the inverse image

$$f^{-1}(V) = \{x \in E : f(x) \in V\}$$

is an open subset of E. If E satisfies the first axiom of countability, then continuity is equivalent to *sequential continuity*, i.e. f is continuous if and only if, for any sequence (x_i) in E and $x \in E$, $x_j \to x$ implies $f(x_j) \to f(x)$ as $j \to \infty$. Note that in our situation, by Theorem 1.2 above, the space $(\mathcal{F}, \mathcal{O}_{\mathcal{F}})$ is always second countable (and thus first countable). Hence we can use sequential continuity to show continuity. As an easy consequence of Theorem 1.3, we conclude the continuity of the union operation.

Corollary 1.5 *The mapping $\cup : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$, $(F, F') \mapsto F \cup F'$ is continuous.*

Proof. It suffices to prove sequential continuity. Let (F_j) and (F'_j) be sequences in $\mathcal{F}$ such that $(F_j, F'_j) \to (F, F')$ in $\mathcal{F} \times \mathcal{F}$, i.e. $F_j \to F$ and $F'_j \to F'$. Let $G \in \mathcal{G}$ with $G \cap (F \cup F') \neq \emptyset$. Then at least one of the two sets, say F, satisfies $G \cap F \neq \emptyset$. We conclude that $G \cap F_j \neq \emptyset$ for a.a. $j \in \mathbb{N}$ and thus $G \cap (F_j \cup F'_j) \neq \emptyset$ for a.a. j. Analogously, let $C \in \mathcal{C}$ such that $C \cap (F \cup F') = \emptyset$. Then $C \cap F = \emptyset$ and $C \cap F' = \emptyset$, which implies that $C \cap F_j = \emptyset$ und $C \cap F'_j = \emptyset$ for a.a. j. This shows $C \cap (F_j \cup F'_j) = \emptyset$ for a.a. j. Now the assertion follows from Theorem 1.3.

Alternatively, one can also use the third criterion in Theorem 1.3 for the proof. □

Example 1.6 In contrast, the mapping $\mathcal{F} \times \mathcal{F} \to \mathcal{F}, (F, F') \mapsto F \cap F'$ is not continuous in general, which can be seen as follows: Let $x_n \in E,\ n \in \mathbb{N}$, such that $x_n \to x$, as $n \to \infty$, and $x_n \neq x$ for all n. Then $\{x_n\} \to \{x\}$, but setting $F_n := \{x_n\}$ and $F := \{x\}$ we see that $\emptyset = F_n \cap F$ does not converge to $F \cap F = \{x\}$.

But this mapping can be shown to have some *semi-continuity* property.

Definition 1.7 Let $(T, \mathcal{O}_T)$ and $(E, \mathcal{O}_E)$ be topological spaces (and as before $\mathcal{F} = \mathcal{F}(E)$, $\mathcal{G} = \mathcal{G}(E)$, $\mathcal{C} = \mathcal{C}(E)$). A mapping $\varphi : T \to \mathcal{F}$ is called

$$\begin{aligned} \textit{upper semi-continuous} &:\Leftrightarrow \quad \forall C \in \mathcal{C} :\ \varphi^{-1}(\mathcal{F}^C) \in \mathcal{O}_T, \\ \textit{lower semi-continuous} &:\Leftrightarrow \quad \forall G \in \mathcal{G} :\ \varphi^{-1}(\mathcal{F}_G) \in \mathcal{O}_T. \end{aligned}$$

Corollary 1.8 *The mapping $\cap : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ is upper semi-continuous.*

Proof. Suppose there is some $C \in \mathcal{C}$ such that the set $\{(F, F') \in \mathcal{F} \times \mathcal{F} : F \cap F' \cap C = \emptyset\}$ is not open in $\mathcal{F} \times \mathcal{F}$. Then there exist $F, F' \in \mathcal{F}$ with $F \cap F' \cap C = \emptyset$ and sequences $(F_i)_i$ and $(F'_i)_i$ in $\mathcal{F}$ such that $F_i \to F$, $F'_i \to F'$ and $F_i \cap F'_i \cap C \neq \emptyset$ for each $i \in \mathbb{N}$. Hence we can choose a point $x_i \in F_i \cap F'_i \cap C$ for each $i \in \mathbb{N}$. Since C is compact, there exists a subsequence $(x_{i_j})_{j \in \mathbb{N}}$ which converges to some $x \in C$. But then necessarily $x \in F \cap F' \cap C$, a contradiction. □

For dealing with measures (in particular probability measures) on the space of closed sets $\mathcal{F}$ a suitable σ-algebra is needed. Recall that every topology generates a Borel σ-algebra. (It is by definition the smallest σ-algebra generated by the open sets.) In the sequel, the Borel σ-algebra $\mathcal{B}(\mathcal{F})$ on $\mathcal{F}$ will always be the one generated by the Fell topology. The following observations are particularly useful.

Remark 1.9 The Borel σ-algebra $\mathcal{B}(\mathcal{F})$ is generated by $\{\mathcal{F}^C : C \in \mathcal{C}\}$ as well as by $\{\mathcal{F}_G : G \in \mathcal{G}\}$. That is, either of these two families suffices, which is no surprise in view of the fact that – in contrast to topologies – σ-algebras are stable with respect to taking complements. Indeed, for every $G \in \mathcal{G}$ there is an ascending sequence (C_n) in $\mathcal{C}$ with $C_n \uparrow G$, and therefore

$$\mathcal{F}_G = \bigcup_{n=1}^{\infty} \mathcal{F}_{C_n} = \bigcup_{n=1}^{\infty} \left(\mathcal{F} \setminus \mathcal{F}^{C_n}\right),$$

showing that $\mathcal{F}_G \in \sigma(\{\mathcal{F}^C : C \in \mathcal{C}\})$. Similarly, for each $C \in \mathcal{C}$, there is a decending sequence of relatively compact open sets G_n such that $G_n \downarrow C$, as $n \to \mathbb{N}$, and therefore

$$\mathcal{F}^C = \bigcup_{n=1}^{\infty} \left(\mathcal{F} \setminus \mathcal{F}_{G_n}\right).$$

Remark 1.10 The family $\mathcal{C}$ of compact sets is measurable, i.e. an element of $\mathcal{B}(\mathcal{F})$. To see this, let $(D_n)_{n\in\mathbb{N}}$ be a growing sequence of relatively compact, open sets in E such that $\bigcup_{n\in\mathbb{N}} D_n = E$. Then, for every compact set $C \subset E$, there is an index $n \in \mathbb{N}$ such that $C \subset D_n \subset \operatorname{cl} D_n$. Hence

$$\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{F}^{E\setminus \operatorname{cl} D_n} = \bigcup_{n=1}^{\infty} \left(\mathcal{F} \setminus \mathcal{F}_{E\setminus \operatorname{cl} D_n}\right) \in \mathcal{B}(\mathcal{F}).$$

For the remainder of this section, we consider the important special case $E = \mathbb{R}^d$ and write $\mathcal{F}^d$ for $\mathcal{F}(\mathbb{R}^d)$.

Theorem 1.11 (i) *The mapping $\mathcal{F}^d \to \mathcal{F}^d$, $F \mapsto -F = F^*$ is continuous.*

(ii) *The mapping $\mathbb{R} \times \mathcal{F}^d \to \mathcal{F}^d$, $(\alpha, F) \mapsto \alpha F$ is continuous on $(\mathbb{R} \setminus \{0\}) \times \mathcal{F}^d$.*

Proof. See Work sheet 2, where we also discuss that assertion (ii) is not true on $\mathbb{R} \times \mathcal{F}^d$. □

Theorem 1.12 (i) *The mapping $\mathcal{F}^d \times \mathcal{F}^d \to \mathcal{F}^d$, $(F, F') \mapsto F \cap F'$ is measurable.*

(ii) *The mappings $\mathcal{F}^d \to \mathcal{F}^d$, $F \mapsto \operatorname{cl}(\mathbb{R}^d \setminus F)$, $F \mapsto \operatorname{cl}\operatorname{conv}(F)$ and $F \mapsto \partial F$ are measurable. (Here $\operatorname{conv}(F)$ denotes the convex hull of F and ∂F its topological boundary.)*

(iii) *The mapping $\mathcal{F}^d \times \mathcal{F}^d \to \mathcal{F}^d$, $(F, F') \mapsto \operatorname{cl}(F + F')$ is measurable. (Here $+$ denotes* Minkowski addition*, i.e., $F + F' := \{x + x' : x \in F, x' \in F'\}$.*

(iv) *The mapping $\mathbb{R}^d \times \mathcal{F}^d \to \mathbb{R}$, $(x, F) \mapsto \mathbb{1}_F(x)$ is measurable.*

Proof. We provide proofs for some of the assertions and leave the remaining ones as exercises, as they require very similar arguments, see also [6].

For a proof of (i), let $\varphi(F_1, F_2) := F_1 \cap F_2$. For any $C \in \mathcal{C}^d$ the set $\varphi^{-1}\left(\mathcal{F}^C\right)$ is open by Corollary 1.8. Therefore, the assertion follows from Remark 1.9.

To prove (ii) and (iii), one can show that the corresponding mappings are lower semi-continuous. For instance, we have for any open set $G \subset \mathbb{R}^d$,

$$\{F \in \mathcal{F}^d : G \cap \operatorname{cl} F^c \neq \emptyset\} = \{F \in \mathcal{F}^d : G \cap F^c \neq \emptyset\} = \{F \in \mathcal{F}^d : G \subset F\}^c.$$

Now observe that $\{F \in \mathcal{F}^d : G \subset F\}$ is closed. Indeed, if $F_i \to F$ and $G \subset F_i$ for all $i \in \mathbb{N}$, then for $x_i := x \in G \subset F_i$ one has $x_i \to x$ and therefore $x \in F$. Hence, $G \subset F$. □

The family $\mathcal{C}^d$ of compact sets is an important subfamily of $\mathcal{F}^d$. In the last part of this section we want to introduce a metric on $\mathcal{C}^d$, called the Hausdorff metric and discuss its relation with the Fell topology (or rather with the trace topology on $\mathcal{C}^d$ induced by the Fell topology).

Definition 1.13 Let $B^d := B(0,1)$ denote the unit ball in $\mathbb{R}^d$. For $B \subset \mathbb{R}^d$ and $\varepsilon > 0$ let

$$B_{\oplus\varepsilon} := B + \varepsilon B^d$$

be the *ε-parallel set* of B.

Remark 1.14 For any $B \in \mathcal{F}^d$, one has

$$B_{\oplus\varepsilon} = \{x \in \mathbb{R}^d : d(x,B) \leq \varepsilon\},$$

where $d(x,B) := \inf\{d(x,y) : y \in B\}$ and $d(\cdot,\cdot)$ denotes the Euclidean metric in $\mathbb{R}^d$. Note that the right hand side makes sense in an arbitrary metric space (E,d) and can therefore be used to define parallel sets more generally. The considerations below can be generalized accordingly.

Definition 1.15 The Hausdorff metric δ on $\mathcal{C}^d \setminus \{\emptyset\}$ is defined by

$$\delta(C,C') := \min\{\varepsilon \geq 0 : C \subset C'_{\oplus\varepsilon}, C' \subset C_{\oplus\varepsilon}\}.$$

Furthermore, one sets $\delta(\emptyset, C) = \delta(C,\emptyset) := \infty,\ C \in \mathcal{C}^d \setminus \{\emptyset\}$, and $\delta(\emptyset,\emptyset) := 0$.

Theorem 1.16 *The Hausdorff metric is a metric on $\mathcal{C}^d \setminus \{\emptyset\}$ as well as on $\mathcal{C}^d$.*

Proof. See Work sheet 2. □

Theorem 1.17 *Let $C, C_1, C_2, \ldots \in \mathcal{C}^d \setminus \{\emptyset\}$. Then $\delta(C_n, C) \to 0$, as $n \to \infty$, implies $C_n \to C$ with respect to the Fell topology. The converse is true, if $C_n \subset K$, $n \in \mathbb{N}$, for some $K \in \mathcal{C}^d$.*

Proof. "⇒": Assume that $\delta(C_n, C) \to 0$ as $n \to \infty$. We show the convergence $C_n \to C$ w.r.t. the Fell topology in $\mathcal{F}^d$ employing criterion (3) in Theorem 1.3. First let $x \in C$. Since $d(x, C_n) \to 0$ as $n \to \infty$, there exists for each $n \in \mathbb{N}$ a point $x_n \in C_n$ such that $x_n \to x$ as $n \to \infty$, showing the first condition. Further, let $(C_{n_j})_j$ be some subsequence and let $x_{n_j} \in C_{n_j}$ such that x_{n_j} converges to some point $x \in \mathbb{R}^d$ as $j \to \infty$. Since $d(C_{n_j}, C) \to 0$, there exists for each $j \in \mathbb{N}$ a point $y_{n_j} \in C$ such that $d(x_{n_j}, y_{n_j}) \to 0$. Hence, $d(y_{n_j}, x) \to 0$, that is, $y_{n_j} \to x$ as $j \to \infty$ and therefore $x \in C$.

"⇐": Assume that $C_n \to C$ as $n \to \infty$ in the Fell topology and that there is some $K \in \mathcal{C}$ such that $C_n \subset K$ for $n \in \mathbb{N}$. We show that $\delta(C_n, C) \to 0$. If $C = \emptyset$, then $C_n = \emptyset$ for a.a. $n \in \mathbb{N}$ (why?) and the convergence w.r.t δ is obvious. Therefore we can assume $C \neq \emptyset$ which implies $C_n \neq \emptyset$ for a.a. $n \in \mathbb{N}$ (why?).

Assume for a contradiction that $\delta(C_n, C) \to 0$ as $n \to \infty$ does not hold. Then there would exist some $\varepsilon > 0$ such that $\delta(C_n, C) \geq \varepsilon$ for infinitely many $n \in \mathbb{N}$. Let $I \subset \mathbb{N}$ be the index set describing all those C_n. Then for each $i \in I$ at least one of the following two cases applies:

(a) $\exists x_i \in C\, \forall y \in C_i : d(x_i, y) \geq \varepsilon$;

(b) $\exists y_i \in C_i\, \forall x \in C : d(y_i, x) \geq \varepsilon$.

If (a) holds for an infinite subset $I' \subset I$, then there is an infinite subset $J \subset I'$ und a subsequence $(x_j)_j$, $j \in J$, in C such that $x_j \to x_0 \in C$, since C is (sequentially)

compact. Since $C_j \to C$ w.r.t. the Fell topology, there are $y_j \in C_j$ such that $y_j \to x_0$. But this implies $d(x_j, y_j) \to 0$ as $j \to \infty$, a contradiction to assertion (a).

If (b) holds for an infinite subset $I' \subset I$, then the (sequential) compactness of K and the relation $C_i \subset K$ implies the existence of an infinite subset $J \subset I'$ and of a subsequence $(y_j)_{j\in J}$ such that $y_j \in C_j \subset K$ and $y_j \to x_0$ for some $x_0 \in K$. Since $C_j \to C$ in $\mathcal{F}^d$, we conclude $x_0 \in C$, which is a contradiction to (b). □

That the converse in Theorem 1.16 does not hold in general can be seen from the following example.

Example 1.18 Let $x, x_n \in \mathbb{R}^d$, $n \in \mathbb{N}$, such that $\|x_n\| \to \infty$, as $n \to \infty$. Then $\{x_n, x\} \to \{x\}$ with respect to the Fell topology on $\mathcal{F}^d$, but $\delta(\{x_n, x\}, \{x\}) \to \infty$.

Let λ_d denote the Lebesgue measure in $\mathbb{R}^d$. For later use we prove a semi-continuity property of the volume.

Lemma 1.19 *The mapping $V_d : \mathcal{C}^d \to \mathbb{R}$, $C \mapsto \lambda_d(C)$ is upper semi-continuous, that is, if $C, C_n \in \mathcal{C}^d$, $n \in \mathbb{N}$ and $C_n \to C$, then $V_d(C) \geq \limsup_{n\to\infty} V_d(C_n)$.*

Proof. Assume that $C_n \to C$ in the Hausdorff metric and let $x \in \mathbb{R}^d$. Then

$$\limsup_{n\to\infty} \mathbf{1}_{C_n}(x) \leq \mathbf{1}_C(x).$$

Observe that for a.a. $n \in \mathbb{N}$ one has $C_n \subset C + B^d$, so that $\mathbf{1}_{C_n} \leq \mathbf{1}_{C+B^d}$, and $\mathbf{1}_{C+B^d}$ is obviously integrable w.r.t. λ_d. Therefore, by Fatou's lemma,

$$\begin{aligned} V_d(C) &= \int \mathbf{1}_C(x)\, \lambda_d(dx) \geq \int \limsup_{n\to\infty} \mathbf{1}_{C_n}(x)\, \lambda_d(dx) \\ &\geq \limsup_{n\to\infty} \int \mathbf{1}_{C_n}(x)\, \lambda_d(dx) = \limsup_{n\to\infty} V_d(C_n), \end{aligned}$$

which completes the proof. □

1.2 Random closed sets

Now we introduce the notion of a random closed set, for which we go back to the general setting. That is, E will as before denote a locally compact Hausdorff space with countable base, $\mathcal{F} = \mathcal{F}(E)$ the family of closed sets in E (equipped with the Fell topology $\mathcal{O}_\mathcal{F}$), and $\mathcal{B}(\mathcal{F})$ will be the Borel σ-algebra in $(\mathcal{F}, \mathcal{O}_\mathcal{F})$. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be some probability space.

Definition 1.20 A *random closed set (RCS)* in E is a measurable mapping $Z : \Omega \to \mathcal{F}$ (where measurability is w.r.t. the σ-algebra $\mathcal{A}$ in Ω and the Borel σ-algebra $\mathcal{B}(\mathcal{F})$ in $\mathcal{F}$). The image measure of $\mathbb{P}$ under Z, i.e. the measure $\mathbb{P}_Z = Z(\mathbb{P}) = \mathbb{P} \circ Z^{-1}$, is called *distribution of Z*.

Note that $\mathbb{P}_Z$ is a measure on $(\mathcal{F}, \mathcal{B}(\mathcal{F}))$. If an RCS satisfies $Z(\Omega) \subset \mathcal{C}$ (or, more generally, $\mathbb{P}_Z(\mathcal{C}) = 1$), then we also speak of a *random compact set*.

Remark 1.21 For random closed sets Z, Z' defined on some probability spaces $(\Omega, \mathcal{A}, \mathbb{P})$ and $(\Omega', \mathcal{A}', \mathbb{P}')$ we use the notation $Z \stackrel{d}{=} Z'$ for equality in distribution, i.e. for the relation $\mathbb{P}_Z = \mathbb{P}'_{Z'}$. For any $A \in \mathcal{B}(\mathcal{F})$ we write

$$\mathbb{P}(Z \in A) := \mathbb{P}_Z(A) = \mathbb{P}(\{\omega \in \Omega \mid Z(\omega) \in A\}).$$

We discuss some examples of random closed sets.

Examples 1.22 (Some basic examples of random closed sets)

(i) For any fixed $F \in \mathcal{F}$, the mapping $Z : \Omega \to \mathcal{F}, \omega \mapsto F$ is an RCS in E.

(ii) If Z_1, Z_2 are random closed sets in E defined on a common probability space, then also $Z_1 \cup Z_2$ and $Z_1 \cap Z_2$ are RCS's in E (cf. Corollaries 1.5 and 1.8). In particular, if $F \in \mathcal{F}$ and Z is an RCS in E, then $Z \cap F$ is an RCS in E.

Examples 1.23 (Basic examples of random closed sets in $E = \mathbb{R}^d$)

(i) If $\xi : \Omega \to \mathbb{R}^d$ is a *random vector* (i.e., a $(\mathcal{A}, \mathcal{B}(\mathbb{R}^d))$-measurable mapping), then $\{\xi\}$ is an RCS in $\mathbb{R}^d$. (To see this, let $G \in \mathcal{G}$. Then $Z^{-1}(\mathcal{F}_G) = \{Z \cap G \neq \emptyset\} = \{\xi \in G\} \in \mathcal{A}$ and so the assertion is clear from Remark 1.9.)

(ii) Let $\xi_n : \Omega \to \mathbb{R}^d$, $n \in \mathbb{N}$, be random vectors such that the set $Z(\omega) := \{\xi_n(\omega) \mid n \in \mathbb{N}\}$ has no accumulation points in $\mathbb{R}^d$ for almost all $\omega \in \Omega$. Then Z is an RCS in $\mathbb{R}^d$.

(iii) Let Z be an RCS in $\mathbb{R}^d$, $t \in \mathbb{R}^d$ and $\vartheta \in SO_d$. Then $Z+t$ and ϑZ are random closed sets. (Here SO_d is the group of rotations in $\mathbb{R}^d$. Translation and rotation of an RCS are defined in the obvious way by $(Z+t)(\omega) := Z(\omega)+t$ and $(\vartheta Z)(\omega) := \vartheta(Z(\omega))$, $\omega \in \Omega$.)

(iv) Let $C_0, C_1 \in \mathcal{C}^d$ be two compact sets, $\Omega = \{0, 1\}$ and $\mathbb{P}$ be defined by $\mathbb{P}(\{0\}) := p$ for some $p \in [0, 1]$. Then an RCS $Z : \Omega \to \mathcal{F}^d$ is given by setting $Z(0) := C$ und $Z(1) := D$. Z describes a random experiment, in which C_0 is chosen with probability p and C_1 with probability $1 - p$.

(v) Let R be a nonnegative random variable. Then $Z : \Omega \to \mathcal{F}^d, \omega \mapsto B(0, R(\omega))$ defines an RCS. Z generates a (closed) ball with random radius R.

Example 1.24 For $n \in \mathbb{N}$, let $\xi_1, \xi_2, \ldots, \xi_n$ be i.i.d. random vectors $\xi_n : \Omega \to \mathbb{R}^d$, which are uniformly distributed on the unit ball $B(0, 1) \subset \mathbb{R}^d$. Then $Z := \operatorname{conv}\{\xi_1, \ldots, \xi_n\}$ defines a random compact set in $\mathbb{R}^d$. Moreover, $Z(\omega) \in \mathcal{K}^d$ almost surely, where $\mathcal{K}^d$ is the family of all compact convex sets in $\mathbb{R}^d$. Z generates a random convex polytope contained in the unit ball with a random number of vertices ($\leq n$).

We will see more examples later on (see also Work sheet 3). In the introduction we formulated the aim to model complex homogeneous structures by composing them from small building blocks. The following definition provides an outlook to one of the fundamental models of this course and gives some motivation for the things to come. For the moment it mainly serves as fundamental scheme to generate unbounded random closed sets.

Definition 1.25 (Marked point process and germ-grain model)
For each $n \in \mathbb{N}$, let ξ_n be a random point in $\mathbb{R}^d$ and Z_n a random element in $\mathcal{C}^d$ (i.e. a random compact set). Let τ be an $(\mathbb{N}_0 \cup \{\infty\})$-valued random variable. Assume that $\operatorname{card}\{n \in \mathbb{N} \colon \xi_n \in C, n \leq \tau\} < \infty$ $\mathbb{P}$-almost surely for each $C \in \mathcal{C}^d$ and that $\mathbb{P}$-a.s. $\xi_m \neq \xi_n$ for $m < n \leq \tau$. Then the collection

$$\Psi := \{(\xi_n, Z_n) \colon 1 \leq n \leq \tau\}$$

of elements of $\mathbb{R}^d \times \mathcal{C}^d$ is called a *marked point process* in $\mathbb{R}^d$ with *mark space* $\mathcal{C}^d$. The associated *germ-grain model* is the union set

$$Z := \bigcup_{n=1}^{\tau} (Z_n + \xi_n). \tag{1.1}$$

Here the ξ_n are called *germs*, the Z_n are the *primary grains* and $Z_n + \xi_n := \{x + \xi_n \colon x \in Z_n\}$ are the *secondary grains*. Moreover, the set $\Phi := \{\xi_n \colon 1 \leq n \leq \tau\}$ will be called a *point process* in $\mathbb{R}^d$. The random variable τ describes the number of (marked) points of Ψ and at the same time the number of points in Φ.

The above definition is preliminary. It will be modified and specified later on. For the moment we ignore all measurability issues. Later on, it will turn out to be more elegant to view (marked) point processes as random measures rather than collections of points. This will for instance allow multiple points and to integrate w.r.t. a point process. A germ-grain model Z can indeed be an unbounded RCS even if all the grains Z_n are bounded. For this τ should be ∞ with probability one. The above finiteness assumption on the number of points falling into compact sets ensures that the points of Φ do not accumulate, which is necessary (but not sufficient) to obtain a closed union set almost surely. We give two specific examples of germ-grain models.

Example 1.26 Let $(\xi_n)_{n\in\mathbb{N}}$ be the points of the lattice $\mathbb{Z}^2$ (in some fixed enumeration) and let $Z_n := B(0, R_n)$, $n \in \mathbb{N}$ be random balls in $\mathbb{R}^2$ centred at 0 whose radii R_n are random variables that are uniformly distributed on the interval $[0, 1]$. Assume that the sequence (R_n) is independent. Then $\Psi = \{(\xi_n, Z_n) : n \in \mathbb{N}\}$ is a marked point process (with $\tau = \infty$) and $Z = \bigcup_{n=1}^{\infty}(Z_n + \xi_n)$ is a germ-grain model. The balls Z_n are the primary grains and the balls $Z_n + \xi_n = B(\xi_n, R_n)$ are the secondary grains. Then Z is an unbounded RCS.

Example 1.27 Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of independent random points that are uniformly distributed in the window $W := [0, 1]^2 \subset \mathbb{R}^2$. Let τ be a Poisson distributed random variable with parameter $\lambda > 0$. Then $\Psi^0 = \{(\xi_n, B(0, 1)) : 1 \leq n \leq \tau\}$ is a marked point process (with deterministic marks $Z_n = B(0, 1) \in \mathcal{C}^2$). The resulting germ-grain model $Z^0 := \bigcup_{n=1}^{\tau}(Z_n + \xi_n)$ is a (bounded) RCS. An unbounded RCS Z can be obtained by considering a sequence $(Z^i)_{i\in\mathbb{N}}$ of independent copies of Z^0 and moving them to the different cells of the square lattice, i.e. $Z := \bigcup_{i=1}^{\infty} Z^i + x_i$, where $(x_i)_{i\in\mathbb{N}}$ are the points of the lattice $\mathbb{Z}^2$ (in some enumeration).

The following notions will turn out to be particularly useful in this respect, as they make precise the idea of homogeneity for a random closed set in $\mathbb{R}^d$: any translated copy of Z should be equal to Z in distribution.

Definition 1.28 Let Z be an RCS in $\mathbb{R}^d$. Z is called *stationary*, if and only if for all translation vectors $t \in \mathbb{R}^d$

$$\mathbb{P}_{Z+t} = \mathbb{P}_Z.$$

Z is called *isotropic*, if and only if for all $\vartheta \in SO_d$

$$\mathbb{P}_{\vartheta Z} = \mathbb{P}_Z.$$

While a bounded RCS may be isotropic (see e.g. Example 1.23(v)), stationarity is a property which necessarily leads to unbounded structures, as can be seen from the next

statement below. But before let us have a look at the unbounded random structures in the Examples 1.26 and 1.27. The RCS Z generated in Example 1.26 is neither stationary nor isotropic. The distribution of Z is easily seen to be invariant with respect to translations by vectors $t \in \mathbb{Z}^2$ with integer coordinates but not by other vectors $t \in \mathbb{R}^2 \setminus \mathbb{Z}^2$. Similarly, rotations by multiples of $\frac{\pi}{2}$ leave the distribution invariant, but not rotations by any other angles. The random closed set Z generated in Example 1.27 is in fact stationary and isotropic, which is not obvious at the moment but will be proved later.

Theorem 1.29 *Let Z be a stationary random compact set in $\mathbb{R}^d$. Then $\mathbb{P}(Z = \emptyset) = 1$.*

Proof. For a contradiction assume that $\mathbb{P}(Z \neq \emptyset) > 0$. Then, by conditioning on the event $Z \neq \emptyset$, we can generate another random compact set Z', the distribution of which is given by $\mathbb{P}_{Z'} := \mathbb{P}_Z(\cdot \mid Z \neq \emptyset) = \mathbb{P}(Z \in \cdot, Z \neq \emptyset)/\mathbb{P}(Z \neq \emptyset)$. Note that $\mathbb{P}(Z' \neq \emptyset) = 1$. The stationarity of Z implies that Z' is stationary as well. Indeed, we have for any $A \in \mathcal{B}(\mathcal{F}^d)$ and any $t \in \mathbb{R}^d$

$$\begin{aligned}\mathbb{P}_{Z'+t}(A) &= \frac{\mathbb{P}(Z + t \in A,\, Z \neq \emptyset)}{\mathbb{P}(Z \neq \emptyset)} = \frac{\mathbb{P}(Z + t \in A,\, Z + t \neq \emptyset)}{\mathbb{P}(Z \neq \emptyset)} \\ &= \frac{\mathbb{P}(Z \in A,\, Z \neq \emptyset)}{\mathbb{P}(Z \neq \emptyset)} = \mathbb{P}_{Z'}(A)\,.\end{aligned}$$

Therefore, it is sufficient to produce a contradiction in the case $\mathbb{P}(Z \neq \emptyset) = 1$. Denote by $\ell(B)$ the lexicographic minimum of a (nonempty!) compact set $B \subset \mathbb{R}^d$. Observe that $\ell(B + x) = \ell(B) + x$, $x \in \mathbb{R}^d$. Since $Z \neq \emptyset$ almost surely, $\ell(Z)$ is a well defined random vector and, by the stationarity, $\ell(Z) + x = \ell(Z + x) \stackrel{d}{=} \ell(Z)$. Denote by $\ell_1(B)$ the first coordinate of $\ell(B)$. Then we have

$$p := \mathbb{P}(\ell_1(Z) \in [0, 1)) = \mathbb{P}(\ell_1(Z) \in [n, n + 1)), \qquad n \in \mathbb{Z},$$

and this implies

$$\begin{aligned}1 = \mathbb{P}(\ell_1(Z) \in \mathbb{R}) &= \mathbb{P}\left(\ell_1(Z) \in \bigcup_{n\in\mathbb{Z}} [n, n + 1)\right) \\ &= \sum_{n\in\mathbb{Z}} \mathbb{P}\left(\ell_1(Z) \in [n, n + 1)\right) = \sum_{n\in\mathbb{Z}} p,\end{aligned}$$

which is impossible. Hence we have to dismiss the initial hypothesis $\mathbb{P}(Z \neq \emptyset) > 0$. □

In the previous statement we have seen that stationarity is a property of unbounded random closed sets. But stationarity is much more than unboundedness. Stationarity can be interpreted as the property that any point $t \in \mathbb{R}^d$ has exactly the same chance of being contained in a random closed set. The probability $\mathbb{P}(t \in Z)$ that a given point $t \in \mathbb{R}^d$ is contained in an RCS Z can also be reinterpreted as the proportion of space covered by Z. This leads to the notion of volume fraction.

Definition 1.30 Let Z be a stationary random closed set. Then the quantity

$$p_Z := \mathbb{E}[\lambda_d(Z \cap [0, 1]^d)]$$

is called *volume fraction* of Z.

It is not difficult to see that $\lambda_d(Z \cap [0,1]^d)$ is a (nonnegative) random variable, see also Work sheet 3. Hence the volume fraction p_Z is a well defined expectation. Moreover, it is indeed a 'fraction', since clearly $\lambda_d(Z(\omega) \cap [0,1]^d) \in [0,1]$ for any $\omega \in \Omega$, and therefore $0 \le p_Z \le 1$. The following statement relates the volume fraction to the probability that a given point is contained in Z.

Theorem 1.31 *Let Z be a stationary RCS with volume fraction p_Z. Then*

$$\mathbb{E}[\lambda_d(Z \cap B)] = p_Z \lambda_d(B), \quad B \in \mathcal{B}^d,$$

and

$$p_Z = \mathbb{P}(t \in Z), \quad t \in \mathbb{R}^d.$$

Proof. See Work sheet 3. □

1.3 The capacity functional

For a (real-valued) random variable ξ it is well known that the distribution function

$$F_\xi(t) := \mathbb{P}(\xi \le t) = \mathbb{P}(\{\xi\} \cap (-\infty, t] \ne \emptyset), \ t \in \mathbb{R},$$

of ξ determines the distribution $\mathbb{P}_\xi$ of ξ uniquely, which is essentially due to the fact that the half-lines $(-\infty, t]$, $t \in \mathbb{R}$ generate the Borel σ-algebra $\mathcal{B}(\mathbb{R})$. We discuss a similar concept for random closed sets. It is based on the observation that the distribution of a RCS Z can be characterized in terms of the hitting probabilities of compact sets.

Definition 1.32 Let Z be an RCS in E. The mapping

$$T_Z : \mathcal{C} \to [0,1], \qquad C \mapsto T_Z(C) := \mathbb{P}(Z \cap C \ne \emptyset),$$

is called *capacity functional* of Z (or *„hitting"-functional* or *Choquet capacity*).

Observe that $0 \le T_Z \le 1$ and $T_Z(\emptyset) = 0$. The power of the capacity functional lies in its ability to determine the distribution of an RCS.

Theorem 1.33 *Let Z, Z' be two random closed sets in E. Then, $Z \stackrel{d}{=} Z'$ if and only if $T_Z = T_{Z'}$.*

Proof. "⇒": $Z \stackrel{d}{=} Z'$ means that $\mathbb{P}(Z \in A) = \mathbb{P}(Z' \in A)$ for each set $A \in \mathcal{B}(\mathcal{F})$. Thus we have in particular, for each $C \in \mathcal{C}$ (and the associated 'hitting set' $\mathcal{F}_C$),

$$T_Z(C) = \mathbb{P}(Z \cap C \ne \emptyset) = \mathbb{P}(Z \in \mathcal{F}_C) = \mathbb{P}(Z' \in \mathcal{F}_C) = \mathbb{P}(Z' \cap C \ne \emptyset) = T_{Z'}(C).$$

"⇐": The family $\{\mathcal{F}^K : K \in \mathcal{C}\}$ is an intersection stable generator of $\mathcal{B}(\mathcal{F})$ (see Remark 1.9), and for such sets $\mathcal{F}^K$ we obviously have

$$\mathbb{P}_Z(\mathcal{F}^K) = \mathbb{P}(Z \in \mathcal{F}^K) = 1 - \mathbb{P}(Z \in \mathcal{F}_K) = 1 - T_Z(K) = 1 - T_{Z'}(K) = \mathbb{P}_{Z'}(\mathcal{F}^K).$$

Therefore, $\mathbb{P}_Z = \mathbb{P}_{Z'}$ follows from the uniqueness theorem for measures. □

In $\mathbb{R}^d$, Theorem 1.33 can for instance be used to derive a criterion for stationarity and isotropy of an RCS in terms of the capacity functional.

Definition 1.34 We call a functional $T : \mathcal{C}^d \to \mathbb{R}$ *translation invariant* if and only if $T(C + t) = T(C)$ for all $t \in \mathbb{R}^d$ and all $C \in \mathcal{C}^d$. T is called *rotation invariant* if and only if $T(\vartheta C) = T(C)$ for each $C \in \mathcal{C}^d$ and all rotations $\vartheta \in SO_d$.

The volume functional $V_d : \mathcal{C}^d \to \mathbb{R}$, $C \mapsto \lambda_d(C)$ is an example of a translation and rotation invariant functional, since the Lebesgue measure of a set does not change under translations and rotations. Also the surface area or the number of edges (if defined, i.e., for polytopes) have these properties. We will get to know more such functionals and we will study them more systematically later on. Now we show that stationarity (isotropy) of an RCS is equivalent to the translation (rotation) invariance of its capacity functional.

Corollary 1.35 *Let Z be an RCS in $\mathbb{R}^d$. Then Z is stationary if and only if T_Z is translation invariant. Z is isotropic if and only if T_Z is rotation invariant.*

Proof. If Z is stationary, then we have for any $t \in \mathbb{R}^d$ and $C \in \mathcal{C}$

$$T_Z(C+t) = \mathbb{P}(Z \cap (C+t) \neq \emptyset) = \mathbb{P}((Z-t) \cap C \neq \emptyset) = \mathbb{P}(Z \cap C \neq \emptyset) = T_Z(C),$$

which shows that T_Z is translation invariant. For the reverse implication assume that T_Z is translation invariant. Then, for any $t \in \mathbb{R}^d$ and $C \in \mathcal{C}$,

$$T_{Z+t}(C) = \mathbb{P}((Z+t) \cap C \neq \emptyset) = \mathbb{P}(Z \cap (C-t) \neq \emptyset) = T_Z(C-t) = T_Z(C).$$

But this means $T_{Z+t} = T_Z$ for any $t \in \mathbb{R}^d$ which, by Theorem 1.33, implies $Z + t \stackrel{d}{=} Z$ for $t \in \mathbb{R}^d$, that is, the stationarity of Z. The second assertion follows with a similar argument. □

We refer to Work sheet 3 for some specific examples of RCS, for which the capacity functional can be computed. We will use it as a tool later on. However, in general it can be difficult to determine the capacity functional. Other functionals like the volume fraction introduced in the previous section or the contact distribution function (to be discussed later) are easier to handle, but in general they do not characterize the distribution of an RCS.

2 Random measures and point processes

In the previous chapter we got to know random closed sets as the basic unifying concept behind essentially all random geometric structures. Now we will study random measures and point processes. In this course they serve as tools to generate specific random sets. Roughly speaking, a random measure is a measure that arises as the outcome of a random experiment. Mathematically, we will define a random measure as a random variable in a suitable space of measures. Similarly, a point process is roughly a point pattern, that is the outcome of a random experiment. We will view point patterns as counting measures and in this way point processes can be interpreted as special random measures whose realizations are counting measures. The natural setting for the definition of random measures and point processes is an arbitrary separable metric space $(\mathbb{X}, \rho)$ as the basic set on which the measures live. Later on $\mathbb{X}$ will be chosen to be $\mathbb{R}^d$ most of the time.

In the subsequent sections we will also get to know Poisson point processes as a special class of particularly nice and easy to handle point processes, and marked point process as a way to add more information to the points of a point process. We conclude with some considerations on the stationarity of random measures and point processes.

2.1 Measures and counting measures

In the sequel, $(\mathbb{X}, \rho)$ will always denote a separable metric space and $\mathcal{B}(\mathbb{X}) =: \mathcal{X}$ will denote its Borel σ-algebra.

Definition 2.1 A set $B \subset \mathbb{X}$ is called *bounded*, if and only if

$$\rho(B) := \sup\{\rho(x, y) \colon x, y \in B\} < \infty.$$

The number $\rho(B)$ is called the *diameter* of B. We denote by $\mathcal{X}_b := \{B \in \mathcal{X} \colon B \text{ bounded}\}$ the family of bounded sets. A set $A \subset \mathbb{X}$ is called *locally finite*, if and only if $\mathrm{card}(A \cap B) < \infty$, for each $B \in \mathcal{X}_b$. Here $\mathrm{card}(A)$ denotes the cardinality of the set A.

Recall that every compact (and also every relatively compact) subset of X is bounded. When we speak of measures on $\mathbb{X}$, we will always mean measures on the Borel σ-algebra $\mathcal{X}$. Recall that a measure μ on $\mathbb{X}$ is called *locally finite*, if and only if $\mu(B) < \infty$ for $B \in \mathcal{X}_b$.

Definition 2.2 We denote by

$$M(\mathbb{X}) := \{\mu \colon \mu \text{ is a locally finite measure on } \mathbb{X}\}$$

the set of all *locally finite measures* on $\mathbb{X}$, by

$$N(\mathbb{X}) := \{\varphi \in M(\mathbb{X}) \colon \varphi(B) \in \mathbb{N}_0 \text{ for } B \in \mathcal{X}_b\}$$

the set of all locally finite *counting measures* on $\mathbb{X}$, and by

$$N_s(\mathbb{X}) := \{\varphi \in N(\mathbb{X}) \colon \varphi(\{x\}) \in \{0, 1\} \text{ for } x \in \mathbb{X}\}$$

the set of all *simple* counting measures on $\mathbb{X}$.

Let $\mathcal{M}(\mathbb{X})$ be the smallest σ-algebra on $M(\mathbb{X})$ for which the mappings $\mu \mapsto \mu(B)$, for $B \in \mathcal{X}$ ('evaluation functionals') are measurable. That is, $\mathcal{M}(\mathbb{X})$ is the smallest σ-algebra, which contains all sets of the form $\{\mu \colon \mu(B) \in C\}$, $B \in \mathcal{X}$, $C \in \mathcal{B}([0, \infty])$. Moreover, let

$$\mathcal{N}(\mathbb{X}) := \mathcal{M}(\mathbb{X}) \cap N(\mathbb{X}) \quad \text{and} \quad \mathcal{N}_s(\mathbb{X}) := \mathcal{M}(\mathbb{X}) \cap N_s(\mathbb{X})$$

be the corresponding trace σ-algebras on $N(\mathbb{X})$ and $N_s(\mathbb{X})$, respectively.

Observe that, for any $\mu \in M(\mathbb{X})$, μ assumes finite values on compact sets.

Definition 2.3 For each fixed $n \in \mathbb{N}$, let $\{B_{n,i} : i \in \mathbb{N}\}$ be a partition of $\mathbb{X}$ into sets from $\mathcal{X}_b$, that is $B_{n,i} \in \mathcal{X}_b$, $B_{n,i} \cap B_{n,j} = \emptyset$ for $i \neq j$ and $\bigcup_i B_{n,i} = \mathbb{X}$. We call a sequence $(\{B_{n,i} : i \in \mathbb{N}\})_{n\in\mathbb{N}}$ of such partitions a *directed sequence of partitions*, if each $B \in \{B_{n,i} : i \in \mathbb{N}\}$ is a union of sets in $\{B_{n+1,i} : i \in \mathbb{N}\}$ and if $\sup\{\rho(B_{n,i}) : i \in \mathbb{N}\} \to 0$, as $n \to \infty$.

For $\mathbb{X} = \mathbb{R}^d$, for instance, the translates $C + t$, $t \in \mathbb{Z}^d$ of the (half-open) unit cube $C := [0, 1)^d$ form a partition of $\mathbb{R}^d$. For each $n \in \mathbb{N}$, the scaled copies $2^{-n}(C + t)$, $t \in \mathbb{Z}^d$ form another partition of $\mathbb{R}^d$ in which all sets have diameter $\rho_n = 2^{-n}\rho(C)$. It is easy to see that these partitions form a directed sequence.

Remark 2.4 In any separable metric space $(\mathbb{X}, \rho)$ there exists a directed sequence of partitions.

Definition 2.5 A measure μ on $\mathbb{X}$ is called *diffuse* or *atom-free*, if $\mu\{x\} := \mu(\{x\}) = 0$ for all $x \in \mathbb{X}$. A point $x \in \mathbb{X}$ is called an *atom* of a measure μ, if and only if $\mu\{x\} > 0$. The quantity $\mu\{x\}$ is also called the *mass* of μ at x.

Examples 2.6 (1) Let $x \in \mathbb{X}$. The *Dirac measure* δ_x at x, defined by

$$\delta_x(B) := \mathbb{1}_B(x), \quad B \in \mathcal{X},$$

has an atom at x.

(2) In $\mathbb{X} = \mathbb{R}^d$ the Lebesgue measure is diffuse as well as any measure with a Lebesgue density.

Remark 2.7 Let $\mu \in M(\mathbb{X})$. For any $B \in \mathcal{X}_b$ and any $r > 0$, μ has only finitely many atoms in B with mass at least r. Therefore, in total μ has at most countably many atoms in B und thus in $\mathbb{X}$.

It is clear that any finite (and possibly weighted) sum of Dirac measures is a counting measure, that is, an element of $N(\mathbb{X})$. The same is true for any countable sum of such measures, provided that the atoms do not accumulate in $\mathbb{X}$. The following statement provides a certain converse. It implies in particular that any counting measure can be written as a (weighted) sum of Dirac measures.

Theorem 2.8 *Any $\mu \in M(\mathbb{X})$ can be represented in the form*

$$\mu = \mu_c + \sum_{i=1}^{\tau} a_i \delta_{x_i},$$

where μ_c is a diffuse measure, $\tau \in \mathbb{N}_0 \cup \{\infty\}$ and, for each i, $a_i > 0$ and $x_i \in \mathbb{X}$. The mappings $\mu \mapsto (\mu_c, \tau)$ and $\mu \mapsto (a_1, x_1, a_2, x_2, \ldots)$ can be chosen to be measurable, where we set $(a_i, x_i) := (0, x_0)$ for $i > \tau$, if necessary, with a fixed $x_0 \in \mathbb{X}$.

Proof. We give the main idea of the proof but we leave out some details and we refer to a result in [4] for one essential step. For $r > 0$, we define a measure μ_r^* by

$$\mu_r^*(B) := \text{card}\{x \in B : \mu(\{x\}) \geq r\} \quad \text{for } B \in \mathcal{X}.$$

Clearly, $\mu_r^* \in N_s(\mathbb{X})$. Let $(\{B_{n,i} : i \in \mathbb{N}\})_{n\in\mathbb{N}}$ be a directed sequence of partitions of $\mathbb{X}$ with sets from $\mathcal{X}_b$. One can show that

$$\mu_r^*(B) = \lim_{n\to\infty} \sum_{i\in\mathbb{N}} \mathbb{1}\{\mu(B_{n,i} \cap B) \geq r\} \quad \text{for } B \in \mathcal{X}_b.$$

Therefore, the mapping $\mu \mapsto \mu_r^*$ is measurable.

Let $y \notin \mathbb{X}$ be a fixed additional element. The proof of Lemma 2.1.6 in [4] shows that there exists a sequence of measurable maps $\zeta_i : N_s(\mathbb{X}) \to \mathbb{X} \cup \{y\}$, $i \in \mathbb{N}$, such that

$$\eta = \sum_{i \in \mathbb{N}} \delta_{\zeta_i(\eta)}, \qquad \text{for any } \eta \in N_s(\mathbb{X}).$$

(That is, the maps ζ_i find an enumeration of the atoms of a counting measure η and ζ_i picks the i-th atom in this enumeration.) We define

$$\zeta_{n,i}(\mu) := \zeta_i\left(\mu^*_{\frac{1}{n}}\right), \qquad i, n \in \mathbb{N}.$$

Then the mapping $\mu \mapsto \zeta_{n,i}(\mu)$ is measurable, and every atom of μ will be contained in the sequence $(\zeta_{n,i}(\mu) : n, i \in \mathbb{N})$ (possibly many times). Passing to a subsequence, we derive an enumeration $(\zeta'_1(\mu), \zeta'_2(\mu), \ldots)$ of the atoms of μ by means of measurable maps $\zeta'_i : M(\mathbb{X}) \to \mathbb{X} \cup \{y\}$, $i \in \mathbb{N}$, chosen in such a way that

with $\tau := \lim_{r \to 0} \mu_r^*(\mathbb{X}) \in \mathbb{N}_0 \cup \{\infty\}$ one has $\zeta'_i(\mu) \in \mathbb{X}$ for $i < \tau + 1$ and $\zeta'_i(\mu) = y$ for $i > \tau$. Now set $x_i := \zeta'_i(\mu)$ and $a_i := \mu(\{x_i\})$ for $i \in \mathbb{N}$, where $\mu(\{y\}) := 0$. Finally, let $\mu_c := \mu - \sum_{i=1}^{\tau} a_i \delta_{x_i}$. Then also the mapping $\mu \mapsto \mu_c$ is measurable and μ_c is diffuse. □

Remark 2.9 In the proof of Theorem 2.8 one uses (for showing the measurability of a_i) that the mapping

$$\mathbb{X} \times M(\mathbb{X}) \to [0, \infty], \qquad (x, \mu) \mapsto \mu(\{x\}),$$

is measurable. More generally, it holds that for any measurable function $f : \mathbb{X} \times \mathbb{X} \to [0, \infty]$ the mapping

$$\mathbb{X} \times M(\mathbb{X}) \to [0, \infty], \qquad (x, \mu) \mapsto \int f(x, y)\, \mu(dy),$$

is measurable. This can be shown for indicators $f = \mathbb{1}_A$ of measurable sets A using a monotone class argument and then extended to the general case by algebraic induction.

Corollary 2.10 The sets $N(\mathbb{X})$ and $N_s(\mathbb{X})$ are measurable subsets of $M(\mathbb{X})$.

Proof. Observe that $\mu \in N(\mathbb{X})$ if and only if $\mu_c = 0$ and $a_i \in \mathbb{N}_0$ for $i \in \mathbb{N}$. Furthermore, $\mu \in N_s(\mathbb{X})$ if and only if $\mu \in N(\mathbb{X})$ and $a_i \in \{0, 1\}$ for $i \in \mathbb{N}$. □

2.2 Random measures and point processes

Having now available σ-algebras on the family $M(\mathbb{X})$ of measures and on the family $N(\mathbb{X})$ of counting measures, we are in a position to define what we mean by a random element in $M(\mathbb{X})$ and $N(\mathbb{X})$, respectively.

Definition 2.11 Let $(\Omega, \mathcal{A}, \mathbb{P})$ be some probability space.

(i) A *random measure* η on $\mathbb{X}$ is a measurable mapping $\eta \colon \Omega \to M(\mathbb{X})$.

(ii) A *point process* Φ on $\mathbb{X}$ is a measurable mapping $\Phi \colon \Omega \to N(\mathbb{X})$.

(iii) A point process Φ is called *simple*, if $\mathbb{P}(\Phi \in N_s(\mathbb{X})) = 1$.

Remarks 2.12 (i) Sometimes a point process on $\mathbb{X}$ is defined to be a random measure Φ with $\mathbb{P}(\Phi \in N(\mathbb{X})) = 1$, which is essentially equivalent to the definition above. It underlines that a point process may be viewed as a random measure with special properties.

(ii) For any $\varphi \in N(\mathbb{X})$ the support of φ is defined by

$$\operatorname{supp} \varphi := \{x \in \mathbb{X} \colon \varphi(\{x\}) > 0\}.$$

Note that there is a one-to-one correspondence between simple counting measures $\varphi \in N_s(\mathbb{X})$ and their support $\operatorname{supp} \varphi$.

(iii) If η ist a random measure, then $\eta(\omega)$ is a measure for each $\omega \in \Omega$ and we write $\eta(\omega, B) := \eta(\omega)(B)$ for any $\omega \in \Omega$ and $B \in \mathcal{X}$. Moreover, we write $\eta(B)$ for the mapping $\omega \mapsto \eta(\omega, B)$ (which is a random variable, as the next item clarifies).

(iv) A mapping $\eta \colon \Omega \to M(\mathbb{X})$ is measurable, if and only if the mapping $\omega \mapsto \eta(\omega, B)$ is measurable for each $B \in \mathcal{X}$, see Work sheet 4.

Remark 2.13 Let Φ be a point process on $\mathbb{X}$. According to Theorem 2.8 (and in particular due to the stated measurability of the involved mappings), there are random variables $\tau \in \mathbb{N} \cup \{\infty\}$, $A_i \in \mathbb{N}$ and $\xi_i \in \mathbb{X}$ (with $\xi_i \neq \xi_j$, for $i < j \leq \tau$) such that $\Phi = \sum_{i=1}^{\tau} A_i \delta_{\xi_i}$. Alternatively, one can split the atoms into several ones with unit masses and represent Φ as a sum of Dirac masses at (not necessarily disjoint) random points. That is, there are random variables $\tau' \in \mathbb{N} \cup \{\infty\}$ and random points ξ_i' in $\mathbb{X}$ (not necessarily disjoint) such that $\Phi = \sum_{i=1}^{\tau'} \delta_{\xi_i'}$. Often point processes Φ are directly defined this way as a sum of (Dirac masses of) randomly generated points.

Let us now discuss some examples of random measures and point processes. In the first example (and also later on), we use the j-dimensional *Hausdorff measure* $\mathcal{H}^j$ on $\mathbb{R}^d$. For the definition we refer to Section 14.5 in [6]. If $B \subset \mathbb{R}^d$ is a convex set, whose affine hull has dimension $k \geq 1$, then $\mathcal{H}^k(B)$ is the k-dimensional Lebesgue measure of B (measured in the affine hull). Moreover, $\mathcal{H}^j(B) = 0$ for $j < k$ and $\mathcal{H}^j(B) = \infty$ für $j > k$. Further, $\mathcal{H}^0$ is the counting measure on $\mathbb{R}^d$. If B is a j-dimensional smooth surface, then $\mathcal{H}^j(B)$ is the (differential geometric) surface area of B.

Example 2.14 Let Z be a random closed set in $\mathbb{R}^d$. Then

$$\eta(B) := \lambda_d(Z \cap B), \quad B \in \mathcal{B}^d,$$

defines a random measure η on $\mathbb{R}^d$. This follows from Theorem 1.12 (iv) and Fubini's Theorem (see also the proof of Theorem 1.31 discussed on Work sheet 3). Similarly, the mapping $B \mapsto \mathcal{H}^{d-1}(\partial Z \cap B)$ defines a random measure on $\mathbb{R}^d$, provided the measure $\mathcal{H}^{d-1}(\partial Z \cap \cdot)$ is locally finite. Here $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure on $\mathbb{R}^d$. The measurability follows from Theorem 1.12 (ii) together with Corollary 2.1.4 in [7].

Example 2.15 Let $\{X_t \colon t \in \mathbb{R}^d\}$ be an $\mathbb{R}_+$-valued random field such that $(\omega, t) \mapsto X_t(\omega)$ is measurable. Then

$$\eta(B) := \int_{\mathbb{R}^d} \mathbb{1}_B(t) X_t \, dt, \quad B \in \mathcal{B}^d$$

defines a random measure on $\mathbb{R}^d$, provided X_t satisfies some integrability conditions.

Example 2.16 Let $m \in \mathbb{N}$ and let $X_1, \dots, X_m$ be i.i.d. random variables in $\mathbb{X}$ with distribution $\mathbb{V} = \mathbb{P}(X_i \in \cdot)$. Then $\Phi := \sum_{i=1}^m \delta_{X_i}$ is a point process, called *Binomial process* with parameters m and $\mathbb{V}$. Indeed, $\Phi(B)$ is measurable for any $B \in \mathcal{X}$, since

$$\Phi(B) = \sum_{i=1}^m \mathbb{1}\{X_i \in B\},$$

showing that Φ is a point process (by Remark 2.12 (iv)). Moreover, we have

$$\mathbb{P}(\Phi(B) = k) = \binom{m}{k} \mathbb{V}(B)^k (1 - \mathbb{V}(B))^{m-k}$$

for $k \in \{0, 1, \dots, m\}$, and $\mathbb{P}(\Phi(B) = k) = 0$ für $k > m$. That is, $\Phi(B) \sim \text{Bin}(m, \mathbb{V}(B))$ for $B \in \mathcal{X}$, justifying the name 'Binomial' process for Φ.

Example 2.17 Let $X_1, X_2, \dots$ be i.i.d. random variables in $\mathbb{X}$ and let τ be an $\mathbb{N}_0$-valued random variable that is independent of the sequence $(X_n)_{n\in\mathbb{N}}$. Then

$$\Phi := \sum_{i=1}^{\tau} \delta_{X_i}$$

is a point process, called *mixed Binomial process* with parameters τ and $\mathbb{V}$. (For a proof that Φ is indeed a point process see also Work sheet 4.)

The above definition of mixed Binomial processes will turn out to be an essential tool for practically constructing examples of point processes. In particular, it can be used to generate the germs for germ-grain models as introduced in Definition 1.25. The assumed independence of the random points will turn out to be very helpful in the further analysis of the generated structures. We are also in the position now to introduce homogeneous Poisson point processes in $\mathbb{R}^d$.

Definition 2.18 A point process $\Phi : \Omega \to N(\mathbb{R}^d)$ on $\mathbb{R}^d$ is called *homogeneous (or stationary) Poisson process* with intensity $\gamma \geq 0$, if and only if the following conditons are satisfied:

(i) For any $m \geq 2$ and pairwise disjoint Borel sets $B_1, \dots, B_m \in \mathcal{B}^d$, the random variables $\Phi(B_1), \dots, \Phi(B_m)$ are stochastically independent.

(ii)
$$\mathbb{P}(\Phi(B) = k) = \frac{\gamma^k (\lambda_d(B))^k}{k!} e^{-\gamma\lambda_d(B)}, \qquad \text{for any } B \in \mathcal{B}^d \text{ and } k \in \mathbb{N}_0.$$

Here we use the convention that $\infty \cdot e^{-\infty} := 0$.

Note that this definition is axiomatic and not constructive. A priori it is not clear, whether processes exist satisfying all these conditions. On the other hand it is very desirable to have such a strong independence as required by (i). It means that the behaviour of the process Φ in one region B_1 does not affect the behaviour of Φ in another region B_2 disjoint from B_1. The homogeneity is induced by the requirement that the (expected) number of points in a set B is proportional to its volume. Indeed, condition (ii) means that the random variables $\Phi(B)$ are Poisson distributed with parameter $\gamma\lambda_d(B)$. This implies in particular that the expectation $\mathbb{E}\Phi(B) = \gamma\lambda_d(B)$ is directly proportional to the volume. Why we require Poisson distributions here is not so obvious at the moment, but

one can show that this is more or less a canonical choice if one wants the other conditions to be satisfied. In the next subsection we will prove the existence of Poisson processes, construct them explicitly using mixed Binomial processes and study their properties.

Before turning to this, we discuss some further concepts for general point processes and random measures. The *distribution* $\mathbb{P}_\eta$ of a random measure η is a measure on $M(\mathbb{X})$ defined by $\mathbb{P}_\eta : \mathcal{M}(\mathbb{X}) \to [0,1]$, $A \mapsto \mathbb{P}(\eta \in A)$. For a point process Φ we have in particular $\mathbb{P}_\Phi(N(\mathbb{X})) = 1$. The following notion for random measures is the appropriate counterpart to the expectation for real valued random variables.

Definition 2.19 The *intensity measure* Θ of a random measure η is defined by

$$\Theta(B) := \mathbb{E}[\eta(B)], \quad B \in \mathcal{X}.$$

It is easy to see that Θ is indeed a measure on $\mathbb{X}$ justifying the terminology. $\Theta(B)$ quantifies how much mass of η a set B contains on average, giving a rough idea where the mass of realizations of η may be located. In the case of a point process Φ, $\Theta(B)$ can be interpreted as the expected number of points of Φ located in B. Let us compute the intensity measure for some examples.

Example 2.20 (i) For a Binomial process Φ as in Example 2.16, we have

$$\Theta(B) = \mathbb{E}\left[\sum_{i=1}^m \delta_{X_i}(B)\right] = \sum_{i=1}^m \mathbb{E}[\mathbb{1}\{X_i \in B\}] = \sum_{i=1}^m \mathbb{P}(X_i \in B) = m\mathbb{V}(B).$$

(ii) For a mixed Binomial process as in Example 2.17, we have

$$\begin{aligned}\Theta(B) &= \sum_{m=0}^\infty \mathbb{E}\left[\mathbb{1}\{\tau = m\}\sum_{i=1}^m \mathbb{1}\{X_i \in B\}\right] = \sum_{m=0}^\infty \mathbb{P}(\tau = m) m\mathbb{V}(B)\\ &= \mathbb{E}[\tau]\mathbb{V}(B).\end{aligned}$$

(iii) For a homogeneous Poisson process Φ in $\mathbb{R}^d$ with intensity $\gamma \geq 0$, the random variable $\Phi(B)$ is Poisson distributed with parameter $\gamma\lambda_d(B)$, for any $B \in \mathcal{B}^d$, and therefore $\Theta(B) = \mathbb{E}[\Phi(B)] = \gamma\lambda_d(B)$, $B \in \mathcal{B}^d$, that is Θ is a multiple of the Lebesgue measure.

We briefly recall some facts about integration with respect to an arbitrary measure. For a measurable function $f\colon \mathbb{X} \to [-\infty, \infty]$, denote by $f^+ := \max\{0, f\}$ and $f^- := -\min\{0, f\}$ the positive and negative part of f, respectively. Obviously, $f = f^+ - f^-$. Let ν be a measure on $\mathbb{X}$. For $f \geq 0$, the integral $\int_\mathbb{X} f\, d\nu$ is always defined (by means of approximation of f from below with step functions) but it may be infinite. For arbitrary f, we define $\int_\mathbb{X} f\, d\nu := \int_\mathbb{X} f^+\, d\nu - \int_\mathbb{X} f^-\, d\nu$, provided the right hand side is not of the form $\infty - \infty$. In case $\int_\mathbb{X} f^+\, d\nu = \int_\mathbb{X} f^-\, d\nu = \infty$ we set $\int_\mathbb{X} f\, d\nu := 0$. We call f ν-integrable, if and only if $\int_\mathbb{X} f^+\, d\nu < \infty$ and $\int_\mathbb{X} f^-\, d\nu < \infty$ holds. Equivalent to this is the condition $\int_\mathbb{X} |f|\, d\nu < \infty$, where $|f| := f^+ + f^-$.

Theorem 2.21 (Campbell's formula) *Let η be a random measure on $\mathbb{X}$ with intensity measure Θ and let $f\colon \mathbb{X} \to [-\infty, \infty]$ be measurable. Then the integral $\int_\mathbb{X} f\, d\eta$ is a $[-\infty, \infty]$-valued random variable. If additionally $f \geq 0$ or f is Θ-integrable, then*

$$\mathbb{E}\left[\int_\mathbb{X} f(x)\, \eta(dx)\right] = \int_\mathbb{X} f(x)\, \Theta(dx). \tag{2.1}$$

Proof. For nonnegative measurable functions $f \geq 0$, the formula can be shown using algebraic induction: For any measurable set $A \subset \mathbb{X}$ and $f = \mathbb{1}_A$, equation (2.1) reduces to $\mathbb{E}[\eta(A)] = \Theta(A)$, which is true by definition of the intensity measure. Then the additivity of expectation, integral and measure implies that formula (2.1) is also true for step functions, that is, functions of the form $f = \sum_{i=1}^{k} c_i \mathbb{1}_{A_i}$, where $k \in \mathbb{N}$, $c_i \geq 0$ and $A_i \in \mathcal{X}$ for $i = 1, \ldots, k$. Indeed, we have

$$\mathbb{E}\left[\int_{\mathbb{X}} f \, d\eta\right] = \mathbb{E}\left[\sum_{i=1}^{k} c_i \eta(A_i)\right] = \sum_{i=1}^{k} c_i \mathbb{E}[\eta(A_i)] = \sum_{i=1}^{k} c_i \Theta(A_i) = \int_{\mathbb{X}} f d\Theta.$$

For a general nonnegative measurable function f, we approximate f from below by a sequence f_n of step functions such that $f_n \uparrow f$ as $n \to \infty$. Then formula (2.1) follows for such f by monotone convergence.

For a general Θ-integrable f one uses the decomposition into positive and negative part. By the assumed integrability, we have $\int_{\mathbb{X}} |f| \, d\Theta = \int_{\mathbb{X}} f^+ \, d\Theta + \int_{\mathbb{X}} f^- \, d\Theta < \infty$. Therefore, formula (2.1) (applied to the nonnegative functions f^+ and f^-) implies that $\mathbb{E}\left[\int_{\mathbb{X}} f^{\pm}(x) \, \eta(dx)\right] < \infty$. Hence, using again (2.1), we infer that

$$\begin{aligned}\mathbb{E}\left[\int_{\mathbb{X}} f \, d\eta\right] &= \mathbb{E}\left[\int_{\mathbb{X}} f^+ \, d\eta - \int_{\mathbb{X}} f^- \, d\eta\right] = \mathbb{E}\left[\int_{\mathbb{X}} f^+ \, d\eta\right] - \mathbb{E}\left[\int_{\mathbb{X}} f^- \, d\eta\right] \\ &= \int_{\mathbb{X}} f^+ \, d\Theta - \int_{\mathbb{X}} f^- \, d\Theta = \int_{\mathbb{X}} f \, d\Theta. \qquad \square\end{aligned}$$

Remark 2.22 If Φ is a point process of the form $\Phi = \sum_{i=1}^{\tau} \delta_{\xi_i}$, then the integral on the left hand side of equation (2.1) reduces to a sum: $\int f d\Phi = \sum_{i=1}^{\tau} f(\xi_i)$. Therefore, Campbell's formula becomes

$$\mathbb{E}\left[\sum_{i=1}^{\tau} f(\xi_i)\right] = \int_{\mathbb{X}} f \, d\Theta.$$

Our last aim in this section is a characterization theorem for distributions of random measures.

Theorem 2.23 *Let η and η' be random measures on $\mathbb{X}$. Then the following assertions are equivalent.*

(i) $\eta \stackrel{d}{=} \eta'$.

(ii) $\int_{\mathbb{X}} f \, d\eta \stackrel{d}{=} \int_{\mathbb{X}} f \, d\eta'$ *for all measurable* $f \geq 0$.

(iii) $\mathbb{E}\left[\exp\left(-\int_{\mathbb{X}} f \, d\eta\right)\right] = \mathbb{E}\left[\exp\left(-\int_{\mathbb{X}} f \, d\eta'\right)\right]$ *for all measurable* $f \geq 0$.

(iv) $(\eta(B_1), \ldots, \eta(B_m)) \stackrel{d}{=} (\eta'(B_1), \ldots, \eta'(B_m))$ *for all* $m \in \mathbb{N}$ *and* $B_1, \ldots, B_m \in \mathcal{X}_b$.

Proof. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious.

(iii) $\Rightarrow$ (iv): Let $m \in \mathbb{N}$ and $B_1, \ldots, B_m \in \mathcal{X}_b$. Applying (iii) to functions of the form $f = c_1 \mathbb{1}_{B_1} + \ldots + c_m \mathbb{1}_{B_m}$, where $c_i \geq 0$, we get

$$\mathbb{E}\left[\exp\left(-\sum_{i=1}^{m} c_i \eta(B_i)\right)\right] = \mathbb{E}\left[\exp\left(-\sum_{i=1}^{m} c_i \eta'(B_i)\right)\right].$$

This equation can be reinterpreted as an equality between the Laplace transforms of the random vectors $X := (\eta(B_1), \dots, \eta(B_m))$ and $X' := (\eta'(B_1), \dots, \eta'(B_m))$. Hence, (iv) follows from the fact that a nonnegative random vector is determined uniquely by its Laplace transform (see e.g. Bemerkung 2.2.6 in [4]). Alternatively, the Cramér-Wold theorem can be used (cf. e.g. [2, Seite 205]).

(iv) $\Rightarrow$ (i): Let $\mathcal{G}$ be the family of all sets $A \in \mathcal{M}(\mathbb{X})$ of the form

$$A = \{\mu \in M(\mathbb{X}) \colon (\mu(B_1), \dots, \mu(B_m)) \in C\},$$

where $m \in \mathbb{N}$, $B_i \in \mathcal{X}_b$, $C \in \mathcal{B}(\mathbb{R}^m)$. $\mathcal{G}$ is $\cap$-stable and generates $\mathcal{M}(\mathbb{X})$, i.e. $\sigma(\mathcal{G}) = \mathcal{M}(\mathbb{X})$. Condition (iv) implies that

$$\mathbb{P}(\eta \in A) = \mathbb{P}(\eta' \in A)$$

for all $A \in \mathcal{G}$ and therefore, by the Uniqueness theorem for measures, the equality in distribution $\eta \stackrel{d}{=} \eta'$ follows. □

Remark 2.24 In the special case that $\eta = \Phi$ is a point process, then another criterion equivalent to those provided in Theorem 2.23 is obtained by requiring condition (iv) only for pairwise disjoint sets:

(iv') $(\eta(B_1), \dots, \eta(B_m)) \stackrel{d}{=} (\eta'(B_1), \dots, \eta'(B_m))$ for all $m \in \mathbb{N}$ and all pairwise disjoint sets $B_1, \dots, B_m \in \mathcal{X}_b$.

See Work sheet 4 or Lemma 2.2.3 in [4].

Remark 2.25 For any random measure η on $\mathbb{X}$, the functional L_η, defined by

$$L_\eta(f) := \mathbb{E}\left[\exp\left(-\int_{\mathbb{X}} f \, d\eta\right)\right]$$

for any measurable function $f : \mathbb{X} \to [0, \infty]$, is called *Laplace functional* of η. Note that this definition comprises in particular the Laplace functional L_Φ of a point processes Φ. By Theorem 2.23, the Laplace functional of a random measure η determines the distribution of η uniquely. This will be particularly useful in the sequel.

2.3 Poisson processes

We introduce Poisson processes in greater generality than in Definition 2.18. We prove their existence, give an explicit construction and discuss some of their properties. Let $(\mathbb{X}, \rho)$ be a separable space and let as before $\mathcal{X}$ denote the Borel σ-algebra on $\mathbb{X}$.

Definition 2.26 Let Θ be a locally finite measure on $\mathbb{X}$. A point process Φ on $\mathbb{X}$ is called *Poisson process with intensity measure* Θ, if and only if the following conditions hold:

(i) $\Phi(B_1), \dots, \Phi(B_m)$ are stochastically independent random variables whenever $m \in \mathbb{N}$, $m \geq 2$ and $B_1, \dots, B_m \in \mathcal{X}$ are pairwise disjoint.

(ii)
$$\mathbb{P}(\Phi(B) = k) = \frac{\Theta(B)^k}{k!} e^{-\Theta(B)}, \quad B \in \mathcal{X},\ k \in \mathbb{N}_0,$$

where $\infty^k e^{-\infty} := 0$.

We write $\Phi \sim \mathrm{Poiss}(\Theta)$ in this case.

At the moment it is not clear whether for given Θ a Poisson process with this intensity measure exists. This will be established in Theorem 2.31 below. Before we discuss some properties. Observe that the homogeneous Poisson process with intensity γ introduced in Definition 2.18 is a special case of the above definition with $\mathbb{X} = \mathbb{R}^d$ and $\Theta = \gamma\lambda_d$.

Remarks 2.27 (i) Note that condition (ii) implies in particular that $\mathbb{E}[\Phi(B)] = \Theta(B)$ for any $B \in \mathcal{X}$. Hence, Θ is indeed the intensity measure of Φ in the sense of Definition 2.19.

(ii) In case $\Theta(B) = \infty$ for some $B \in \mathcal{X}$ (which may happen for unbounded sets), one has $\mathbb{P}(\Phi(B) = k) = 0$ for each $k \in \mathbb{N}_0$, and therefore

$$\mathbb{P}(\Phi(B) = \infty) = 1.$$

For convenience, we will write $\Phi(B) \sim \mathrm{Poiss}(\infty)$, even though this degenerated random variable is not really Poisson distributed. The above notation $\Phi \sim \mathrm{Poiss}(\Theta)$ is very suggestive as it implies that $\Phi(B) \sim \mathrm{Poiss}(\Theta(B))$ for any $B \in \mathcal{X}$.

(iii) Assume that Φ and Ψ are two Poisson processes with the same locally finite intensity measure Θ. Then Φ und Ψ have the same distribution. This follows immediately from Definition 2.26 (ii) and Remark 2.24. Hence, (in case it exists) a Poisson process with intensity measure Θ is unique in the sense of distribution.

Note that for any point process (or random measure) Φ and any measurable set $B \in \mathcal{X}$, also the restriction Φ_B of Φ to B is again a point process (random measure) (see Work sheet 5). Moreover, the Poisson property is preserved by restrictions:

Remark 2.28 If Φ is a Poisson process on $\mathbb{X}$ with intensity measure Θ and $B_1, B_2, \ldots$ are pairwise disjoint elements of $\mathcal{X}$, then the restrictions $\Phi_{B_1}, \Phi_{B_2}, \ldots$ are independent Poisson processes with intensity measures $\Theta_{B_1}, \Theta_{B_2}, \ldots$ This can be shown using (the proof of) Theorem 2.23 and Remark 2.24, see Worksheet 5.

The next statement provides a characterization of Poisson processes in terms of the Laplace functional introduced in Remark 2.25. It will be particularly useful for proving the existence of Poisson processes. Recall that the Laplace functional L_Φ of a point process Φ is defined by

$$L_\Phi(f) := \mathbb{E}\left[\exp\left(-\int_{\mathbb{X}} f \, d\Phi\right)\right]$$

for any measurable function $f : \mathbb{X} \to [0, \infty]$.

Theorem 2.29 *Let Φ be a point process on $\mathbb{X}$ and Θ a locally finite measure on $\mathbb{X}$. Then the following assertions are equivalent.*

(i) $\Phi \sim \mathrm{Poiss}(\Theta)$.

(ii) *For any measurable function $f : \mathbb{X} \to [0, \infty]$,*

$$L_\Phi(f) = \exp\left(-\int_{\mathbb{X}} (1 - e^{-f(x)}) \, \Theta(dx)\right). \tag{2.2}$$

Proof. Assume that $\Phi \sim \mathrm{Poiss}(\Theta)$. To show that this implies (ii), we first prove (2.2) for step functions. For any $B \in \mathcal{X}$ with $\Theta(B) < \infty$ and $c \geq 0$, we have

$$\mathbb{E}[\exp(-c\Phi(B))] = e^{-\Theta(B)} \sum_{k=0}^{\infty} \frac{\Theta(B)^k}{k!} e^{-ck} = e^{-\Theta(B)} e^{(\Theta(B)e^{-c})} = e^{-\Theta(B)(1-e^{-c})}. \quad (2.3)$$

This implies for any step function $f := c_1 \mathbb{1}_{B_1} + \ldots + c_m \mathbb{1}_{B_m}$ with $c_i \geq 0$ and pairwise disjoint sets $B_1, \ldots, B_m \in \mathcal{X}_b$, that

$$\begin{aligned} L_\Phi(f) &= \mathbb{E}\left[\exp\left(-\int_{\mathbb{X}} f d\Phi\right)\right] = \mathbb{E}\left[\exp\left(-\sum_{i=1}^m c_i \Phi(B_i)\right)\right] \\ &= \mathbb{E}\left[\prod_{i=1}^m \exp(-c_i\Phi(B_i))\right] = \prod_{i=1}^m \mathbb{E}[\exp(-c_i\Phi(B_i))] \\ &= \prod_{i=1}^m \exp(-\Theta(B_i)(1-e^{-c_i})) = \exp\left(-\sum_{i=1}^m \Theta(B_i)(1-e^{-c_i})\right), \end{aligned}$$

where in the second line we have used the independence of the $\Phi(B_i)$ (recall that the B_i are disjoint) and the last but one equality is due to (2.3). We conclude that

$$L_\Phi(f) = \exp\left(-\sum_{i=1}^m \int_{B_i} (1-e^{-c_i})\, d\Theta\right) = \exp\left(-\int_{\mathbb{X}} (1-e^{-f})\, d\Theta\right),$$

where in the last step we have used again the disjointness of the sets B_i. For measurable functions $f\colon \mathbb{X} \to [0,\infty]$ we approximate f as usual from below by a sequence of step functions f_n such that $f_n \uparrow f$. Then (2.2) follows by monotone convergence. This shows that (i) implies (ii).

For a proof of the reverse implication assume that (2.2) holds. Let $B \in \mathcal{X}$ with $\Theta(B) < \infty$. Then $\Phi(B)$ is a nonnegative random variable, whose Laplace transform is given by

$$L_{\Phi(B)}(c) = \mathbb{E}\left[\exp\left(-c\Phi(B)\right)\right] = \mathbb{E}\left[\exp\left(-c\int \mathbb{1}_B d\Phi\right)\right] = L_\Phi(c\mathbb{1}_B)$$

for any $c \geq 0$. Hence, by (2.2),

$$\begin{aligned} L_{\Phi(B)}(c) &= \exp\left(-\int_{\mathbb{X}} \left(1-e^{-c\mathbb{1}_B(x)}\right)\, \Theta(dx)\right) = \exp\left(-\int_B \left(1-e^{-c}\right)\, \Theta(dx)\right) \\ &= \exp\left(-\left(1-e^{-c}\right)\Theta(B)\right) \end{aligned}$$

From (2.3) it is clear, that a Poisson random variable $\xi \sim \mathrm{Poiss}(\Theta(B))$ has exactly the same Laplace transform. Hence, by the uniqueness theorem for the Laplace transform, we conclude that $\Phi(B) \sim \mathrm{Poiss}(\Theta(B))$ for any $B \in \mathcal{X}$ with $\Theta(B) < \infty$. For $\Theta(B) = \infty$, the computations above show that $\mathbb{E}\left[\exp\left(-c\Phi(B)\right)\right] = 0$ for any $c \geq 0$ and therefore $\Phi(B) = \infty$ $\mathbb{P}$-almost surely. This shows that Φ satisfies the second condition of the definition of a Poisson process. It remains to show the independence.

For this let $B_1, \ldots, B_m \in \mathcal{X}_b$ pairwise disjoint and consider the random vector $X := (\Phi(B_1), \ldots, \Phi(B_m))$. Then, again by (2.2) (applied to $f := \sum_{i=1}^m c_i \mathbb{1}_{B_i}$), we infer that the Laplace transform L_X of X at the vector $c = (c_1, \ldots, c_m)$ with $c_i \geq 0$ is given by

$$L_X(c) = \mathbb{E}\left[\exp\left(-\sum_{i=1}^m c_i \Phi(B_i)\right)\right] = \mathbb{E}\left[\exp\left(-\int_{\mathbb{X}} f\, d\Phi\right)\right]$$

$$
\begin{aligned}
&= \exp\left(-\int_{\mathbb{X}} \left(1 - \exp\left(-\sum_{i=1}^{m} c_i \mathbb{1}_{B_i}(x)\right)\right) \Theta(dx)\right) \\
&= \exp\left(-\sum_{i=1}^{m} \int_{B_i} \left(1 - e^{-c_i}\right) \Theta(dx)\right) \\
&= \exp\left(-\sum_{i=1}^{m} \left(1 - e^{-c_i}\right) \Theta(B_i)\right) \\
&= \prod_{i=1}^{m} \mathbb{E}\left[e^{-c_i \Phi(B_i)}\right] = \prod_{i=1}^{m} L_{\Phi(B_i)}(c_i).
\end{aligned}
$$

Hence the Laplace transform of X is the product of the Laplace transforms of its components, from which the independence of these components $\Phi(B_1), \ldots, \Phi(B_m)$ is transparent (cf. e.g. in [4, Bem. 2.2.6] or [5, Satz 15.6 and Übung 15.1.2]). The general case (in which some of the B_i may be unbounded and hence $\Theta(B_i) = \infty$ might occur for some i) can be deduced from this. □

Now we can make the first step towards proving the existence of Poisson processes. We will show that the mixed Binomial processes of Example 2.17 provide a scheme to explicitly construct Poisson processes. All we have to do is to ensure that the parameter τ, which controls the number of points of the process, is a Poisson variable.

Theorem 2.30 *Let $\tau \sim \mathrm{Poiss}(c)$ for some $c \geq 0$ and let $\mathbb{V}$ be some distribution on $\mathbb{X}$. Let Φ be the mixed Binomial process with parameters τ and $\mathbb{V}$ as in Example 2.17, i.e., $\Phi = \sum_{i=1}^{\tau} \delta_{X_i}$ with and $X_i \sim \mathbb{V}$ for $i \in \mathbb{N}$ and $\tau, X_1, X_2, \ldots$ stochastically independent. Then $\Phi \sim \mathrm{Poiss}(c\mathbb{V})$.*

Proof. We use the criterion provided by Theorem 2.29 and show that the Laplace functional of Φ is that of a Poisson process. For any measurable function $f : \mathbb{X} \to [0, \infty]$, we have

$$
\begin{aligned}
L_\Phi(f) &= \mathbb{E}\left[\exp\left(-\sum_{k=1}^{\tau} f(X_k)\right)\right] = \sum_{m=0}^{\infty} \mathbb{E}\left[\mathbb{1}\{\tau = m\} \exp\left(-\sum_{k=1}^{m} f(X_k)\right)\right] \\
&= \sum_{m=0}^{\infty} \mathbb{P}(\tau = m) \mathbb{E}\left[\prod_{k=1}^{m} \exp(-f(X_k))\right] = \sum_{m=0}^{\infty} \mathbb{P}(\tau = m) \prod_{k=1}^{m} \mathbb{E}\left[\exp(-f(X_k))\right],
\end{aligned}
$$

where we have used the independence of $\tau, X_1, X_2, \ldots$ for the last steps. Recalling that $\tau \sim \mathrm{Poiss}(c)$ and $X_i \sim \mathbb{V}$, we infer that

$$
\begin{aligned}
L_\Phi(f) &= \sum_{m=0}^{\infty} e^{-c} \frac{c^m}{m!} \left(\int_{\mathbb{X}} e^{-f(x)} \, \mathbb{V}(dx)\right)^m = e^{-c} \exp\left(c \int_{\mathbb{X}} e^{-f} \, d\mathbb{V}\right) \\
&= \exp\left(-c\left(1 - \int_{\mathbb{X}} e^{-f} \, d\mathbb{V}\right)\right) = \exp\left(-\int_{\mathbb{X}} (1 - e^{-f}) \, d(c\mathbb{V})\right).
\end{aligned}
$$

Hence $L_\Phi(f)$ is of the form given in (2.2) and so, by Theorem 2.29, $\Phi \sim \mathrm{Poiss}(c\mathbb{V})$. □

Observe that Theorem 2.30 implies the existence of a Poisson process Φ with intensity measure Θ for any finite measure Θ on $\mathbb{X}$. Indeed, let $c := \Theta(\mathbb{X})$ and $\mathbb{V} := c^{-1}\Theta$. Let Φ be

the mixed Binomial process with parameters $\tau \sim \mathrm{Poiss}(c)$ and $\mathbb{V}$. Then, by Theorem 2.30, $\Phi \sim \mathrm{Poiss}(c\mathbb{V}) = \mathrm{Poiss}(\Theta)$. This is also one of the main ideas for proving the existence of Poisson processes in general, which we state next.

Theorem 2.31 *For any locally finite measure Θ on $\mathbb{X}$ there exists a point process Φ on $\mathbb{X}$ such that*

$$\Phi \sim \mathrm{Poiss}(\Theta).$$

Proof. We consider a partition $\mathbb{X} = \bigcup_{i=1}^{\infty} \mathbb{X}_i$ of $\mathbb{X}$ into measurable sets $\mathbb{X}_i$ such that $\Theta(\mathbb{X}_i) < \infty$ for each $i \in \mathbb{N}$. Set $c_i := \Theta(\mathbb{X}_i) \geq 0$ and let $\mathbb{V}_i := (c_i)^{-1}\Theta_{\mathbb{X}_i}$ whenever $c_i > 0$. For $c_i = 0$, let $\mathbb{V}_i$ be an arbitrary probability measure on $\mathbb{X}_i$. Then we have $\Theta = \sum_{i=1}^{\infty} c_i \cdot \mathbb{V}_i$. Let $\Phi_1, \Phi_2, \ldots$ be independent mixed Binomial processes with parameters $\tau_i \sim \mathrm{Poiss}(c_i)$ and $\mathbb{V}_i$. Employing again Theorem 2.29, it is easy to check that $\Phi := \sum_{i=1}^{\infty} \Phi_i \sim \mathrm{Poiss}(\Theta)$. Indeed, we have, for any measurable $f \geq 0$,

$$\begin{aligned} L_\Phi(f) &= \mathbb{E}\left[\exp\left(-\int f\, d\Phi\right)\right] = \mathbb{E}\left[\exp\left(-\sum_{i=1}^{\infty}\int f\, d\Phi_i\right)\right] \\ &= \mathbb{E}\left[\prod_{i=1}^{\infty}\exp\left(-\int f\, d\Phi_i\right)\right] = \prod_{i=1}^{\infty}\mathbb{E}\left[\exp\left(-\int f\, d\Phi_i\right)\right] = \prod_{i=1}^{\infty} L_{\Phi_i}(f), \end{aligned}$$

which is due to the assumed independence of the Φ_i. Since $\Phi_i \sim \mathrm{Poiss}(c_i\mathbb{V}_i)$, Theorem 2.29 implies that

$$\begin{aligned} L_\Phi(f) &= \prod_{i=1}^{\infty}\exp\left(-\int \left(1-e^{-f}\right)\, d(c_i\mathbb{V}_i)\right) \\ &= \exp\left(-\sum_{i=1}^{\infty}\int \left(1-e^{-f}\right)\, d(c_i\mathbb{V}_i)\right) = \exp\left(-\int \left(1-e^{-f}\right)\, d\Theta\right). \end{aligned}$$

This completes the proof. □

We point out that beside the Laplace functional there are other ways to characterize Poisson processes, e.g. in terms of the celebrated Mecke formula. We will not discuss this here but refer to [6] or [3].

Recall from Definition 2.11 that a point process Φ is simple if and only if its realizations are almost surely simple counting measures, i.e. elements of $N_s(\mathbb{X})$. We end this section by showing that Poisson processes are simple (provided their intensity measure has no atoms).

Theorem 2.32 *Let Φ be a Poisson process on $\mathbb{X}$ with intensity measure Θ. Then Φ is simple if and only if Θ is diffuse.*

Proof. If Θ has an atom, that is, $\Theta(\{x\}) > 0$ for some $x \in \mathbb{X}$, then condition (ii) in Definition 2.26 implies that $\mathbb{P}(\Phi(\{x\}) = k) > 0$ for $k \in \mathbb{N}$ and hence Φ is not simple. This shows that simplicity of Φ implies that Θ is diffuse.

For a proof of the reverse implication, suppose that $\Theta(\{x\}) = 0$ for all $x \in \mathbb{X}$. Assume for a contradiction that Φ is not simple, i.e., $\mathbb{P}(\Phi \in N_s) < 1$. Then there exists a measurable, bounded set $C \in \mathcal{C}$ such that

$$\alpha := \mathbb{P}(\Phi_C \text{ is not simple }) > 0.$$

This implies in particular $\theta := \Theta(C) > 0$. We use now that the range of a finite diffuse measure is an interval. An elementary proof of this (not completely obvious but easy to believe) fact can be found in [1, Lemma 9.1]. Due to this fact, we can, for each $k \in \mathbb{N}$, partition C into k pairwise disjoint sets $C_1^{(k)}, \ldots, C_k^{(k)}$ such that

$$\Theta(C_i^{(k)}) = \frac{\theta}{k}, \quad i = 1, \ldots, k.$$

Then it must be true for at least one of the sets $C_i^{(k)}$ that $\mathbb{P}(\Phi(C_i^{(k)}) > 1) \geq \alpha/k$. Due to the definition of a Poisson process, this implies

$$\frac{\alpha}{k} \leq 1 - e^{-\Theta(C_i^{(k)})} \left(1 + \Theta(C_i^{(k)})\right).$$

Hence $\alpha \leq k(1 - e^{-\theta/k}) - \theta e^{-\theta/k}$, and this is true for each $k \in \mathbb{N}$. But, as $k \to \infty$, the right hand side of this inequality converges to $\theta - \theta = 0$ (which can be seen e.g. by L'Hopital's rule), which is a contradiction to $\alpha > 0$. □

2.4 Markings and marked point processes

Now we discuss the concept of marked point processes in more detail. In Definition 1.25 we gave already a preliminary definition. Given a point process Φ on $\mathbb{X}$, we want to attach *marks* to the points of Φ. In Definition 1.25, the marks are compact sets (grains) that we attach to the points of Φ to build the germ-grain model. In Example 1.27 the marks are positive random variables, which describe random radii of balls that we center at the points of Φ. In general, a mark can be almost anything, e.g. a color, a data vector representing measurements carried out at the locations described by the points of Φ or even another point process representing a cluster of points spread around a point of Φ. The basic idea is that we choose a mark randomly from a given mark space $\mathbb{Y}$, i.e., a mark should be a random variable with values in $\mathbb{Y}$. We also want to allow the freedom that the distribution of the mark depends on the point to which we attach it. This can be conveniently described with the help of probability kernels.

Definition 2.33 Let $(\mathbb{Y}, \mathcal{Y})$ be a separable metric space with σ-algebra $\mathcal{Y}$. Let K be a stochastic kernel from $\mathbb{X}$ to $\mathbb{Y}$, that is, $K \colon \mathbb{X} \times \mathcal{Y} \to [0, 1]$ is a function satisfying the following properties:

(i) For each $x \in \mathbb{X}$, $K(x, \cdot)$ is a probability measure on $(\mathbb{Y}, \mathcal{Y})$.

(ii) The mapping $x \mapsto K(x, C)$ is measurable for each $C \in \mathcal{Y}$.

For $k \in \mathbb{N} \cup \{\infty\}$ and any counting measure $\varphi := \sum_{i=1}^{k} \delta_{x_i} \in N(\mathbb{X})$, we denote by $K^*(\varphi, \cdot)$ the distribution of the point process $\sum_{i=1}^{k} \delta_{(x_i, Y_i)}$ in $\mathbb{X} \times \mathbb{Y}$. Here $Y_1, Y_2, \ldots$ are independent random elements in $\mathbb{Y}$ with distributions $K(x_1, \cdot), K(x_2, \cdot), \ldots$, where we set $x_i := x_0$ for $i > k$ (and some $x_0 \in \mathbb{X}$) if necessary.

In other words, the above definition says that

$$K^*(\varphi, A) = \int \mathbb{1}\left\{\sum_{j=1}^{k} \delta_{(x_j, y_j)} \in A\right\} \left(\otimes_{i \in \mathbb{N}} K(x_i, \cdot)\right) \left(d\,(y_i)_{i \in \mathbb{N}}\right),$$

for $A \in \mathcal{N}(\mathbb{X} \times \mathbb{Y})$. One can show that (see Work sheet 6, Question 1) that K^* is a stochastic kernel from $N(\mathbb{X})$ to $N(\mathbb{X} \times \mathbb{Y})$.

Definition 2.34 Let Φ be a point process on $\mathbb{X}$. A point process Ψ on $\mathbb{X} \times \mathbb{Y}$ is called *K-marking* of Φ if and only if

$$\mathbb{P}(\Psi \in A|\Phi) = K^*(\Phi, A) \quad \mathbb{P}\text{-almost surely}$$

for all $A \in \mathcal{N}(\mathbb{X} \times \mathbb{Y})$. In this setting, $\mathbb{Y}$ is called *mark space*. In case $K(x, \cdot) = \mathbb{Q}$ for all $x \in \mathbb{X}$, where $\mathbb{Q}$ is some fixed distribution on $\mathbb{Y}$, the K-marking Ψ is also called an *independent $\mathbb{Q}$-marking* of Φ and $\mathbb{Q}$ is called *mark distribution*. In general, it makes sense to call $K(x, \cdot)$ the *mark distribution at $x \in \mathbb{X}$*. Ψ is also called a *marked point process*.

The point process Ψ is a K-marking of Φ, if and only if the joint distribution of Ψ and Φ is given by

$$\mathbb{P}((\Psi, \Phi) \in \cdot) = \int \int \mathbb{1}\{(\psi, \varphi) \in \cdot\}\, K^*(\varphi, d\psi)\, \mathbb{P}_\Phi(d\varphi).$$

In particular, the distribution of Ψ is given by

$$\begin{aligned} \mathbb{P}_\Psi = \mathbb{P}(\Psi \in \cdot) &= \int \int \mathbb{1}\{\psi \in \cdot\}\, K^*(\varphi, d\psi)\, \mathbb{P}_\Phi(d\varphi) \\ &= \mathbb{E} \int \mathbb{1}\{\psi \in \cdot\}\, K^*(\Phi, d\psi). \end{aligned} \tag{2.4}$$

If Ψ is an independent $\mathbb{Q}$-marking of Φ, then $\mathbb{P}_\Psi = \mathbb{P}_\Phi \otimes \mathbb{Q}$.

Remark 2.35 The assumptions on $(\mathbb{Y}, \mathcal{Y})$ ensure that $\mathbb{X} \times \mathbb{Y}$ is a separable metric space as well (w.r.t. the product metric).

We got to know the intensity measure as an important characteristic of a point process taking the role of the expectation. Clearly, the intensity measure of a marked point process Ψ is a measure on $\mathbb{X} \times \mathbb{Y}$. The following statement relates it to the intensity measure Θ of the underlying point process Φ.

Theorem 2.36 *If Ψ is a K-marking of a point process Φ with intensity measure Θ, then the intensity measure of Ψ is given by*

$$(\Theta \otimes K)(C) := \int_{\mathbb{X}} \int_{\mathbb{Y}} \mathbb{1}_C(x, y)\, K(x, dy)\, \Theta(dx), \quad C \in \mathcal{X} \otimes \mathcal{Y}.$$

Proof. By definition of K^*, we have for any counting measure $\varphi = \sum_{i=1}^k \delta_{x_i} \in N(\mathbb{X})$ and any set $C \in \mathcal{X} \otimes \mathcal{Y}$,

$$\begin{aligned} \int \psi(C)\, K^*(\varphi, d\psi) &= \int \sum_{j=1}^k \mathbb{1}_C(x_j, y_j)\, (\otimes_{i \in \mathbb{N}} K(x_i, \cdot))\, \left(d\,(y_i)_{i \in \mathbb{N}}\right) \\ &= \sum_{j=1}^k \int \mathbb{1}_C(x_j, y_j)\, (\otimes_{i \in \mathbb{N}} K(x_i, \cdot))\, \left(d\,(y_i)_{i \in \mathbb{N}}\right) \\ &= \sum_{j=1}^k \int \mathbb{1}_C(x_j, y)\, K(x_j, dy) \\ &= \sum_{j=1}^k g_C(x_j) = \int g_C\, d\varphi, \end{aligned}$$

where the function $g_C : \mathbb{X} \to \mathbb{R}$ is given by

$$g_C(x) := \int \mathbb{1}_C(x,y) K(x,dy), \quad x \in \mathbb{X}.$$

Noting that g_C is non-negative and measurable, we conclude from Campbell's Theorem 2.21 that

$$\begin{aligned}\mathbb{E}\left[\Psi(C)\right] &= \mathbb{E}\left[\int \psi(C)\, K^*(\Phi, d\psi)\right] = \mathbb{E}\left[\int g_C \, d\Phi\right] = \int g_C(x)\, \Theta(dx) \\ &= \iint \mathbb{1}_C(x,y)\, K(x,dy)\, \Theta(dx).\end{aligned}$$

Hence $\mathbb{E}\left[\Psi(\cdot)\right] = \Theta \otimes K$ as claimed. □

Note that in the above statement $\Theta \otimes K$ is not a product measure in the usual sense as K is a probability kernel and not a measure on $\mathbb{Y}$. However, in special case of an independent marking, the intensity measure of Ψ becomes a product measure.

Corollary 2.37 *If Ψ is an independent $\mathbb{Q}$-marking of a point process Φ on $\mathbb{X}$ with intensity measure Θ, then the intensity measure of Ψ is the product measure $\Theta \otimes \mathbb{Q}$.*

Of particular interest in the sequel will be the case that the basic point process Φ is a Poisson process. Fortunately, the Poisson property is preserved by markings, which can be seen from looking at the corresponding Laplace functionals.

Theorem 2.38 *Let Φ be a Poisson process on $\mathbb{X}$ and let Ψ be a K-marking of Φ. Then Ψ is a Poisson process with intensity measure Θ^* given by $\Theta^* = \Theta \otimes K$.*

Proof. We claim that for any measurable function $f\colon \mathbb{X} \times \mathbb{Y} \to [0,\infty]$,

$$L_\Psi(f) := \mathbb{E}\left[\exp\left(-\int_{\mathbb{X}\times\mathbb{Y}} f \, d\Psi\right)\right] = \mathbb{E}\left[\exp\left(-\int_{\mathbb{X}} f^* \, d\Phi\right)\right] = L_\Phi(f^*),$$

where f^* is given by $f^*(x) := -\ln \int_{\mathbb{Y}} e^{-f(x,y)}\, K(x,dy)$ and $-\ln(0) := \infty$. In Theorem 2.29 we have derived an explicit expression for the latter Laplace functional. Recalling that Θ is the intensity measure of Φ, we infer that

$$\begin{aligned}L_\Psi(f) &= \exp\left(-\int_{\mathbb{X}} (1 - e^{-f^*})\, d\Theta\right) \\ &= \exp\left(-\int_{\mathbb{X}} \left(1 - \int_{\mathbb{Y}} e^{-f(x,y)}\, K(x,dy)\right)\, \Theta(dx)\right) \\ &= \exp\left(-\int_{\mathbb{X}}\int_{\mathbb{Y}} (1 - e^{-f(x,y)})\, K(x,dy)\, \Theta(dx)\right) \\ &= \exp\left(-\int_{\mathbb{X}\times\mathbb{Y}} (1 - e^{-f})\, d(\Theta \otimes K)\right).\end{aligned}$$

Applying now again Theorem 2.29, we conclude that $\Psi \sim \mathrm{Poiss}(\Theta^*)$.

It remains to verify the above claim. By (2.4), we have

$$\mathbb{E}\left[\exp\left(-\int f \, d\Psi\right)\right] = \mathbb{E}\left[\int \exp\left(-\int f \, d\psi\right)\, K^*(\Phi, d\psi)\right]. \tag{2.5}$$

Using the above notation for the description of K^*, the inner integral on the right hand side can be written as follows:

$$\begin{aligned}
&\int \exp\left(-\int f\,d\psi\right)\, K^*(\varphi, d\psi) \\
&\quad = \int \exp\left(-\sum_{j=1}^{k} f(x_j, y_j)\right)\, \left(\otimes_{i\in\mathbb{N}} K(x_i,\cdot)\right)\left(d\,(y_i)_{i\in\mathbb{N}}\right)) \\
&\quad = \int \prod_{j=1}^{k} e^{-f(x_j,y_j)}\, \left(\otimes_{i\in\mathbb{N}} K(x_i,\cdot)\right)\left(d\,(y_i)_{i\in\mathbb{N}}\right)) \\
&\quad = \prod_{j=1}^{k} \int e^{-f(x_j,y_j)}\, K(x_j, dy_j) \\
&\quad = \prod_{j=1}^{k} \int e^{-f(x_j,y)}\, K(x_j, dy) \\
&\quad = \exp\left[\sum_{j=1}^{k} \ln \int e^{-f(x_j,y)}\, K(x_j, dy)\right] \\
&\quad = \exp\left[-\int_{\mathbb{X}} \left(-\ln \int_{\mathbb{Y}} e^{-f(x,y)}\, K(x, dy)\right)\, \varphi(dx)\right] \\
&\quad = \exp\left[-\int_{\mathbb{X}} f^*(x)\, \varphi(dx)\right].
\end{aligned}$$

Inserting this last expression into equation (2.5) proves the claim. □

As a special application of the marking procedure, we consider thinnings of a point process, a construction where the marks of the points are used to decide whether these points are kept or not. A thinning of a point process on $\mathbb{X}$ is again a point process on $\mathbb{X}$.

Definition 2.39 Let $p : \mathbb{X} \to [0,1]$ be a measurable function and K be the probability kernel from $\mathbb{X}$ to $\mathbb{Y} := \{0,1\}$ given by

$$K(x,\cdot) := (1-p(x))\delta_0 + p(x)\delta_1. \tag{2.6}$$

If Ψ is the resulting K-marking of Φ, then the point process

$$\Phi_p := \Psi(\cdot \times \{1\})$$

is called *p-thinning* of Φ.

As a consequence of the previous Theorem 2.38, we infer that the thinning operation preserves the Poisson property.

Corollary 2.40 *Let K be as in* (2.6)*, and let Ψ be a K-marking of a Poisson process Φ on $\mathbb{X}$ with intensity measure Θ. Then $\Phi_p := \Psi(\cdot \times \{1\})$ and $\Phi - \Phi_p := \Psi(\cdot \times \{0\})$ are independent Poisson processes with intensity measures Θ_1 and Θ_0 given by*

$$\Theta_1(B) = \int_B p(x)\, \Theta(dx) \quad \text{and} \quad \Theta_0(B) = \int_B (1-p(x))\, \Theta(dx)$$

respectively, for $B \in \mathcal{X}$.

Proof. According to Theorem 2.38, Ψ is a Poisson process on $\mathbb{X} \times \{0,1\}$. Remark 2.28 implies that the restrictions of Ψ to the sets $\mathbb{X} \times \{1\}$ and $\mathbb{X} \times \{0\}$, respectively, are independent Poisson processes. Since Φ_p and $\Phi - \Phi_p$ are the images of these restrictions under the projection onto $\mathbb{X}$, these two point processes are still independent Poisson processes. Furthermore, we infer from Theorem 2.36 that, for any $B \in \mathcal{X}$,

$$\Theta_1(B) = \mathbb{E}[\Phi_p(B)] = \mathbb{E}[\Psi(B \times \{1\})] = \int_{\mathbb{X}} \mathbb{1}_B(x) K(x, \{1\})\, \Theta(dx) = \int_B p(x)\, \Theta(dx).$$

A similar computation for Θ_0 completes the proof. □

For further examples of marked point processes we refer to Work sheet 6. It is also worth to recall Examples 1.26 and 1.27 in the light of the new (refined) definition of a marked point process.

2.5 Stationarity

Stationarity is a concept that we have already seen for random closed sets in $\mathbb{R}^d$: an RCS Z is stationary if and only if any translated copy $Z+t$, $t \in \mathbb{R}^d$ is equal to Z in distribution. There is a natural analogue of stationarity for point processes and random measures. It requires to explain what we mean by a 'translation' of a measure. Let $\mu \in M(\mathbb{R}^d)$ and let $x \in \mathbb{R}^d$ be a translation vector. We define the measure $\mu + x$ by

$$(\mu + x)(A) := \mu(A - x), \qquad A \in \mathcal{B}(\mathbb{R}^d).$$

More formally, the *translation* $\mu + x$ of a measure μ is the image measure of μ under the translation $T_x : \mathbb{R}^d \to \mathbb{R}^d$, $z \mapsto z + x$, that is, $\mu + x := T_x(\mu)$. Observe that this naturally induces a mapping $T_x : M(\mathbb{R}^d) \to M(\mathbb{R}^d)$, $\mu \mapsto \mu + x$. For a counting measure $\varphi \in N(\mathbb{R}^d)$ with $\varphi = \sum_{i=1}^k \delta_{x_i}$, for instance, the translation $T_x\varphi$ (or $\varphi + x$) is given by

$$(T_x\varphi)(A) = \varphi(T_x^{-1}(A)) = \sum_{i=1}^k \delta_{x_i}(T_x^{-1}(A)) = \sum_{i=1}^k \delta_{T_x(x_i)}(A) = \left(\sum_{i=1}^k \delta_{T_x(x_i)}\right)(A),$$

for $A \in \mathcal{B}^d$, that is,

$$T_x\varphi = \sum_{i=1}^k \delta_{T_x(x_i)} \qquad (\text{or } \varphi + x = \sum_{i=1}^k \delta_{x_i + x}).$$

In particular, we have $T_x\delta_0 = \delta_x$. Integration with respect to a translated measure $T_x\varphi$ can be expressed as an integration with respect to φ. For any measurable function $f : \mathbb{R}^d \to [0, \infty]$ we have

$$\int f(y)\, (T_x\varphi)(dy) = \int f(y + x)\, \varphi(dy) = \int f \circ T_x\, d\varphi.$$

It is important to note that the mapping $M(\mathbb{R}^d) \to M(\mathbb{R}^d)$, $\mu \mapsto T_x\mu$ is measurable (and similarly the corresponding mapping on $N(\mathbb{R}^d)$).

In the same way, we define the *rotation* of a measure $\mu \in M(\mathbb{R}^d)$. For any matrix $\rho \in \mathrm{SO}_d$ let $T_\rho : \mathbb{R}^d \to \mathbb{R}^d$, $y \mapsto \rho y$ be the corresponding rotation and define the rotation of μ as the image measure $T_\rho\mu$ of μ with respect to T_ρ (where $T_\rho\mu$ is given by $(T_\rho\mu)(A) = \mu(T_\rho^{-1}(A))$, $A \in \mathcal{B}^d$ and $T_\rho^{-1}(A) := \{x \in \mathbb{R}^d : \rho x \in A\}$).

Now it is clear how to define the translation and rotation of a point process (or random measure) Φ. For any $x \in \mathbb{R}^d$, $\Phi + x$ is defined by $(\Phi + x)(\omega) := \Phi(\omega) + x$, $\omega \in \Omega$. And for any $\rho \in \mathrm{SO}_d$, $T_\rho\Phi$ is defined by $(T_\rho\Phi)(\omega) := T_\rho(\Phi(\omega))$, $\omega \in \Omega$.

Definition 2.41 A point process (or random measure) Φ in $\mathbb{R}^d$ is called *stationary*, if and only if $\Phi + x \stackrel{d}{=} \Phi$ for all $x \in \mathbb{R}^d$. Furthermore, Φ is called *isotropic*, if and only if $T_\rho \Phi \stackrel{d}{=} \Phi$ for all $\rho \in \mathrm{SO}_d$.

Note that, by Theorem 2.23, stationarity of Φ is equivalent to the relation

$$\int f(y+x)\Phi(dy) = \int f(y)\Phi(dy) \tag{2.7}$$

being satisfied for any measurable function $f : \mathbb{R}^d \to [0,\infty]$ and any $x \in \mathbb{R}^d$. A similar remark applies to isotropy.

Let Φ be a stationary (isotropic) point process in $\mathbb{R}^d$ with locally finite intensity measure $\Theta = \mathbb{E}\Phi$. Then Θ is translation (rotation) invariant, i.e. $\Theta(A+x) = \Theta(A)$ for any $x \in \mathbb{R}^d$ ($\Theta(\rho A) = \Theta(A)$ for any $\rho \in \mathrm{SO}_d$) and any $A \in \mathcal{B}^d$. This can be seen by applying (2.7) to the indicator functions of sets $B \in \mathcal{B}^d$.

The translation invariance implies that $\Theta = \gamma\lambda_d$ for some $\gamma \geq 0$, that is, the intensity measure is a multiple of the Lebesgue measure. The proportionality factor γ is called the *intensity* of Φ. If Φ is a stationary Poisson process in $\mathbb{R}^d$ with intensity $\gamma > 0$, then Θ is diffuse (and not the null process) and hence, by Theorem 2.32, Φ is simple. Therefore, by Theorem 2.8, Φ has a representation of the form $\sum_{n\in\mathbb{N}} \delta_{\xi_n}$, where $\xi_1, \xi_2, \ldots$ are pairwise different random points in $\mathbb{R}^d$.

In the previous section we have introduced the notion of a K-marking of a point process Φ on $\mathbb{X}$. The result was a special (marked) point process Ψ on the product space $\mathbb{X} \times \mathbb{Y}$. For $\mathbb{X} = \mathbb{R}^d$, we extend the definition of stationarity to marked point processes. In view of the interpretation of the second component $\mathbb{Y}$ as the space of marks, it is natural, to explain translations in $\mathbb{R}^d \times \mathbb{Y}$ as translations of the first component.

For $x \in \mathbb{R}^d$, let $T_x : \mathbb{R}^d \times \mathbb{Y} \to \mathbb{R}^d \times \mathbb{Y}$ be defined by $T_x(z,y) := (z+x, y)$. Then we define similarly as before, the translation $T_x\mu$ of a measure μ on $\mathbb{R}^d \times \mathbb{Y}$ as the image measure of μ w.r.t. T_x. It is clear that T_x (as a mapping from $M(\mathbb{R}^d \times \mathbb{Y})$ to itself) is measurable. A point process Ψ on $\mathbb{R}^d \times \mathbb{Y}$ is called *stationary*, if and only if $T_x\Psi \stackrel{d}{=} \Psi$ for each $x \in \mathbb{R}^d$.

The above observation that the intensity measure of a stationary point process on $\mathbb{R}^d$ is a multiple of the Lebesgue measure extends to point processes on $\mathbb{R}^d \times \mathbb{Y}$ as follows.

Theorem 2.42 *Let Ψ be a stationary point process on $\mathbb{R}^d \times \mathbb{Y}$ such that $\mathbb{E}\,\Psi(\cdot \times \mathbb{Y})$ is locally finite. Then there exists a constant $\gamma \geq 0$ and a probability measure $\mathbb{Q}$ on $\mathbb{Y}$, such that $\mathbb{E}\,\Psi = \gamma\lambda_d \otimes \mathbb{Q}$.*

Proof. For fixed $A \in \mathcal{Y}$, we consider the measure Λ_A on $\mathbb{R}^d$ given by $\Lambda_A(B) := \mathbb{E}\Psi(B \times A)$, $B \in \mathcal{B}^d$. According to the assumptions, Λ_A is locally finite and translation invariant. Therefore, we can find some constant $c(A) \geq 0$ such that $\Lambda_A(B) = c(A)\lambda_d(B)$ for $B \in \mathcal{B}(\mathbb{R}^d)$. It is easy to check that $c(\cdot)$ is a measure on $\mathbb{Y}$. Furthermore, for the choice $A = \mathbb{Y}$ and $B = [0,1]^d$, we have in particular $\mathbb{E}[\Psi([0,1]^d \times \mathbb{Y})] = \Lambda_{\mathbb{Y}}([0,1]^d) = c(\mathbb{Y}) < \infty$. If $c(\mathbb{Y}) > 0$, then let $\mathbb{Q}(A) = c(\mathbb{Y})^{-1}c(A)$, which defines a probability measure $\mathbb{Q}$ on $\mathbb{Y}$. Otherwise let $\mathbb{Q}$ be an arbitrary probability measure on $\mathbb{Y}$. In both cases we infer that

$$\mathbb{E}[\Psi(B \times A)] = c(\mathbb{Y})\mathbb{Q}(A)\lambda_d(B) = c(\mathbb{Y})(\lambda_d \otimes \mathbb{Q})(B \times A),$$

for any $A \in \mathcal{Y}$ and $B \in \mathcal{B}^d$, from which the assertion follows. □

Observe that the condition in Theorem 2.42 that $\mathbb{E}\,\Psi(\cdot \times \mathbb{Y})$ is locally finite implies the local finiteness of the intensity measure of Ψ but is stronger. The derived product

form of the intensity measure for such stationary processes raises the question, whether every such process Ψ can be obtained as an independent $\mathbb{Q}$-marking of some stationary point process Φ. At least in the case of Poisson processes this can be answered in the affirmative.

Theorem 2.43 *Let Ψ be a stationary marked Poisson process on $\mathbb{R}^d \times \mathbb{Y}$ with intensity measure $\gamma\lambda_d \otimes \mathbb{Q}$, where $\gamma > 0$ and $\mathbb{Q}$ is some probability measure on $\mathbb{Y}$. Then Ψ is equal in distribution to an independent $\mathbb{Q}$-marking of a stationary Poisson process on $\mathbb{R}^d$ with intensity γ.*

Proof. This is a direct consequence of Theorem 2.38 and the fact that Poisson processes are uniquely determined by their intensity measure. □

3 The Boolean model

In Definition 1.25, we introduced a germ-grain model in $\mathbb{R}^d$ as the union set Z of a marked point process Ψ. In Section 2.4, we got to know markings, as a special construction that generates marked point processes, and studied some of their properties. In general, any point process Ψ on a product space $\mathbb{X} \times \mathbb{Y}$ may be called a marked point process on $\mathbb{X}$ (with mark space $\mathbb{Y}$). However, when we speak of a marked point process in the sequel, we will usually mean a process Ψ for which the intensity measure of the underlying point process (the *process of the germs*) is locally finite. This is equivalent to assuming that $\mathbb{E}\Psi(\cdot \times \mathbb{Y})$ is locally finite (and implies that the intensity measure of Ψ is also locally finite).

If we fix $\mathbb{X}$ to be $\mathbb{R}^d$ and choose the mark space $\mathbb{Y}$ to be $\mathcal{C}^d$, the space of compact subsets of $\mathbb{R}^d$, then we recover the situation of Definition 1.25. Such marked point processes Ψ are also called *germ-grain processes*.

In the sequel we will be interested in the stationary situation, i.e. we assume throughout that Ψ is stationary in the sense of the definition given before Theorem 2.42 on page 32. By Theorem 2.42, the intensity measure of Ψ is then of the form $\Theta^* = \mathbb{E}\,\Psi = \gamma\lambda_d \otimes \mathbb{Q}$, for some constant γ, which we will call the *intensity* of Ψ, and some probability measure $\mathbb{Q}$ on $\mathcal{C}^d$, which we will call the *grain distribution* or *shape distribution* in the sequel. A random compact set Z_0 with distribution $\mathbb{Q}$ will be called the *typical grain* of Ψ.

Since Ψ is in particular a point process on $\mathbb{R}^d \times \mathcal{C}^d$, by Theorem 2.8 and Remark 2.13, it can be represented in the form

$$\Psi = \sum_{n\in\mathbb{N}} \delta_{(\xi_n, Z_n)} \tag{3.1}$$

where ξ_n are random points in $\mathbb{R}^d$ and Z_n are random elements of $\mathcal{C}^d$. The resulting *germ-grain model* is the union set

$$Z := \bigcup_{n\in\mathbb{N}} Z_n + \xi_n. \tag{3.2}$$

We will be particularly interested in the situation when the underlying germ process Φ is a Poisson process (according to the above assumptions, a stationary Poisson process with intensity measure $\Theta = \gamma\lambda_d$). Then, by Theorem 2.38, Ψ is a Poisson process too and, by Theorem 2.43, it is equal in distribution to an independent $\mathbb{Q}$-marking of Φ with intensity measure $\Theta^* = \gamma\lambda_d \otimes \mathbb{Q}$. This also implies, that the sets Z_n in (3.1) can be chosen as an i.i.d. sequence of random sets with distribution $\mathbb{Q}$ which is also independent of the sequence (ξ_n). In this situation, the union set Z in (3.2) is called a *(stationary) Boolean model* or, more precisely, the *stationary Boolean model determined by* $(\gamma, \mathbb{Q})$, as this data characterizes the germ-grain process Ψ and thus Z uniquely.

In this chapter we will in particular study the properties of Boolean models. First of all, we will address the question under which conditions a Boolean model Z is a random closed set. This will enable us to look at functionals already available for RCS such as the capacity functional or the volume fraction. Later we introduce further geometric functionals (in particular intrinsic volumes) for which we will have to impose some additional geometric restrictions regarding the grain distribution $\mathbb{Q}$.

Before we start this programme, we will first explore yet another idea how to generate germ-grain models, namely by employing the concept of a particle process.

3.1 Particle processes

Let

$$\mathcal{F}' := \mathcal{F}^d \setminus \{\emptyset\} \quad \text{and} \quad \mathcal{C}' := \mathcal{C}^d \setminus \{\emptyset\}.$$

Recall from Section 1.1 that $\mathcal{F}'$ is in particular a separable topological space and (equipped with the Hausdorff metric) a complete metric space.

Essentially, we want to define a particle process as a point process on $\mathcal{C}'$, i.e. as a measurable mapping from some probability space to the family of locally finite counting measures on $\mathcal{C}'$. For technical reasons, it is advisable to use a slightly stronger notion of local finiteness here: We will call a measure φ on $\mathcal{F}'$ *locally finite* if and only if $\varphi(\mathcal{K}) < \infty$ for any compact set $\mathcal{K} \subset \mathcal{F}'$. (Note that this definition is more restrictive than the one based on bounded sets (in $\mathcal{C}'$ and w.r.t. the Hausdorff metric δ). The above condition implies that $\varphi(\mathcal{H}) < \infty$ for any bounded measurable subset $\mathcal{H} \subset \mathcal{C}'$. (why? - see also Work sheet 7, Question 4). The converse is not true, unfortunately. Note that φ is locally finite if and only if

$$\varphi(\mathcal{F}_C) < \infty \text{ for any } C \in \mathcal{C}^d. \tag{3.3}$$

(why? - see also Work sheet 7, Question 3)

Definition 3.1 A *particle process* Φ in $\mathbb{R}^d$ is a point process on $\mathcal{C}'$.

By Theorem 2.8 and Remark 2.13, any particle process Φ in $\mathbb{R}^d$ can be represented in the form

$$\Phi = \sum_{i=1}^{\tau} \delta_{C_i},$$

where C_i are random compact sets and τ is an $\mathbb{N}_0 \cup \{\infty\}$-valued random variable. The sets C_i should be viewed as the analogues of the secondary grains in a germ-grain process. To relate the concept of a particle process to that of a germ-grain process, we need to separate the 'shape' and the 'position' of the secondary grains. For this purpose the idea of a center function is needed.

Definition 3.2 A *center function* $c\colon \mathcal{C}' \to \mathbb{R}^d$ is a measurable mapping satisfying

$$c(K + x) = c(K) + x, \quad \text{for any } K \in \mathcal{C}', x \in \mathbb{R}^d.$$

c is also called *translation covariant* because of this property.

Example 3.3 For $K \in \mathcal{C}'$, let $c(K)$ be the center of the (uniquely determined) *circumball* $B(K)$ of K, i.e. the smallest ball containing K. Since

$$B(K + x) = B(K) + x, \quad \text{for any } K \in \mathcal{C}', x \in \mathbb{R}^d,$$

the mapping $c : \mathcal{K}' \to \mathbb{R}^d$ is indeed translation covariant. One can show that $B(\cdot)$ and $c(\cdot)$ are continuous with respect to the Hausdorff metric (see Worksheet 7) and hence also w.r.t. the Fell topology. In particular, c is measurable.

If we fix some center function c, then under certain assumptions a particle process Φ of the form

$$\Phi = \sum_{i=1}^{\tau} \delta_{K_i}$$

gives rise to a marked point process Ψ on $\mathbb{R}^d$ with mark space $\mathcal{C}'$ defined by

$$\Psi := \int_{\mathcal{C}'} \mathbb{1}\{(c(K), K - c(K)) \in \cdot\}\, \Phi(dK) = \sum_{i=1}^{\tau} \delta_{(c(K_i), K_i - c(K_i))}.$$

Ψ is thus obtained as the image of Φ under the mapping

$$T : \mathcal{C}' \to \mathbb{R}^d \times \mathcal{C}', \quad K \mapsto (c(K), K - c(K)).$$

More precisely, in this construction the marks are necessarily contained in the family

$$\mathcal{C}_0 := \{K \in \mathcal{C}' : c(K) = 0\}$$

of all compact sets that are centred at the origin.

We extend the concept of stationarity to particle processes in the obvious way. For a translation vector $x \in \mathbb{R}^d$, let $T_x : \mathcal{F}^d \to \mathcal{F}^d$ be defined by $F \mapsto T_x(F) = F + x$. For any $\varphi \in N(\mathcal{F}^d)$, let $T_x\varphi$ be the image measure of φ w.r.t. T_x.

Remark 3.4 The mapping $N(\mathcal{F}^d) \times \mathbb{R}^d \to N(\mathcal{F}^d)$, $(\varphi, x) \mapsto T_x\varphi$ is measurable.

Definition 3.5 A particle process Φ on $\mathbb{R}^d$ is called *stationary*, if and only if

$$T_x\Phi \stackrel{d}{=} \Phi, \quad \text{for any } x \in \mathbb{R}^d.$$

The following statement relates stationary particle processes to stationary germ-grain processes.

Theorem 3.6 *Let Φ be a stationary particle process on $\mathbb{R}^d$ with locally finite intensity measure $\Theta \neq 0$. Let $c\colon \mathcal{C}' \to \mathbb{R}^d$ be some center function. Then*

$$\Psi(\cdot) := \int_{\mathcal{C}'} \mathbb{1}\{(c(C), C - c(C)) \in \cdot\}\, \Phi(dC) \tag{3.4}$$

is a stationary marked point process with mark space $\mathcal{C}_0 = \{C \in \mathcal{C}' \colon c(C) = 0\}$. Moreover, there exist a unique number $\gamma > 0$ and a unique probability measure $\mathbb{Q}$ on $\mathcal{C}'$ satisfying $\mathbb{Q}(\mathcal{C}_0) = 1$ such that

$$\Theta(\mathcal{H}) = \gamma \int_{\mathbb{R}^d} \int_{\mathcal{C}'} \mathbb{1}_{\mathcal{H}}(K + x)\, \mathbb{Q}(dK)\, dx, \quad \mathcal{H} \in \mathcal{B}(\mathcal{C}'). \tag{3.5}$$

Proof. We first show that $\Psi(\cdot \times \mathcal{C}')$ is a point process on $\mathbb{R}^d$. To see the local finiteness, we show that $\mathbb{E}\Psi(B \times \mathcal{C}') < \infty$ for any bounded set $B \in \mathcal{B}^d$. Let $W_0 := [0,1)^d$ and let $z_1, z_2, \ldots$ be an enumeration of $\mathbb{Z}^d$. Then $\bigcup_i (W_0 + z_i) = \mathbb{R}^d$ is a disjoint decomposition of $\mathbb{R}^d$. Set

$$\tilde{\gamma} := \mathbb{E}\Psi(W_0 \times \mathcal{C}') = \mathbb{E}\left[\int_{\mathcal{C}'} \mathbb{1}\{c(C) \in W_0\}\, \Phi(dC)\right].$$

By Campbell's Theorem 2.21 and the stationarity of Φ (which implies that Θ is translation

invariant), we have

$$\begin{aligned}
\tilde{\gamma} &= \int_{\mathcal{C}'} \mathbb{1}\{c(C) \in W_0\}\, \Theta(dC) \\
&\leq \sum_{i=1}^{\infty} \int_{\mathcal{C}'} \mathbb{1}\{C \cap (W_0 + z_i) \neq \emptyset, c(C) \in W_0\}\, \Theta(dC) \\
&= \sum_{i=1}^{\infty} \int_{\mathcal{C}'} \mathbb{1}\{(C + z_i) \cap (W_0 + z_i) \neq \emptyset, c(C + z_i) \in W_0\}\, \Theta(dC) \\
&= \sum_{i=1}^{\infty} \int_{\mathcal{C}'} \mathbb{1}\{C \cap W_0 \neq \emptyset, c(C) \in W_0 - z_i\}\, \Theta(dC) \\
&= \int_{\mathcal{C}'} \mathbb{1}\{C \cap W_0 \neq \emptyset\}\, \Theta(dC) \leq \int_{\mathcal{C}'} \mathbb{1}\{C \cap W \neq \emptyset\}\, \Theta(dC) < \infty,
\end{aligned}$$

where $W = [0,1]^d$ is the closure of W_0. The finiteness of the last expression is due to the assumed local finiteness of Θ and the compactness of $\mathcal{F}_K$ for any $K \in \mathcal{C}^d$. The stationarity of Φ implies

$$\begin{aligned}
\mathbb{E} \int_{\mathcal{C}'} \mathbb{1}\{c(C) \in W_0 + x\}\, \Phi(dC) &= \mathbb{E} \int_{\mathcal{C}'} \mathbb{1}\{c(C - x) \in W_0\}\, \Phi(dC) \\
&= \mathbb{E} \int_{\mathcal{C}'} \mathbb{1}\{c(C) \in W_0\}\, \Phi(dC) = \tilde{\gamma}
\end{aligned}$$

for any $x \in \mathbb{R}^d$. For any bounded set $B \in \mathcal{B}^d$ there exist $x_1, \ldots, x_m \in \mathbb{R}^d$, such that $B \subset \bigcup_{i=1}^m (W_0 + x_i)$ and therefore

$$\begin{aligned}
\mathbb{E} \int_{\mathcal{C}'} \mathbb{1}\{c(C) \in B\}\, \Phi(dC) &\leq \mathbb{E} \int_{\mathcal{C}'} \sum_{i=1}^{m} \mathbb{1}\{c(C) \in W_0 + x_i\}\, \Phi(dC) \\
&= \sum_{i=1}^{m} \mathbb{E} \int_{\mathcal{C}'} \mathbb{1}\{c(C) \in W_0 + x_i\}\, \Phi(dC) = m\tilde{\gamma} < \infty.
\end{aligned}$$

Hence $\mathbb{E}\Psi(B \times \mathcal{C}') < \infty$, which implies that $\Psi(B \times \mathcal{C}') < \infty$ almost surely. Since this holds for any bounded set B, we conclude that $\Psi(\cdot \times \mathcal{C}')$ is almost surely locally finite and thus a point process on $\mathbb{R}^d$. Moreover, $\mathbb{E}\Psi(\cdot \times \mathcal{C}')$ locally finite, and therefore Ψ is a marked point process on $\mathbb{R}^d \times \mathcal{C}'$.

Next we show that Ψ is stationary. For any measurable mapping $f\colon \mathbb{R}^d \times \mathcal{C}' \to [0, \infty)$

and $x \in \mathbb{R}^d$ we have, by (3.4), that

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathcal{C}'} f\, d(T_x\Psi) &= \int_{\mathbb{R}^d\times\mathcal{C}'} f(y+x,C)\ \Psi(d(y,C))\\ &= \int_{\mathcal{C}'} f(c(K)+x, K-c(K))\ \Phi(dK)\\ &= \int_{\mathcal{C}'} f(c(K+x), K+x-c(K+x))\ \Phi(dK)\\ &= \int_{\mathcal{C}'} f(c(K), K-c(K))\ (T_x\Phi)(dK)\\ &\stackrel{d}{=} \int_{\mathcal{C}'} f(c(K), K-c(K))\ \Phi(dK) = \int_{\mathbb{R}^d\times\mathcal{C}'} f\, d\Psi,\end{aligned}$$

where we have used the stationarity of Φ in the last line. By Theorem 2.23, we conclude that $\Psi \stackrel{d}{=} T_x\Psi$. Now Theorem 2.42 implies $\mathbb{E}\,\Psi = \gamma\lambda_d \otimes \mathbb{Q}$ for some constant $\gamma \geq 0$ and some probability measure $\mathbb{Q}$ on $\mathcal{C}'$. It is also clear from the definition that $\mathbb{Q}$ is concentrated on $\mathcal{C}_0$, i.e. $\mathbb{Q}(\mathcal{C}_0) = 1$.

Let $T : \mathbb{R}^d \times \mathcal{C}_0 \to \mathcal{C}'$ be defined by $T(x, C) = C + x$. The mapping T is bijective with $T^{-1}(C) = (c(C), C - c(C))$. Therefore, $\Psi = T^{-1}\Phi$ and $\Phi = T\Psi$, and thus

$$\begin{aligned}\Theta = \mathbb{E}\Phi = \mathbb{E}T(\Psi) = T(\mathbb{E}\Psi) = T(\tilde{\gamma}\lambda_d \otimes \mathbb{Q})\\ = \gamma \int\int \mathbb{1}\{K + x \in \cdot\}\, dx\, \mathbb{Q}(dK).\end{aligned}$$

Moreover, since $\lambda_d(W_0) = 1$,

$$\gamma = \gamma(\lambda_d \otimes \mathbb{Q})(W_0 \times \mathcal{C}_0) = \mathbb{E}\Psi(W_0 \times \mathcal{C}_0) = \mathbb{E}\int \mathbb{1}\{c(C) \in W_0\}\, \Phi(dC) = \tilde{\gamma}.$$

This shows in particular the uniqueness of γ and that $\gamma > 0$ since $\Theta \neq 0$. The uniqueness of $\mathbb{Q}$ can be seen from Remark 3.7 below. □

Remark 3.7 Under the assumptions of Theorem 3.6 we have for any set $B \in \mathcal{B}^d$ with $0 < \lambda_d(B) < \infty$

$$\gamma = \frac{1}{\lambda_d(B)}\mathbb{E}\left[\int_{\mathcal{C}'} \mathbb{1}_B(c(K))\ \Phi(dK)\right].$$

Indeed, setting $\mathcal{H} := \{K \in \mathcal{C}' \colon c(K) \in B\}$, equation (3.5) implies

$$\begin{aligned}\mathbb{E}\left[\int_{\mathcal{C}'} \mathbb{1}\{c(K) \in B\}\ \Phi(dK)\right] &= \Theta(\mathcal{H})\\ &= \gamma \int_{\mathcal{C}_0}\int_{\mathbb{R}^d} \mathbb{1}\{c(K+x) \in B\}\, dx\, \mathbb{Q}(dK)\\ &= \gamma \int_{\mathcal{C}_0} \lambda_d(B - c(K))\ \mathbb{Q}(dK)\\ &= \gamma \int_{\mathcal{C}_0} \lambda_d(B)\ \mathbb{Q}(dK) = \gamma\lambda_d(B).\end{aligned}$$

A similar argument shows that

$$\mathbb{Q}(\mathcal{H}) = \frac{1}{\gamma\lambda_d(B)}\mathbb{E}\left[\int_{\mathcal{C}'} \mathbb{1}_B(c(K))\mathbb{1}_{\mathcal{H}}(K - c(K))\,\Phi(dK)\right]$$

for any measurable set $\mathcal{H} \subset \mathcal{C}'$, from which the uniqueness of $\mathbb{Q}$ is clear.

Definition 3.8 For any stationary particle process Φ with locally finite intensity measure, the number γ and the measure $\mathbb{Q}$ in Theorem 3.6 are called *intensity* and *shape distribution* of Φ (with respect to c), respectively.

Remark 3.9 The intensity γ can be interpreted as the 'number of centers per unit volume' or more precisely, as the 'number of particles whose center lies in a given set of unit volume'. From this interpretation it is intuitively clear that the intensity does not depend on the chosen center function. We refer to Work sheet 7 for a proof. In contrast, the shape distribution $\mathbb{Q}$ clearly depends on the choice of the center function. If $\tilde{c}$ is another center function and $\tilde{\mathbb{Q}}$ the corresponding shape distribution, then already the resulting family $\tilde{\mathcal{C}}_0$ of shapes (i.e. of sets with center 0) may be different. It can be shown that the distribution $\mathbb{Q}'$ can be obtained from $\mathbb{Q}$ as the image measure under a suitable transformation, see Work sheet 7.

Remark 3.10 Not every probability measure $\mathbb{Q}$ on $\mathcal{C}$ can arise as the shape distribution of some process Φ under the assumptions of Theorem 3.6. For any $C \in \mathcal{C}'$, we have

$$\begin{aligned}\infty > \Theta(\mathcal{F}_C) &= \gamma \int_{\mathcal{C}'} \int_{\mathbb{R}^d} \mathbb{1}\{(K + x) \cap C \neq \emptyset\}\, dx\, \mathbb{Q}(dK)\\ &= \gamma \int_{\mathcal{C}_0} \lambda_d(K + C^*)\, \mathbb{Q}(dK),\end{aligned}$$

where $C^* := \{y \in \mathbb{R}^d : -y \in C\}$ is the point reflection of C at the origin. Note that $(K + x) \cap C \neq \emptyset$ if and only if $x \in K + C^*$. Therefore,

$$\int_{\mathcal{C}_0} \lambda_d(K + C^*)\, \mathbb{Q}(dK) < \infty, \quad C \in \mathcal{C}^d \tag{3.6}$$

is a necessary condition for $\mathbb{Q}$ to be a shape distribution.

Theorem 3.11 *Let Φ and c be as in Theorem 3.6. Assume additionally that Φ is a Poisson process. Then*

$$\Psi := \int_{\mathcal{C}'} \mathbb{1}\{(c(K), K - c(K)) \in \cdot\}\, \Phi(dK)$$

is a Poisson process on $\mathbb{R}^d \times \mathcal{C}'$.

Proof. We use the mapping theorem for Poisson processes, which has been discussed on Work sheet 5, Problem 3, and apply it to the mapping T from above, that is, $T(K) = (c(K), K - c(K))$ for $K \in \mathcal{C}'$. It is easy to see that for any bounded set $\mathcal{H} \subset \mathbb{R}^d \times \mathcal{C}'$ (bounded w.r.t. the product metric) the preimage $T^{-1}(\mathcal{H})$ is also bounded in $\mathcal{C}'$ (w.r.t. the Hausdorff metric δ), and for any bounded set $\mathcal{H}' \in \mathcal{B}(\mathcal{C}')$, $\Phi(\mathcal{H}') < \infty$ almost surely (which is implied by the local finiteness of Φ). We note, in particular, that

$$\begin{aligned}\mathbb{E}\, T(\Phi)(B \times \mathcal{C}') &= \gamma \int_{\mathcal{C}'} \int_{\mathbb{R}^d} \mathbb{1}\{c(K + x) \in B\}\, dx\, \mathbb{Q}(dK)\\ &= \gamma \int_{\mathcal{C}'} \lambda_d(B)\, \mathbb{Q}(dK) < \infty,\end{aligned}$$

for any bounded $B \in \mathcal{B}^d$, so that $\mathbb{E}\Psi(\cdot \times \mathcal{C}')$ is locally finite. □

In the case of Poisson processes there is also a converse to the above statement. A stationary marked Poisson process can be represented as a particle process, provided that the mark distribution $\mathbb{Q}$ is a valid grain distribution for a particle process, see Remark 3.10.

Theorem 3.12 *Let Ψ be a Poisson process on $\mathbb{R}^d \times \mathcal{C}'$ with intensity measure $\gamma\lambda_d \otimes \mathbb{Q}$ for some $\gamma > 0$ and some probability measure $\mathbb{Q}$ on $\mathcal{C}'$. Assume that $\mathbb{Q}$ satisfies condition (3.6). Then*

$$\Phi(\cdot) := \int_{\mathbb{R}^d \times \mathcal{C}'} \mathbb{1}\{K + x \in \cdot\}\, \Psi(d(x, K))$$

is a stationary Poisson particle process on $\mathbb{R}^d$.

Proof. As a Poisson process, Ψ is uniquely determined (in distribution) by its intensity measure. Hence Ψ is stationary. Condition (3.6) ensures that Φ is a particle process on $\mathbb{R}^d$ (see also Work sheet 7, Problem 2). Applying again the mapping theorem, we see that Φ is stationary and a Poisson process. □

3.2 The Boolean model

Now let Φ be a particle process in $\mathbb{R}^d$ with locally finite intensity measure and let Ψ be the corresponding marked point process (with respect to a given center function c) with mark space $\mathcal{C}_0$. Recall that Ψ is called a germ-grain process. By Theorem 2.8 and Remark 2.13, we have representations

$$\Phi = \sum_{i=1}^{\tau} \delta_{C_i} \qquad \text{and} \qquad \Psi = \sum_{i=1}^{\tau} \delta_{(x_i, K_i)}$$

where $\tau \in \mathbb{N}_0 \cup \{\infty\}$ is a random variable and $C_1, C_2, \ldots$ are random elements in $\mathcal{C}'$. Further, $x_i = c(C_i)$ and $K_i = C_i - x_i$, for any $i \in \mathbb{N}$. It is clear that Φ is stationary if and only if Ψ is stationary.

For given Φ or Ψ the union set

$$Z := \bigcup_{K \in \Phi} K = \bigcup_{i=1}^{\tau} C_i = \bigcup_{i=1}^{\tau} (K_i + x_i)$$

is called the generated *germ-grain model*. Recall that the definition of a particle process includes a finiteness assumption on compact subsets of $\mathcal{C}'$, which implies in particular that $\Phi(\mathcal{F}_C)$ is finite a.s. for any $C \in \mathcal{C}^d$, see (3.3). This allows to conclude that Z is closed (almost surely) and thus a random closed set.

Lemma 3.13 *Let Φ be a particle process. Then the generated germ-grain model Z is closed almost surely and thus a random closed set.*

Proof. Let B_n, $n \in \mathbb{N}$ be an increasing sequence of compact sets in $\mathbb{R}^d$ such that $B_n \subset \operatorname{int}(B_{n+1})$ and $\bigcup_n B_n = \mathbb{R}^d$, e.g. $B_n = nB^d$. Let Ω_0 be the set of all $\omega \in \Omega$ such that $\Phi(\omega)(\mathcal{F}_{B_n}) < \infty$ for all $n \in \mathbb{N}$. Then $\mathbb{P}(\Omega_0) = 1$ and it suffices to show that $Z(\omega)$ is closed for all $\omega \in \Omega_0$.

Therefore let $\omega \in \Omega_0$ and let $(z_i)_{i \in \mathbb{N}}$ be a sequence of points in $Z(\omega)$ which converges to some $z \in \mathbb{R}^d$ as $i \to \infty$. Then there exists some $n \in \mathbb{N}$ such that $z_i \in B_n$ for all $i \in \mathbb{N}$.

Since $\Phi(\omega)(\mathcal{F}_{B_n}) < \infty$, we find $N \in \mathbb{N}$ and particles C_j, $j \in \{1, \dots, N\}$ in the support of $\Phi(\omega)$ such that

$$Z(\omega) \cap B_n = \left(\bigcup_{j=1}^{N} C_j \right) \cap B_n.$$

We infer that $z_i \in \bigcup_{j=1}^{N} C_j \in \mathcal{C}'$ for all $i \in \mathbb{N}$ and thus

$$z \in \bigcup_{j=1}^{N} C_j \subset Z(\omega),$$

which shows that $Z(\omega)$ is closed. □

We reconsider and extend the Examples 1.26 and 1.27 from above and clarify that they define indeed germ-grain models.

Example 3.14 Let $(\xi_n)_{n\in\mathbb{N}}$ be the points of the lattice $\mathbb{Z}^2$ (in some enumeration).

(a) In Ex. 1.26 we considered an i.i.d. sequence of random variables R_n, $n \in \mathbb{N}$, uniformly distributed on the interval $[0, 1]$ and generated balls $Z_n := B(0, R_n)$ in $\mathbb{R}^2$ centred at the origin 0 and with random radius R_n. Then

$$\Phi = \sum_{n\in\mathbb{N}} \delta_{\xi_n + Z_n} \tag{3.7}$$

defines a particle process in $\mathbb{R}^2$. Indeed, for any $C \in \mathcal{C}'$, we have

$$\begin{aligned} \Phi(\mathcal{F}_C) &= \#\{\xi_n : \xi_n + Z_n \cap C \neq \emptyset\} \\ &\leq \#\{\xi_n : B(\xi, 1) \cap C \neq \emptyset\} = \#(\mathbb{Z}^2 \cap (C + B(0, 1))) < \infty, \end{aligned}$$

which implies that Φ is a.s. finite on compact sets. Due to the independence of the R_n, the associated marked point process $\Psi = \sum_{n\in\mathbb{N}} \delta_{(\xi_n, Z_n)}$ is an independent $\mathbb{Q}$-marking, where $\mathbb{Q}$ is the uniform distribution on the set of balls with radius between 0 and 1. The underlying 'germ process' is deterministic, it is the counting measure on $\mathbb{Z}^2$. Note that Ψ is not stationary. A germ at the origin is more likely than a germ at $(1/2, 1/2)$.

(b) In (a) the generated balls may overlap. Now we discuss a model that avoids such overlaps. For each $n \in \mathbb{N}$, let $S_n := \min\{R_n, \min\{d(\xi_n, \xi_j) - R_j : j = 1, \dots, n-1\}\}$ (where R_n is as in part (a)) and set $Z_n := B(0, S_n)$. Observe that balls may touch but they will never overlap. The random variables of the sequence (S_n) are stochastically dependent (and therefore also the primary grains of the sequence (Z_n)). Again Φ as in (3.7) defines a particle process, but the associated germ-grain process Ψ is not an independent $\mathbb{Q}$-marking anymore. It is not even a marking, as the marks are dependent on each other. (This exemplifies that not every marked point process is a marking.)

Example 3.15 In Ex. 1.27, we considered a sequence $(\xi_n)_{n\in\mathbb{N}}$ of independent random points uniformly distributed in $W := [0, 1]^2 \subset \mathbb{R}^2$ and a random variable $\tau \sim \mathrm{Poiss}(\gamma)$ with some parameter $\gamma > 0$. Meanwhile we already know that $\Phi = \sum_{i=1}^{\tau} \delta_{\xi_n}$ is a (homogeneous) Poisson point process on W with intensity γ. If we use this as a germ process and attach unit balls to the germs, then, since everything is finite here, a germ-grain process Ψ is generated by $\Psi := \sum_{n=1}^{\tau} \delta_{(\xi_n, B^2)}$ and a germ-grain model by $Z := \bigcup_{n=1} \xi_n + B_2$.

This was extended to a process on $\mathbb{R}^2$ by placing independent copies of Ψ in the cells of the square lattice. But now we can generate this models on the whole plane $\mathbb{R}^2$ directly by starting with a Poisson process on $\mathbb{R}^2$ for the germs. (Recall from the proof of Theorem 2.31 that this generates indeed the same process). This leads to Boolean models discussed in detail below.

Now let Φ be a Poisson particle process with locally finite intensity measure Θ. Then the generated germ-grain model

$$Z := \bigcup_{K \in \Phi} K$$

is called *Boolean model*. Note that the definition of a Boolean model includes the local finiteness assumption on Θ. If Φ is stationary, then Z is stationary, too (why? - see Work sheet 8). In this case, Φ is uniquely determined in distribution by its intensity measure Θ, and hence by the intensity γ and the shape distribution $\mathbb{Q}$ of the associated marked point process Ψ (given a center function), see Theorem 3.6. A random closed set Z_0 with distribution $\mathbb{Q}$ is called *typical grain* of Z. Recall from Remark 3.10, that the local finiteness of Θ implies the relation (3.6). Setting $C = \{0\}$ in this equation, we see in particular that

$$\mathbb{E}\lambda_d(Z_0) = \int_{\mathcal{C}'} \lambda_d(K)\mathbb{Q}(dK) < \infty, \tag{3.8}$$

i.e., the typical grain Z_0 of Z has finite expected (and thus almost sure) volume. To understand the geometry of Boolean models Z, we first look at their capacity functionals T_Z, cf. Definition 1.32. By Corollary 1.35, stationarity of Z is equivalent to the translation invariance of T_Z.

Theorem 3.16 *Let Φ be a Poisson particle process with locally finite intensity measure Θ. Let Z be the resulting Boolean model. Then the following assertions hold.*

(a) *For any $C \in \mathcal{C}^d$,*

$$T_Z(C) = 1 - \exp(-\Theta(\mathcal{F}_C)). \tag{3.9}$$

(b) *The distribution of the Boolean model Z determines uniquely the distribution of the generating Poisson particle process Φ.*

(c) *If Φ is stationary and determined by $(\gamma, \mathbb{Q})$, then*

$$\begin{aligned} T_Z(C) &= 1 - \exp\left(-\gamma \int_{\mathcal{C}_0^d} \lambda_d(K + C^*)\, \mathbb{Q}(dK)\right) \\ &= 1 - \exp\left(-\gamma \mathbb{E}\lambda_d(Z_0 + C^*)\right). \end{aligned} \tag{3.10}$$

where Z_0 is the typical grain of Φ, i.e. an RCS with distribution $\mathbb{Q}$.

(d) *If Z is a Boolean model and stationary (as a random closed set), then the generating particle process Φ is necessarily stationary and hence, Z is a stationary Boolean model.*

Proof. (a) For any $C \in \mathcal{C}^d$, we have

$$T_Z(C) = \mathbb{P}(Z \cap C \neq \emptyset) = 1 - \mathbb{P}(Z \cap C = \emptyset) = 1 - \mathbb{P}(\Phi(\mathcal{F}_C) = 0).$$

Now the Poisson property of Φ implies $\mathbb{P}(\Phi(\mathcal{F}_C) = 0) = \exp(-\Theta(\mathcal{F}_C))$ from which the assertion is clear.
(b) Since Φ is a Poisson process, its distribution is uniquely determined by its intensity measure Θ, cf. Remark 2.27 (iii). Therefore it is enough to show that the distribution of Z determines Θ. By the formula in (a), Z determines Θ on sets $\mathcal{F}_C$, $C \in \mathcal{C}^d$ via the capacity functional. Noting that any locally finite measure μ on $\mathcal{F}'$ is uniquely determined by its values $\mu(\mathcal{F}_C)$ for $C \in \mathcal{C}^d$ completes the proof, see also Work sheet 8.
(c) By Theorem 3.6, we have

$$\begin{aligned}\Theta(\mathcal{F}_C) &= \gamma \int \int \mathbb{1}\{(K + x) \cap C \neq \emptyset\}\, \lambda_d(dx)\, \mathbb{Q}(dK) \\ &= \gamma \int \int \mathbb{1}\{x \in C + K^*\}\, \lambda_d(dx)\, \mathbb{Q}(dK) \\ &= \gamma \int \lambda_d(K + C^*)\, \mathbb{Q}(dK).\end{aligned}$$

Inserting this into the formula in (a) proves the assertion.
(d) Recall that a random closed set Z is stationary if and only if T_Z is translation invariant. Due to the assertion in (b), the latter implies that Θ is translation invariant. This shows the stationarity of Φ. □

Setting $C = \{0\}$ in (3.10), we infer for the volume fraction p_Z of a stationary Boolean model Z that

$$p_Z = \mathbb{P}(0 \in Z) = T_Z(\{0\}) = 1 - \exp(-\gamma \mathbb{E}\lambda_d(Z_0)). \tag{3.11}$$

This is a neat and simple formula. It is no surprise that p_Z depends on both, the intensity γ and the shape distribution $\mathbb{Q}$, but the latter affects the volume fraction only via the expected volume of the shapes, while the shape itself is irrelevant.

Recall that an RCS Z in $\mathbb{R}^d$ is called isotropic if and only if $\vartheta Z \stackrel{d}{=} Z$ for any $\vartheta \in \mathrm{SO}_d$. Isotropy of a stationary Boolean model can be achieved by requiring the typical grain to be isotropic.

Theorem 3.17 *Let Z be a stationary Boolean model with typical grain Z_0. If Z_0 is isotropic then Z is isotropic.*

Proof. For any $\vartheta \in \mathrm{SO}_d$, the mapping $\mathcal{F}^d \to \mathcal{F}^d, A \mapsto \vartheta A$ is continuous (and thus measurable) and therefore, ϑZ is an RCS. Moreover, we infer from equation (3.10) that for any $C \in \mathcal{C}^d$,

$$\begin{aligned}T_{\vartheta Z}(C) &= \mathbb{P}((\vartheta Z) \cap C \neq \emptyset) = \mathbb{P}(Z \cap (\vartheta^{-1}C) \neq \emptyset) = T_Z(\vartheta^{-1}C) \\ &= 1 - \exp\big(-\gamma \mathbb{E}[\lambda_d(Z_0 + (\vartheta^{-1}C)^*)]\big) \\ &= 1 - \exp\big(-\gamma \mathbb{E}[\lambda_d(Z_0 + \vartheta^{-1}C^*)]\big) \\ &= 1 - \exp\big(-\gamma \mathbb{E}[\lambda_d(\vartheta Z_0 + C^*)]\big) = T_Z(C),\end{aligned}$$

where we have used the isotropy of Z_0 in the last step. □

In general, the converse is not true, i.e. without additional assumptions the isotropy of Z does not imply the isotropy of the typical grain, see also Work sheet 8. Another

functional of interest is the covariance function, which describes the probability that two given points both belong to a random closed set Z. In the stationary case, one of the points can be fixed (e.g. to be the origin) and then the covariance function is a mapping from $\mathbb{R}^d$ to $[0,1]$.

Definition 3.18 The *covariance function* of a stationary RCS Z is defined by

$$C(x) := \mathbb{P}(0 \in Z, x \in Z), \quad x \in \mathbb{R}^d.$$

Remarks 3.19 (i) The stationarity of Z implies that

$$C(x) = \mathbb{P}(y \in Z, x + y \in Z), \quad x, y \in \mathbb{R}^d.$$

(ii) For an interpretation of the covariance function (and a justification for the terminology), consider the random field $Y(x) := \mathbb{1}_Z(x)$, $x \in \mathbb{R}^d$. Then

$$\mathrm{Cov}(Y(0), Y(x)) = \mathbb{E}[Y(0)Y(x)] - \mathbb{E}[Y(0)]\mathbb{E}[Y(x)] = C(x) - p_Z^2.$$

The following statement provides a formula for the covariance function of a stationary Boolean model Z in terms of its volume fraction p_Z and the so called *covariagram* of the typical grain Z_0.

Theorem 3.20 *For any stationary Boolean model Z, the covariance function is given by*

$$C(x) = 2p_Z - 1 + (1 - p_Z)^2 \exp\big(\gamma C_0(x)\big), \qquad x \in \mathbb{R}^d,$$

where

$$C_0(x) := \mathbb{E}[\lambda_d(Z_0 \cap (Z_0 - x))] = \mathbb{E}[\lambda_d(Z_0 \cap (Z_0 + x))]$$

is the expected covariogram *of Z_0 at x.*

Proof. The proof is discussed on Work sheet 8. □

One of the consequences of this statement is that the local behaviour of Z at locations that are far away from each other is almost independent.

Corollary 3.21 For any stationary Boolean model it holds

$$\lim_{\|x\| \to \infty} C(x) = p_Z^2,$$

that is,

$$\lim_{\|x\| \to \infty} \mathbb{P}(0 \in Z, x \in Z) = \mathbb{P}(0 \in Z)\mathbb{P}(x \in Z).$$

Furthermore, if $\lambda_d(\partial Z_0) = 0$ almost surely, then

$$\lim_{x \to 0} C(x) = p_Z.$$

The first statement means that the events $\{0 \in Z\}$ and $\{x \in Z\}$ are asymptotically independent, as the distance between 0 and x tends to ∞, while the second statement clarifies that the covariance function is continuous at 0, as $C(0) = \mathbb{P}(0 \in Z) = p_Z$.

Proof. Since the assumed local finiteness of Θ implies $\mathbb{E}[\lambda^d(Z_0)] < \infty$, see (3.8), we have, by dominated convergence,

$$\lim_{\|x\|\to\infty} C_0(x) = \lim_{\|x\|\to\infty} \mathbb{E}\left[\lambda^d(Z_0 \cap (Z_0+x))\right] = \mathbb{E}\left[\lim_{\|x\|\to\infty} \lambda^d(Z_0 \cap (Z_0+x))\right] = 0.$$

Inserting this into the formula derived in Theorem 3.20 (and noting that the expression on the right hand side of this formula is a continuous function of $C_0(x)$) shows the first limit. For the second limit observe that

$$\begin{aligned}
\lim_{x\to 0} C_0(x) = \lim_{x\to 0} \mathbb{E}\left[\lambda^d(Z_0 \cap (Z_0+x))\right] &= \mathbb{E}\left[\lim_{x\to 0} \lambda^d(Z_0 \cap (Z_0+x))\right] \\
&= \mathbb{E}\left[\lim_{x\to 0} \int \mathbb{1}_{Z_0}(y)\mathbb{1}_{Z_0+x}(y)\, \lambda^d(dy)\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}^d} \lim_{x\to 0} \mathbb{1}_{Z_0}(y)\mathbb{1}_{Z_0+x}(y)\, \lambda^d(dy)\right] \\
&= \mathbb{E}\left[\lambda^d(\mathrm{int}Z_0)\right] = \mathbb{E}\left[\lambda^d(Z_0)\right],
\end{aligned}$$

where we have used again the theorem of dominated convergence (2 times). Moreover, in the last line we have used that $\lim_{x\to 0} \mathbb{1}_{Z_0+x}(y) = 1$ for any point $y \in \mathrm{int}Z_0$ and $\lim_{x\to 0} \mathbb{1}_{Z_0+x}(y) = 0$ for any point $y \in Z_0^c$. The limit for $y \in \partial Z_0$ does not exist but ∂Z_0 is a null set by assumption and thus negligible for the integration. Taking into account (3.11), we conclude that

$$\lim_{x\to 0} C(x) = 2p_Z - 1 + (1-p_Z)^2 \exp\left(\gamma \mathbb{E}[\lambda^d(Z_0)]\right) = 2p_Z - 1 + 1 - p_Z = p_Z. \quad \square$$

Example 3.22 To illustrate that the expected covariogram can be computed explicitly in certain situations (and hence the covariance function), we consider the case that the typical grain is a ball in $\mathbb{R}^3$ with random radius R. Assume that $Z_0 \stackrel{d}{=} B(0,R)$ for some nonnegative random variable R satisfying $\mathbb{E}(R^3) < \infty$ (compare with (3.8)). Denote by F the distribution of R. Then

$$\mathbb{E}[\lambda^3(Z_0)] = \mathbb{E}\left[\frac{4}{3}\pi R^3\right] = \frac{4}{3}\pi \int_0^\infty r^3\, F(dr).$$

Moreover, parameterizing the intersection $B(0,R) \cap B(x,R)$ by means of slices perpendicular to the direction x, we get

$$\begin{aligned}
C_0(x) &= 2\mathbb{E} \int \mathbb{1}\{\|x\|/2 \le t \le R\}\pi(R^2 - t^2)\, dt \\
&= \frac{4}{3}\pi \int_{\frac{\|x\|}{2}}^\infty r^3 \left(1 - \frac{3\|x\|}{4r} + \frac{\|x\|^3}{16r^3}\right) F(dr).
\end{aligned}$$

In the sequel, we denote by $\mathcal{K}^d$ the family of all convex, compact subsets of $\mathbb{R}^d$. The elements of $\mathcal{K}^d$ are called *convex bodies*. Furthermore, we set

$$\mathcal{K}' := \mathcal{K}^d \setminus \{\emptyset\}.$$

Another concept to quantify shape and size information of a random closed set Z are contact distribution functions. They provide information about how far a 'typical' point

in the complement of Z is away from Z. More precisely, it measures for a any distance r how likely it is for a point in the complement of Z to have distance to Z at most r. This gives e.g. an impression whether a random structure Z has a few large holes or rather many small ones. Using different structuring elements one can also obtain shape information, e.g. whether holes are round or elongated in some direction.

Definition 3.23 Let $B \in \mathcal{K}^d$ such that $0 \in B$. For a stationary RCS Z in $\mathbb{R}^d$, the mapping

$$r \mapsto H_B(r) := 1 - \mathbb{P}(Z \cap rB = \emptyset \mid 0 \notin Z), \qquad r \geq 0,$$

is called *contact distribution function* of Z (with respect to the *gauge body* or *structuring element* B).

Frequently used structuring elements B are Euclidean balls and segments.

Definition 3.24 If $B = B^d$, then the function H_{B^d} is called *spherical contact distribution function*. If $B = [0, u]$ is a closed segment with endpoints 0 and u, where $u \in \mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : ||x|| = 1\}$, then the function $H_B = H_{[0,u]}$ is called *linear contact distribution function* (in direction u).

Remark 3.25 In case $\dim(B) \leq d - 1$ (e.g. if B is a segment $[0, u]$), it can happen that

$$H_B(\infty) := \lim_{r\to\infty} H_B(r) < 1.$$

If $\mathbb{P}(0 \notin Z) > 0$, then the requirement $0 \in B$ implies that

$$H_B(r) = 1 - \frac{\mathbb{P}(Z \cap rB = \emptyset, 0 \notin Z)}{\mathbb{P}(0 \notin Z)} = 1 - \frac{\mathbb{P}(Z \cap rB = \emptyset)}{\mathbb{P}(0 \notin Z)} = 1 - \frac{1 - T_Z(rB)}{1 - p_Z}. \quad (3.12)$$

Note that due to (3.8) we have $\mathbb{P}(0 \notin Z) = 1 - p_Z = \exp\big(-\gamma\mathbb{E}[\lambda_d(Z_0)]\big) > 0$ in case Z is a stationary Boolean model.

Remark 3.26 For a motivation of the terminology 'contact distribution function' consider the random variable

$$R := \inf\{r > 0 : Z \cap rB \neq \emptyset\}$$

where we set $\inf \emptyset := \infty$ as usual. R may be interpreted as the 'contact radius', i.e. the radius r for which a growing 'ball' rB of radius r first touches Z. Observe that

$$\{R \leq r\} = \{Z \cap rB \neq \emptyset\},$$

where the inclusion "'$\subseteq$"' follows from the fact that both sets are closed (and the other inclusion "'$\supseteq$"' is obvious). Moreover, $\{R = 0\} = \{0 \in Z\}$. Therefore,

$$H_B(r) = \mathbb{P}(R \leq r \mid R > 0).$$

For a stationary Boolean model Z, an explicit formula for the contact distribution function H_B can be given in terms of the intensity γ and the typical grain Z_0 .

Theorem 3.27 *Let Z be a stationary Boolean model with intensity γ and typical grain Z_0. Then for any structuring element $B \in \mathcal{K}'$, one has*

$$1 - H_B(r) = \exp\big(-\gamma\mathbb{E}[\lambda_d((Z_0 + rB^*) \setminus Z_0)]\big).$$

Proof. Combining (3.12), (3.11) and Theorem 3.16, we infer that

$$\begin{aligned} 1 - H_B(r) &= \frac{\exp\big(-\gamma\mathbb{E}[\lambda_d(Z_0 + rB^*)]\big)}{\exp\big(-\gamma\mathbb{E}[\lambda_d(Z_0)]\big)} \\ &= \exp\big(-\gamma(\mathbb{E}[\lambda_d(Z_0 + rB^*)] - \mathbb{E}[\lambda_d(Z_0)])\big) \\ &= \exp\big(-\gamma\mathbb{E}[\lambda_d((Z_0 + rB^*) \setminus Z_0)]\big). \end{aligned}$$

For the last equality note that $Z_0 \subseteq Z_0 + rB^*$. □

Example 3.28 Assume that the typical grain Z_0 of a stationary Boolean model Z in $\mathbb{R}^d$ satisfies $Z_0 \stackrel{d}{=} B(0, R_0)$, where R_0 is some nonnegative random variable. Then Theorem 3.27 yields for the spherical contact distribution function of Z

$$\begin{aligned} H_{B^d}(r) &= 1 - \exp\big(-\gamma\mathbb{E}[\lambda_d((R_0 + r)B^d) \setminus R_0B^d)]\big) \\ &= 1 - \exp\big(-\gamma\kappa_d\mathbb{E}[(R_0 + r)^d - R_0^d]\big) \\ &= 1 - \exp\left(-\gamma\kappa_d\mathbb{E}\left[\sum_{j=0}^{d-1}\binom{d}{j}R_0^j r^{d-j}\right]\right) \\ &= 1 - \exp\left(-\gamma\kappa_d\sum_{j=1}^{d-1}\binom{d}{j}r^{d-j}\mathbb{E}[R_0^j]\right). \end{aligned}$$

Here $\kappa_d := \lambda_d(B^d)$ is the volume of the unit ball in $\mathbb{R}^d$. We will return to this example later on. See also Work sheet 8 for an application.

3.3 Intrinsic volumes and the Steiner formula

For analyzing the geometry of complex random structures, geometric descriptors are needed which are on the one hand able to quantify certain geometric properties and on the other hand easy to compute. We will now discuss a family of geometric functionals, called intrinsic volumes, which turn out to be very useful in this respect. They share many nice properties with the well known volume such as translation invariance, additivity and a nice scaling behaviour. Here we introduce intrinsic volumes for convex bodies. Later we will also discuss extensions to more general sets.

In the sequel, we write $V_d(K) := \lambda_d(K)$ for the d-dimensional volume of a Borel set $K \subset \mathbb{R}^d$, and we denote by $S_{d-1}(K)$ its *surface area*, which is formally defined as the $(d-1)$-dimensional Hausdorff measure of the boundary ∂K of K. If ∂K is sufficiently regular (i.e. rectifiable, which is in particular satisfied for the boundaries of convex sets), then $S_{d-1}(K)$ coincides with the *outer Minkowski content* of K, i.e.

$$S_{d-1}(K) = M_{d-1}(K) := \lim_{\varepsilon \searrow 0}\frac{1}{\varepsilon}(V_d(K_{\oplus\varepsilon}) - V_d(K)). \tag{3.13}$$

Recall from Def. 1.13 and Remark 1.14 that $K_{\oplus\varepsilon} := \{x \in \mathbb{R}^d \colon d(x, K) \le \varepsilon\}$ is the ε-parallel set of K of radius $\varepsilon \ge 0$ and that, for any $K \in \mathcal{C}^d$ and $\varepsilon \ge 0$,

$$K_{\oplus\varepsilon} = K + \varepsilon B^d := \{x + y \colon x \in K,\ y \in \varepsilon B^d\},$$

see also Work sheet 3, Problem 1 for a proof of this equality. We are interested in the *parallel volume* $V_d(K + \varepsilon B^d)$ of K as a function of the parallel radius ε. First note

that if (3.13) holds, then the existence of the limit in this formula may be reinterpreted as follows

$$V_d(K_{\oplus\varepsilon}) = V_d(K) + \varepsilon S_{d-1}(K) + o(\varepsilon),$$

as $\varepsilon \searrow 0$, i.e. the remainder term on the right hand side is small compared to ε, as $\varepsilon \searrow 0$. To get some idea of the remainder term for convex sets, we look at two examples.

Example 3.29 Let $d = 2$ and let $K = rB^2$ be a ball in $\mathbb{R}^2$ of radius $r > 0$. Then

$$\begin{aligned} V_2(K + \varepsilon B^2) = V_2((r+\varepsilon)B^2) &= \pi(r+\varepsilon)^2 = \pi r^2 + 2\pi r\varepsilon + \pi\varepsilon^2 \\ &= V_2(K) + \varepsilon S_1(K) + \pi\varepsilon^2. \end{aligned}$$

Example 3.30 Let $K \subset \mathbb{R}^2$ be a convex polytope. (Think first of a rectangle or a triangle and make a sketch of some parallel sets.) It is not difficult to see that

$$V_2(K + \varepsilon B^2) = V_2(K) + \varepsilon S_1(K) + \pi\varepsilon^2$$

As for the ball, we find that the parallel volume is a polynomial of degree 2 in ε and the linear coefficient is the boundary length.

One can ask for which sets K such a polynomial behaviour can be observed for the parallel volume. It turns out that this is a general phenomenon for convex bodies $K \in \mathcal{K}^d$ in any dimension d. The parallel volume is a polynomial in ε of degree at most d, as the next statement shows. We will be particularly interested in the coefficients of this polynomial, which are called intrinsic volumes and turn out to be important geometric characteristics of the body K. Before we introduce some useful constants.

Definition 3.31 For any $d \in \mathbb{N}$ let

$$\kappa_d := V_d(B^d) = \frac{\pi^{d/2}}{\Gamma((d/2)+1)} \quad \text{and} \quad \omega_d := S_{d-1}(B^d)$$

be the volume and surface area of the unit ball B^d in $\mathbb{R}^d$, respectively. For convenience, we also set $\kappa_0 := 1$ and $\omega_0 := 0$.

Note that $\kappa_1 = 2, \kappa_2 = \pi, \kappa_3 = \frac{4}{3}\pi, \ldots$. Moreover, we have for the surface area of B^d

$$\omega_d = S_{d-1}(B^d) = \lim_{\varepsilon\searrow 0} \frac{1}{\varepsilon}\left(V_d((1+\varepsilon)B^d) - V_d(B^d)\right) = \kappa_d \lim_{\varepsilon\searrow 0} \frac{1}{\varepsilon}\left((1+\varepsilon)^d - 1\right) = d\kappa_d.$$

Theorem 3.32 (Steiner formula) *For $K \in \mathcal{K}^d$ there are uniquely determined numbers $V_0(K), \ldots, V_d(K) \geq 0$ such that, for each $\varepsilon \geq 0$,*

$$V_d(K + \varepsilon B^d) = \sum_{j=0}^{d} \kappa_{d-j}\varepsilon^{d-j} V_j(K). \tag{3.14}$$

Proof. As the case $K = \emptyset$ is trivial, we assume that $K \neq \emptyset$. For $x \in \mathbb{R}^d$ denote by $p(K, x)$ the nearest point in K to x, i.e., $p(K, x)$ is the (unique!) point in K that satisfies $\|x - p(K, x)\| = d(x, K)$. Observe that, for $x \in K$, $p(K, x) = x$ and, for $x \notin K$, $p(K, x) \in \partial K$. Due to the convexity of K, the point $p(K, x) \in K$ is uniquely determined (why?). $p(K, x)$ is called the *metric projection* of x onto K.

Every point $y \in \partial K$ is the image $p(K,x) = y$ of some point $x \notin K$ under the metric projection onto K and then the vector

$$u(K,x) := \frac{x - p(K,x)}{d(x,K)} \in \mathbb{S}^{d-1}, \quad x \notin K,$$

is called *(unit) outer normal* of K at $y = p(K,x)$. The notion is motivated by the observation that

$$K \subset \{z \in \mathbb{R}^d : \langle z - y, u(K,x)\rangle \leq 0\},$$

that is, K is completely contained in one of the half spaces bounded by the hyperplane

$$H_{u(K,x)} := \{z \in \mathbb{R}^d : \langle z - y, u(K,x)\rangle = 0\}.$$

(A *support hyperplane* of K is defined as a hyperplane H such that $K \cap H \neq \emptyset$ and $K \subset H^-$, where H^- is one of the two half spaces bounded by H. Thus, $H_{u(K,x)}$ is a support hyperplane of K.) We note that the set $N(K,y)$ of all outer normals of K at a given boundary point $y \in \partial K$ (and with the origin 0 added) forms a closed convex cone, i.e. a closed convex set that is closed with respect to positive linear combinations. $N(K,y)$ is called the *normal cone* of K at y.

We first consider the case that K is a *polytope*, that is, K is a bounded intersection of finitely many half spaces or, equivalently, the convex hull of finitely many points. For a polytope K we define a *face* of K as the intersection of K with a support hyperplane. A face F of K is called a *j-face* for some $j \in \{0,\dots,d-1\}$ if and only if the dimension $\dim F := \dim(\text{aff } F)$ of F equals j. (Here aff F denotes the affine hull of F, i.e. the smallest affine subspace containing F.) For any $j = 0,\dots,d-1$, we denote by $\mathcal{F}_j(K)$ the family of all j-faces of K. Note that the families $\mathcal{F}_j(K)$ are finite. We observe that the boundary of any polytope is the disjoint union of the relative interiors of its faces. Furthermore, given a face $F \in \bigcup_{j=0}^{d-1} \mathcal{F}_j(K)$ of K, the normal cones $N(K,y)$ are the same at all points $y \in \operatorname{relint} F$. Therefore, it makes sense to speak of the *normal cone* $N(K,F)$ of a polytope K at a face F. Further, we define the *outer angle* of K at F by

$$\gamma(F,K) := \frac{\mathcal{H}^{d-j-1}(N(K,F) \cap \mathbb{S}^{d-1})}{\mathcal{H}^{d-j-1}(L \cap \mathbb{S}^{d-1})}, \quad F \in \mathcal{F}_j(K),$$

where L denotes the orthogonal complement of the direction space of aff$F$ and $\mathcal{H}^k$ is the k-dimensional Hausdorff measure in $\mathbb{R}^d$. Due to the direct proportionality between the surface area of $N(K,F) \cap \mathbb{S}^{d-1}$ and the corresponding $(d-j)$-dimensional volume of $N(K,F) \cap B^d$, we have

$$\gamma(F,K) = \frac{\mathcal{H}^{d-j}(N(K,F) \cap B^d)}{\mathcal{H}^{d-j}(L \cap B^d)} = \frac{\mathcal{H}^{d-j}(N(K,F) \cap B^d)}{\kappa_{d-j}}.$$

The above considerations imply that each point $x \in \mathbb{R}^d \setminus K$ has a unique representation $x = y + z$ with $y = p(K,x) \in \partial K$ and $z \in N(K,y)$, and if $K \in \mathcal{K}^d$ is a polytope then the point y in this representation is in the relative interior of a unique face $F \in \bigcup_{j=0}^{d-1} \mathcal{F}_j(K)$.

Therefore, we can decompose the parallel volume as follows.

$$V_d(K+\varepsilon B^d)-V_d(K)=\int_{\mathbb{R}^d}\mathbb{1}\{x\in(K+\varepsilon B^d)\setminus K\}\,dx$$

$$=\sum_{j=0}^{d-1}\sum_{F\in\mathcal{F}_j(K)}\int_{\mathbb{R}^d}\mathbb{1}\{x\in(K+\varepsilon B^d)\setminus K,\ p(K,x)\in\operatorname{relint}F\}\,dx$$

$$=\sum_{j=0}^{d-1}\sum_{F\in\mathcal{F}_j(K)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbb{1}\left\{z\in N(K,F),||z||\le\varepsilon,y\in\operatorname{relint}F\right\}\,\mathcal{H}^{d-j}(dz)\,\mathcal{H}^j(dy)$$

$$=\sum_{j=0}^{d-1}\sum_{F\in\mathcal{F}_j(K)}\mathcal{H}^j(\operatorname{relint}F)\int_{\mathbb{R}^d}\mathbb{1}\left\{z\in N(K,F),||z||\le 1\right\}\,\mathcal{H}^{d-j}(dz)\cdot\varepsilon^{d-j}$$

$$=\sum_{j=0}^{d-1}\sum_{F\in\mathcal{F}_j(K)}\mathcal{H}^j(\operatorname{relint}F)\mathcal{H}^{d-j}(N(K,F)\cap B^d)\frac{\kappa_{d-j}}{\kappa_{d-j}}\cdot\varepsilon^{d-j}$$

$$=\sum_{j=0}^{d-1}\sum_{F\in\mathcal{F}_j(K)}\mathcal{H}^j(F)\cdot\gamma(F,K)\cdot\kappa_{d-j}\cdot\varepsilon^{d-j},$$

Setting now

$$V_j(K):=\sum_{F\in\mathcal{F}_j(K)}\mathcal{H}^j(F)\cdot\gamma(F,K),\quad j\in\{0,1,\dots,d\},\tag{3.15}$$

we obtain (3.14) for a polytope $K\in\mathcal{K}^d\setminus\{\emptyset\}$.

For general convex bodies we use approximation by polytopes. For any $K\in\mathcal{K}^d\setminus\{\emptyset\}$ there are polytopes K_n, $n\in\mathbb{N}$ such that $\delta(K_n,K)\to 0$ as $n\to\infty$ (see Work sheet 10). By (3.14), we have for any $n\in\mathbb{N}$ and $\varepsilon=k\in\{1,\dots,d+1\}$,

$$V_d(K_n+kB^d)=\sum_{j=0}^{d}\kappa_{d-j}k^{d-j}V_j(K_n).$$

For fixed $n\in\mathbb{N}$ this provides a system of $d+1$ linear equations which can be solved for the $d+1$ variables $V_0(K_n),\dots,V_d(K_n)$. Observe that this system has a unique solution (why?), which can be written in the form

$$V_j(K_n)=\sum_{k=1}^{d+1}\alpha_{jk}V_d(K_n+kB^d),\quad j=0,\dots,d.$$

Here the coefficients $\alpha_{jk}\in\mathbb{R}$ do not depend on K_n. Now we set

$$V_j(K):=\sum_{k=1}^{d+1}\alpha_{jk}V_d(K+kB^d),\quad K\in\mathcal{K}^d\setminus\{\emptyset\}.\tag{3.16}$$

Since the volume V_d as well as the Minkowski addition are continuous (w.r.t. δ), we conclude that $V_j(K_n)\to V_j(K)$, as $n\to\infty$. Hence V_j is a continuous functional on $\mathcal{K}^d\setminus\{\emptyset\}$ w.r.t. the Hausdorff metric. Since the equation

$$V_d(K_n+\varepsilon B^d)=\sum_{j=0}^{d}\kappa_{d-j}\varepsilon^{d-j}V_j(K_n)$$

is valid for the functionals V_j defined by (3.16), the continuity of the V_j implies that (3.14) holds also for the limit set K of the sequence (K_n). □

Definition 3.33 The numbers $V_0(K), \ldots, V_d(K)$ are called *intrinsic volumes* of the set K.

Remark 3.34 The coefficients κ_{d-j} in (3.14) are inserted for convenience. They standardize the intrinsic volumes in a particular way, leading amongst others to the assertions (i) and (iii) in Remark 3.35 below. More importantly, but more difficult to show is the fact that this way the V_j become independent of the dimension of the ambient space. That is, if $K \subset \mathbb{R}^k \subset \mathbb{R}^d$, then the $V_j(K)$ assume the same values regardless of whether they are determined in $\mathbb{R}^k$ or in $\mathbb{R}^d$. (Note that the parallel sets of K in $\mathbb{R}^k$ and $\mathbb{R}^d$ are different. This is also the reason why these functionals are called 'intrinsic' volumes.

Remarks 3.35 (i) The coefficient $V_d(K)$ on the right hand side of (3.14) equals $\lambda_d(K)$, which is transparent from setting $\varepsilon = 0$. This shows that the notation is consistent.

(ii) Remark (i) implies that

$$S_{d-1}(K) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(V_d(K + \varepsilon B^d) - V_d(K)) = \kappa_1 V_{d-1}(K) = 2V_{d-1}(K).$$

In case $K \in \mathcal{K}^d$ satisfies $\dim K = d$ (meaning that K has interior points), the coefficient $V_{d-1}(K)$ equals half the surface area of K. In case $\dim K \leq d-1$, $V_{d-1}(K)$ is the $(d-1)$-dimensional Lebesgue measure of K. This can also be interpreted as half the surface area, if each of the two 'sides' of K is considered as a surface of K and included in the computation of the surface area.

(iii) Whenever $K \neq \emptyset$, one has $V_0(K) = 1$ (see Worksheet 9). Furthermore, for the empty set $V_0(\emptyset) = \ldots = V_d(\emptyset) = 0$ holds. The coefficient $V_0(K)$ is also called *Euler characteristic* of K.

Theorem 3.36 *Let $j \in \{0, \ldots, d\}$. Then the j-th intrinsic volume $V_j : \mathcal{K}^d \to \mathbb{R}$ has the following properties:*

(i) *V_j is invariant with respect to rigid motions, that is,*

$$V_j(gK) = V_j(K), \quad K \in \mathcal{K}^d, g \in \mathrm{G}_d,$$

where G_d is the group of rigid motions of $\mathbb{R}^d$ (any composition of rotations and translations).

(ii) *V_j is homogeneous of degree j, that is,*

$$V_j(\alpha K) = \alpha^j V_j(K), \quad K \in \mathcal{K}^d, \alpha > 0.$$

(iii) *V_j is continuous on $\mathcal{K}^d$ w.r.t. the Hausdorff metric.*

(iv) *V_j is additive, that is,*

$$V_j(K \cup L) = V_j(K) + V_j(L) - V_j(K \cap L), \quad K, L, K \cup L \in \mathcal{K}^d.$$

Proof. (i) and (iii) follow directly from equation (3.16) and the corresponding properties of the volume V_d.

(ii) For any $\alpha > 0$ and $K \in \mathcal{K}^d$, the Steiner formula (3.14) and the scaling properties of the volume imply that

$$\sum_{j=0}^{d} \kappa_{d-j} \varepsilon^{d-j} V_j(\alpha K) = V_d(\alpha K + \varepsilon B^d) = \alpha^d V_d(K + \frac{\varepsilon}{\alpha} B^d)$$

$$= \alpha^d \sum_{j=0}^{d} \kappa_{d-j} \varepsilon^{d-j} \alpha^{-(d-j)} V_j(K) = \sum_{j=0}^{d} \kappa_{d-j} \varepsilon^{d-j} \alpha^j V_j(K).$$

A comparison of the coefficients shows that $V_j(\alpha K) = \alpha^j V_j(K)$ as claimed.

(iv) The idea of proof is as follows. Let $K, L \in \mathcal{K}^d$ with $K \cup L \in \mathcal{K}^d$. Assume that $K \neq \emptyset$ and $L \neq \emptyset$. By looking at the metric projections $p(K, \cdot)$, $p(L, \cdot)$ and $p(K \cap L, \cdot)$, one can show that

$$\mathbb{1}_{(K \cup L) + \varepsilon B^d} + \mathbb{1}_{(K \cap L) + \varepsilon B^d} = \mathbb{1}_{K + \varepsilon B^d} + \mathbb{1}_{L + \varepsilon B^d}.$$

Integration over these functions allows to conclude that $K \mapsto V_d(K + \varepsilon B^d)$ is additive. Then one can use equation (3.16) to deduce the assertion. For the details we refer to Work sheet 9. □

For further examples on the computation of intrinsic volumes see Work sheet 9. The intrinsic volumes can be generalized in many ways. We have introduced them as functionals on $\mathcal{K}^d$. Later we will extend them to sets that can be represented as finite unions of convex sets. In fact, these functionals can be introduced for far more general classes of sets. Like V_d and V_{d-1}, which give volume and surface area of a set, all intrinsic volumes encode specific geometric information of the sets. In particular, the intrinsic volumes V_j with $j \leq d-2$ may be interpreted in terms of curvature or e.g. in terms of projections of the set to linear subspaces. We will return to this later on.

We also mention briefly (and without giving any proofs) a different line of generalization. The Steiner formula is just a special case of a more general polynomial expansion of the volume of Minkowski sums. There exists a uniquely determined nonnegative and symmetric functional $V \colon (\mathcal{K}^d \setminus \{\emptyset\})^d \to \mathbb{R}$ such that

$$V_d(t_1 K_1 + \ldots + t_m K_m) = \sum_{i_1, \ldots, i_d = 1}^{m} t_{i_1} \cdots t_{i_d} V(K_{i_1}, \ldots, K_{i_d}) \tag{3.17}$$

for all $m \in \mathbb{N}$, $t_1, \ldots, t_m \geq 0$ and $K_1, \ldots, K_m \in \mathcal{K}^d \setminus \{\emptyset\}$, which is 'multilinear' in its arguments.

Definition 3.37 The nonnegative real number $V(K_1, \ldots, K_d)$ is called *mixed volume* of the bodies $K_1, \ldots, K_d$. For $K, L \in \mathcal{K}^d \setminus \{\emptyset\}$ and $j \in \{0, \ldots, d\}$ we introduce the abbreviation

$$V(K[j], L[d-j]) := V(\underbrace{K, \ldots, K}_{j}, \underbrace{L, \ldots, L}_{d-j}).$$

Remark 3.38 If we choose $m = 2$, $t_1 = 1$, $t_2 = t$, $K_1 = K$ and $K_2 = L$ in equation (3.17), then we obtain

$$V_d(K + tL) = \sum_{j=0}^{d} \binom{d}{j} t^{d-j} V(K[j], L[d-j]).$$

This formula is also called Steiner formula for the structuring element L. A comparison of the case $L = B^d$ with the Steiner formula in (3.14) shows that

$$V_j(K) = \binom{d}{j} \kappa_{d-j}^{-1} V(K[j], B^d[d-j]), \quad j = 0, \dots, d.$$

3.4 The Boolean model with convex grains

As a first application of intrinsic volumes, we look again at a stationary Boolean model Z with intensity γ and typical grain Z_0. We assume now additionally that $\mathbb{P}(Z_0 \in \mathcal{K}^d \setminus \{\emptyset\}) = 1$, that is, the grains are convex bodies. Note that this is a serious restriction on the choice of grains. However, it is in line with our initial aim to generate complex structures from simple building blocks. Restricting the complexity of grains allows in return a much finer analysis of the geometry to the resulting union sets. Observe that the convexity assumption on the typical grain implies that intrinsic volumes $V_j(Z_0)$ are (almost surely) well defined.

Recall from Definition 3.24 the spherical contact distribution function H_{B^d} of Z. Using the Steiner formula, H_{B^d} can be expressed in terms of the *expected intrinsic volumes* $\mathbb{E}V_j(Z_0)$ of Z_0.

Theorem 3.39 *Let Z be a stationary Boolean model with intensity γ and typical grain Z_0 satisfying $\mathbb{P}(Z_0 \in \mathcal{K}^d \setminus \{\emptyset\}) = 1$. Then the spherical contact distribution function of Z is given by*

$$H_{B^d}(r) = 1 - \exp\left(-\gamma \sum_{j=0}^{d-1} \kappa_{d-j} r^{d-j} \mathbb{E}V_j(Z_0)\right), \quad r \geq 0.$$

Proof. By Theorem 3.27, we have for any $B \in \mathcal{K}^d$ with $0 \in B$

$$1 - H_B(r) = \exp\left(-\gamma \left(\mathbb{E}[\lambda_d(Z_0 + rB^*)] - \mathbb{E}[\lambda_d(Z_0)]\right)\right).$$

For $B = B^d$ the Steiner formula 3.32 implies that the parallel volume on the right is given by

$$\mathbb{E}[\lambda_d(Z_0 + rB^d)] = \sum_{j=0}^{d} \kappa_{d-j} r^{d-j} \mathbb{E}[V_j(Z_0)],$$

from which the assertion is clear. □

Remark 3.40 Recall that the definition of a Boolean model implies that condition (3.6) is satisfied, which means in particular that $\mathbb{E}[\lambda_d(Z_0 + rB^d)] < \infty$ holds for any $r \geq 0$. The latter condition is also sufficient for (3.6) (why?). If the typical grain is almost surely convex then, by the Steiner formula, it is also equivalent to assuming

$$\mathbb{E}[V_j(Z_0)] < \infty, \quad j = 0, \dots, d.$$

Remark 3.41 Theorem 3.39 implies in particular that the spherical contact distribution function is determined by the numbers $\gamma\mathbb{E}[V_j(Z_0)]$, $j = 0, \dots, d-1$. Recall from (3.11) that the volume fraction is determined by $\gamma\mathbb{E}[V_d(Z_0)]$.

Also for the special case of linear contact distribution functions, a representation in terms of intrinsic volumes is possible, in fact the functional V_{d-1} (i.e. the surface area) suffices in this case.

Theorem 3.42 *Let Z be a stationary Boolean model with intensity γ and typical grain Z_0 satisfying $\mathbb{P}(Z_0 \in \mathcal{K}^d \setminus \{\emptyset\}) = 1$. Then, for any $u \in \mathbb{S}^{d-1}$,*

$$1 - H_{[0,u]}(r) = \exp\left(-\gamma r \mathbb{E}[V_{d-1}(Z_0|u^\perp)]\right), \qquad r > 0,$$

where $K|u^\perp$ denotes the orthogonal projection of a set $K \subset \mathbb{R}^d$ onto the hyperplane $u^\perp := \operatorname{span}(u)^\perp$. Moreover, if Z_0 isotropic, then

$$1 - H_{[0,u]}(r) = \exp\left(-\gamma \frac{2\kappa_{d-1}}{d\kappa_d} r \mathbb{E}[V_{d-1}(Z_0)]\right).$$

Proof. By Theorem 3.27, we have for $B = [0, u]$

$$\begin{aligned} 1 - H_{[0,u]}(r) &= \exp\left(-\gamma \mathbb{E}[\lambda_d((Z_0 + r[0,u]^*) \setminus Z_0)]\right) \\ &= \exp\left(-\gamma \mathbb{E}[V_d(Z_0 + r[0,-u]) - V_d(Z_0)]\right). \end{aligned}$$

Therefore, to prove the first formula, it suffices to show that, for any $K \in \mathcal{K}^d$,

$$V_d(K + r[0,u]) = V_d(K) + r V_{d-1}(K|u^\perp), \quad r \geq 0.$$

This is rather obvious from a sketch of the set $K + r[0,u]$. For a formal proof let $f(y) := \max\{\lambda \in \mathbb{R} \colon y + \lambda u \in K\}$ for any $y \in u^\perp$. Observe that any point $z \in (K + rB) \setminus K$ has a unique representation $z = y + \lambda u$ with $y \in K|u^\perp$ and $f(y) < \lambda \leq f(y) + r$. Therefore, by Fubini's Theorem, we infer for $B = [0,u]$

$$\begin{aligned} V_d((K + rB) \setminus K) &= \int_{u^\perp} \int_{\mathbb{R}} \mathbb{1}\{y + \lambda u \in (K + rB) \setminus K\} \, d\lambda \, dy \\ &= \int_{u^\perp} \int_{\mathbb{R}} \mathbb{1}\{y \in K|u^\perp\} \mathbb{1}\{f(y) < \lambda \leq f(y) + r\} \, d\lambda \, dy \\ &= \int_{u^\perp} \mathbb{1}\{y \in K|u^\perp\} r \, dy \\ &= r V_{d-1}(K|u^\perp), \end{aligned}$$

completing the proof of the first formula. The second formula is a consequence of a projection formula from integral geometry (to be discussed in the next chapter). It says that, for any $K \in \mathcal{K}^d$,

$$\int_{SO_d} V_{d-1}(K|\varrho u^\perp) \, \nu(d\varrho) = \frac{2\kappa_{d-1}}{d\kappa_d} V_{d-1}(K).$$

Here ν denotes the uniform distribution on SO_d (known as the Haar probability measure), that is, on the left hand side of the formula we average over all possible projections onto $(d-1)$-dimensional linear subspaces. In the isotropic case we have, for any $u \in \mathbb{S}^{d-1}$ and any $\rho \in SO_d$,

$$\mathbb{E} V_{d-1}(Z_0|u^\perp) = \mathbb{E} V_{d-1}(Z_0|\varrho u^\perp)$$

and therefore, by the projection formula,

$$\mathbb{E} V_{d-1}(Z_0|u^\perp) = \mathbb{E} \int_{SO_d} V_{d-1}(Z_0|\varrho u^\perp) \, \nu(d\varrho) = \frac{2}{d\kappa_d} \mathbb{E}\kappa_{d-1} V_{d-1}(Z_0). \qquad \square$$

Example 3.43 (How far can one see in a (Boolean) forest?) We model the forest (or rather a horizontal cut through it at eye level) as a planar Boolean model where the grains (representing the trunks of trees) are balls with random radii. For this purpose let $d = 2$ and let the typical grain be $Z_0 := B(0, R)$, where R is a nonnegative random variable. According to Remark 3.26, we can interpret $1 - H_{[0,u]}(r)$ as the probability, that the 'ball' $r[0, u]$ has no contact with Z (given that $0 \notin Z$). If we are located at the origin, then this can be viewed as the probability that we can see at least distance r in direction u.

From the isotropic case in Theorem 3.42 we infer that, for any $u \in \mathbb{S}^{d-1}$,

$$\begin{aligned}\mathbb{P}\left([0, ru] \cap Z = \emptyset | 0 \notin Z\right) &= 1 - H_{[0,u]}(r) \\ &= \exp\left(-\gamma r \frac{2\kappa_1}{2\kappa_2} \mathbb{E}[V_1(B(0, R))]\right) \\ &= \exp\left(-\gamma r \frac{2}{\pi} \mathbb{E}[\pi R]\right) = \exp(-2\gamma r \mathbb{E}[R]).\end{aligned}$$

Choose for instance $\mathbb{E}[R] = 0,2\text{m}$ and $\gamma = 0,01\text{m}^{-2}$ (i.e. average diameter 40cm and one tree per 100m^2). Then we have $1 - H_{[0,u]}(r) = \exp(-0,004\text{m}^{-1} \cdot r)$ and therefore the probability that we can see, say at least $r = 50\text{m}$ in direction u is

$$1 - H_{[0,u]}(50) = e^{-0.2} \approx 0,8187.$$

Remark 3.44 For a general structuring element B, i.e. a set $B \in \mathcal{K}^d$ such that $0 \in B$, the contact distribution function can be expressed in terms of the mixed volume, cf. Def. 3.37 and Remark 3.38:

$$1 - H_B(r) = \exp\left(-\gamma \sum_{j=0}^{d-1} r^{d-j} \binom{d}{j} \mathbb{E}\big[V(Z_0[j], B^*[d-j])\big]\right), \quad r \geq 0.$$

This follows directly from Theorem 3.27 and Remark 3.38. In general it is not possible to express these functions in terms of intrinsic volumes alone. However, if the typical grain is isotropic, then the above formula simplifies and one gets

$$1 - H_B(r) = \exp\left(-\gamma \sum_{j=0}^{d-1} \frac{\kappa_j \kappa_{d-j}}{\binom{d}{j} \kappa_d} r^{d-j} V_{d-j}(B) \mathbb{E}[V_j(Z_0)]\right), \quad r \geq 0.$$

This can be deduced from the principal kinematic formula which will be discussed in the next chapter.

4 Densities of intrinsic volumes

The aim in this chapter is to employ intrinsic volumes for a further analysis of Boolean models (and more general germ-grain models). Even under the assumption of convex grains, the union set Z of a germ-grain model is typically far from being convex. Therefore, it will be necessary to extend these functionals to more general sets. The natural class of sets for this are unions of convex sets to which intrinsic volumes can indeed be extended, see Section 4.3. Then intrinsic volumes of Z (intersected with a bounded convex window W) are well defined random variables. In the case of stationarity, the distribution will not depend on the position of the window and (if viewed in the right way) not even on the shape of W but only on its size (in terms of the volume). This leads to so called *densities* of the functionals V_j, describing roughly the 'j-th intrinsic volume of Z per unit volume'. For stationary Boolean models, explicit formulas for these densities can be derived in terms of the given data of the underlying particle process, i.e. the intensity γ and the distribution $\mathbb{Q}$ of the typical grain. More precisely, these formulas will only involve γ and the expected intrinsic volumes $\mathbb{E}V_i(Z_0)$, $i \in \{0, \dots, d\}$ of Z_0. We start by analyzing further the expected intrinsic volumes (and more general expected functionals f) of a particle process. These will also be called 'densities'.

4.1 Geometric densities of particle processes

Let Φ be a stationary particle process in $\mathbb{R}^d$ with locally finite intensity measure Θ. As before we denote by γ the intensity and by $\mathbb{Q}$ the shape distribution of Φ with respect to some fixed center function $c\colon \mathcal{C}' \to \mathbb{R}^d$. We write $\mathcal{C}_0$ for the family of compact sets C with $c(C) = 0$. First we explain what we mean by the 'density' of a functional on $\mathcal{C}'$ with respect to Φ.

Definition 4.1 For a particle process Φ as above and a translation invariant, measurable functional $f : \mathcal{C}' \to \mathbb{R}$ which is either nonnegative or $\mathbb{Q}$-integrable the number

$$\gamma_f := \gamma_f(\Phi) := \gamma \int_{\mathcal{C}_0} f \, d\mathbb{Q}$$

is called *f-density* of Φ.

Note that even though Φ is a random object, the density $\gamma_f(\Phi)$ of Φ is deterministic. By $\mathbb{Q}$-integrability we mean that $\int_{\mathcal{C}'} |f| \, d\mathbb{Q}$ exists and is finite, which implies in particular that γ_f is finite. For nonnegative f in contrast, γ_f is nonnegative but may be infinite. Let Z_0 be the typical grain of Φ. Then obviously

$$\gamma_f = \gamma \mathbb{E}[f(Z_0)].$$

Note that in the literature f-densities are often defined without the factor γ. This should be kept in mind when comparing formulas in different sources. Examples of functionals f of the above form are for instance volume and surface area, which are clearly defined for all compact sets, translation invariant and nonnegative, and, in the case of the volume, the measurability is clear from Lemma 1.19. (In the case of the surface area measurability can also be shown.) The number of connected components or the number of elements are further valid geometric functionals of the above form. Also the intrinsic volumes V_j may be viewed as functionals f of the above form, although strictly speaking they are not defined on nonconvex compact sets. But if $\mathbb{Q}$ is concentrated on a subfamily of $\mathcal{C}'$ (such as $\mathcal{K}'$) then, of course, it does not matter how f is defined outside this subfamily

and the f-density is still defined. Note that the intrinsic volumes are translation invariant, measurable and nonnegative functionals on $\mathcal{K}'$, see Theorem 3.36 (i) and (3.16).

The following statement provides different representations of f-densities, giving also some motivation for the term 'density'.

Theorem 4.2 *Let Φ be a stationary particle process as above. Let $f\colon \mathcal{C}' \to \mathbb{R}$ be a measurable and translation invariant. Assume that either $f \geq 0$ or $\int_{\mathcal{C}'} |f|\, d\mathbb{Q} < \infty$. Then the following assertions hold.*

(i) *For any set $B \in \mathcal{B}^d$ with $0 < \lambda_d(B) < \infty$,*

$$\gamma_f = \frac{1}{\lambda_d(B)} \mathbb{E}\left[\int_{\mathcal{C}'} \mathbb{1}\{c(C) \in B\} f(C)\, \Phi(dC)\right].$$

(ii) *For any window $W \in \mathcal{K}^d$ such that $V_d(W) > 0$,*

$$\gamma_f = \lim_{r\to\infty} \frac{1}{V_d(rW)} \mathbb{E}\left[\int_{\mathcal{C}'} \mathbb{1}\{C \subset rW\} f(C)\, \Phi(dC)\right].$$

(iii) *Suppose that*

$$\int_{\mathcal{C}'} |f(C)| \lambda_d(C + B^d)\, \mathbb{Q}(dC) < \infty. \tag{4.1}$$

Then, for any window $W \in \mathcal{K}^d$ such that $V_d(W) > 0$

$$\gamma_f = \lim_{r\to\infty} \frac{1}{V_d(rW)} \mathbb{E}\left[\int_{\mathcal{C}'} \mathbb{1}\{C \cap rW \neq \emptyset\} f(C)\, \Phi(dC)\right].$$

Proof. (i) By Campbell's formula 2.21, Theorem 3.6 and the translation invariance of f, we have

$$\begin{aligned}
\mathbb{E}\left[\int_{\mathcal{C}'} \mathbb{1}\{c(C) \in B\} f(C)\, \Phi(dC)\right] &= \int_{\mathcal{C}'} \mathbb{1}\{c(C) \in B\} f(C)\, \Theta(dC) \\
&= \gamma \int_{\mathcal{C}_0} \int_{\mathbb{R}^d} \mathbb{1}\{c(K + x) \in B\} f(K + x)\, dx\, \mathbb{Q}(dK) \\
&= \gamma \int_{\mathcal{C}_0} \int_{\mathbb{R}^d} \mathbb{1}\{c(K) + x \in B\} f(K)\, dx\, \mathbb{Q}(dK) \\
&= \gamma \int_{\mathcal{C}_0} \lambda_d(B) f(K)\, \mathbb{Q}(dK) = \lambda_d(B)\gamma_f,
\end{aligned}$$

from which assertion (i) is clear.

(ii) Similarly as in (i), we infer that

$$\mathbb{E}\left[\int_{\mathcal{C}'} \mathbb{1}\{C \subset rW\} f(C)\, \Phi(dC)\right] = \gamma \int_{\mathcal{C}_0} f(K) \lambda_d(\{x \in \mathbb{R}^d\colon K + x \subset rW\})\, \mathbb{Q}(dK). \tag{4.2}$$

Without loss of generality, we can assume that $0 \in \operatorname{int} W$ (why?). Then there exists a measurable function $\rho\colon \mathcal{C}' \to [0, \infty)$ such that

$$K \subset \rho(K)W, \quad K \in \mathcal{C}'.$$

For fixed $K \in \mathcal{C}'$ let $r > \rho(K)$. Then we have

$$\left(1 - \frac{\rho(K)}{r}\right)^d = \frac{(r-\rho(K))^d}{r^d} = \frac{V_d((r-\rho(K))W)}{V_d(rW)}$$
$$= V_d(rW)^{-1} \int_{\mathbb{R}^d} \mathbb{1}\{x \in (r-\rho(K))W\}\, dx$$
$$\leq V_d(rW)^{-1} \int_{\mathbb{R}^d} \mathbb{1}\{K + x \subset rW\}\, dx \leq 1.$$

For the first inequality above note that, for any $x \in (r-\rho(K))W$, we have

$$K + x \subset K + (r-\rho(K))W \subset \rho(K)W + (r-\rho(K))W = rW.$$

(Why does the second inequaltity hold?) Multiplying $V_d(rW)^{-1}$ in (4.2), we get

$$\frac{1}{V_d(rW)} \mathbb{E}\left[\int_{\mathcal{C}'} \mathbb{1}\{C \subset rW\} f(C)\, \Phi(dC)\right]$$
$$= \gamma \int_{\mathcal{C}_0} f(K) \underbrace{\frac{\lambda_d(\{x \in \mathbb{R}^d \colon K + x \subset rW\})}{\lambda_d(rW)}}_{\uparrow 1,\ r\to\infty} \mathbb{Q}(dK).$$

By the above considerations, the second factor in the integrand on the right is bounded from below and above by $(1-\rho(K)/r)^d$ and 1, respectively. Moreover, the lower bound converges monotonically to 1, as $r \to \infty$. Therefore, by monotone convergence, the right hand side converges to γ_f in case $f \geq 0$. In case of an integrable f, the assertion follows by dominated convergence.

(iii) See Work sheet 10. □

Observe that in particular the second formula in the above statement provides a useful approach to estimating the density $\gamma_f(\Phi)$ from observations of the particle process in a given (large) window. It does not rely on information of the process outside the window.

4.2 Geometric densities of germ-grain models

Our next aim is to introduce densities of geometric functionals for germ-grain models in a similar fashion as done for particle processes in the previous section. As germ-grain models are union sets of many particles and form complex structures, the definition and computation of geometric functionals seems much harder even if the grains are simple. It will therefore be of great help, if the functionals behave nicely under union formation, which leads us to the notion of additivity of a functional. In this section we will first study additive functionals of stationary random closed sets whose realizations are unions of convex bodies and then apply this to germ-grain models. We start by defining the appropriate class of sets and then recall the notion of additivity.

Definition 4.3 (i) The *convex ring* $\mathcal{R}^d$ is the family of all sets in $\mathbb{R}^d$ that can be represented as a finite union of convex bodies.

(ii) The *extended convex ring* $\mathcal{S}^d$ is the family of all sets $M \subset \mathbb{R}^d$ such that

$$M \cap K \in \mathcal{R}^d, \quad \text{for any } K \in \mathcal{K}^d.$$

(iii) A functional $f\colon \mathcal{K}^d \setminus \{\emptyset\} \to \mathbb{R}$ (or $f\colon \mathcal{K}^d \to \mathbb{R}$) is called *additive*, if

$$f(K \cup L) + f(K \cap L) = f(K) + f(L), \tag{4.3}$$

for all $K, L \in \mathcal{K}^d \setminus \{\emptyset\}$ such that $K \cup L \in \mathcal{K}^d$. If the empty set is included in the domain of f, then one requires additionally that $f(\emptyset) = 0$. A functional $f\colon \mathcal{R}^d \to \mathbb{R}$ is called *additive*, if and only if $f(\emptyset) = 0$ and equation (4.3) holds for all $K, L \in \mathcal{R}^d$.

Elements of the convex ring are also called *polyconvex*. Observe that the convex ring $\mathcal{R}^d$ is closed with respect to unions and intersections. The intrinsic volumes V_j are examples of additive functionals on $\mathcal{K}^d$. Later we will see that they can be extended to $\mathcal{R}^d$ and are also additive (by definition) on this family.

Our programme for the remainder of this chapter can be summarized as follows: Let Z be a stationary random closed set in the extended convex ring.

(i) For translation invariant, additive functionals $f\colon \mathcal{R}^d \to \mathbb{R}$ we will show (under suitable integrability conditons) the existence of the limit

$$\delta_f := \lim_{r\to\infty} \frac{\mathbb{E}[f(Z \cap rW)]}{V_d(rW)} \tag{4.4}$$

for windows $W \in \mathcal{K}^d$ such that $V_d(W) > 0$. This limit will be called the f-density of Z.

(ii) We will derive some alternative representations of δ_f (analogous to Theorem 4.2), which also justify to call the limit δ_f a density of Z.

(iii) For the special case of Boolean models, we will look more closely at V_i-densities (that is, densities of intrinsic volumes) and we will derive formulas that express the δ_{V_i} in terms of the V_j-densities γ_{V_j} of the associated particle processes.

We need some auxiliary results regarding additive functionals. The first one, the so called inclusion-exclusion principle, is a direct consequence of the additivity. For any union of sets, the value of f can be computed from the values of f at mutual intersections of the single sets.

Lemma 4.4 *(Inclusion-exclusion principle) Let $f\colon \mathcal{R}^d \to \mathbb{R}$ be additive. Then, for any $m \in \mathbb{N}$ and all $K_1, \dots K_m \in \mathcal{R}^d$,*

$$f(K_1 \cup \ldots \cup K_m) = \sum_{j=1}^{m} (-1)^{j-1} \sum_{1\le i_1<\ldots<i_j\le m} f(K_{i_1} \cap \ldots \cap K_{i_j}).$$

Proof. By induction (see Work sheet 10). □

The formula is extremely useful for computing functionals. Recall that, by definition, any set in $\mathcal{R}^d$ can be represented as a finite union of convex sets and that any intersection of convex sets is again convex (possibly empty). The formula implies that knowing f on convex sets is sufficient for knowing f on the whole convex ring $\mathcal{R}^d$.

In order to formulate the next statement we introduce some further notation.

- Set $C := [0,1]^d$, $C^0 := [0,1)^d$, $\partial^+ C := C \setminus C^0$.
- For $z \in \mathbb{Z}^d$, let $C_z := C + z$, $C_z^0 := C^0 + z$ and $\partial^+ C_z := \partial^+ C + z$.

- For $f\colon \mathcal{R}^d \to \mathbb{R}$ let

$$f(K,z) := f(K \cap C_z) - f(K \cap \partial^+ C_z).$$

Observe that the half-open cubes C_z^0, $z \in \mathbb{Z}^d$, are pairwise disjoint and form a partition of $\mathbb{R}^d$. The positive boundary $\partial^+ C_z$ of a cube is polyconvex as it is a union of $d-1$-dimensional cubes. Therefore the sets $K \cap C_z$ and $K \cap \partial^+ C_z$ are polyconvex for any $K \in \mathcal{R}^d$. Hence, the function f is well defined on $\mathcal{R}^d \times \mathbb{Z}^d$. The following statement roughly says that an additive functional f can be decomposed into a sum of its contributions on the cells C_z of the partition, given by the functions $f(\cdot, z)$.

Lemma 4.5 *If $f\colon \mathcal{R}^d \to \mathbb{R}$ is additive, then*

$$f(K) = \sum_{z \in \mathbb{Z}^d} f(K,z), \quad \textit{for any } K \in \mathcal{R}^d.$$

Proof. Let $<$ denote the strict lexicographical ordering on $\mathbb{Z}^d$, which is defined as follows: two points $(z_1, \dots, z_d) \in \mathbb{Z}^d$ and $(y_1, \dots, y_d) \in \mathbb{Z}^d$ satisfy the relation

$$(z_1, \dots, z_d) < (y_1, \dots, y_d)$$

if and only if there exists some $k \in \{1, \dots, d\}$ such that $z_1 = y_1, \dots, z_{k-1} = y_{k-1}$ and $z_k < y_k$. Observe that

$$\partial^+ C_z = C_z \cap \bigcup_{y\colon z<y} C_y, \quad z \in \mathbb{Z}^d.$$

By Lemma 4.4, we infer that

$$\begin{aligned}
\sum_{z \in \mathbb{Z}^d} f(K \cap \partial^+ C_z) &= \sum_{z \in \mathbb{Z}^d} f\left(\bigcup_{y: z<y} (K \cap C_z \cap C_y) \right) \\
&= \sum_{z \in \mathbb{Z}^d} \sum_{j=1}^{\infty} (-1)^{j-1} \sum_{z<y_1<\ldots<y_j} f(K \cap C_z \cap C_{y_1} \cap \ldots \cap C_{y_j}) \\
&= -\sum_{k=2}^{\infty} (-1)^{k-1} \sum_{z_1<\ldots<z_k} f(K \cap C_{z_1} \cap \ldots \cap C_{z_k}).
\end{aligned}$$

Note that the sum in the second line is in fact finite, since $C_z \cap C_{y_j} \neq \emptyset$ for only finitely many j and $f(\emptyset) = 0$ by definition. Applying again Lemma 4.4, we conclude that

$$\begin{aligned}
f(K) &= f\left(\bigcup_{z \in \mathbb{Z}^d} (K \cap C_z) \right) \\
&= \sum_{z \in \mathbb{Z}^d} f(K \cap C_z) + \sum_{k=2}^{\infty} (-1)^{k-1} \sum_{z_1<\ldots<z_k} f(K \cap C_{z_1} \cap \ldots \cap C_{z_k}) \\
&= \sum_{z \in \mathbb{Z}^d} f(K \cap C_z) - \sum_{z \in \mathbb{Z}^d} f(K \cap \partial^+ C_z) \\
&= \sum_{z \in \mathbb{Z}^d} \big(f(K \cap C_z) - f(K \cap \partial^+ C_z) \big) \\
&= \sum_{z \in \mathbb{Z}^d} f(K,z).
\end{aligned}$$

□

Definition 4.6 A functional $f\colon \mathcal{R}^d \to \mathbb{R}$ is called *conditionally bounded*, if and only if for each $K \in \mathcal{K}'$, f is bounded on the family $\{L \in \mathcal{K}' \colon L \subset K\}$.

Observe that the volume V_d is conditionally bounded. We will see later that also the other intrinsic volumes are (additive functionals on $\mathcal{R}^d$ and) conditionally bounded. This can for instance be deduced from (3.16). The following statement is rather surprising. The imposed conditions on f imply that the limit on the left (which should be compared with the limit (4.4) that we want to use to define densities) does not depend on the chosen window W, as the right hand side is independent of W. It is an important step towards proving the existence of f-densities.

Lemma 4.7 *If $f\colon \mathcal{R}^d \to \mathbb{R}$ is translation invariant, additive and conditionally bounded, then*

$$\lim_{r\to\infty} \frac{f(rW)}{V_d(rW)} = f(C) - f(\partial^+ C),$$

for any set $W \in \mathcal{K}^d$ such that $V_d(W) > 0$.

Proof. Without loss of generality we can assume that $0 \in \operatorname{int} W$. By Lemma 4.5, we have

$$f(rW) = \sum_{z\in\mathbb{Z}^d} f(rW, z), \quad r > 0. \tag{4.5}$$

Define

$$Z_r^1 := \{z \in \mathbb{Z}^d : C_z \cap rW \neq \emptyset, C_z \not\subset rW\} \quad \text{and} \quad Z_r^2 := \{z \in \mathbb{Z}^d : C_z \subset rW\}.$$

Wir claim that (see Work sheet 10, Problem 3 for the proof)

$$\lim_{r\to\infty} \frac{\operatorname{card} Z_r^1}{V_d(rW)} = 0 \quad \text{and} \quad \lim_{r\to\infty} \frac{\operatorname{card} Z_r^2}{V_d(rW)} = 1. \tag{4.6}$$

Since f is translation invariant and conditionally bounded, we have

$$\begin{aligned} |f(rW, z)| &= |f(rW \cap C_z) - f(rW \cap \partial^+ C_z)| \\ &= |f((rW - z) \cap C) - f((rW - z) \cap \partial^+ C)| \le b \end{aligned}$$

for some constant $b \ge 0$ which is independent of z, r and W (why?). Therefore,

$$V_d(rW)^{-1} \left| \sum_{z\in Z_r^1} f(rW, z) \right| \le V_d(rW)^{-1} \operatorname{card}(Z_r^1) b$$

and, by (4.6), the right hand side converges to 0, as $r \to \infty$. Hence, taking into account that $f(\emptyset) = 0$, we conclude from (4.5) that

$$\lim_{r\to\infty} \frac{f(rW)}{V_d(rW)} = \lim_{r\to\infty} V_d(rW)^{-1} \sum_{z\in Z_r^2} f(rW, z) = (f(C) - f(\partial^+ C)) \lim_{r\to\infty} \frac{\operatorname{card} Z_r^2}{V_d(rW)},$$

from which the assertion is clear by (4.6). For the second equality we have used that, the translation invariance of f, we have for any $C_z \subset rW$,

$$f(rW, z) = f(rW \cap C_z) - f(rW \cap \partial^+ C_z) = f(C_z) - f(\partial^+ C_z) = f(C) - f(\partial^+ C).$$

□

The next statement is the main result of this section. It shows the existence of f-densities for germ-grain models (cf. (4.4)) and provides an alternative representation. Thus we reach two of the aims formulated at the beginning of the section.

Theorem 4.8 *Let Ψ be a stationary marked point process with finite intensity γ and a shape distribution $\mathbb{Q}$ that is concentrated on $\mathcal{K}'$. Assume that the intensity measure of the associated particle process is locally finite. Let Z be the generated germ-grain model. Let $f\colon \mathcal{R}^d \to \mathbb{R}$ be a measurable, translation invariant, additive and conditionally bounded functional satisfying*

$$\mathbb{E}[2^{N_C}] < \infty, \tag{4.7}$$

where $C = [0,1]^d$ and the random variable N_M is defined for any set $M \in \mathcal{B}^d$ by

$$N_M := \int_{\mathbb{R}^d \times \mathcal{C}'} \mathbb{1}\{(K+x) \cap M \neq \emptyset\}\, \Psi(d(x,K)).$$

Then the f-density δ_f of Z exists and is given by

$$\delta_f := \lim_{r\to\infty} \frac{\mathbb{E}[f(Z\cap rW)]}{V_d(rW)} = \mathbb{E}[f(Z\cap C^d) - f(Z \cap \partial^+ C^d)].$$

Proof. Consider the functional $f^*\colon \mathcal{R}^d \to \mathbb{R}\cup\{\pm\infty\}$ defined by $f^*(M) := \mathbb{E}[f(Z\cap M)]$. It is immediate from the assumptions on f that f^* is measurable, translation invariant and additive (taking into account the additivity of the expectation). If we show that f^* is conditionally bounded and assumes finite values for all $M \in \mathcal{R}^d$, then the assertion follows by applying Lemma 4.7 to f^*.

To see that f^* is indeed conditionally bounded, let first $W \in \mathcal{K}^d$ with $W \subset C$. Let $K_1, K_2, \ldots, K_{N_W}$ be those convex particles of the associated particle process Φ that intersect W. Then, by Lemma 4.4, we have for any realization of Φ

$$f(Z\cap W) = f\left(W \cap \bigcup_{j=1}^{N_W} K_j\right) = \sum_{j=1}^{N_W} (-1)^{j-1} \sum_{1\le i_1<\ldots<i_j\le N_W} f(W\cap K_{i_1}\cap\ldots\cap K_{i_j}).$$

Note that any of the sets on the right is contained in C. Thus, since f is conditionally bounded, there exists some constant b_C such that for all sets $W\cap K_{i_1}\cap\ldots\cap K_{i_j}$,

$$|f(W\cap K_{i_1}\cap\ldots\cap K_{i_j})| \le b_C.$$

By the triangle inequality, this implies

$$|f(Z\cap W)| \le b_C 2^{N_W} \le b_C 2^{N_C}. \tag{4.8}$$

Now let $W \in \mathcal{K}^d$ be arbitrary, i.e. we drop the assumption $W \subset C$. Then we find $z_1,\ldots,z_m \in \mathbb{Z}^d$ such that $W = \bigcup_{i=1}^m (W\cap C_{z_i})$. Using again Lemma 4.4, we infer that

$$\begin{aligned} f(Z\cap W) &= f\left(\bigcup_{i=1}^m (Z\cap W\cap C_{z_i})\right) \\ &= \sum_{j=1}^m (-1)^{j-1} \sum_{1\le i_1<\ldots<i_j\le m} f\left(Z\cap W\cap C_{z_{i_1}}\cap\cdots\cap C_{z_{i_j}}\right) \\ &= \sum_{j=1}^m (-1)^{j-1} \sum_{1\le i_1<\ldots<i_j\le m} f\left((Z-z_{i_1})\cap(W-z_{i_1})\cap C\cap C_{z_{i_2}-z_{i_1}}\cap\cdots\cap C_{z_{i_j}-z_{i_1}}\right), \end{aligned}$$

where we have employed the translation invariance of f in the last step. For each summand on the right, the set $\overline{W} := (W - z_{i_1}) \cap C \cap C_{z_{i_2}-z_{i_1}} \cap \cdots \cap C_{z_{i_j}-z_{i_1}}$ is convex and a subset of C. Hence, since Z is stationary, we are again in the situation of the first case. Applying (4.8) to $\overline{W}$ and noting that there are at most 2^m summands on the right, we conclude that $|f(Z \cap W)| \leq 2^m b_C 2^{N_C}$ for each realization of Z and therefore

$$\mathbb{E}|f(Z \cap W)| \leq 2^m b_C \mathbb{E}[2^{N_C}],$$

for each $W \in \mathcal{K}^d$. Note that the right hand side depends on W only via m. Now let $K \subset W$ be convex. Then $K = \bigcup_{i \in I}(K \cap C_{z_i})$ for some index set $I \subset \{1, \ldots, m\}$ and hence, by the previous argument, we get

$$\mathbb{E}|f(Z \cap K)| \leq 2^{|I|} b_C \mathbb{E}[2^{N_C}] \leq 2^m b_C \mathbb{E}[2^{N_C}].$$

Due to the assumption (4.7), this implies that f^* is conditionally bounded. Finally, let $M \in \mathcal{R}^d$ and let $W_1, \ldots, W_n \in \mathcal{K}^d$ such that $M = \bigcup_{i=1}^n W_i$. Then, by the inclusion-exclusion principle,

$$\mathbb{E}|f(Z \cap M)| \leq \sum_{j=1}^{n} (-1)^{j-1} \sum_{1 \leq i_1 < \ldots < i_j \leq n} \mathbb{E}|f(Z \cap W_{i_1} \cap \cdots \cap W_{i_j})|.$$

By the above arguments, each of the summands on the right is finite, implying the finiteness of the left hand side. But this means that $f^*(M)$ assumes a finite value. □

Remark 4.9 Let f be a translation invariant and additive functional on $\mathcal{R}^d$. The proof of Theorem 4.8 shows that f is conditionally bounded if and only if f is bounded on convex subsets of the unit cube C (or of some other fixed convex set).

4.3 Integral Geometry

Let $f\colon \mathcal{K}^d \to \mathbb{R}$ be additive. An additive functional $\tilde{f}\colon \mathcal{R}^d \to \mathbb{R}$ is called *additive extension* of f, if and only if $\tilde{f}|_{\mathcal{K}^d} = f$. Observe that due to the inclusion-exclusion principle, there is at most one additive extension of f (why?). Therefore, it is safe to use the same letter for f and its additive extension, that is, we will write $f := \tilde{f}$. The first statement ensures that intrinsic volumes (which we introduced as functionals on $\mathcal{K}^d$ via the Steiner formula (Theorem 3.32) have an additive extension to the convex ring.

Theorem 4.10 *For each $j \in \{0, \ldots, d\}$, V_j has a unique additive extension.*

For the proof of this theorem we will use the following auxiliary result.

Lemma 4.11 *Let $f\colon \mathcal{K}^d \to \mathbb{R}$ be continuous and additive. Assume that*

$$\sum_{i=1}^{m} \alpha_i \mathbb{1}_{K_i} = 0$$

for some $m \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$, $K_1, \ldots, K_m \in \mathcal{K}^d$. Then

$$\sum_{i=1}^{m} \alpha_i f(K_i) = 0.$$

Proof. Assume for a contradiction that the assertion is false. Then there are $m \in \mathbb{N}$, $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ and $K_1, \dots, K_m \in \mathcal{K}^d$ such that

$$\sum_{i=1}^{m} \alpha_i \mathbb{1}_{K_i} = 0, \tag{4.9}$$

but

$$\sum_{i=1}^{m} \alpha_i f(K_i) = a \neq 0. \tag{4.10}$$

Assume that m is minimal for such α_i and K_i to exist.

Let $H \subset \mathbb{R}^d$ be a hyperplane (and H^+, H^- the two closed half-spaces bounded by H) such that $K_1 \subset \operatorname{int} H^+$. Multiplying $\mathbb{1}_{H^-}$ and $\mathbb{1}_H$, respectively, in (4.9), we obtain

$$\sum_{i=1}^{m} \alpha_i \mathbb{1}_{K_i \cap H^-} = 0, \quad \sum_{i=1}^{m} \alpha_i \mathbb{1}_{K_i \cap H} = 0.$$

Since $K_1 \cap H^- = K_1 \cap H = \emptyset$, both sums have at most $m-1$ nonzero summands. Therefore, the assumed minimality of m implies that

$$\sum_{i=2}^{m} \alpha_i f(K_i \cap H^-) = \sum_{i=2}^{m} \alpha_i f(K_i \cap H) = 0.$$

Observe that both sums do not change if the summation is started at $i = 1$, since $f(\emptyset) = 0$. Applying (4.10) to the sets $K_i = (K_i \cap H^+) \cup (K_i \cap H^-)$, the additivity of f implies

$$\sum_{i=1}^{m} \alpha_i f(K_i \cap H^+) = a. \tag{4.11}$$

Observe now that there exists a sequence $(H_j)_{j\in\mathbb{N}}$ of hyperplanes such that $K_1 = \bigcap_{j=1}^{\infty} H_j^+$ and $K_1 \subset \operatorname{int} H_j^+$, $j \in \mathbb{N}$. (One can for instance choose a dense sequence $\{x_j : j \in \mathbb{N}\}$ in $\mathbb{R}^d \setminus K$ and take for each x_j a hyperplane orthogonal to $x_j - p(K_1, x_j)$ separating x_j and K_1.) Iterating the above idea, we infer that, for each $k \in \mathbb{N}$,

$$\sum_{i=1}^{m} \alpha_i f\left(K_i \cap \bigcap_{j=1}^{k} H_j^+\right) = a.$$

In case $K_i \cap K_1 \neq \emptyset$, we have $K_i \cap \bigcap_{j=1}^{k} H_j^+ \to K_i \cap K_1$, as $k \to \infty$. And in case $K_i \cap K_1 = \emptyset$, we have $K_i \cap \bigcap_{j=1}^{k} H_j^+ = \emptyset$ for k sufficiently large. Hence, the continuity of f and $f(\emptyset) = 0$ imply that

$$\sum_{i=1}^{m} \alpha_i f(K_i \cap K_1) = a.$$

Now repeat this construction with the sets $K_i' := K_i \cap K_1$ and approximate K_2' in this way, and so on up to m. We end up with

$$\mathbb{1}_{K_1 \cap \dots \cap K_m} \sum_{i=1}^{m} \alpha_i = \sum_{i=1}^{m} \alpha_i \mathbb{1}_{K_1 \cap \dots \cap K_m} = 0 \tag{4.12}$$

and

$$f(K_1 \cap \ldots \cap K_m) \sum_{i=1}^{m} \alpha_i = \sum_{i=1}^{m} \alpha_i f(K_1 \cap \ldots \cap K_m) = a \neq 0. \tag{4.13}$$

Equation (4.12) implies either $\sum_{i=1}^{m} \alpha_i = 0$ or $K_1 \cap \ldots \cap K_m = \emptyset$, which are both in contradiction to (4.13). □

Now we are ready to prove the existence of the additive extension for intrinsic volumes.

Proof of Theorem 4.10. Set $f := V_j$ and let V be the real vector space spanned by indicator functions of convex bodies, i.e., the set of all linear combinations of such functions. Observe that the mapping $B \mapsto \mathbb{1}_B$ is additive on $\mathcal{R}^d$ and recall from Worksheet 10, Problem 3 that for any $K \in \mathcal{R}^d$ and any representation $K = \bigcup_{i=1}^{m} K_i$ with $K_1, \ldots, K_m \in \mathcal{K}^d$ we have

$$\mathbb{1}_K = \sum_{k=1}^{m} (-1)^{k-1} \sum_{1 \leq i_1 < \ldots < i_k \leq m} \mathbb{1}_{K_{i_1} \cap \ldots \cap K_{i_k}}.$$

Hence $\mathbb{1}_K \in V$, since on the right we have a linear combination of indicators of convex sets. For any function $h = \sum_{i=1}^{m} \alpha_i \mathbb{1}_{K_i} \in V$, we define $\hat{f}(h) := \sum_{i=1}^{m} \alpha_i f(K_i)$. By Lemma 4.11, the mapping $\hat{f} : V \to \mathbb{R}$ is well defined (why?). Moreover, $\hat{f}$ is linear and $\hat{f}(\mathbb{1}_K) = f(K)$ for $K \in \mathcal{K}^d$. We set $\tilde{f}(K) := \hat{f}(\mathbb{1}_K)$ for $K \in \mathcal{R}^d$. Note that $\tilde{f}$ coincides with f on $\mathcal{K}^d$, i.e. it is an extension of f to $\mathcal{R}^d$. It remains to show that $\tilde{f}$ is additive. Employing the linearity of $\hat{f}$ and the additivity of the indicator function, we infer that, for any $K, M \in \mathcal{R}^d$,

$$\begin{aligned}
\tilde{f}(K \cup M) + \tilde{f}(K \cap M) &= \hat{f}(\mathbb{1}_{K \cup M}) + \hat{f}(\mathbb{1}_{K \cap M}) \\
&= \hat{f}(\mathbb{1}_{K \cup M} + \mathbb{1}_{K \cap M}) \\
&= \hat{f}(\mathbb{1}_K + \mathbb{1}_M) \\
&= \hat{f}(\mathbb{1}_K) + \hat{f}(\mathbb{1}_M) \\
&= \tilde{f}(K) + \tilde{f}(M).
\end{aligned}$$

Hence, $\tilde{f}$ is an additive extension of $f = V_j$. This completes the proof. □

Remark 4.12 The proof of Theorem 4.10 applies to any continuous, additive functional $f \colon \mathcal{K}^d \to \mathbb{R}$. That is, any such functional has an additive extension. In fact, the range $\mathbb{R}$ of f can even be replaced by an arbitrary topological vector space.

Remark 4.13 The set family $\mathcal{R}^d$ can be represented as a countable union of closed subsets of $\mathcal{C}^d$ and is thus measurable. If $f \colon \mathcal{R}^d \to \mathbb{R}$ is measurable on $\mathcal{K}^d$ and additive, then f is measurable on $\mathcal{R}^d$. For details we refer to [6, Theorem 14.4.4].

Remark 4.14 The additive extension of V_j is also called *j-th intrinsic volume* and denoted by V_j. Beside being additive, the functional $V_j \colon \mathcal{R}^d \to \mathbb{R}$ is also homogeneous of degree j and motion invariant. This can be deduced from the corresponding properties of V_j on $\mathcal{K}^d$, cf. Theorem 3.36, by noting that any $K \in \mathcal{R}^d$ has a representation $K = \bigcup_{i=1}^{m} K_i$ with $K_i \in \mathcal{K}^d$ and that

$$V_j(K) = V_j\left(\bigcup_{i=1}^{m} K_i\right) = \sum_{k=1}^{m} (-1)^{k-1} \sum_{1 \leq i_1 < \ldots < i_k \leq m} V_j(K_{i_1} \cap \ldots \cap K_{i_k}).$$

Furthermore, we still have $V_d(K) = \lambda_d(K)$ for $K \in \mathcal{R}^d$. If $\mathrm{cl}(\mathrm{int}\, K) = K$, then $V_{d-1}(K) = \frac{1}{2}\mathcal{H}^{d-1}(\partial K)$ (without proof). The functional $V_0 = \chi$ is called *Euler characteristic*. It is $\mathbb{Z}$-valued. In dimension $d = 2$, we have $V_0(K) = k(K) - l(K)$, where $k(K)$ denotes the number of connected components of K and $l(K)$ is the number of holes of K, that is, the number of bounded connected components of $\mathbb{R}^2 \setminus K$.

Example 4.15 For a (filled) square, the Euler characteristic is 1, and for a union of three disjoint squares it is 3. If we remove from the interior of a square a smaller open square, then the Euler characteristic of the resulting set is 0. Also the boundary of this set is polyconvex and has Euler characteristic 0. If we remove two disjoint open squares from the interior of a square, then the Euler characteristic of the resulting set is -1, the boundary of this set has characteristic 0.

Another important result in connection with intrinsic volumes is Hadwiger's theorem. It provides a way to introduce intrinsic volumes axiomatically. It says that any functional on $\mathcal{K}^d$ satisfying certain properties is necessarily a linear combination of intrinsic volumes.

Theorem 4.16 (Hadwiger's characterisation theorem) *Let $\Psi\colon \mathcal{K}' \to \mathbb{R}$ be an additive, continuous and motion invariant functional. Then there exist $c_0, \dots, c_d \in \mathbb{R}$ such that*

$$\Psi = \sum_{i=0}^{d} c_i V_i.$$

We will not discuss a proof of this theorem here, although we will use it later for proving the Crofton formula. We refer instead to [6, Theorem 14.4.6] for a proof. If one requires in Theorem 4.16 additionally that the functional Ψ is homogeneous of degree j, for some $j \in \{0, \dots, d\}$ (cf. Theorem 3.36) then Ψ is necessarily a constant multiple of the j-th intrinsic volume, i.e. $\Psi = cV_j$ for some $c \in \mathbb{R}$.

There are numerous interesting relations between the intrinsic volumes of different sets. One can for instance relate the intrinsic volumes of a set K to the intrinsic volumes of its projections onto lower dimensional subspaces or of its sections with lower dimensional affine spaces. One can think, for instance, of the orthogonal projections of a body $K \subset \mathbb{R}^3$ onto planes or of the intersections of K with lines. Another type of relation is provided by kinematic formulas, for which the intersection of K with a 'randomly moving' second body is studied. In order to discuss such relations, we introduce some further notation. Let

- $G(d, q)$ denote the family of all q-dimensional linear subspaces of $\mathbb{R}^d$;
- $A(d, q)$ denote the family of all q-dimensional affine subspaces of $\mathbb{R}^d$.

The elements of $A(d, k)$ will also be called *q-planes* or *q-flats*. We recall that SO_d is the group of rotations and G_d the group of rigid motions of $\mathbb{R}^d$. Observe that SO_d leaves the family $G(d, q)$ invariant in that linear q-spaces are mapped to linear q-spaces by rotations. Similarly, G_d (and SO_d) leave $A(d, k)$ invariant. Our aim in the sequel is to study integrals of the form

$$\int_{A(d,q)} V_j(K \cap F)\, dF, \quad K \in \mathcal{K}^d \qquad \text{(Crofton integral)}$$

and

$$\int_{\mathrm{G}_d} V_j(K \cap gM)\, dg, \quad K, M \in \mathcal{K}^d \qquad \text{(kinematic integral)},$$

where dF means that we integrate over all q-flats with respect to some 'uniform' measure on $A(d,q)$ and similarly dg means integration with respect to some uniform measure on G_d. To make this precise, we discuss what it means for group to *operate on a space* and invariance w.r.t. such an operation.

Recall that a topological group is a group G which is equipped with a topology such that the map $(x,y) \mapsto xy$ (the group operation) and the map $x \mapsto x^{-1}$ (inversion of group elements) are continuous. We assume throughout that G is locally compact and second countable.

Definition 4.17 Let G be a topological group with identity element e and let E be a topological space. An *operation* φ of G on E is a mapping $\varphi : G \times E \to E$, $(g,x) \mapsto \varphi(g,x)$, satisfying:

(i) $\varphi(g', \varphi(g,x)) = \varphi(g'g, x)$, for all $g, g' \in G,\ x \in E$.

(ii) $\varphi(e,x) = x$, for all $x \in E$.

If for all $x, y \in E$ there is a $g \in G$ such that $\varphi(g,x) = y$, then φ is said to *operate transitively* on E. If φ is continuous (recall that E and G are topological spaces), then φ is said to *operate continuously* on E.

Often one simply writes gx for $\varphi(g,x)$, if the operation φ is clear from the context. Then condition (i) reads for instance as follows

$$g'(gx) = (g'g)x.$$

Remark 4.18 (i) Every group G operates on itself via (left) group multiplication, that is, setting $E = G$ and $\varphi(g,x) := gx$ for $g, x \in G$ defines an operation of G on G.

(ii) If G acts on E then due to condition (i)

$$g^{-1}(gx) = (g^{-1}g)(x) = ex = x \quad \text{and} \quad g(g^{-1}x) = x.$$

Therefore, the mapping $E \to E, x \mapsto gx$ is bijective (for each $g \in G$) and the inverse mapping is given by $x \mapsto g^{-1}x$.

Example 4.19 (i) The group SO_d operates on $\mathbb{S}^{d-1}$ and on $G(d,q)$ (by applying rotations to points and linear q-spaces, respectively, in the obvious way). In both cases the operation is transitive. In the second case this just means that any linear q-space can be transformed into any other linear q-space by a rotation.

(ii) The group G_d operates transitively on $A(d,q)$ via $gF := \{gx : x \in F\}$ for $g \in \mathrm{G}_d$ and $F \in A(d,q)$.

Definition 4.20 Suppose that G operates continuously on E. A measure ν on E is called *G-invariant* (or *invariant* if G is clear from the context), if and only if

$$\nu(gA) = \nu(A), \quad \text{for all sets } A \in \mathcal{B}(E) \text{ and all } g \in G.$$

Equivalently, one can require that for all measurable $f\colon E \to [0,\infty)$ and $g \in G$,

$$\int_E f(gx)\, \nu(dx) = \int_E f(x)\, \nu(dx).$$

In case $G = E$ this property of ν is also called *left invariance*. (In this situation ν is called *right invariant* if and only if $\nu(Ag) = \nu(A),\ A \in \mathcal{B}(E),\ g \in G$.)

For illustration of the concept, let us look at some examples.

Example 4.21 (i) If G is a countable group (operating on itself), then the counting measure on G is G-invariant.

(ii) For $G = \mathrm{G}_d$ operating on $\mathbb{R}^d$, the Lebesgue measure λ_d on $\mathbb{R}^d$ is G_d-invariant. Any constant multiple of λ_d is also G_d-invariant but are there more G_d-invariant measures on $\mathbb{R}^d$?

It is well known from basic measure theory that λ_d is (up to constants) the only translation invariant measure on $\mathbb{R}^d$, and it is the unique one satisfying $\lambda_d([0,1]^d) = 1$. Since invariance w.r.t. all rigid motions is a stronger requirement than invariance with respect to translations, λ_d must also be the only G_d-invariant measure on $\mathbb{R}^d$ (giving the unit cube mass zero). Hence there cannot be any other G_d-invariant measures on $\mathbb{R}^d$. We are interested in finding also invariant measures on G_d, SO_d, $G(d,k)$ and $A(d,k)$ and see their uniqueness, for which the following general statement is useful. Recall that a measure μ on a locally compact space E is called a *Radon measure* if and only if $\nu(A) < \infty$ for any compact set $A \subset E$.

Theorem 4.22 *Let G be a locally compact topological group with countable base. Then there exists a nonzero left invariant Radon measure ν on G. The measure ν is unique up to constants, that is, if ν' is another left invariant Radon measure then $\nu' = c\nu$ for some constant $c \geq 0$.*

The measure ν in Theorem 4.22 (and any of its nonzero multiples) is called *Haar measure* of G. Often there is a natural normalization, e.g. if the measure is finite then one can normalize its total mass to 1. For a proof of Theorem 4.22 and a more detailed discussion of the topic we refer to [6, Chapter 13]. In our applications the existence of the Haar measure will be clear as one can give a direct construction in each case. But the uniqueness will be inferred from the above statement. Here we use this statement to see the existence and uniqueness of Haar measures on G_d, SO_d as well as the existence of invariant measures on $A(d,k)$ and $G(d,k)$.

Theorem 4.23 *On the group SO_d there is a uniquely determined SO_d-invariant probability measure ν.*

Idea of proof. For $d = 2$, the rotation group can be parametrized by the angle via the map $T : [0, 2\pi) \to \mathrm{SO}_2, \alpha \mapsto T(\alpha)$. This map allows to transfer the topology from the interval $[0, 2\pi)$ to SO_2 (which makes SO_2 compact). The image measure of the Lebesgue measure $\lambda_1(\cdot \cap [0, 2\pi))$ on $[0, 2\pi)$ is an SO_2-invariant measure on SO_2, which can be normalized to a probability measure ν. For $d > 2$, one can parametrize SO_d using orthogonal matrices and a measure defined on the space of d-tuples of linearly independent unit vectors, see e.g. [6, Thm. 13.2.9]. The uniqueness follows from Theorem 4.22. □

Theorem 4.24 *On the group G_d of rigid motions there is a unique (up to constants) left invariant Radon measure μ.*

Proof. Every rigid motion is the composition of a rotation and a translation. Consider the mapping

$$\gamma \colon \mathbb{R}^d \times SO_d \to \mathrm{G}_d,\ \gamma(x, \vartheta) := T_x \circ \vartheta$$

where $T_x(y) := y + x$. Observe that γ is bijective (!). One can use γ to transfer the topology from $\mathbb{R}^d \times \mathrm{SO}_d$ to G_d. With respect to this topology, G_d is a locally compact topological group satisfying the assumption of Theorem 4.22. Application of this theorem

yields the existence and uniqueness (up to a constant) of a left invariant Radon measure. One of its multiples can directly be related to $\lambda_d \otimes \nu$ on $\mathbb{R}^d \times \mathrm{SO}_d$ using γ:

$$\mu(\cdot) := \int_{\mathbb{R}^d} \int_{\mathrm{SO}_d} \mathbb{1}\{\underbrace{T_x \circ \vartheta}_{=\gamma(x,\vartheta)} \in \cdot\} \, \nu(d\vartheta) \lambda_d(dx). \tag{4.14}$$

Indeed, the right hand side defines a Radon measure on G_d which is G_d-invariant. Note that μ is normalized such that $\mu(\{g \in \mathrm{G}_d \colon \gamma^{-1}(g) \in [0,1]^d \times \mathrm{SO}_d\}) = 1$. □

Theorem 4.25 *On $G(d,q)$ there is a unique SO_d-invariant probability measure ν_q.*

Proof. Fix some q-space $L_q \in G(d,q)$. Consider the surjective (but not injective) mapping

$$\beta_q \colon \mathrm{SO}_d \to G(d,q), \ \vartheta \mapsto \vartheta L_q.$$

Choose on $G(d,q)$ the coarsest topology such that β_q is continuous, that is, a set $A \subset G(d,q)$ is open if and only if $\beta_q^{-1}(A)$ is open in SO_d. Let

$$\nu_q(A) := \int_{SO_d} \mathbb{1}\{\underbrace{\vartheta L_q}_{=\beta_q(\vartheta)} \in A\} \, \nu(d\vartheta), \qquad A \in \mathcal{B}(G(d,q)). \tag{4.15}$$

Obviously, this defines a measure on $G(d,q)$, which is finite (and hence a Radon measure), since ν is a probability measure. In fact, $\nu_q(G(d,q)) = \nu(\beta^{-1}(G(d,q))) = \nu(\mathrm{SO}_d) = 1$. It is easy to see that ν_q is SO_d-invariant. The uniqueness can be deduced from Theorem 4.22. □

Theorem 4.26 *On $A(d,q)$ there is a unique G_d-invariant Radon measure μ_q such that*

$$\mu_q(\{F \in A(d,q) \colon F \cap B^d \neq \emptyset\}) = \kappa_{d-q}.$$

For any measurable function $f \colon A(d,q) \to [0,\infty]$,

$$\int_{A(d,q)} f \, d\mu_q = \int_{G(d,q)} \int_{L^\perp} f(L+y) \, \lambda_{d-q}(dy) \nu_q(dL). \tag{4.16}$$

Idea of proof. Fix $L_q \in G(d,q)$ and consider the surjective (but not injective) mapping

$$\gamma_q \colon L_q^\perp \times SO_d \to A(d,q), \ (x,\vartheta) \mapsto \vartheta(L_q + x).$$

Define

$$\mu_q(\cdot) := \int_{SO_d} \int_{L_q^\perp} \mathbb{1}\{\underbrace{\vartheta(L_q+x)}_{=\gamma_q(x,\vartheta)} \in \cdot\} \, \lambda_{d-q}(dx) \nu(d\vartheta). \tag{4.17}$$

Similarly as in the proof of the previous theorem, it can be shown that μ_q is a Radon measure on $A(d,q)$ and G_d-invariant. The uniqueness can again be deduced from Theorem 4.22. □

After this small excursion to invariant measures, we have now in particular specified a 'uniform' measure μ_q on $A(d,q)$, and therefore we can study Crofton integrals, that is, integrals over q-flats with respect to the measure μ_q. Recall that the volume of the unit ball in $\mathbb{R}^m$ is given by the formula

$$\kappa_m = \frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2}+1)},$$

where Γ is the Gamma function defined by $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\,dt,\ x > 0$. It satisfies the relations $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. For convenience, we introduce for any $s, r \in \mathbb{N}_0$ the constant

$$c_s^r := \frac{r!\kappa_r}{s!\kappa_s} \qquad \text{and we set} \qquad c_{s_1,\dots,s_k}^{r_1,\dots,r_k} := \prod_{j=1}^k c_{s_j}^{r_j},$$

for any $k \in \mathbb{N}$ and $s_1, \dots, s_k, r_1, \dots, r_k \in \mathbb{N}_0$.

Theorem 4.27 (Crofton formula) *Let $K \in \mathcal{K}'$, $k \in \{1, \dots, d-1\}$ and $j \in \{0, \dots, k\}$. Then*

$$\int_{A(d,k)} V_j(K \cap F)\,\mu_k(dF) = c_{j,d}^{k,d-k+j} V_{d-k+j}(K). \tag{4.18}$$

In the two cases $k = d$ and $k = 0$ the above formula is also valid provided that we set $\mu_d := \delta_{\mathbb{R}^d}$ and $\mu_0 := \lambda_d$, and identify $A(d,0)$ with $\mathbb{R}^d$.

Proof. Let the functional $\psi\colon \mathcal{K}' \to [0,\infty)$ be defined by

$$\psi(K) := \int_{A(d,k)} V_j(K \cap F)\,\mu_k(dF), \quad K \in \mathcal{K}'.$$

Our aim is to apply Hadwiger's theorem, for which we need to show that ψ has the required properties. First note that, for each fixed K, the integrand is bounded (by a constant that is independent of F). Indeed, by the Steiner formula (3.14), we have

$$V_d(K + B^d) \geq V_d((K \cap F) + B^d) = \sum_{i=0}^d \kappa_{d-i} V_i(K \cap F)$$

and thus $V_j(K \cap F) \leq c_j \mathbb{1}\{K \cap F \neq \emptyset\}$ for some constant c_j independent of $F \in A(d,k)$. Also note that $\mathcal{F}_K \cap A(d,k)$ is compact in $A(d,k)$ (and that μ_k is a Radon measure). Hence $\psi(K) < \infty$ for any $K \in \mathcal{K}'$ (i.e., ψ is indeed a functional from $\mathcal{K}'$ to $[0,\infty)$). The additivity of V_j implies that ψ is additive. Moreover, we have for any $g \in \mathrm{G}_d$,

$$\psi(gK) = \int_{A(d,k)} V_j(K \cap g^{-1}F)\,\mu_k(dF) = \psi(K),$$

by the G_d-invariance of μ_q, showing that ψ is motion invariant. The functional ψ ist also continuous. Let $K, K_n \in \mathcal{K}'$ such that $K_n \to K$ as $n \to \infty$. We claim that this implies $K_n \cap F \to K \cap F$ for μ_q-almost all $F \in A(d,k)$ (but not for all F!). For a proof of the claim we refer to the proof of Theorem 5.1.2 in [6]. Taking this and the continuity of V_j into account, the convergence $\psi(K_n) \to \psi(K)$ follows by dominated convergence.

Therefore, we can apply Hadwiger's theorem 4.16 and infer that

$$\psi = \sum_{r=0}^d c_r V_r,$$

for suitable constants $c_0, \dots, c_d \in \mathbb{R}$. By (4.16), we have for any $\alpha > 0$,:

$$\begin{aligned}
\psi(\alpha K) &= \int_{G(d,k)} \int_{L^\perp} V_j(\alpha K \cap (L + x))\lambda_{d-k}(dx)\nu_k(dL) \\
&= \alpha^j \int_{G(d,k)} \int_{L^\perp} V_j(K \cap (L + \alpha^{-1}x))\lambda_{d-k}(dx)\nu_k(dL) \\
&= \alpha^j \alpha^{d-k} \int_{G(d,k)} \int_{L^\perp} V_j(K \cap (L + y))\lambda_{d-k}(dy)\nu_k(dL) = \alpha^{d+j-k}\psi(K),
\end{aligned}$$

where we have applied the substitution $\alpha^{-1}x = y$ in the last line. Among the intrinsic volumes only V_{d+j-k} has the right degree of homogeneity, and thus $c_r = 0$ for $r \neq d+j-k$. We conclude that

$$\int_{A(d,k)} V_j(K \cap F)\mu_k(dF) = cV_{d-k+j}(K), \quad K \in \mathcal{K}',$$

for some constant $c > 0$ which still needs to be determined. This can be done by inserting some particular set for K, for instance the unit ball B^d, into the formula and comparing both sides. After some computations one finds that $c = c_{j,d}^{k,d-k+j}$. □

Remark 4.28 Setting $j = 0$ and $m = d - k$ in Theorem 4.27 leads to the relation

$$V_m(K) = c_{m,d-m}^{0,d} \int_{A(d,d-m)} V_0(K \cap F)\, \mu_{d-m}(dF)$$

and thus to another geometric interpretation of the intrinsic volumes. Observe that the integrand on the right is just the indicator $\mathbb{1}\{K \cap F \neq \emptyset\}$. In case $m = 1$, we get for instance

$$V_1(K) = c_{1,1}^{0,d} \int_{A(d,1)} \mathbb{1}\{K \cap F \neq \emptyset\}\, \mu_1(dF).$$

Thus $V_1(K)$ can be interpreted as the measure of all lines hitting the set K.

Remark 4.29 Another interpretation arises from equation (4.16), which implies that

$$\begin{aligned} V_m(K) &= c_{m,d-m}^{0,d} \int_{G(d,d-m)} \int_{L^\perp} V_0(K \cap (L+y))\lambda_m(dy)\nu_{d-m}(dL) \\ &= c_{m,d-m}^{0,d} \int_{G(d,d-m)} \int_{L^\perp} \mathbb{1}\{y \in K|L^\perp\}\lambda_m(dy)\nu_{d-m}(dL) \\ &= c_{m,d-m}^{0,d} \int_{G(d,d-m)} \lambda_m(K|L^\perp)\nu_{d-m}(dL) \\ &= c_{m,d-m}^{0,d} \int_{G(d,m)} \lambda_m(K|L)\nu_m(dL). \end{aligned}$$

Here $K|L^\perp$ denotes the image of K under the orthogonal projection onto $L^\perp$. (For the validity of the last equality we refer to Work sheet 12.) In case $m = 1$, L is a line and the integrand $\lambda_1(K|L)$ is the *width of K in direction L*. (It is the minimal distance between two parallel hyperplanes orthogonal to L such that K is between them.) Since the integral is over all possible directions, $V_1(K)$ may be interpreted as the *average width of K* (up to a constant).

One of the direct consequences of the last formula is the monotonicity of intrinsic volumes.

Corollary 4.30 For each $j \in \{0, \dots, d\}$, V_j is monotone on $\mathcal{K}^d$, i.e., for any $K, M \in \mathcal{K}^d$, $K \subset M$ implies $V_j(K) \leq V_j(M)$.

Remark 4.31 Observe that both sides of the Crofton formula are additive. Therefore, the formula does also hold for $K \in \mathcal{R}^d$.

One application of the Crofton formula are so-called K_0-*isotropic random* q-*flats*, where $K_0 \in \mathcal{K}'$ and $q \in \{0, \dots, d-1\}$. These random flats can be used to treat problems from the field of *Geometric probabilities* such as Buffon's needle problem or the question, what is the probability that a random line that is conditioned to hit the unit ball B^d, also hits a given subset $K \subset B^d$. We refer to Work sheet 11, Problem 3 for an example.

The next theorem is a general statement on kinematic integrals. It will be specialized afterwards to give the Principal kinematic formula.

Theorem 4.32 *If a functional $f\colon \mathcal{K}' \to \mathbb{R}$ is additive and continuous, then*

$$\int_{\mathrm{G}_d} f(K \cap gM)\, \mu(dg) = \sum_{k=0}^{d} f_{d-k}(K) V_k(M), \quad \text{for } K, M \in \mathcal{K}', \tag{4.19}$$

where

$$f_{d-k}(K) := \int_{A(d,k)} f(K \cap F)\, \mu_k(dF), \quad \text{for any } K \in \mathcal{K}' \text{ and } k = 0, \dots, d.$$

Idea of proof. For the details we refer to Theorem 5.1.2 in [6]. Similar considerations as in the proof of Theorem 4.27 yield that the mapping $g \mapsto f(K \cap gM)$ is measurable for all $K, M \in \mathcal{K}'$. For fixed $K \in \mathcal{K}'$, we define $\psi : \mathcal{K}' \to \mathbb{R}$ by

$$\psi(M) = \int_{\mathrm{G}_d} f(K \cap gM)\, \mu(dg), \quad M \in \mathcal{K}'.$$

Then ψ is additive, continuous and motion invariant due to the motion invariance of μ. Therefore, Hadwiger's theorem 4.16 implies that

$$\psi = \sum_{i=0}^{d} f_{d-i}(K) V_i$$

for suitable constants $f_{d-i}(K)$ (which depend on K).

To determine these constants, let $k \in \{0, \dots, d\}$ and $L_k \in G(d,k)$. Further, let $C \subset L_k$ be the k-dimensional unit cube in L_k with center 0. Noting that $V_i(C) = 0$ for $i > k$, we infer on the one hand that, for $r > 0$,

$$\psi(rC) = \sum_{i=0}^{k} f_{d-i}(K) r^i V_i(C),$$

which implies that $f_{d-k}(K) = \lim_{r\to\infty} r^{-k}\psi(rC)$. On the other hand, a direct computation shows (cf. the proof of [6, Thm 5.1.2]) that

$$\frac{1}{r^k}\psi(rC) = \frac{1}{r^k}\int_{\mathrm{G}_d} f(K \cap grC)\, \mu(dg) \overset{r\to\infty}{\longrightarrow} \int_{A(d,k)} f(K \cap F)\, \mu_k(dF). \qquad \square$$

Theorem 4.33 (Principal kinematic formula) *For any $K, M \in \mathcal{K}'$ and any $j \in \{0, \dots, d\}$, it holds that*

$$\int_{\mathrm{G}_d} V_j(K \cap gM)\, \mu(dg) = \sum_{k=j}^{d} c_{j,d}^{k,d-k+j} V_k(K) V_{d-k+j}(M).$$

Proof. We apply Theorem 4.32 with $f := V_j$. In this case, the Crofton formula (4.18) yields for any $j \leq k$:

$$f_{d-k}(K) = \int_{A(d,k)} V_j(K \cap F)\, \mu_k(dF) = c_{j,d}^{k,d-k+j} V_{d-k+j}(K).$$

For $j > k$, this integral vanishes, as the set $K \cap F$ is at most k-dimensional. □

Note that also the Principal kinematic formula remains valid for $K, M \in \mathcal{R}^d$.

Remark 4.34 For any $K, M \in \mathcal{K}'$, we have

$$\int_{\mathbb{R}^d} V_d(K \cap (M + x))\, dx = V_d(K)V_d(M) \text{ and}$$

$$\int_{\mathbb{R}^d} V_{d-1}(K \cap (M + x))\, dx = V_{d-1}(K)V_d(M) + V_d(K)V_{d-1}(M),$$

that is, for $j = d$ and $j = d - 1$, the Principal kinematic formula remains true if the integration over the rotations is omitted. This is not true in general for other indices j.

Analogously to random q-flats, in the field of *Geometric probabilities* one studies randomly moving bodies. For $A_0 \in \mathcal{B}(\mathrm{G}_d)$ an A_0-*isotropic random motion* is defined as a G_d-valued random variable $\tilde{g}$ with distribution $\frac{\mu(A_0 \cap \cdot)}{\mu(A_0)}$. For convex bodies $K_0, M \in \mathcal{K}^d$, one can for instance consider the set $A_0 := \{g \in \mathrm{G}_d : K_0 \cap gM \neq \emptyset\}$, which allows to study the probability that the randomly moving body $\tilde{g}M$ intersects a second fixed body $K \subset K_0$. The Principal kinematic formula turns out to be useful for computing such probabilities (see Worksheet 11, Problem 4).

. . . (to be continued) . . .

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