Solutions for Work Sheet 3

Problem 1 (On the Hausdorff metric – Part 2)

Let $(\mathbb{X}, \|\cdot\|)$ be a normed vector space (over $\mathbb{R}$ or $\mathbb{C}$), and recall from Problem 4 on Work sheet 2 that the Hausdorff metric δ on $\mathcal{C}(\mathbb{X}) \setminus \{\varnothing\}$ is defined as

$$\delta(C, C') := \inf\big\{\varepsilon \geqslant 0 \ : \ C \subset C'_{\oplus\varepsilon},\ C' \subset C_{\oplus\varepsilon}\big\}, \qquad C, C' \in \mathcal{C}(\mathbb{X}) \setminus \{\varnothing\},$$

and that we put $\delta(\varnothing, C) = \delta(C, \varnothing) := \infty$, $C \in \mathcal{C}(\mathbb{X}) \setminus \{\varnothing\}$, as well as $\delta(\varnothing, \varnothing) := 0$. Note that if we denote by $B_{\mathbb{X}} = B(0, 1)$ the closed unit ball around the origin in $\mathbb{X}$, then

$$B_{\oplus\varepsilon} = B + \varepsilon B_{\mathbb{X}}, \qquad B \subset \mathbb{X},\ \varepsilon \geqslant 0.$$

Let $C, C', D, D' \in \mathcal{C}(\mathbb{X}) \setminus \{\varnothing\}$, and show that

a) $\delta\big(\mathrm{conv}(C), \mathrm{conv}(D)\big) \leqslant \delta(C, D)$,

b) $\delta(C + C', D + D') \leqslant \delta(C, D) + \delta(C', D')$, and

c) $\delta(C \cup C', D \cup D') \leqslant \max\big\{\delta(C, D), \delta(C', D')\big\}$,

where $\mathrm{conv}(C)$ denotes the convex hull of C.

Proposed solution:

a) First observe that if $K, L \subset \mathbb{X}$ are convex, then $K + L$ is also convex: Indeed, letting $\lambda \in [0, 1]$ and $x, y \in K + L$ we can write $x = x_K + x_L$ and $y = y_K + y_L$ to find that

$$\lambda x + (1 - \lambda)y = \lambda x_K + (1 - \lambda)y_K + \lambda x_L + (1 - \lambda)y_L \in K + L.$$

Now, let $C, D \in \mathcal{C}(\mathbb{X}) \setminus \{\varnothing\}$. Clearly, $C \subset D_{\oplus\delta(C,D)} = D + \delta(C, D)\, B_{\mathbb{X}}$, so $C \subset \mathrm{conv}(D) + \delta(C, D)\, B_{\mathbb{X}}$ and, since the set on the right hand side is convex by the preliminary observation above, we have that

$$\mathrm{conv}(C) \subset \mathrm{conv}(D) + \delta(C, D)\, B_{\mathbb{X}}.$$

Similarly, it follows that $\mathrm{conv}(D) \subset \mathrm{conv}(C) + \delta(C, D)\, B_{\mathbb{X}}$. We conclude that

$$\delta\big(\mathrm{conv}(C), \mathrm{conv}(D)\big) = \inf\Big\{\varepsilon \geqslant 0 \ : \ \mathrm{conv}(C) \subset \mathrm{conv}(D) + \varepsilon B_{\mathbb{X}},\ \mathrm{conv}(D) \subset \mathrm{conv}(C) + \varepsilon B_{\mathbb{X}}\Big\} \leqslant \delta(C, D).$$

b) It follows from $C \subset D + \delta(C, D)\, B_{\mathbb{X}}$ and $C' \subset D' + \delta(C', D')\, B_{\mathbb{X}}$ that

$$C + C' \subset D + D' + \big(\delta(C, D) + \delta(C', D')\big) B_{\mathbb{X}}.$$

Similarly, $D + D' \subset C + C' + \big(\delta(C, D) + \delta(C', D')\big) B_{\mathbb{X}}$. Therefore, $\delta(C + C', D + D') \leqslant \delta(C, D) + \delta(C', D')$.

c) As above, we find that $C \cup C' \subset \big(D + \delta(C, D)\, B_{\mathbb{X}}\big) \cup \big(D' + \delta(C', D')\, B_{\mathbb{X}}\big)$. Put

$$\widetilde{\varepsilon} := \max\big\{\delta(C, D),\ \delta(C', D')\big\}.$$

Then,

$$C \cup C' \subset (D + \widetilde{\varepsilon} B_{\mathbb{X}}) \cup (D' + \widetilde{\varepsilon} B_{\mathbb{X}}) = (D \cup D') + \widetilde{\varepsilon} B_{\mathbb{X}},$$

and therefore $\delta(C \cup C', D \cup D') \leqslant \widetilde{\varepsilon} = \max\big\{\delta(C, D), \delta(C', D')\big\}$.

Problem 2 (Random closed sets)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space.

a) Let $\xi : \Omega \to \mathbb{R}^d$ be a random vector. Show that $Z_1 = \{\xi\}$ is a random closed set.

b) Let $\xi_1, \xi_2, \ldots$ be a sequence of random vectors (in $\mathbb{R}^d$) such that $\{\xi_k(\omega) : k \in \mathbb{N}\}$ has no accumulation point in $\mathbb{R}^d$ (for each $\omega \in \Omega$). Show that $Z_2 := \{\xi_k : k \in \mathbb{N}\}$ is a random closed set.

c) Let $R : \Omega \to (0, \infty)$ be a positive random variable and $\xi : \Omega \to \mathbb{R}^d$ a random vector.

   (i) Show that the "random closed ball" Z_3 with center ξ and radius R, that is, the mapping $Z_3 : \Omega \to \mathcal{F}^d$, $Z_3(\omega) := B\big(\xi(\omega), R(\omega)\big)$, is a random closed set.

   (ii) Show that $\mathbb{E}\big[\lambda^d(Z_3 + K)\big] < \infty$ holds for every compact set $K \subset \mathbb{R}^d$ if, and only if, $\mathbb{E}[R^d] < \infty$.

d) Find a random closed set Z_4 such that the set $G := \big\{x \in \mathbb{R}^d \,:\, \mathbb{P}(x \in Z_4) \neq 0\big\}$ is open.

e) Let Z_5 be a stationary random closed set in $\mathbb{R}^d$ with volume fraction $p_{Z_5} = \mathbb{E}\big[\lambda^d(Z_5 \cap [0,1]^d)\big]$. Let $B \in \mathcal{B}^d$, and show that

$$\mathbb{E}\Big[\lambda^d(Z_5 \cap B)\Big] = p_{Z_5} \cdot \lambda^d(B) \qquad \text{and} \qquad p_{Z_5} = \mathbb{P}\big(t \in Z_5\big), \quad t \in \mathbb{R}^d.$$

Proposed solution:

a) It suffices to show that the map $\varphi : \big(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\big) \to \big(\mathcal{F}, \mathcal{B}(\mathcal{F})\big) = \big(\mathcal{F}^d, \mathcal{B}(\mathcal{F}^d)\big)$, $x \mapsto \{x\}$ is measurable. For $C \in \mathcal{C}^d$ we have

$$\varphi^{-1}(\mathcal{F}^C) = \Big\{x \in \mathbb{R}^d \,:\, \{x\} \in \mathcal{F}^C\Big\} = \big\{x \in \mathbb{R}^d \,:\, \{x\} \cap C = \varnothing\big\} = \big\{x \in \mathbb{R}^d \,:\, x \notin C\big\} \in \mathcal{B}(\mathbb{R}^d).$$

As $\{\mathcal{F}^C : C \in \mathcal{C}^d\}$ generates the Borel-σ-field of $\mathcal{F} = \mathcal{F}^d$, the measurability of φ follows.

b) As $\{\xi_k(\omega) : k \in \mathbb{N}\}$ has no accumulation point (for each $\omega \in \Omega$), the set is closed and hence in $\mathcal{F} = \mathcal{F}^d$. We have that, for every $C \in \mathcal{C}^d$,

$$Z_2^{-1}(\mathcal{F}^C) = \big\{Z \in \mathcal{F}^C\big\} = \{Z \cap C = \varnothing\} = \Big\{\bigcup_{j=1}^{\infty} \{\xi_j\} \cap C = \varnothing\Big\} = \bigcap_{j=1}^{\infty} \Big\{\{\xi_j\} \cap C = \varnothing\Big\} \in \mathcal{A},$$

so $Z_2 : \Omega \to \mathcal{F}^d$ is measurable and hence a random set.

c) (i) We have, for all $C \in \mathcal{C}^d$,

$$Z_3^{-1}\big(\mathcal{F}^C\big) = \big\{Z_3 \in \mathcal{F}^C\big\} = \big\{Z \cap C = \varnothing\big\} = \big\{R < \mathrm{dist}(\xi, C)\big\} \in \mathcal{A},$$

since $(\mathbb{R}, \mathbb{R}^d) \ni (x, y) \mapsto \mathbb{1}\{x < \mathrm{dist}(y, C)\}$ is measurable. Indeed, notice that $\mathbb{R}^d \ni y \mapsto \mathrm{dist}(y, C)$ is measurable (in fact, upper semi-continuous) since

$$\big\{y \in \mathbb{R}^d \,:\, \mathrm{dist}(y, C) < r\big\}$$

is an open set for each $r \in \mathbb{R}$. Hence, $Z_3 = B(\xi, R)$ is a random closed set.

(ii) "$\Longrightarrow$" Denoting by κ_d the volume of the d-dimensional unit ball and choosing $K = \{0\} \in \mathcal{C}^d$, we obtain

$$\mathbb{E}[R^d] = \kappa_d^{-1}\, \mathbb{E}\big[R^d \cdot \lambda^d\big(B(0,1)\big)\big] = \kappa_d^{-1}\, \mathbb{E}\big[\lambda^d(Z_3)\big] = \kappa_d^{-1}\, \mathbb{E}\big[\lambda^d(Z_3 + \{0\})\big] < \infty.$$

"$\Longleftarrow$ " Let $K \in \mathcal{C}^d$, and choose $N \in \mathbb{N}$ such that $K \subset B(0, N)$. Then,

$$\begin{aligned}
\mathbb{E}\Big[\lambda^d(Z_3 + K)\Big] \leqslant \mathbb{E}\Big[\lambda^d\big(Z_3 + B(0,N)\big)\Big] &= \mathbb{E}\Big[\lambda^d\big(B(\xi, R + N)\big)\Big] \\
&= \kappa_d\, \mathbb{E}\big[(R + N)^d\big] \\
&= \kappa_d\, \mathbb{E}\left[\sum_{k=0}^{d} \binom{d}{k} R^k\, N^{d-k}\right] \\
&= \kappa_d \sum_{k=0}^{d} \binom{d}{k} N^{d-k}\, \mathbb{E}[R^k] \\
&\leqslant \kappa_d \cdot \max\big\{1,\, \mathbb{E}[R^d]\big\} \cdot (N+1)^d \\
&< \infty.
\end{aligned}$$

d) Let R be uniformly distributed over the interval $[0,1]$, and take $Z_4 := B(0,R)$ as well as $x \in \mathbb{R}^d$. Then

$$\mathbb{P}(x \in Z_4) = \mathbb{P}\big(R \geqslant \|x\|\big) = \begin{cases} 0, & \|x\| \geqslant 1, \\ 1 - \|x\|, & \|x\| < 1. \end{cases}$$

Hence, G is the open unit ball.

e) We first prove measurability of $\Omega \ni \omega \mapsto \lambda^d\big(Z_5(\omega) \cap B\big)$. Note that if $F \in \mathcal{F}^d$, then $Z_5 \cap F$ is a random closed set. Since $\mathbb{R}^d \times \mathcal{F}^d \ni (x, F') \mapsto \mathbb{1}\{x \in F'\}$ is measurable by Theorem 1.12, the mapping

$$\mathbb{R}^d \times \Omega \ni (x, \omega) \mapsto \mathbb{1}\big\{x \in Z_5(\omega) \cap F\big\}$$

is measurable, and Fubini's theorem yields the measurability of

$$\Omega \ni \omega \mapsto \lambda^d\big(Z_5(\omega) \cap F\big) = \int_{\mathbb{R}^d} \mathbb{1}\big\{x \in Z_5(\omega) \cap F\big\}\, \mathrm{d}x.$$

Next, consider the collection

$$\mathcal{D} = \big\{B \in \mathcal{B}^d \ : \ \Omega \ni \omega \mapsto \lambda^d\big(Z_5(\omega) \cap B\big) \text{ is measurable}\big\}.$$

Notice that $\mathcal{D}$ contains the π-system $\mathcal{F}^d$ which generates $\mathcal{B}^d$, and $\mathcal{D}$ is easily seen to be a Dynkin system. By the monotone class theorem, we have $\mathcal{D} = \mathcal{B}^d$.

Now, let $B \in \mathcal{B}^d$ and $t \in \mathbb{R}^d$. Notice that since Z_5 is stationary, we have

$$\mathbb{P}(t \in Z_5) = \mathbb{P}\big(t \in Z_5 - (x - t)\big) = \mathbb{P}(x \in Z_5)$$

for every $x \in \mathbb{R}^d$. Thus the volume fraction satisfies

$$p_{Z_5} = \mathbb{E}\Big[\lambda^d(Z_5 \cap [0,1]^d)\Big] = \mathbb{E}\int_{\mathbb{R}^d} \mathbb{1}_{Z_5 \cap [0,1]^d}(x)\, \mathrm{d}x = \int_{[0,1]^d} \mathbb{P}(x \in Z_5)\, \mathrm{d}x = \mathbb{P}(t \in Z_5)$$

by Fubini's theorem, and similarly

$$\mathbb{E}\Big[\lambda^d(Z_5 \cap B)\Big] = \mathbb{E}\int_{\mathbb{R}^d} \mathbb{1}_{Z_5 \cap B}(x)\, \mathrm{d}x = \int_B \mathbb{P}(x \in Z_5)\, \mathrm{d}x = p_{Z_5} \cdot \lambda^d(B).$$

Problem 3 (Properties of the capacity functional)

Let Z be a random closed set in $\mathbb{R}^d$, defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and denote by T_Z the capacity functional of Z. Prove the following assertions.

a) $0 \leqslant T_Z \leqslant 1$ as well as $T_Z(\varnothing) = 0$, and

b) for any sets $C, C_1, C_2, \ldots \in \mathcal{C}^d$ with $C_n \searrow C$ it holds that $T_Z(C_n) \to T_Z(C)$, as $n \to \infty$.

Proposed solution:

a) This first claim immediately follows from the definition of T_Z as a probability. The empty set is never hit by any set, meaning that $T_Z(\varnothing) = \mathbb{P}\big(Z \cap \varnothing \neq \varnothing\big) = 0$ [which should be distinguished from the event $Z = \varnothing$, which may very well have a positive probability].

b) Let $C, C_1, C_2, \ldots \in \mathcal{C}^d$ with $C_n \searrow C$, that is, $C_n \supset C_{n+1}$ for each $n \in \mathbb{N}$ and $C = \bigcap_{n=1}^\infty C_n$. We have

$$T_Z(C) = \mathbb{P}\big(Z \cap C \neq \varnothing\big) = \mathbb{P}\Big(Z \cap \bigcap_{n=1}^\infty C_n \neq \varnothing\Big) = \mathbb{P}\Big(Z \in \bigcap_{n=1}^\infty \mathcal{F}_{C_n}\Big) = \lim_{n\to\infty} \mathbb{P}\big(Z \in \mathcal{F}_{C_n}\big) = \lim_{n\to\infty} T_Z(C_n),$$

where the second to last equality uses $\mathcal{F}_{C_n} \supset \mathcal{F}_{C_{n+1}}$, $n \in \mathbb{N}$, and the continuity from above of the measure $\mathbb{P}(Z \in \cdot\,)$.

Problem 4 (Capacity functionals – Examples)

Compute the capacity functional of the following random sets $\widetilde{Z}$ in $\mathbb{R}^d$ which are assumed to be defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

a) $\widetilde{Z} = F$, for some fixed closed set $F \in \mathcal{F}^d$,

b) $\widetilde{Z} = \{\xi\}$, where ξ is a random element of $\mathbb{R}^d$.

Let $Z, Z_1, Z_2, \dots$ be independent and identically distributed random closed sets in $\mathbb{R}^d$, defined on $(\Omega, \mathcal{A}, \mathbb{P})$. Let N be an $\mathbb{N}_0$-valued random variable which is independent of $(Z_n)_{n\in\mathbb{N}}$. Write $p_k := \mathbb{P}(N = k)$, $k \in \mathbb{N}_0$, and put $Z^* := \bigcup_{n=1}^N Z_n$. Denote by $G(s) := \sum_{k=0}^\infty p_k \cdot s^k$, for $s \in [0, 1]$, the generating function of N.

c) Prove that Z^* is a random closed set with $T_{Z^*}(\cdot) = 1 - G\big(1 - T_Z(\cdot)\big)$.

d) Show that if $N \sim \text{Po}(\lambda)$ for some $\lambda > 0$, then $T_{Z^*}(\cdot) = 1 - e^{-\lambda \cdot T_Z(\cdot)}$.

Proposed solution:

a) For any $C \in \mathcal{C}^d$, we have $T_{\widetilde{Z}}(C) = \mathbb{P}\big(\widetilde{Z} \cap C \neq \varnothing\big) = \mathbb{P}\big(F \cap C \neq \varnothing\big) = \mathbb{1}\big\{F \cap C \neq \varnothing\big\}$.

b) For any $C \in \mathcal{C}^d$, we have $T_{\widetilde{Z}}(C) = \mathbb{P}\big(\widetilde{Z} \cap C \neq \varnothing\big) = \mathbb{P}\big(\{\xi\} \cap C \neq \varnothing\big) = \mathbb{P}(\xi \in C)$.

c) Z^* is well-defined since $Z^*(\omega) = \bigcup_{n=1}^{N(\omega)} Z_n(\omega)$ is a closed set as a finite union of closed sets (for each $\omega \in \Omega$). Moreover, $Z^* : \Omega \to \mathcal{F} = \mathcal{F}^d$ is measurable as

$$Z^{*\,-1}\left(\mathcal{F}^C\right) = \bigcup_{k=0}^{\infty} \left(\{N = k\} \cap \left\{ \bigcup_{n=1}^{k} Z_n \in \mathcal{F}^C \right\}\right) = \bigcup_{k=0}^{\infty} \left(\{N = k\} \cap \bigcap_{n=1}^{k} \left\{Z_n \in \mathcal{F}^C\right\}\right) \in \mathcal{A}.$$

For any $C \in \mathcal{C}^d$, we have (using the independence properties)

$$\begin{aligned}
T_{Z^*}(C) = \mathbb{P}\left(\bigcup_{n=1}^{N} Z_n \cap C \neq \varnothing\right) &= \sum_{k=0}^{\infty} p_k \cdot \mathbb{P}\left(\bigcup_{n=1}^{k} Z_n \cap C \neq \varnothing\right) \\
&= \sum_{k=0}^{\infty} p_k \cdot \Big(1 - \mathbb{P}\big(Z_1 \cap C = \varnothing, \dots, Z_k \cap C = \varnothing\big)\Big) \\
&= 1 - \sum_{k=0}^{\infty} p_k \cdot \big(\mathbb{P}(Z \cap C = \varnothing)\big)^k \\
&= 1 - \sum_{k=0}^{\infty} p_k \cdot \big(1 - T_Z(C)\big)^k \\
&= 1 - G\big(1 - T_Z(C)\big).
\end{aligned}$$

d) Recall that for a Poisson random variable we have $p_k = \frac{\lambda^k}{k!} \cdot e^{-\lambda}$. Therefore, part c) yields that for any $C \in \mathcal{C}^d$

$$T_{Z^*}(C) = 1 - \sum_{k=0}^{\infty} \big(1 - T_Z(C)\big)^k \cdot \frac{\lambda^k}{k!} \cdot e^{-\lambda} = 1 - e^{-\lambda} \cdot \exp\Big(\lambda \cdot \big(1 - T_Z(C)\big)\Big) = 1 - e^{-\lambda \cdot T_Z(C)}.$$