Solutions for Work Sheet 9

Problem 1 (The Euler characteristic)

Use the Steiner formula to prove that, for any $K \in \mathcal{K}^d \setminus \{\varnothing\}$, we have $V_0(K) = 1$.

Proposed solution: Let $K \in \mathcal{K}^d \setminus \{\varnothing\}$ and write $B^d := B(0,1) \subset \mathbb{R}^d$ for the d-dimensional unit ball. As the set K is compact, we find $r > 0$ such that $K \subset r \cdot B^d$. Fix $x \in K$ and observe that

$$(x + \varepsilon \cdot B^d) \subset K + \varepsilon \cdot B^d \subset r \cdot B^d + \varepsilon \cdot B^d, \qquad \varepsilon > 0.$$

Using that the volume functional is monotone, we obtain

$$V_d(x + \varepsilon \cdot B^d) \leqslant V_d(K + \varepsilon \cdot B^d) \leqslant V_d(r \cdot B^d + \varepsilon \cdot B^d),$$

and hence

$$\varepsilon^d \kappa_d = V_d(x + \varepsilon \cdot B^d) \leqslant V_d(K + \varepsilon \cdot B^d) \leqslant (r + \varepsilon)^d \, V_d(B^d) = \sum_{j=0}^{d} \binom{d}{j} r^{d-j} \, \varepsilon^j \, \kappa_d,$$

where κ_d denotes the volume of B^d. Upon dividing this inequality by $\varepsilon^d \kappa_d$ and observing that

$$\lim_{\varepsilon \to \infty} \frac{1}{\varepsilon^d \kappa_d} \sum_{j=0}^{d} \binom{d}{j} r^{d-j} \, \varepsilon^j \, \kappa_d = \lim_{\varepsilon \to \infty} \sum_{j=0}^{d} \binom{d}{j} r^{d-j} \, \varepsilon^{j-d} = 1,$$

we get

$$\lim_{\varepsilon \to \infty} \frac{V_d(K + \varepsilon \cdot B^d)}{\varepsilon^d \kappa_d} = 1.$$

By the Steiner formula,

$$1 = \lim_{\varepsilon \to \infty} \frac{V_d(K + \varepsilon \cdot B^d)}{\varepsilon^d \kappa_d} = \lim_{\varepsilon \to \infty} \frac{1}{\varepsilon^d \kappa_d} \sum_{j=0}^{d} \kappa_{d-j} \, \varepsilon^{d-j} \, V_j(K) = V_0(K).$$

Problem 2 (Additivity of intrinsic volumes)

a) Let $K, L \in \mathcal{K}^d \setminus \{\varnothing\}$ such that $K \cup L \in \mathcal{K}^d$. Prove that, for any $\varepsilon \geqslant 0$,

$$\mathbb{1}_{(K \cup L) + \varepsilon \cdot B^d} + \mathbb{1}_{(K \cap L) + \varepsilon \cdot B^d} = \mathbb{1}_{K + \varepsilon \cdot B^d} + \mathbb{1}_{L + \varepsilon \cdot B^d}.$$

b) Conclude from part a) that V_j is additive (for each $j \in \{0, \ldots, d\}$).

Proposed solution:

a) For any $M \in \mathcal{K}^d \setminus \{\varnothing\}$ and $y \in \mathbb{R}^d$ we have

$$\mathbb{1}_{M+\varepsilon\cdot B^d}(y) = \mathbb{1}\big\{\mathrm{dist}(y, M) \leqslant \varepsilon\big\} = \mathbb{1}\big\{\|p(M, y) - y\| \leqslant \varepsilon\big\}.$$

Therefore, we need to prove that, for any $x \in \mathbb{R}^d$,

$$\mathbb{1}\big\{\|p(K\cup L, x) - x\| \leqslant \varepsilon\big\} + \mathbb{1}\big\{\|p(K\cap L, x) - x\| \leqslant \varepsilon\big\} = \mathbb{1}\big\{\|p(K, x) - x\| \leqslant \varepsilon\big\} + \mathbb{1}\big\{\|p(L, x) - x\| \leqslant \varepsilon\big\}.$$

Let $x \in \mathbb{R}^d$. Since the previous line is symmetric in K and L, we may assume without loss of generality that $p(K\cup L, x) \in K$. We conclude that

$$\mathrm{dist}(x, K) \leqslant \| \underbrace{p(K\cup L, x)}_{\in K} - x\| = \mathrm{dist}(x, K\cup L) \leqslant \mathrm{dist}(x, K),$$

so, by definition of the metric projection, $p(K, x) = p(K\cup L, x)$ which, in turn, implies

$$\mathbb{1}\big\{\|p(K\cup L, x) - x\| \leqslant \varepsilon\big\} = \mathbb{1}\big\{\|p(K, x) - x\| \leqslant \varepsilon\big\}. \tag{1}$$

We proceed by distinguishing two cases:

In **case 1**, we assume that $p(L, x) \in K\cap L$. It follows that

$$\mathrm{dist}(x, L) \leqslant \mathrm{dist}(x, K\cap L) \leqslant \|p(L, x) - x\| = \mathrm{dist}(x, L),$$

hence $\mathrm{dist}(x, K\cap L) = \|p(L, x) - x\|$ and therefore $p(K\cap L, x) = p(L, x)$. Together with (1), the claim follows.

In **case 2**, we assume that $p(L, x) \in L \setminus K$. As $K\cup L$ is convex, we have

$$\big[p(K, x),\, p(L, x)\big] := \big\{\alpha\cdot p(K, x) + (1-\alpha)\cdot p(L, x) \,:\, \alpha \in [0, 1]\big\} \subset K\cup L.$$

From $p(K, x) = p(K\cup L, x)$ and $p(L, x) \notin K\cap L$ it follows that $\mathrm{dist}(x, K) = \mathrm{dist}(x, K\cup L) < \mathrm{dist}(x, L)$ and hence $p(K, x) \notin L$. Thus, we have $p(K, x) \in K\setminus L$, that is, we can find a minimal $\lambda \in (0, 1)$ such that $\lambda\cdot p(K, x) + (1-\lambda)\cdot p(L, x) \in K\cap L$. Therefore,

$$\begin{aligned}
\mathrm{dist}(x, K\cap L) \leqslant \big\|\lambda\cdot p(K, x) + (1-\lambda)\cdot p(L, x) - x\big\| &\leqslant \lambda\cdot\|p(K, x) - x\| + (1-\lambda)\cdot\|p(L, x) - x\| \\
&= \lambda\cdot\mathrm{dist}(x, K) + (1-\lambda)\cdot\mathrm{dist}(x, L) \\
&< \mathrm{dist}(x, L) \\
&\leqslant \mathrm{dist}(x, K\cap L),
\end{aligned}$$

a contradiction. We conclude that only the first case (in which the claim follows) can occur.

b) Part a) implies that, for $K, L \in \mathcal{K}^d$ with $K\cup L \in \mathcal{K}^d$ and $\varepsilon \geqslant 0$, we have

$$\begin{aligned}
V_d\big((K\cup L) + \varepsilon\cdot B^d\big) + V_d\big((K\cap L) + \varepsilon\cdot B^d\big) &= \int_{\mathbb{R}^d} \Big(\mathbb{1}_{(K\cup L)+\varepsilon\cdot B^d}(x) + \mathbb{1}_{(K\cap L)+\varepsilon\cdot B^d}(x)\Big)\mathrm{d}\lambda^d(x) \\
&= \int_{\mathbb{R}^d} \Big(\mathbb{1}_{K+\varepsilon\cdot B^d}(x) + \mathbb{1}_{L+\varepsilon\cdot B^d}(x)\Big)\mathrm{d}\lambda^d(x) \\
&= V_d(K + \varepsilon\cdot B^d) + V_d(L + \varepsilon\cdot B^d).
\end{aligned}$$

Notice that the equation holds trivially if any of the two sets is empty. The Steiner formula implies that, for every $\varepsilon \geqslant 0$,

$$\sum_{j=0}^{d} \kappa_{d-j}\,\varepsilon^{d-j}\,V_j(K\cup L) + \sum_{j=0}^{d} \kappa_{d-j}\,\varepsilon^{d-j}\,V_j(K\cap L) = \sum_{j=0}^{d} \kappa_{d-j}\,\varepsilon^{d-j}\,V_j(K) + \sum_{j=0}^{d} \kappa_{d-j}\,\varepsilon^{d-j}\,V_j(L).$$

From the observation that

$$\sum_{j=0}^{d} \kappa_{d-j}\,\varepsilon^{d-j}\Big(V_j(K\cup L) + V_j(K\cap L)\Big) = \sum_{j=0}^{d} \kappa_{d-j}\,\varepsilon^{d-j}\Big(V_j(K) + V_j(L)\Big), \qquad \varepsilon \geqslant 0,$$

and a comparison of coefficients, the claim follows.

Problem 3 (Intrinsic volumes of the unit ball)

Calculate the intrinsic volumes V_j ($j \in \{0, \dots, d\}$) of the d-dimensional unit ball B^d.

Proposed solution: On the one hand, we have that, for each $r \geqslant 0$,

$$V_d(B^d + r \cdot B^d) = (1+r)^d \, \kappa_d = \sum_{j=0}^{d} \binom{d}{j} r^{d-j} \, \kappa_d.$$

On the other hand, the Steiner formula yields

$$V_d(B^d + r \cdot B^d) = \sum_{j=0}^{d} \kappa_{d-j} \, r^{d-j} \, V_j(B^d).$$

A comparison of the coefficients gives

$$V_j(B^d) = \binom{d}{j} \frac{\kappa_d}{\kappa_{d-j}}.$$

Problem 4 (A bound on the intrinsic volumes)

Consider a set $W \in \mathcal{K}^d$ which contains a ball of radius r, that is, there exists some $x \in \mathbb{R}^d$ and some $r > 0$ such that $x + r \cdot B^d \subset W$. Prove that, for any $j \in \{0, \dots, d\}$,

$$V_j(W) \leqslant \frac{(2^d - 1) \cdot V_d(W)}{\kappa_{d-j} \, r^{d-j}}.$$

Proposed solution: The claim is trivially true for $j = d$ (since $\kappa_0 := 1$ and $2^d - 1 \geqslant 1$). Let $j \in \{0, \dots, d-1\}$. By the Steiner formula,

$$(2^d - 1) \cdot V_d(W) = V_d(2 \cdot W) - V_d(W) \geqslant V_d(W + r \cdot B^d) - V_d(W) = \sum_{k=0}^{d-1} \kappa_{d-k} \, r^{d-k} \, V_k(W) \geqslant \kappa_{d-j} \, r^{d-j} \, V_j(W),$$

and the claim follows immediately.