# Work Sheet 11

## Instructions for week 11 (June $29^{th}$ to July $3^{rd}$):

- Work through Section 4.3 of the lecture notes.
- Answer the control questions 1) to 6), and solve Problems 1 to 4 of the exercises below.
- Please hand in your solutions to the exercises for correction until the morning of Monday, July $6^{th}$. For the procedure, please have a look at the general information in Ilias. The submission of solutions is voluntary.

## Control questions to monitor your progress:

1) Verify that each additive functional $f : \mathcal{K}^d \to \mathbb{R}$ has at most one additive extension to the convex ring.
2) Check that the mapping $\hat{f} : V \to \mathbb{R}$ in the proof of Theorem 4.10 is well-defined. The problem is that a function $h \in V$ may have different representations as a sum of indicators. How does Lemma 4.11 help to resolve this?
3) Verify in Example 4.15 that the discussed sets are polyconvex (by representing them as unions of convex sets) and have the claimed Euler characteristic. What is the Euler characteristic of the letter "$B$"?
4) Verify that the measure $\mu$ defined in Equation (4.14) of the lecture notes is a Radon measure and $G_d$-invariant.
5) Compute the average width (cf. Remark 4.29) of a segment of length 1 in $\mathbb{R}^2$.
6) Convince yourself that the integral formula in Remark 4.29 does indeed imply that the intrinsic volumes are monotone on $\mathcal{K}^d$ (cf. Corollary 4.30). For which intrinsic volumes does the monotonicity carry over to the convex ring?

## Exercises for week 11:

**Problem 1 (Slicing a Boolean model)**

Let $\Phi$ be a stationary Poisson particle process in $\mathbb{R}^d$, defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, with locally finite intensity measure $\Theta \neq 0$. Let $c : \mathcal{C}' \to \mathbb{R}^d$ be a center function, and let $\gamma > 0$ and $\mathbb{Q}$ be as in Theorem 3.6. Denote by $Z$ the corresponding Boolean model. Let $k \in \{1, \ldots, d-1\}$ and consider a $k$-dimensional plane $H \subset \mathbb{R}^d$ with $0 \in H$.

a) Prove that $\Phi_H(\cdot) := \int_{\mathcal{C}^d} \mathbb{1}\{C \cap H \neq \varnothing\} \cdot \mathbb{1}\{C \cap H \in \cdot\} \mathrm{d}\Phi(C)$ is a Poisson particle process in $H$ and that $\Phi_H$ is stationary in the sense that $T_x \Phi_H \overset{d}{=} \Phi_H$ for every $x \in H$.

b) Prove that, subject to identifying $H$ with $\mathbb{R}^k$, the random closed set $Z \cap H$ is a Boolean model in $H$.

**Problem 2 (On the condition in Theorem 4.8)**

Let $\gamma > 0$, and let $\mathbb{Q}$ be a probability measure on $\mathcal{C}'$ such that

$$\int_{\mathcal{C}'} \lambda^d(K + C)\, \mathrm{d}\mathbb{Q}(K) < \infty, \qquad C \in \mathcal{C}^d.$$

Consider a Poisson process $\Psi$ in $\mathbb{R}^d \times \mathcal{C}'$ with intensity measure $\gamma \cdot \lambda^d \otimes \mathbb{Q}$. Further, let $C \in \mathcal{C}'$ and define

$$N_C := \int_{\mathbb{R}^d \times \mathcal{C}'} \mathbb{1}\big\{(K + x) \cap C \neq \varnothing\big\}\, \mathrm{d}\Psi(x, K).$$

Prove that

$$\mathbb{E}\big[r^{N_C}\big] < \infty, \qquad r \in \mathbb{R}.$$

**Problem 3 (Random $q$-flats)**

Let $K, K_0 \in \mathcal{K}^d$ with $K \subset K_0$ and $V_d(K_0) > 0$. Let $q \in \{0, \ldots, d-1\}$ and

$$A_{K_0} := \big\{E \in A(d, q) : E \cap K_0 \neq \varnothing\big\}.$$

An $A(d, q)$-valued random element $X_q$ with distribution $\frac{1}{\mu_q(A_{K_0})} \cdot \mu_q(\,\cdot \cap A_{K_0})$ is called an isotropic random $q$-flat through $K_0$. Here $\mu_q$ denotes the $G_d$-invariant measure on $A(d, q)$ from Theorem 4.26.

a) Calculate the probability $\mathbb{P}(X_q \cap K \neq \varnothing)$ in terms of the intrinsic volumes of $K$ and $K_0$.

b) Let $d = 2$, $e \in \mathbb{R}^2$ with $\|e\| = 1$, and $0 < r \leqslant 1$. Determine the probability that a random line (that is, a random 1-plane) through $B^2 := B(0, 1) \subset \mathbb{R}^2$ intersects the line segment $[-r \cdot e, r \cdot e]$.

c) Let $d = 2$ and let $X_1$ be a random line through $B^2$. Calculate the probability $p$ of the line segment $X_1 \cap B^2$ being longer than $\sqrt{3}$.

d) Let $d = 2$ and let $X_1$ be a random line through $B^2$. Denote by $a$ the side length of an equilateral triangle $T_a$ and assume that the center of the largest circle contained in $T_a$ is the origin. Find a value for $a$ such that $\mathbb{P}(X_1 \cap T_a \neq \varnothing) = p$, where $p$ is as in part c).

Compare the results from c) and d) with the discussion on Bertrand's paradox in Problem 1 of Work sheet 1.

**Problem 4 (Randomly moving convex bodies)**

Let $M, K, K_0 \in \mathcal{K}^d$ with $K \subset K_0$ and $V_d(K_0) > 0$. Define

$$G_{K_0,M} := \{g \in G_d : gM \cap K_0 \neq \varnothing\}.$$

Let $\alpha = \alpha_{K_0,M}$ be a $G_d$-valued random element with distribution $\frac{1}{\mu(G_{K_0,M})} \cdot \mu(\,\cdot \cap G_{K_0,M})$, where $\mu$ is the invarinat measure on $G_d$ from Theorem 4.24. Then, $\alpha M$ is called a $G_{K_0,M}$-isotropic randomly moving convex body.

a) Calculate the probability $\mathbb{P}(\alpha M \cap K \neq \varnothing)$ in terms of the intrinsic volumes of $M$, $K$, and $K_0$.

b) Let $d = 2$, $e \in \mathbb{R}^2$ with $\|e\| = 1$, $0 < r \leqslant 1$, $K_0 = B^2$, and $M = [0,1]^2 \subset \mathbb{R}^2$. Determine the probability

$$\mathbb{P}\Big(\alpha\big([0,1]^2\big) \cap [-r \cdot e, r \cdot e] \neq \varnothing\Big).$$

**Hint:** You may use the formula $V_i\big([0,1]^d\big) = \binom{d}{i}$.

c) Let $d = 2$, $e \in \mathbb{R}^2$ with $\|e\| = 1$, $0 < r \leqslant 1$, $K_0 = B^2$, and $M = [0, a \cdot e_1]$, where $e_1 = \binom{1}{0} \in \mathbb{R}^2$ and $a > 0$. Calculate the probability

$$\mathbb{P}\Big(\alpha\big([0, a \cdot e_1]\big) \cap [-r \cdot e, r \cdot e] \neq \varnothing\Big).$$

d) In part c), what happens if we let $a \to \infty$?

**The solutions to these problems will be uploaded on July 6th.**