Solutions for Work Sheet 11

Problem 1 (Slicing a Boolean model)

Let Φ be a stationary Poisson particle process in $\mathbb{R}^d$, defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, with locally finite intensity measure $\Theta \neq 0$. Let $c : \mathcal{C}' \to \mathbb{R}^d$ be a center function, and let $\gamma > 0$ and $\mathbb{Q}$ be as in Theorem 3.6. Denote by Z the corresponding Boolean model. Let $k \in \{1, \dots, d-1\}$ and consider a k-dimensional plane $H \subset \mathbb{R}^d$ with $0 \in H$.

a) Prove that $\Phi_H(\cdot) := \int_{\mathcal{C}^d} \mathbb{1}\{C \cap H \neq \varnothing\} \cdot \mathbb{1}\{C \cap H \in \cdot\} \mathrm{d}\Phi(C)$ is a Poisson particle process in H and that Φ_H is stationary in the sense that $T_x \Phi_H \overset{d}{=} \Phi_H$ for every $x \in H$.

b) Prove that, subject to identifying H with $\mathbb{R}^k$, the random closed set $Z \cap H$ is a Boolean model in H.

Proposed solution: Write $\mathcal{C}^{(H)} := \{C \in \mathcal{C}^d : C \subset H\}$ and $\mathcal{F}^{(H)} := \{F \in \mathcal{F}^d : F \subset H\}$. Then, $\mathcal{F}^{(H)} \in \mathcal{B}(\mathcal{F}^d)$ and therefore also $\mathcal{C}^{(H)} = \mathcal{F}^{(H)} \cap \mathcal{C}^d \in \mathcal{B}(\mathcal{F}^d)$. Indeed, since $H \in \mathcal{F}^d$, the set H^c is open and hence $\mathcal{F}^{(H)} = (\mathcal{F}_{H^c})^c \in \mathcal{B}(\mathcal{F}^d)$.

a) The mapping theorem for point processes (Problem 3 of Work sheet 5) applied to the map $T : \mathcal{C}' \to \mathcal{C}^{(H)}$, $T(C) := C \cap H$ implies that

$$\int_{\mathcal{C}^d} \mathbb{1}\{C \cap H \in \cdot\} \mathrm{d}\Phi(C)$$

is a Poisson process in $\mathcal{C}^{(H)}$. Restricting this process to $\{C \cap H : C \in \mathcal{C}^d\} \setminus \{\varnothing\}$, it follows that Φ_H is also a Poisson process. Let $x \in H$. Using the stationarity of Φ, we obtain

$$\begin{aligned}
T_x \Phi_H(\cdot) = \int_{\mathcal{C}^{(H)}} \mathbb{1}\{C + x \in \cdot\} \mathrm{d}\Phi_H(C) &= \int_{\mathcal{C}^d} \mathbb{1}\big\{(C \cap H) + x \in \cdot\big\} \mathbb{1}\{C \cap H \neq \varnothing\} \mathrm{d}\Phi(C) \\
&= \int_{\mathcal{C}^d} \mathbb{1}\big\{(C + x) \cap H \in \cdot\big\} \mathbb{1}\big\{(C + x) \cap H \neq \varnothing\big\} \mathrm{d}\Phi(C) \\
&\overset{d}{=} \int_{\mathcal{C}^d} \mathbb{1}\{C \cap H \in \cdot\} \mathbb{1}\{C \cap H \neq \varnothing\} \mathrm{d}\Phi(C) \\
&= \Phi_H(\cdot).
\end{aligned}$$

b) First notice that $Z \cap H$ is a random closed set in H as, for any $C \in \mathcal{C}^{(H)}$, we have

$$\begin{aligned}
\Big\{Z \cap H \in \big(\mathcal{F}^{(H)}\big)^C\Big\} = \Big\{Z \cap H \in \mathcal{F}^{(H)} \setminus \mathcal{F}_C^{(H)}\Big\} = \Omega \setminus \Big\{Z \cap H \in \mathcal{F}_C^{(H)}\Big\} &= \Omega \setminus \big\{Z \in \mathcal{F}_{H \cap C}\big\} \\
&= \Omega \setminus \big\{Z \in \mathcal{F}_C\big\} \\
&= \big\{Z \in \mathcal{F}^C\big\} \\
&\in \mathcal{A}.
\end{aligned}$$

To prove that $Z \cap H$ is a Boolean model, we calculate the capacity functional. For $C \in \mathcal{C}^{(H)}$ we have, by

Theorem 3.16 and Theorem 3.6,

$$\begin{aligned}T_{Z\cap H}(C) = \mathbb{P}(Z\cap H\cap C \neq \varnothing) = T_Z(H\cap C)\\ &= 1-\exp\Big(-\gamma\int_{\mathcal{C}^d}\int_{\mathbb{R}^d}\mathbb{1}\big\{(K+x)\cap H\cap C\neq\varnothing\big\}\,\mathrm{d}\lambda^d(x)\,\mathrm{d}\mathbb{Q}(K)\Big)\\ &= 1-\exp\Big(-\gamma\int_{\mathcal{C}^d}\int_{\mathbb{R}^d}\mathbb{1}\big\{(K+x)\cap H\in\mathcal{F}_C^{(H)}\big\}\,\mathrm{d}\lambda^d(x)\,\mathrm{d}\mathbb{Q}(K)\Big).\end{aligned}$$

Consider the intensity measure of Φ_H,

$$\begin{aligned}\Theta_H(\mathcal{H}) = \mathbb{E}\big[\Phi_H(\mathcal{H})\big] &= \mathbb{E}\Big[\int_{\mathcal{C}^d}\mathbb{1}\{C\cap H\neq\varnothing\}\cdot\mathbb{1}\{C\cap H\in\cdot\}\,\mathrm{d}\Phi(C)\Big]\\ &= \gamma\int_{\mathcal{C}^d}\int_{\mathbb{R}^d}\mathbb{1}\big\{(K+x)\cap H\in\mathcal{H}\big\}\,\mathrm{d}\lambda^d(x)\,\mathrm{d}\mathbb{Q}(K),\qquad \mathcal{H}\in\mathcal{B}(\mathcal{F}^d)\cap(\mathcal{F}^{(H)})',\end{aligned}$$

on $(\mathcal{F}^{(H)})'$ and note that Θ_H is locally finite, as

$$\Theta_H(\mathcal{F}_C^{(H)}) = \Theta(\mathcal{F}_C) < \infty,\qquad C\in\mathcal{C}^{(H)},$$

by the local finiteness of Θ. Now, if Z_H is a Boolean model corresponding to Φ_H (in H), then the capacity functional is (by Theorem 3.16)

$$\begin{aligned}T_{Z_H}(C) &= 1-\exp\Big(-\Theta_H(\mathcal{F}_C^{(H)})\Big)\\ &= 1-\exp\Big(-\gamma\int_{\mathcal{C}^d}\int_{\mathbb{R}^d}\mathbb{1}\big\{(K+x)\cap H\in\mathcal{F}_C^{(H)}\big\}\,\mathrm{d}\lambda^d(x)\,\mathrm{d}\mathbb{Q}(K)\Big)\\ &= T_{Z\cap H}(C),\qquad C\in\mathcal{C}^{(H)}.\end{aligned}$$

As the capacity functional determines the distribution of a random closed set, we have $Z_H \overset{d}{=} Z\cap H$.

Problem 2 (On the condition in Theorem 4.8)

Let $\gamma > 0$, and let $\mathbb{Q}$ be a probability measure on $\mathcal{C}'$ such that

$$\int_{\mathcal{C}'}\lambda^d(K+C)\,\mathrm{d}\mathbb{Q}(K) < \infty,\qquad C\in\mathcal{C}^d.$$

Consider a Poisson process Ψ in $\mathbb{R}^d\times\mathcal{C}'$ with intensity measure $\gamma\cdot\lambda^d\otimes\mathbb{Q}$. Further, let $C\in\mathcal{C}'$ and define

$$N_C := \int_{\mathbb{R}^d\times\mathcal{C}'}\mathbb{1}\big\{(K+x)\cap C\neq\varnothing\big\}\,\mathrm{d}\Psi(x,K).$$

Prove that

$$\mathbb{E}\big[r^{N_C}\big] < \infty,\qquad r\in\mathbb{R}.$$

Proposed solution: We apply the mapping theorem for Poisson processes to the function

$$T:\mathbb{R}^d\times\mathcal{C}'\to\mathcal{C}',\quad (x,K)\mapsto T(x,K) := K+x.$$

Then, we have

$$N_C = \int_{\mathbb{R}^d\times\mathcal{C}'}\mathbb{1}\big\{(K+x)\in\mathcal{F}_C\big\}\,\mathrm{d}\Psi(x,K) = \int_{\mathbb{R}^d\times\mathcal{C}'}\mathbb{1}\big\{T(x,K)\in\mathcal{F}_C\big\}\,\mathrm{d}\Psi(x,K) = \Phi(\mathcal{F}_C\cap\mathcal{C}'),$$

where $\Phi = T(\Psi)$ is a stationary Poisson process with intensity measure

$$\Lambda(\cdot) = \big(T(\gamma\cdot\lambda^d\otimes\mathbb{Q})\big)(\cdot) := (\gamma\cdot\lambda^d\otimes\mathbb{Q})\big(T^{-1}(\cdot)\big) = \gamma\int_{\mathcal{C}'}\int_{\mathbb{R}^d}\mathbb{1}\{K+x\in\cdot\}\,\mathrm{d}\lambda^d(x)\,\mathrm{d}\mathbb{Q}(K).$$

We have $\Lambda(\mathcal{F}_C \cap \mathcal{C}') < \infty$ for each $C \in \mathcal{C}^d$ (using the integrability assumption), and therefore

$$\begin{aligned}
\mathbb{E}\left[r^{N_C}\right] = \mathbb{E}\left[r^{\Phi(\mathcal{F}_C \cap \mathcal{C}')}\right] = \sum_{k=0}^{\infty} r^k \, \mathbb{P}\big(\Phi(\mathcal{F}_C \cap \mathcal{C}') = k\big) &= \sum_{k=0}^{\infty} r^k \cdot \frac{e^{-\Lambda(\mathcal{F}_C \cap \mathcal{C}')} \cdot \Lambda(\mathcal{F}_C \cap \mathcal{C}')^k}{k!} \\
&= e^{-\Lambda(\mathcal{F}_C \cap \mathcal{C}')} \, e^{r \cdot \Lambda(\mathcal{F}_C \cap \mathcal{C}')} \\
&= e^{(r-1) \cdot \Lambda(\mathcal{F}_C \cap \mathcal{C}')} \\
&= \left(e^{\Lambda(\mathcal{F}_C \cap \mathcal{C}')}\right)^{r-1} \\
&< \infty, \qquad r \in \mathbb{R}.
\end{aligned}$$

Problem 3 (Random q-flats)

Let $K, K_0 \in \mathcal{K}^d$ with $K \subset K_0$ and $V_d(K_0) > 0$. Let $q \in \{0, \dots, d-1\}$ and

$$A_{K_0} := \big\{E \in A(d,q) : E \cap K_0 \neq \varnothing\big\}.$$

An $A(d,q)$-valued random element X_q with distribution $\frac{1}{\mu_q(A_{K_0})} \cdot \mu_q(\,\cdot \cap A_{K_0})$ is called an isotropic random q-flat through K_0. Here μ_q denotes the G_d-invariant measure on $A(d,q)$ from Theorem 4.26.

a) Calculate the probability $\mathbb{P}(X_q \cap K \neq \varnothing)$ in terms of the intrinsic volumes of K and K_0.

b) Let $d = 2$, $e \in \mathbb{R}^2$ with $\|e\| = 1$, and $0 < r \leqslant 1$. Determine the probability that a random line (that is, a random 1-plane) through $B^2 := B(0,1) \subset \mathbb{R}^2$ intersects the line segment $[-r \cdot e, r \cdot e]$.

c) Let $d = 2$ and let X_1 be a random line through B^2. Calculate the probability p of the line segment $X_1 \cap B^2$ being longer than $\sqrt{3}$.

d) Let $d = 2$ and let X_1 be a random line through B^2. Denote by a the side length of an equilateral triangle T_a and assume that the center of the largest circle contained in T_a is the origin. Find a value for a such that $\mathbb{P}(X_1 \cap T_a \neq \varnothing) = p$, where p is as in part c).

Compare the results from c) and d) with the discussion on Bertrand's paradox in Problem 1 of Work sheet 1.

Proposed solution:

a) We have, by the Crofton formula (Theorem 4.27),

$$\begin{aligned}
\mathbb{P}(X_q \cap K \neq \varnothing) = \frac{1}{\mu_q(A_{K_0})} \cdot \mu_q(A_K \cap A_{K_0}) = \frac{\mu_q(A_K)}{\mu_q(A_{K_0})} &= \frac{\int_{A(d,q)} \mathbb{1}\{F \cap K \neq \varnothing\} \, \mathrm{d}\mu_q(F)}{\int_{A(d,q)} \mathbb{1}\{F \cap K_0 \neq \varnothing\} \, \mathrm{d}\mu_q(F)} \\
&= \frac{\int_{A(d,q)} V_0(F \cap K) \, \mathrm{d}\mu_q(F)}{\int_{A(d,q)} V_0(F \cap K_0) \, \mathrm{d}\mu_q(F)} \\
&= \frac{c_{0,d}^{q,d-q} \cdot V_{d-q}(K)}{c_{0,d}^{q,d-q} \cdot V_{d-q}(K_0)} \\
&= \frac{V_{d-q}(K)}{V_{d-q}(K_0)}.
\end{aligned}$$

b) Denote by X_1 the random line through B^2. With part a) and Remark 3.36 (ii), we get

$$\mathbb{P}\Big(X_1 \cap [-r \cdot e, r \cdot e] \neq \varnothing\Big) = \frac{V_1\big([-r \cdot e, r \cdot e]\big)}{V_1(B^2)} = \frac{2r}{\frac{1}{2} \cdot 2\pi} = \frac{2r}{\pi}.$$

c) By the geometric observations in Problem 1 on Work sheet 1, the random chord $X_1 \cap B^2$ of B^2 is longer than $\sqrt{3}$ precisely when the chord does not intersect the ball $B\big(0, \frac{1}{2}\big)$. By part a) and Remark 3.36 (ii), we obtain

$$p := \mathbb{P}\Big(V_1(X_1 \cap B^2) \geqslant \sqrt{3}\Big) = \mathbb{P}\Big((X_1 \cap B^2) \cap B(0, \tfrac{1}{2}) = \varnothing\Big) = 1 - \mathbb{P}\Big(X_1 \cap B(0, \tfrac{1}{2}) \neq \varnothing\Big)$$
$$= 1 - \frac{V_1\big(B(0, \frac{1}{2})\big)}{V_1(B^2)}$$
$$= 1 - \frac{\frac{1}{2} \cdot \pi}{\frac{1}{2} \cdot 2\pi}$$
$$= \frac{1}{2}.$$

d) By part a) and Remark 3.36 (ii), we have

$$\mathbb{P}\big(X_1 \cap T_a \neq \varnothing\big) = \frac{V_1(T_a)}{V_1(B^2)} = \frac{\frac{1}{2} \cdot 3a}{\frac{1}{2} \cdot 2\pi} = \frac{3a}{2\pi}.$$

Thus, we have $\mathbb{P}\big(X_1 \cap T_a \neq \varnothing\big) = p$ if, and only if, $a = \frac{\pi}{3}$.

Problem 4 (Randomly moving convex bodies)

Let $M, K, K_0 \in \mathcal{K}^d$ with $K \subset K_0$ and $V_d(K_0) > 0$. Define

$$G_{K_0,M} := \big\{g \in G_d : gM \cap K_0 \neq \varnothing\big\}.$$

Let $\alpha = \alpha_{K_0,M}$ be a G_d-valued random element with distribution $\frac{1}{\mu(G_{K_0,M})} \cdot \mu(\,\cdot\, \cap G_{K_0,M})$, where μ is the invarinat measure on G_d from Theorem 4.24. Then, αM is called a $G_{K_0,M}$-isotropic randomly moving convex body.

a) Calculate the probability $\mathbb{P}(\alpha M \cap K \neq \varnothing)$ in terms of the intrinsic volumes of M, K, and K_0.

b) Let $d = 2$, $e \in \mathbb{R}^2$ with $\|e\| = 1$, $0 < r \leqslant 1$, $K_0 = B^2$, and $M = [0,1]^2 \subset \mathbb{R}^2$. Determine the probability

$$\mathbb{P}\Big(\alpha\big([0,1]^2\big) \cap [-r \cdot e, r \cdot e] \neq \varnothing\Big).$$

Hint: You may use the formula $V_i\big([0,1]^d\big) = \binom{d}{i}$.

c) Let $d = 2$, $e \in \mathbb{R}^2$ with $\|e\| = 1$, $0 < r \leqslant 1$, $K_0 = B^2$, and $M = [0, a \cdot e_1]$, where $e_1 = \binom{1}{0} \in \mathbb{R}^2$ and $a > 0$. Calculate the probability

$$\mathbb{P}\Big(\alpha\big([0, a \cdot e_1]\big) \cap [-r \cdot e, r \cdot e] \neq \varnothing\Big).$$

d) In part c), what happens if we let $a \to \infty$?

Proposed solution:

a) Since $G_{K,M} \subset G_{K_0,M}$, the principal kinematic formula (Theorem 4.33) yields

$$\mathbb{P}\big(\alpha M \cap K \neq \varnothing\big) = \frac{\mu(G_{K,M} \cap G_{K_0,M})}{\mu(G_{K_0,M})} = \frac{\mu(G_{K,M})}{\mu(G_{K_0,M})} = \frac{\int_{G_d} \mathbb{1}\{gM \cap K \neq \varnothing\}\,\mathrm{d}\mu(g)}{\int_{G_d} \mathbb{1}\{gM \cap K_0 \neq \varnothing\}\,\mathrm{d}\mu(g)}$$
$$= \frac{\int_{G_d} V_0(gM \cap K)\,\mathrm{d}\mu(g)}{\int_{G_d} V_0(gM \cap K_0)\,\mathrm{d}\mu(g)}$$
$$= \frac{\sum_{k=0}^{d} c_{0,d}^{k,d-k} \cdot V_k(K) \cdot V_{d-k}(M)}{\sum_{k=0}^{d} c_{0,d}^{k,d-k} \cdot V_k(K_0) \cdot V_{d-k}(M)}.$$

b) Part a), Problem 3 on Work sheet 9, and the hint give

$$\begin{aligned}
\mathbb{P}\Big(\alpha\big([0,1]^2\big) \cap [-r\cdot e, r\cdot e] \neq \varnothing\Big) &= \frac{\sum_{k=0}^{2} c_{0,2}^{k,2-k} \cdot V_k\big([-r\cdot e, r\cdot e]\big) \cdot V_{2-k}\big([0,1]^2\big)}{\sum_{k=0}^{2} c_{0,2}^{k,2-k} \cdot V_k(B^2) \cdot V_{2-k}\big([0,1]^2\big)} \\
&= \frac{c_{0,2}^{0,2} \cdot V_0\big([-r\cdot e, r\cdot e]\big) \cdot V_2\big([0,1]^2\big) + c_{0,2}^{1,1} \cdot V_1\big([-r\cdot e, r\cdot e]\big) \cdot V_1\big([0,1]^2\big)}{\sum_{k=0}^{2} c_{0,2}^{k,2-k} \cdot \binom{2}{k} \cdot \frac{\kappa_2}{\kappa_{2-k}} \cdot \binom{2}{2-k}} \\
&= \frac{1 + \frac{8r}{\pi}}{\sum_{k=0}^{2} \kappa_k \cdot \binom{2}{k}} \\
&= \frac{1 + \frac{8r}{\pi}}{1 + 4 + \pi},
\end{aligned}$$

where we used that

$$c_{0,2}^{k,2-k} = \frac{k! \cdot \kappa_k}{0! \cdot \kappa_0} \cdot \frac{(2-k)! \cdot \kappa_{2-k}}{2! \cdot \kappa_2} = \frac{k! \cdot \kappa_k \cdot (2-k)! \cdot \kappa_{2-k}}{2\kappa_2} = \frac{\kappa_k \cdot \kappa_{2-k}}{\binom{2}{2-k} \cdot \kappa_2},$$

and therefore $c_{0,2}^{0,2} = 1$, $c_{0,2}^{1,1} = \frac{2}{\pi}$. We also used that the volume of a line segment in $\mathbb{R}^2$ is 0.

c) By part a), we have

$$\begin{aligned}
\mathbb{P}\Big(\alpha\big([0, a\cdot e_1]\big) \cap [-r\cdot e, r\cdot e] \neq \varnothing\Big) &= \frac{\sum_{k=0}^{2} c_{0,2}^{k,2-k} \cdot V_k\big([-r\cdot e, r\cdot e]\big) \cdot V_{2-k}\big([0, a\cdot e_1]\big)}{\sum_{k=0}^{2} c_{0,2}^{k,2-k} \cdot V_k(B^2) \cdot V_{2-k}\big([0, a\cdot e_1]\big)} \\
&= \frac{c_{0,2}^{1,1} \cdot V_1\big([-r\cdot e, r\cdot e]\big) \cdot V_1\big([0, a\cdot e_1]\big)}{c_{0,2}^{1,1} \cdot V_1(B^2) \cdot V_1\big([0, a\cdot e_1]\big) + c_{0,2}^{2,0} \cdot V_2(B^2) \cdot V_0\big([0, a\cdot e_1]\big)} \\
&= \frac{\frac{2}{\pi} \cdot 2r \cdot a}{\frac{2}{\pi} \cdot \frac{1}{2} \cdot 2\pi \cdot a + 1 \cdot \pi \cdot 1} \\
&= \frac{\frac{4r\cdot a}{\pi}}{2a + \pi} \\
&= \frac{4r \cdot a}{2\pi \cdot a + \pi^2}.
\end{aligned}$$

d) Apparently, $\mathbb{P}\big(\alpha\big([0, a\cdot e_1]\big) \cap [-r\cdot e, r\cdot e] \neq \varnothing\big) \to \frac{4r}{2\pi}$ as $a \to \infty$.