Masterthesis

External DLA

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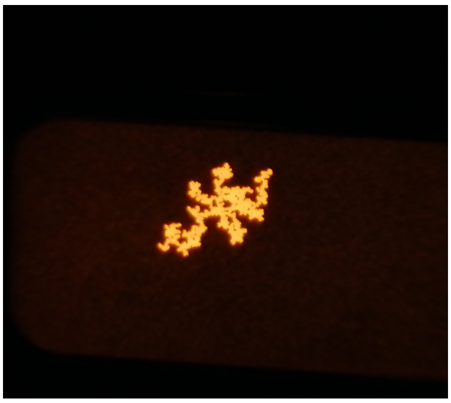
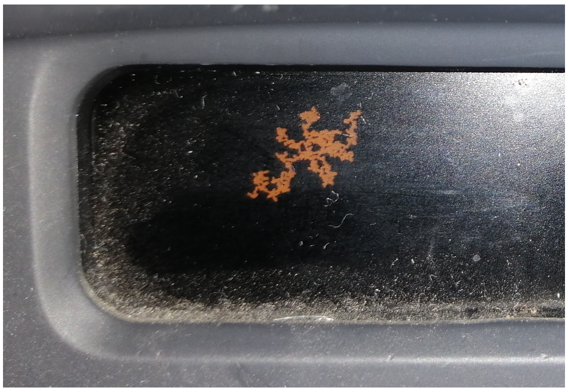
1 Einleitung

External DLA beschreibt einen stochastischen Prozess, welcher zumindest in ähnlicher Form in natürlichen Prozessen beobachtbar ist. Er ähnelt zum Beispiel der fraktalen Gestalt eines sich kreisförmig ausbreitenden Risses einer Glasscheibe, oder eines Risses eines Kristallfluids wie in LCD Displays in alten Autoradios (siehe Fotos). Er kann auch in Schneeflocken oder in elektrostatischen Anhaftungen an Metallen beobachtet werden. Die Formalisierung solcher Prozesse ist sehr aktuell und die sehr konstruktive Definition erlaubten bisher nur mühsame Folgerungen über Struktur und Verhalten des Prozesses. Wir werden uns Modelle auf $\mathbb{Z}^2$, sowie auf anderen Graphen, darunter auch fraktale Graphen, anschauen, und außerdem versuchen, eine Approximation der bisherigen Definition zu finden, die grundsätzlich handlicher ist und auf einfachere Weise zu Erkenntnissen führt. Wir werden außerdem diese Arbeit mit einigen Python Simulationen begleiten. Der Code ist frei verfügbar auf Github.

2 Preliminaries

We prepare this script with the following preliminaries.

Graphs. In this paper we will be interested in the graph $(\mathbb{Z}^d, E)$ with the canonical graph structure, which is $x, y \in \mathbb{Z}^d$ form an edge (e.q. $(x, y) \in E$) if and only if $\exists! i \in \{1, \ldots, d\}$ such that $|x_i - y_i| = 1$ and $x_j = y_j$ for all $j \neq i$. For a point $x \in \mathbb{Z}^d$ its set of *neighbours* is defined as $N(x) := \{y \in \mathbb{Z}^d \mid (x, y) \in E\}$ and the canonical graph structure as defined above basically means that $N(x) = \{x + re_i \mid r \in \{-1, 1\}$ and $i \in \{1, \ldots, d\}\}$, e_i being the usual unit vectors of $\mathbb{Z}^d$. For a set $A \subset \mathbb{Z}^d$ the *edgeset* ∂A of A is defined as $\partial A := \{y \in \mathbb{Z}^d \mid \exists x \in A : \ (x, y) \in E\}$. From now on insted of $(\mathbb{Z}^d, E)$ we will write $\mathbb{Z}^d$.

Random Walks. Let $(S_n)_n$ be a random walk on G and denote $\mathbb{P}_x$ the probability measure of the random walk started at x. That means precisely

$$\mathbb{P}_x(A) := \mathbb{P}_{S_n}(A|S_0 = x) = \mathbb{P}(S_n \in A|S_0 = x)$$

for any subset $A \subset G$. We define the *hitting time* of A as

$$T(A) := \min\{n \geq 0 \mid S_n \in A\},$$

and for the special case $T(\{x\}) =: T(x)$ for every $x \in G$. The *heat kernel* of the random walk S_n is defined to be

$$p_n(x, y) := \mathbb{P}_x(S_n = y)$$

and following the *Green function* as

$$G(x, y) := \sum_{n \geq 0} p_n(x, y).$$

Similarily for a subset $A \subset G$ the *killed* or *stopped Green function* is defined as

$$G_A(x, y) := \sum_{n \geq 0} \mathbb{P}_x(S_n = y, T(A) > n).$$

3 Incremental Aggregation

In this paper we will look at stochastic processes on the set of finite subsets of $\mathbb{Z}^d$, where we start with a one point set at $(0,0)$ and incrementally add a point on the edgeset of the current cluster according to some distribution. What we get is a randomly, point-by-point growing connected cluster which here we will call *Incremental Aggregation*. Define

$$\mathcal{P}_C := \{A \subset \mathbb{Z}^d \mid \text{A is finite and connected}\}, \tag{3.1}$$

the set of finite and connected subsets of $\mathbb{Z}^d$. Furthermore we will be interested in distributions on those sets, so for $A \in \mathcal{P}_C$ we define

$$\mathcal{D}_A := \{\mu : \mathbb{Z}^d \to [0,1] \mid \mu(y) = 0, \forall y \notin A \text{ and } \sum_{y \in A} y = 1\}. \tag{3.2}$$

So every element in $\mathcal{D}_A$ naturally defines a distribution on the elements of A. Now we define incremental aggregation as follows.

Definition 3.1. Let $\mu = (\mu_A)_{A \in \mathcal{P}_C}$ be a family of distributions with $\mu_A \in \mathcal{D}_A$ for all $A \in \mathcal{P}_C$. *Incremental Aggregation* (*with distribution* μ) is a stochastic process $(\mathcal{E}_n)_{n \in \mathbb{N}_0}$ which evolves as follows. The process starts with one point $\mathcal{E}_0 = \{(0,0)\}$. Knowing the process $\mathcal{E}_n$ at time n, let y_n be a random point on $\partial \mathcal{E}_n \in \mathcal{P}_C$ with distribution

$$\mathbb{P}(y_n = y \mid \mathcal{E}_n) := \mu_{\partial \mathcal{E}_n}(y), \quad y \in \mathbb{Z}^d. \tag{3.3}$$

We then define $\mathcal{E}_{n+1} := \mathcal{E}_n \cup \{y_n\}$.

Remark 3.1.

4 External DLA

External DLA is a model of Incremental Aggregation as defined above using a very natural distribution, called the *harmonic measure*.

Definition 4.1. (*Harmonic Measure*)

Remembering the definitions in (2), especially the heat kernel $p_n(x, y) := \mathbb{P}_x(S_n = y)$ of a random walk, the *hitting distribution* of elements in A with *hitting position* $S_{T(A)}$ is

$$H_A(x, y) := p_{T(A)}(x, y), \quad y \in A,$$

and for the special case $x = o := (0, 0)$ we define

$$h_A(y) := H_A(o, y) = \mathbb{P}_o(S_{T(A)} = y), \quad y \in A.$$

Thus, $h_A(y)$ is the probability of hitting A for the first time at y starting from o. h_A is called the *harmonic measure* (*from* o).

Lemma 4.2. harmonic measure := hearmonic measure from infinity. Why does this exist?

Definition 4.3. (*External Diffusion Limited Aggregate*, *External DLA*)

Incremental Aggregation with the harmonic measure h as its distribution we define here, and in literature is known as *Exernal Diffusion Limited Aggregate*, short *External DLA*.

Remark 4.1. contenu...

5 Line Process

In the following we will look at a process which is the approach of a simple approximation of external DLA on $\mathbb{Z}^2$. The idea is to let particles move on straight lines coming from infinity and add to the cluster when hitting it. Obviously in most cases particles cannot move completely straight on $\mathbb{Z}^2$. Therefore we will consider points in $\mathbb{Z}^2$ as the centers of unit squares and let the particles move on straight lines in the full plane $\mathbb{R}^2$. We consider the line hitting a point in $\mathbb{Z}^2$ if and only if it cuts its unit square as defined in the following.

Definition 5.1. Define

$$G_{sq} := \{[k - \frac{1}{2}, k + \frac{1}{2}) \times [l - \frac{1}{2}, l + \frac{1}{2}) \subset \mathbb{R}^2 \mid k, l \in \mathbb{Z}\}, \tag{5.1}$$

note that $\mathbb{R}^2 = \bigcup_{s \in G_{sq}} s$ and $s_1 \cap s_2 = \emptyset$ for all $s_1 \neq s_2 \in G_{sq}$. The canonical function

$$sq : \mathbb{Z}^2 \to G_{sq}, \quad (k, l) \to [k - \frac{1}{2}, k + \frac{1}{2}) \times [l - \frac{1}{2}, l + \frac{1}{2}) \tag{5.2}$$

is bijective and intuitively identifies points in $\mathbb{Z}^2$ with squares in $\mathbb{R}^2$. In the following when using a point $p \in \mathbb{Z}^2$ it will reference the point in $\mathbb{Z}^2$ or the corresponding square in $\mathbb{R}^2$ respecting the context. This bijection also naturally defines a graph structure on G_{sq}, which is two squares $s_1, s_2 \in G_{sq}$ form an edge if and only if $sq^{-1}(s_1)$ and $sq^{-1}(s_2)$ form an edge in G. We call $L \subset \mathbb{R}^2$ a *line* if and only if there exist $a, b \in \mathbb{R}^2$ such that $L = \{a + tb \in \mathbb{R}^2 \mid t \in \mathbb{R}\}$. We may accordingly sometimes reference a line L with $L_{a,b}$. The set of lines we will call $\mathcal{L} := \{L_{a,b} \mid a, b \in \mathbb{R}\}$. For the following context we say a line L *hits* a point $p \in \mathbb{Z}^2$ if and only if $L \cap sq(p) \neq \emptyset$.

BILD Linie durch squares "hitting"
Gradenmaß μ_0

Definition 5.2. Let $L = L_{a,b} \in \mathcal{L}$ be a line. For $A \in \mathcal{P}_C$ we define $L_A := \{p \in A \mid L \text{ hits } p\}$ which is the set of points in A which the line L hits. Here suppose $L_A \neq \emptyset$. We want to define a total ordered relation $<_{line}$ on L_A which will help us construct the distribution for the Line Process. We choose two points $(x_1, x_2) \neq (y_1, y_2) \in L_A$ and devide the definition of a relation into three cases, which are the line having zero, positive or negative gradient:
Case 1, $b = 0$:

$$(x_1, x_2) <_{line} (y_1, y_2) \quad :\Leftrightarrow \quad x_1 < y_1$$

Case 2, $b > 0$:

$$(x_1, x_2) <_{line} (y_1, y_2) \quad :\Leftrightarrow \quad \begin{cases} x_1 < y_1, & \text{if } x_1 \neq y_1, \\ x_2 < y_2, & \text{if } x_1 = y_1. \end{cases}$$

Case 3, $b < 0$:

$$(x_1, x_2) <_{line} (y_1, y_2) \quad :\Leftrightarrow \quad \begin{cases} x_1 < y_1, & \text{if } x_1 \neq y_1, \\ x_2 > y_2, & \text{if } x_1 = y_1. \end{cases}$$

It is easy to see that this well-defines a relation on L_A. In the following we will quickly proove that this relation is totally ordered.

Lemma 5.3. For a line $L = L_{a,b} \in \mathcal{L}$ and $A \in \mathcal{P}_C$ with $L_A \neq \emptyset$ the relation $<_{line}$ on L_A is totally ordered.

Beweis. We will only proove the case where $b > 0$, since the other cases work very similair (Note, that in the case $b = 0$ we have $x_2 = y_2$). Let $b > 0$. For antisymmetry let $(x_1, x_2) <_{line} (y_1, y_2)$ and $(y_1, y_2) <_{line} (x_1, x_2)$. Suppose $x_1 \neq y_1$, then $x_1 < y_1$ and $y_1 < x_1$, therefore $x_1 = y_1$ by antisymmetry of the standard order $<$ in $\mathbb{R}$, a contradiction, hence $x_1 = y_1$. But then we have $x_2 < y_2$ and $y_2 < x_2$ and therefore also $x_2 = y_2$. For transitivity let $(x_1, x_2) <_{line} (y_1, y_2)$ and $(y_1, y_2) <_{line} (z_1, z_2)$. We find four cases. In case $x_1 \neq y_1$ and $y_1 \neq z_1$ we get $x_1 < z_1$ by transitivity of $<$, hence $(x_1, x_2) <_{line} (z_1, z_2)$. In case $x_1 \neq y_1$ and $y_1 = z_1$ we get $x_1 < y_1 = z_1$, therefore $(x_1, x_2) <_{line} (z_1, z_2)$. In case $x_1 = y_1$ and $y_1 \neq z_1$ we get $x_1 = y_1 < z_1$, similair as the last case. In the last case $x_1 = y_1 = z_1$ we get $x_2 < y_2$ and $y_2 < z_2$ and again by transitivity of $<$ we get $x_2 < z_2$, hence $(x_1, x_2) <_{line} (z_1, z_2)$ again. At last, connexity quickly follows from the connexity of $<$ by looking at the two cases $x_1 \neq y_1$ and $x_1 = y_1$. $\square$

Remark 5.1. The relatoin $<_{line}$ on L_A shall basically order the hitpoints of a line L with a finite connected set A from left-bottom to right-top in the case of positive gradient, and left-top to right-bottom in the case of negative gradient of the line. In case of zero gradient we get equal second coordinates for all points in L_A and therefore order them naturally by their first coordinate. This order will permit us to know which points L hits first and last when moving on the line facing A.

Definition 5.4. *Line Hitting Distribution*
Choose $A \in \mathcal{P}_C$, a finite and connected subset of $\mathbb{Z}^2$. We will define a distribution μ_A on $\mathbb{Z}^2$ like in the following. Let $L = L_{a,b}$ be a random line according to the line measure μ_0.

Definition 5.5. *Line Process*
contenu...

6 Notizen

diffusion in fractals: $d_f < d$ (space $\mathbb{Z}^d$)

Literatur

[1] N. Henze. *Maß und Wahrscheinlichkeitstheorie (Stochastik II)*. Karlsruher Institut für Technologie, Karlsruhe, 2010

Erklärung

Hiermit versichere ich, dass ich diese Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt, die wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht und die Satzung des Karlsruher Instituts für Technologie zur Sicherung guter wissenschaftlicher Praxis in der jeweils gültigen Fassung beachtet habe.

Karlsruhe, den 10. März 2020