Some classical problems in random geometry

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Abstract This chapter is intended as a first introduction to selected topics in random geometry. It aims at showing how classical questions from recreational mathematics can lead to the modern theory of a mathematical domain at the interface of probability and geometry. Indeed, in each of the four sections, the starting point is a historical practical problem from geometric probability. We show that the solution of the problem, if any, and the underlying discussion are the gateway to the very rich and active domain of integral and stochastic geometry, which we describe at a basic level. In particular, we explain how to connect Buffon's needle problem to integral geometry, Bertrand's paradox to random tessellations, Sylvester's four-point problem to random polytopes and Jeffrey's bicycle wheel problem to random coverings. The results and proofs selected here have been especially chosen for non-specialist readers. They do not require much prerequisite knowledge on stochastic geometry but nevertheless comprise many of the main results on these models.

Introduction: geometric probability, integral geometry, stochastic geometry

Geometric probability is the study of geometric figures, usually from the Euclidean space, which have been randomly generated. The variables coming from these random spatial models can be classical objects from Euclidean geometry, such as a point, a line, a subspace, a ball, a convex polytope and so on.

It is commonly accepted that geometric probability was born in 1733 with Buffon's original investigation of the falling needle. Subsequently, several open questions appeared including Sylvester's four-point problem in 1864, Bertrand's paradox

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related to a random chord in the circle in 1888 and Jeffreys's bicycle wheel problem in 1946. Until the beginning of the twentieth century, these questions were all considered as recreational mathematics and there was a very thin theoretical background involved which may explain why the several answers to Bertrand's question were regarded as a paradox.

After a course by G. Herglotz in 1933, W. Blaschke developed a new domain called *integral geometry* in his papers *Integralgeometrie* in 1935-1937, see e.g. [17]. It relies on the key idea that the mathematically natural probability models are those that are invariant under certain transformation groups and it provides mainly formulas for calculating expected values, i.e. integrals with respect to rotations or translations of random objects. Simultaneously, the modern theory of probability based on measure theory and Lebesgue's integral was introduced by S. N. Kolmogorov in [74].

During and after the Second World War, people with an interest in applications in experimental science - material physics, geology, telecommunications, etc.- realized the significance of random spatial models. For instance, in the famous foreword to the first edition of the reference book [31], D. G. Kendall narrates his own experience during the War and how his Superintendent asked him about the strength of a sheet of paper. This question was in fact equivalent to the study of a random set of lines in the plane. Similarly, J. L. Meijering published a first work on the study of crystal aggregates with random tessellations while he was working for the Philips company in 1953 [84]. In the same way, C. Palm who was working on telecommunications at Ericsson Technics proved a fundamental result in the one-dimensional case about what is nowadays called the Palm measure associated with a stationary point process [96]. All of these examples illustrate the general need to rigorously define and study random spatial models.

We traditionally consider that the expression *stochastic geometry* dates back to 1969 and was due to D. G. Kendall and K. Krickeberg at the occasion of the first conference devoted to that topic in Oberwolfach. In fact, I. Molchanov and W. S. Kendall note in the preface of [135] that H. L. Frisch and J. M. Hammersley had already written the following lines in 1963 in a paper on percolation: *Nearly all extant percolation theory deals with regular interconnecting structures, for lack of knowledge of how to define randomly irregular structures. Adventurous readers may care to rectify this deficiency by pioneering branches of mathematics that might be called stochastic geometry or statistical topology*.

For more than 50 years, a theory of stochastic geometry has been built in conjunction with several domains, including

- the theory of point processes and queuing theory, see notably the work of J. Mecke [81], D. Stoyan [124], J. Neveu [95], D. Daley [35] and [42],
- convex and integral geometry, see e.g. the work of R. Schneider [112] and W. Weil [129] as well as their common reference book [114],
- the theory of random sets, mathematical morphology and image analysis, see the work of D. G. Kendall [69], G. Matheron [80] and J. Serra [117],
- combinatorial geometry, see the work of R. V. Ambartzumian [3].

It is worth noting that this development has been simultaneous with the research on spatial statistics and analysis of real spatial data coming from experimental science, for instance the work of B. Matérn in forestry [79] or the numerous papers in geostatistics, see e.g. [134].

In this introductory lecture, our aim is to describe some of the best-known historical problems in geometric probability and explain how solving these problems and their numerous extensions has induced a whole branch of the modern theory of stochastic geometry. We have chosen to embrace the collection of questions and results presented in this lecture under the general denomination of *random geometry*. In Section 1, Buffon's needle problem is used to introduce a few basic formulas from integral geometry. Section 2 contains a discussion around Bertrand's paradox which leads us to the construction of random lines and the first results on selected models of random tessellations. In Section 3, we present some partial answers to Sylvester's four-point problem and then derive from it the classical models of random polytopes. Finally, in Section 4, Jeffrey's bicycle wheel problem is solved and is the front door to more general random covering and continuum percolation.

We have made the choice to keep the discussion as non-technical as possible and to concentrate on the basic results and detailed proofs which do not require much prerequisite knowledge on the classical tools used in stochastic geometry. Each topic is illustrated by simulations which are done using *Scilab 5.5*. This chapter is intended as a foretaste of some of the topics currently most active in stochastic geometry and naturally encourages the reader to go beyond it and carry on learning with reference to books such as [31, 114, 135].

Notation and convention. The Euclidean space $\mathbb{R}^d$ of dimension $d \geq 1$ and with origin denoted by o is endowed with the standard scalar product $\langle \cdot, \cdot \rangle$, the Euclidean norm $\| \cdot \|$ and the Lebesgue measure V_d. The set $B_r(x)$ is the Euclidean ball centered at $x \in \mathbb{R}^d$ and of radius $r > 0$. We denote by $\mathbb{B}^d$ (resp. $\mathbb{S}^{d-1}$, $\mathbb{S}_+^{d-1}$) the unit ball (resp. the unit sphere, the unit upper half-sphere). The Lebesgue measure on $\mathbb{S}^{d-1}$ will be denoted by σ_d. We will use the constant $\kappa_d = V_d(\mathbb{B}^d) = \frac{1}{d}\sigma_d(\mathbb{S}^{d-1}) = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$. Finally, a convex compact set of $\mathbb{R}^d$ (resp. a compact intersection of a finite number of closed half-spaces of $\mathbb{R}^d$) will be called a d-dimensional *convex body* (resp. *convex polytope*).

1 From Buffon's needle to integral geometry

In this section, we describe and solve the four century-old needle problem due to Buffon and which is commonly considered as the very first problem in geometric probability. We then show how the solution to Buffon's original problem and to one of its extensions constitutes a premise to the modern theory of integral geometry. In particular, the notion of intrinsic volumes is introduced and two classical integral formulas involving them are discussed.

1.1 Starting from Buffon's needle

In 1733, Georges-Louis Leclerc, comte de Buffon, raised a question which is nowadays better known as Buffon's needle problem. The solution, published in 1777 [20], is certainly a good candidate for the first-ever use of an integral calculation in probability theory. First and foremost, its popularity since then comes from being the first random experiment which provides an approximation of π.

Buffon's needle problem can be described in modern words in the following way: a needle is dropped *at random* onto a parquet floor which is made of parallel strips of wood, each of same width. What is the probability that it falls across a vertical line between two strips?

Let us denote by D the width of each strip and by ℓ the length of the needle. We assume for the time being that $\ell \le D$, i.e. that only one crossing is possible. The randomness of the experiment is described by a couple of real random variables, namely the distance R from the needle's mid-point to the closest vertical line and the angle Θ between a horizontal line and the needle.

The chosen probabilistic model corresponds to our intuition of a *random* drop: the variables R and Θ are assumed to be independent and both uniformly distributed on $(0, D/2)$ and $(-\pi/2, \pi/2)$ respectively.

Now there is intersection if and only if $2R \le \ell\cos(\Theta)$. Consequently, we get

$$p = \frac{2}{\pi D}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\frac{1}{2}\ell\cos(\theta)} \mathrm{d}r\mathrm{d}\theta = \frac{2\ell}{\pi D}.$$

This remarkable identity leads to a numerical method for calculating an approximate value of π. Indeed, repeating the experiment n times and denoting by S_n the number of hits, we can apply Kolmogorov's law of large numbers to show that $\frac{2\ell n}{DS_n}$ converges almost surely to π with an error estimate provided by the classical central limit theorem.

In 1860, Joseph-Émile Barbier provided an alternative solution for Buffon's needle problem, see [6] and [73, Chapter 1]. We describe it below as it solves at the same time the so-called *Buffon's noodle problem*, i.e. the extension of Buffon's needle problem when the needle is replaced by any rigid planar curve of class C^1.

Let us denote by p_k, $k \ge 0$, the probability of exactly k crossings between the vertical lines and the needle. Henceforth, the condition $\ell \le D$ is not assumed to be fulfilled any longer as it would imply trivially that $p = p_1$ and $p_k = 0$ for every $k \ge 2$. We denote by $f(\ell) = \sum_{k\ge 1} kp_k$ the mean number of crossings. The function f has the interesting property of being additive, i.e. if two needles of respective lengths ℓ_1 and ℓ_2 are pasted together at one of their endpoints and in the same direction, then the total number of crossing is obviously the sum of the numbers of crossings of the first needle and of the second one. This means that $f(\ell_1 + \ell_2) = f(\ell_1) + f(\ell_2)$. Since the function f is increasing, we deduce from its additivity that there exists a positive constant α such that $f(\ell) = \alpha\ell$.

More remarkably, the additivity property still holds when the two needles are not in the same direction. This implies that for any rigid finite polygonal line $\mathscr{C}$, the

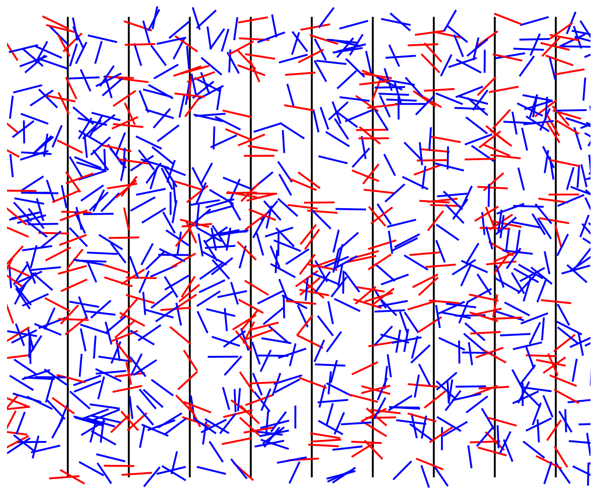

Fig. 1 Simulation of Buffon's needle problem with the particular choice $\ell/D = 1/2$: over 1000 samples, 316 were successful (red), 684 were not (blue).

mean number of crossings with the vertical lines of a noodle with same shape as $\mathscr{C}$, denoted by $f(\mathscr{C})$ with a slight abuse of notation, satisfies

$$f(\mathscr{C}) = \alpha \mathscr{L}(\mathscr{C}) \tag{1}$$

where $\mathscr{L}(\cdot)$ denotes the arc length. Using both the density of polygonal lines in the space of piecewise C^1 rigid planar curves endowed with the topology of uniform convergence and the continuity of the functions f and $\mathscr{L}$ on this space, we deduce that the formula (1) holds for any piecewise C^1 rigid planar curve.

It remains to make the constant α explicit, which we do when replacing $\mathscr{C}$ by the circle of diameter D. Indeed, almost surely, the number of crossings of this noodle with the vertical lines is 2, which shows that $\alpha = \frac{2}{\pi D}$. In particular, when K is a convex body of $\mathbb{R}^2$ with diameter less than D and $p(K)$ denotes the probability that K intersects one of the vertical lines, we get that

$$p(K) = \frac{1}{2} f(\partial K) = \frac{\mathscr{L}(\partial K)}{\pi D}.$$

Further extensions of Buffon's needle problem with more general needles and lattices can be found in [18]. In the next subsection, we are going to show how to derive the classical Cauchy-Crofton's formula from similar ideas.