# Work Sheet 5

## Instructions for week 5 (May $18^{th}$ to May $22^{nd}$):

- Work through Section 2.3 of the lecture notes.
- Answer the control questions 1) to 4), and solve Problems 1 to 4 of the exercises below.
- Please hand in your solutions to the exercises for correction until the morning of Monday, May $25^{th}$. For the procedure, please have a look at the general information in Ilias. The submission of solutions is voluntary.

## Control questions to monitor your progress:

1) Let $\Phi$ be a Poisson process in a separable metric space $(\mathbb{X}, \rho)$ with intensity measure $\Theta := c \cdot \delta_x$ for some fixed $x \in \mathbb{X}$ and $c > 0$. What is the distribution of the random variable $\Phi(B)$, for $B \in \mathcal{X}$?
2) Give an example of a Poisson process that is not simple.
3) Convince yourself that Remark 2.27 $(iii)$ is true, that is, simply by using Remark 2.24 and Definition 2.26 show that the distribution of a Poisson process is uniquely determined by its intensity measure. Can you also infer this fact from Theorem 2.29 and Remark 2.25?
4) In the proof of Theorem 2.29 a calculation shows that if $\Phi$ is a Poisson process with intensity measure $\Theta$, then

$$L_\Phi(f) = \exp\left(-\int_{\mathbb{X}} \left(1 - e^{-f}\right) \mathrm{d}\Theta\right)$$

for all step functions $f = c_1 \mathbb{1}_{B_1} + \ldots + c_m \mathbb{1}_{B_m}$ ($c_j \geqslant 0$ and $B_1, \ldots, B_m \in \mathcal{X}_b$ pairwise disjoint). Can you conclude that the equality holds for every measurable function $f : \mathbb{X} \to [0, \infty]$?

## Exercises for week 5:

**Problem 1 (Simulating point processes)**

Use your favorite programming language to do the following simulations (we suggest the use of `R` or Python).

a) Simulate a binomial process with parameters $m = 50$ and $\mathbb{V} = \mathcal{U}\big([0,1]^2\big)$.

b) Simulate a homogeneous Poisson process in $[0,1]^2$ with intensity $\gamma = 50$, or equivalently, simulate a mixed binomial process with parameters $\tau \sim Po(50)$ and $\mathbb{V} = \mathcal{U}\big([0,1]^2\big)$.

c) Simulate a Poisson process in $[0,1]^3$ with intensity measure $\Theta = 100 \cdot \big(B(0.5, 0.5) \otimes B(1, 1.5) \otimes B(2, 1)\big)$, where $B(\alpha, \beta)$ denotes the beta distribution with parameters $\alpha, \beta > 0$.

d) Visit the website Morphometry.org and try to replicate the results in parts a) and b) using the 'morphometer'.

**Note:** Python-code for the simulations will be made available in Ilias together with the solutions to this problem sheet. The website morphometry.org can be accessed at any time. It also includes more complicated processes and visualizations of other objects from stochastic geometry. Those who are interested in the simulation of point patterns or in the analysis of spatial data should take a look at the `R`-package `spatstat` in connection with the book

A. Baddeley, E. Rubak, and R. Turner (2016); *Spatial Point Patterns: Methodology and Applications with R;* CRC Press, Taylor & Francis Group; Boca Raton.

Those who use Python might want to have a look at the `PySAL`-package and its components.

**Problem 2 (Some properties of the Poisson process)**

Let $(\mathbb{X}, \rho)$ be a separable metric space, and let $\Phi$ be a Poisson process in $\mathbb{X}$, defined on a probability space $\big(\Omega, \mathcal{A}, \mathbb{P}\big)$, with intensity measure $\Theta \in M(\mathbb{X})$.

a) Prove that $\mathrm{Cov}\big(\Phi(A), \Phi(B)\big) = \Theta\big(A \cap B\big)$ for any sets $A, B \in \mathcal{X}$.

b) Let $B \in \mathcal{X}$. Show that the map $M(\mathbb{X}) \to M(\mathbb{X})$, $\mu \mapsto \mu_B := \mu \llcorner B := \mu_{|_B}$ is measurable. Further, prove that the restriction $\Phi_B$ of $\Phi$ onto $B$ is a Poisson process in $\mathbb{X}$ with intensity measure $\Theta_B$.

c) Let $B_1, B_2, \ldots \in \mathcal{X}$ be pairwise disjoint. Prove that $\Phi_{B_1}, \Phi_{B_2}, \ldots$ are independent Poisson processes with intensity measures $\Theta_{B_1}, \Theta_{B_2}, \ldots$.

**Hint for part c):** Construct independent Poisson processes $\widetilde{\Phi}_1, \widetilde{\Phi}_2, \ldots$ with $\widetilde{\Phi}_j(\mathbb{X} \setminus B_j) = 0$ almost surely (for each $j \in \mathbb{N}$) such that $\widetilde{\Phi}_j \stackrel{d}{=} \Phi_{B_j}$ (for each $j \in \mathbb{N}$) and $\sum_{j=1}^{\infty} \widetilde{\Phi}_j \stackrel{d}{=} \Phi$, then verify independence via its definition.

**Problem 3 (A mapping theorem for point processes)**

Let $(\mathbb{X}, \mathcal{X})$ and $(\mathbb{Y}, \mathcal{Y})$ be separable metric spaces with their corresponding Borel $\sigma$-fields. Let $\Phi$ be a point process in $\mathbb{X}$, defined on a probability space $\big(\Omega, \mathcal{A}, \mathbb{P}\big)$, and consider a measurable map $g : \mathbb{X} \to \mathbb{Y}$ for which the preimage of any bounded subset of $\mathbb{Y}$ is bounded in $\mathbb{X}$.

a) Prove that $\widetilde{\Phi} := \Phi \circ g^{-1}$ is a point process in $\mathbb{Y}$.

b) Now, assume that $\Psi$ is a Poisson process in $\mathbb{X}$ with intensity measure $\Theta \in M(\mathbb{X})$. Show that $\widetilde{\Psi} = \Psi \circ g^{-1}$ is a Poisson process in $\mathbb{Y}$ with intensity measure $\widetilde{\Theta} := \Theta \circ g^{-1} \in M(\mathbb{Y})$.

**Problem 4 (Superposition)**

Let $\Phi_1, \ldots, \Phi_n$ be independent point processes in a separable metric space $(\mathbb{X}, \rho)$ with intensity measures $\Theta_1, \ldots, \Theta_n$ and Laplace functionals $L_1, \ldots, L_n$.

a) Denote by $\Theta$ and $L$ the intensity measure and Laplace functional (respectively) of the superposition $\Phi := \Phi_1 + \ldots + \Phi_n$. Express $\Theta$ and $L$ in terms of $\Theta_1, \ldots, \Theta_n$ and $L_1, \ldots, L_n$, respectively.

b) Use the Laplace functional to prove that the superposition of independent Poisson processes $\Psi_1, \ldots, \Psi_n$ (with intensity measures $\Theta_1, \ldots, \Theta_n$) is itself a Poisson process.

**The solutions to these problems will be uploaded on May 25th.**
**Note that there will be no MS-Teams discussion on May 21st. Please ask your question in the Ilias forum or keep them for the MS-Teams discussion on May 28th.**