Karlsruhe Institute of Technology

Institute of Stochastics

Stochastic Geometry | Summer term 2020

PD. Dr. Steffen Winter
Steffen Betsch, M.Sc.

Solutions for Work Sheet 12

Problem 1 (Concerning Remark 4.29)

Let $m \in \{0, \dots, d\}$, $K \in \mathcal{K}^d$, and let $f : \mathcal{K}^d \to \mathbb{R}$ be a measurable map. Prove that

$$\int_{G(d,d-m)} f(K|L^\perp)\,\mathrm{d}\nu_{d-m}(L) = \int_{G(d,m)} f(K|L)\,\mathrm{d}\nu_m(L),$$

where $K|L$ (or $K|L^\perp$) denotes the orthogonal projection of K onto L (or onto $L^\perp$), and where ν_q is the SO_d-invariant probability measure on $G(d,q)$ (for $q \in \{0, \dots, d\}$) from Theorem 4.25.

Proposed solution: Equation (4.15) of the lecture notes, applied to some fixed $L_{d-m} \in G(d, d-m)$, gives

$$\int_{G(d,d-m)} f(K|L^\perp)\,\mathrm{d}\nu_{d-m}(L) = \int_{\mathrm{SO}_d} f\big(K|(\vartheta L_{d-m})^\perp\big)\,\mathrm{d}\nu(\vartheta)$$

where ν is the SO_d-invariant probability measure on SO_d from Theorem 4.23. Since $L_{d-m}^\perp \in G(d,m)$, we similarly obtain

$$\int_{G(d,m)} f(K|L)\,\mathrm{d}\nu_m(L) = \int_{\mathrm{SO}_d} f\big(K|\vartheta(L_{d-m}^\perp)\big)\,\mathrm{d}\nu(\vartheta).$$

Thus, it suffices to show that $(\vartheta L)^\perp = \vartheta(L^\perp)$ for all $\vartheta \in \mathrm{SO}_d$ and all $L \in G(d, d-m)$. To this end, let $y \in \vartheta(L^\perp)$ and $z \in L^\perp$ such that $y = \vartheta z$. For each $x \in L$ we have

$$\big\langle y, \vartheta x \big\rangle = \big\langle \vartheta z, \vartheta x \big\rangle = \langle z, x\rangle = 0,$$

so $y \in (\vartheta L)^\perp$, and therefore $(\vartheta L)^\perp \supset \vartheta(L^\perp)$. Here $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^d$. For the converse inclusion, let $y \in (\vartheta L)^\perp$. Then, $\big\langle y, \vartheta x \big\rangle = 0$ for every $x \in L$, so

$$\big\langle \vartheta^{-1} y, \vartheta^{-1}\vartheta x \big\rangle = 0, \quad x \in L.$$

Consequently, $\vartheta^{-1} y \in L^\perp$, that is, $y \in \vartheta(L^\perp)$.

Problem 2 (Steiner's formula as a special case of the principal kinematic formula)

Show that Steiner's formula (Theorem 3.33) follows from the principal kinematic formula (Theorem 4.33).

Proposed solution: Let $K \in \mathcal{K}^d \setminus \{\varnothing\}$ and $\varepsilon > 0$. We first consider the left hand side of the principal kinematic formula for $j = 0$ and $M = \varepsilon \cdot B^d$, where B^d is the unit ball in $\mathbb{R}^d$. With a decomposition of the invariant measure μ on G_d as in Equation (4.14) of the lecture notes, this gives

$$\begin{aligned}\int_{G_d} V_0\big(K \cap g(\varepsilon \cdot B^d)\big)\, \mathrm{d}\mu(g) &= \int_{\mathbb{R}^d} \int_{\mathrm{SO}_d} V_0\Big(K \cap \big(\vartheta(\varepsilon \cdot B^d) + x\big)\Big)\, \mathrm{d}\nu(\vartheta)\, \mathrm{d}\lambda^d(x) \\ &= \int_{\mathbb{R}^d} \mathbb{1}\{\underbrace{K \cap (\varepsilon \cdot B^d + x) \neq \varnothing}_{\iff x \in K + \varepsilon \cdot B^d}\}\, \mathrm{d}\lambda^d(x) \\ &= V_d(K + \varepsilon \cdot B^d),\end{aligned}$$

where ν is the SO_d-invariant probability measure on SO_d from Theorem 4.23. For the right hand side of the principal kinematic formula, we obtain

$$\begin{aligned}\sum_{k=0}^{d} c_{0,d}^{k,d-k} \cdot V_k(K) \cdot V_{d-k}(\varepsilon \cdot B^d) &= \sum_{k=0}^{d} c_{0,d}^{k,d-k} \cdot V_k(K) \cdot \varepsilon^{d-k} \cdot V_{d-k}(B^d) \\ &= \sum_{k=0}^{d} \frac{k! \cdot \kappa_k \cdot (d-k)! \cdot \kappa_{d-k}}{d! \cdot \kappa_d} \cdot V_k(K) \cdot \varepsilon^{d-k} \cdot \binom{d}{d-k} \cdot \frac{\kappa_d}{\kappa_k} \\ &= \sum_{k=0}^{d} \kappa_{d-k} \cdot V_k(K) \cdot \varepsilon^{d-k},\end{aligned}$$

where we used Problem 3 of Work sheet 9 to calculate the intrinsic volumes of B^d.

Problem 3 (Geometric densities of the Boolean model)

Let Z be a stationary and isotropic Boolean model in $\mathbb{R}^3$ with intensity parameter $\gamma > 0$ and with a distribution $\mathbb{Q}$ of the typical grain which is concentrated on $\mathcal{K}^3 \setminus \{\varnothing\}$.

a) Determine the densities $\delta_0, \ldots, \delta_3$ in terms of $\gamma_0, \ldots, \gamma_3$.

b) Assume that

$$\delta_0 = 0.34, \quad \delta_1 = 0.1, \quad \delta_2 = 0.11, \quad \delta_3 = 0.52.$$

Determine the intensity γ.

c) Let $M \in \mathcal{K}_0$, where $\mathcal{K}_0$ is defined via the center of the circumball of the convex and compact sets, and $\mathbb{Q}(\cdot) := \int_{\mathrm{SO}_3} \mathbb{1}\{\vartheta M \in \cdot\}\mathrm{d}\nu(\vartheta)$, where ν is the SO_3-invariant probability measure on SO_3 from Theorem 4.23. Assume that

$$\delta_0 = 10, \quad \delta_1 = 20, \quad \delta_2 = 0, \quad \delta_3 = 0.$$

Calculate the intensity γ. Given these values, what can you say about the set M?

Proposed solution:

a) From Theorem 4.35 and Equation (4.20) of the lecture notes, we obtain

$$\begin{aligned}\delta_3 &= 1 - e^{-\gamma_3}, \\ \delta_2 &= e^{-\gamma_3} \cdot \gamma_2, \\ \delta_1 &= e^{-\gamma_3}\left(\gamma_1 - \frac{1}{2} \cdot c_1^3 \cdot (c_3^2 \cdot \gamma_2)^2\right) = e^{-\gamma_3} \cdot \left(\gamma_1 - \frac{\pi}{8} \cdot \gamma_2^2\right), \\ \delta_0 &= e^{-\gamma_3}\left(\gamma_0 - \frac{1}{2} \cdot c_0^3\big(c_3^1\gamma_1 \cdot c_3^2\gamma_2 + c_3^2\gamma_2 \cdot c_3^1\gamma_1\big) + \frac{1}{6} \cdot c_0^3 \cdot (c_3^2 \cdot \gamma_2)^3\right) \\ &= e^{-\gamma_3} \cdot \left(\gamma_0 - \frac{1}{2} \cdot \gamma_1 \cdot \gamma_2 + \frac{\pi}{48} \cdot \gamma_2^3\right).\end{aligned}$$

b) The formulae from part a) give

$$\begin{aligned}
\gamma_3 &= -\log\left(1-\delta_3\right),\\
\gamma_2 &= \frac{\delta_2}{1-\delta_3},\\
\gamma_1 &= \frac{\delta_1}{1-\delta_3} + \frac{\pi}{8}\left(\frac{\delta_2}{1-\delta_3}\right)^2,\\
\gamma_0 &= \frac{\delta_0}{1-\delta_3} + \frac{1}{2}\left(\frac{\delta_1}{1-\delta_3} + \frac{\pi}{8}\left(\frac{\delta_2}{1-\delta_3}\right)^2\right)\cdot\frac{\delta_2}{1-\delta_3} - \frac{\pi}{48}\left(\frac{\delta_2}{1-\delta_3}\right)^3.
\end{aligned}$$

Using that $\gamma_0 = \gamma\cdot\mathbb{E}\left[V_0(Z_0)\right] = \gamma$, the given values for δ_j and the formula above yield

$$\gamma \approx 0.73.$$

c) The formulae from part b) and the given values for δ_j yield

$$\gamma_3 = 0, \quad \gamma_2 = 0, \quad \gamma_1 = 20, \quad \gamma_0 = 10.$$

In particular, $\gamma = \gamma_0 = 10$. Using that the intrinsic volumes are invariant under rotations, we observe that

$$\gamma_j = \gamma\int_{\mathcal{K}^3\setminus\{\varnothing\}} V_j(K)\,\mathrm{d}\mathbb{Q}(K) = \gamma\int_{\mathrm{SO}_3} V_j(\vartheta M)\,\mathrm{d}\nu(\vartheta) = \gamma\cdot V_j(M), \quad j=1,2,3.$$

Hence, we have

$$V_1(M) = 2, \quad V_2(M) = 0, \quad \text{and} \quad V_3(M) = 0.$$

Therefore (implicitly using that the given center function is the center of the circumball), we must have $M = [-e, e]$ for some $e\in\mathbb{R}^2$ with $\|e\| = 1$.

Problem 4 (Concerning Remark 4.39)

Let $W\in\mathcal{K}^d$ with $V_d(W) > 0$, and let Z be a stationary Boolean model with typical grain Z_0 such that $\mathbb{E}\left[\left(\lambda^d(Z_0)\right)^2\right] < \infty$. Denote by p_Z the volume fraction of Z (see Definition 1.30 and Theorem 1.31).

a) Prove that $\int_{\mathbb{R}^d}\left(C(x) - p_Z^2\right)\mathrm{d}x < \infty$.

Hint: You may use that $e^t - 1 \leqslant t\cdot e^t$ for each $t \geqslant 0$.

b) Denote by C the covariance function of Z from Definition 3.18. Prove that

$$\sigma_{d,d} := \lim_{r\to\infty}\frac{\mathrm{Var}\left(V_d(Z\cap r\cdot W)\right)}{\lambda^d(r\cdot W)} = \int_{\mathbb{R}^d}\left(C(x) - p_Z^2\right)\mathrm{d}x.$$

Proposed solution:

a) The formula of the covariance function from Theorem 3.20, the definition of C_0, the hint, and Fubini's theorem give

$$\begin{aligned}
\int_{\mathbb{R}^d}\left(C(x) - p_Z^2\right)\mathrm{d}x &= \int_{\mathbb{R}^d}\left(2p_Z - 1 - p_Z^2 + (1-p_Z)^2\cdot e^{\gamma C_0(x)}\right)\mathrm{d}x\\
&= \int_{\mathbb{R}^d}(1-p_Z)^2\cdot\left(e^{\gamma\cdot C_0(x)} - 1\right)\mathrm{d}x\\
&= \int_{\mathbb{R}^d}(1-p_Z)^2\cdot\left(e^{\gamma\cdot\mathbb{E}\left[\lambda^d\left(Z_0\cap(Z_0-x)\right)\right]} - 1\right)\mathrm{d}x\\
&\leqslant \int_{\mathbb{R}^d}(1-p_Z)^2\cdot\gamma\cdot\mathbb{E}\Big[\lambda^d\big(Z_0\cap(Z_0-x)\big)\Big]\cdot e^{\gamma\cdot\mathbb{E}\left[\lambda^d\left(Z_0\cap(Z_0-x)\right)\right]}\,\mathrm{d}x\\
&\leqslant \int_{\mathbb{R}^d}(1-p_Z)^2\cdot\gamma\cdot\mathbb{E}\Big[\lambda^d\big(Z_0\cap(Z_0-x)\big)\Big]\cdot e^{\gamma\cdot\mathbb{E}[\lambda^d(Z_0)]}\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\leqslant (1-p_Z)^2 \cdot \gamma \int_{\mathbb{R}^d} \mathbb{E}\left[\int_{\mathbb{R}^d} \mathbb{1}\big\{y \in Z_0 \cap (Z_0 - x)\big\}\,\mathrm{d}y\right] \cdot e^{\gamma \cdot \mathbb{E}[\lambda^d(Z_0)]}\,\mathrm{d}x \\
&= (1-p_Z)^2 \cdot \gamma \int_{\mathbb{R}^d} \mathbb{E}\left[\int_{\mathbb{R}^d} \mathbb{1}\{y \in Z_0\} \cdot \mathbb{1}\big\{x \in (Z_0 - y)\big\}\,\mathrm{d}y\right] \cdot e^{\gamma \cdot \mathbb{E}[\lambda^d(Z_0)]}\,\mathrm{d}x \\
&= (1-p_Z)^2 \cdot \gamma \cdot \mathbb{E}\left[\int_{\mathbb{R}^d} \mathbb{1}\{y \in Z_0\} \cdot \lambda^d(Z_0 - y)\,\mathrm{d}y\right] \cdot e^{\gamma \cdot \mathbb{E}[\lambda^d(Z_0)]} \\
&= (1-p_Z)^2 \cdot \gamma \cdot \mathbb{E}\Big[\big(\lambda^d(Z_0)\big)^2\Big] \cdot e^{\gamma \cdot \mathbb{E}[\lambda^d(Z_0)]} \\
&< \infty.
\end{aligned}
$$

b) We have

$$
\begin{aligned}
\operatorname{Var}\big(V_d(Z \cap W)\big) &= \mathbb{E}\Big[\big(V_d(Z \cap W)\big)^2\Big] - \Big(\mathbb{E}\big[V_d(Z \cap W)\big]\Big)^2 \\
&= \mathbb{E}\left[\int_{\mathbb{R}^d} \mathbb{1}\{x \in Z \cap W\}\,\mathrm{d}x \int_{\mathbb{R}^d} \mathbb{1}\{y \in Z \cap W\}\,\mathrm{d}y\right] \\
&\quad - \mathbb{E}\left[\int_{\mathbb{R}^d} \mathbb{1}\{x \in Z \cap W\}\,\mathrm{d}x\right] \cdot \mathbb{E}\left[\int_{\mathbb{R}^d} \mathbb{1}\{y \in Z \cap W\}\,\mathrm{d}y\right] \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbb{P}\big(x, y \in Z\big) \cdot \mathbb{1}\{x, y \in W\}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbb{P}\big(x \in Z\big) \cdot \mathbb{P}\big(y \in Z\big) \cdot \mathbb{1}\{x, y \in W\}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Big(\underbrace{\mathbb{P}\big(x, y \in Z\big)}_{= \mathbb{P}(0,\, y - x \in Z) = C(y-x)} - p_Z^2 \Big) \cdot \mathbb{1}\{x, y \in W\}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Big(C(y) - p_Z^2\Big) \cdot \mathbb{1}\{x, y + x \in W\}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d} \Big(C(y) - p_Z^2\Big) \cdot \lambda^d\big(W \cap (W - y)\big)\,\mathrm{d}y. \qquad (1)
\end{aligned}
$$

As $\lambda^d = V_d$ is continuous on $\mathcal{K}^d$ (by Theorem 3.36), the convergence $W \cap \big(W - \frac{1}{r} \cdot y\big) \to W$ (as $r \to \infty$) with respect to the Hausdorff metric implies

$$
\lim_{r \to \infty} \frac{\lambda^d\big(r \cdot W \cap (r \cdot W - y)\big)}{\lambda^d(r \cdot W)} = \lim_{r \to \infty} \frac{\lambda^d\big(W \cap (W - \frac{1}{r} \cdot y)\big)}{\lambda^d(W)} = 1.
$$

Furthermore, we have

$$
\frac{\lambda^d\big(r \cdot W \cap (r \cdot W - y)\big)}{\lambda^d(r \cdot W)} = \frac{\lambda^d\big(W \cap (W - \frac{1}{r} \cdot y)\big)}{\lambda^d(W)} \leqslant 1
$$

and, as in part a),

$$
C(y) - p_Z^2 = (1-p_Z)^2 \cdot \Big(e^{\gamma \cdot \mathbb{E}\left[\lambda^d\left(Z_0 \cap (Z_0 - y)\right)\right]} - 1\Big),
$$

which is greater than or equal to 0 (since $p_Z \leqslant 1$). We conclude that

$$
\left| \Big(C(y) - p_Z^2\Big) \cdot \frac{\lambda^d\big(r \cdot W \cap (r \cdot W - y)\big)}{\lambda^d(r \cdot W)} \right| \leqslant C(y) - p_Z^2,
$$

where the right hand side is integrable with respect to λ^d by part a). Hence we can replace W in (1) by $r \cdot W$, let $r \to \infty$, and apply dominated convergence to obtain

$$
\lim_{r \to \infty} \frac{\operatorname{Var}\big(V_d(Z \cap r \cdot W)\big)}{\lambda^d(r \cdot W)} = \int_{\mathbb{R}^d} \Big(C(y) - p_Z^2\Big)\,\mathrm{d}y.
$$