Institute of Stochastics

Stochastic Geometry | Summer term 2020
PD. Dr. Steffen Winter
Steffen Betsch, M.Sc.

Work Sheet 12

Instructions for week 12 (July 6^{th} to July 10^{th}):

- Work through Section 4.4 of the lecture notes.
- Answer the control questions 1) to 3), and solve Problems 1 to 4 of the exercises below.
- Please hand in your solutions to the exercises for correction until the morning of Monday, July 13^{th}. For the procedure, please have a look at the general information in Ilias. The submission of solutions is voluntary.

Control questions to monitor your progress:

1) Specify (4.20) and the formula in Theorem 4.35 for a Boolean model in $\mathbb{R}$ and obtain expressions for δ_0 and δ_1 in terms of the densities $\gamma = \gamma_0$ and γ_1. What does it mean for Z to be isotropic in this case?
2) Verify the equation for the intensity γ in terms of the densities δ_j in Remark 4.38.
3) In the proof of Theorem 4.40, why is it enough to consider functions f such that $f(x_1, \dots, x_n) = 0$ whenever $x_i = x_j$ for some $i \neq j$?

Exercises for week 12:

Problem 1 (Concerning Remark 4.29)

Let $m \in \{0, \dots, d\}$, $K \in \mathcal{K}^d$, and let $f : \mathcal{K}^d \to \mathbb{R}$ be a measurable map. Prove that

$$\int_{G(d,d-m)} f(K|L^\perp) \, \mathrm{d}\nu_{d-m}(L) = \int_{G(d,m)} f(K|L) \, \mathrm{d}\nu_m(L),$$

where $K|L$ (or $K|L^\perp$) denotes the orthogonal projection of K onto L (or onto $L^\perp$), and where ν_q is the SO_d-invariant probability measure on $G(d,q)$ (for $q \in \{0, \dots, d\}$) from Theorem 4.25.

Problem 2 (Steiner's formula as a special case of the principal kinematic formula)

Show that Steiner's formula (Theorem 3.33) follows from the principal kinematic formula (Theorem 4.33).

Problem 3 (Geometric densities of the Boolean model)

Let Z be a stationary and isotropic Boolean model in $\mathbb{R}^3$ with intensity parameter $\gamma > 0$ and with a distribution $\mathbb{Q}$ of the typical grain which is concentrated on $\mathcal{K}^3 \setminus \{\varnothing\}$.

a) Determine the densities $\delta_0, \dots, \delta_3$ in terms of $\gamma_0, \dots, \gamma_3$.

b) Assume that

$$\delta_0 = 0.34, \quad \delta_1 = 0.1, \quad \delta_2 = 0.11, \quad \delta_3 = 0.52.$$

Determine the intensity γ.

c) Let $M \in \mathcal{K}_0$, where $\mathcal{K}_0$ is defined via the center of the circumball of the convex and compact sets, and $\mathbb{Q}(\cdot) := \int_{SO_3} \mathbb{1}\{\vartheta M \in \cdot\} \mathrm{d}\nu(\vartheta)$, where ν is the SO_3-invariant probability measure on SO_3 from Theorem 4.23. Assume that

$$\delta_0 = 10, \quad \delta_1 = 20, \quad \delta_2 = 0, \quad \delta_3 = 0.$$

Calculate the intensity γ. Given these values, what can you say about the set M?

Problem 4 (Concerning Remark 4.39)

Let $W \in \mathcal{K}^d$ with $V_d(W) > 0$, and let Z be a stationary Boolean model with typical grain Z_0 such that $\mathbb{E}\big[\big(\lambda^d(Z_0)\big)^2\big] < \infty$. Denote by p_Z the volume fraction of Z (see Definition 1.30 and Theorem 1.31).

a) Prove that $\int_{\mathbb{R}^d} \big(C(x) - p_Z^2\big) \, \mathrm{d}x < \infty$.

Hint: You may use that $e^t - 1 \leqslant t \cdot e^t$ for each $t \geqslant 0$.

b) Denote by C the covariance function of Z from Definition 3.18. Prove that

$$\sigma_{d,d} := \lim_{r \to \infty} \frac{\mathrm{Var}\big(V_d(Z \cap r \cdot W)\big)}{\lambda^d(r \cdot W)} = \int_{\mathbb{R}^d} \Big(C(x) - p_Z^2\Big) \, \mathrm{d}x.$$

The solutions to these problems will be uploaded on July 13th.