Karlsruher Institut für Technologie

Masterthesis

External DLA

Tillmann Tristan Bosch

10. March 2020

Supervisor: PD. Dr. Steffen Winter

Faculty of Mathematics

Karlsruhe Institute of Technology

Contents

1 Introduction

2 Preliminaries

2.1 Symbols

Let $d \in \mathbb{N}$ and $q \in \{0, \dots, d\}$.

$\mathbb{N} = \{1, 2, 3, \dots\}$, the set of natural numbers (without 0)
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$
$\mathcal{B}^d$, d-dimensional Borel-σ-algebra of $\mathbb{R}^d$
$\mathcal{K}^d$, the set of convex and compact sets in $\mathbb{R}^d$
$B_d(x, r) = \{y \in \mathbb{R}^d \mid |x - y| \leq r\}$, the d-dimensional closed ball of radius r around x
$B_r := B_2(r, 0)$
$S_{d-1}(r, x) = \partial B_d(r, x)$, the $(d-1)$-dimensional surface of the d-dimensional ball
$A(d, q)$, the set of q-dimensional affine subspaces of $\mathbb{R}^d$
$\mathcal{A}(d, q)$, the σ-algebra of $A(d, q)$, as constructed later in the paper
$\mathcal{G} := A(2, 1)$, the set of lines in the real plane
$SO_d := \{\nu \in \mathbb{R}^{d \times d} \mid \nu\nu^\top = I_d \text{ and } \det \nu = 1\}$, identify $SO_2 = \{\nu_\beta := e^{i\beta} \in \mathbb{C} \mid \beta \in [0, 2\pi)\}$
$G_d := \{\varphi : \mathbb{R}^d \to \mathbb{R}^d, x \mapsto \nu x + b \mid \nu \in SO_d, b \in \mathbb{R}^d\}$, the set of euclidean motions
$\mathcal{P}_f^d$, the set of finite subsets of $\mathbb{Z}^d$
$\mathcal{P}_f := \mathcal{P}_f^2$

2.2 Basic structures

2.2.1 Graphs

Let $d \in \mathbb{N}$. We will be interested in the undirected graph $(\mathbb{Z}^d, E)$ with its canonical graph structure, which is two vertices (or points) $x = (x_1, \dots, x_d), y = (y_1, \dots, y_d) \in \mathbb{Z}^d$ form an edge (e.g. $\{x, y\} \in E$) if and only if there exists exactly one $i \in \{1, \dots, d\}$ such that $|x_i - y_i| = 1$ and $x_j = y_j$ for all $j \neq i$. For a point $x \in \mathbb{Z}^d$ its set of *neighbours* is defined as

$$N(x) := \{y \in \mathbb{Z}^d \mid \{x, y\} \in E\}.$$

For a set $A \subset \mathbb{Z}^d$ the *outer boundary* ∂A of A is defined as

$$\partial A := \{y \in \mathbb{Z}^d \setminus A \mid \exists x \in A : \ \{x, y\} \in E\}$$

and the closure $\bar{A}$ of A as

$$\bar{A} := A \cup \partial A.$$

Instead of $(\mathbb{Z}^d, E)$ we will write $\mathbb{Z}^d$ from now on.

2.2.2 Random Walks

Definition 2.2.1. Throughout the paper let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and for our space of interest $\mathbb{Z}^d$ we will always use the discrete σ-algebra which is the power set of $\mathbb{Z}^d$. A family $(S_n)_{n \in \mathbb{N}_0}$ of measurable functions $S_n : \Omega \to \mathbb{Z}^d$ is called a *random walk on* $\mathbb{Z}^d$ (*starting at* $x \in \mathbb{Z}^d$) if and only if $S_0 = x$ a.s. and

$$\mathbb{P}(S_n = y \mid S_{n-1} = z) = \frac{1}{|N(z)|} = \frac{1}{2d}, \quad \text{for all } y \in N(z) \text{ and } z \in \mathbb{Z}^d.$$

Note that $|N(z)| = 2d$ for all $z \in \mathbb{Z}^d$ since every point has two neighbours in the direction of every dimensional component. We can therefore conclude easily that $\mathbb{P}(S_n = y \mid S_{n-1} = z) = 0$ for all $y \notin N(z)$ and $z \in \mathbb{Z}^d$. For $x, y \in \mathbb{Z}^d$ we introduce the short notation

$$\mathbb{P}_x(S_n = y) := \mathbb{P}(S_n = y \mid S_0 = x).$$

So a random walk can be understood as a particle starting from some point x and moving randomly on the grid choosing its next step uniformly from its neighbours. For the following let $(S_n)_{n \in \mathbb{N}}$ be a random walk on $\mathbb{Z}^d$ starting at $x \in \mathbb{Z}^d$.

Definition 2.2.2. Let $A \subset \mathbb{Z}^d$. We define the *hitting times* of A by

$$T_A := \min\{n \geq 0 \mid S_n \in A\} \text{ and } T_A^+ := \min\{n \geq 1 \mid S_n \in A\},$$

and $T_y := T_{\{y\}}$ and $T_y^+ := T_{\{y\}}^+$ for $y \in \mathbb{Z}^d$.

Definition 2.2.3. A random walk with origin in $x \in \mathbb{Z}^d$ is called *recurrent* if

$$\mathbb{P}_x(T_x^+ < \infty) = 1$$

and *transient* if

$$\mathbb{P}_x(T_x^+ < \infty) < 1.$$

Lemma 2.2.1. A random walk on $\mathbb{Z}^d$ is recurrent if $d \leq 2$ and transient if $d \geq 3$.

Proof. Proofs of this result are presented in [4] Satz 5.1 or [8] Korollar 2.6.6. □

Lemma 2.2.2. The following two statements are equivalent:

(i) $\mathbb{P}_x(T_x^+ < \infty) = 1$ for all $x \in \mathbb{Z}^2$

(ii) $\mathbb{P}_x(T_A^+ < \infty) = 1$ for all $x \in \mathbb{Z}^2$ and $A \subset \mathbb{Z}^2$

Proof. The direction (ii) to (i) is clear. For the other direction choose $x \in \mathbb{Z}^2$ and $A \subset \mathbb{Z}^2$. We know that in general for any point $y \in \mathbb{Z}^2$ we have $\mathbb{P}_x(T_y^+ < \infty) > 0$ for a random walk in $\mathbb{Z}^d$ for any dimension $d \in \mathbb{N}$. By Lemma 2.2.1 we know that a random walk on $\mathbb{Z}^2$ is recurrent. For that case it is proved in [8] Satz 2.6.9 that even $\mathbb{P}_x(T_y^+ < \infty) = 1$ holds for any $y \in \mathbb{Z}^2$. If we choose $y \in A$ we get

$$1 = \mathbb{P}_x(T_y^+ < \infty) \leq \mathbb{P}_x(T_A^+ < \infty),$$

which completes the proof. □

2.2.3 Expressions

If for $A \in \mathcal{F}$ we have $\mathbb{P}(A) = 1$ we will say that „A holds $\mathbb{P}$-a.s.“, or short „A holds a.s.“ (A holds almost surely). Another short expression, if for a set of logical statements $(A_t)_{t \in I}$ with $I \in \{\mathbb{N}, \mathbb{R}\}$ we say „A_t holds for large t“ it shall mean that there exists $T \in I$ such that A_t holds for all $t > T$. Here we mean that a logical statement holds if and only if the statement is true.

3 Incremental Aggregate

3.1 Definition

In this paper we will look at stochastic processes on the set of finite subsets of $\mathbb{Z}^d$, where we start with the one point set $\{0\}$ and incrementally add a point of the outer boundary of the current cluster according to some distribution. What we get is a randomly, point by point growing connected cluster which we will call *incremental aggregate*. Define

$$\mathcal{P}_f^d := \{A \subset \mathbb{Z}^d \mid \text{A is finite}\},$$

the set of finite subsets of $\mathbb{Z}^d$. Furthermore we will be interested in distributions on those sets, so for $A \in \mathcal{P}_f^d$ we define

$$\mathcal{D}_A := \{\mu : \mathbb{Z}^d \to [0,1] \mid \mu(y) = 0 \text{ for all } y \notin A \text{ and } \sum_{y \in A} \mu(y) = 1\},$$

the set of distributions on A. Now we define an *incremental aggregate* as follows.

Definition 3.1.1. Let $\mu = (\mu_A)_{A \in \mathcal{P}_f^d}$ be a family of distributions with $\mu_A \in \mathcal{D}_A$ for all $A \in \mathcal{P}_f^d$. An *incremental aggregate* (*with distribution* μ) is a stochastic process $(\mathcal{E}_n)_{n \in \mathbb{N}}$ which evolves as follows. The process starts with one point $\mathcal{E}_1 = \{0\}$ (define $y_1 := 0$) at the origin of $\mathbb{Z}^d$. Knowing the process $\mathcal{E}_n$ at time n, let y_{n+1} be a random point in $\partial\mathcal{E}_n \in \mathcal{P}_f^d$ with distribution

$$\mathbb{P}(y_{n+1} = y \mid \mathcal{E}_n) := \mu_{\partial\mathcal{E}_n}(y), \quad y \in \mathbb{Z}^d. \tag{3.1}$$

We then define $\mathcal{E}_{n+1} := \mathcal{E}_n \cup \{y_{n+1}\}$ and the limit cluster as $\mathcal{E}_\infty := \bigcup_{n \in \mathbb{N}} \mathcal{E}_n$.

The above defines a Markov chain whose state space is the set of finite and connected subsets of $\mathbb{Z}^d$. For all incremental aggregates in this paper we will focus on the 2-dimensional case $\mathbb{Z}^2$ and will always identify $\mathbb{R}^2$ with $\mathbb{C}$ for a more convenient notation.

Remark 3.1.1. It is worth to mention that the distribution of a new added point y_{n+1} is not depending on the order of how previous points where added to the cluster since the harmonic measure only looks at the boundary of the current cluster and therefore doesn't consider any information about orderings of cluster points.

3.2 Notion of Fractal Dimension and Growth Rate

The notion of fractal dimension is usually used for sets with uncountable cardinality like continuous curves or surfaces. One way of defining a fractal dimension for curves like for example the cost line of New Zealand (see Figure 1) would be to look at the limit relation between the minimum number of boxes (squares) which we need to cover the cost line and the side length of these boxes. The following definition is motivated by [7] page 160. If for $\varepsilon > 0$ $N(\varepsilon)$ is the mininum number of boxes with side length ε which

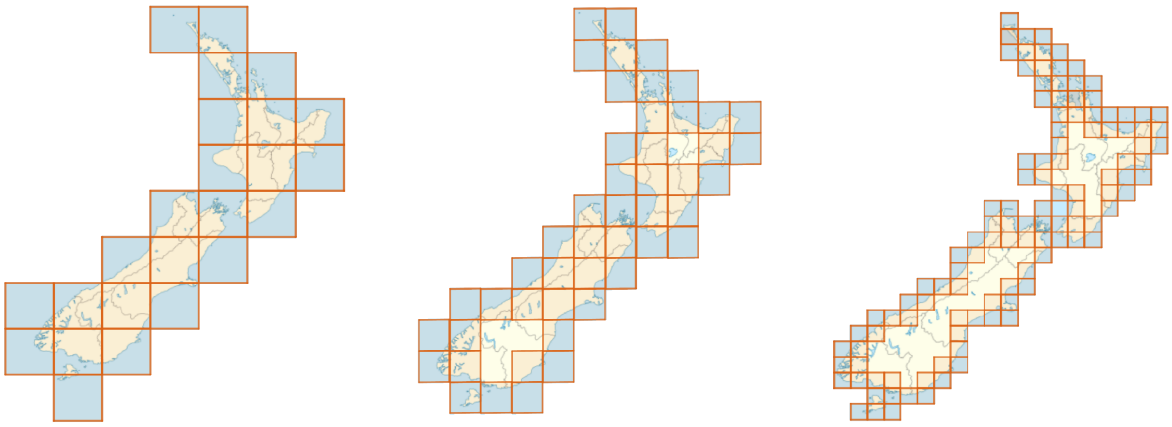

Figure 1: Box-covering of New Zealands outer cost with decreasing box sizes ε

we need to cover the cost line, then the so called *box-dimension* d_b is a constant such that $N(\varepsilon)$ grows as fast as ε^{-d_b} for letting ε tend to zero, so

$$d_b := -\lim_{\varepsilon \to 0} \frac{\ln(N(\varepsilon))}{\ln(\varepsilon)}. \tag{3.1}$$

This definition makes sense in many contexts, for example is the box-dimension of straight line segments 1 and of squares 2 and so on, so in those cases equal to the topological dimension. As with incremental aggregates we are dealing with finite point sets, this approach of defining a fractal dimension for our clusters is not senseful. It is not difficult to show, that the box-dimension of any finite set is 0. A helpful detail about the situation with finite sets as the ones we are looking at is that each point of the cluster can actually be interpreted and identified with a unique square since the cluster is living on the grid $\mathbb{Z}^2$. We will precise that in chapter 5. So instead of decreasing the sizes of the boxes with which we cover our set of interest, we leave the size of the boxes constant and increase the size of our set by adding points and looking at the limit cluster $\mathcal{E}_\infty$. Defining the radius of a finite set $A \in \mathcal{P}_f^d$ (with $|\cdot|$ the euclidean norm) as

$$\mathrm{rad}(A) := \max_{x \in A} |x|,$$

we therefore can identify the relation between the geometrical sizes of the boxes and the cluster as follows. For $n \in \mathbb{N}$ define $\varepsilon_n := \frac{1}{\mathbb{E}[\mathrm{rad}(\mathcal{E}_n)]}$. Before ε_n would have expressed the geometrical relation between one box and the whole cluster which we describe now by the fraction $\frac{1}{\mathbb{E}[\mathrm{rad}(\mathcal{E}_n)]}$ since a box size is now constantly 1 and the clusters geometrical size we can interpret by $\mathbb{E}[\mathrm{rad}(\mathcal{E}_n)]$ up to a constant. The minimum number of boxes with size 1 we need to cover $\mathcal{E}_n$ is always n, so $N(\varepsilon_n) = n$ for all $n \in \mathbb{N}$ and therefore we can rewrite the definition of the box-dimension (3.1) by replacing ε_n with $\frac{1}{\mathbb{E}[\mathrm{rad}(\mathcal{E}_n)]}$ and

define the (*discrete*) *fractal dimension* of $\mathcal{E}_\infty$ as

$$d_f := \liminf_{n\to\infty} \frac{\ln(n)}{\ln(\mathbb{E}[\mathrm{rad}(\mathcal{E}_n)])} \tag{3.2}$$

Why we choose the limes inferior here we will see later. Another way of tackling the intuition for a fractal dimension of discrete sets is by considering subsets of the d-dimensional ball $B_d(0,m)$ of radius $m \geq 0$, which is motivated by [5] Part II page 98 and [3] 2.6 page 82. If we take a finite subset M of that ball, we could assign it the dimension k if its cardinality is of order m^k. If we apply the same argument to the clusters $\mathcal{E}_n$ and balls with radius $\mathbb{E}[\mathrm{rad}(\mathcal{E}_n)]$ for all $n \in \mathbb{N}$, we get that the dimension of $\mathcal{E}_\infty$ could be interpreted as a constant k which fulfills that $n = |\mathcal{E}_n|$ grows as fast as $\mathbb{E}[\mathrm{rad}(\mathcal{E}_n)]^k$ for letting n tend to infinity. We therefore get the same definition as in (3.2). This way of defining a fractal dimension for incremental aggregates strongly correlates with the growth rate of the aggregate which shall indicate how the radius of the cluster evolves while increasing the particle number. We can define the growth rate by looking for the smallest exponent α such that there exists a constant $c > 0$ with

$$\mathbb{E}[\mathrm{rad}(\mathcal{E}_n)] \leq cn^\alpha$$

for large n. Rewriting this we come to the equivalent inequality

$$\frac{\ln(\mathbb{E}[\mathrm{rad}(\mathcal{E}_n)])}{\ln(n)} - \frac{\ln(c)}{\ln(n)} \leq \alpha$$

for large n and we could finally define the growth rate α_f of an incremental aggregate as the smallest value satisfying this inequality, so

$$\alpha_f := \limsup_{n\to\infty} \frac{\ln(\mathbb{E}[\mathrm{rad}(\mathcal{E}_n)])}{ln(n)}. \tag{3.3}$$

Since $\mathrm{rad}(\mathcal{E}_n) \leq n$ for all $n \in \mathbb{N}$ a.s., we get

$$\alpha_f \leq 1.$$

We further chose the limes inferior in the definition (3.2) to simply have

$$d_f = \frac{1}{\alpha_f} \tag{3.4}$$

which includes the cases $d_f = \infty$ or $\alpha_f = \infty$ ($\frac{1}{0} := \infty$). We therefore get

$$d_f \geq 1.$$

In $\mathbb{Z}^d$ for some $d \in \mathbb{N}$ we can fill the cube with center 0 and side length n with an order of $n^{\frac{1}{d}}$ points. Therefore we can fill the ball of radius n with an order of $n^{\frac{1}{d}}$ points aswell, so there is a constant $c > 0$ such that $cn^{\frac{1}{d}} \leq \mathrm{rad}(\mathcal{E}_n)$ for all $n \in \mathbb{N}$ a.s.. Hence

$$d_f \leq \liminf_{n\to\infty} \frac{\ln(n)}{\ln(cn^{\frac{1}{d}})} = d.$$

So in total we get the trivial boundaries

$$1 \leq d_f \leq d \tag{3.5}$$

and

$$\frac{1}{d} \leq \alpha_f \leq 1$$

for any incremental aggregate in $\mathbb{Z}^d$.

4 External Diffusion Limited Aggregate

4.1 Definition

External DLA is a model of an incremental aggregate as defined above using a very natural family of distributions, called the *harmonic measures*.

Definition 4.1.1. (*harmonic measure*) Let $A \subset \mathbb{Z}^d$. The hitting probability of A is the function

$$H_A : \mathbb{Z}^d \times A \to [0,1], \quad (x,y) \mapsto H_A(x,y) := \mathbb{P}_x(S_{T_A^+} = y).$$

In literature you can find the same definition where T_A is used instead of T_A^+. Since in the following for finite sets $A \in \mathcal{P}_f^d$ the limit $|x| \to \infty$ of $H_A(x,y)$ is of interest, T_A^+ is chosen for convenience. In fact, for a fixed element $x \in \mathbb{Z}^d$ the function $H_A(x,\cdot)$ defines a measure on A with total mass $\mathbb{P}_x(T_A^+ < \infty)$ and it can be adapted to a probability measure by conditioning the random walk to hit A in finite time. Define

$$\bar{H}_A : \mathbb{Z}^d \times A \to [0,1], \quad (x,y) \mapsto \bar{H}_A(x,y) := \mathbb{P}_x(S_{T_A^+} = y \mid T_A^+ < \infty),$$

so for fixed $x \in \mathbb{Z}^d$ the function $\bar{H}_A(x,\cdot)$ defines a probability measure on A. Indeed this definition is motivated by [3] (Chapter 2, Definition 2.1) and in the same chapter it is proved, that for finite sets $A \in \mathcal{P}_f^d$ the limit

$$\lim_{|x|\to\infty} \bar{H}_A(x,y) =: h_A(y)$$

exists for each $y \in A$. The function $h_A : A \to [0,1]$ is called the *harmonic measure of* A. For an element $y \in A$, h_A can be interpreted as the probability that a random walk starting at „infinity“ hits A the first time at y.

Definition 4.1.2. (*external diffusion limited aggregate*) *External diffusion limited aggregate* (on $\mathbb{Z}^d$), short *external DLA* or just *DLA*, is an incremental aggregate with the family of harmonic measures $(h_A)_{A\in\mathcal{P}_f^d}$ as distribution.

Remark 4.1.1. If we look at the 2-dimensional case, by Lemma 2.2.2, we have that $\mathbb{P}_x(T_A^+ < \infty) = 1$ for any $x \in \mathbb{Z}^2$, and therefore get $\bar{H}_A = H_A$. So in two dimensions the harmonic measure of $A \subset \mathbb{Z}^2$ can be written as

$$h_A(y) = \lim_{|x|\to\infty} \mathbb{P}_x(S_{T_A^+} = y), \quad y \in A.$$

4.2 Fractal Dimension and Growth Rate of External DLA in $\mathbb{Z}^2$

There is not yet a rigorous proof on the exact fractal dimension of external DLA. There are rather few rigorously proved results and it seems that it is very hard to prove such results on DLA. Looking at computer simulations of DLA in $\mathbb{Z}^2$ it seems that the clusters

are relatively sparse and they appear to have a noninteger fractal dimension. In [6] DLA is observed in a magnetic aggregation context, and empirically they find a fractal dimension of around 1.8. Other simulations seem to suggest a value a little less than 1.7 for d_f in two dimensions ([3] page 83). There is also a theory that predicts

$$d_f = \frac{d^2+1}{d+1}$$

in $\mathbb{Z}^d$ which seems to agree fairly well with simulations ([3] page 83). There are only few rigorously proved results and we will see one of them in the following. First we will proof some lemmas. For that define two random functions

$$r : \mathbb{N} \to [0,\infty), \quad n \mapsto \mathrm{rad}(\mathcal{E}_n)$$

and

$$T : [0,\infty) \to \mathbb{N}, \quad s \mapsto \min\{j \in \mathbb{N} \mid r(j) \geq s\}.$$

Then it is easy to show that for all $n \in \mathbb{N}$, $s \in [0,\infty)$ and $\omega \in \Omega$

(i) $r(\omega)$ and $T(\omega)$ grow monotonously

(ii) $T(\omega)(r(\omega)(n)) \leq n$

(iii) $r(\omega)(T(\omega)(s)) \geq s$

hold.

Lemma 4.2.1. For both random functions r and T we have that

$$r(\omega)(n) \to \infty \text{ for } n \to \infty$$

and

$$T(\omega)(s) \to \infty \text{ for } s \to \infty$$

for all $\omega \in \Omega$.

Proof. Since we are moving on the grid $\mathbb{Z}^d$ we have that for a ball $B_d(0,n)$ with radius $n \geq 0$ we have a finite number $N := |B_d(0,n) \cap \mathbb{Z}^d|$ and for any $\omega \in \Omega$ get

$$r(\omega)(2N) \geq n.$$

Therefore for all $\omega \in \Omega$ and any $n \in \mathbb{N}$ we can find a $M \in \mathbb{N}$ such that $r(\omega)(M) \geq n$, and since $r(\omega)$ grows monotonously we get $r(\omega)(n) \to \infty$ for $n \to \infty$ for all $\omega \in \Omega$. Very similarly we can argument for T. □

Lemma 4.2.2. Let $a > 0$ and $h : [0,\infty) \to [0,\infty)$ be a bijective, multiplicative and monotonously growing function. Then the following are equivalent:

(i) $\exists c > 0 : \mathbb{P}(r(n) \leq ch(n) \text{ for large } n) = 1$

(ii) $\exists c > 0 : \mathbb{P}(T(as) \geq ch^{-1}(s) \text{ for large } s) = 1$

Proof. $\Rightarrow$: For $c > 0$ define

$$A_c := \{r(n) \leq ch(n) \text{ for large } n\}$$

and

$$B_c := \{T(as) \geq ch^{-1}(s) \text{ for large } s\}.$$

Choose $c > 0$ such that $\mathbb{P}(A_c) = 1$. Take $\omega \in A_c$ and choose $N \in \mathbb{N}$ such that $r(\omega)(n) \leq ch(n)$ for all $n > N$. Hence there exists $\tilde{c} > 0$ such that $\tilde{c}h^{-1}(r(\omega)(n)) \leq n$ for all $n > N$. By Lemma 4.2.1 we can choose $M \in \mathbb{N}$ big enough such that $T(\omega)(aM) > N$, hence $T(\omega)(as) \geq T(\omega)(aM) > N$ for all $s > M$ since $T(\omega)$ grows monotonously. Hence we can write $\tilde{c}h^{-1}(r(\omega)(T(\omega)(as))) \leq T(\omega)(as)$ for all $s > M$ and since $r(T(\omega)(as)) \leq as$ we finally get $\tilde{c}h^{-1}(a)h^{-1}(s) = \tilde{c}h^{-1}(as) \leq T(\omega)(as)$ for all $s > M$, hence $\omega \in B_{\tilde{c}h^{-1}(a)}$, where we used the multiplicativity of h and that h^{-1} falls monotonously. We therefore get $A_c \subset B_{\tilde{c}h^{-1}(a)}$, hence $\mathbb{P}(B_{\tilde{c}h^{-1}(a)}) = 1$.
$\Leftarrow$: The proof for this direction works analogously and needs Lemma 4.2.1 aswell. □

Lemma 4.2.3. Let $n \in \mathbb{N}$ and $T_1, \ldots, T_n$ be independent geometrically distributed random variables with parameter $p < \frac{1}{2}$. Let $Y := T_1 + \cdots + T_n$, then for every $a \in [2p, 1)$ we have

$$\mathbb{P}(Y \leq \frac{an}{p}) \leq (ae^2)^n.$$

Proof. The moment generating function of Y is

$$\mathbb{E}[e^{tY}] = (pe^t)^n(1 - e^t(1-p))^{-n} = p^n(e^{-t} - (1-p))^{-n}.$$

By Chebyshev for any random variable X we know the inequality

$$\mathbb{P}(X \geq x) \leq \inf_{t>0} \frac{\mathbb{E}[e^{tX}]}{e^{tx}}$$

and can therefore follow that for any $t > 0$

$$\mathbb{P}(Y \leq \frac{an}{p}) = \mathbb{P}(-Y \geq -\frac{an}{p}) \leq \exp(\frac{ant}{p})\mathbb{E}[e^{-tY}] = \exp(\frac{ant}{p})p^n(e^t - (1-p))^{-n}.$$

Choose $t = \ln(\frac{a(1-p)}{a-p})$ (note that $t > 0$), then

$$\begin{aligned}\mathbb{P}(Y \leq \frac{an}{p}) &\leq (\frac{a(1-p)}{a-p})^{\frac{an}{p}} p^n(1-p)^{-n}(\frac{p}{a-p})^{-n} \\ &= (\frac{a}{a-p})^{\frac{an}{p}}(1-p)^{\frac{an}{p}}(1-p)^{-n}(\frac{1}{a-p})^{-n}\end{aligned}$$

$$\begin{aligned}
&\leq (1+\frac{p}{a-p})^{\frac{an}{p}}(1-p)^n(1-p)^{-n}(a-p)^n \\
&\leq (1+\frac{p}{a-p})^{\frac{2(a-p)n}{p}}a^n \\
&\leq (ae^2)^n.
\end{aligned}$$

□

Definition 4.2.1. For $x \in \mathbb{Z}^2$ and $r \in [1,\infty)$ define

$$\mathcal{P}_r^x := \{A \in \mathcal{P}_f \mid x \in A, r = \max_{y\in A}|x-y| \text{ and } A \text{ is connected}\}.$$

The following theorem gives a bound on the harmonic measure which will be necessary in the proof we present here for the growth rate of DLA. The theorem is proved by [3] in Theorem 2.5.2 and we state it slightly different here to generalize it easier afterwards.

Theorem 4.2.1. There exists a constant $c > 0$ such that for all $r \in [1,\infty)$

$$h_A(0) \leq cr^{-\frac{1}{2}} \quad \text{for all } A \in \mathcal{P}_r^0.$$

Proof. If $A \in \mathcal{P}_r^0$ then $0 \in A$ and $r = \text{rad}(A)$ and for that case the proof is presented for all dimensions $d \in \mathbb{N}$ in [3] Theorem 2.5.2. The proof is very technical and requires various results of other theorems and lemmas which are also to find in [3]. □

Proposition 4.2.1. Theorem 4.2.1 can be generalized to the following. There exists a $c > 0$ such that for all $y \in \mathbb{Z}^2$ and $s, r \in [1,\infty)$ with $s \geq r$ we have

$$h_A(y) \leq cr^{-\frac{1}{2}} \quad \text{for all } A \in \mathcal{P}_s^y.$$

Proof. For $y \in \mathbb{Z}^2$ define the translation function

$$\Phi_y : \mathbb{Z}^2 \to \mathbb{Z}^2, x \mapsto x + y.$$

Recall that in $\mathbb{Z}^2$ the harmonic measure of $A \subset \mathbb{Z}^2$ can be written as

$$h_A(z) = \lim_{|x|\to\infty} \mathbb{P}_x(S_{T_A^+} = z), \quad z \in A,$$

(see Remark 4.1.1). Since the distribution of random walks in $\mathbb{Z}^2$ is invariant under translation we can conlude that

$$\begin{aligned}
h_A(z) &= \lim_{|x|\to\infty} \mathbb{P}_x(S_{T_A^+} = z) \\
&= \lim_{|x|\to\infty} \mathbb{P}_{\Phi_y(x)}(S_{T^+_{\Phi_y(A)}} = \Phi_y(z)) \\
&= h_{\Phi_y(A)}(\Phi_y(z))
\end{aligned}$$

for all $z \in A$. By Theorem 4.2.1 we know that there exists a $c > 0$ such that for all $s \in [1,\infty)$

$$h_A(0) \leq cs^{-\frac{1}{2}} \quad \text{for all } A \in \mathcal{P}_s^0.$$

Therefore for $s, r \in \mathbb{R}$ with $s \geq r$ we get

$$h_A(0) = h_{\Phi_y(A)}(\Phi_y(0)) = h_{\Phi_y(A)}(y) \leq cs^{-\frac{1}{2}} \leq cr^{-\frac{1}{2}} \quad \text{for all } A \in \mathcal{P}_s^0,$$

and since $A \in \mathcal{P}_s^0 \Leftrightarrow \Phi_y(A) \in \mathcal{P}_s^{\Phi_y(0)}$ we get

$$h_A(y) \leq cr^{-\frac{1}{2}} \quad \text{for all } A \in \mathcal{P}_s^y.$$

Since c was chosen independently of y, s and r, this completes the proof. □

Theorem 4.2.2. For the growth rate of external DLA in $\mathbb{Z}^2$ as defined in 3.2 (3.3) we have

$$\alpha_f \leq \frac{2}{3}. \tag{4.1}$$

Proof. For $c > 0$ define

$$A_c := \{\omega \in \Omega \mid \operatorname{rad}(\mathcal{E}_n(\omega)) \leq cn^{\frac{2}{3}} \text{ for large } n\}$$

and

$$D_c := \{\omega \in \Omega \mid T(\omega)(2n) \geq cn^{\frac{3}{2}} \text{ for large } n\}.$$

If we have a $c > 0$ such that

$$\mathbb{P}(A_c) = 1, \tag{4.2}$$

then we have

$$\begin{aligned}
\alpha_f &= \limsup_{n\to\infty} \frac{\ln(\mathbb{E}[\operatorname{rad}(\mathcal{E}_n)])}{\ln(n)} \\
&= \limsup_{n\to\infty} \frac{\ln(\int_\Omega \operatorname{rad}(\mathcal{E}_n) d\mathbb{P})}{\ln(n)} \\
&= \limsup_{n\to\infty} \frac{\ln(\int_{A_c} \operatorname{rad}(\mathcal{E}_n) d\mathbb{P})}{\ln(n)} \\
&\overset{(+)}{\leq} \limsup_{n\to\infty} \frac{\ln(\int_{A_c} cn^{\frac{2}{3}} d\mathbb{P})}{\ln(n)} \\
&= \limsup_{n\to\infty} \frac{\ln(cn^{\frac{2}{3}})}{\ln(n)} \\
&= \frac{2}{3} \limsup_{n\to\infty} \frac{\ln(c^{\frac{3}{2}}) + \ln(n)}{\ln(n)} = \frac{2}{3} \cdot 1 = \frac{2}{3}
\end{aligned}$$

If we choose

$$h : [0, \infty) \to [0, \infty), x \mapsto x^{\frac{2}{3}},$$

and $a = 2$, then by Lemma 4.2.2 we can show (4.2) if we find a constant $c > 0$ such that

$$\mathbb{P}(D_c) = 1. \tag{4.3}$$

Note that h is bijective, multiplicative and monotonously growing. Lets first argument why the inequality at (+) indeed holds. We define ...

So now we will try to prove (4.3). For $n \in \mathbb{N}$ write $\mathcal{E}_n = \{y_1, \dots, y_n\}$ according to the definition in 3.1.1, where y_j is the j-th point added to the cluster. Let $\beta > 0$ which will be determined lateron. For $n \in \mathbb{N}$ let $\tilde{m}_n := \beta n^{\frac{3}{2}}$ and define

$$V_n := \{\omega \in \Omega \mid T(\omega)(2n) < \tilde{m}_n\}.$$

Further define the set of realised random walk paths of length n with starting point in $\tilde{B}_n := \{x \in \mathbb{Z}^2 \mid d(x, \partial B_n) \leq 1 \text{ and } |x| \geq n\}$

$$Z_n := \{[z] := \{z_1, \dots, z_n\} \subset \mathbb{Z}^2 \mid z_1 \in \tilde{B}_n \text{ and } z_i \in N(z_{i-1}) \text{ for all } i \in \{2, \dots, n\}\},$$

for $[z] \in Z_n$ define events

$$W_n([z]) := \{\omega \in \Omega \mid \exists j_1 < \cdots < j_n \leq \tilde{m}_n \text{ such that } y_{j_i}(\omega) = z_i \text{ for all } i \in \{1, \dots, n\}\}$$

and the union of these events

$$W_n := \bigcup_{[z] \in Z_n} W_n([z]).$$

With $\tilde{B}_n$ we mean that the random walks in Z_n start „on the boundary“ of B_n. We will quickly prove that

$$V_n \subset W_n \text{ for all } n \in \mathbb{N}.$$

Let $n \in \mathbb{N}$ and $\omega \in V_n$, then $T(\omega)(2n) < \tilde{m}_n$. For $m_n := \max\{j \in \mathbb{N} \mid j \leq \tilde{m}_n\}$ we therefore have $\mathrm{rad}(\mathcal{E}_{m_n}(\omega)) \geq 2n$ (Note that $T(\omega)(2n) \in \mathbb{N}$). Therefore since $\mathcal{E}_{m_n}(\omega)$ is connected there must exist indices $j_1 < \cdots < j_n$ s.t. $y_{j_1}(\omega) \in \tilde{B}_n$ and $[z_0] := [y_{j_1}(\omega), \dots, y_{j_n}(\omega)] \subset \mathcal{E}_{m_n}(\omega)$, since $\max_{x \in [z_0]} |x| \leq 2n$. Therefore $j_n \leq m_n \leq \tilde{m}_n$ and therefore $\omega \in W_n([z_0]) \subset W_n$ which proofs the inclusion.

For a sequence of events $(A_n)_{n \in \mathbb{N}}$ recall that

$$\limsup_{n \to \infty} A_n := \bigcap_{n \in \mathbb{N}} \bigcup_{i \geq n} A_i = \{\omega \in \Omega \mid \omega \in A_n \text{ for infinitely many } n \in \mathbb{N}\}.$$

We will use the Lemma of Borel-Cantelli on the sequence of events $(W_n)_{n \in \mathbb{N}}$. If we can show that

$$\sum_{n \in \mathbb{N}} \mathbb{P}(W_n) < \infty, \tag{4.4}$$

then with $V_n \subset W_n$ for all $n \in \mathbb{N}$ and Borel-Cantelli we get

$$\mathbb{P}(\limsup_{n\to\infty} V_n) \leq \mathbb{P}(\limsup_{n\to\infty} W_n) = 0.$$

Since

$$(\limsup_{n\to\infty} V_n)^C = \{\omega \in \Omega \mid \exists N \in \mathbb{N} \text{ s.t. } \omega \in V_n^C \text{ for all } n > N\} = D_\beta$$

we can then conclude that $\mathbb{P}(D_\beta) = 1$ and have finished the proof. So we want to show (4.4).

For $n \in \mathbb{N}$, $[z] \in Z_n$ and $i \in \{1, \dots, n\}$ we define random variables

$$\tau_i : \Omega \to \mathbb{N}^\infty, \tau_i(\omega) = j :\Leftrightarrow \begin{cases} y_j(\omega) = z_i, & j < \infty, \\ z_i \notin \mathcal{E}_\infty(\omega), & j = \infty, \end{cases}$$

so τ_i is either the index j such that the j-th added point to the cluster is equal to z_i, or infinity if the final cluster $\mathcal{E}_\infty$ doesn't contain z_i. τ_i is measurable because for $j \in \mathbb{N}$ we have

$$\tau_i^{-1}(j) = \{y_j = z_i\} = y_j^{-1}(z_i) \in \mathcal{F}$$

and for $j = \infty$ we have

$$\begin{aligned} \tau_i^{-1}(\infty) &= \{z_i \notin \mathcal{E}_\infty = \bigcup_{k\in\mathbb{N}} \mathcal{E}_k\} \\ &= \{z_i \notin \mathcal{E}_k \text{ for all } k \in \mathbb{N}\} \\ &= \bigcap_{k\in\mathbb{N}} \{z_i \notin \mathcal{E}_k\} \\ &= \bigcap_{k\in\mathbb{N}} \bigcap_{l=1}^{k} \{y_l \neq z_i\} \\ &= \bigcap_{k\in\mathbb{N}} \bigcap_{l=1}^{k} y_l^{-1}(\mathbb{Z}^2 \setminus \{z_i\}) \in \mathcal{F}. \end{aligned}$$

EVTL UNNÖTIG We also define

$$U_{[z]}^n = \{\tau_1 < \tau_2 < \cdots < \tau_n < \infty\}$$

and

$$u_{[z]}^n := \mathbb{P}(U_{[z]}^n)$$

which play an important role in proof later. EVTL UNNÖTIG Further for $i \in \{1, \dots, n-1\}$ we define waiting times

$$\sigma_i : \Omega \to \mathbb{N}^\infty, \omega \to \begin{cases} \tau_{i+1}(\omega) - \tau_i(\omega), & \text{if } \tau_{i+1}(\omega) < \infty \text{ and } \tau_i(\omega) < \infty, \\ \infty, & \text{else,} \end{cases}$$

so σ_i is the waiting time between adding z_i and z_{i+1} to the cluster. We quickly argument that σ_i is measurable aswell. For $j \in \mathbb{N}$ we have

$$\sigma_i^{-1}(j) = \{\tau_{i+1} - \tau_i = j\} = \bigcup_{k\in\mathbb{N}} \{\tau_{i+1} = k\} \cap \{\tau_i = k - j\} \in \mathcal{F}$$

and

$$\sigma_i^{-1}(\infty) = \{\tau_{i+1} = \infty\} \cup \{\tau_i = \infty\} \in \mathcal{F},$$

since τ_{i+1} and τ_i are measurable.

Let $i \in \{1, \dots, n-1\}$. We will now prove that the distribution of σ_i is bounded by that of a geometrically distributed random variable with parameter

$$p_n := c_1 n^{-\frac{1}{2}}$$

for some constant $c_1 > 0$, which is the same constant for all $n \in \mathbb{N}$. We will want to use Proposition 4.2.1 here, for which we need to create terms where the harmonic measure appears. For that we need probabilities which condition on a given cluster which we will develop now. We first define some helpful sets. For $m \in \mathbb{N}$ define

$$C_m := \{\mathcal{E}_m(\omega) \mid \omega \in \Omega\}$$

the set of realized clusters of size m and the disjoint union of them as $C := \bigcup_{m\in\mathbb{N}} C_m$. Further define another disjoint partition of C with sets

$$E_m := \{\mathcal{E} \in C \mid m - 1 \leq \mathrm{rad}(\mathcal{E}) \leq m\}$$

such that $C = \bigcup_{m\in\mathbb{N}} E_m$ aswell. Since $|z_1| \geq n$ and z_i is reachable by z_1 in $i-1$ steps we need some minmal amount of time steps such that it is possible for the cluster to contain z_i. So there exists a $n_0 \in \mathbb{N}$ such that

$$\mathbb{P}(\tau_i = j) \begin{cases} > 0, & \text{for } j \geq n_0, \\ = 0, & \text{for } j < n_0. \end{cases}$$

Further if we choose $j \geq n_0$, then $\mathcal{E}_{j-1}$ must have some minimal radius if we want it to be possible that the next added point is z_i, similarly argumented as above. So there exists a $m_0 \in \mathbb{N}_0$ such that

$$\mathbb{P}(\mathcal{E}_{j-1} = \mathcal{E}, \tau_i = j) \begin{cases} > 0, & \text{for } \mathcal{E} \in E_m \text{ and } m \geq m_0, \\ = 0, & \text{for } \mathcal{E} \in E_m \text{ and } m < m_0. \end{cases}$$

Having made that clear we can use the law of total probability twice in the following. For $k \in \mathbb{N}_0$ get

$$\mathbb{P}(\sigma_i > k) = \sum_{n_0 \leq j \leq \infty} \mathbb{P}(\tau_i = j)\mathbb{P}(\sigma_i > k \mid \tau_i = j)$$

$$= \mathbb{P}(\tau_i = \infty) + \sum_{n_0 \leq j < \infty} \mathbb{P}(\tau_i = j)\mathbb{P}(\sigma_i > k \mid \tau_i = j),$$

note that $\mathbb{P}(\sigma_i > k \mid \tau_i = \infty) = 1$ since $\{\tau_i = \infty\} \subset \{\sigma_i = \infty\} \subset \{\sigma_i > k\}$. Further for $n_0 \leq j < \infty$ we get

$$\mathbb{P}(\sigma_i > k \mid \tau_i = j) = \sum_{m_0 \leq m} \sum_{\mathcal{E} \in E_m} \mathbb{P}(\tau_i = j \mid \mathcal{E}_{j-1} = \mathcal{E})\mathbb{P}(\sigma_i > k \mid \mathcal{E}_{j-1} = \mathcal{E}, \tau_i = j).$$

We now state that there exists a constant $c_1 > 0$ such that for all $k \in \mathbb{N}$, $n_0 \leq j < \infty$, $m_0 \leq m$ and $\mathcal{E} \in E_m$ we have

$$\mathbb{P}(\sigma_i > k \mid \mathcal{E}_{j-1} = \mathcal{E}, \tau_i = j) \geq (1 - c_1 n^{-\frac{1}{2}})^k$$

which we prove by induction. For shorter expressions write $\gamma_{ij} := \{\mathcal{E}_{j-1} = \mathcal{E}\} \cap \{\tau_i = j\}$. For $k = 1$ by Proposition 4.2.1 there exists a $c > 0$ such that

$$\begin{aligned}
\mathbb{P}(\sigma_i > 1 \mid \gamma_{ij}) &\geq \mathbb{P}(\tau_{j+1} \neq z_{i+1} \mid \gamma_{ij}) \\
&= 1 - h_{\mathcal{E} \cup \{z_i\}}(z_{i+1}) \\
&\geq 1 - cn^{-\frac{1}{2}} \\
&= (1 - cn^{-\frac{1}{2}})^1.
\end{aligned}$$

Note that we can use Proposition 4.2.1 since ... Note that the harmonic measure doesn't care about the order of how points where added to form the current cluster, as mentioned in Remark 3.1.1. Now let the statement be true for some $k = l$. Then again by Proposition 4.2.1 there exists a $c > 0$ such that

$$\begin{aligned}
\mathbb{P}(\sigma_i > l + 1 \mid \gamma_{ij}) &= \mathbb{P}(\sigma_i > l, y_{l+1} \neq z_{i+1} \mid \gamma_{ij}) \\
&= \mathbb{P}(\sigma_i > l \mid \gamma_{ij})\mathbb{P}(y_{l+1} \neq z_{i+1} \mid \gamma_{ij}, \sigma_i > l) \\
&\overset{(+)}{\geq} (1 - c_1 n^{-\frac{1}{2}})^l \mathbb{P}(y_{l+1} \neq z_{i+1} \mid \gamma_{ij}, \sigma_i > l) \\
&\overset{(++)}{\geq} (1 - c_1 n^{-\frac{1}{2}})^l (1 - c_1 n^{-\frac{1}{2}}) \\
&= (1 - c_1 n^{-\frac{1}{2}})^{l+1},
\end{aligned}$$

where in $(+)$ we used the induction assumption. In order to show $(++)$ we would need to split the probability by conditioning on all clusters such that ...
INDUCTION END
Now choose β such that $4e^2\beta c_1 < 1$ und choose $N \in \mathbb{N}$ such that $\beta c_1 \geq 2p_n$ for all $n > N$. If we then define $a := \beta c_1$ and $Y := \sum_{i=1}^{n-1} \sigma_i = \tau_n - \tau_1$ we can use Lemma 4.2.3 and get

$$PROBLEM \mathbb{P}(\tau_n \leq \tilde{m}_n) \leq \mathbb{P}(\tau_n - \tau_1 \leq \tilde{m}_n) = \mathbb{P}(Y \leq \frac{an}{p_n}) \leq (e^2 a)^{n-1} \quad \text{for all } n > N$$

and since $W_n([z]) \subset \{\tau_n \leq \tilde{m}_n\}$ we get

$$\mathbb{P}(W_n([z])) \leq (e^2 a)^n \quad \text{for all } n > N.$$

Counting the elements in Z_n we have less or equal $c_2 n$ points in $\tilde{B}_n$ for some constant $c_2 > 0$ as starting points and 4^{n-1} possibilities for the next $n-1$ steps of a random walk of length n. So $|Z_n| \leq c_2 n 4^{n-1}$ and therefore

$$\sum_{[z]\in Z_n} \mathbb{P}(W_n([z])) \leq c_2 n 4^{n-1}(e^2 a)^{n-1} = c_2 n (4e^2 a)^{n-1} \quad \text{for all } n > N$$

and finally

$$\sum_{n>N} \mathbb{P}(W_n) \leq \sum_{n>N} c_2 n (4e^2 \beta c_1)^{n-1} =: r_\beta.$$

Since β was chosen such that $4e^2\beta c_1 < 1$, r_β is finite and therefore in total we get

$$\sum_{n\in\mathbb{N}} \mathbb{P}(W_n) < \infty,$$

which completes the proof. □

Remark 4.2.1. With this theorem and 3.4 we conclude that the fractal dimension of external DLA is at least $\frac{3}{2}$. In the next chapter we will look at another incremental aggregate which tries to approximate DLA and compare simulations of both to get a empirical comparision of their growth rates.

5 Line Hitting Aggregate

5.1 Motivation

In the following we will look at a process which is the approach of a simple approximation of external DLA on $\mathbb{Z}^2$. The idea is to let particles move on straight lines coming from infinity and add them to the cluster where they hit it. A very similar approach was done by [9] as a so called ballistic model, where they let particles move on straight lines to the cluster, but just from a small amount of directions. Our approach here extends this idea, as we will try to choose a random direction out of all directions in the plane in a most senseful fair way. Anyway we will see, that empirically we find similar results for the fractal dimension of this cluster as in [9].

Obviously in most cases particles cannot move completely straight on $\mathbb{Z}^2$. Therefore we will consider points in $\mathbb{Z}^2$ as the centers of unit squares in $\mathbb{R}^2$ and let the particles move on straight lines in the full plane $\mathbb{R}^2$. We consider a line hitting a point in $\mathbb{Z}^2$ if and only if it intersects with its unit square as defined in the following. Note that we identify $\mathbb{R}^2$ with $\mathbb{C}$ here.

Definition 5.1.1. Define

$$\mathbb{C}_{sq} := \{[k-\frac{1}{2}, k+\frac{1}{2}] + [l-\frac{1}{2}, l+\frac{1}{2}]i \subset \mathbb{C} \mid k, l \in \mathbb{Z}\}, \tag{5.1}$$

note that $\mathbb{C} = \bigcup_{s \in \mathbb{C}_{sq}} s$. The canonical function

$$sq : \mathbb{Z}^2 \to \mathbb{C}_{sq}, \quad (k,l) \to [k-\frac{1}{2}, k+\frac{1}{2}] + [l-\frac{1}{2}, l+\frac{1}{2}]i \tag{5.2}$$

is bijective and intuitively identifies points in $\mathbb{Z}^2$ with squares in $\mathbb{C}$ which is p is the center of the square $sq(p)$ for all $p \in \mathbb{Z}^2$. In the following when using a point $p \in \mathbb{Z}^2$ it will reference the point in $\mathbb{Z}^2$ or the corresponding square in $\mathbb{C}$ respecting the context. This bijection also naturally defines a graph structure on $\mathbb{C}_{sq}$, which is two squares $s_1, s_2 \in \mathbb{C}_{sq}$ form an edge if and only if $sq^{-1}(s_1)$ and $sq^{-1}(s_2)$ form an edge in $\mathbb{Z}^2$. For the following we say a line g *hits* a point $p \in \mathbb{Z}^2$ if and only if $g \cap sq(p) \neq \emptyset$ (see in *Figure* 2).

Definition 5.1.2. Let $g = g_{\alpha,p} \in \mathcal{G}$ and $A \in \mathcal{P}_f$. We define

$$g \cap A := \{p \in A \mid g \text{ hits } p\}$$

which is the subset of all points in A which are hit by g (see *Figure* 3). For the following we suppose $g \cap A \neq \emptyset$. We will define a total ordered relation $\lhd$ on $g \cap A$ which shall be defined equivalently for all $g \in \mathcal{G}$ and $A \in \mathcal{P}_f$ with $g \cap A \neq \emptyset$. We choose two points $x, y \in g \cap A$ and split the definition of the relation $\lhd$ into four cases, depending on

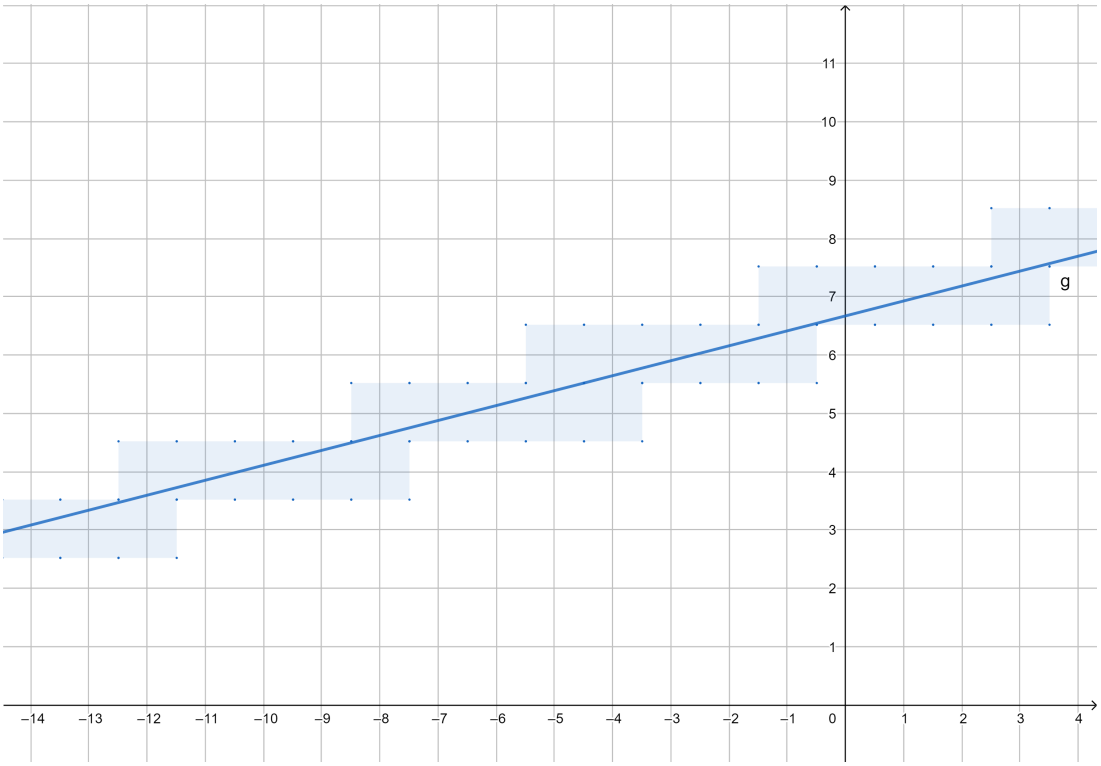

Figure 2: g hits squares around points

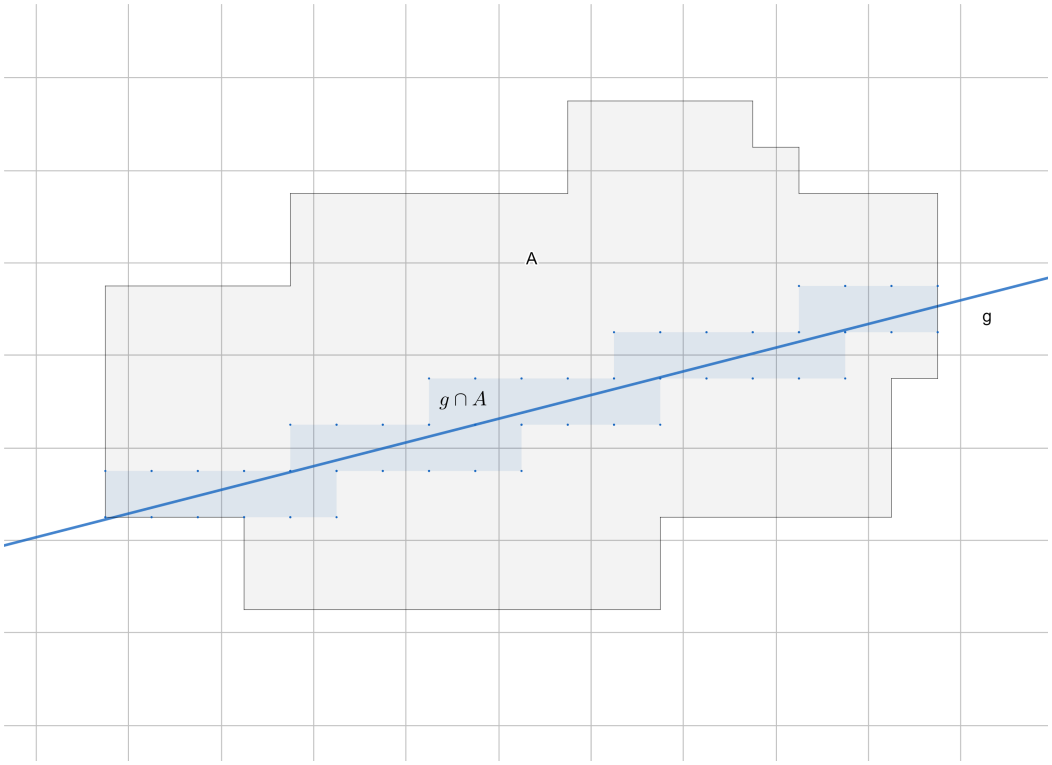

Figure 3: g hits squares in A

whether the line g goes from left-bottom to right-top, left-top to right-bottom, parallel to the x-axis and parallel to the y-axis. Denote the real part of x with $\mathrm{Re}(x)$ and the imaginary one with $\mathrm{Im}(x)$.

Case 1 : g is parallel to the x-axis $(\Leftrightarrow \quad \alpha = \frac{\pi}{2})$

$$x \lhd y \quad :\Leftrightarrow \quad \mathrm{Re}(x) < \mathrm{Re}(y)$$

Case 2 : g is parallel to the y-axis $(\Leftrightarrow \quad \alpha = 0)$

$$x \lhd y \quad :\Leftrightarrow \quad \mathrm{Im}(x) < \mathrm{Im}(y)$$

Case 3 : g is going from left-bottom to right-top $(\Leftrightarrow \quad \alpha \in (\frac{\pi}{2}, \pi))$

$$x \lhd y \quad :\Leftrightarrow \quad \begin{cases} \mathrm{Re}(x) < \mathrm{Re}(y), & \text{if } \mathrm{Re}(x) \neq \mathrm{Re}(y), \\ \mathrm{Im}(x) < \mathrm{Im}(y), & \text{if } \mathrm{Re}(x) = \mathrm{Re}(y). \end{cases}$$

Case 4 : g is going from left-top to right-bottom $(\Leftrightarrow \quad \alpha \in (0, \frac{\pi}{2}))$

$$x \lhd y \quad :\Leftrightarrow \quad \begin{cases} \mathrm{Re}(x) < \mathrm{Re}(y), & \text{if } \mathrm{Re}(x) \neq \mathrm{Re}(y), \\ \mathrm{Im}(x) > \mathrm{Im}(y), & \text{if } \mathrm{Re}(x) = \mathrm{Re}(y). \end{cases}$$

It is easy to see that this relation on $g \cap A$ is well-defined. In the following we will prove that this relation is strictly and totally ordered.

Lemma 5.1.1. For a line $g = g_{\alpha,p} \in \mathcal{G}$ and $A \in \mathcal{P}_f$ with $g \cap A \neq \emptyset$ the relation $\lhd$ on $g \cap A$ is totally ordered.

Proof. We will only proove the case where g is going from left-bottom to right-top, which is *Case* 3 of the definition. In this case we have $\alpha \in (\frac{\pi}{2}, \pi)$. Note, that the proof for *Case* 4 will work very similar and in the case of g being parallel to one of the axes (*Case* 1 or 2), all properties for a totally ordered relation follow directly from the totally ordered relation $<$ on $\mathbb{R}$. So let $\alpha \in (\frac{\pi}{2}, \pi)$.

Antisymmetry : For antisymmetry let $x \lhd y$ and $y \lhd x$. Suppose $\mathrm{Re}(x) \neq \mathrm{Re}(y)$, then $\mathrm{Re}(x) < \mathrm{Re}(y)$ and $\mathrm{Re}(y) < \mathrm{Re}(x)$, a contradiction because of the total order $<$ in $\mathbb{R}$. So $\mathrm{Re}(x) = \mathrm{Re}(y)$. But then we have $\mathrm{Im}(x) < \mathrm{Im}(y)$ and $\mathrm{Im}(y) < \mathrm{Im}(x)$ and therefore also $\mathrm{Im}(x) = \mathrm{Im}(y)$, hence $x = y$.

Transitivity : For transitivity let $x \lhd y$ and $y \lhd z$. We find four cases. In case $\mathrm{Re}(x) \neq \mathrm{Re}(y)$ and $\mathrm{Re}(y) \neq \mathrm{Re}(z)$ we get $\mathrm{Re}(x) < \mathrm{Re}(z)$ by transitivity of $<$, hence $x \lhd z$. In case $\mathrm{Re}(x) \neq \mathrm{Re}(y)$ and $\mathrm{Re}(y) = \mathrm{Re}(z)$ we get $\mathrm{Re}(x) < \mathrm{Re}(y) = \mathrm{Re}(z)$, therefore $x \lhd z$. In case $\mathrm{Re}(x) = \mathrm{Re}(y)$ and $\mathrm{Re}(y) \neq \mathrm{Re}(z)$ we get $\mathrm{Re}(x) = \mathrm{Re}(y) < \mathrm{Re}(z)$, similar

as the last case. In the last case $\mathrm{Re}(x) = \mathrm{Re}(y) = \mathrm{Re}(z)$ we get $\mathrm{Im}(x) < \mathrm{Im}(y)$ and $\mathrm{Im}(y) < \mathrm{Im}(z)$ and again by transitivity of $<$ we get $\mathrm{Im}(x) < \mathrm{Im}(z)$, hence $x \lhd z$ again.

Connexity : Connexity is given since for any two points $x, y \in g \cap A$ we have either $\mathrm{Re}(x) \neq \mathrm{Re}(y)$ or $\mathrm{Re}(x) = \mathrm{Re}(y)$ and therefore either $x \lhd y$ or $y \lhd x$.

□

Remark 5.1.1. The relation $\lhd$ on $g \cap A$ basically orders the hitting points of g with A from left to right (or bottom to top in case of a line parallel to the y-axis). This order allows us to identify the outermost hitting points which are the minimum and maximum of $g \cap A$ with respect to $\lhd$. To clarify, we define $\min(g \cap A) := x_0$ if and only if $x_0 \lhd x$ for all $x \in g \cap A, x \neq x_0$, analogously $\max(g \cap A)$. This means when moving on g facing A coming from infinity this order allows to know where in A the line g hits first when „entering“ A and where it hits last when „leaving“ A. What we want to do next is to choose a line randomly out of all lines which hit the current cluster. This is isn‘t an obvious task and we will have to develop a most fair underlying distribution on lines in the next section where we will touch basic aspects of Integral Geometry.

5.2 Integral Geometry

In the next section we want to define an approximation for External DLA. This approximation will be an incremental aggregate for which distribution definition we need some concepts and results from Integral Geometry which we will discuss and develop in this section. In the process we want to define we will want to choose a random line out of all lines which intersect with the current cluster of the aggregate. This random choosing is not obvious since most of the time the cluster will be strongly non symmetric and it is even less obvious how to actually get a realization of a random line when simulating with Python. In our case we are looking for a parametrisation of lines in the plane and a reasonable way of choosing parameters randomly.

We will introduce a possible solution for this problem first through the abstract and general concepts of integral geometry and later through a simple parametrisation for the case of lines in the plane which goes hand in hand with the general result.

5.2.1 General results

In the general context we are in $\mathbb{R}^d$ for $d \in \mathbb{N}$ and consider q-dimensional affine subspaces where $q \in \{0, \ldots, d\}$, short q-flats in $\mathbb{R}^d$. The set of q-flats in $\mathbb{R}^d$ is denoted by $A(d, q)$. Later we will be interested in choosing random lines in the real plane (i.e. 1-flats in $\mathbb{R}^2$). In order to get a probability measure on some set of q-flats, we first need a measure and a σ-algebra on $A(d, q)$ in total.

Definition 5.2.1. For $B \in \mathcal{B}^d$ define

$$[B]_{d,q} := \{F \in A(d, q) \mid F \cap B \neq \emptyset\}.$$

If the context is clear, we will only write $[B]$ instead of $[B]_{d,q}$.

Definition 5.2.2. The σ-algebra $\mathcal{A}(d,q)$ on $A(d,q)$ is defined by

$$\mathcal{A}(d,q) := \sigma(\{[K] \mid K \in \mathcal{K}^d\}).$$

Theorem 5.2.1. On $A(d,q)$ there exists a unique G_d-invariant Radon measure μ_q such that

$$\mu_q(A_{B_d(1,0)}) = \kappa_{d-q}, \tag{5.1}$$

where $\kappa_n := \lambda_n(B_n(1,0))$ is the n-dimensional Lebesque meausure of the n-dimensional unit ball for $n \in \mathbb{N}$, and $\kappa_0 := 1$.

Proof. [1] Theorem 4.26

MAKE PROOF CLEARER □

5.2.2 Construction in the plane: Isotropic lines

For our special case we choose $d = 2$ and $q = 1$, thus lines in the plane. We denote this set of lines by $\mathcal{G}$. The following construction in this chapter is completely motivated by [2] 2.1.1. Firstly we propose a parametrisation of lines which works as follows. Every line can be uniquely determined by an angle $\alpha \in [0, \pi)$ and a real number $p \in \mathbb{R}$. Let $\langle \cdot, \cdot \rangle$ be the standard scalar product on $\mathbb{R}^2$, respectively used for values in $\mathbb{C}$ as we identify $\mathbb{R}^2$ with $\mathbb{C}$ as $\mathbb{R}$-vectorspaces. Let

$$e_\alpha := e^{\alpha i} = \cos(\alpha) + \sin(\alpha)i$$

and

$$s_\alpha := -\sin(\alpha) + \cos(\alpha)i$$

be the unit vectors 1 and i rotated by α counterclockwise. Lets consider the representation $x = \langle x, e_\alpha \rangle e_\alpha + \langle x, s_\alpha \rangle s_\alpha$ for $x \in \mathbb{C}$. Since e_α and s_α form a base of $\mathbb{C}$ as a $\mathbb{R}$-vectorspace, the parameters $\langle x, e_\alpha \rangle$ and $\langle x, s_\alpha \rangle$ are unique for each x. It thus is easy to realize that $g_{\alpha,p} := \{x \in \mathbb{C} \mid \langle x, e_\alpha \rangle = p\}$ defines a line (compare with *Figure* 4) and that every line has a unique pair of α and p for such a representation. In words, $g_{\alpha,p}$ contains all points which have length p in direction of e_α. With $\Phi := [0, \pi) \times \mathbb{R}$ this naturally defines a bijection

$$\chi : \Phi \to \mathcal{G}, \quad (\alpha, p) \mapsto g_{\alpha,p}.$$

We take the subspace Borel-σ-algebra $\mathcal{B}_\Phi := \mathcal{B}^2 \cap \Phi$ on Φ and define the σ-algebra $\mathfrak{G}$ on $\mathcal{G}$ by $\mathfrak{G} := \chi(\mathcal{B}_\Phi)$. This works well since χ is a bijection. We want to show in the following that this way of defining a σ-algebra on $\mathcal{G}$ makes sense as it is indeed equivalent to the general context as defined above. To do that it is convenient to use a special generator set for the Borel-σ-algebra $\mathcal{B}_\Phi$.

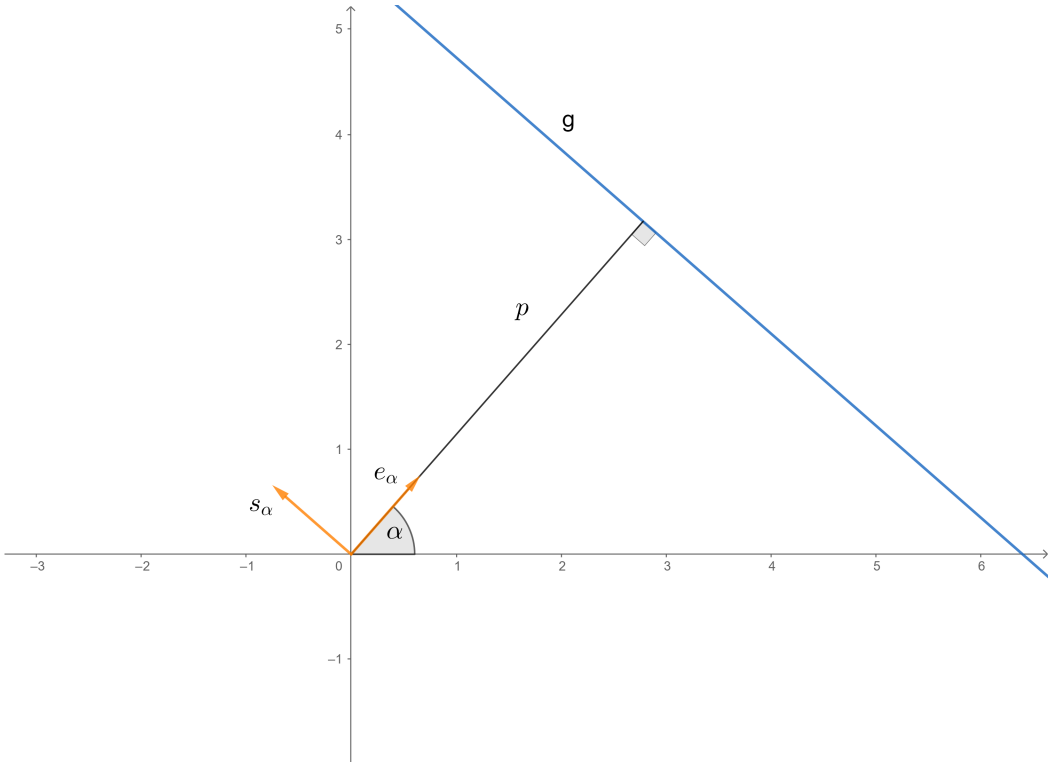

Figure 4: Line parameters α and p

Lemma 5.2.1. Define

$$\mathcal{R}_+ := \{[\alpha, \beta] \times (0, b] \mid 0 \leq \alpha < \beta < \pi, b \geq 0\},$$

$$\mathcal{R}_- := \{[\alpha, \beta] \times [b, 0) \mid 0 \leq \alpha < \beta < \pi, b \leq 0\},$$

$$\mathcal{R}_0 := \{[\alpha, \beta] \times \{0\} \mid 0 \leq \alpha < \beta < \pi\},$$

and

$$\mathcal{R} := \mathcal{R}_+ \cup \mathcal{R}_- \cup \mathcal{R}_0.$$

Then $\sigma(\mathcal{R}) = \mathcal{B}_\Phi$.

Proof. We show that $\sigma(\mathcal{R})$ contains all rectangles in Φ of the form $[\alpha, \beta] \times (a, b]$ with $a, b > 0$, $[\alpha, \beta] \times [a, b)$ with $a, b < 0$ and $[\alpha, \beta] \times [a, b]$ with $0 \in [a, b]$. First let $a, b > 0$ and $R = [\alpha, \beta] \times (a, b]$, then

$$R = ([\alpha, \beta] \times (0, b]) \setminus ([\alpha, \beta] \times (0, a])$$

and therefore $R \in \sigma(\mathcal{R}_+) \subset \sigma(\mathcal{R})$. Similarly it works if $a, b < 0$. If $0 \in [a, b]$ then we can write $R = [\alpha, \beta] \times [a, b]$ with three components $R = [\alpha, \beta] \times [a, 0) \cup [\alpha, \beta] \times \{0\} \cup [\alpha, \beta] \times (0, b]$ which lie in $\mathcal{R}_-, \mathcal{R}_0$ and $\mathcal{R}_+$ respectively. Therefore $R \in \sigma(\mathcal{R})$ aswell. By measure theory the above described rectangles form a generator set of $\mathcal{B}_\Phi$, which completes the proof. □

Lemma 5.2.2. We have $\mathcal{A}(2,1) = \mathfrak{G}$.

Proof. We will consider generators of these σ-algebras. By Lemma 5.2.1 we know that $\sigma(\mathcal{R}) = \mathcal{B}_\Phi$ and since χ is a bijection, we have $\chi(\sigma(\mathcal{R})) = \sigma(\chi(\mathcal{R}))$ and finally $\mathfrak{G} = \sigma(\chi(\mathcal{R}))$. For $\tilde{A} := \{[K] \mid K \in \mathcal{K}^2\}$ we have by definition $\mathcal{A}(2,1) = \sigma(\tilde{A})$.

$\subset$: Let $K \in \mathcal{K}^2$. We will show that $\chi^{-1}([K])$ is a closed set in Φ. If that is the case we have $\chi^{-1}([K]) \in \mathcal{B}_\Phi$, therefore $[K] \in \chi(\mathcal{B}_\Phi) = \mathfrak{G}$ and finally $\mathcal{A}(2,1) = \sigma(\tilde{A}) \subset \mathfrak{G}$. To show that $\chi^{-1}([K])$ is closed let $(\alpha_0, p_0) \in \Phi \setminus \chi^{-1}([K])$. Then $\chi(\alpha_0, p_0) \notin [K]$ and therefore $\chi(\alpha_0, p_0) \cap K = \emptyset$. Since K is closed we can find small values $\tilde{\alpha}, \tilde{p} > 0$ such that $\chi(\alpha, p) \cap K = \emptyset$ for all $(\alpha, p) \in [\alpha_0, \alpha_0 + \tilde{\alpha}] \times [p_0, p_0 + \tilde{p}] =: R$. Hence we have $R \subset \Phi \setminus \chi^{-1}([K])$, so $\Phi \setminus \chi^{-1}([K])$ is open. Hence $\chi^{-1}([K])$ is closed.

$\supset$: For this inclusion we will show that $\chi(R) \in \mathcal{A}(2,1)$ for all $R \in \mathcal{R}$. First let $R = [\alpha, \beta] \times (0, b] \in \mathcal{R}_+$ for some $b > 0$. Define

$$S := \{pe_\gamma \in \mathbb{C} \mid (\gamma, p) \in R\}$$

Furthermore for $n \in \mathbb{N}$ define

$$A_n := \{tns_\beta \mid t \in [0,1]\} \text{ and } B_n := \{-tns_\alpha \mid t \in [0,1]\},$$

the segments from 0 to ns_β and 0 to $-ns_\alpha$ (*Figure* 5). We will show now that

$$\chi(R) = [\bar{S}] \setminus (\bigcup_{n\in\mathbb{N}} [A_n] \cup \bigcup_{n\in\mathbb{N}} [B_n]) =: \tilde{S},$$

where $\bar{S}$ is the closure of S (note that $\bar{S} = S \cup \{0\}$). Let $(\gamma, p) \in R$. Then $pe_\gamma \in \chi(\gamma, p) \cap S$ and therefore $\chi(\gamma, p) \in [\bar{S}]$. Assume that there exits an $n \in \mathbb{N}$ such that $\chi(\gamma, p) \cap A_n \neq \emptyset$. Then $\beta + \frac{\pi}{2} - \gamma < \frac{\pi}{2}$, hence $\beta < \gamma$, a contradiction. Similarly argument for any B_n, so we finally have $\chi(\gamma, p) \notin [A_n]$ and $\chi(\gamma, p) \notin [B_n]$ for any $n \in \mathbb{N}$. Hence $\chi(\gamma, p) \in \tilde{S}$ and therefore $\chi(R) \subset \tilde{S}$.

Now let $(\gamma, p) \in \Phi$ such that $\chi(\gamma, p) \in \tilde{S}$. Assume that $\gamma \notin [\alpha, \beta]$ then with a similar argument as in the first inclusion it is easy to see that there must be an $n \in \mathbb{N}$ such that $\chi(\gamma, p) \cap A_n \neq \emptyset$ or $\chi(\gamma, p) \cap B_n \neq \emptyset$, a contradiction. Therefore $\gamma \in [\alpha, \beta]$. Now assume $p \notin (0, b]$. If $p > b$ then $\chi(\gamma, b) \cap B_b = \emptyset$ and since $\bar{S} \subset B_b$ it is $\chi(\gamma, p) \notin [\bar{S}]$, a contradiction. If $p < 0$ then, since the angle between the segments A_n and B_n opposite of S is strictly smaller than π, $\chi(\gamma, p)$ must intersect with A_n or B_n for some $n \in \mathbb{N}$, again a contradiction. Note that $p \neq 0$ since $[\{0\}] \subset [A_1]$. Thus we have $p \in (0, b]$. Therefore we have $(\gamma, p) \in R$ and finally $\tilde{S} \subset \chi(R)$.

It is left to show that $\tilde{S} \in \mathcal{A}(2,1)$. All the segments A_n and B_n are compact and convex for all $n \in \mathbb{N}$, and since S is bounded and convex as a circle segment with angle smaller than π, $\bar{S}$ is compact and convex. Finally $\tilde{S} \in \mathcal{A}(2,1)$.

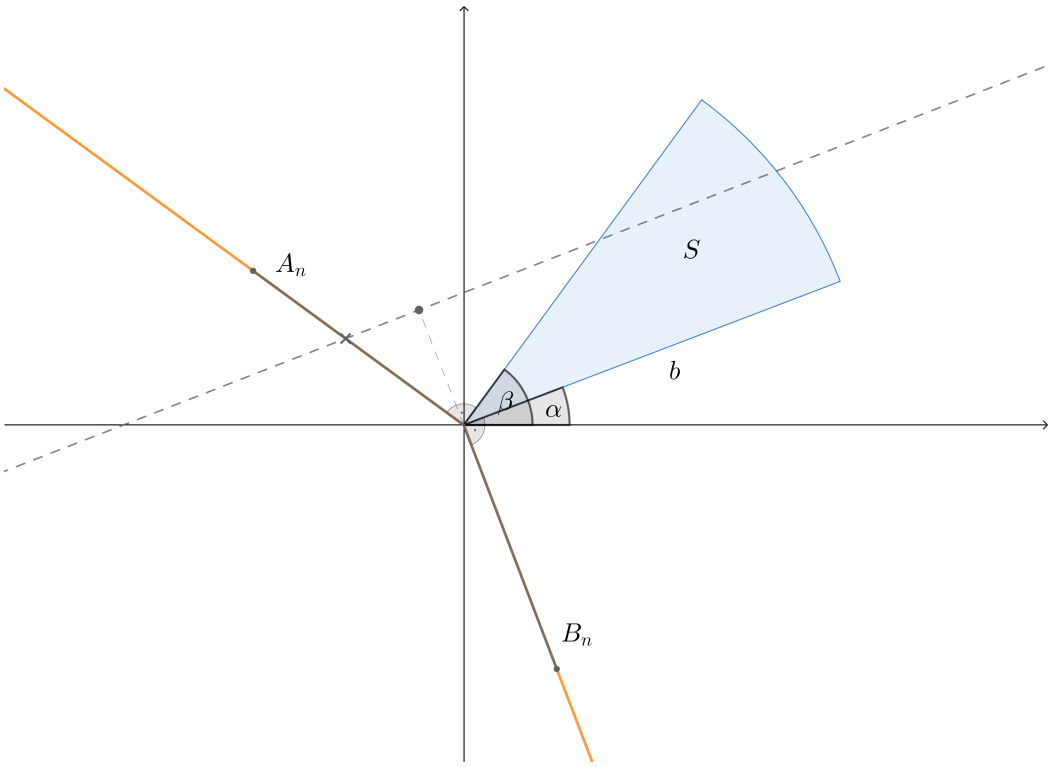

Figure 5: S and the sets A_n and B_n

In total we get $\mathcal{R}_+ \subset \mathcal{A}(2,1)$. $\mathcal{R}_- \subset \mathcal{A}(2,1)$ can be shown analogously. So for the last case let $R = [\alpha, \beta] \times \{0\} \in \mathcal{R}_0$. Define the line segment

$$T := \{(1-t)s_\alpha + ts_\beta \mid t \in [0,1]\}.$$

Then $[\{0\}] \cap [T] \in \mathcal{A}(2,1)$. We show that $\chi(R) = [\{0\}] \cap [T]$. Let $(\gamma, 0) \in R$. Then $\chi(\gamma, 0)$ contains 0 and since γ is in between α and β, $\chi(\gamma, 0)$ must intersect with T (*Figure* 6). For the other inclusion let $(\gamma, p) \in \Phi$ such that $\chi(\gamma, p) \in [\{0\}] \cap [T]$. Since $0 \in \chi(\gamma, p)$ it must be $p = 0$ and since it intersects with T its angle must lay in between α and β. All in all this completes the proof. □

Definition 5.2.3. A $\mathcal{F}$-$\mathfrak{G}$-measurable function $g : \Omega \to \mathcal{G}$ is called a *random line*.

Definition 5.2.4. We define the measure $\mu := \lambda_{2|\Phi} \circ \chi^{-1}$ on $(\mathcal{G}, \mathfrak{G})$ where $\lambda_{2|\Phi}$ is the 2-dimensional Lebesgue measure restricted to Φ. We say a measure ν on $(\mathcal{G}, \mathfrak{G})$ is locally finite if for any $K \in \mathcal{K}^2$ we have $\nu([K]) < \infty$.

Lemma 5.2.3. μ is locally finite and G_2-invariant.

Proof. Let $K \in \mathcal{K}^2$ and since K is compact choose $r > 0$ such that $K \subset B_r$. Then we have $[K] \subset A_{B_r}$ and

$$\mu([K]) \leq \mu(A_{B_r}) = \lambda_{2|\Phi}(\chi^{-1}(A_{B_r})) = \lambda_{2|\Phi}([0,\pi) \times [-r, r]) = 2\pi r < \infty,$$

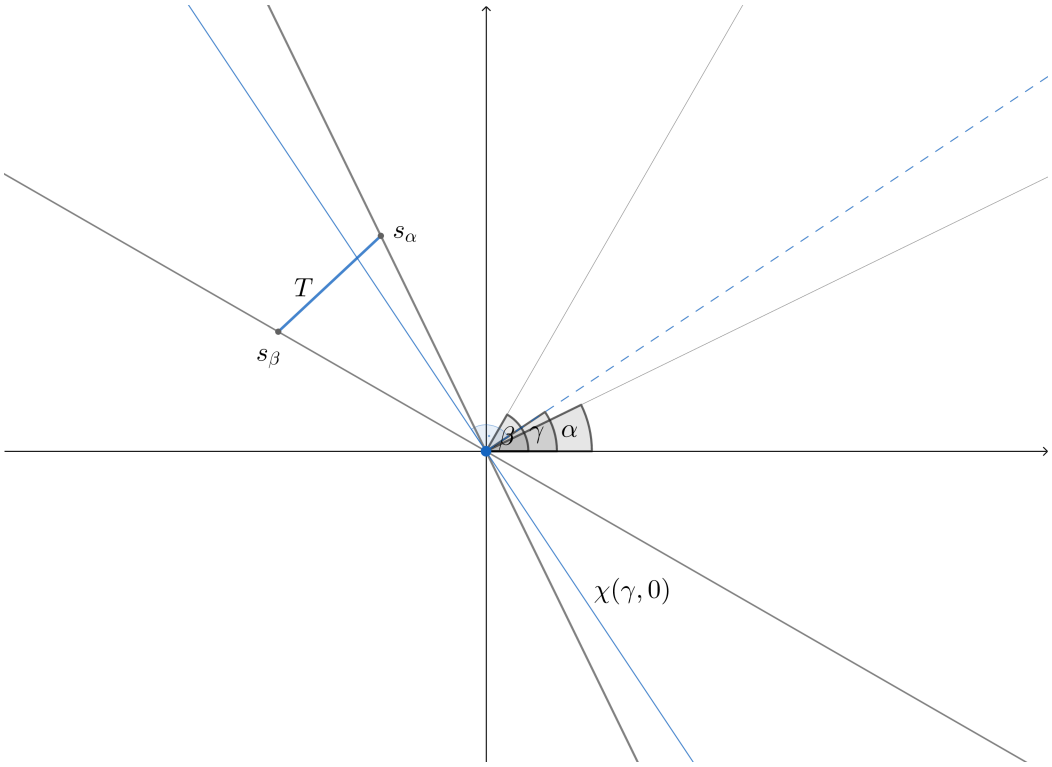

Figure 6: Line in $[\{0\}] \cap [T]$

hence μ is locally finite. To show that μ is G_2-invariant, that means euclidean motion invariant, we must show it is translation and rotation invariant. First we clarify what exactly translation and rotation mean for lines. We denote x *modulo* r as $(x)_r$. For $b \in \mathbb{C}$ and $\beta \in [0, 2\pi)$ we define

$$T_b : \Phi \to \Phi, \quad (\alpha, p) \mapsto (\alpha, p + \langle e_\alpha, b \rangle) \tag{5.2}$$

and

$$D_\beta : \Phi \to \Phi, \quad (\alpha, p) \mapsto ((\alpha + \beta)_\pi, \delta((\alpha + \beta)_{2\pi})p), \tag{5.3}$$

where

$$\delta : [0, 2\pi) \to \{-1, 1\}, \quad \gamma \to \begin{cases} 1, & \gamma \in [0, \pi) \\ -1, & \gamma \in [\pi, 2\pi) \end{cases}.$$

It is easy to see that both functions all well-defined. T_b defines a translation by b and D_β a rotation by β. Lets proof that first. Let $(\alpha, p) \in \Phi$, then

$$\begin{aligned} x \in \chi(\alpha, p) + b &\Leftrightarrow x - b \in \chi(\alpha, p) \\ &\Leftrightarrow \langle e_\alpha, x - b \rangle = p \\ &\Leftrightarrow \langle e_\alpha, x \rangle = p + \langle e_\alpha, b \rangle \end{aligned}$$

$$\begin{aligned}&\Leftrightarrow x \in \chi(\alpha, p + \langle e_\alpha, b\rangle)\\ &\Leftrightarrow x \in \chi(T_b(\alpha, p))\end{aligned}$$

and therefore $\chi(\alpha,p)+b = \chi(T_b(\alpha,p))$. Hence $T_b(\alpha,p)$ are indeed the parameters for the by b translated line. For the rotation lets devide it into two cases. First let $(\alpha+\beta)_{2\pi} \in [0,\pi)$, then $\delta((\alpha+\beta)_{2\pi}) = 1$ and $(\alpha+\beta)_\pi = \alpha+\beta$ and therefore

$$D_\beta(\alpha,p) = (\alpha+\beta, p).$$

In the second case with $(\alpha+\beta)_{2\pi} \in [\pi, 2\pi)$ we have $\delta((\alpha+\beta)_{2\pi}) = -1$ and $(\alpha+\beta)_\pi = \alpha+\beta-\pi$ and therefore

$$D_\beta(\alpha,p) = (\alpha+\beta-\pi, -p).$$

In the second case we have to carefully understand the parametrisation of $\mathcal{G}$, but finally we can see that $D_\beta(\alpha,p)$ are indeed the parameters of the by β rotated line.

We will further show now, that μ is invariant in respect to both these functions. Let $A \in \mathfrak{G}$, $b \in \mathbb{C}$ and $\nu_\beta \in SO_2$ for some $\beta \in [0, 2\pi)$. We will understand $A+b = \{g+b \in \mathcal{G} \mid g \in A\}$ and $\nu_\beta A = \{\nu_\beta g \in \mathcal{G} \mid g \in A\}$ pointwise, and $g+b = \{x+b \mid x \in g\}$ and $\nu_\beta g = \{\nu_\beta x \mid x \in g\}$ pointwise aswell. We furthermore define $A_p := \{\alpha \in [0,\pi) \mid (\alpha,p) \in \chi^{-1}(A)\}$ and $A_\alpha := \{p \in \mathbb{R} \mid (\alpha,p) \in \chi^{-1}(A)\}$ for $(\alpha,p) \in \Phi$. For a translation we get

$$\begin{aligned}\mu(A+b) &= \int_{\mathfrak{G}} \mathbb{1}_{A+b}(g)\mu(dg)\\ &= \int_{\mathfrak{G}} \mathbb{1}_{A}(g-b)\mu(dg)\\ &= \int_{\mathfrak{G}} \mathbb{1}_{A}(g-b)\lambda_{2|\Phi}(\chi^{-1}(dg))\\ &= \int_{\chi^{-1}(\mathfrak{G})} \mathbb{1}_{\chi^{-1}(A)}(\chi^{-1}(g-b))\lambda_{2|\Phi}(d(\chi^{-1}(g)))\\ &\overset{(5.2)}{=} \int_{\Phi} \mathbb{1}_{\chi^{-1}(A)}(\alpha, p-\langle e_\alpha, b\rangle)\lambda_{2|\Phi}(d(\alpha,p))\\ &= \int_0^\pi \int_{\mathbb{R}} \mathbb{1}_{A_\alpha}(p-\langle e_\alpha,b\rangle)\lambda_1(dp)\lambda_{1|[0,\pi)}(d\alpha)\\ &= \int_0^\pi \int_{\mathbb{R}} \mathbb{1}_{A_\alpha+\langle e_\alpha,b\rangle}(p)\lambda_1(dp)\lambda_{1|[0,\pi)}(d\alpha)\\ &= \int_0^\pi \lambda_1(A_\alpha+\langle e_\alpha,b\rangle)\lambda_{1|[0,\pi)}(d\alpha)\\ &\overset{(+)}{=} \int_0^\pi \lambda_1(A_\alpha)\lambda_{1|[0,\pi)}(d\alpha)\\ &= \dots\\ &= \mu(A),\end{aligned}$$

and for a rotation we get

$$\mu(\nu_\beta A) = \int_{\mathcal{G}} \mathbb{1}_{\nu_\beta A}(g)\mu(dg)$$

$$
\begin{aligned}
&= \int_{\mathcal{G}} \mathbb{1}_A(\nu_{-\beta} g)\lambda_{2|\Phi}(\chi^{-1}(dg)) \\
&= \int_{\chi^{-1}(\mathcal{G})} \mathbb{1}_{\chi^{-1}(A)}(\chi^{-1}(\nu_{-\beta} g))\lambda_{2|\Phi}(d(\chi^{-1}(g))) \\
&\overset{(5.3)}{=} \int_{\Phi} \mathbb{1}_{\chi^{-1}(A)}((\alpha-\beta)_\pi, \delta((\alpha-\beta)_{2\pi})p)\lambda_{2|\Phi}(\alpha, p) \\
&= \int_{\Phi} \mathbb{1}_{\chi^{-1}(A)}((\alpha-\beta)_\pi, \delta((\alpha-\beta)_{2\pi})p)\lambda_{2|\Phi}(\alpha, p) \\
&= \int_0^{2\pi} \int_{\mathbb{R}} \mathbb{1}_{\chi^{-1}(A)}((\alpha-\beta)_\pi, \delta((\alpha-\beta)_{2\pi})p)\lambda_1(dp)\lambda_{1|[0,\pi)}(d\alpha) \\
&= \int_0^{2\pi} \int_{\mathbb{R}} \mathbb{1}_{A_{(\alpha-\beta)_\pi}}(\delta((\alpha-\beta)_{2\pi})p)\lambda_1(dp)\lambda_{1|[0,\pi)}(d\alpha) \\
&\overset{(+)}{=} \int_0^{2\pi} \int_{\mathbb{R}} \mathbb{1}_{A_{(\alpha-\beta)_\pi}}(p)\lambda_1(dp)\lambda_{1|[0,\pi)}(d\alpha) \\
&= \int_{\mathbb{R}} \int_0^{2\pi} \mathbb{1}_{\chi^{-1}(A)}((\alpha-\beta)_\pi, p)\lambda_{1|[0,\pi)}(d\alpha)\lambda_1(dp) \\
&= \int_{\mathbb{R}} \int_0^{2\pi} \mathbb{1}_{A_p}((\alpha-\beta)_\pi)\lambda_{1|[0,\pi)}(d\alpha)\lambda_1(dp) \\
&\overset{(+)}{=} \int_{\mathbb{R}} \int_0^{2\pi} \mathbb{1}_{A_p}(\alpha)\lambda_{1|[0,\pi)}(d\alpha)\lambda_1(dp) \\
&= \int_{\mathbb{R}} \int_0^{2\pi} \mathbb{1}_{\chi^{-1}(A)}(\alpha, p)\lambda_{1|[0,\pi)}(d\alpha)\lambda_1(dp) \\
&= \dots \\
&= \mu(A).
\end{aligned}
$$

where in (+) we used the translation and rotation invariance of the Lebesgue measure. This completes the proof. □

By 5.2.1 we know that μ is, up to a factor, the only euclidean motion invariant measure on $\mathcal{G}$. Since it is locally finite, for $K \in \mathcal{K}^2$ we can define a probability measure on $\mathcal{G}$ by

$$\mathbb{P}_\mu^K(A) := \frac{\mu(A \cap [K])}{\mu([K])}, \quad A \in \mathfrak{G}.$$

Definition 5.2.5. Let $K \in \mathcal{K}^2$. A random line $g : \Omega \to \mathcal{G}$ is called *K-isotropic* if

$$\mathbb{P}(g \in A) = \mathbb{P}_\mu^K(A), \quad A \in \mathfrak{G}.$$

Lemma 5.2.4. Let $M, K \in \mathcal{K}^2$ with $M \subset K$. Let f be a random K-isotropic and g be a random M-isotropic line. Then for all $A \in \mathfrak{G}$ we have

$$\mathbb{P}(f \in A \mid f \in [M]) = \mathbb{P}(g \in A).$$

Proof. Note that since $M \subset K$ it is $[M] \subset [K]$. For $A \in \mathfrak{G}$ we therefore directly get

$$\mathbb{P}(f \in A \mid f \in [M]) = \frac{\mathbb{P}(f \in A \cap [M])}{\mathbb{P}(f \in [M])}$$

$$
\begin{aligned}
&= \frac{\mu(A \cap [M] \cap [K])}{\mu([K])} \frac{\mu([K])}{\mu([M] \cap [K])} \\
&= \frac{\mu(A \cap [M])}{\mu([M])} \\
&= \mathbb{P}(g \in A).
\end{aligned}
$$

□

If we choose a simple convex set such as $K = B_r$ the ball around the origin with radius r then choosing random K-isotropic lines becomes a very intuitive and easy realizable task as the following lemma shows.

Lemma 5.2.5. Let $K = B_r \in \mathcal{K}^2$ and let (α, p) be uniformly distributed in $\tilde{\Phi} := [0, \pi) \times [-r, r] = \chi^{-1}([K]) \subset \Phi$. Then $\chi(\alpha, p)$ is a random K-isotropic line.

Proof. For $A \in \mathfrak{G}$ we get

$$
\begin{aligned}
\mathbb{P}(\chi(\alpha, p) \in A) = \mathbb{P}((\alpha, p) \in \chi^{-1}(A)) &= \frac{\lambda_{2|\tilde{\Phi}}(\chi^{-1}(A) \cap \chi^{-1}([K])))}{\lambda_{2|\tilde{\Phi}}(\chi^{-1}([K]))} \\
&= \frac{\lambda_{2|\Phi}(\chi^{-1}(A \cap [K]))}{\lambda_{2|\Phi}(\chi^{-1}([K]))} = \frac{\mu(A \cap [K])}{\mu([K])}.
\end{aligned}
$$

□

Remark 5.2.1. Both lemmas 5.2.4 and 5.2.5 give a help for realizing K-isotropic lines for complicated sets K. Lemma 5.2.4 tells us that we if we are looking for a K-isotropic line, we can actually take a convex, compact set B which contains K and realize B-isotropic lines. If we realize such a line and it happens that it intersects K, we know that its distribution is equal to trying to realize K-isotropic lines directly. And how to realize B-isotropic lines? Lemma 5.2.5 tells us that if we choose $B = B_r$ a ball with a big enough radius such that it contains K, then realizing B-isotropic lines comes by choosing the line parameters α and p uniformly in $[0, \pi)$ and $[-r, r]$. Finally we have a practicable process of choosing random K-isotropic lines, even if K happens to be very asymmetric and complicated. This gives the base to define a new Incremental Aggregate in the next section which tries to approximate external DLA.

5.3 Definition

We are now able to choose a line randomly out of all lines hitting a cluster in a fair and agreeable way, which are K-isotropic lines as presented in Definition 5.2.5. All in all we have realized the mathematical structure to define the incremental aggregate as we have planned it in the motivation in the beginning of this chapter. Recall that for a bounded subset $A \subset \mathbb{C}$ the convex hull $conv(A)$ of A is defined to be the smallest convex set containing A, formally

$$
conv(A) := \bigcap_{A \subset K \in \mathcal{K}^2} K \in \mathcal{K}^2.
$$

For a set $A \in \mathcal{P}_f$ we define

$$conv(A) := conv(\bigcup_{p \in A} sq(p))$$

and since $\bigcup_{p\in A} sq(p)$ is a bounded set we have $conv(A) \in \mathcal{K}^2$.

Definition 5.3.1. (*random line hitting distribution*) Let $A \in \mathcal{P}_f$ and $K := conv(A) \in \mathcal{K}^2$. For $x \in \mathbb{Z}^2$ and $g \in \mathcal{G}$ define

$$\gamma_A(g) := \frac{1}{2}\mathbb{1}\{|g \cap A| \geq 2\} + \mathbb{1}\{|g \cap A| = 1\},$$

$$\tilde{\mu}_A(x, g) := \gamma_A(g) \begin{cases} \mathbb{1}\{x \in \{\min(g \cap A), \max(g \cap A)\}\}, & \text{if } g \cap A \neq \emptyset, \\ 0, & \text{if } g \cap A = \emptyset. \end{cases} \tag{5.1}$$

and

$$\mu_A(x) := \frac{1}{\mathbb{P}^K_\mu([A])} \int_{\mathcal{G}} \tilde{\mu}_A(x, g)\ \mathbb{P}^K_\mu(dg). \tag{5.2}$$

We quickly show that $\mu_A \in \mathcal{D}_A$. For all $x \in \mathbb{Z}^2 \setminus A$ and $g \in \mathcal{G}$ we have $\tilde{\mu}_A(x, g) = 0$ and therefore $\mu_A(x) = 0$. Furthermore for all $g \in \mathcal{G}$ we have

$$\sum_{x\in A} \tilde{\mu}_A(x, g) = \begin{cases} \frac{1}{2}2, & |g \cap A| \geq 2, \\ 1, & |g \cap A| = 1, \\ 0, & |g \cap A| = 0. \end{cases} \quad = \mathbb{1}\{g \cap A \neq \emptyset\}$$

and therefore

$$\sum_{x\in A} \mu_A(x) = \frac{1}{\mathbb{P}^K_\mu([A])} \int_{\mathcal{G}} \mathbb{1}\{g \cap A \neq \emptyset\}\ \mathbb{P}^K_\mu(dg) = \frac{\mathbb{P}^K_\mu([A])}{\mathbb{P}^K_\mu([A])} = 1.$$

Hence, the family of distributions $(\mu_A)_{A\in\mathcal{P}_f}$ defines an incremental aggregate.

Definition 5.3.2. (*line hitting aggregate*) An incremental aggregate with the random line hitting distribution we call *line hitting aggregate*, short *LHA*.

Remark 5.3.1. Note, that the indicator function in (5.1) is well-defined, since a line g chosen by the distribution $\mathbb{P}^K_\mu$ certainly hits $K = conv(A)$ and it is easy to understand that g hits $conv(A)$ if and only if g hits A.

5.4 Fractal Dimension and Growth Rate of LHA

6 Python Simulation of the Line Hitting Aggregate

In this section we will present a simulation for the line hitting aggregate and approximate its fractal dimension empirically. The simulation is written in *python* and images are created with *pygame*. If you want to check and try the code yourself, feel welcomed to checkout this repository on github:

LINK

Each vertex in $\mathbb{Z}^2$ can be naturally identified with values in $\mathbb{C}$ and is represented in this way in this code. Every value in $\mathbb{C}$ with integer components can be represented by squares as presented in Definition 5.1.1, and each square will be represented by exactly one pixel on a monitor screen, so every picture created by the simulation consists of pixels each naturally representing exactly one vertex in $\mathbb{Z}^2$. The aim of this simulation is to create realizations of LHA as close as possible to its mathematical definition, so the only sources of error are external python packages and unexact machine calculation.

Note that this software contains also a simulation for external DLA. This simulation doesn't claim to be mathematically precise, but it is good enough to play around and create images with it, and get a vague visual comparison of the two aggregation models.

6.1 Images and Empirical Fractal Dimension

6.2 Code

In the following we will describe the code and claim that it is a precise implementation of LHA, excluding errors from python packages and machine calculation. In the simulation we are using imports for systemic purposes (**pygame, sys, json, time, datetime, os, shutil**) and for mathematical purposes (**random, math.sin, math.cos, math.pi, math.log**). Only the imports for mathematical purposes are used for the calculation of a realization of LHA. So on the side of imports only these packages will have an effect on the mathematical preciseness of the realised process. Other than that every calculations preciseness is bounded to some extent by the preciseness of machine calculation. We will ignore a study on that in this simulation.

When using the software you will have the possibility to enter some parameters in the *parameters.json* file which include colorings of background and particle pixels, how many particles you want the cluster to have at the end and how big the created image shall be (more on that you can find in the README file). An example of this file looks like this:

```
{
"aggregate": "lha",
"cluster_size": 2000,
"background_color": "ffffff",
"particle_color": ["2b1c8f", "cc0000"],
"color_generation_size": 500,
"image_width": 300,
"image_height": 300
}
```

If correct parameters are set, you can run a shell script which in turn will run a single python file *main.py* which reads the entered parameters and executes first the mathematical process which needs the mathematical imports and second the data creation of the calculated process which needs the systemic imports as described above. The created data will contain an image of the created cluster, a json file with the parameters saved for later lookup and another json file containing all calculated fractal dimension values and all particle coordinates for later lookup aswell.

We will have a look now on what happens on the mathematical side. We will show code directly here to make things clearer. Note that in the original code here and there you will find comments or other irrelevant differences, which we leave away here.

The main executed python file *main.py* looks like this:

```
if __name__ == "__main__":

    with open("parameters.json") as json_file:
        data = json.load(json_file)

    if data["aggregate"] == "lha":
        aggregate = lha.Line_Hitting_Aggregate()
    elif data["aggregate"] == "dla":
        aggregate = dla.External_DLA()

    aggregate.run_process(data["cluster_size"] - 1)
    cd.export_data(lha, data)
```

Note again that you can also run an external DLA simulation but we do not claim that it is mathematically precise and therefore won't argument that in this paper. The parameters in *parameters.json* are read and depending on which process (LHA or DLA) shall be simulated, the according object gets initiated and its *run_process* method gets executed. The init method of *Line_Hitting_Aggregate* looks like the following:

```
class Line_Hitting_Aggregate:
    def __init__(self):
        self.particles = [0]
        self.boundary_set = self.get_neighbours(0)
        self.cluster_radius = 0
        self.fractal_dimension_values = [1.5]
```

We init the particle set *particles* as a set containg 0, init the boundary set which in every iteration of the process will contain all neighbours of all particles in the particle set, set the initial cluster radius to 0 and init a list *fractal_dimension_values* which will

contain all values of fractal dimensions we calculate during the process, more on that later. We init this list to contain one value 1.5 to ensure that its length is equal to the amount of particles which is helpful for later calculation. We choose 1.5 since it is most neutral (since the fractal dimension lies in between 1 and 2 by 3.2 (3.5)). Anyway this first value is not relevant and will not go into any calculation of later fractal dimension values. As neighbours we only consider the neighbour coordinates up, down, left and right as defined in 2.2.1.

After initializing a *Line_Hitting_Aggregate* object the code executes its *run_process* method, which looks like the following:

```
def run_process(self, iterations):

    for k in range(iterations):
        line_hits_cluster = False

        while not line_hits_cluster:
            random_line = geom.Line(self.get_random_line())

            if self.line_hits_cluster(random_line):
                line_hits_cluster = True
                next_particle = self.get_next_particle(random_line)

                self.particles.append(next_particle)
                self.actualize_boundary_set(next_particle)
                self.actualize_cluster_radius(next_particle)
                self.add_fractal_dimension_value()
```

This method will directly run a loop with $data[\text{"}cluster\_size\text{"}] - 1$ iterations. In each iteration exactly one particle will be added to the particle set, so at the end we end up with exactly $data[\text{"}cluster\_size\text{"}]$ particles since we started with one particle.

If we call K the current cluster it is now important to understand again, how choosing random K-isotropic lines works. In 5.2.1 we have argumented that a K-isotropic line can be realized by considering a circle B which contains the current cluster K and choose a B-isotropic line which intersects with the circle. If it happens that this line intersects with K, by 5.2.4 we know that we have realised a random K-isotropic line. The method *get_random_line*

```
def get_random_line(self):
    alpha = pi * random.random()
    radius = self.cluster_radius + 2
    p = 2 * radius * random.random() - radius
    return (alpha, p)
```

produces uniformly chosen line parameters, and using $cluster\_radius + 2$ here ensures that the line is hitting a surrounding circle B. The method $random.random()$ produces an uniformly chosen value in $[0, 1)$. The fact that $random.random()$ cannot choose the value 1 doesn't disturb the distribution of the line since $\{1\}$ is a zero set for the 2-dimensional Lebesque-measure. By 5.2.5, choosing these line parameters uniformly indeed realises a random B-isotropic line.

The method *line_hits_cluster*

```
def line_hits_cluster(self, line):
    for k in range(len(self.particles)):
        if line.intersects_with_polygon(self.get_square(self.particles[-k])):
            return True
    return False
```

bases on the method *intersects_with_polygon* which evaluates whether a line intersects with a given polygon or not (the implemented geometry classes *Line* and *Polygon* you can find in the file *geometry.py*). Here the polygon is a square given by the method

```
def get_square(self, pos):
    return geom.Polygon([pos + 1/2 * (1+1j), pos + 1/2 * (1-1j), pos + 1/2 * (-1-1j), pos + 1/2 * (-1+1j)])
```

which in total implements the definitions 5.1.1 and 5.1.2.

If the condition is satisfied that a chosen line hits the cluster, we enter the if-condition and then choose the next particle with the method *get_next_particle*

```
def get_next_particle(self, line):
    hit_positions = self.get_boundary_hit_positions(line)
    min_position = self.get_min(line.alpha, hit_positions)
    max_position = self.get_max(line.alpha, hit_positions)
    return random.choice([min_position, max_position])
```

which first collects all particle squares which are hit by the line, calculates minimum and maximum and then chooses uniformly between minimum and maximum with *random.choice*. The methods *get_min* and *get_max* base on the method *is_lower*

```
def is_lower(self, alpha, x, y):
    x_0, x_1 = x.real, x.imag
    y_0, y_1 = y.real, y.imag

    if alpha == pi/2: # Case 1
        return x_0 < y_0
    elif alpha == 0: # Case 2
        return x_1 < y_1
    elif pi/2 < alpha < pi: # Case 3
        if x_0 == y_0:
            return x_1 < y_1
        else:
            return x_0 < y_0
    elif 0 < alpha < pi/2: # Case 4
        if x_0 == y_0:
            return x_1 > y_1
        else:
            return x_0 < y_0
```

which implements the total ordered relation on all squares which got hit by the line as defined in 5.1.2. Minimum and maximum are then canonically calculated with respect to this relation.

After having realised the next particle, values like the particle set, the boundary set and the cluster radius get actualized and finally a new fractal dimension value is added to the fractal dimension values list. The method *add_fractal_dimension_value*

```
def add_fractal_dimension_value(self):
    radius = max(self.cluster_radius, 2)
    value = log(len(self.particles))/log(radius)
    self.fractal_dimension_values.append(value)
```

devides the log of the current amount of particles in the cluster by the log of the current radius of the cluster as motivated by the definition 3.2 (3.2) of the discrete fractal dimension and adds this value to the *fractal_dimension_values* list. Note that *log* in python is the natural logarithm with base e. Also note that choosing the maximum of the cluster radius and 2 only serves the purpose to avoid problems in the very beginning of the simulation when the cluster radius is 1 and therefore creating division by zero in the subsequent division ($\log(1) = 0$).

All in all this procedure should implement the distribution of the line hitting aggregate as mathematically defined. At the end its preciseness is depending on the preciseness of the distributions given by the python methods *random.random* and *random.choice*, and machine calculation in general. Eventhough we do not claim to have implemented a mathematically precise simulation for external DLA, feel free to also try that one out and create beautiful pictures of the two incremental aggregates.

7 Outlook

In this paper we have looked at a general definition of incremental aggregates in $\mathbb{Z}^d$ and the notion of defining a fractal dimension for such discrete clusters. We have looked at two such incremental aggregates, the external diffusion limited aggregate and the line hitting aggregate, and have proved a rigorous result for the fractal dimension of external DLA.
CONTINUE

I hope that I have raised interest in the topic of incremental aggregates with this paper. I feel delighted to watch at the beautiful realizations of the aggregates and I hope the reader can feel similar looking at them and realizing images him- and herself.

References

[1] Daniel Hug, Günter Last, Steffen Winter. *Stochastic Geometry, Lecture Notes (summer term 2020)*. Institute of Technology, Karlsruhe, 2020

[2] Franz Sackmann. *Zufällige Geraden, Staatsexamensarbeit*. University Karlsruhe (TH), Karlsruhe, 2007

[3] Gregory F. Lawler *Intersections of Random Walks*. University of Chicago, 1996

[4] Norbert Henze *Irrfahrten - Faszination der Random Walks* Institute of Technology, Karlsruhe, 2018

[5] Uta Freiberg, Ben Hambly, Michael Hinz and Steffen Winter *Fractal Geometry and Stochastics VI* 2020

[6] J. Gonzalez-Gutierrez, J. L. Carrillo-Estrada and J. C. Ruiz-Suarez *Aggregation and dendritic growth in a magnetic granular system* 2013

[7] Egbert Brieskorn *Felix Hausdorff zum Gedächtnis, Band I, vieweg Verlag* 1996 page 160

[8] Prof. Dr. Stefan Tappe *Markov-Ketten, lecture script summer semester 2018* Abteilung für Mathematische Stochastik, Albert-Ludwigs-Universität Freiburg

[9] D. Bensimon, B. Shraiman and S. Liang *On the ballistic model of aggregation* Physics Letter, University of Chicago 1984

Erklärung

Hiermit versichere ich, dass ich diese Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt, die wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht und die Satzung des Karlsruher Instituts für Technologie zur Sicherung guter wissenschaftlicher Praxis in der jeweils gültigen Fassung beachtet habe.

Karlsruhe, den 10. März 2020