Work Sheet 13

Instructions for week 13 (July 13^{th} to July 17^{th}):

- Work through Sections 5.1 and 5.2 of the lecture notes.
- Answer the control questions 1) to 5), and solve Problems 1 to 3 of the exercises below.
- Please hand in your solutions to the exercises for correction until the morning of Monday, July 20^{th}. For the procedure, please have a look at the general information in Ilias. The submission of solutions is voluntary.

Please take notice of the following announcements for next week (the last week of the lecture time):

- There will be no new material in the lecture notes, but we prepare a 14th work sheet which contains two exercises for simulations of and inference for the Boolean model.
- Steffen Winter will offer a discussion session via MS-Teams on July 22^{nd} at 16:00. If you want to participate in the discussion, please prepare your questions and send an e-mail to steffen.winter@kit.edu to announce your participation. Please also join the 'Team' corresponding to the exercise class via the link in our Ilias course.

Control questions to monitor your progress:

1) Provide examples of tessellations in $\mathbb{R}^2$ (and $\mathbb{R}^3$) for which
   - all cells are isometric and which is face-to-face,
   - all cells are isometric but which is not face-to-face,
   - the cells are isometric to one of two given polytopes.
2) Convince yourself that Voronoi tessellations are face-to-face.

   Hint: A formal proof is given in Theorem 10.2.1 of the book 'Stochastic and Integral Geometry' by R. Schneider and W. Weil, which appeared 2008 in the 'Probability and Its Applications' series of Springer.
3) Determine the total face contents $L_0, \dots, L_3$ of the unit cube $W = [0,1]^3 \subset \mathbb{R}^3$ and compare them to the intrinsic volumes $V_0, \dots, V_3$ of W.

   Hint: Recall that $V_j\big([0,1]^3\big) = \binom{3}{j}$, for $j = 0, 1, 2, 3$.
4) Let $X^{(k)}$ be the process of k-faces of a stationary random tessellation. Verify that $X^{(k)}$ is stationary.
5) Specify the general formulae in Theorem 5.13 for $d = 2$ and $d = 3$.

Exercises for week 13:

Problem 1

Let $K \in \mathcal{K}^3$ be such that $K \subset [0,1]^3$, and let X_1 be a random line (that is, a random 1-flat) through $[0,1]^3$, as defined in Problem 3 of Work sheet 11, on a given probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Suppose that, for a realization $X_1(\omega)$ ($\omega \in \Omega$), you can observe whether $X_1(\omega)$ intersects K, and, if so, that you can measure the length of $X_1(\omega) \cap K$. Construct an unbiased estimator for the surface area and the volume of K.

Problem 2 (Lemma 5.2)

Let $\mathfrak{m}$ be a tessellation in $\mathbb{R}^d$ and $K \in \mathfrak{m}$.

a) Prove that one can find finitely many cells $K_1, \dots, K_\ell \in \mathfrak{m} \setminus \{K\}$ such that $K_j \cap K \neq \varnothing$ ($j = 1, \dots, \ell$) and

$$\partial K = \bigcup_{j=1}^{\ell} (K_j \cap K).$$

b) Prove that K is a polytope.

Problem 3 (Example 5.6)

Let $\varphi \in N_s(\mathbb{R}^d)$ such that $\varphi(\mathbb{R}^d) > 0$. For $x \in \varphi$ define the Voronoi cell of x as

$$C(\varphi, x) = \Big\{ z \in \mathbb{R}^d \, : \, \|z - x\| \leqslant \|z - y\| \text{ for each } y \in \varphi \Big\}.$$

Prove that all Voronoi cells are bounded if $\mathrm{conv}(\varphi) = \mathbb{R}^d$, that is, if the convex hull of the points in φ is $\mathbb{R}^d$.

The solutions to these problems will be uploaded on July 20th.