Work Sheet 7

Instructions for week 7 (June 1^{st} to June 5^{th}):

- Work through the introductory part of Chapter 3 as well as Section 3.1 of the lecture notes.
- Answer the control questions 1) to 4), and solve Problems 1 to 4 of the exercises below.
- Please hand in your solutions to the exercises for correction until the morning of Monday, June 8^{th}. For the procedure, please have a look at the general information in Ilias. The submission of solutions is voluntary.

Control questions to monitor your progress:

1) Consider the probability space specified by $\Omega := \{\omega_1, \dots, \omega_m\}$, $\mathcal{A} := \mathcal{P}(\{\omega_1, \dots, \omega_m\})$, $m \geqslant 3$, and
$$\mathbb{P}(\omega_j) = \tfrac{1}{m}, \qquad j = 1, \dots, m.$$
For each $j \in \{1, \dots, m\}$, let $C_j = B(0, j) \subset \mathbb{R}^d$. Define $X_1, \dots, X_k : \Omega \to \mathcal{C}'$ through
$$X_i(\omega_j) = C_j, \qquad i = 1, \dots, k, \quad j = 1, \dots, m.$$
Put $\Phi = \sum_{i=1}^k \delta_{X_i}$. Can you verify that Φ is a particle process in $\mathbb{R}^d$? What is the probability that X_1 is contained in $[-3, 3]^d$? If you where to observe $Z := \bigcup_{i \in \mathbb{N}} X_i$, what would you see?

2) Give two examples of a center function in the sense of Definition 3.2 (other than the center of the circumball of a compact set).

3) Let $(E, \mathcal{O}_E)$ be a locally compact Hausdorff space with a countable base of topology. Consider the collection of closed subsets of E, denoted by $\mathcal{F}(E)$, endowed with the Fell-topology and the corresponding Borel-σ-field $\mathcal{B}(\mathcal{F}(E))$, and $\mathcal{F}' = \mathcal{F}(E) \setminus \{\varnothing\}$. Let Θ be a measure on $\big(\mathcal{F}', \mathcal{B}(\mathcal{F}(E)) \cap \mathcal{F}'\big)$. Convince yourself that Θ is locally finite, in the sense that $\Theta(\mathcal{H}) < \infty$ for every compact subset of the (locally compact) space $\big(\mathcal{F}', \mathcal{O}_{\mathcal{F}(E)} \cap \mathcal{F}'\big)$, if, and only if, $\Theta(\mathcal{F}_C) < \infty$ for every $C \in \mathcal{C}(E)$.

   (**Hint:** Use parts (2) and (3) of Theorem 1.2)

4) Denote by $\mathcal{C}' = \mathcal{C}^d \setminus \{\varnothing\}$ the collection of non-empty compact subsets of $\mathbb{R}^d$. Endow $\mathcal{C}'$ with the topology induced by the Hausdorff metric δ and write $\mathcal{B}(\mathcal{C}')$ for the corresponding Borel-σ-field. Can you verify that a locally finite measure Θ on $\mathcal{F}' = \mathcal{F}^d \setminus \{\varnothing\}$ satisfies $\Theta(\mathcal{H}) < \infty$ for every bounded set $\mathcal{H} \in \mathcal{B}(\mathcal{C}')$?

   (**Hint:** If $\mathcal{H}$ is a (measurable) bounded subset of $(\mathcal{C}', \delta)$, then (by definition of the Hausdorff metric) there exists a compact set $C \in \mathcal{C}'$ such that $K \subset C$ for each $K \in \mathcal{H}$. Thus, $\mathcal{H} \subset \mathcal{F}^d_C \cap \mathcal{C}'$.)

Exercises for week 7:

Problem 1 (Estimation of the intensity of a Poisson process)

Let Φ be a Poisson process in $\mathbb{R}^d$ with intensity measure $\gamma \cdot \lambda^d$, where $\gamma > 0$.

a) Prove that $\widehat{\gamma}_B := \dfrac{\Phi(B)}{\lambda^d(B)}$, for bounded $B \in \mathcal{B}(\mathbb{R}^d)$ such that $\lambda^d(B) > 0$, is an unbiased estimator for γ.

b) Let $W_1, W_2, \ldots \in \mathcal{B}(\mathbb{R}^d)$ be Borel sets such that $0 < \lambda^d(W_n) < \infty$, $n \in \mathbb{N}$, and $\lambda^d(W_n) \overset{n\to\infty}{\longrightarrow} \infty$. Prove that the estimator $\widehat{\gamma}$ is weakly consistent, in the sense that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\Big(\big|\widehat{\gamma}_{W_n} - \gamma\big| > \varepsilon\Big) = 0.$$

c) Let $W_1, W_2, \ldots \in \mathcal{B}(\mathbb{R}^d)$ such that $0 < \lambda^d(W_n) < \infty$, $n \in \mathbb{N}$, and $\sup_{n\in\mathbb{N}} \frac{n^{1+\delta}}{\lambda^d(W_n)} < \infty$ for some $\delta > 0$. Prove that the estimator $\widehat{\gamma}$ is strongly consistent, in the sense that

$$\lim_{n\to\infty} \widehat{\gamma}_{W_n} = \gamma \quad \mathbb{P}\text{-almost surely.}$$

Hint: Recall from probability theory that if random variables $X, \{X_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ satisfy $\sum_{n=1}^{\infty} \mathbb{P}\big(|X_n - X| > \varepsilon\big) < \infty$, for every $\varepsilon > 0$, then $X_n \to X$ $\mathbb{P}$-almost surely.

d) Simulate a homogeneous Poisson process in $[-25, 25]^3$ with intensity $\gamma = 0.005$, that is, simulate a mixed binomial process with parameters $\tau \sim Po(50^3 \cdot 0.005)$ and $V = \mathcal{U}\big([-25, 25]^3\big)$. Proceed to estimate the intensity on the growing observation windows $W_n := B(0, n) \subset \mathbb{R}^3$, for $n = 1, \ldots, 25$, and see how the estimator behaves.

Problem 2 (Particle processes)

Defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, let Ψ be a stationary point process in $\mathbb{R}^d \times \mathcal{C}'$ such that $\mathbb{E}\big[\Psi(\cdot \times \mathcal{C}')\big]$ is locally finite. Let $\gamma \geqslant 0$ and the probability measure $\mathbb{Q}$ on $\mathcal{C}'$ be as in Theorem 2.42 (that is, $\mathbb{E}\Psi = \gamma \cdot \lambda^d \otimes \mathbb{Q}$) and assume that

$$\int_{\mathcal{C}'} \lambda^d(K + C)\, \mathrm{d}\mathbb{Q}(K) < \infty, \qquad C \in \mathcal{C}'.$$

Prove that

$$\Phi := \int_{\mathbb{R}^d \times \mathcal{C}'} \mathbb{1}\big\{K + x \in \cdot\,\big\}\, \mathrm{d}\Psi(x, K)$$

is a particle process.

Problem 3 (Continuity of the circumball mapping)

a) Let $C \in \mathcal{C}^d$ be non-empty, that is, $C \in \mathcal{C}' = \mathcal{C}^d \setminus \{\varnothing\}$. Denote by

$$B(x, r) = \big\{y \in \mathbb{R}^d : \|y - x\| \leqslant r\big\}, \qquad x \in \mathbb{R}^d,\ r \geqslant 0,$$

the ball in $\mathbb{R}^d$ of radius r around x. Prove that among all balls that contain C there exists a unique ball $B(C)$ with smallest radius (the circumball of C).

b) Prove that the map $r : \mathcal{C}' \to [0, \infty)$, $C \mapsto r(C)$, where $r(C)$ denotes the radius of the circumball $B(C)$, is continuous with respect to the Hausdorff metric.

c) Prove that the map $c : \mathcal{C}' \to \mathbb{R}^d$, $C \mapsto c(C)$, where $c(C)$ denotes the center of the circumball $B(C)$, is continuous with respect to the Hausdorff metric.

d) Conclude that the map $\mathcal{C}' \to \mathcal{C}'$, $C \mapsto B(C)$ is continuous with respect to the Hausdorff metric.

Problem 4 (On Theorem 3.6 and Remark 3.9)

Let Φ be a stationary particle process in $\mathbb{R}^d$ with locally finite intensity measure $\Theta \neq 0$. Let $c : \mathcal{C}' \to \mathbb{R}^d$ be some center function (in the sense of Definition 3.2). As in the lecture, we define $\mathcal{C}_0 := \{C \in \mathcal{C}' : c(C) = 0\}$. By Theorem 3.6 there exists a unique $\gamma > 0$ and a unique probability measure $\mathbb{Q}$ on $\mathcal{C}'$ such that $\mathbb{Q}(\mathcal{C}_0) = 1$ and

$$\Theta(\mathcal{H}) = \gamma \int_{\mathbb{R}^d} \int_{\mathcal{C}'} \mathbb{1}_{\mathcal{H}}(C + x)\, \mathrm{d}\mathbb{Q}(C)\, \mathrm{d}x, \qquad \mathcal{H} \in \mathcal{B}(\mathcal{C}').$$

a) Prove that γ does not depend on the choice of c.

b) Let $c' : \mathcal{C}' \to \mathbb{R}^d$ be another center function, and $\mathbb{Q}_{c'}$ the corresponding shape distribution. Also put $\mathcal{C}_{0,c'} := \{C \in \mathcal{C}' : c'(C) = 0\}$. Prove that

$$\mathbb{Q} = \mathbb{Q}_{c'} \circ T^{-1},$$

where $T : \mathcal{C}_{0,c'} \to \mathcal{C}_0$, $C \mapsto C - c(C)$.

c) Find a center function c' such that $\mathbb{Q} \neq \mathbb{Q}_{c'}$.

The solutions to these problems will be uploaded on June 8th.
Feel free to ask your questions about the exercises in the optional MS-Teams discussion on June 4th (09:15 h).