Work Sheet 3

Instructions for week 3 (May 4^{th} to May 8^{th}):

- Please answer the little questionnaire that we have posted in Ilias. It asks you for some feedback regarding the course material and how well you get along with it.
- Work through Sections 1.2 and 1.3 of the lecture notes.
- Answer the control questions 1) to 6), and solve Problems 1 to 4 of the exercises below.
- Please hand in your solutions to the exercises for correction until the morning of Monday, May 11^{th}. For the procedure, please have a look at the general information in Ilias. The submission of solutions is voluntary.

Control questions to monitor your progress:

1) In Example 1.23 (ii) the union set of a sequence of random points is considered. Why is the additional assumption that the sequence has no accumulation points (almost surely) necessary to generate a random closed set?
2) Check that the set generated in Example 1.23 (iv) is indeed a random closed set, that is, verify that the mapping Z is measurable.
3) Let $(E, \mathcal{O}_E)$ be a second countable, locally compact Hausdorff space. Let $(\Omega, \mathcal{A})$ be a measurable space, and let $Z : \Omega \to \mathcal{F} = \mathcal{F}(E)$ be a random closed set in E. The selection theorem of Kuratowski and Ryll-Nardzewski states that there exists an $\big(\mathcal{A}, \mathcal{B}(E)\big)$-measurable selection of Z, that is, there exists a measurable map $\vartheta : \Omega \to E$ such that $\vartheta(\omega) \in Z(\omega)$ for each $\omega \in \Omega$.

   Can you think of possible useful applications of this theorem?
4) See if you can construct some simple examples for both stationary and isotropic random closed sets.
5) Construct a random closed set Z with capacity functional given through $T_Z(C) = \mathbb{1}\{C \neq \varnothing\}$ for each $C \in \mathcal{C}$.
6) Can you prove the second assertion from Corollary 1.33, namely that a random closed set Z is isotropic if, and only if, its capacity functional T_Z is rotation invariant?

Exercises for week 3:

Problem 1 (On the Hausdorff metric – Part 2)

Let $(\mathbb{X}, \|\cdot\|)$ be a normed vector space (over $\mathbb{R}$ or $\mathbb{C}$), and recall from Problem 4 on Work sheet 2 that the Hausdorff metric δ on $\mathcal{C}(\mathbb{X}) \setminus \{\varnothing\}$ is defined as

$$\delta(C, C') := \inf\big\{\varepsilon \geqslant 0 \ : \ C \subset C'_{\oplus\varepsilon},\ C' \subset C_{\oplus\varepsilon}\big\}, \qquad C, C' \in \mathcal{C}(\mathbb{X}) \setminus \{\varnothing\},$$

and that we put $\delta(\varnothing, C) = \delta(C, \varnothing) := \infty$, $C \in \mathcal{C}(\mathbb{X}) \setminus \{\varnothing\}$, as well as $\delta(\varnothing, \varnothing) := 0$. Note that if we denote by $B_{\mathbb{X}} = B(0, 1)$ the closed unit ball around the origin in $\mathbb{X}$, then

$$B_{\oplus\varepsilon} = B + \varepsilon B_{\mathbb{X}}, \qquad B \subset \mathbb{X},\ \varepsilon \geqslant 0.$$

Let $C, C', D, D' \in \mathcal{C}(\mathbb{X}) \setminus \{\varnothing\}$, and show that

a) $\delta\big(\mathrm{conv}(C), \mathrm{conv}(D)\big) \leqslant \delta(C, D)$,

b) $\delta(C + C', D + D') \leqslant \delta(C, D) + \delta(C', D')$, and

c) $\delta(C \cup C', D \cup D') \leqslant \max\big\{\delta(C, D), \delta(C', D')\big\}$,

where $\mathrm{conv}(C)$ denotes the convex hull of C.

Problem 2 (Random closed sets)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space.

a) Let $\xi : \Omega \to \mathbb{R}^d$ be a random vector. Show that $Z_1 = \{\xi\}$ is a random closed set.

b) Let $\xi_1, \xi_2, \ldots$ be a sequence of random vectors (in $\mathbb{R}^d$) such that $\{\xi_k(\omega) : k \in \mathbb{N}\}$ has no accumulation point in $\mathbb{R}^d$ (for each $\omega \in \Omega$). Show that $Z_2 := \{\xi_k : k \in \mathbb{N}\}$ is a random closed set.

c) Let $R : \Omega \to (0, \infty)$ be a positive random variable and $\xi : \Omega \to \mathbb{R}^d$ a random vector.

  (i) Show that the "random closed ball" Z_3 with center ξ and radius R, that is, the mapping $Z_3 : \Omega \to \mathcal{F}^d$, $Z_3(\omega) := B\big(\xi(\omega), R(\omega)\big)$, is a random closed set.

  (ii) Show that $\mathbb{E}\big[\lambda^d(Z_3 + K)\big] < \infty$ holds for every compact set $K \subset \mathbb{R}^d$ if, and only if, $\mathbb{E}[R^d] < \infty$.

d) Find a random closed set Z_4 such that the set $G := \big\{x \in \mathbb{R}^d \ : \ \mathbb{P}(x \in Z_4) \neq 0\big\}$ is open.

e) Let Z_5 be a stationary random closed set in $\mathbb{R}^d$ with volume fraction $p_{Z_5} = \mathbb{E}\big[\lambda^d(Z_5 \cap [0, 1]^d)\big]$. Let $B \in \mathcal{B}^d$, and show that

$$\mathbb{E}\Big[\lambda^d(Z_5 \cap B)\Big] = p_{Z_5} \cdot \lambda^d(B) \qquad \text{and} \qquad p_{Z_5} = \mathbb{P}\big(t \in Z_5\big), \quad t \in \mathbb{R}^d.$$

Problem 3 (Properties of the capacity functional)

Let Z be a random closed set in $\mathbb{R}^d$, defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and denote by T_Z the capacity functional of Z. Prove the following assertions.

a) $0 \leqslant T_Z \leqslant 1$ as well as $T_Z(\varnothing) = 0$, and

b) for any sets $C, C_1, C_2, \ldots \in \mathcal{C}^d$ with $C_n \searrow C$ it holds that $T_Z(C_n) \to T_Z(C)$, as $n \to \infty$.

Problem 4 (Capacity functionals – Examples)

Compute the capacity functional of the following random sets $\widetilde{Z}$ in $\mathbb{R}^d$ which are assumed to be defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

a) $\widetilde{Z} = F$, for some fixed closed set $F \in \mathcal{F}^d$,

b) $\widetilde{Z} = \{\xi\}$, where ξ is a random element of $\mathbb{R}^d$.

Let $Z, Z_1, Z_2, \dots$ be independent and identically distributed random closed sets in $\mathbb{R}^d$, defined on $(\Omega, \mathcal{A}, \mathbb{P})$. Let N be an $\mathbb{N}_0$-valued random variable which is independent of $(Z_n)_{n\in\mathbb{N}}$. Write $p_k := \mathbb{P}(N = k)$, $k \in \mathbb{N}_0$, and put $Z^* := \bigcup_{n=1}^{N} Z_n$. Denote by $G(s) := \sum_{k=0}^{\infty} p_k \cdot s^k$, for $s \in [0,1]$, the generating function of N.

c) Prove that Z^* is a random closed set with $T_{Z^*}(\cdot) = 1 - G\big(1 - T_Z(\cdot)\big)$.

d) Show that if $N \sim \mathrm{Po}(\lambda)$ for some $\lambda > 0$, then $T_{Z^*}(\cdot) = 1 - e^{-\lambda \cdot T_Z(\cdot)}$.

The solutions to these problems will be uploaded on May 11th.
Feel free to ask your questions about the exercises in the optional MS-Teams discussion on May 7th (09:15 h).