Work Sheet 2

Instructions for week 2 (April 27^{th} to May 1^{st}):

- Work through the remaining part of Section 1.1 in the lecture notes starting with Example 1.4.
- Answer the control questions 1) to 5), and solve Problems 1 to 4 of the exercises below.
- Please hand in your solutions to the exercises for correction until the morning of Monday, May 4^{th}. For the procedure, please have a look at the general information in Ilias. The submission of solutions is voluntary.

Control questions to monitor your progress:

1) Why is the Fell topology also called "hit-or-miss topology"?
2) Let $f : T \to \widetilde{T}$ be a map between two topological spaces $(T, \mathcal{O}_T)$ and $(\widetilde{T}, \mathcal{O}_{\widetilde{T}})$. Can you find a simple proof for the fact that if f is continuous then f is also sequentially continuous, that is, $f(t_n) \to f(t)$ in $\widetilde{T}$ (as $n \to \infty$) for each convergent sequence $(t_n)_{n\in\mathbb{N}}$ in T with $t_n \to t$?
   (**Hint:** Work with the definition of continuity in topological spaces.)

   Note that if T is first countable, then sequential continuity of f implies continuity of f. This fact is a little harder to prove, but can you still do it?
   (**Hint:** Assume that f is not continuous; take an open set $V \subset \widetilde{T}$ such that $U := f^{-1}(V)$ is not open; take a point p on the topological boundary of U and construct a sequence $p_n \in T \setminus U$ with $p_n \to p$ using that T is first countable; conclude that f cannot be sequentially continuous.)
3) Let $\varphi : T \to \mathcal{F}(E)$ be a map between a topological space $(T, \mathcal{O}_T)$ and the space $\mathcal{F}(E)$ of closed subsets of a second countable locally compact Hausdorff space $(E, \mathcal{O}_E)$. Convince yourself that either form of semi-continuity introduced in Definition 1.7 implies measurability of φ with respect to the corresponding Borel-σ-fields.
   (**Hint:** Use Remark 1.9.)
4) Consider the collection $\mathcal{C}^d$ of compact subsets of $\mathbb{R}^d$ endowed with the Hausdorff metric as defined in Definition 1.15 of the lecture (see also Problem 4 below). What is the Hausdorff distance $\delta(C, D)$ between the compact sets C and D in the following examples?
   - $d = 1, \quad C = [-2, 1], \quad D = [3, 9]$.
   - $d = 1, \quad C = [-2, -1] \cup [10, 12] \cup \{15\}, \quad D = \{-10\} \cup \{1\} \cup \{11\} \cup [20, 25]$.
   - $d = 2, \quad C = [-1, 1]^2, \quad D = [3, 5] \times [2, 4]$.
   - $d = 2, \quad C = B\big((0, 0), 1\big), \quad D = B\big((3, 4), 2\big)$.
   - $C = [-1, 1]^d, \quad D = B\big((5, \dots, 5), 2\big), \quad$ (calculate in dependence on the dimension d).
5) From the proof of Theorem 1.17: Let $C, C_1, C_2, \ldots \in \mathcal{C}^d \setminus \{\varnothing\}$ with $C_n \subset K$ for some fixed $K \in \mathcal{C}^d$. Assume that $C_n \to C$ (as $n \to \infty$) with respect to the Fell topology.
   - Why is $C_n = \varnothing$ for all but finitely many $n \in \mathbb{N}$ whenever $C = \varnothing$?
   - Why is $C_n \neq \varnothing$ for all but finitely many $n \in \mathbb{N}$ whenever $C \neq \varnothing$?

Exercises for week 2:

Problem 1 (Convergence with respect to the Fell topology – Examples, Part 1)

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^d$ and $x \in \mathbb{R}^d$. Further, let $(r_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers and $r > 0$. Denote by $B(x, r)$ the closed ball of radius r around x.

a) If $x_n \to x$ and $r_n \to r$ (as $n \to \infty$), then $B(x_n, r_n) \to B(x, r)$ in $(\mathcal{F}, \mathcal{O}_{\mathcal{F}})$, as $n \to \infty$.

b) If $\|x_n\| \to \infty$, then $\{x_n\} \to \varnothing$ in $(\mathcal{F}, \mathcal{O}_{\mathcal{F}})$, as $n \to \infty$.

Problem 2 (Continuity with respect to the Fell topology)

Denote by $(\mathcal{F}^d, \mathcal{O}_{\mathcal{F}^d})$ the space of closed subsets of $\mathbb{R}^d$ endowed with the Fell topology. Prove that the following maps are continuous:

a) $a : \mathbb{R}^d \times \mathcal{F}^d \to \mathcal{F}^d, \quad a(x, F) := F + x$.

b) $r : \mathcal{F}^d \to \mathcal{F}^d, \quad r(F) := F^* := -F$.

c) $e : (0, \infty) \times \mathcal{F}^d \to \mathcal{F}^d, \quad e(\alpha, F) := \alpha F$.

Also prove that the map $\widetilde{e} : [0, \infty) \times \mathcal{F}^d \to \mathcal{F}^d$, $\widetilde{e}(\alpha, F) := \alpha F$ is not continuous.

Hint: By Theorem 1.2, $(\mathcal{F}^d, \mathcal{O}_{\mathcal{F}^d})$ is second countable (hence first countable), and functions defined on first countable spaces are continuous if, and only if, they are sequentially continuous. Apply this fact together with Theorem 1.3.

Problem 3 (Convergence with respect to the Fell topology – Examples, Part 2)

Denote by $(\mathcal{F}^d, \mathcal{O}_{\mathcal{F}^d})$ the space of closed subsets of $\mathbb{R}^d$ endowed with the Fell topology. For $u \in \mathbb{R}^d$ with $\|u\| = 1$, and $r \geqslant 0$, write

$$H_{u,r} := \left\{ x \in \mathbb{R}^d : \langle x, u \rangle = r \right\} .$$

Let u_n be a sequence in $\mathbb{R}^d$ with $\|u_n\| = 1$ for each $n \in \mathbb{N}$, and let r_n be a sequence in $[0, \infty)$.

a) Show that if $u_n \to u$ and $r_n \to r$, then $H_{u_n,r_n} \to H_{u,r}$ in $(\mathcal{F}^d, \mathcal{O}_{\mathcal{F}^d})$, as $n \to \infty$.

b) Show that if $u_n \to u$ and $r_n \to \infty$, then $H_{u_n,r_n} \to \varnothing$ in $(\mathcal{F}^d, \mathcal{O}_{\mathcal{F}^d})$, as $n \to \infty$.

Consider the sequence $(P_n)_{n\in\mathbb{N}}$ of paraboloids $P_n = \left\{ z \in \mathbb{R}^d \,\middle|\, \frac{z_1^2 + \ldots + z_{d-1}^2}{n} = z_d \right\}$, and $e_d = (0, \ldots, 0, 1) \in \mathbb{R}^d$.

c) Show that $P_n \to H_{e_d,0}$ in $(\mathcal{F}^d, \mathcal{O}_{\mathcal{F}^d})$, as $n \to \infty$.

Problem 4 (On the Hausdorff metric – Part 1)

Let $(\mathbb{X}, d)$ be a metric space, and recall from the lecture that, for any set $B \subset \mathbb{X}$ and $\varepsilon \geqslant 0$,

$$B_{\oplus\varepsilon} := \left\{ x \in \mathbb{X} \,:\, \operatorname{dist}(x, B) \leqslant \varepsilon \right\}$$

denotes the ε-parallel set of B. Here, $\operatorname{dist}(x, B) := \inf_{y\in B} d(x, y)$ is the distance from x to B with respect to the metric d on $\mathbb{X}$. Also recall that the Hausdorff metric δ on $\mathcal{C}(\mathbb{X}) \setminus \{\varnothing\}$ is defined as

$$\delta(C, C') := \inf\left\{ \varepsilon \geqslant 0 \;:\; C \subset C'_{\oplus\varepsilon},\; C' \subset C_{\oplus\varepsilon} \right\}, \qquad C, C' \in \mathcal{C}(\mathbb{X}) \setminus \{\varnothing\},$$

and that we put $\delta(\varnothing, C) = \delta(C, \varnothing) := \infty$, $C \in \mathcal{C}(\mathbb{X}) \setminus \{\varnothing\}$, as well as $\delta(\varnothing, \varnothing) := 0$. Prove the following assertions:

a) $\delta(C, C') = \max\left\{ \sup_{x\in C} \operatorname{dist}(x, C'),\; \sup_{y\in C'} \operatorname{dist}(y, C) \right\}, \quad C, C' \in \mathcal{C}(\mathbb{X}) \setminus \{\varnothing\}$,

b) δ is a metric on $\mathcal{C}(\mathbb{X}) \setminus \{\varnothing\}$, and

c) δ is also a metric on $\mathcal{C}(\mathbb{X})$.

The solutions to these problems will be uploaded on May 4th.
Feel free to ask your questions about the exercises in the optional MS-Teams discussion on April 30th (09:15 h).