Karlsruhe Institute of Technology

Institute of Stochastics

Stochastic Geometry | Summer term 2020

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Solutions for Work Sheet 11

Instructions for week 11 (June 29^{th} to July 3^{rd}):

- Work through Section ... of the lecture notes.
- Answer the control questions 1) to ...), and solve Problems 1 to 4 of the exercises below.
- Please hand in your solutions to the exercises for correction until the morning of Monday, July 6^{th}. For the procedure, please have a look at the general information in Ilias. The submission of solutions is voluntary.

Control questions to monitor your progress:

1)

Exercises for week 11:

Problem 1 (Slicing a Boolean model)

Let Φ be a stationary Poisson particle process in $\mathbb{R}^d$ with locally finite intensity measure $\Theta \neq 0$. Let $c : \mathcal{C}' \to \mathbb{R}^d$ be a center function, and let $\gamma > 0$ and $\mathbb{Q}$ be as in Theorem 3.6. Denote by Z the corresponding Boolean model. Let $k \in \{1, \ldots, d-1\}$ and consider a k-dimensional plane $H \subset \mathbb{R}^d$ with $0 \in H$.

a) Prove that $\Phi_H(\cdot) := \int_{\mathcal{C}^d} \mathbb{1}\{C \cap H \neq \varnothing\} \cdot \mathbb{1}\{C \cap H \in \cdot\} \mathrm{d}\Phi(C)$ is a Poisson particle process in H and that Φ_H is stationary in the sense that $T_x \Phi_H \overset{d}{=} \Phi_H$ for every $x \in H$.

b) Prove that, subject to identifying H with $\mathbb{R}^k$, the random closed set $Z \cap H$ is a Boolean model in H.

Proposed solution: Write $\mathcal{C}^{(H)} := \{C \in \mathcal{C}^d : C \subset H\}$ and $\mathcal{F}^{(H)} := \{F \in \mathcal{F}^d : F \subset H\}$. Then, $\mathcal{F}^{(H)} \in \mathcal{B}(\mathcal{F}^d)$ and therefore also $\mathcal{C}^{(H)} = \mathcal{F}^{(H)} \cap \mathcal{C}^d \in \mathcal{B}(\mathcal{F}^d)$. Indeed, since $H \in \mathcal{F}^d$, the set H^c is open and hence $\mathcal{F}^{(H)} = \big(\mathcal{F}_{H^c}\big)^c \in \mathcal{B}(\mathcal{F}^d)$.

a) The mapping theorem for point processes (see Problem ... of Work sheet ...) applied to the map $T : \mathcal{C}' \to \mathcal{C}^{(H)}$, $T(C) := C \cap H$ implies that

$$\int_{\mathcal{C}^d} \mathbb{1}\{C \cap H \in \cdot\} \mathrm{d}\Phi(C)$$

is a Poisson process in $\mathcal{C}^{(H)}$. Restricting this process to $\{C \cap H : C \in \mathcal{C}^d\} \setminus \{\varnothing\}$, it follows that Φ_H is also a Poisson process. Let $x \in H$. Using the stationarity of Φ, we obtain

$$\begin{aligned}
T_x \Phi_H(\cdot) = \int_{\mathcal{C}^{(H)}} \mathbb{1}\{C + x \in \cdot\} \mathrm{d}\Phi_H(C) &= \int_{\mathcal{C}^d} \mathbb{1}\big\{(C \cap H) + x \in \cdot\big\} \mathbb{1}\{C \cap H \neq \varnothing\} \mathrm{d}\Phi(C) \\
&= \int_{\mathcal{C}^d} \mathbb{1}\big\{(C + x) \cap H \in \cdot\big\} \mathbb{1}\big\{(C + x) \cap H \neq \varnothing\big\} \mathrm{d}\Phi(C) \\
&\overset{d}{=} \int_{\mathcal{C}^d} \mathbb{1}\{C \cap H \in \cdot\} \mathbb{1}\{C \cap H \neq \varnothing\} \mathrm{d}\Phi(C) \\
&= \Phi_H(\cdot).
\end{aligned}$$

b) First notice that $Z \cap H$ is a random closed set in H as, for any $C \in \mathcal{C}^{(H)}$, we have

$$\begin{aligned}
\big\{Z \cap H \in \big(\mathcal{F}^{(H)}\big)^C\big\} = \big\{Z \cap H \in \mathcal{F}^{(H)} \setminus \mathcal{F}_C^{(H)}\big\} = \Omega \setminus \big\{Z \cap H \in \mathcal{F}_C^{(H)}\big\} &= \Omega \setminus \big\{Z \in \mathcal{F}_{H \cap C}\big\} \\
&= \Omega \setminus \big\{Z \in \mathcal{F}_C\big\} \\
&= \big\{Z \in \mathcal{F}^C\big\} \\
&\in \mathcal{A}.
\end{aligned}$$

To prove that $Z \cap H$ is a Boolean model, we calculate the capacity functional. For $C \in \mathcal{C}^{(H)}$ we have, by Theorem 3.16 and Theorem 3.6,

$$\begin{aligned}
T_{Z \cap H}(C) = \mathbb{P}\big(Z \cap H \cap C \neq \varnothing\big) = T_Z(H \cap C) \\
&= 1 - \exp\left(-\gamma \int_{\mathcal{C}^d} \int_{\mathbb{R}^d} \mathbb{1}\big\{(K + x) \cap H \cap C \neq \varnothing\big\} \mathrm{d}\lambda^d(x) \, \mathrm{d}\mathbb{Q}(K)\right) \\
&= 1 - \exp\left(-\gamma \int_{\mathcal{C}^d} \int_{\mathbb{R}^d} \mathbb{1}\big\{(K + x) \cap H \in \mathcal{F}_C^{(H)}\big\} \mathrm{d}\lambda^d(x) \, \mathrm{d}\mathbb{Q}(K)\right).
\end{aligned}$$

Consider the intensity measure of Φ_H,

$$\begin{aligned}
\Theta_H(\mathcal{H}) = \mathbb{E}\big[\Phi_H(\mathcal{H})\big] &= \mathbb{E}\left[\int_{\mathcal{C}^d} \mathbb{1}\{C \cap H \neq \varnothing\} \cdot \mathbb{1}\{C \cap H \in \cdot\} \mathrm{d}\Phi(C)\right] \\
&= \gamma \int_{\mathcal{C}^d} \int_{\mathbb{R}^d} \mathbb{1}\big\{(K + x) \cap H \in \mathcal{H}\big\} \mathrm{d}\lambda^d(x) \, \mathrm{d}\mathbb{Q}(K), \qquad \mathcal{H} \in \mathcal{B}\big(\mathcal{F}^{(H)}\big) \cap (\mathcal{F}^{(H)})',
\end{aligned}$$

on $\mathcal{F}^{(H)}$ and note that Θ_H is locally finite, as

$$\Theta_H\big(\mathcal{F}_C^{(H)}\big) = \Theta(\mathcal{F}_C) < \infty, \qquad C \in \mathcal{C}^{(H)},$$

by the local finiteness of Θ. Now, if Z_H is a Boolean model corresponding to Φ_H (in H), then the capacity functional is (by Theorem 3.16)

$$\begin{aligned} T_{Z_H}(C) &= 1 - \exp\Big(-\Theta_H(\mathcal{F}_C^{(H)})\Big) \\ &= 1 - \exp\Big(-\gamma \int_{\mathcal{C}^d}\int_{\mathbb{R}^d} \mathbb{1}\big\{(K+x)\cap H \in \mathcal{F}_C^{(H)}\big\}\,\mathrm{d}\lambda^d(x)\,\mathrm{d}\mathbb{Q}(K)\Big) \\ &= T_{Z\cap H}(C), \qquad C \in \mathcal{C}^{(H)}. \end{aligned}$$

As the capacity functional determines the distribution of a random closed set, we have $Z_H \overset{d}{=} Z \cap H$.

Problem 2 (On the condition in Theorem 4.8)

Let $\gamma > 0$, and let $\mathbb{Q}$ be a probability measure on $\mathcal{C}'$ such that

$$\int_{\mathcal{C}'} \lambda^d(K+C)\,\mathrm{d}\mathbb{Q}(K) < \infty, \qquad C \in \mathcal{C}^d.$$

Consider a Poisson process Ψ in $\mathbb{R}^d \times \mathcal{C}'$ with intensity measure $\gamma \cdot \lambda^d \otimes \mathbb{Q}$. Further, let $C \in \mathcal{C}'$ and define

$$N_C := \int_{\mathbb{R}^d\times\mathcal{C}'} \mathbb{1}\big\{(K+x)\cap C \neq \varnothing\big\}\,\mathrm{d}\Psi(x,K).$$

Prove that

$$\mathbb{E}\big[r^{N_C}\big] < \infty, \qquad r \in \mathbb{R}.$$

Proposed solution: We apply the mapping theorem for Poisson processes to the function

$$T : \mathbb{R}^d \times \mathcal{C}' \to \mathcal{C}', \quad (x,K) \mapsto T(x,K) := K + x.$$

Then, we have

$$N_C = \int_{\mathbb{R}^d\times\mathcal{C}'} \mathbb{1}\big\{(K+x) \in \mathcal{F}_C\big\}\,\mathrm{d}\Psi(x,K) = \int_{\mathbb{R}^d\times\mathcal{C}'} \mathbb{1}\big\{T(x,K) \in \mathcal{F}_C\big\}\,\mathrm{d}\Psi(x,K) = \Phi(\mathcal{F}_C),$$

where $\Phi = T(\Psi)$ is a stationary Poisson process with intensity measure

$$\Lambda(\cdot) = \big(T(\gamma\cdot\lambda^d\otimes\mathbb{Q})\big)(\cdot) := (\gamma\cdot\lambda^d\otimes\mathbb{Q})\big(T^{-1}(\cdot)\big) = \gamma\int_{\mathcal{C}'}\int_{\mathbb{R}^d} \mathbb{1}\{K+x\in\cdot\}\,\mathrm{d}\lambda^d(x)\,\mathrm{d}\mathbb{Q}(K).$$

By Theorem... (Satz 1.1.2), we have $\Lambda(\mathcal{F}_C) < \infty$ (using that $\mathcal{F}_C$ is compact and Λ is locally finite), and therefore

$$\begin{aligned} \mathbb{E}\big[r^{N_C}\big] = \mathbb{E}\Big[r^{\Phi(\mathcal{F}_C)}\Big] = \sum_{k=0}^{\infty} r^k\,\mathbb{P}\big(\Phi(\mathcal{F}_C)=k\big) &= \sum_{k=0}^{\infty} r^k \cdot \frac{e^{-\Lambda(\mathcal{F}_C)}\cdot\Lambda(\mathcal{F}_C)^k}{k!} \\ &= e^{-\Lambda(\mathcal{F}_C)}\,e^{r\cdot\Lambda(\mathcal{F}_C)} \\ &= e^{(r-1)\cdot\Lambda(\mathcal{F}_C)} \\ &= \big(e^{\Lambda(\mathcal{F}_C)}\big)^{r-1} \\ &< \infty, \qquad r\in\mathbb{R}. \end{aligned}$$

Problem 3 (Random q-flats)

Let $K, K_0 \in \mathcal{K}^d$ with $K \subset K_0$ and $V_d(K_0) > 0$. Let $q \in \{0,\dots,d-1\}$ and

$$A_{K_0} := \big\{E \in A(d,q) : K_0 \cap E \neq \varnothing\big\}.$$

An $A(d,q)$-valued random element X_q with distribution $\frac{1}{\mu_q(A_{K_0})}\cdot\mu_q(\,\cdot\,\cap A_{K_0})$ is called an isotropic random q-flat through K_0.

a) Calculate the probability $\mathbb{P}(X_q \cap K \neq \varnothing)$ in terms of the intrinsic volumes of K and K_0.

b) Let $d = 2$, $e \in \mathbb{R}^2$ with $\|e\| = 1$, and $0 < r \leqslant 1$. Determined the probability the a random 1-plane in $B^2 := B(0,1) \subset \mathbb{R}^2$ intersects the line segment $[-r \cdot e, r \cdot e]$.

c) Let $d = 2$ and let X_1 be a random 1-plane in B^2. Calculate the probability p of the line segment $X_1 \cap B^2$ being longer than $\sqrt{3}$.

d) Let $d = 2$ and let X_1 be a random 1-plane in B^2. Denote by a the side length of an equilateral triangle T_a and assume that the center of the largest circle contained in T_a is the origin. Find a value for a such that $\mathbb{P}(X_1 \cap T_a \neq \varnothing) = p$, where p is as in part c).

e) Interpret the results from c) and d) in the context of Bertrand's paradox (see Problem 1 of Work sheet 1).

Proposed solution:

a) We have, by the Crofton formula (Theorem ...),

$$\begin{aligned}
\mathbb{P}(X_q \cap K \neq \varnothing) = \frac{1}{\mu_q(A_{K_0})} \cdot \mu_q(A_K \cap A_{K_0}) &= \frac{\mu_q(A_K)}{\mu_q(A_{K_0})} \\
&= \frac{\int_{A(d,q)} \mathbb{1}\{K \cap F \neq \varnothing\} \,\mathrm{d}\mu_q(F)}{\int_{A(d,q)} \mathbb{1}\{K_0 \cap F \neq \varnothing\} \,\mathrm{d}\mu_q(F)} \\
&= \frac{\int_{A(d,q)} V_0(F \cap K) \,\mathrm{d}\mu_q(F)}{\int_{A(d,q)} V_0(F \cap K_0) \,\mathrm{d}\mu_q(F)} \\
&= \frac{c_{0,d}^{q,d-q} \cdot V_{d-q}(K)}{c_{0,d}^{q,d-q} \cdot V_{d-q}(K_0)} \\
&= \frac{V_{d-q}(K)}{V_{d-q}(K_0)}.
\end{aligned}$$

b) Denote by X_1 the random 1-plane in B^2. With part a) and Remark 3.36 (ii), we get

$$\mathbb{P}\Big(X_1 \cap [-r \cdot e, r \cdot e] \neq \varnothing\Big) = \frac{V_1\left([-r \cdot e, r \cdot e]\right)}{V_1(B^2)} = \frac{2r}{\frac{1}{2} \cdot 2\pi} = \frac{2r}{\pi}.$$

c) By the geometric observations in Problem 1 on Work sheet 1, the random chord $X_1 \cap B^2$ of B^2 is longer than $\sqrt{3}$ precisely when the chord does not intersect the ball $B\big(0, \frac{1}{2}\big)$. By part a) and Remark 3.36 (ii), we obtain

$$\begin{aligned}
p := \mathbb{P}\Big(V_1(X_1 \cap B^2) \geqslant \sqrt{3}\Big) = \mathbb{P}\Big((X_1 \cap B^2) \cap B\big(0, \tfrac{1}{2}\big) \neq \varnothing\Big) &= \mathbb{P}\Big(X_1 \cap B\big(0, \tfrac{1}{2} \neq \varnothing\big)\Big) \\
&= \frac{V_1\left(B\big(0, \frac{1}{2}\big)\right)}{V_1(B^2)} \\
&= \frac{\frac{1}{2} \cdot \pi}{\frac{1}{2} \cdot 2\pi} \\
&= \frac{1}{2}.
\end{aligned}$$

d) By part a) and Remark 3.36 (ii), we have

$$\mathbb{P}\big(X_1 \cap T_a \neq \varnothing\big) = \frac{V_1(T_a)}{V_1(B^2)} = \frac{\frac{1}{2} \cdot 3a}{\frac{1}{2} \cdot 2\pi} = \frac{3a}{2\pi}.$$

Thus, we have $\mathbb{P}\big(X_1 \cap T_a \neq \varnothing\big) = p$ if, and only if, $a = \frac{\pi}{3}$.

e)

Problem 4 (Randomly moving convex bodies)

Let $M, K, K_0 \in \mathcal{K}^d$ with $K \subset K_0$ and $V_d(K_0) > 0$. Define

$$G_{K_0,M} := \{g \in G_d : K_0 \cap gM \neq \varnothing\}.$$

A G_d-valued random element $\alpha = \alpha_{K_0,M}$ with distribution $\frac{1}{\mu(G_{K_0,M})} \cdot \mu(\,\cdot\, \cap G_{K_0,M})$ is called a $G_{K_0,M}$-isotropic randomly moving convex body.

a) Calculate the probability $\mathbb{P}(\alpha M \cap K \neq \varnothing)$ in terms of the intrinsic volumes of M, K, and K_0.

b) Let $d = 2$, $e \in \mathbb{R}^2$ with $\|e\| = 1$, $0 < r \leqslant 1$, and $K_0 = B^2$. Determine the probability

$$\mathbb{P}\Big(\alpha\big([0,1]^2\big) \cap [-r\cdot e, r\cdot e] \neq \varnothing\Big).$$

Hint: You may use the formula $V_i\big([0,1]^d\big) = \binom{d}{i}$.

c) Let $d = 2$, $e \in \mathbb{R}^2$ with $\|e\| = 1$, $0 < r \leqslant 1$, $K_0 = B^2$, and $a > 0$. Calculate the probability

$$\mathbb{P}\Big(\alpha\big([0,a]\big) \cap [-r\cdot e, r\cdot e] \neq \varnothing\Big).$$

d) In part c), what happens if we let $a \to \infty$?

Proposed solution:

a) Since $G_{K,M} \subset G_{K_0,M}$, the principle kinematic formula (Theorem ...) yields

$$\begin{aligned}
\mathbb{P}\big(\alpha M \cap K \neq \varnothing\big) = \frac{\mu(G_{K,M} \cap G_{K_0,M})}{\mu(G_{K_0,M})} = \frac{\mu(G_{K,M})}{\mu(G_{K_0,M})} &= \frac{\int_{G_d} \mathbb{1}\{gM \cap K \neq \varnothing\}\,\mathrm{d}\mu(g)}{\int_{G_d} \mathbb{1}\{gM \cap K_0 \neq \varnothing\}\,\mathrm{d}\mu(g)} \\
&= \frac{\int_{G_d} V_0(gM \cap K)\,\mathrm{d}\mu(g)}{\int_{G_d} V_0(gM \cap K_0)\,\mathrm{d}\mu(g)} \\
&= \frac{\sum_{k=0}^{d} c_{0,d}^{k,d-k} \cdot V_k(K) \cdot V_{d-k}(M)}{\sum_{k=0}^{d} c_{0,d}^{k,d-k} \cdot V_k(K_0) \cdot V_{d-k}(M)}.
\end{aligned}$$

b) Part a), Problem 3 on Work sheet 9, and the hint give

$$\begin{aligned}
\mathbb{P}\Big(\alpha\big([0,1]^2\big) \cap [-r\cdot e, r\cdot e] \neq \varnothing\Big) &= \frac{\sum_{k=0}^{2} c_{0,2}^{k,2-k} \cdot V_k\big([-r\cdot e, r\cdot e]\big) \cdot V_{2-k}\big([0,1]^2\big)}{\sum_{k=0}^{2} c_{0,2}^{k,2-k} \cdot V_k(B^2) \cdot V_{2-k}\big([0,1]^2\big)} \\
&= \frac{c_{0,2}^{0,2} \cdot V_0\big([-r\cdot e, r\cdot e]\big) \cdot V_2\big([0,1]^2\big) + c_{0,2}^{1,1} \cdot V_1\big([-r\cdot e, r\cdot e]\big) \cdot V_1\big([0,1]^2\big)}{\sum_{k=0}^{2} c_{0,2}^{k,2-k} \cdot \binom{2}{k} \cdot \frac{\kappa_2}{\kappa_{2-k}} \cdot \binom{2}{2-k}} \\
&= \frac{1 + \frac{4r}{\pi}}{\sum_{k=0}^{2} \kappa_k \cdot \binom{2}{k}} \\
&= \frac{1 + \frac{4r}{\pi}}{1 + 4 + \pi} \\
&\approx 0.12 + 0.156 \cdot r,
\end{aligned}$$

where we also used that $c_{0,2}^{0,2} = 1$, $c_{0,2}^{1,1} = \frac{1!\cdot\kappa_1}{0!\cdot\kappa_0} \cdot \frac{1!\cdot\kappa_1}{2!\cdot\kappa_2} = \frac{2}{\pi}$ as well as

$$c_{0,2}^{k,2-k} = \frac{k!\cdot\kappa_k}{0!\cdot\kappa_0} \cdot \frac{(2-k)!\cdot\kappa_{2-k}}{2!\cdot\kappa_2} = \frac{k!\cdot\kappa_k \cdot (2-k)!\cdot\kappa_{2-k}}{2\kappa_2} = \frac{\kappa_k \cdot \kappa_{2-k}}{\binom{2}{2-k} \cdot \kappa_2}.$$

c) By part a), we have

$$
\begin{aligned}
\mathbb{P}\Big(\alpha\big([0,a]\big)\cap[-r\cdot e, r\cdot e]\neq\varnothing\Big) &= \frac{\sum_{k=0}^{2} c_{0,2}^{k,2-k}\cdot V_k\big([-r\cdot e, r\cdot e]\big)\cdot V_{2-k}\big([0,a]\big)}{\sum_{k=0}^{2} c_{0,2}^{k,2-k}\cdot V_k(B^2)\cdot V_{2-k}\big([0,a]\big)} \\
&= \frac{c_{0,2}^{1,1}\cdot V_1\big([-r\cdot e, r\cdot e]\big)\cdot V_1\big([0,a]\big)}{c_{0,2}^{1,1}\cdot V_1(B^2)\cdot V_1\big([0,a]\big)+c_{0,2}^{2,0}\cdot V_2(B^2)\cdot V_0\big([0,a]\big)} \\
&= \frac{\frac{2}{\pi}\cdot 2r\cdot a}{\frac{2}{\pi}\cdot\frac{1}{2}\cdot 2\pi\cdot a+1\cdot\pi\cdot 1} \\
&= \frac{\frac{4r\cdot a}{\pi}}{2a+\pi} \\
&= \frac{4r\cdot a}{2\pi\cdot a+\pi^2}.
\end{aligned}
$$

d) Apparently, $\mathbb{P}\big(\alpha\big([0,a]\big)\cap[-r\cdot e, r\cdot e]\neq\varnothing\big)\to\frac{4r}{2\pi}$ as $a\to\infty$.