Solutions for Work Sheet 4

Problem 1 (Measurability of measure-valued functions)

Let $M(\mathbb{X})$ be the set of all locally finite measures on a separable metric space $(\mathbb{X}, \rho)$ with Borel-σ-field $\mathcal{X} = \mathcal{B}(\mathbb{X})$. For $B \in \mathcal{X}$, denote by $\pi_B : M(\mathbb{X}) \to [0, \infty]$ the map $\mu \mapsto \pi_B(\mu) := \mu(B)$. Let $\mathcal{M}(\mathbb{X})$ be the smallest σ-field on $M(\mathbb{X})$ for which all maps π_B, $B \in \mathcal{X}$, are measurable. Furthermore, let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space.

a) Consider a map $\eta : \Omega \to M(\mathbb{X})$. Show the equivalence of the following statements:

   (i) η is a random measure on E.

   (ii) For all sets $B \in \mathcal{X}$, $\eta(B) : \Omega \to [0, \infty]$ is a random variable.

b) Prove that the mapping $\mathcal{S} : M(\mathbb{X}) \times M(\mathbb{X}) \to M(\mathbb{X}), (\mu, \nu) \mapsto \mu + \nu$ is measurable.

c) Verify that either part a) and part b) imply that the sum of two random measures is itself a random measure.

Proposed solution:

a) We show the two implications separately:

(i) $\Rightarrow$ (ii) Let η be a random measure on $\mathbb{X}$. The maps

$$\pi_B : M(\mathbb{X}) \to [0, \infty], \quad \mu \mapsto \mu(B), \quad B \in \mathcal{X},$$

are measurable by definition of the σ-field $\mathcal{M}(\mathbb{X})$. Therefore, $\eta(B) = \pi_B \circ \eta$ is measurable.

(ii) $\Rightarrow$ (i) The σ-field $\mathcal{M}(\mathbb{X})$ is generated by $\big\{\{\mu \in M(\mathbb{X}) : \mu(B) \in A\} : B \in \mathcal{X},\ A \in \mathcal{B}(\overline{\mathbb{R}})\big\}$. If $\eta(B)$ is a random variable (and hence measurable) for every $B \in \mathcal{X}$, then

$$\eta^{-1}\Big(\big\{\mu \in M(\mathbb{X}) : \mu(B) \in A\big\}\Big) = \big\{\omega \in \Omega : \eta(\omega, B) \in A\big\} = \big\{\eta(B) \in A\big\} \in \mathcal{A},$$

for all $B \in \mathcal{X}$ and $A \in \mathcal{B}(\overline{\mathbb{R}})$. Thus, η is a random measure.

b) Since $\mathcal{M}(\mathbb{X}) = \sigma\big(\{M_{B,r} : B \in \mathcal{X}, r \geqslant 0\}\big)$, where

$$M_{B,r} := \pi_B^{-1}\big((-\infty, r]\big) = \big\{\mu \in M(\mathbb{X}) : \mu(B) \leqslant r\big\},$$

it suffices to show that $\mathcal{S}^{-1}(M_{B,r}) \in \mathcal{M}(\mathbb{X}) \otimes \mathcal{M}(\mathbb{X})$, for all $B \in \mathcal{X}$, $r \geqslant 0$. Notice that

$$\begin{aligned}
\mathcal{S}^{-1}(M_{B,r}) &= \big\{(\mu, \nu) \in M(\mathbb{X}) \times M(\mathbb{X}) : (\mu + \nu) \in M_{B,r}\big\} \\
&= \big\{(\mu, \nu) \in M(\mathbb{X}) \times M(\mathbb{X}) : \mu(B) + \nu(B) \leqslant r\big\} \\
&= \big\{(\mu, \nu) \in M(\mathbb{X}) \times M(\mathbb{X}) : \pi_B(\mu) + \pi_B(\nu) \leqslant r\big\} \\
&\in \mathcal{M}(\mathbb{X}) \otimes \mathcal{M}(\mathbb{X}),
\end{aligned}$$

where we used that $\pi_B : M(\mathbb{X}) \to [0, \infty]$, and hence also $M(\mathbb{X}) \times M(\mathbb{X}) \to [0, \infty]^2$, $(\mu, \nu) \mapsto \big(\pi_B(\mu), \pi_B(\nu)\big)$ and $M(\mathbb{X}) \times M(\mathbb{X}) \to [0, \infty]$, $(\mu, \nu) \mapsto \pi_B(\mu) + \pi_B(\nu)$, are measurable functions.

c) Assume that $\eta_1, \eta_2 : \Omega \to M(\mathbb{X})$ are random measures on $\mathbb{X}$ (in particular, they are measurable maps). Then $\eta := \eta_1 + \eta_2 = \mathcal{S}(\eta_1, \eta_2)$ is measurable by part b). Moreover, $\eta \in M(\mathbb{X})$ (i.e., η is locally finite), since we have $\eta(B) = \eta_1(B) + \eta_2(B) < \infty$, for every set $B \in \mathcal{X}_b$, as $\eta_1, \eta_2 \in M(\mathbb{X})$. Alternatively, the measurability follows from part a) when considering that $\eta(B) = \eta_1(B) + \eta_2(B)$ is a random variable, for every $B \in \mathcal{X}$, since $\eta_1(B), \eta_2(B)$ are random variables by part a).

Problem 2 (Measurability of point processes)

Let $X_1, X_2, \dots$ be random elements of a separable metric space $(\mathbb{X}, \rho)$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ (where the X_j need not be independent and can have different distributions). Let τ be an $\mathbb{N}_0$-valued random variable. Prove that

$$\Phi := \sum_{j=1}^{\tau} \delta_{X_j}$$

is a point process, that is, the map $\Phi : \Omega \to N(\mathbb{X})$ is measurable.

Note: If $X_1, X_2, \dots$ are i.i.d. random elements of $\mathbb{X}$ and τ is independent of $(X_j)_{j\in\mathbb{N}}$, then the process Φ is the mixed binomial process from Example 2.17.

Proposed solution: Let $B \in \mathcal{X}$. For each $j \in \mathbb{N}$ the map

$$\Omega \ni \omega \mapsto \mathbb{1}\big\{\tau(\omega) \geqslant j\big\} \cdot \mathbb{1}\big\{X_j(\omega) \in B\big\}$$

is measurable. Thus,

$$\Omega \ni \omega \mapsto \Phi(\omega, B) = \sum_{j=1}^{\tau(\omega)} \delta_{X_j(\omega)}(B) = \sum_{j=1}^{\infty} \mathbb{1}\big\{j \leqslant \tau(\omega)\big\} \cdot \delta_{X_j(\omega)}(B) = \sum_{j=1}^{\infty} \mathbb{1}\big\{\tau(\omega) \geqslant j\big\} \cdot \mathbb{1}\big\{X_j(\omega) \in B\big\}$$

is measurable as a sum and limit of measurable functions. Part a) of Problem 1 implies that $\Phi : \Omega \to M(\mathbb{X})$ is measurable, and we trivially have $\Phi(\omega, \cdot\,) \in N(\mathbb{X})$ for each $\omega \in \Omega$.

Problem 3

Let $\big(\Omega, \mathcal{A}, \mathbb{P}\big)$ be the probability space which underlies the following random elements. Let X_1, X_2 be uniformly distributed over the open unit disc $B^\circ(0,1) := \big\{x \in \mathbb{R}^2 : \|x\| < 1\big\}$, let X_3 be uniformly distributed over the discrete set $\big\{(0,0), (0,1), (1,0), (1,1)\big\}$, and let X_4 be uniformly distributed over the segment $[-1,1] \times \{0\}$. Define the point process $\Phi := \sum_{j=1}^{4} \delta_{X_j}$. Calculate $\mathbb{E}\big[\Phi(B)\big]$ for any $B \in \mathcal{B}(\mathbb{R}^2)$. Apply this knowledge about the intensity measure to calculate

$$\mathbb{E}\left[\int_{[-1,1]^2} \big(x^2 + y^2\big)\, \Phi\big(\mathrm{d}(x,y)\big)\right].$$

Proposed solution: Apparently,

$$\mathbb{E}\big[\Phi(B)\big] = \sum_{j=1}^{4} \mathbb{E}\Big[\delta_{X_j}(B)\Big] = \sum_{j=1}^{4} \mathbb{E}\Big[\mathbb{1}\{X_j \in B\}\Big] = \sum_{j=1}^{4} \mathbb{P}\big(X_j \in B\big) = \sum_{j=1}^{4} \mathbb{P}^{X_j}(B).$$

In particular, we have

$$\begin{aligned}
\mathbb{P}^{X_1}(\cdot) &= \mathbb{P}^{X_2}(\cdot) = \frac{1}{\pi}\Big(\lambda^2|_{B^\circ(0,1)}\Big)(\cdot),\\
\mathbb{P}^{X_3}(\cdot) &= \frac{1}{4}\,\delta_{(0,0)}(\cdot) + \frac{1}{4}\,\delta_{(0,1)}(\cdot) + \frac{1}{4}\,\delta_{(1,0)}(\cdot) + \frac{1}{4}\,\delta_{(1,1)}(\cdot),\\
\mathbb{P}^{X_4}(\cdot) &= \frac{1}{2}\,\lambda^1 \circ I^{-1}(\cdot), \quad \text{where} \quad I : \begin{cases} [-1,1] & \to [-1,1] \times \{0\}, \\ t & \mapsto \quad (t, 0). \end{cases}
\end{aligned}$$

With Campbell's formula (Theorem 2.21), we obtain

$$
\begin{aligned}
\mathbb{E}\left[\int_{[-1,1]^2} (x^2+y^2)\,\Phi\big(\mathrm{d}(x,y)\big)\right] &= \int_{[-1,1]^2} (x^2+y^2)\,(\mathbb{E}\Phi)\big(\mathrm{d}(x,y)\big) \\
&= \frac{2}{\pi}\int_{[-1,1]^2} (x^2+y^2)\left(\lambda^2|_{B^\circ(0,1)}\right)\big(\mathrm{d}(x,y)\big) + \frac{1}{4}\cdot 0 + \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot 2 \\
&\quad + \frac{1}{2}\int_{[-1,1]^2} (x^2+y^2)\left(\lambda^1\circ I^{-1}\right)\big(\mathrm{d}(x,y)\big) \\
&= \frac{2}{\pi}\int_0^1\int_0^{2\pi} \left(r^2\cdot r\right)\mathrm{d}\varphi\,\mathrm{d}r + 1 + \frac{1}{2}\int_{-1}^1 x^2\,\mathrm{d}x \\
&= 4\int_0^1 r^3\,\mathrm{d}r + 1 + \frac{1}{2}\left[\frac{1}{3}x^3\right]_{-1}^1 \\
&= 4\left[\frac{1}{4}r^4\right]_0^1 + 1 + \frac{1}{2}\left(\frac{1}{3}+\frac{1}{3}\right) \\
&= \frac{7}{3}.
\end{aligned}
$$

Problem 4 (Equality in distribution of point processes – A proof of Remark 2.24)

Let Φ and Φ' be point processes on a separable metric space $(\mathbb{X},\rho)$. Prove that the following are equivalent:

(iv) For all $m\in\mathbb{N}$ and any $B_1,\dots,B_m\in\mathcal{X}_b$,

$$\big(\Phi(B_1),\dots,\Phi(B_m)\big) \overset{d}{=} \big(\Phi'(B_1),\dots,\Phi'(B_m)\big).$$

(iv') For all $m\in\mathbb{N}$ and any pairwise disjoint $B_1,\dots,B_m\in\mathcal{X}_b$,

$$\big(\Phi(B_1),\dots,\Phi(B_m)\big) \overset{d}{=} \big(\Phi'(B_1),\dots,\Phi'(B_m)\big).$$

Proposed solution: The implication $(iv)\implies(iv')$ holds trivially, so we only have to prove that (iv') implies (iv). Let $m\in\mathbb{N}$ and take some arbitrary sets $B_1,\dots,B_m\in\mathcal{X}_b$. We divide the set $B_1\cup\ldots\cup B_m$ into disjoint parts such that any of the sets $B_1,\dots,B_m$ can be recovered from this partition, and then apply (iv') to these disjoint parts.

Let $\ell_1,\dots,\ell_m\in\mathbb{N}_0$. For an index set $J\subset[m]:=\{1,\dots,m\}$, we define $D_J:=\bigcap_{j\in J}B_j\cap\bigcap_{j\notin J}B_j^c$. It holds that $B_i=\bigcup_{J\subset[m]\,:\,i\in J}D_J$, for every $i\in[m]$. Indeed,

- '$\subset$': let $x\in B_i$, and set $J:=\big\{j\in[m]:x\in B_j\big\}$. Then, $i\in J$ and $x\in D_J$, that is, $x\in\bigcup_{J\subset[m]\,:\,i\in J}D_J$, and
- '$\supset$': if $x\in D_J$ for some $J\subset[m]$ with $i\in J$, then by definition of D_J, $x\in B_i$.

We put

$$M:=\left\{(k_J)_{J\subset[m]\,:\,J\neq\varnothing}\in\mathbb{N}_0^{2^m-1}\;\middle|\;\forall j\in[m]:\ell_j=\sum\nolimits_{J\subset[m]\,:\,j\in J}k_J\right\}.$$

The set M parameterizes all possible distributions of the total mass of a point process Φ on $B_1\cup\ldots\cup B_m$, for which $\Phi(B_j)=\ell_j$, $j\in[m]$, to the partition $(D_J)_{J\subset[m],\,J\neq\varnothing}$. We then have

$$\bigcap_{j=1}^m\big\{\mu\in N(\mathbb{X}):\pi_{B_j}(\mu)=\ell_j\big\}=\dot{\bigcup}_{k\in M}\bigcap\nolimits_{J\subset[m]\,:\,J\neq\varnothing}\big\{\mu\in N(\mathbb{X}):\mu(D_J)=k_J\big\}.$$

For the prove of this fact, we consider the two inclusions separately:

'$\subset$': Let $\mu \in \bigcap_{j=1}^{m} \{\mu \in N(\mathbb{X}) : \pi_{B_j}(\mu) = \ell_j\}$, i.e., $\mu(B_i) = \ell_i$, for $i \in [m]$. Put

$$k_J := \mu(D_J) \in \mathbb{N}_0, \quad \text{for } J \subset [m],\ J \neq \varnothing.$$

For $j \in [m]$, we have

$$\sum_{J \subset [m] \,:\, j \in J} k_J = \sum_{J \subset [m] \,:\, j \in J} \mu(D_J) = \mu\left(\dot{\bigcup}_{J \subset [m] \,:\, j \in J} D_J\right) = \mu(B_j) = \ell_j,$$

so $(k_J)_{J \subset [m] \,:\, J \neq \varnothing} \in M$, and, for all $J \subset [m]$ with $J \neq \varnothing$, we have $\mu(D_J) = k_J$.

'$\supset$': Now, let $\mu \in \dot{\bigcup}_{k \in M} \bigcap_{J \subset [m] \,:\, J \neq \varnothing} \{\mu \in N(\mathbb{X}) : \mu(D_J) = k_J\}$. Then, we find $k \in M$ such that $\mu(D_J) = k_J$, for all $J \subset [m]$ with $J \neq \varnothing$, and $\sum_{J \subset [m] \,:\, j \in J} k_J = \ell_j$, $j \in [m]$. Therefore,

$$\mu(B_j) = \mu\left(\dot{\bigcup}_{J \subset [m] \,:\, j \in J} D_J\right) = \sum_{J \subset [m] \,:\, j \in J} \mu(D_J) = \sum_{J \subset [m] \,:\, j \in J} k_J = \ell_j, \quad j \in [m],$$

and thus $\mu \in \bigcap_{j=1}^{m} \{\mu \in N(\mathbb{X}) : \pi_{B_j}(\mu) = \ell_j\}$.

We conclude that

$$\begin{aligned}
\mathbb{P}\Big(\Phi(B_1) = \ell_1, \dots, \Phi(B_m) = \ell_m\Big) &= \mathbb{P}\left(\Phi \in \bigcap_{j=1}^{m} \{\mu \in N(\mathbb{X}) : \pi_{B_j}(\mu) = \ell_j\}\right) \\
&= \sum_{k \in M} \mathbb{P}\left(\Phi \in \bigcap_{J \subset [m] \,:\, J \neq \varnothing} \{\mu \in N(\mathbb{X}) : \mu(D_J) = k_J\}\right) \\
&\overset{(iv')}{=} \sum_{k \in M} \mathbb{P}\left(\Phi' \in \bigcap_{J \subset [m] \,:\, J \neq \varnothing} \{\mu \in N(\mathbb{X}) : \mu(D_J) = k_J\}\right) \\
&= \mathbb{P}\Big(\Phi'(B_1) = \ell_1, \dots, \Phi'(B_m) = \ell_m\Big),
\end{aligned}$$

which immediately implies (iv).