Karlsruhe Institute of Technology

Institute of Stochastics

Stochastic Geometry | Summer term 2020
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Work Sheet 1

Instructions for week 1 (April 20^{th} to April 24^{th}):

- Watch the introductory video on the lecture to learn where random geometric structures appear in technology and real life, and try to come up with answers to the control questions 1) and 2).
- Carefully read and elaborate on Section 1.1 of Pierre Calka's "Some classical problems in random geometry" on Buffon's needle problem. Answer the control questions 3) and 4), and try to solve Problem 1 on Bertrand's paradox.
- Work through Section 1.1 of the lecture notes up to (and including) Theorem 1.3 and its proof. Answer the control question 5), and solve Problems 2 and 3 of the exercises below.
- Please hand in your solutions to the exercises for correction until the morning of Monday, April 27^{th}. For the procedure, please have a look at the general information in Ilias. The submission of solutions is voluntary.

Control questions to monitor your progress:

1) Find at least three examples of complex real world structures, that are not mentioned in the introduction slides, exhibiting some or all of the features mentioned on slide 4 (the slide called "Some observations").
2) Random point patterns are of major interest in the analysis of spatial data. Come up with three applications where the understanding of random or mildly structured point patterns could be essential.
3) In the given reference on Buffon's needle problem it is stated that the needle crosses one of the lines precisely when $2R \leqslant \ell \cos(\Theta)$ [using the author's notation]. Can you verify this claim with a simple calculation? (Hint: Use the geometric definition of the cosine in a right-angled triangle.)
4) In the context of Buffon's needle problem, it is explained how one can approximate π by (randomly) letting needles fall onto a field of equidistant lines and counting the number of crossings of the lines by the needles. Can you devise a similar approximation of π by letting a coin fall onto such strips on the floor repeatedly? (Hint: Consider the coin as a 'noodle'.)
5) Let $(E, \mathcal{O}_E)$ be a locally compact Hausdorff space with a countable base of the topology, and denote by $\mathcal{F}$ and $\mathcal{C}$ the closed and compact subsets of E, respectively. For $A \subset E$, let

$$\mathcal{F}_A := \{F \in \mathcal{F} : F \cap A \neq \varnothing\}, \qquad \mathcal{F}^{\subset A} := \{F \in \mathcal{F} : F \subset A\},$$

and $\mathcal{F}^{\subset A}_{A_1,\dots,A_k} = \mathcal{F}^{\subset A} \cap \mathcal{F}_{A_1} \cap \ldots \cap \mathcal{F}_{A_k}$. Convince yourself that

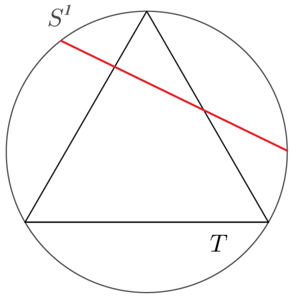
$$\tilde{\tau} := \left\{ \mathcal{F}^{\subset E \setminus C}_{G_1,\dots,G_k} \ : \ C \in \mathcal{C},\ G_1, \dots, G_k \in \mathcal{O}_E,\ k \in \mathbb{N}_0 \right\}$$

is the base of a unique topology on $\mathcal{F}$ and that this topology is exactly the Fell topology introduced in the lecture.

Exercises for week 1:

Problem 1 (Bertrand's paradox)

Consider the unit circle $S^1 \subset \mathbb{R}^2$ and an equilateral triangle T whose vertices lie on S^1. What is the probability that a randomly chosen circular chord is longer than the edges of the triangle? Does the probability depend on the choice of your model for this situation?



Problem 2 (Theorem 1.2)

Let $(E, \mathcal{O}_E)$ be a locally compact Hausdorff space with a countable base of the topology. Denote by $\mathcal{F}$ the collection of closed subsets of E and write $\mathcal{C}$ for the collection of compact subsets of E. For $A \subset E$, let $\mathcal{F}^A := \{F \in \mathcal{F} : F \cap A = \varnothing\}$ and $\mathcal{F}_A := \{F \in \mathcal{F} : F \cap A \neq \varnothing\}$. Prove the following assertions.

(T) The topology $\mathcal{O}_E$ has a countable base $\mathcal{D}$ which consists of open, relatively compact sets such that any set $O \in \mathcal{O}_E$ is the union of all sets $D \in \mathcal{D}$ that satisfy $\mathrm{cl} D \subset O$.

(1) The space $(\mathcal{F}, \mathcal{O}_\mathcal{F})$ is a compact Hausdorff space with a countable base of the topology (in particular, it is metrizable by Urysohn's metrization theorem).

(2) The subspace $\mathcal{F}' := \mathcal{F} \setminus \{\varnothing\}$ with the subspace topology is locally compact.

(3) The family $\{\mathcal{F}^C \mid C \in \mathcal{C}\}$ is a neighborhood base of $\varnothing$.

Reminder [neighborhood bases]:
A neighborhood of a point x in a topological space $(T, \mathcal{O}_T)$ is a set $V \subset T$ such that there exists an open set $O \in \mathcal{O}_T$ with $x \in O \subset V$. Denote the collection of all neighborhoods of x by $\mathcal{N}(x)$. A neighborhood base (or local base) of x is a collection $\mathfrak{B}(x) \subset \mathcal{N}(x)$ such that for every $V \in \mathcal{N}(x)$ there exists a set $B \in \mathfrak{B}(x)$ with $B \subset V$.

Problem 3 (Theorem 1.3)

Let $(F_i)_{i \in \mathbb{N}}$ be a sequence in $\mathcal{F}$, and $F \in \mathcal{F}$. Consider the following properties:

(1) $F_i \longrightarrow F$ in $(\mathcal{F}, \mathcal{O}_\mathcal{F})$, as $i \to \infty$.

(2) (a) $G \in \mathcal{G}$, $G \cap F \neq \varnothing \implies G \cap F_i \neq \varnothing$ for all $i \in \mathbb{N}$ except finitely many,
(b) $C \in \mathcal{C}$, $C \cap F = \varnothing \implies C \cap F_i = \varnothing$ for all $i \in \mathbb{N}$ except finitely many.

(3) (α) For each $x \in F$ and all but finitely many $i \in \mathbb{N}$ there exist some $x_i \in F_i$ such that $x_i \longrightarrow x$, as $i \to \infty$.
(β) For any subsequence $\left(F_{i_k}\right)_{k \in \mathbb{N}}$ and points $x_{i_k} \in F_{i_k}$ such that $x_{i_k} \overset{k \to \infty}{\longrightarrow} x$, we have $x \in F$.

Show that (2) and (3) are equivalent [the equivalence of (1) and (2) was discussed in the lecture].

The solutions to these problems will be uploaded on April 27th.
Feel free to ask your questions about the exercises in the optional MS-Teams discussion on April 23rd (09:45 h).