Masterthesis

External DLA

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2.1 Namings

Here we list all namings which will be used in this paper. Let $d \in \mathbb{N}$ and $q \in \{0, \ldots, d\}$.

$\mathbb{N} = \{1, 2, 3, \ldots\}$	set of natural numbers (without 0)
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$	
$\mathcal{K}^d$	set of convex and compact sets in $\mathbb{R}^d$
$B_d(r, x) = \{y \in \mathbb{R}^d \mid |x - y| \leq r\}$	d-dimensional closed ball of radius r around x
$S_{d-1}(r, x) = \partial B_d(r, x)$	$(d-1)$-dimensional surface of the d-dimensional ball
$A(d, q)$	set of q-dimensional affine subspaces of $\mathbb{R}^d$
$\mathcal{A}(d, q)$	σ-algebra of $A(d, q)$, as constructed later in the paper
$\mathcal{G} := A(2, 1)$	set of lines in the real plane
$\mathfrak{G} := \mathcal{A}(2, 1)$	
G_d	set of euclidean motions in $\mathbb{R}^d$ (rotations and translations)

2.2 Basic structures

We prepare this script with the following preliminaries. Let $d \in \mathbb{N}$.

***Graphs*.**   We will be interested in the graph $(\mathbb{Z}^d, E)$ with its canonical graph structure, which is two vertices (or points) $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{Z}^d$ form an edge (e.q. $(x, y) \in E$) if and only if there exists exactly one $i \in \{1, \ldots, d\}$ such that $|x_i - y_i| = 1$ and $x_j = y_j$ for all $j \neq i$. For a point $x \in \mathbb{Z}^d$ its set of *neighbours* is defined as

$$N(x) := \{y \in \mathbb{Z}^d \mid (x, y) \in E\}.$$

For a set $A \subset \mathbb{Z}^d$ the *outer boundary* ∂A of A is defined as

$$\partial A := \{y \in \mathbb{Z}^d \setminus A \mid \exists x \in A : \ (x, y) \in E\}$$

Instead of $(\mathbb{Z}^d, E)$ we will write $\mathbb{Z}^d$ from now on.

***Probability Space*.**   Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space which we will base on in this paper. For our space of interest $\mathbb{Z}^d$ we will always use the discrete σ-Algebra which is the power set of $\mathbb{Z}^d$. If for $A \in \mathcal{F}$ we have $\mathbb{P}(A) = 1$ we will say that "A holds $\mathbb{P}$-*a.s.*", or short "A holds *a.s.*" (almost sure).

***Random Walk*.**   A family $(S_n)_{n \in \mathbb{N}}$ of measurable functions $S_n : \Omega \to \mathbb{Z}^d$ is called a *Random Walk on* $\mathbb{Z}^d$ *(starting at* $x \in \mathbb{Z}^d$*)* if and only if $S_0 = x$ *a.s.* and $S_n \in N(S_{n-1})$ *a.s.* for all $n \geq 1$. It further shall hold that

$$\mathbb{P}(S_n = y) = \frac{1}{|N(S_{n-1})|} = \frac{1}{2d} \quad \text{for all } y \in N(S_{n-1}) \quad a.s.$$

Note that $|N(y)| = 2d$ for all $y \in \mathbb{Z}^d$ since every point has two neighbours in every dimension. So a Random Walk can be understood as a particle starting from some point x and moving randomly on the grid choosing its next step uniformly from its neighbours. Further define

$$\mathbb{P}_x(S_n \in A) := \mathbb{P}(S_n \in A | S_0 = x)$$

for any subset $A \subset G$. We define the *hitting times* of A

$$T_A := \min\{n \geq 0 \mid S_n \in A\} \text{ and } T_A^+ := \min\{n \geq 1 \mid S_n \in A\},$$

$T_x := T_{\{x\}}$ and $T_x^+ := T_{\{x\}}^+$ for one element sets and $x \in \mathbb{Z}^d$. The *heat kernel* of the random walk S_n is defined to be

$$p_n(x, y) := \mathbb{P}_x(S_n = y)$$

and the *Green function* as

$$G(x, y) := \sum_{n \geq 0} p_n(x, y).$$

G is well-defined and finite since $\mathbb{Z}^2$ is transient. Similarily for a subset $A \subset G$ the *killed* or *stopped Green function* is defined as

$$G_A(x, y) := \sum_{n \geq 0} \mathbb{P}_x(S_n = y, T_A > n).$$

3 Incremental Aggregate

In this paper we will look at stochastic processes on the set of finite subsets of $\mathbb{Z}^d$, where we start with a one point set at $(0,0)$ and incrementally add a point on the outer boundary of the current cluster according to some distribution. What we get is a randomly, point-by-point growing connected cluster which here we will call *Incremental Aggregate*. Define

$$\mathcal{P}_f := \{A \subset \mathbb{Z}^d \mid \text{A is finite}\}, \tag{3.1}$$

the set of finite subsets of $\mathbb{Z}^d$. Furthermore we will be interested in distributions on those sets, so for $A \in \mathcal{P}_f$ we define

$$\mathcal{D}_A := \{\mu : \mathbb{Z}^d \to [0,1] \mid \mu(y) = 0 \text{ for all } y \notin A \text{ and } \sum_{y \in A} \mu(y) = 1\}, \tag{3.2}$$

the set of distributions on A. Now we define *Incremental Aggregate* as follows.

Definition 3.0.1. Let $\mu = (\mu_A)_{A \in \mathcal{P}_f}$ be a family of distributions with $\mu_A \in \mathcal{D}_A$ for all $A \in \mathcal{P}_f$. *Incremental Aggregate* (*with distribution* μ) is a stochastic process $(\mathcal{E}_n)_{n \in \mathbb{N}_0}$ which evolves as follows. The process starts with one point $\mathcal{E}_0 = \{(0,0)\}$ in the origin of $\mathbb{Z}^d$. Knowing the process $\mathcal{E}_n$ at time n, let y_n be a random point in $\partial \mathcal{E}_n \in \mathcal{P}_f$ with distribution

$$\mathbb{P}(y_n = y \mid \mathcal{E}_n) := \mu_{\partial \mathcal{E}_n}(y), \quad y \in \mathbb{Z}^d. \tag{3.3}$$

We then define $\mathcal{E}_{n+1} := \mathcal{E}_n \cup \{y_n\}$.

Remark 3.0.1.

4 External DLA

External DLA is a model of Incremental Aggregate as defined above using a very natural distribution, called the *harmonic measure*.

Definition 4.0.1. (*Harmonic Measure*) Let $A \in \mathcal{P}_f$. Remembering the definitions in (2), especially the heat kernel $p_n(x, y) = \mathbb{P}_x(S_n = y)$ of a random walk, for $x \in \mathbb{Z}^2$ the *harmonic measure* (from x) of A is

$$h_A(x, y) := \mathbb{1}\{y \in A\}p_{T_A}(x, y) = \mathbb{1}\{y \in A\}\mathbb{P}_x(S_{T_A} = y), \quad \text{for } y \in \mathbb{Z}^2.$$

We now define the *harmonic measure* (from infinity) of A as the family $h = (h_A)_{A \in \mathcal{P}_f}$ with

$$h_A(y) := \lim_{|x| \to \infty} h_A(x, y), \quad y \in \mathbb{Z}^2.$$

This is well-define because ... CONTINUE

Definition 4.0.2. (*External Diffusion Limited Aggregate*) *External Diffusion Limited Aggregate*, short *External DLA*, is a incremental aggregate with the harmonic measure h as distribution.

Remark 4.0.1. contenu...

5 Integral Geometry

In the next section we want to define an approximation for External DLA. This approximation will be a incremental aggregate for which definition of its distribution we need some concepts and results from Integral Geometry which we will discuss and develop in this section. The process we want to define bases on choosing a random line out of all lines which cut the current cluster of the aggregate. This random choosing is not obvious since the cluster most of times will be strongly not symmetric and it is even less obvious how to actually get a realisation of a random line when simulating with Python. In our case we are looking for a parametrisation of lines in the real plane and a reasonable way of choosing random parameters.

We will introduce a possible solution for this problem first through the abstract and general way of integral geometry and later through a simple parametrisation for the case of lines in the real plane which goes hand in hand with the general result.

5.1 General result

In the general context we are in $\mathbb{R}^d$ for $d \in \mathbb{N}$ and consider q-dimensional affine subspaces where $q \in \{0, \ldots, d\}$, short q-flats in $\mathbb{R}^d$. This set of q-flats in $\mathbb{R}^d$ is denoted by $A(d, q)$. Later we will be interested in choosing random lines in the real plane (i.e. 1-flats in $\mathbb{R}^2$). In order to get a probability measure on some set of q-flats, we need a measure and a σ-algebra on $A(d, q)$ in total.

Definition 5.1.1. For $K \in \mathcal{K}^d$ define $A_K := \{F \in A(d, q) \mid F \cap K \neq \emptyset\}$. Then the σ-algebra $\mathcal{A}(d, q)$ on $A(d, q)$ shall be defined by

$$\mathcal{A}(d, q) := \sigma(\{A_K \mid K \in \mathcal{K}^d\}).$$

After constructions which involve typical results in integral geometry like intrinsic volumes, Steiner Formula and the Crofton Formula we get the follwing theorem. Let $G_d = \{\phi : \mathbb{R}^d \to \mathbb{R}^d, \ x \mapsto Ax + b \mid b \in R^d, A \in \mathbb{R}^{d \times d} \text{ with } A^T A = I\}$ be the set of euclidean motions in $\mathbb{R}^d$.

Theorem 5.1.1. On $A(d, q)$ there exists a unique G_d-invariant Radon measure μ_q such that

$$\mu_q(A_{B_d(1,0)}) = \kappa_{d-q}, \tag{5.1}$$

where $\kappa_n := \lambda_n(B_n(1, 0))$ is the n-dimensional Lebesque meausure of the n-dimensional unit ball for $n \in \mathbb{N}$, and $\kappa_0 := 1$.

Proof. [2] Theorem 4.26 $\qquad \square$

5.2 Construction in the real plane

For our case we choose $d = 2$ and $q = 1$, thus lines in the real plane, denoted by the set $\mathcal{G}$. The following construction is completely motivated by [3] 2.1.1. Firstly we propose a parametrisation of lines which works as following. Every line can be uniquely determined by an angle $\alpha \in [0, \pi)$ and a real number $p \in \mathbb{R}$. Let $e_\alpha := \binom{\cos(\alpha)}{\sin(\alpha)}$ be the unit vector $\binom{1}{0}$ turned by α counterclockwise. It is easy to realize that $g_{\alpha,p} := \{x \in \mathbb{R}^2 \mid \langle x, e_\alpha \rangle = p\}$ defines a line and that every line has a unique pair of α and p for such a representation. With $\Phi := [0, \pi) \times \mathbb{R}$ this naturally defines a bijection

$$\chi : \Phi \to \mathcal{G}, \quad (\alpha, p) \mapsto g_{\alpha,p}.$$

We take the subspace Borel-σ-algebra $\mathcal{B}_\Phi := \mathcal{B}^2 \cap \Phi$ on Φ and define the σ-algebra $\mathfrak{G}$ on $\mathcal{G}$ by $\mathfrak{G} := \chi(\mathcal{B}_\Phi)$. This works well since Φ is a bijection. The next lemma shows that this way of defining a σ-algebra on $\mathcal{G}$ makes sense and for our special case indeed is equivalent to the general context as defined above.

Lemma 5.2.1. We have $\mathfrak{G} = \mathcal{A}(2, 1)$.

Proof. To proof this it is helpful to consider generators of these σ-algebras. Define the set of closed rectangles in Φ as $R := \{[a, b] \times [c, d] \subset \Phi \mid 0 \leq a < b < \pi, c < d \in \mathbb{R}\}$. Then by measure theory we know that $\mathcal{B}_\Phi = \sigma(R)$ and since χ is a bijection, we have $\chi(\sigma(R)) = \sigma(\chi(R))$, and finally $\mathfrak{G} = \sigma(\chi(R))$. For $\tilde{A} := \{A_K \mid K \in \mathcal{K}^2\}$ we have by definition $\mathcal{A}(2, 1) = \sigma(\tilde{A})$. We will show $\tilde{A} = \chi(R)$ which will complete the proof.

$\subset$: $\square$

6 Line Hitting Aggregate

In the following we will look at a process which is the approach of a simple approximation of external DLA on $\mathbb{Z}^2$. The idea is to let particles move on straight lines coming from infinity and add to the cluster when hitting it. Obviously in most cases particles cannot move completely straight on $\mathbb{Z}^2$. Therefore we will consider points in $\mathbb{Z}^2$ as the centers of unit squares and let the particles move on straight lines in the full plane $\mathbb{R}^2$. We consider a line hitting a point in $\mathbb{Z}^2$ if and only if it intersects with its unit square as defined in the following.

Definition 6.0.1. Define

$$\mathbb{R}^2_{sq} := \{[k - \frac{1}{2}, k + \frac{1}{2}] \times [l - \frac{1}{2}, l + \frac{1}{2}] \subset \mathbb{R}^2 \mid k, l \in \mathbb{Z}\}, \tag{6.1}$$

note that $\mathbb{R}^2 = \bigcup_{s \in \mathbb{R}^2_{sq}} s$. The canonical function

$$sq : \mathbb{Z}^2 \to \mathbb{R}^2_{sq}, \quad (k, l) \to [k - \frac{1}{2}, k + \frac{1}{2}] \times [l - \frac{1}{2}, l + \frac{1}{2}] \tag{6.2}$$

is bijective and intuitively identifies points in $\mathbb{Z}^2$ with squares in $\mathbb{R}^2$, which is p is the center of square $sq(p)$ for all $p \in \mathbb{Z}^2$. In the following when using a point $p \in \mathbb{Z}^2$ it will reference the point in $\mathbb{Z}^2$ or the corresponding square in $\mathbb{R}^2$ respecting the context. This bijection also naturally defines a graph structure on $\mathbb{R}^2_{sq}$, which is two squares $s_1, s_2 \in \mathbb{R}^2_{sq}$ form an edge if and only if $sq^{-1}(s_1)$ and $sq^{-1}(s_2)$ form an edge in $\mathbb{Z}^2$. For the following context we say a line L *hits* a point $p \in \mathbb{Z}^2$ if and only if $L \cap sq(p) \neq \emptyset$.

   BILD Linie durch squares "hitting"

Definition 6.0.2. Geradenmaß μ_0

Definition 6.0.3. Let $L = L_{a,b} \in \mathcal{L}$ be a line. For a finite set $A \in \mathcal{P}_f$ we define

$$L_A := \{p \in A \mid L \text{ hits } p\}$$

   which is the subset of points in A are hit by the line L hits. For the following we suppose $L_A \neq \emptyset$. We want to define a total ordered relation $<_{line}$ on L_A. We choose two points $(x_1, x_2), (y_1, y_2) \in L_A$ and devide the definition of a relation into four cases, which are the line going from left-bottom to right-top, left-top to right-bottom, parallel to the x-axis and parallel to the y-axis. Write $b = (b_1, b_2)$.

Case 1 : L is parallel to the x-axis $\Leftrightarrow$ $b_2 = 0$

$$(x_1, x_2) <_{line} (y_1, y_2) \quad :\Leftrightarrow \quad x_1 < y_1$$

Case 2 : L is parallel to the y-axis $\quad\Leftrightarrow\quad b_1 = 0$

$$(x_1, x_2) <_{line} (y_1, y_2) \quad :\Leftrightarrow \quad x_2 < y_2$$

Case 3 : L is going from left-bottom to right-top $\quad\Leftrightarrow\quad b_1 b_2 > 0$

$$(x_1, x_2) <_{line} (y_1, y_2) \quad :\Leftrightarrow \quad \begin{cases} x_1 < y_1, & \text{if } x_1 \neq y_1, \\ x_2 < y_2, & \text{if } x_1 = y_1. \end{cases}$$

Case 4 : L is going from left-top to right-bottom $\quad\Leftrightarrow\quad b_1 b_2 < 0$

$$(x_1, x_2) <_{line} (y_1, y_2) \quad :\Leftrightarrow \quad \begin{cases} x_1 < y_1, & \text{if } x_1 \neq y_1, \\ x_2 > y_2, & \text{if } x_1 = y_1. \end{cases}$$

It is easy to see that this well-defines a relation on L_A. In the following we will quickly proove that this relation is totally ordered.

Lemma 6.0.1. For a line $L = L_{a,b} \in \mathcal{L}$ and $A \in \mathcal{P}_f$ with $L_A \neq \emptyset$ the relation $<_{line}$ on L_A is totally ordered.

Proof. We will only proove the case where L is going from left-bottom to right-top, which is *Case 3* of the definition. In this case we have $b_1 b_2 > 0$. Note, that the proof for *Case 4* will work very similair. And in the case of L being parallel to one of the axes (*Case 1 or 2*), all properties for a totally ordered relation follow directly from the totally ordered relation $<$ on $\mathbb{R}$. So let $b_1 b_2 > 0$.

Antisymmetry : For antisymmetry let $(x_1, x_2) <_{line} (y_1, y_2)$ and $(y_1, y_2) <_{line} (x_1, x_2)$. Suppose $x_1 \neq y_1$, then $x_1 < y_1$ and $y_1 < x_1$, therefore $x_1 = y_1$ by antisymmetry of the standard order $<$ in $\mathbb{R}$, a contradiction, hence $x_1 = y_1$. But then we have $x_2 < y_2$ and $y_2 < x_2$ and therefore also $x_2 = y_2$.

Transitivity : For transitivity let $(x_1, x_2) <_{line} (y_1, y_2)$ and $(y_1, y_2) <_{line} (z_1, z_2)$. We find four cases. In case $x_1 \neq y_1$ and $y_1 \neq z_1$ we get $x_1 < z_1$ by transitivity of $<$, hence $(x_1, x_2) <_{line} (z_1, z_2)$. In case $x_1 \neq y_1$ and $y_1 = z_1$ we get $x_1 < y_1 = z_1$, therefore $(x_1, x_2) <_{line} (z_1, z_2)$. In case $x_1 = y_1$ and $y_1 \neq z_1$ we get $x_1 = y_1 < z_1$, similair as the last case. In the last case $x_1 = y_1 = z_1$ we get $x_2 < y_2$ and $y_2 < z_2$ and again by transitivity of $<$ we get $x_2 < z_2$, hence $(x_1, x_2) <_{line} (z_1, z_2)$ again.

Connexity : Connexity is given since for any two points $(x_1, x_2), (y_1, y_2) \in L_A$ we have either $x_1 \neq y_1$ or $x_1 = y_1$ and therefore either $(x_1, x_2) <_{line} (y_1, y_2)$ or $(y_1, y_2) <_{line} (x_1, x_2)$. $\qquad\square$

Remark 6.0.1. The relation $<_{line}$ on L_A basically orders the hitpoints of a line L with a finite set A from left to right (or bottom to top in case of a line parallel to the y-axis). This order allows us to identify the outest hitting points of A by L. This means when moving on a line L facing A this order allows us to know where in A the line L hits first when „entering" A and where it hits last when „leaving" A which will be the two points $min_{<_{line}} L_A$ and $max_{<_{line}} L_A$ (or the other way around).

Definition 6.0.4. *Random Line Hitting Distribution*

Choose $A \in \mathcal{P}_f$. We define a distribution μ_A on $\mathbb{Z}^2$ as in the following. Let $L = L_{a,b}$ be a random line according to the line measure μ_0 with the condition that L hits A. Then define μ_A with distribution $\mu_A \sim U(\{min_{<_{line}} L_A, max_{<_{line}} L_A\})$, which chooses uniformly an element out of $\{min_{<_{line}} L_A, max_{<_{line}} L_A\}$. We call this distribution the *Random Line Hitting Distribution (of A)*.

Definition 6.0.5. *Line Hitting Aggregate*

Incremental Aggregate with the Random Line Hitting Distribution as its distribution we will call here *Line Hitting Aggregate*, short *LHA*.

7 Questions

$\mathcal{K}' = ?$

References

[1] N. Henze. *Maß und Wahrscheinlichkeitstheorie (Stochastik II)*. Karlsruher Institut für Technologie, Karlsruhe, 2010

[2] Daniel Hug, Günter Last, Steffen Winter. *Stochastic Geometry, Lecture Notes (summer term 2020)*. Institute of Technology, Karlsruhe, 2020

[3] Franz Sackmann. *Zufällige Geraden, Staatsexamensarbeit*. University Karlsruhe (TH), Karlsruhe, 2007

Erklärung

Hiermit versichere ich, dass ich diese Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt, die wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht und die Satzung des Karlsruher Instituts für Technologie zur Sicherung guter wissenschaftlicher Praxis in der jeweils gültigen Fassung beachtet habe.

Karlsruhe, den 10. März 2020