Stochastic Geometry

Lecture in the summer term 2020

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Institute of Stochastics (KIT)

A short and informal introduction

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- provides tools (in particular geometric functionals) to describe, quantify and classify the geometry of these models.

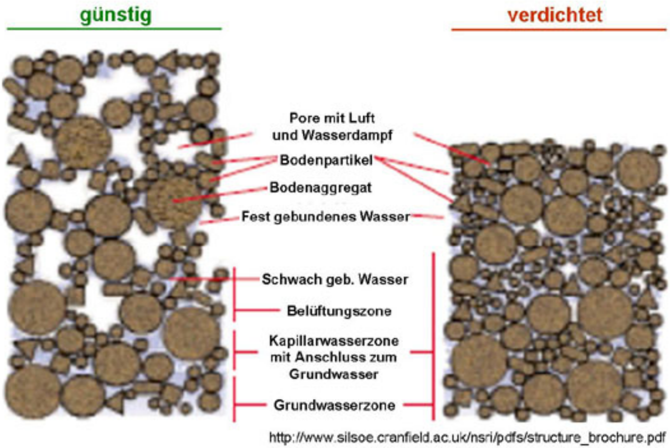
Real-world examples of complex structures

Bread

Real-world examples of complex structures

Froth (of beer)

Real-world examples of complex structures



Soil

Real-world examples of complex structures

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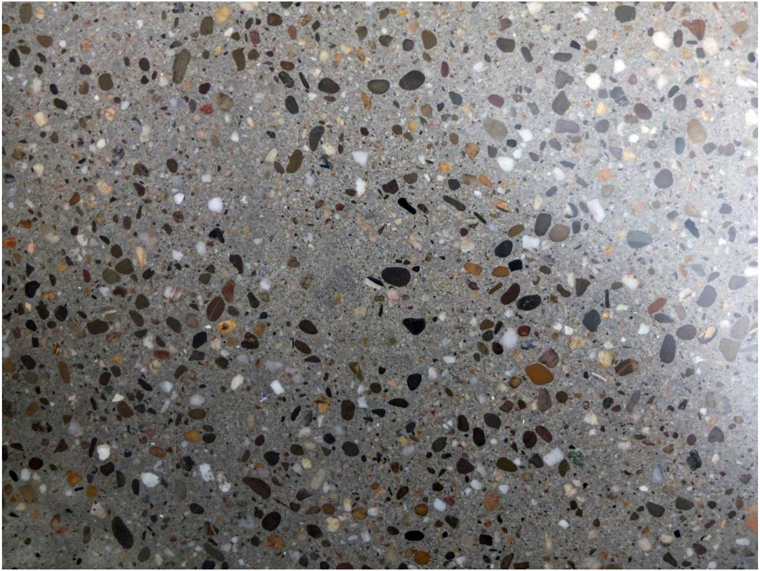
Gravel

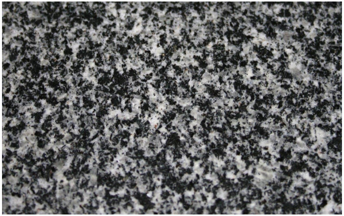
Real-world examples of complex structures



Concrete

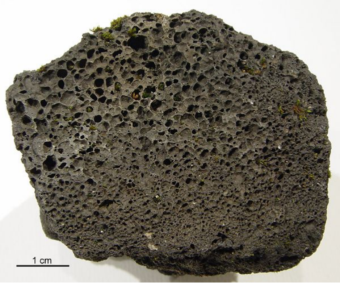
Real-world examples of complex structures

Rock (granite)



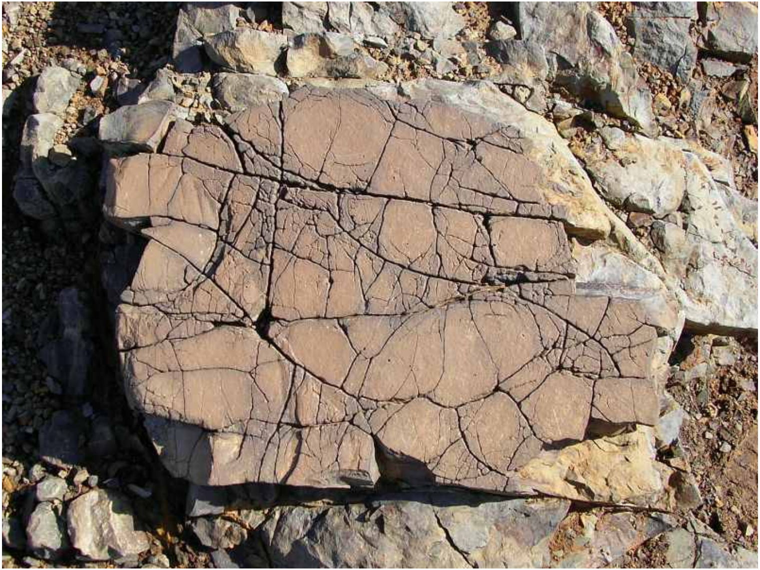
Rock (tonalite)

Rock (drift *Geschiebe*)



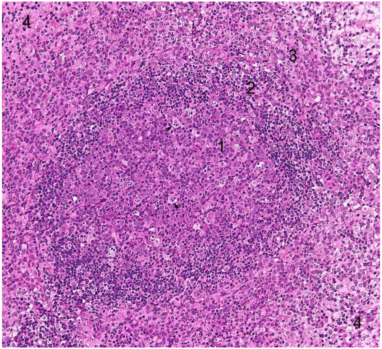
Rock (lava)

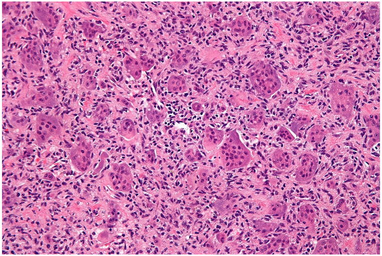
Real-world examples of complex structures



Rock (madeira)

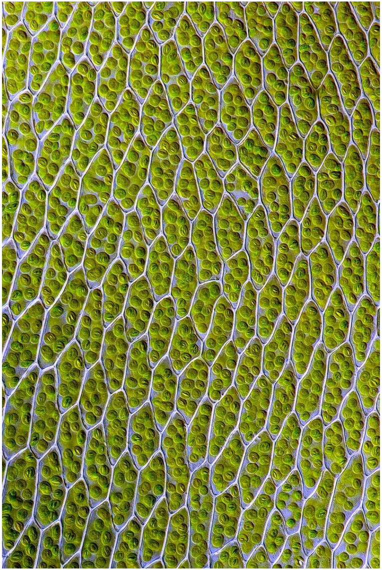
Real-world examples of complex structures



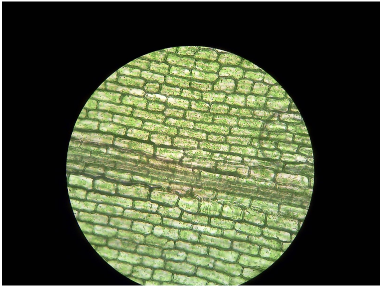


Cellular tissue (human)

Real-world examples of complex structures



Cellular tissue (moss)



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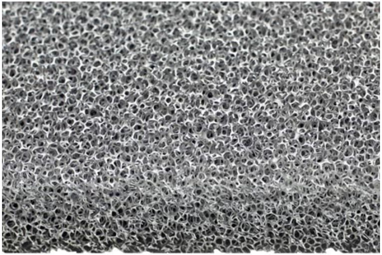
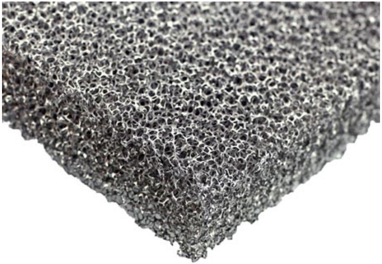
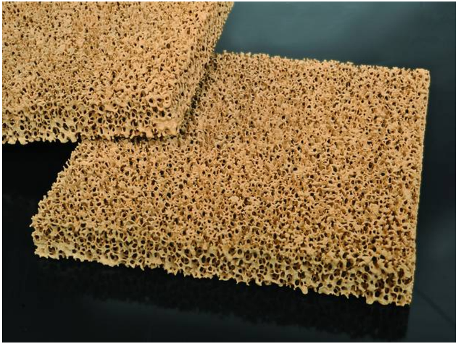




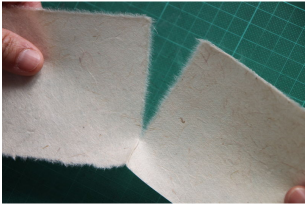
Metal foam (aluminium)

Real-world examples of complex structures

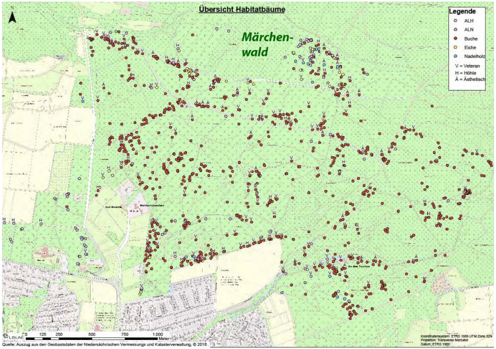
Ceramic foam (as used in catalytic converters)

Real-world examples of complex structures



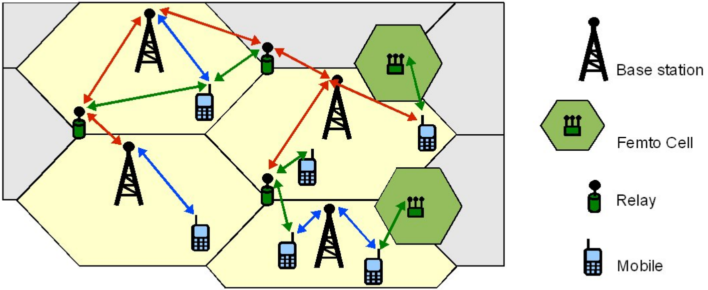
Fiber structure of paper

Real-world examples of complex structures



Forest

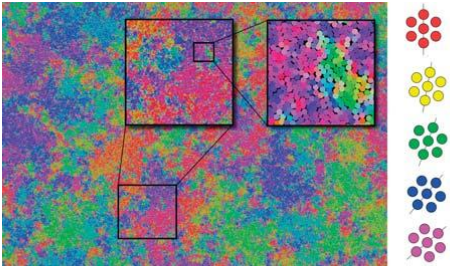
Real-world examples of complex structures



Source: https://www.nari.ee.ethz.ch/wireless/research/research_tracks.html

Modelling of telecommunication networks

Real-world examples of complex structures



Physics models for liquids (disordered ball packing)

Real-world examples of complex structures

Source: https://www.nationalgeographic.co.uk/photography/2018/03/la-drops-96-milllion-shade-balls-its-reservoirs?image=1shadeballsgallery

Disordered ball packing

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- Typically the exact local structure is not in the focus of interest. More important are global properties or properties 'in the mean' such as
  - the volume fraction of a substance or of the pores space,
  - the mean surface area,
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- In many cases only 1D or 2D sections of 3D-structures are observable.

Aims of the course

We want to provide models for such structures and tools to analyse them. The real-world structures shown only serve as a motivation. In the focus of the lecture are mathematical models and their properties, mathematical tools to study them as well as the probabilistic and geometric background of such modelling.

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- tessellations – partitions of the space into cells (in particular Poisson hyperplane tessellations and Poisson-Voronoi tessellations)
- geometric functionals such as intrinsic volumes (volume, surface area, Euler characteristic, ...), the number of vertices or edges of polytopes, ...

Two simple models of random sets

Random ball: In the plane $\mathbb{R}^2$ choose a *random point* X in the unit interval $W = [0,1]^2$ (i.e. a random vector (X_1, X_2) whose components are random variables (RV) with values in $[0,1]$) and let R be a nonnegative RV. Then the closed ball $B(X,R)$ with centre X and radius R is an example of a *random closed set*, whose properties are determined by the distribution of the RV's X and R.

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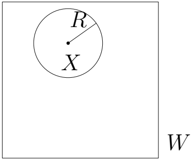

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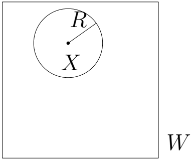

Random segment: Let L be a nonnegative RV and Φ be a $[0,\pi)$-valued RV. Then the set

$$[(L\cos(\Phi), L\sin(\Phi)), (-L\cos(\Phi), -L\sin(\Phi))]$$

is a line segment centred at the origin with length $2L$ and direction Φ.

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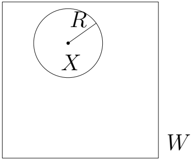

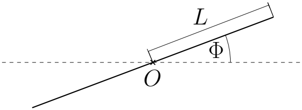

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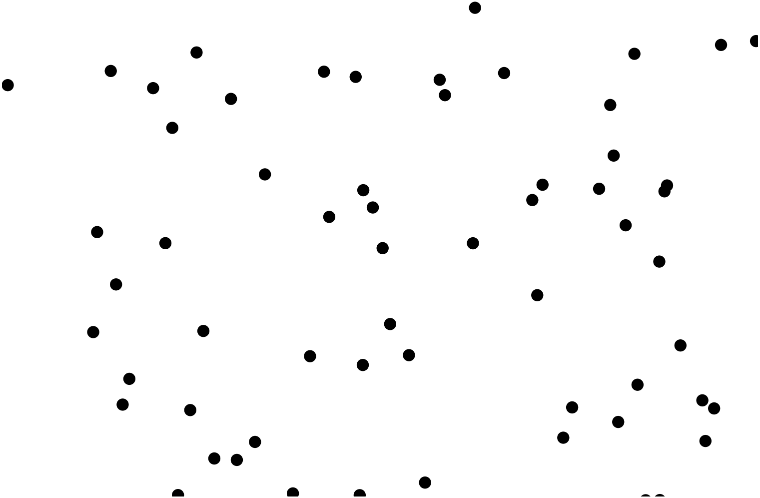
Some realization of a germ-grain model



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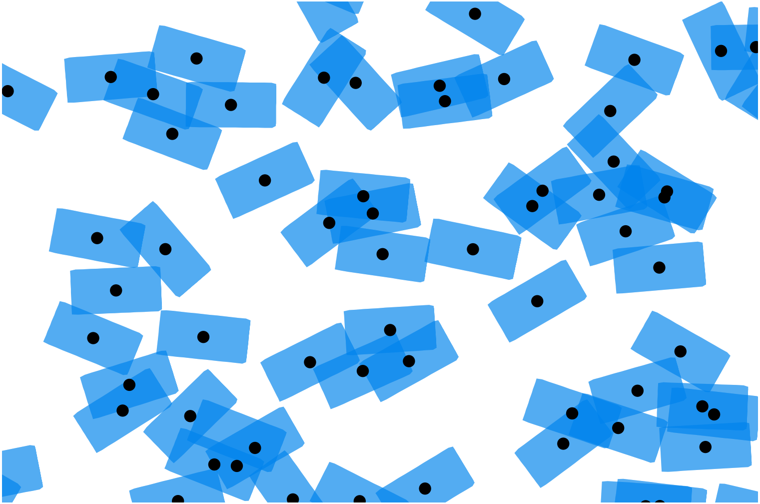

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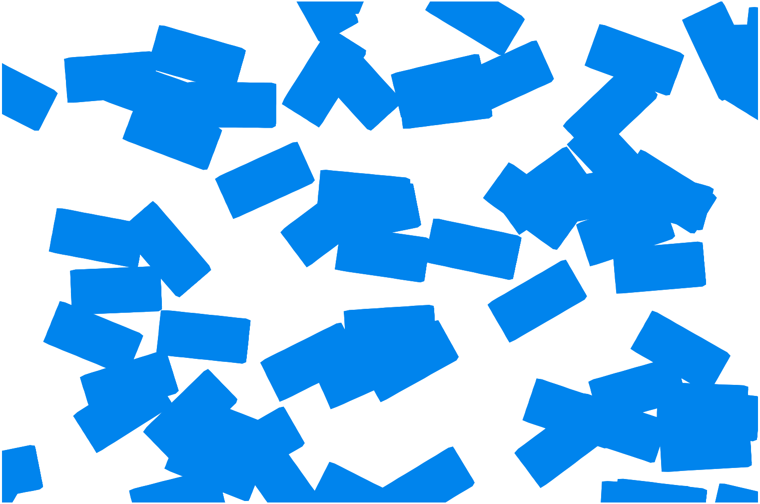

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