Institute of Stochastics

Stochastic Geometry | Summer term 2020
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Work Sheet 4

Instructions for week 4 (May 11^{th} to May 15^{th}):

- Work through Sections 2.1 and 2.2 of the lecture notes.
- Answer the control questions 1) to 5), and solve Problems 1 to 4 of the exercises below.
- Please hand in your solutions to the exercises for correction until the morning of Monday, May 18^{th}. For the procedure, please have a look at the general information in Ilias. The submission of solutions is voluntary.

Control questions to monitor your progress:

1) Recall Definition 2.3 of a directed sequence of partitions. Can you imagine a construction for such a directed sequence of partitions in an arbitrary separable metric space (cf. Remark 2.4)?

   (**Hint:** Contemplate that for any $n \in \mathbb{N}$ the separable metric space has a countable base of its topology consisting of sets of diameter at most $1/n$. Construct the required partitions via suitable intersections.)

2) Let $(\mathbb{X}, \rho)$ be a separable metric space, and let $\mu \in M(\mathbb{X})$ and $r > 0$. Why does μ have at most finitely many atoms with mass at least r in any bounded set $B \in \mathcal{X}_b$? Conclude from this fact that μ can have at most countably many atoms in $\mathbb{X}$.

3) Let $(\mathbb{X}, \rho)$ be a separable metric space, and let $x, x_1, x_2, \ldots \in \mathbb{X}$ such that $x_n \to x$ as $n \to \infty$. Do we have $\sum_{j=1}^{\infty} \delta_{x_j} \in N(\mathbb{X})$?

4) In the proof of Corollary 2.10, verify that the stated equivalences do indeed imply that $N(\mathbb{X})$ and $N_s(\mathbb{X})$ are measurable subsets of $M(\mathbb{X})$, that is, $N(\mathbb{X}), N_s(\mathbb{X}) \in \mathcal{M}(\mathbb{X})$.

5) Can you verify that the intensity measure (Definition 2.19) of any random measure is itself a measure on the underlying separable metric space?

Exercises for week 4:

Problem 1 (Measurability of measure-valued functions)

Let $M(\mathbb{X})$ be the set of all locally finite measures on a separable metric space $(\mathbb{X}, \rho)$ with Borel-σ-field $\mathcal{X} = \mathcal{B}(\mathbb{X})$. For $B \in \mathcal{X}$, denote by $\pi_B : M(\mathbb{X}) \to [0, \infty]$ the map $\mu \mapsto \pi_B(\mu) := \mu(B)$. Let $\mathcal{M}(\mathbb{X})$ be the smallest σ-field on $M(\mathbb{X})$ for which all maps π_B, $B \in \mathcal{X}$, are measurable. Furthermore, let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space.

a) Consider a map $\eta : \Omega \to M(\mathbb{X})$. Show the equivalence of the following statements:

  (i) η is a random measure on E.

  (ii) For all sets $B \in \mathcal{X}$, $\eta(B) : \Omega \to [0, \infty]$ is a random variable.

b) Prove that the mapping $\mathcal{S} : M(\mathbb{X}) \times M(\mathbb{X}) \to M(\mathbb{X})$, $(\mu, \nu) \mapsto \mu + \nu$ is measurable.

c) Verify that either part a) and part b) imply that the sum of two random measures is itself a random measure.

Problem 2 (Measurability of point processes)

Let $X_1, X_2, \ldots$ be random elements of a separable metric space $(\mathbb{X}, \rho)$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ (where the X_j need not be independent and can have different distributions). Let τ be an $\mathbb{N}_0$-valued random variable. Prove that

$$\Phi := \sum_{j=1}^{\tau} \delta_{X_j}$$

is a point process, that is, the map $\Phi : \Omega \to N(\mathbb{X})$ is measurable.

Note: If $X_1, X_2, \ldots$ are i.i.d. random elements of $\mathbb{X}$ and τ is independent of $(X_j)_{j \in \mathbb{N}}$, then the process Φ is the mixed binomial process from Example 2.17.

Problem 3

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be the probability space which underlies the following random elements. Let X_1, X_2 be uniformly distributed over the open unit disc $B^\circ(0,1) := \{x \in \mathbb{R}^2 : \|x\| < 1\}$, let X_3 be uniformly distributed over the discrete set $\{(0,0), (0,1), (1,0), (1,1)\}$, and let X_4 be uniformly distributed over the segment $[-1, 1] \times \{0\}$. Define the point process $\Phi := \sum_{j=1}^4 \delta_{X_j}$. Calculate $\mathbb{E}\left[\Phi(B)\right]$ for any $B \in \mathcal{B}(\mathbb{R}^2)$. Apply this knowledge about the intensity measure to calculate

$$\mathbb{E}\left[\int_{[-1,1]^2} \left(x^2 + y^2\right) \Phi\big(\mathrm{d}(x, y)\big)\right].$$

Problem 4 (Equality in distribution of point processes – A proof of Remark 2.24)

Let Φ and Φ' be point processes on a separable metric space $(\mathbb{X}, \rho)$. Prove that the following are equivalent:

(iv) For all $m \in \mathbb{N}$ and any $B_1, \ldots, B_m \in \mathcal{X}_b$,

$$\big(\Phi(B_1), \ldots, \Phi(B_m)\big) \overset{d}{=} \big(\Phi'(B_1), \ldots, \Phi'(B_m)\big).$$

(iv') For all $m \in \mathbb{N}$ and any pairwise disjoint $B_1, \ldots, B_m \in \mathcal{X}_b$,

$$\big(\Phi(B_1), \ldots, \Phi(B_m)\big) \overset{d}{=} \big(\Phi'(B_1), \ldots, \Phi'(B_m)\big).$$

The solutions to these problems will be uploaded on May 18th.
Feel free to ask your questions about the exercises in the optional MS-Teams discussion on May 14th (09:15 h).