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Masterthesis

External DLA

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2 Preliminaries

2.1 Namings

Here we list all namings which will be used in this paper. Let $d \in \mathbb{N}$ and $q \in \{0, \dots, d\}$.

$\mathbb{N} = \{1, 2, 3, \dots\}$	set of natural numbers (without 0)
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$	
$\mathcal{K}^d$	set of convex and compact sets in $\mathbb{R}^d$
$B_d(r, x) = \{y \in \mathbb{R}^d \mid \lvert x - y\rvert \leq r\}$	d-dimensional closed ball of radius r around x
$S_{d-1}(r, x) = \partial B_d(r, x)$	$(d-1)$-dimensional surface of the d-dimensional ball
$A(d, q)$	set of q-dimensional affine subspaces of $\mathbb{R}^d$
$\mathcal{A}(d, q)$	σ-algebra of $A(d, q),$ as constructed later in the paper
$\mathcal{G} := A(2, 1)$	set of lines in the real plane
$\mathfrak{G} := \mathcal{A}(2, 1)$	

2.2 Basic structures

We prepare this script with the following preliminaries. Let $d \in \mathbb{N}$.

Graphs. We will be interested in the graph $(\mathbb{Z}^d, E)$ with its canonical graph structure, which is two vertices (or points) $x = (x_1, \dots, x_d), y = (y_1, \dots, y_d) \in \mathbb{Z}^d$ form an edge (e.q. $(x, y) \in E$) if and only if there exists exactly one $i \in \{1, \dots, d\}$ such that $|x_i - y_i| = 1$ and $x_j = y_j$ for all $j \neq i$. For a point $x \in \mathbb{Z}^d$ its set of *neighbours* is defined as

$$N(x) := \{y \in \mathbb{Z}^d \mid (x, y) \in E\}.$$

For a set $A \subset \mathbb{Z}^d$ the *outer boundary* ∂A of A is defined as

$$\partial A := \{y \in \mathbb{Z}^d \setminus A \mid \exists x \in A : \ (x, y) \in E\}$$

Instead of $(\mathbb{Z}^d, E)$ we will write $\mathbb{Z}^d$ from now on.

Probability Space. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space which we will base on in this paper. For our space of interest $\mathbb{Z}^d$ we will always use the discrete σ-Algebra which is the power set of $\mathbb{Z}^d$. If for $A \in \mathcal{F}$ we have $\mathbb{P}(A) = 1$ we will say that "A holds $\mathbb{P}$-*a.s.*", or short "A holds *a.s.*" (almost sure).

Random Walk. A family $(S_n)_{n \in \mathbb{N}}$ of measurable functions $S_n : \Omega \to \mathbb{Z}^d$ is called a *Random Walk on* $\mathbb{Z}^d$ (*starting at* $x \in \mathbb{Z}^d$) if and only if $S_0 = x$ *a.s.* and $S_n \in N(S_{n-1})$ *a.s.* for all $n \geq 1$. It further shall hold that

$$\mathbb{P}(S_n = y) = \frac{1}{|N(S_{n-1})|} = \frac{1}{2d} \quad \text{for all } y \in N(S_{n-1}) \quad a.s.$$

Note that $|N(y)| = 2d$ for all $y \in \mathbb{Z}^d$ since every point has two neighbours in every dimension. So a Random Walk can be understood as a particle starting from some point x and moving randomly on the grid choosing its next step uniformly from its neighbours. Further define

$$\mathbb{P}_x(S_n \in A) := \mathbb{P}(S_n \in A | S_0 = x)$$

for any subset $A \subset G$. We define the *hitting times* of A

$$T_A := \min\{n \geq 0 \mid S_n \in A\} \text{ and } T_A^+ := \min\{n \geq 1 \mid S_n \in A\},$$

$T_x := T_{\{x\}}$ and $T_x^+ := T_{\{x\}}^+$ for one element sets and $x \in \mathbb{Z}^d$. The *heat kernel* of the random walk S_n is defined to be

$$p_n(x, y) := \mathbb{P}_x(S_n = y)$$

and the *Green function* as

$$G(x,y) := \sum_{n\geq 0} p_n(x,y).$$

G is well-defined and finite since $\mathbb{Z}^2$ is transient. Similarily for a subset $A \subset G$ the *killed* or *stopped Green function* is defined as

$$G_A(x,y) := \sum_{n\geq 0} \mathbb{P}_x(S_n = y, T_A > n).$$

3 Incremental Aggregate

In this paper we will look at stochastic processes on the set of finite subsets of $\mathbb{Z}^d$, where we start with a one point set at $(0,0)$ and incrementally add a point on the outer boundary of the current cluster according to some distribution. What we get is a randomly, point-by-point growing connected cluster which here we will call *Incremental Aggregate*. Define

$$\mathcal{P}_f := \{A \subset \mathbb{Z}^d \mid \text{A is finite}\}, \tag{3.1}$$

the set of finite subsets of $\mathbb{Z}^d$. Furthermore we will be interested in distributions on those sets, so for $A \in \mathcal{P}_f$ we define

$$\mathcal{D}_A := \{\mu : \mathbb{Z}^d \to [0,1] \mid \mu(y) = 0 \text{ for all } y \notin A \text{ and } \sum_{y \in A} \mu(y) = 1\}, \tag{3.2}$$

the set of distributions on A. Now we define *Incremental Aggregate* as follows.

Definition 3.0.1. Let $\mu = (\mu_A)_{A \in \mathcal{P}_f}$ be a family of distributions with $\mu_A \in \mathcal{D}_A$ for all $A \in \mathcal{P}_f$. *Incremental Aggregate* (*with distribution* μ) is a stochastic process $(\mathcal{E}_n)_{n \in \mathbb{N}_0}$ which evolves as follows. The process starts with one point $\mathcal{E}_0 = \{(0,0)\}$ in the origin of $\mathbb{Z}^d$. Knowing the process $\mathcal{E}_n$ at time n, let y_n be a random point in $\partial\mathcal{E}_n \in \mathcal{P}_f$ with distribution

$$\mathbb{P}(y_n = y \mid \mathcal{E}_n) := \mu_{\partial\mathcal{E}_n}(y), \quad y \in \mathbb{Z}^d. \tag{3.3}$$

We then define $\mathcal{E}_{n+1} := \mathcal{E}_n \cup \{y_n\}$.

Remark 3.0.1.

4 External DLA

External DLA is a model of Incremental Aggregate as defined above using a very natural distribution, called the *harmonic measure*.

Definition 4.0.1. (*Harmonic Measure*) Let $A \in \mathcal{P}_f$. Remembering the definitions in (2), especially the heat kernel $p_n(x,y) = \mathbb{P}_x(S_n = y)$ of a random walk, for $x \in \mathbb{Z}^2$ the *harmonic measure* (from x) of A is

$$h_A(x,y) := \mathbb{1}\{y \in A\} p_{T_A}(x,y) = \mathbb{1}\{y \in A\} \mathbb{P}_x(S_{T_A} = y), \quad \text{for } y \in \mathbb{Z}^2.$$

We now define the *harmonic measure* (from infinity) of A as the family $h = (h_A)_{A \in \mathcal{P}_f}$ with

$$h_A(y) := \lim_{|x| \to \infty} h_A(x,y), \quad y \in \mathbb{Z}^2.$$

This is well-define because ... CONTINUE

Definition 4.0.2. (*External Diffusion Limited Aggregate*) *External Diffusion Limited Aggregate*, short *External DLA*, is a incremental aggregate with the harmonic measure h as distribution.

Remark 4.0.1. contenu...

5 Integral Geometry

In the next section we want to define an approximation for External DLA. This approximation will be a incremental aggregate for which definition of its distribution we need some concepts and results from Integral Geometry which we will discuss and develop in this section. The process we want to define bases on choosing a random line out of all lines which cut the current cluster of the aggregate. This random choosing is not obvious and it is even less obvious how to actually get a realisation of a random line when simulating with Python. In our case we are looking for a parametrisation of lines in the real plane and a reasonable way of choosing random parameters.

We will introduce a possible solution for this problem first through the abstract concepts and constructions of integral geometry. Later we find a simple parametrisation for the simple case of lines in the real plane and proof, that it goes hand in hand with the general construction.

5.1 Intrinsic Volumes

A useful concept to measure intrinsic geometrical properties of Borel-sets $K \subset \mathbb{R}^d$ are *intrinsic volumes*. We define the d-th intrinsic volume of K as $V_d(K) := \lambda_d(K)$. Furthermore define $S_{d-1}(K)$ to be the *surface area* of $V_d(K)$, which is formally defined as the Hausdorff-measure of ∂K. For the following theorem define

$$\kappa_d := V_d(B_d) \text{ for } d > 0, \text{ and } \kappa_0 := 1$$

where we can calculate $V_d(B_d) = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$ with the *Gamma function* $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt,\ x > 0$.

Lemma 5.1.1. If $K \subset \mathcal{K}^d$, then S_{d-1} is equal to the *outer Minkowski content*, i.e.

$$S_{d-1}(K) = M_{d-1}(K) := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon}(V_d(K_{\oplus\varepsilon}) - V_d(K)),$$

where $K_{\oplus\varepsilon} = \{x \in \mathbb{R}^d \mid d(x, K) \leq \varepsilon\}$. It is easy to show, that $K_{\oplus\varepsilon} = K + \varepsilon B_d := \{x + y \in \mathbb{R}^d \mid x \in K, y \in \varepsilon B_d\}$, where $B_d := \{x \in \mathbb{R}^d \mid d(0, x) \leq 1\}$.

Proof. □

Theorem 5.1.1. (*Steiner Formula*) For $K \in \mathcal{K}^d$ there exist uniquely determined numbers $V_0(K), \ldots, V_d(K) \in \mathbb{R}$, such that for each $\varepsilon \geq 0$

$$V_d(K + \varepsilon B_d) = \sum_{j=0}^{d} \kappa_{d-j}\varepsilon^{d-j}V_j(K). \tag{5.1}$$

Proof. [2] Theorem 3.32 □

Definition 5.1.1. $V_0(K), \dots, V_d(K)$ are called *intrinsic volumes* of K.

Remark 5.1.1. (i) What we get are functions $V_j : \mathcal{K}^d \to \mathbb{R}$ for the j-intrinsic volume V_j. It can be shown that every V_j can be uniquely extended to a function $V_j : \mathcal{R}^d \to \mathbb{R}$, where $\mathcal{R}^d := \{\bigcup_{j=1}^n K_j \subset \mathbb{R}^d \mid n \in \mathbb{N}_0, K_j \in \mathcal{K}^d\}$ ([2] Theorem 4.10).

(ii) The coefficients κ_{d-j} are chosen such that the V_j become independent of the dimension of the underlying space. This means that V_j will assign the same value for K if K is considered to be subset of $\mathbb{R}^d$ or $\mathbb{R}^{\tilde{d}}$ for $d < \tilde{d}$, although the unit balls B_d and $B_{\tilde{d}}$ are different in those two spaces. This is why the V_j are called *intrinsic* volumes (REFERENCE).

(iii) For $\varepsilon = 0$ the right side of equation (5.1) reduces to $V_d(K)$ which shows a consistency of the notation.

(iv) With Lemma (5.1.1) and equation (5.1) we get $S_{d-1}(K) = \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon}(V_d(K+\varepsilon B_d) - V_d(K) = \kappa_1 V_{d-1}(K) = 2V_{d-1}(K)$, which will be a useful result.

(v) It can be shown that $V_0(\emptyset) = \dots = V_d(\emptyset) = 0$ and $V_0(K) = 1$ if $K \neq \emptyset$ (REFERENCE).

5.2 Random q-flats

An interesting set in Integral Geometry is the set of q-dimensional affine subspaces in the d-dimensional real space, short $A(d,q)$ the set of q-flats in $\mathbb{R}^d$. Later we will be interested in choosing random lines in the real plane (i.e. 1-flats in $\mathbb{R}^2$). For that we need a suitable σ-algebra and measure on $A(d,q)$ which we'll define now in this section. Let G_d be the set of rigid motions (euclidean motions, which are concatenations of translation and rotation) in $\mathbb{R}^d$.

Definition 5.2.1. For $K \in \mathcal{K}^d$ define $A_K := \{F \in A(d,q) \mid F \cap K \neq \emptyset\}$. Then the σ-algebra $\mathcal{A}(d,q)$ on $A(d,q)$ shall be defined by

$$\mathcal{A}(d,q) := \sigma(\{A_K \mid K \in \mathcal{K}^d\}).$$

Theorem 5.2.1. On $A(d,q)$ there exists a unique G_d-invariant Radon measure μ_q such that

$$\mu_q(A_{B_d(1,0)}) = \kappa_{d-q} \tag{5.1}$$

Proof. [2] Theorem 4.26 □

Theorem 5.2.2. (*Crofton formula*) Let $K \in \mathcal{K}^d \setminus \{\emptyset\}$, $k \in \{1, \dots, d-1\}$ and $j \in \{0, \dots, k\}$. Then

$$\int_{A(d,k)} V_j(K \cap F)\mu_k(dF) = c_{j,d}^{k,d-k+j} V_{d-k+j}(K), \tag{5.2}$$

where $c_s^r := \frac{r!\kappa_r}{s!\kappa_s}$ for $s, r \in \mathbb{N}_0$ and $c_{s_1,\dots,s_k}^{r_1,\dots,r_k} := \prod_{j=1}^k c_{s_j}^{r_j}$.

Proof. [2] Theorem 4.27 □

Definition 5.2.2. Let $K_0 \in \mathcal{K}^d$ with $V_d(K_0) > 0$. Let $q \in \{0, \dots, d-1\}$. A $A(d,q)$-valued random element X_q with distribution $\frac{\mu_q(\,\cdot\, \cap A_{K_0})}{\mu_q(A_{K_0})}$ is called an *isotropic random q-flat* through K_0.

Lemma 5.2.1. Let $K, K_0 \in \mathcal{K}^d$ with $K \subset K_0$ and $V_d(K_0) > 0$. Let $q \in \{0, \dots, d-1\}$ and X_q be an isotropic random q-flat through K_0. Then

$$\mathbb{P}(X_q \cap K \neq \emptyset) = \frac{V_{d-q}(K)}{V_{d-q}(K_0)}. \tag{5.3}$$

Proof. We directly get

$$\begin{aligned}
\mathbb{P}(X_q \cap K \neq \emptyset) = \mathbb{P}(X_q \in A_K) &= \frac{\mu_q(A_K \cap A_{K_0})}{\mu_q(A_{K_0})} \\
&= \frac{\mu_q(A_K)}{\mu_q(A_{K_0})} = \frac{\int_{A(d,q)} \mathbb{1}\{F \cap K \neq \emptyset\} \mu_q(dF)}{\int_{A(d,q)} \mathbb{1}\{F \cap K_0 \neq \emptyset\} \mu_q(dF)} \\
&= \frac{\int_{A(d,q)} V_0(F \cap K) \mu_q(dF)}{\int_{A(d,q)} V_0(F \cap K_0) \mu_q(dF)} \overset{Crofton}{=} \frac{c_{0,d}^{q,d-q} V_{d-q}(K)}{c_{0,d}^{q,d-q} V_{d-q}(K_0)} \\
&= \frac{V_{d-q}(K)}{V_{d-q}(K_0)}.
\end{aligned}$$

□

Remark 5.2.1. Taking the situation from the last Lemma, with Remark (5.1.1) (iv) and $q = 1$ we get

$$\mathbb{P}(X_1 \cap K \neq \emptyset) = \frac{V_{d-1}(K)}{V_{d-1}(K_0)} = \frac{S_{d-1}(K)}{S_{d-1}(K_0)}.$$

For $d = 2$ we basically can interpretate, that the propability of a line X_1 which intersects K_0 also interesects K can be calculated by deviding the boundary length of K by the boundary length of K_0. This seems to be a convenient result although it may not be completely intuitive.

5.3 Constructions in the real plane

From now on consider the case $d = 2$ and $q = 1$ which is looking at lines in the real plane. For some $K_0 \in \mathcal{K}^2$ we will be interested in choosing a random line out of all lines which intersect with K_0. We have looked at exactly this situation in Lemma (5.2.1). We will look at some examples in this section, argue about why the measure μ_1 is senseful to be used and what parametrisations on lines could be helpful to actually calculate realisations for random lines equivalently to μ_1. In the next chapter we'll define an Incremental Aggregate where we'll use our insights here to realise random lines. Note that every line

$g \in \mathcal{G}$ has a form $g = g_{a,b} := \{a + tb \in \mathbb{R}^2 \mid t \in \mathbb{R}^2\}$ for some vectors $a, b \in \mathbb{R}^2$ with $b \neq (0,0)$. From now on for $r > 0$ and $x \in \mathbb{R}^2$ define $B_r(x) := \{y \in R^2 \mid d(x,y) \leq r\}$ and $B_r := B_r(0)$. Let furthermore g be an isotropic random 1-flat through K_0. Recap the definition of κ_j, which is the j-dimensional Lebesgue measure of the j-dimensional unit ball, for our use here $\kappa_0 = 1$, $\kappa_1 = 2$ and $\kappa_2 = \pi$. For the next examples we always use Lemma (5.2.1) and Remark (5.2.1) (iii) and choose $K, K_0 \in \mathcal{K}^2$ with $K \subset K_0$.

Example 5.3.1. Let $0 < r < R$, $K_0 := B_R$ and $K := B_r$. We get

$$\mathbb{P}(g \cap B_r \neq \emptyset) = \frac{V_1(B_r)}{V_1(B_R)} = \frac{\frac{1}{2}S_1(B_r)}{\frac{1}{2}S_1(B_R)} = \frac{2\pi r}{2\pi R} = \frac{r}{R}.$$

Example 5.3.2. Let $0 < r \leq \frac{R}{\sqrt{2}}$, $K_0 := B_R$ and $K := [-r,r]^2$. We get

$$\mathbb{P}(g \cap [-r,r]^2 \neq \emptyset) = \frac{S_1([-r,r]^2)}{S_1(B_R)} = \frac{8r}{2\pi R} = \frac{4}{\pi}\frac{r}{R}.$$

Example 5.3.3. Let $0 < r \leq R$, $K_0 := B_R$ and $K := [-r,r]$. We get

$$\mathbb{P}(g \cap [-r,r] \neq \emptyset) = \frac{V_1([-r,r])}{V_1(B_R)} = \frac{\lambda_1([-r,r])}{\frac{1}{2}S_1(B_R)} = \frac{2r}{\pi R} = \frac{2}{\pi}\frac{r}{R}.$$

Example 5.3.4. Let $K_0 := B_1$ and $K := T_a$ a equilateral triangle with side length $a = \frac{\pi}{3}$ centered at $(0,0)$. We get

$$\mathbb{P}(g \cap T_a \neq \emptyset) = \frac{V_1(T_a)}{V_1(B_1)} = \frac{3a}{2\pi} = \frac{1}{2}.$$

Remark 5.3.1. We still haven't answered why μ_1 is a senseful measure to be used to calculate probabilities for situations like in the last examples. And yet it is not obvious how to actually simulate a realisation of a random line with μ_1. For now we can calculate the probability that a random line g hits some convex set K in the base set K_0. What we need is some form of parametrisation of this random choosing so we can actually end up for example with one angle defining the rotation of g to the x-axis and a real number defining the interesection value of g with the y-axis. If we wouldn't know about μ_1 for now, we could choose an angle and a real number intuitively the following way: Choose an angle α uniformly in $[0,\pi)$. Then consider all lines in the plane which are rotated counterclockwise by α starting at the x-axis and which intersect K. Take the set of intersection values of these lines with the y-axis and choose a value y_0 uniformly out of it, so finally α and y_0 define a unique line and we have a realisation of a random line which interesects K. This procedure may sound balanced, but it is only if K_0 is strongly symmetric. At this point it is important to remember what exactly we want. Out of all lines which intersect K_0 we are looking for the probability that a random chosen line intersects K and finally calculate a realisation. Imagine $K_0 = [-a,a] \times [-b,b]$ with $a < b$. Then it makes sense that there are "more" lines through the longer side $[-b,b]$ than through the shorter side $[-a,a]$. If we now choose the angle α uniformly, the shorter side is overweighted and the larger side underweighted, or in other words, the angle area around 0 should be less likely to be chosen than the angle area around $\frac{\pi}{2}$. So this naive attempt of choosing the angle for our line uniformly doesn't consider asymmetries of K_0 and does therefore not hold the senseful idea, that longer sides of K_0 are hit by more lines than shorter ones.

Definition 5.3.1.

6 Line Hitting Aggregate

In the following we will look at a process which is the approach of a simple approximation of external DLA on $\mathbb{Z}^2$. The idea is to let particles move on straight lines coming from infinity and add to the cluster when hitting it. Obviously in most cases particles cannot move completely straight on $\mathbb{Z}^2$. Therefore we will consider points in $\mathbb{Z}^2$ as the centers of unit squares and let the particles move on straight lines in the full plane $\mathbb{R}^2$. We consider a line hitting a point in $\mathbb{Z}^2$ if and only if it intersects with its unit square as defined in the following.

Definition 6.0.1. Define

$$\mathbb{R}^2_{sq} := \{[k - \frac{1}{2}, k + \frac{1}{2}] \times [l - \frac{1}{2}, l + \frac{1}{2}] \subset \mathbb{R}^2 \mid k, l \in \mathbb{Z}\}, \tag{6.1}$$

note that $\mathbb{R}^2 = \bigcup_{s \in \mathbb{R}^2_{sq}} s$. The canonical function

$$sq : \mathbb{Z}^2 \to \mathbb{R}^2_{sq}, \quad (k, l) \to [k - \frac{1}{2}, k + \frac{1}{2}] \times [l - \frac{1}{2}, l + \frac{1}{2}] \tag{6.2}$$

is bijective and intuitively identifies points in $\mathbb{Z}^2$ with squares in $\mathbb{R}^2$, which is p is the center of square $sq(p)$ for all $p \in \mathbb{Z}^2$. In the following when using a point $p \in \mathbb{Z}^2$ it will reference the point in $\mathbb{Z}^2$ or the corresponding square in $\mathbb{R}^2$ respecting the context. This bijection also naturally defines a graph structure on $\mathbb{R}^2_{sq}$, which is two squares $s_1, s_2 \in \mathbb{R}^2_{sq}$ form an edge if and only if $sq^{-1}(s_1)$ and $sq^{-1}(s_2)$ form an edge in $\mathbb{Z}^2$. For the following context we say a line L *hits* a point $p \in \mathbb{Z}^2$ if and only if $L \cap sq(p) \neq \emptyset$.

BILD Linie durch squares "hitting"

Definition 6.0.2. Geradenmaß μ_0

Definition 6.0.3. Let $L = L_{a,b} \in \mathcal{L}$ be a line. For a finite set $A \in \mathcal{P}_f$ we define

$$L_A := \{p \in A \mid L \text{ hits } p\}$$

which is the subset of points in A are hit by the line L hits. For the following we suppose $L_A \neq \emptyset$. We want to define a total ordered relation $<_{line}$ on L_A. We choose two points $(x_1, x_2), (y_1, y_2) \in L_A$ and devide the definition of a relation into four cases, which are the line going from left-bottom to right-top, left-top to right-bottom, parallel to the x-axis and parallel to the y-axis. Write $b = (b_1, b_2)$.

Case 1 : $\quad L$ is parallel to the x-axis $\quad \Leftrightarrow \quad b_2 = 0$

$$(x_1, x_2) <_{line} (y_1, y_2) \quad :\Leftrightarrow \quad x_1 < y_1$$

Case 2 : L is parallel to the y-axis $\Leftrightarrow$ $b_1 = 0$

$$(x_1, x_2) <_{line} (y_1, y_2) \quad :\Leftrightarrow \quad x_2 < y_2$$

Case 3 : L is going from left-bottom to right-top $\Leftrightarrow$ $b_1 b_2 > 0$

$$(x_1, x_2) <_{line} (y_1, y_2) \quad :\Leftrightarrow \quad \begin{cases} x_1 < y_1, & \text{if } x_1 \neq y_1, \\ x_2 < y_2, & \text{if } x_1 = y_1. \end{cases}$$

Case 4 : L is going from left-top to right-bottom $\Leftrightarrow$ $b_1 b_2 < 0$

$$(x_1, x_2) <_{line} (y_1, y_2) \quad :\Leftrightarrow \quad \begin{cases} x_1 < y_1, & \text{if } x_1 \neq y_1, \\ x_2 > y_2, & \text{if } x_1 = y_1. \end{cases}$$

It is easy to see that this well-defines a relation on L_A. In the following we will quickly proove that this relation is totally ordered.

Lemma 6.0.1. For a line $L = L_{a,b} \in \mathcal{L}$ and $A \in \mathcal{P}_f$ with $L_A \neq \emptyset$ the relation $<_{line}$ on L_A is totally ordered.

Proof. We will only proove the case where L is going from left-bottom to right-top, which is *Case 3* of the definition. In this case we have $b_1 b_2 > 0$. Note, that the proof for *Case 4* will work very similair. And in the case of L being parallel to one of the axes (*Case 1 or 2*), all properties for a totally ordered relation follow directly from the totally ordered relation $<$ on $\mathbb{R}$. So let $b_1 b_2 > 0$.

Antisymmetry : For antisymmetry let $(x_1, x_2) <_{line} (y_1, y_2)$ and $(y_1, y_2) <_{line} (x_1, x_2)$. Suppose $x_1 \neq y_1$, then $x_1 < y_1$ and $y_1 < x_1$, therefore $x_1 = y_1$ by antisymmetry of the standard order $<$ in $\mathbb{R}$, a contradiction, hence $x_1 = y_1$. But then we have $x_2 < y_2$ and $y_2 < x_2$ and therefore also $x_2 = y_2$.

Transitivity : For transitivity let $(x_1, x_2) <_{line} (y_1, y_2)$ and $(y_1, y_2) <_{line} (z_1, z_2)$. We find four cases. In case $x_1 \neq y_1$ and $y_1 \neq z_1$ we get $x_1 < z_1$ by transitivity of $<$, hence $(x_1, x_2) <_{line} (z_1, z_2)$. In case $x_1 \neq y_1$ and $y_1 = z_1$ we get $x_1 < y_1 = z_1$, therefore $(x_1, x_2) <_{line} (z_1, z_2)$. In case $x_1 = y_1$ and $y_1 \neq z_1$ we get $x_1 = y_1 < z_1$, similair as the last case. In the last case $x_1 = y_1 = z_1$ we get $x_2 < y_2$ and $y_2 < z_2$ and again by transitivity of $<$ we get $x_2 < z_2$, hence $(x_1, x_2) <_{line} (z_1, z_2)$ again.

Connexity : Connexity is given since for any two points $(x_1, x_2), (y_1, y_2) \in L_A$ we have either $x_1 \neq y_1$ or $x_1 = y_1$ and therefore either $(x_1, x_2) <_{line} (y_1, y_2)$ or $(y_1, y_2) <_{line} (x_1, x_2)$. □

Remark 6.0.1. The relation $<_{line}$ on L_A basically orders the hitpoints of a line L with a finite set A from left to right (or bottom to top in case of a line parallel to the y-axis). This order allows us to identify the outest hitting points of A by L. This means when moving on a line L facing A this order allows us to know where in A the line L hits first when „entering“ A and where it hits last when „leaving“ A which will be the two points $min_{<_{line}} L_A$ and $max_{<_{line}} L_A$ (or the other way around).

Definition 6.0.4. *Random Line Hitting Distribution*

Choose $A \in \mathcal{P}_f$. We define a distribution μ_A on $\mathbb{Z}^2$ as in the following. Let $L = L_{a,b}$ be a random line according to the line measure μ_0 with the condition that L hits A. Then define μ_A with distribution $\mu_A \sim U(\{min_{<_{line}} L_A, max_{<_{line}} L_A\})$, which chooses uniformly an element out of $\{min_{<_{line}} L_A, max_{<_{line}} L_A\}$. We call this distribution the *Random Line Hitting Distribution* (*of* A).

Definition 6.0.5. *Line Hitting Aggregate*

Incremental Aggregate with the Random Line Hitting Distribution as its distribution we will call here *Line Hitting Aggregate*, short *LHA*.

7 Questions

$\mathcal{K}' =?$

References

[1] N. Henze. *Maß und Wahrscheinlichkeitstheorie (Stochastik II)*. Karlsruher Institut für Technologie, Karlsruhe, 2010

[2] Daniel Hug, Günter Last, Steffen Winter. *Stochastic Geometry, Lecture Notes (summer term 2020)*. Institute of Technology, Karlsruhe, 2020

[3] Franz Sackmann. *Zufällige Geraden, Staatsexamensarbeit.* University Karlsruhe (TH), Karlsruhe, 2007

Erklärung

Hiermit versichere ich, dass ich diese Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt, die wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht und die Satzung des Karlsruher Instituts für Technologie zur Sicherung guter wissenschaftlicher Praxis in der jeweils gültigen Fassung beachtet habe.

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