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Solutions for Work Sheet 2

Problem 1 (Convergence with respect to the Fell topology – Examples, Part 1)

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^d$ and $x \in \mathbb{R}^d$. Further, let $(r_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers and $r > 0$. Denote by $B(x,r)$ the closed ball of radius r around x.

a) If $x_n \to x$ and $r_n \to r$ (as $n \to \infty$), then $B(x_n, r_n) \to B(x,r)$ in $(\mathcal{F}, \mathcal{O}_{\mathcal{F}})$, as $n \to \infty$.

b) If $\|x_n\| \to \infty$, then $\{x_n\} \to \varnothing$ in $(\mathcal{F}, \mathcal{O}_{\mathcal{F}})$, as $n \to \infty$.

Proposed solution:

a) We verify condition (3) from Theorem 1.3 (see also Problem 3 on Work Sheet 1). We can write any $y \in B(x,r)$ as $y = x + \lambda \cdot r \cdot u$ with $\lambda \in [0,1]$ and $u \in \mathbb{R}^d$ such that $\|u\| = 1$. Put $y_n := x_n + \lambda \cdot r_n \cdot u$. Apparently, $y_n \in B(x_n, r_n)$, and we have

$$\|y - y_n\| \leqslant \|x - x_n\| + \lambda \cdot |r_n - r| \cdot \|u\| \to 0 \quad (\text{as } n \to \infty),$$

so the first part of (3) is verified. To prove the second part, let $(n_k)_{k\in\mathbb{N}}$ be any increasing sequence in $\mathbb{N}$, and let $y_{n_k} \in B(x_{n_k}, r_{n_k})$ (for each $k \in \mathbb{N}$) be such that $y_{n_k} \to y$ (as $k \to \infty$) for some $y \in \mathbb{R}^d$. Notice that the distance of the point $y_{n_k} \in B(x_{n_k}, r_{n_k})$ from $B(x,r)$ is at most $\|x - x_{n_k}\| + |r_{n_k} - r|$. Thus, $\operatorname{dist}\big(y, B(x,r)\big) = 0$ and as $B(x,r)$ is closed, we conclude that $y \in B(x,r)$.

b) We use part (2) of Theorem 1.3. As there exists no open set which has non-empty intersection with $\varnothing$, the first part of (2) is trivially satisfied. To verify the second part of (2), let C be any compact set in $\mathbb{R}^d$ (they obviously all satisfy $C \cap \varnothing = \varnothing$) and note that, as $\|x_n\| \to \infty$, we find an index $n_0 \in \mathbb{N}$ such that $x_n \notin C$ for all $n \geqslant n_0$ (using that C is bounded). Thus, $C \cap \{x_n\} = \varnothing$ for each $n \geqslant n_0$.

Problem 2 (Continuity with respect to the Fell topology)

Denote by $(\mathcal{F}^d, \mathcal{O}_{\mathcal{F}^d})$ the space of closed subsets of $\mathbb{R}^d$ endowed with the Fell topology. Prove that the following maps are continuous:

a) $a : \mathbb{R}^d \times \mathcal{F}^d \to \mathcal{F}^d, \quad a(x,F) := F + x$.

b) $r : \mathcal{F}^d \to \mathcal{F}^d, \quad r(F) := F^* := -F$.

c) $e : (0,\infty) \times \mathcal{F}^d \to \mathcal{F}^d, \quad e(\alpha, F) := \alpha F$.

Also prove that the map $\widetilde{e} : [0,\infty) \times \mathcal{F}^d \to \mathcal{F}^d$, $\widetilde{e}(\alpha,F) := \alpha F$ is not continuous.

Proposed solution:

a) Consider a convergent sequence $(x_n, F_n) \to (x, F)$ in $\mathbb{R}^d \times \mathcal{F}^d$, as $n \to \infty$. We apply part (3) of Theorem 1.3 to show that $a(x_n, F_n) \to a(x,F)$ which implies sequential continuity and hence continuity of a:

- Let $y + x \in a(x,F) = F + x$ and choose $y_n \in F_n$ so that $y_n \to y$ as $n \to \infty$ [using part (3) of Theorem 1.3 for the convergence $F_n \to F$]. Then $y_n + x_n \in a(x_n, F_n) = F_n + x_n$ and $y_n + x_n \to y + x$ (as $n \to \infty$).

- Now let $(n_k)_{k\in\mathbb{N}}$ be any increasing sequence in $\mathbb{N}$, and $z_{n_k} \in a(x_{n_k}, F_{n_k}) = F_{n_k} + x_{n_k}$ such that $z_{n_k} \to z$, as $k \to \infty$, for some $z \in \mathbb{R}^d$. Then, we have $z_{n_k} - x_{n_k} \in F_{n_k}$ with $z_{n_k} - x_{n_k} \to z - x \in F$ [where $z - x \in F$ follows from the fact that $F_n \to F$], and hence $z \in a(x, F) = F + x$.

  By part (3) of Theorem 1.3, we obtain $a(x_n, F_n) = F_n + x_n \to F + x = a(x, F)$ (as $n \to \infty$).

b) Let $(F_n)_{n\in\mathbb{N}}$ be a convergent sequence in $(\mathcal{F}^d, \mathcal{O}_{\mathcal{F}^d})$ with limit set F. We prove that $r(F_n) = -F_n \to -F = r(F)$ (as $n \to \infty$) using condition (3) of Theorem 1.3:

- If $F = \varnothing$ then $r(F) = -F = \varnothing$ and the first part of condition (3) is trivially true, so assume $F \neq \varnothing$. Let $x \in r(F) = -F$. Apparently, $y := -x \in F$ and the convergence $F_n \to F$ yields the existence of an index $n_0 \in \mathbb{N}$ and elements $y_n \in F_n$ for each $n \geqslant n_0$ such that $y_n \to y$ as $n \to \infty$. As $y_n \in F_n$, we have $x_n := -y_n \in r(F_n) = -F_n$, and clearly $x_n = -y_n \to -y = x$ (as $n \to \infty$).
- Now let $(n_k)_{k\in\mathbb{N}}$ be any increasing sequence in $\mathbb{N}$, and $z_{n_k} \in r(F_{n_k}) = -F_{n_k}$ such that $z_{n_k} \to z$, as $k \to \infty$, for some $z \in \mathbb{R}^d$. Then, $-z_{n_k} \in F_{n_k}$ and $-z_{n_k} \to -z$ (as $k \to \infty$). From the convergence $F_n \to F$ (as $n \to \infty$) we infer that $-z \in F$, and thus $z \in r(F) = -F$.

c) Let $(F_n)_{n\in\mathbb{N}}$ be a convergent sequence in $(\mathcal{F}^d, \mathcal{O}_{\mathcal{F}^d})$ with limit set F, and let $(\alpha_n)_{n\in\mathbb{N}}$ be a convergent sequence in $(0, \infty)$ with limit $\alpha \in (0, \infty)$. We prove that $e(\alpha_n, F_n) = \alpha_n F_n \to \alpha F = e(\alpha, F)$ (as $n \to \infty$) using condition (3) of Theorem 1.3:

- If $F = \varnothing$ then $e(\alpha, F) = \alpha F = \varnothing$ and there is nothing to prove. Thus, assume that $F \neq \varnothing$ and let $x \in e(\alpha, F) = \alpha F$. Apparently, $y := \frac{1}{\alpha} x \in F$ and the convergence $F_n \to F$ yields the existence of an index $n_0 \in \mathbb{N}$ and elements $y_n \in F_n$ for each $n \geqslant n_0$ such that $y_n \to y$ as $n \to \infty$. As $y_n \in F_n$, we have $x_n := \alpha_n y_n \in e(\alpha_n, F_n) = \alpha_n F_n$, and $x_n = \alpha_n y_n \to \alpha y = x$ (as $n \to \infty$).
- Let $(n_k)_{k\in\mathbb{N}}$ be any increasing sequence in $\mathbb{N}$, and $z_{n_k} \in e(\alpha_{n_k}, F_{n_k}) = \alpha_{n_k} F_{n_k}$ such that $z_{n_k} \to z$, as $k \to \infty$, for some $z \in \mathbb{R}^d$. Then, $\frac{1}{\alpha_{n_k}} z_{n_k} \in F_{n_k}$ and $\frac{1}{\alpha_{n_k}} z_{n_k} \to \frac{1}{\alpha} z$ (as $k \to \infty$). From the convergence $F_n \to F$ (as $n \to \infty$) we infer that $\frac{1}{\alpha} z \in F$, and thus $z \in e(\alpha, F) = \alpha F$.

To prove that the map $\widetilde{e}$ is not continuous, consider the case $d = 1$ and the sets $F_n := [0, n] \in \mathcal{F}^1$ as well as $\alpha_n := \frac{1}{n} \in (0, \infty)$. Then, $F_n \to F := [0, \infty)$ in $(\mathcal{F}^1, \mathcal{O}_{\mathcal{F}^1})$ and $\alpha_n \to \alpha := 0$ (both as $n \to \infty$). However, $\widetilde{e}(\alpha_n, F_n) = \alpha_n F_n = [0, 1]$ does not converge to $\widetilde{e}(\alpha, F) = \alpha F = \{0\}$, so $\widetilde{e}$ is not sequentially continuous.

Problem 3 (Convergence with respect to the Fell topology – Examples, Part 2)

Denote by $(\mathcal{F}^d, \mathcal{O}_{\mathcal{F}^d})$ the space of closed subsets of $\mathbb{R}^d$ endowed with the Fell topology. For $u \in \mathbb{R}^d$ with $\|u\| = 1$, and $r \geqslant 0$, write

$$H_{u,r} := \left\{ x \in \mathbb{R}^d : \langle x, u \rangle = r \right\}.$$

Let u_n be a sequence in $\mathbb{R}^d$ with $\|u_n\| = 1$ for each $n \in \mathbb{N}$, and let r_n be a sequence in $[0, \infty)$.

a) Show that if $u_n \to u$ and $r_n \to r$, then $H_{u_n, r_n} \to H_{u,r}$ in $(\mathcal{F}^d, \mathcal{O}_{\mathcal{F}^d})$, as $n \to \infty$.

b) Show that if $u_n \to u$ and $r_n \to \infty$, then $H_{u_n, r_n} \to \varnothing$ in $(\mathcal{F}^d, \mathcal{O}_{\mathcal{F}^d})$, as $n \to \infty$.

Consider the sequence $(P_n)_{n\in\mathbb{N}}$ of paraboloids $P_n = \left\{ z \in \mathbb{R}^d \;\middle|\; \frac{z_1^2 + \ldots + z_{d-1}^2}{n} = z_d \right\}$, and $e_d = (0, \ldots, 0, 1) \in \mathbb{R}^d$.

c) Show that $P_n \to H_{e_d, 0}$ in $(\mathcal{F}^d, \mathcal{O}_{\mathcal{F}^d})$, as $n \to \infty$.

Proposed solution:

a) Notice that $H_{u_n, r_n} = H_{u_n, 0} + r_n u_n$. By the continuity of the map in part a) of Problem 2, it suffices to prove $H_{u_n, 0} \to H_{u, 0}$ in $(\mathcal{F}^d, \mathcal{O}_{\mathcal{F}^d})$. We use condition (3) of Theorem 1.3. For the first part, fix some $x \in H_{u,0}$. Let x_n be the orthogonal projection of x onto $H_{u_n, 0}$, that is, $x_n = x - \langle x, u_n \rangle u_n \in H_{u_n, 0}$. From

$$\big\| \langle x, u_n \rangle u_n \big\| \to \big| \langle x, u \rangle \big| = 0$$

we see that $x_n \to x$. For the second part, let $x_{n_k} \to x$ with $x_{n_k} \in H_{u_{n_k}, 0}$ for each $k \in \mathbb{N}$. We have

$$\langle x, u \rangle = \lim_{k\to\infty} \langle x_{n_k}, u_{n_k} \rangle = 0,$$

and conclude that $x \in H_{u,0}$.

b) From Theorem 1.2 (see also Problem 2 on Work Sheet 1) we know that $\{\mathcal{F}^C \mid C \in \mathcal{C}\}$ is a neighborhood base of $\varnothing$. Let $C \in \mathcal{C}$ be arbitrary. There exists some $R > 0$ such that $C \subset B(0, R)$, and since $r_n \to \infty$, we find $n_0 \in \mathbb{N}$ such that $r_n > R$, for $n \geqslant n_0$. Therefore, $H_{u_n,r_n} \in \mathcal{F}^C$, for $n \geqslant n_0$. We conclude that $H_{u_n,r_n} \to \varnothing$.

c) We use part (3) of Theorem 1.3. To prove the first part, let $x = (x_1, \dots, x_{d-1}, 0) \in H_{e_d,0}$ be arbitrary. Choose

$$z^{(n)} = \left(\left(1 + \tfrac{1}{n}\right) x_1, \dots, \left(1 + \tfrac{1}{n}\right) x_{d-1}, \frac{\left(1+\frac{1}{n}\right)^2}{n} \left(x_1^2 + \dots + x_{d-1}^2\right) \right).$$

From

$$\frac{\left(z_1^{(n)}\right)^2 + \dots + \left(z_{d-1}^{(n)}\right)^2}{n} = \frac{\left(1 + \frac{1}{n}\right)^2 \left(x_1^2 + \dots + x_{d-1}^2\right)}{n} = z_d^{(n)}$$

we conclude that $z^{(n)} \in P_n$. Moreover, $\left\|z^{(n)} - x\right\|^2 = \sum_{j=1}^{d-1} \frac{x_j^2}{n^2} + \frac{(1+1/n)^4}{n^2} \left(x_1^2 + \dots + x_{d-1}^2\right)^2 \to 0$, as $n \to \infty$. For the proof of the second part, let $z^{(n_k)} \in P_{n_k}$ such that $z^{(n_k)} \to z$ as $k \to \infty$. By definition of P_{n_k}, we have

$$z_d^{(n_k)} = \frac{\left(z_1^{(n_k)}\right)^2 + \dots + \left(z_{d-1}^{(n_k)}\right)^2}{n_k} \to 0, \quad \text{as } k \to \infty,$$

where we use that $\left(z_\ell^{(n_k)}\right)^2 \to z_\ell^2 < \infty$, $\ell \in \{1, \dots, d-1\}$. Consequently, $z_d = 0$ and $z \in H_{e_d,0}$.

Problem 4 (On the Hausdorff metric – Part 1)

Let $(\mathbb{X}, d)$ be a metric space, and recall from the lecture that, for any set $B \subset \mathbb{X}$ and $\varepsilon \geqslant 0$,

$$B_{\oplus\varepsilon} := \left\{x \in \mathbb{X} \,:\, \operatorname{dist}(x, B) \leqslant \varepsilon\right\}$$

denotes the ε-parallel set of B. Here, $\operatorname{dist}(x, B) := \inf_{y \in B} d(x, y)$ is the distance from x to B with respect to the metric d on $\mathbb{X}$. Also recall that the Hausdorff metric δ on $\mathcal{C}(\mathbb{X}) \setminus \{\varnothing\}$ is defined as

$$\delta(C, C') := \inf\left\{\varepsilon \geqslant 0 \;:\; C \subset C'_{\oplus\varepsilon},\, C' \subset C_{\oplus\varepsilon}\right\}, \qquad C, C' \in \mathcal{C}(\mathbb{X}) \setminus \{\varnothing\},$$

and that we put $\delta(\varnothing, C) = \delta(C, \varnothing) := \infty$, $C \in \mathcal{C}(\mathbb{X}) \setminus \{\varnothing\}$, as well as $\delta(\varnothing, \varnothing) := 0$. Prove the following assertions:

a) $\delta(C, C') = \max\left\{ \sup_{x \in C} \operatorname{dist}(x, C'),\ \sup_{y \in C'} \operatorname{dist}(y, C)\right\}, \quad C, C' \in \mathcal{C}(\mathbb{X}) \setminus \{\varnothing\}$,

b) δ is a metric on $\mathcal{C}(\mathbb{X}) \setminus \{\varnothing\}$, and

c) δ is also a metric on $\mathcal{C}(\mathbb{X})$.

Proposed solution:

a) Fix any $C, C' \in \mathcal{C}(\mathbb{X}) \setminus \{\varnothing\}$. First, let $\varepsilon \geqslant 0$ be such that $C \subset C'_{\oplus\varepsilon}$ and $C' \subset C_{\oplus\varepsilon}$. By definition, $\operatorname{dist}(x, C') \leqslant \varepsilon$ for every $x \in C$ and $\operatorname{dist}(y, C) \leqslant \varepsilon$ for every $y \in C'$, so we conclude that

$$\max\left\{ \sup_{x \in C} \operatorname{dist}(x, C'),\ \sup_{y \in C'} \operatorname{dist}(y, C)\right\} \leqslant \varepsilon.$$

Since we chose an arbitrary ε with the above property, it follows that

$$\max\left\{ \sup_{x \in C} \operatorname{dist}(x, C'),\ \sup_{y \in C'} \operatorname{dist}(y, C)\right\} \leqslant \delta(C, C').$$

To prove the converse inequality, put $\varepsilon_1 := \sup_{x \in C} \operatorname{dist}(x, C')$, $\varepsilon_2 := \sup_{y \in C'} \operatorname{dist}(y, C)$, as well as $\varepsilon := \max\{\varepsilon_1, \varepsilon_2\}$. Since $\operatorname{dist}(x, C') \leqslant \varepsilon_1 \leqslant \varepsilon$ for every $x \in C$, we have $C \subset C'_{\oplus\varepsilon}$. Similarly, we find $C' \subset C_{\oplus\varepsilon_2} \subset C_{\oplus\varepsilon}$. We conclude that

$$\delta(C, C') \leqslant \varepsilon = \max\{\varepsilon_1, \varepsilon_2\} = \max\left\{ \sup_{x \in C} \operatorname{dist}(x, C'),\ \sup_{y \in C'} \operatorname{dist}(y, C)\right\}.$$

b) Let $C, C', D \in \mathcal{C}(\mathbb{X}) \setminus \{\varnothing\}$. Apparently, $\delta(C, C') \geqslant 0$ by definition, and we have

$$\delta(C, C') = 0 \quad \Longleftrightarrow \quad C \subset C'_{\oplus 0} = C' \text{ and } C' \subset C_{\oplus 0} = C \quad \Longleftrightarrow \quad C = C'.$$

Moreover, $\delta(C, C') = \delta(C', C)$ as the definition of δ is symmetric in the two sets. It remains to prove the triangle inequality for which we use the representation from part a). Indeed, observe that for any $x \in C$, we have

$$\operatorname{dist}(x, C') \leqslant d(x, z) + \operatorname{dist}(z, C') \leqslant d(x, z) + \delta(D, C')$$

for every $z \in D$, so by taking $\inf_{z \in D}$ we obtain

$$\operatorname{dist}(x, C') \leqslant \operatorname{dist}(x, D) + \delta(D, C') \leqslant \delta(C, D) + \delta(D, C'),$$

and hence $\sup_{x \in C} \operatorname{dist}(x, C') \leqslant \delta(C, D) + \delta(D, C')$. Similarly, $\sup_{y \in C'} \operatorname{dist}(y, C) \leqslant \delta(C', D) + \delta(D, C)$ and part a) implies that the triangle inequality holds.

c) Extending the definition of δ to $\varnothing$ as recalled above, it is immediate that δ remains non-negative, the identity of indiscernibles holds, and δ is symmetric. To verify the triangle inequality, let $C, C', D \in \mathcal{C}(\mathbb{X})$. If all three sets were non-empty, the inequality follows from b). If $C = C' = D = \varnothing$, then $\delta(C, C') = \delta(C, D) = \delta(D, C') = 0$ and the triangle inequality holds trivially. If one of the three sets is empty and another is non-empty, one of the terms on the right hand side of the inequality will necessarily be ∞ and so it is also trivially satisfied.