Solutions for Work Sheet 6

Problem 1 (Some advanced properties of the Poisson process)

Let $(\mathbb{X}, \rho)$ be a separable metric space, and let $(\Omega, \mathcal{A}, \mathbb{P})$ be the probability space which underlies the random quantities that appear in the following.

a) Let Φ be a Poisson process in $\mathbb{X}$ with intensity measure Θ. For $A \in \mathcal{X}$ such that $0 < \Theta(A) < \infty$, and $k \in \mathbb{N}$, we have

$$\mathbb{P}\Big(\Phi_A \in \cdot \;\Big|\; \Phi(A) = k\Big) = \mathbb{P}\bigg(\sum_{j=1}^{k} \delta_{X_j} \in \cdot\bigg)$$

with independent $X_1, \ldots, X_k \sim \frac{\Theta_A}{\Theta(A)}$.

b) Let Ψ be a point process in $\mathbb{X}$. Then Ψ is a Poisson process with intensity measure $\lambda \in M(\mathbb{X})$ if, and only if,

$$\mathbb{E}\Big[F\big(\Psi_A\big)\Big] = e^{-\lambda(A)} \left(F(\mathbf{0}) + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{A^k} F\bigg(\sum_{j=1}^{k} \delta_{y_j}\bigg) \, \mathrm{d}\lambda^k(y_1, \ldots, y_k)\right)$$

for all $A \in \mathcal{X}$ with $\lambda(A) < \infty$, and all measurable $F : N(\mathbb{X}) \to [0, \infty]$.

Proposed solution:

a) By Problem 2 b) on Sheet 5, Φ_A is a Poisson process with intensity measure Θ_A. By Theorem 2.30 of the lecture notes, we have

$$\Phi_A \overset{d}{=} \sum_{j=1}^{\tau} \delta_{X_j},$$

where $X_j \sim \frac{\Theta_A}{\Theta(A)}$, $\tau \sim \mathrm{Po}\big(\Theta(A)\big)$, and where $(X_j)_{j \in \mathbb{N}}$ and τ are independent. In particular, $\tau \overset{d}{=} \Phi(A)$, and we conclude that, for $k \in \mathbb{N}$,

$$\begin{aligned}\mathbb{P}\Big(\Phi_A \in \cdot \;\Big|\; \Phi(A) = k\Big) = \mathbb{P}\bigg(\sum_{j=1}^{\tau} \delta_{X_j} \in \cdot\,, \tau = k\bigg) \Big/ \mathbb{P}(\tau = k) &= \mathbb{P}\bigg(\sum_{j=1}^{k} \delta_{X_j} \in \cdot\,, \tau = k\bigg) \Big/ \mathbb{P}(\tau = k) \\ &= \mathbb{P}\bigg(\sum_{j=1}^{k} \delta_{X_j} \in \cdot\bigg).\end{aligned}$$

b) We show the two implications separately:

'$\Longrightarrow$' Assume that Ψ is a Poisson process with intensity measure λ, and let $A \in \mathcal{X}$ with $\lambda(A) < \infty$ and $F : N(\mathbb{X}) \to [0, \infty]$ a measurable function. If $\lambda(A) = 0$, the statement is trivial, so we assume

$\lambda(A) \in (0,\infty)$. Then, using part a) of the exercise, we have

$$\begin{aligned}\mathbb{E}\Big[F(\Psi_A)\Big] &= \sum_{k=0}^{\infty} \mathbb{P}\Big(\Psi(A)=k\Big)\cdot \mathbb{E}\Big[F(\Psi_A)\;\Big|\;\Psi(A)=k\Big] \\ &= e^{-\lambda(A)}\,F(\mathbf{0}) + e^{-\lambda(A)} \sum_{k=1}^{\infty} \frac{\lambda(A)^k}{k!} \int_{A^k} F\Big(\sum_{j=1}^{k} \delta_{y_j}\Big)\,\mathrm{d}\left(\frac{\lambda}{\lambda(A)}\right)^k (y_1,\dots,y_k) \\ &= e^{-\lambda(A)} \left(F(\mathbf{0}) + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{A^k} F\Big(\sum_{j=1}^{k} \delta_{y_j}\Big)\,\mathrm{d}\lambda^k(y_1,\dots,y_k)\right).\end{aligned}$$

'$\Longleftarrow$' Now, assume that the stated equation holds for all sets $A \in \mathcal{X}$ with $\lambda(A) < \infty$ and all measurable $F : N(\mathbb{X}) \to [0,\infty]$. Thus, for arbitrary $B \in \mathcal{X}$ with $\lambda(B) < \infty$ and $k \in \mathbb{N}_0$, we define the function $F(\mu) := \mathbb{1}\{\mu(B) = k\}$ to conclude that

$$\begin{aligned}\mathbb{P}\Big(\Psi(B)=k\Big) = \mathbb{E}\Big[F(\Psi_B)\Big] &= e^{-\lambda(B)} \left(F(\mathbf{0}) + \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \int_{B^\ell} F\Big(\sum_{j=1}^{\ell} \delta_{y_j}\Big)\,\mathrm{d}\lambda^\ell(y_1,\dots,y_\ell)\right) \\ &= e^{-\lambda(B)}\,\frac{\lambda(B)^k}{k!}.\end{aligned}$$

For $B \in \mathcal{X}$ with $\lambda(B) = \infty$, we choose $C_i \in \mathcal{X}$ with $\lambda(C_i) < \infty$ and $C_i \nearrow \mathbb{X}$, and observe that, for $k \in \mathbb{N}_0$,

$$\mathbb{P}\Big(\Psi(B) \leqslant k\Big) \leqslant \liminf_{i\to\infty} \mathbb{P}\Big(\Psi(B\cap C_i) \leqslant k\Big) = \liminf_{i\to\infty} \sum_{j=0}^{k} e^{-\lambda(B\cap C_i)}\,\frac{\lambda(B\cap C_i)^j}{j!} = 0,$$

so $\mathbb{P}\big(\Psi(B)=\infty\big) = 1$. Now, let $B_1,\dots,B_k \in \mathcal{X}$ be pairwise disjoint with $\lambda(B_j) < \infty$, $j = 1,\dots,k$, and let $n_1,\dots,n_k \in \mathbb{N}_0$, $n := \sum_{j=1}^{k} n_j$. If $n = 0$, i.e., if $n_j = 0$ for $j = 1,\dots,k$, then

$$\mathbb{P}\Big(\Psi(B_1)=0,\dots,\Psi(B_k)=0\Big) = \exp\left(-\lambda\Big(\bigcup\nolimits_{j=1}^{k} B_j\Big)\right) = \prod_{j=1}^{k} e^{-\lambda(B_j)} = \prod_{j=1}^{k} \mathbb{P}\Big(\Psi(B_j)=0\Big).$$

If $n \geqslant 1$, we define $\widetilde{F}(\mu) := \mathbb{1}\{\mu(B_1) = n_1,\dots,\mu(B_k) = n_k\}$ and $B := \bigcup_{j=1}^{k} B_j$ (note: $\lambda(B) < \infty$), and conclude that

$$\begin{aligned}\mathbb{P}\Big(\Psi(B_1)=n_1,\dots,\Psi(B_k)=n_k\Big) = \mathbb{E}\Big[\widetilde{F}(\Psi_B)\Big] &= \frac{e^{-\lambda(B)}}{n!} \int_{B^n} \widetilde{F}\Big(\sum_{j=1}^{n} \delta_{y_j}\Big)\,\mathrm{d}\lambda^n(y_1,\dots,y_n) \\ &= \frac{e^{-\lambda(B)}}{n!} \binom{n}{n_1,\dots,n_k} \prod_{j=1}^{k} \lambda(B_j)^{n_j} \\ &= \prod_{j=1}^{k} \mathbb{P}\Big(\Psi(B_j)=n_j\Big).\end{aligned}$$

Thus follows the independence of $\Psi(B_1),\dots,\Psi(B_k)$. For general pairwise disjoint $B_1,\dots,B_k \in \mathcal{X}$, choose a sequence $A_i \in \mathcal{X}$ of pairwise disjoint sets with $\lambda(A_i) < \infty$ and $\bigcup_{i\in\mathbb{N}} A_i = \mathbb{X}$. Then, $\Psi(B_j \cap A_i)$, $j = 1,\dots,k$ and $i \in \mathbb{N}$, are independent and, therefore, so are $\Psi(B_j) = \sum_{i=1}^{\infty} \Psi\big(B_j \cap A_i\big)$, for $j = 1,\dots,k$.

Problem 2

Let $m \in \mathbb{N}$ and $\gamma_1, \dots, \gamma_m > 0$ such that $\gamma_j \neq \gamma_k$ for all $j, k \in \{1, \dots, m\}$, $j \neq k$. Further, let $p_1, \dots, p_m \in (0, 1]$ with $\sum_{j=1}^m p_j = 1$. Let X be a discrete random variable taking values in $\{\gamma_1, \dots, \gamma_m\}$ such that

$$\mathbb{P}(X = \gamma_j) = p_j, \quad j = 1, \dots, m.$$

Consider a point process Φ in $\mathbb{R}^d$ whose distribution is specified by the conditional distributions

$$\mathbb{P}\Big(\Phi(B) = n \,\Big|\, X = \gamma_j\Big) = \frac{\big(\gamma_j \cdot \lambda^d(B)\big)^n}{n!} \cdot e^{-\gamma_j \cdot \lambda^d(B)},$$

where $B \in \mathcal{B}(\mathbb{R}^d)$, $n \in \mathbb{N}_0$, and $j = 1, \dots, m$. Prove the following assertions.

a) The process Φ is stationary.

b) The process Φ is not a Poisson process for $m \geqslant 2$.

Proposed solution:

a) First of all, notice that the process Φ is simple. Indeed, for any $x \in \mathbb{R}^d$, we have

$$\mathbb{P}\Big(\Phi(\{x\}) \geqslant 1\Big) = \sum_{n=1}^{\infty} \sum_{j=1}^{m} p_j \cdot \mathbb{P}\Big(\Phi(\{x\}) = n \,\Big|\, X = \gamma_j\Big) = \sum_{j=1}^{m} p_j \sum_{n=1}^{\infty} \frac{\big(\gamma_j \cdot \lambda^d(\{x\})\big)^n}{n!} \cdot e^{-\gamma_j \cdot \lambda^d(\{x\})} = 0,$$

so $\mathbb{P}^\Phi\big(N_s(\mathbb{R}^d)\big) = \mathbb{P}\big(\Phi \in N_s(\mathbb{R}^d)\big) = 1$. In order to prove that Φ is stationary, let $B \in \mathcal{B}(\mathbb{R}^d)$ and $t \in \mathbb{R}^d$. We have

$$\begin{aligned}
\mathbb{P}\Big((\Phi + t)(B) = 0\Big) = \mathbb{P}\Big(\Phi(B - t) = 0\Big) &= \sum_{j=1}^{m} p_j \cdot \mathbb{P}\Big(\Phi(B-t) = 0 \,\Big|\, X = \gamma_j\Big) \\
&= \sum_{j=1}^{m} p_j \cdot \frac{\big(\gamma_j \cdot \lambda^d(B-t)\big)^0}{0!} \cdot e^{-\gamma_j \cdot \lambda^d(B-t)} \\
&= \sum_{j=1}^{m} p_j \cdot \frac{\big(\gamma_j \cdot \lambda^d(B)\big)^0}{0!} \cdot e^{-\gamma_j \cdot \lambda^d(B)} \\
&= \mathbb{P}\Big(\Phi(B) = 0\Big).
\end{aligned}$$

Hence (by the hint), the measures $\mathbb{P}^{\Phi + t}$ and $\mathbb{P}^\Phi$ agree on a generating π-system of $N_s(\mathbb{R}^d)$ and are thus identical, that is, $\Phi + t \stackrel{d}{=} \Phi$.

b) Choose any set $B \in \mathcal{B}(\mathbb{R}^d)$ with $0 < \lambda^d(B) < \infty$. The intensity measure of Φ evaluated in B satisfies

$$\begin{aligned}
\mathbb{E}\big[\Phi(B)\big] = \sum_{n=0}^{\infty} n \cdot \mathbb{P}\big(\Phi(B) = n\big) &= \sum_{n=0}^{\infty} n \sum_{j=1}^{m} p_j \cdot \mathbb{P}\Big(\Phi(B) = n \,\Big|\, X = \gamma_j\Big) \\
&= \sum_{n=0}^{\infty} n \sum_{j=1}^{m} p_j \cdot \frac{\big(\gamma_j \cdot \lambda^d(B)\big)^n}{n!} \cdot e^{-\gamma_j \cdot \lambda^d(B)} \\
&= \sum_{j=1}^{m} p_j \cdot e^{-\gamma_j \cdot \lambda^d(B)} \sum_{n=1}^{\infty} \frac{\big(\gamma_j \cdot \lambda^d(B)\big)^n}{(n-1)!} \\
&= \sum_{j=1}^{m} p_j \cdot e^{-\gamma_j \cdot \lambda^d(B)} \cdot \gamma_j \cdot \lambda^d(B) \cdot e^{\gamma_j \cdot \lambda^d(B)} \\
&= \lambda^d(B) \cdot \mathbb{E}[X],
\end{aligned}$$

hence if Φ was a Poisson process, then we ought to have

$$\mathbb{P}\big(\Phi(B) = 0\big) = e^{-\lambda^d(B) \cdot \mathbb{E}[X]}.$$

By definition of Φ, however, we have

$$\mathbb{P}\big(\Phi(B)=0\big) = \sum_{j=1}^{m} p_j \cdot \mathbb{P}\Big(\Phi(B)=0 \,\Big|\, X=\gamma_j\Big) = \sum_{j=1}^{m} p_j \cdot e^{-\gamma_j \cdot \lambda^d(B)} = \mathbb{E}\left[e^{-\lambda^d(B)\cdot X}\right].$$

Now, notice that since the function $[0,\infty) \ni x \mapsto \varphi(x) := e^{-\lambda^d(B)\cdot x}$ is convex, and since neither φ is linear nor are $\gamma_1 = \ldots = \gamma_m$, Jensen's inequality yields

$$\mathbb{P}\big(\Phi(B)=0\big) = \mathbb{E}\left[e^{-\lambda^d(B)\cdot X}\right] > e^{-\lambda^d(B)\cdot \mathbb{E}[X]},$$

so Φ cannot be a Poisson process.

Problem 3 (Some properties of p-thinnings)

Let $(\mathbb{X},\rho)$ be a separable metric space. Let $p : \mathbb{X} \to [0,1]$ be measurable, and let Φ be a point process in $\mathbb{X}$ and Φ_p the p-thinning of Φ, both defined on some probability space $(\Omega,\mathcal{A},\mathbb{P})$.

a) Prove that, for any measurable $g : \mathbb{X} \to [0,\infty]$, the Laplace functional of Φ_p is given through

$$L_{\Phi_p}(g) := \mathbb{E}\left[\exp\left(-\int_{\mathbb{X}} g(x)\,\mathrm{d}\Phi_p(x)\right)\right] = \mathbb{E}\left[\exp\left(\int_{\mathbb{X}} \log\Big(1-p(x)\big[1-e^{-g(x)}\big]\Big)\,\mathrm{d}\Phi(x)\right)\right].$$

b) Let $B \in \mathcal{X}$. Can we interpret Φ_B as a p-thinning of Φ? If so, how do we have to choose p?

c) Prove that, for any $n \in \mathbb{N}$, Φ can be written as a superposition of identically distributed point processes $\Phi_1,\ldots,\Phi_n$ such that $\mathbb{P}\big(\Phi(\mathbb{X}) \geqslant 1\big) > 0$ implies

$$\mathbb{P}\big(\Phi_k(\mathbb{X}) \geqslant 1\big) > 0, \quad k = 1,\ldots,n.$$

d) Prove that c) is not necessarily true if we require $\Phi_1,\ldots,\Phi_n$ to be independent.

e) Prove that if Φ is a Poisson process with intensity measure $\Theta \in M(\mathbb{X})$ then Φ is infinitely divisible, that is, for each $n \in \mathbb{N}$ we find i.i.d. point processes $\Phi_1,\ldots,\Phi_n$ such that $\Phi \overset{d}{=} \Phi_1 + \ldots + \Phi_n$.

Proposed solution: Recall that by Theorem 2.8 there exist $\mathbb{X}$-valued random elements $\xi_1,\xi_2,\ldots$ as well as an $\mathbb{N}_0 \cup \{\infty\}$-valued random variable τ such that $\Phi = \sum_{j=1}^{\tau} \delta_{\xi_j}$. Let K be the stochastic kernel from $\mathbb{X}$ to $\{0,1\}$ given through

$$K(x,\cdot) := \big(1-p(x)\big)\cdot\delta_0 + p(x)\cdot\delta_1, \quad x \in \mathbb{X}.$$

Further, let Ψ be a K-marking of Φ, that is, $\Psi = \sum_{j=1}^{\tau} \delta_{(\xi_j,Y_j)}$, where, conditional on $\tau,\xi_1,\xi_2,\ldots$, the random variables $Y_1,Y_2,\ldots$ are independent with $Y_j \sim K(\xi_j,\cdot)$, $j \in \mathbb{N}$. By Definition 2.39, the p-thinning Φ_p of Φ is the point process $\Psi\big(\,\cdot\times\{1\}\big)$. Observe that

$$\Phi_p(\cdot) = \Psi\big(\,\cdot\times\{1\}\big) = \sum_{j=1}^{\tau} \delta_{(\xi_j,Y_j)}\big(\,\cdot\times\{1\}\big) = \sum_{j=1}^{\tau} \delta_{\xi_j}(\cdot)\,\mathbb{1}\big\{Y_j = 1\big\}.$$

a) For measurable $f : \mathbb{X}\times\{0,1\} \to [0,\infty]$ put $f^*(x) := -\log\big(\int_{\mathbb{X}} \exp\big(-f(x,y)\big)\,K(x,\mathrm{d}y)\big)$. In the proof of Theorem 2.38 it was shown that

$$\mathbb{E}\left[\exp\left(-\int_{\mathbb{X}} f(x,y)\,\mathrm{d}\Psi(x,y)\right)\right] = \mathbb{E}\left[\exp\left(-\int_{\mathbb{X}} f^*(x)\,\mathrm{d}\Phi(x)\right)\right].$$

We define

$$f(x,y) = \begin{cases} g(x), & \text{if } y = 1,\\ 0, & \text{else.}\end{cases}$$

It follows that $f^*(x) = -\log\big[p(x)\cdot\exp\big(-g(x)\big) + \big(1-p(x)\big)\big]$, and, by choice of f,

$$\mathbb{E}\left[\exp\left(-\int_{\mathbb{X}} f(x,y)\,\mathrm{d}\Psi(x,y)\right)\right] = \mathbb{E}\left[\exp\left(-\int_{\mathbb{X}} g(x)\,\mathrm{d}\Phi_p(x)\right)\right].$$

We conclude that

$$\mathbb{E}\Big[\exp\Big(-\int_{\mathbb{X}} g(x)\,\mathrm{d}\Phi_p(x)\Big)\Big] = \mathbb{E}\Big[\exp\Big(-\int_{\mathbb{X}} -\log\Big(p(x)\cdot\exp\big(-g(x)\big) + \big(1-p(x)\big)\Big)\mathrm{d}\Phi(x)\Big)\Big]$$
$$= \mathbb{E}\Big[\exp\Big(\int_{\mathbb{X}} \log\Big(1 - p(x)\big[1-e^{-g(x)}\big]\Big)\mathrm{d}\Phi(x)\Big)\Big].$$

b) Yes, such an interpretation is possible. Indeed, choose $p(x) = \mathbb{1}_B(x)$ and $Y_j = \mathbb{1}\{\xi_j \in B\}$, for each $j \in \mathbb{N}$. Then,

$$\Phi_p = \sum_{j=1}^{\tau} \delta_{\xi_j}\,\mathbb{1}\{Y_j = 1\} = \sum_{j=1}^{\tau} \delta_{\xi_j}\,\mathbb{1}_B(\xi_j) = \Phi_B.$$

c) Let $Z_1, Z_2, \dots$ be independent random variables which are independent of Φ and which are uniformly distributed over $\{1, \dots, n\}$. Then,

$$\Phi = \sum_{j=1}^{\tau} \delta_{\xi_j} = \sum_{k=1}^{n} \underbrace{\sum_{j=1}^{\tau} \mathbb{1}\big\{Z_j = k\big\}\cdot\delta_{\xi_j}}_{=:\,\Phi_k} = \sum_{k=1}^{n} \Phi_k.$$

As the sequence $(Z_j)_{j\in\mathbb{N}}$ is independent of Φ and we have

$$\big\{\Phi_k(\mathbb{X}) \geqslant 1\big\} \subset \big\{\Phi(\mathbb{X}) \geqslant 1\big\} = \{\tau \geqslant 1\},$$

it follows that

$$\mathbb{P}\Big(\Phi_k(\mathbb{X}) \geqslant 1\Big) = \mathbb{P}\Big(\Phi_k(\mathbb{X}) \geqslant 1,\, \tau \geqslant 1\Big) \geqslant \mathbb{P}\Big(Z_1 = k,\, \tau \geqslant 1\Big) = \tfrac{1}{n}\cdot\mathbb{P}\big(\tau \geqslant 1\big) > 0.$$

d) Consider $\Phi := \delta_x$ for some fixed $x \in \mathbb{X}$. Suppose there exist $n \in \mathbb{N}$, $n \geqslant 2$, and i.i.d. point processes $\Phi_1, \dots, \Phi_n$ such that $\delta_x = \Phi_1 + \ldots + \Phi_n$ almost surely. Then, for $k \in \{1, \dots, n\}$, we have $\Phi_k\big(\mathbb{X}\setminus\{x\}\big) = 0$ as well as $\Phi_k\big(\{x\}\big) \in \{0, 1\}$ almost surely. Therefore, $\Phi_k\big(\{x\}\big) \sim Bern(p_k)$ for some $p_k \in [0, 1]$. As $\Phi_1, \dots, \Phi_n$ have the same distribution, $p_1 = \ldots = p_n =: p$. Moreover, we have

$$0 = \mathbb{P}\Big(\delta_x\big(\{x\}\big) = 2\Big) = \mathbb{P}\Big((\Phi_1 + \ldots + \Phi_n)\big(\{x\}\big) = 2\Big) = \binom{n}{2} p^2\,(1-p)^{n-2},$$

hence $p \in \{0, 1\}$. However, neither for $p = 1$ nor for $p = 0$ do we have $\Phi_1 + \ldots + \Phi_n = \delta_x$.

e) Denote by $\mathcal{U}$ the uniform distribution over $\{1, \dots, n\}$, and let Ψ be an independent $\mathcal{U}$-marking of Φ. By Corollary 2.37 and Theorem 2.38, Ψ is a Poisson process with intensity measure $\Theta\otimes\mathcal{U}$. Thus,

$$\Phi_1 := \Psi\big(\,\cdot\times\{1\}\big),\ \dots,\ \Phi_n := \Psi\big(\,\cdot\times\{n\}\big)$$

are also Poisson processes, each of them having intensity measure $\frac{1}{n}\cdot\Theta$, as

$$(\Theta\otimes\mathcal{U})\big(B\times\{k\}\big) = \tfrac{1}{n}\cdot\Theta(B), \qquad B \in \mathcal{X},\ k \in \{1, \dots, n\}.$$

To prove that $\Phi_1, \dots, \Phi_n$ are independent, we write

$$\Phi_k = \Psi\big(\,\cdot\times\{k\}\big) = \Psi_{\mathbb{X}\times\{k\}}\big(\,\cdot\times\{k\}\big).$$

As the sets $\mathbb{X}\times\{k\}$, $k = 1, \dots, n$, are disjoint, the processes $\Psi_{\mathbb{X}\times\{1\}}, \dots, \Psi_{\mathbb{X}\times\{n\}}$ (and hence $\Phi_1, \dots, \Phi_n$) are independent by Problem 2 c) of Sheet 5. By Problem 4 b) on Sheet 5, $\Phi_1 + \ldots + \Phi_n$ is a Poisson process with intensity measure $\sum_{k=1}^{n} \frac{1}{n}\cdot\Theta = \Theta$, and the claim follows.