Worksheet 13

Wednesday, April 17, 2024

1. **Linear Program.** Consider the following linear program:

$$\begin{aligned} \text{maximize} \quad & x_1 + x_2 - x_3 \\ \text{subject to} \quad & x_1 - x_2 = 3 \\ & x_2 + x_3 \leq 1 \\ & x_2, x_3 \geq 0 \end{aligned}$$

Does this linear program have an optimum? If so, what are the optimal values of x_1, x_2, x_3? (No need to show your work, but explain briefly.)

Solution LP1 has an optimum, because (1) LP1 is feasible with $(x_1, x_2, x_3) = (3, 0, 0)$; and (2) the feasible region is bounded between $0 \leq x_2 \leq 1$ and $0 \leq x_3 \leq 1$ and $3 \leq x_1 \leq 4$ (recall $x_1 = x_2 + 3$). The optimum values are $(x_1, x_2, x_3) = (4, 1, 0)$, giving an objective value of 5.

2. **Linear Programs and Simplex.** Given the following linear program:

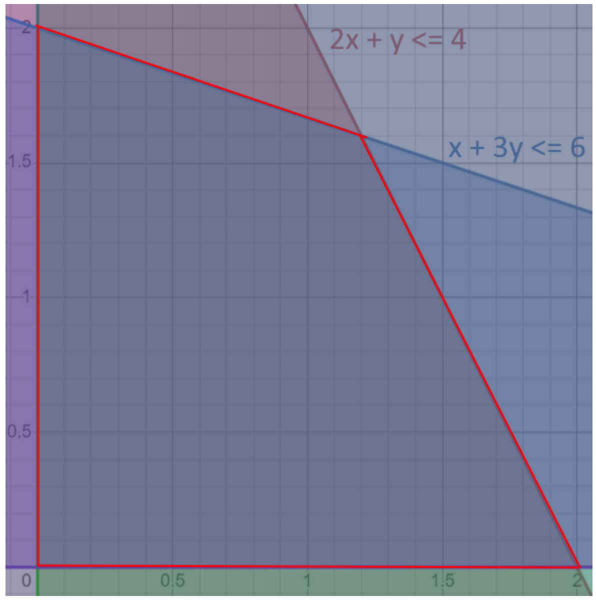
$$\begin{aligned} \text{maximize} \quad & x + y \\ \text{subject to} \quad & 2x + y \leq 4 \\ & x + 3y \leq 6 \\ & x, y \geq 0 \end{aligned}$$

(a) Plot the feasible region.

(b) Show all the vertices in the order of being visited by the simplex method with the values at those points, assuming we start at $(0, 0)$ and move along x-axis first.

Solution

(a) The feasible solution is the red-outlined region in the below graph:



(b) Let $f(x,y) = x + y$.
   (a) $(0,0)$, $f(0,0) = 0$
   (b) $(2,0)$, $f(2,0) = 2$
   (c) $(1.2,1.6)$, $f(1.2,1.6) = 2.8$

3. Max Flow. Following are two variations of the maximum flow problem. Show that they can be reformulated to linear programs.

(a) Each edge e has not only a capacity, but also a lower bound $b_e \geq 0$ on the flow it must carry.

(b) The outgoing flow from each node u is not the same as the incoming flow, but is smaller by a factor of $(1 - \varepsilon_u)$, where ε_u is a loss coefficient associated with node u.

Solution

(a)

$$\begin{aligned}
\text{maximize} \quad & \sum_{(s,v)\in E} f_{s,v} \\
\text{subject to} \quad & b_e \leq f_e \leq c_e \quad \forall e \in E \\
& \sum_{(u,v)\in E} f_{u,v} = \sum_{(v,w)\in E} f_{v,w} \quad \forall v \in V - \{s,t\}
\end{aligned}$$

(b)

$$\begin{aligned}
\text{maximize} \quad & \sum_{(s,v)\in E} f_{s,v} \\
\text{subject to} \quad & 0 \leq f_e \leq c_e \quad \forall e \in E \\
& (1-\varepsilon_v) \sum_{(u,v)\in E} f_{u,v} = \sum_{(v,w)\in E} f_{v,w} \quad \forall v \in V - \{s,t\}
\end{aligned}$$

4. **Cargo Plane.** A cargo plane can carry a maximum weight of 100 tons and a maximum volume of 60 cubic meters. There are three materials to be transported, and the cargo company may choose to carry any amount of each, up to the maximum available limits given below.

   - Material 1 has a density of 2 tons/cubic meter, a maximum available amount of 40 cubic meters, and a revenue of $1,000 per cubic meter.
   - Material 2 has a density of 1 ton/cubic meter, a maximum available amount of 30 cubic meters, and a revenue of $1,200 per cubic meter.
   - Material 3 has a density of 3 tons/cubic meter, a maximum available amount of 20 cubic meters, and a revenue of $12,000 per cubic meter.

   Write a linear program to maximize revenue while obeying these constraints.

   Solution Let q_i denote the quantity (in qubic meters) of material i. The linear program is:

$$
\begin{aligned}
\text{maximize} \quad & 1000q_1 + 1200q_2 + 12000q_3 \\
\text{subject to} \quad & 2q_1 + q_2 + 3q_3 \leq 100 \\
& q_1 + q_2 + q_3 \leq 60 \\
& q_1 \leq 40 \\
& q_2 \leq 30 \\
& q_3 \leq 20 \\
& q_1, q_2, q_3 \geq 0
\end{aligned}
$$

5. **Change-Making Linear Program.** This is a continuation of worksheet 12's problem 4a. You are given an unlimited supply of coins of denominations $v_1, \ldots, v_n \in \mathbb{N}$ and a value $W \in \mathbb{N}$. Your goal is to make change for W using the minimum number of coins, that is, find the smallest set of coins whose total value is W.

   (a) Formulate a linear programming model for this problem.

   (b) Even when a problem involves integer variables such as the number of coins, the intuitive linear program may give a non-integral solution. This is a problem since it doesn't make much sense to divide a coin into fractions in the real world. Give an example input that causes your linear program to give a non-integral output.

   (c) Explain how the problem of non-integral solutions can be solved if the constraints are allowed to take on a different form.

   Solution

   (a) Let x_i be the number of coins of denomination v_i used to reach W.

$$
\begin{aligned}
\text{minimize} \quad & \sum_i x_i \\
\text{subject to} \quad & \sum_i v_i x_i = W \\
& x_i \geq 0 \quad \forall i \in \{1, 2, \ldots n\}
\end{aligned}
$$

(b) Let $W = 10$, $v_1 = 2$, and $v_2 = 30$. This linear program would output $\frac{1}{3}$. It cannot distinguish between the valid optimal solution $x_1 = 5, x_2 = 0$ and the invalid (but even smaller-valued) solution $x_1 = 0, x_2 = \frac{1}{3}$.

(c) If we could constrain the x_i values to be non-negative integers instead of non-negative real numbers, the optimal solution would always be an integer, since the objective function is a sum of the (now integral) x_i values. One way to rewrite the final constraint is $x_i \in \mathbb{N}$.

This is called integer linear programming, which unfortunately cannot be solved as computationally easily as linear programming (ILP). For certain problems such as bipartite matching, the LP always gives an integral (and thus truly optimal) solution. For other problems, such as vertex cover, the LP can be rounded to integer values to give a reasonable approximation to the ILP. For this problem, unfortunately, that does not appear to be the case.