CMPSC 465 Spring 2024 | Data Structures & Algorithms Mehrdad Mahdavi and David Koslicki | HW 1

1. (5 pts.) Problem 1

I understand the course policies.

2. (36 pts.) Problem 2

(a) $f = O(g)$: $\lim_{n\to\infty} \frac{6n\cdot 2^n + n^{100}}{3^n} = \lim_{n\to\infty} \frac{6n}{1.5^n} + \lim_{n\to\infty} \frac{n^{100}}{3^n} = 0$, now since $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, we have $f = O(g)$.

(b) $f = \Theta(g)$: $\lim_{n\to\infty} \frac{\log 2n}{\log 3n} = \lim_{n\to\infty} \frac{\log 2 + \log n}{\log 3 + \log n} = 1$.

(c) $f = \Omega(g)$: $\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt[3]{n}} = \infty$

(d) $f = \Omega(g)$: $\lim_{n\to\infty} \frac{\frac{n^2}{\log n}}{n(\log n)^4} = \lim_{n\to\infty} \frac{n}{(\log n)^5} = \infty$

(e) $f = \Theta(g)$: n^2 dominates both $n\log n$, and $(\log n)^5$ terms, which implies that $f = \Theta(n^2), g = \Theta(n^2)$,

(f) $f = O(g)$: Let $k = \log_2 n$. Then $f(n) = k^k$ and $g(n) = 2^{k^2} = (2^k)^k$. Note that $2^k \geq k$ for all $k \geq 1$. Therefore, $(2^k)^k \geq k^k$ for all $k \geq 1$ and $f = O(g)$. Alternatively, taking the log of both functions (a valid manipulation since logarithms are increasing functions) gives $\log_2 f(n) = (\log_2 n)(\log_2 \log_2 n)$ and $\log_2 g(n) = (\log_2 n)^2$. Because $\log_2 \log_2 n \leq \log_2 n$, $(\log_2 n)(\log_2 \log_2 n) \leq (\log_2 n)^2$.

(g) $f = \Theta(g)$: We can write $f = n\log(n^{20}) = 20n\log n = \Theta(n\log n)$, and $g = \log(3n!) = \Theta(\log(n!))$. Thus, it remains to show that $\log(n!) = \Theta(n\log n)$, but we know this is true, as we saw the proof in Worksheet 1, problem 2(i).

(h) $f = \Theta(g)$: we can write $8\log n = \log(n^8) < \log(n^9 + \log n) < \log(n^{10}) = 10\log n \Rightarrow f = \Theta(\log n)$ and $g = \log 2n = \log 2 + \log n = \Theta(\log n)$. Another possible set of bounds for f are $\log(n^9) < \log(n^9 + \log n) < \log(2n^9)$.

(i) $f = \Omega(g)$: Let $k = \lfloor \sqrt{n} \rfloor$, then we have $f(n) \geq 8^{k^2} \cdot k^4$, $g(n) = k!$. Now, note that $8^k \geq k$ for all $k \geq 1$, and so $8^{k^2} = (8^k)^k \geq k^k$ for all $k \geq 1$. From Worksheet 1, we know that $k^k \geq k!$ and so $8^{k^2} \geq k!$ which also implies that $8^{k^2} \cdot k^4 \geq k!$ for all $k \geq 1$. Thus, it follows from the definition of O that $k! = O(8^{k^2} \cdot k^4)$ (taking $c = 1$ and $n_0 = 1$); it follows from the definition of Omega that $f = \Omega(g)$.

3. (15 pts.) Problem 3

By the formula for the sum of a partial geometric series, for $c \neq 1$: $S(k) := \sum_{i=0}^{k} c^i = \frac{1-c^{k+1}}{1-c}$. Thus,

- If $c > 1$, we have $c^{k+1} > 1$, and then $S(k) = \frac{c^{k+1}-1}{c-1}$. Now we can write

$$\lim_{k\to\infty} \frac{S(k)}{c^k} = \lim_{k\to\infty} \frac{c - \frac{1}{c^k}}{c-1} = \frac{c}{c-1} - \frac{1}{c-1} \lim_{k\to\infty} \frac{1}{c^k}.$$

  Since $c > 1$, we can conclude $\lim_{k\to\infty} \frac{1}{c^k} = 0$. Therefore we have $\lim_{k\to\infty} \frac{S(k)}{c^k} = \frac{c}{c-1}$. Note that $0 < \frac{c}{c-1} < \infty$, therefore $S(k) = \Theta(c^k)$.

- If $c = 1$, then every term in the sum is 1. Thus, $S(k) = k + 1 = \Theta(k)$.

- If $c < 1$, then $\frac{1}{1-c} > \frac{1-c^{k+1}}{1-c} = S(k) > 1$. Thus, $S(k) = \Theta(1)$.

4. (20 pts.) Problem 4

(a) $f(n)$ is a degree d polynomial. We have $\lim_{n\to\infty} \frac{f(n)}{n^k} = 0$, for $k > d$. So, we can conclude that for $k \geq d$, $f(n) = O(n^k)$. On the other hand, for $k < d$, $\lim_{n\to\infty} \frac{f(n)}{n^k} = \infty$, which means that $f(n) = \Omega(n^k)$ in this case. Also for $k = d$, we have $\lim_{n\to\infty} \frac{f(n)}{n^k} = a_d$, so $f(n) = \Theta(n^k)$

(b) Part (b) is a special case of part (c); same proof works. Alternatively, this problem can also be solved by remembering that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

(c) Since $k \leq n$ every term in the sum is at most n, so

$$\sum_{k=1}^{n} k^j = 1^j + \cdots + n^j \leq n^j + \cdots + n^j = \sum_{k=1}^{n} n^j = n^{j+1}.$$

We do something similar for the lower bound. The only additional idea is that we only look at the second half of the sum. The smallest element in the second half of the sum corresponds to $k = n/2$ (assuming without loss of generality that n is even). Then,

$$\sum_{k=1}^{n} k^j \geq \sum_{k=n/2}^{n} k^j \geq \sum_{k=n/2}^{n} (\tfrac{n}{2})^j \geq (\tfrac{n}{2})^{j+1} = \frac{1}{2^{j+1}} \cdot n^{j+1}.$$

(d) First we show the upper bound. $\sum_{i=1}^{n} \sum_{j\neq i, j=1}^{n} ij \leq (\sum_{i=1}^{n} i)^2 = (\frac{n(n+1)}{2})^2 = O(n^4)$. For the lower bound, we can write:

$$\sum_{i=1}^{n} \sum_{j\neq i, j=1}^{n} ij \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} i \sum_{j=\lceil \frac{n}{2} \rceil+1}^{n} j \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \frac{n}{2} \sum_{j=\lceil \frac{n}{2} \rceil+1}^{n} \left(\frac{n}{2}+1\right) \quad \geq \tfrac{n}{2}(\tfrac{n}{2}+1) \cdot (\frac{n}{2})^2 \geq \frac{n(n+2)}{4} \cdot \frac{n^2}{4} = \Omega(n^4).$$

Alternatively, the summation can be expanded into $(\sum_{i=1}^{n} i)^2 - \sum_{i=1}^{n} i^2$, then simplified using the standard summations from 1 to n. This will give a polynomial in n with degree 4, which has been shown to be $\Theta(n^4)$. The intuition for this approach is that the only thing stopping the original summation from being easily decoupled is the condition $j \neq i$, which excludes the same products as the new i^2 summation subtracts.

5. (24 pts.) Problem 5

1. Based on the definition, $f(n) = O(g(n))$ means that there exist positive constants c and n_0 such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$. Thus, $0 \leq \frac{1}{c} f(n) \leq g(n)$ for all $n \geq n_0$. As $\frac{1}{c}$ is positive, given the definition of the Ω-notation, we can deduce that $f(n) = O(g(n))$ means the same as $g(n) = \Omega(f(n))$.
2. According to the definition of Θ-notation, we need to find two constants c_1 and c_2 such that $0 \leq c_1(f(n)+g(n)) \leq \max(f(n),g(n)) \leq c_2(f(n)+g(n))$ for all $n \geq n_0$ where n_0 is a positive constant. Given that $f(n)$ and $g(n)$ are asymptotically non-negative, we can make the following conclusions:
   - $\max(f(n),g(n)) \leq f(n)+g(n)$
   - $\max(f(n),g(n))$ comprises at least half of $f(n)+g(n)$, meaning $\max(f(n),g(n)) \geq 0.5(f(n)+g(n))$
   - $0.5(f(n)+g(n)) \geq 0$

   As such, c_1 could be 0.5 and c_2 could be 1. Therefore, $\max(f(n),g(n)) = \Theta(f(n)+g(n))$.

3. We know that one can directly translate between logarithms of different bases using the following fundamental identity:

$$\log_a n = \frac{\log_b n}{\log_b a}$$

So we can get $\log_a n = \frac{1}{\log_b a} \log_b n$, where $\log_b a$ is a constant. Therefore, $\log_a n = \Theta(\log_b n)$

Rubric:

Problem 1

Assign full credit if "I understand the course policies" is written. Subtract one point if they forgot to mention their collaborators or if they do not write "none" for the collaborators.

Problem 2

Each part is 4 pts: 2 pt for identifying right relation between f and g and 2 pts for a reasonable explanation.

Problem 3

Case $c > 1$: 6 pts
Case $c = 1$: 3 pts
Case $c < 1$: 6 pts

Problem 4

Each part is worth 5 pts.
part a: showing only one case is worth 2.5 pts.
part b-c-d: showing only upper bound is worth 2 pts, and showing only lower bound is worth 3 pts.

Problem 5

1. This part is worth 8 points.
   5 points: the inference from $0 \leq f(n) \leq cg(n)$ to $0 \leq \frac{1}{c} f(n) \leq g(n)$
   3 points: emphasize $\frac{1}{c}$ is positive
2. This part is worth 8 points.
   3 points: correctly prove the upper bound
   4 points: correctly prove the lower bound
   1 point: emphasize the lower bound i.e. $c_1(f(n) + g(n))$ is non-negative
3. This part worth 8 points, students should get 4 point if they show $\log_a n = \frac{\log_b n}{\log_b a}$, and get the rest 4 point if they show $\log_a n = \frac{1}{\log_b a} \log_b n \leq c \log_b n$