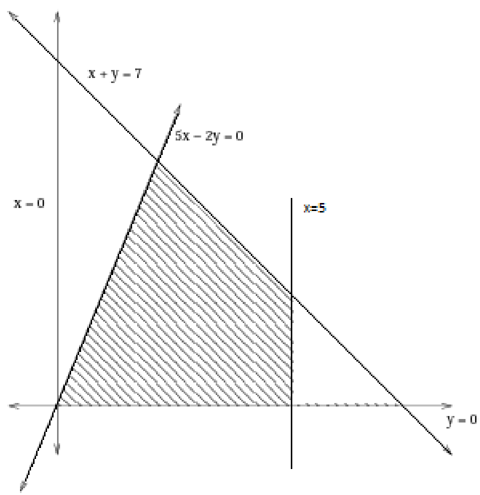
1. (25 pts.) **Solving a Linear Program.**

(a) The feasible solution is the shaded region in the below graph:



(b) The optimal solution is achieved in the upper right corner of the feasible region at the point $(5,2)$, and has a value of $5x+3y=31$.
Choosing x, it is bounded when $x=5$. Moving from the origin:
$z_2 = y$
$z_1 = 5-x$
$x = 5-z_1$
substituting:
max $25-5z_1+3z_2$
$z_1 >= 0$
$z_1 <= 5$
$z_2 - z_1 <= 2$
$z_2 >= 0$
$-25+2z_2+5z_1 <= 0$
Choosing z_2, it is bounded when $z_2 = 2$
$w_1 = z_1$
$z_2 = 2-w_2+w_1$
substituting:
max $31-2w_1-3w_2$
Since the coefficients are all negative, simplex is done and the optimum value is 31. Setting w_1 and w_2 to 0 gives $z_1 = 0$ and $z_2 = 2$, giving $y = 2$ and $x = 5$.

(c) The dual linear program is:

$$\begin{aligned} \text{minimize} \quad & 7b + 5c \\ \text{subject to} \quad & 5a + b + c \geq 5 \\ & -2a + b \geq 3 \\ & a \leq 0 \\ & b \geq 0 \\ & c \geq 0 \end{aligned}$$

This is found by multiplying the first three constraints (the constraints that are for more than just non-negativity) by a, b, and c, respectively. Note that $a \leq 0$ comes from the fact that the first constraint uses $\geq$ instead of $\leq$ in the primal. Note that the counts of variables and interesting constraints flipped from 2 variables with 3 constraints in the primal to 3 variables with 2 constraints in the dual.

The dual linear program is minimized at $a = 0$, $b = 3$, $c = 2$, with a value of 31. Therefore, the optimal values of the primal and dual objective functions are equal, and strong duality is confirmed to hold for this problem (as it does for all linear programs).

2. (25 pts.) **Spaceship.** Let x_1 be the number of oxidizer units provided for compartment 1, and define similarly x_2 for compartment 2 and x_3 for compartment 3. The probability that all units fail in compartment 1 is $(0.35)^{x_1}$, for compartment 2 is $(0.45)^{x_2}$ and for compartment 3 is $(0.3)^{x_3}$. The probability that all units fail in all compartments is $(0.35)^{x_1}(0.45)^{x_2}(0.3)^{x_3}$. The exponential function is non-linear, so we take logarithm (which preserves ordering of numbers) to get the linear program

$$\text{Minimize } \log(0.35)x_1 + \log(0.45)x_2 + \log(0.3)x_3$$

$$\text{subject to} \begin{cases} 40x_1 + 50x_2 + 30x_3 \leq 400 & \text{space constraint (cu in.)} \\ 20x_1 + 15x_2 + 10x_3 \leq 100 & \text{weight constraint (lb)} \\ 30x_1 + 35x_2 + 25x_3 \leq 300 & \text{cost constraint (\$1000)} \\ \log(0.35)x_1, \log(0.45)x_2, \log(0.3)x_3 \leq \log(0.05) & \text{reliability constraints (log scale)} \end{cases}.$$

(Note that the reliability constraints imply the non-negativity constraints that $x_1, x_2, x_3 \geq 0$.)

3. (25 pts.) **Vertex Cover.**

(a) min $x_1 + x_2 + ... + x_n$, where $|V| = n$
$x_1, ..., x_n <= 1$
$x_1, ..., x_n >= 0$
$x_i + x_j >= 1$ for every edge {i,j} in E
A node can either be selected or not selected, and for each edge, at least one of the nodes must be selected. The goal is to minimize the total number selected, with each node being worth the same.

(b) A graph has 3 nodes, and all of the nodes are connected to each other, creating a triangle shape. The optimum solution is at vertex $x_1, x_2, x_3 = 0.5$ with optimum value 1.5.

4. (25 pts.) **NP and EXP.** The solution to a NP problem with a input of n bits can be verified in poly-time, so the solution, y, must have $y \in \{0,1\}^{p(n)}$, where $p(n)$ is a polynomial, or else just reading the solution could not be done in poly-time. So, any NP problem can be solved in exponential time by iterating over all possible solutions and checking whether the solution is correct using a poly-time verifier. So the running time will be $O(2^{p(n)}q(n, p(n)))$, where $q(n, p(n))$ is the running time of the poly-time verifier. Then, there exists some integer k where $p(n) = O(n^k)$ and $q(n, p(n)) = O(n^k)$ since they are polynomials, giving a running time of $O(2^{n^{k+1}})$, so NP $\subseteq$ EXP.

5. (0 pts.) **Acknowledgments.**

(a) I did not work in a group.
(b) I did not consult with anyone other than my group members.
(c) I did not consult any non-class materials.

Rubric:

Problem 1, 25 pts

(a) 5 pts for correct plot

(b) 10 pts: 5 points for showing simplex steps, 5 points for correct answer

(c) 10 pts: 6 for correct dual, 4 for the optimal variable values.

Problem 2, 25 pts
5 pts for the objective function, 5 pts per constraint.

Problem 3, 25 pts

(a) 15 pts for correct LP

(b) 10 pts for valid instance

Problem 4, 25 pts

(a) 8 pts solution must have poly length

(b) 10 pts correct algorithm

(c) 7 pts correct final running time