FIR Filter Design by Window Function

Basic Types of Filters

An ideal Lowpass filter allows to pass all frequency components of a signal below a designated cutoff frequency ,Fc, and rejects all frequency components of a signal above Fc.

An ideal Highpass filter allows to pass all frequency components of a signal above a designated cutoff frequency, Fc, and rejects all frequency components below Fc.

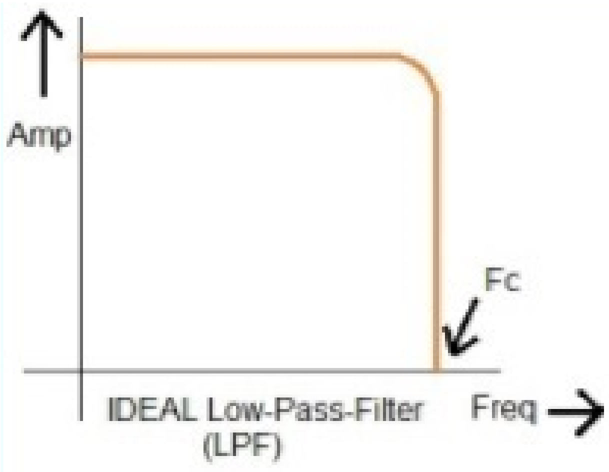


Fig.1 LPF

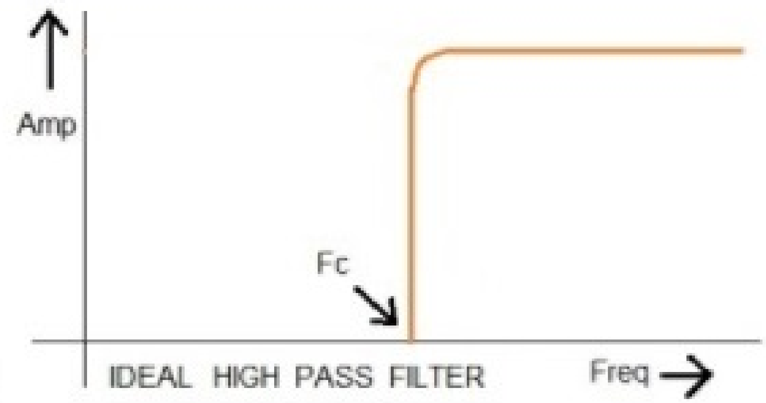


Fig.2 HPF

Basic Types of Filters

An ideal Bandpass filter allows to pass all frequency components of a signal within a certain range and rejects all frequency components outside of that range.

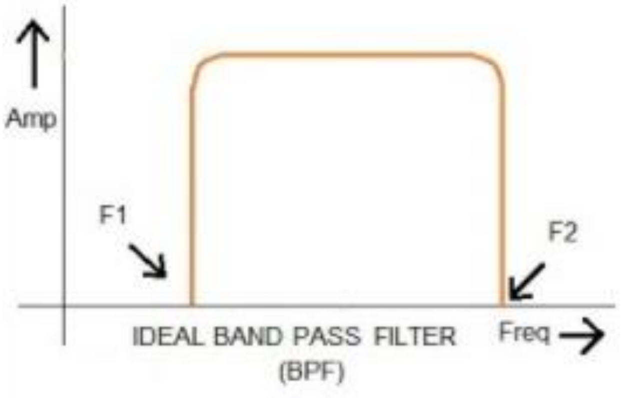


Fig.3 BPF

An ideal Bandstop filter rejects all frequency components of a signal within a certain range and allows to pass all frequency components outside of that range.

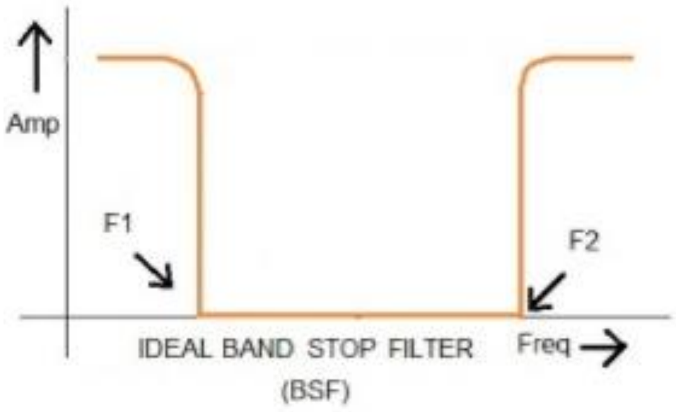


Fig.4 BSF

FIR Filter

- FIR (Finite Impulse Response) filter is a filter whose impulse response is of finite duration.
- For causal system,

$$h[n]=0 \quad \text{for } n<0$$

Summary of ideal impulse responses for standard FIR filters.

Filter Type	Ideal Impulse Response $h(n)$ (noncausal FIR coefficients)
Lowpass:	$h(n)=\begin{cases}\frac{\Omega_c}{\pi} & n=0\\ \frac{\sin(\Omega_c n)}{n\pi} \text{ for } n\neq 0 & -M\leq n\leq M\end{cases}$
Highpass:	$h(n)=\begin{cases}\frac{\pi-\Omega_c}{\pi} & n=0\\ -\frac{\sin(\Omega_c n)}{n\pi} \text{ for } n\neq 0 & -M\leq n\leq M\end{cases}$
Bandpass:	$h(n)=\begin{cases}\frac{\Omega_H-\Omega_L}{\pi} & n=0\\ \frac{\sin(\Omega_H n)}{n\pi}-\frac{\sin(\Omega_L n)}{n\pi} \text{ for } n\neq 0 & -M\leq n\leq M\end{cases}$
Bandstop:	$h(n)=\begin{cases}\frac{\pi-\Omega_H+\Omega_L}{\pi} & n=0\\ -\frac{\sin(\Omega_H n)}{n\pi}+\frac{\sin(\Omega_L n)}{n\pi} \text{ for } n\neq 0 & -M\leq n\leq M\end{cases}$

Causal FIR filter coefficients: shifting $h(n)$ to the right by M samples.
Transfer function:

$$H(z)=b_0+b_1z^{-1}+b_2z^{-2}+\cdots b_{2M}z^{-2M}$$

where $b_n=h(n-M)$, $n=0,1,\cdots,2M$

Consider the Ideal Impulse Response for Standard Lowpass FIR Filter

- Lowpass:

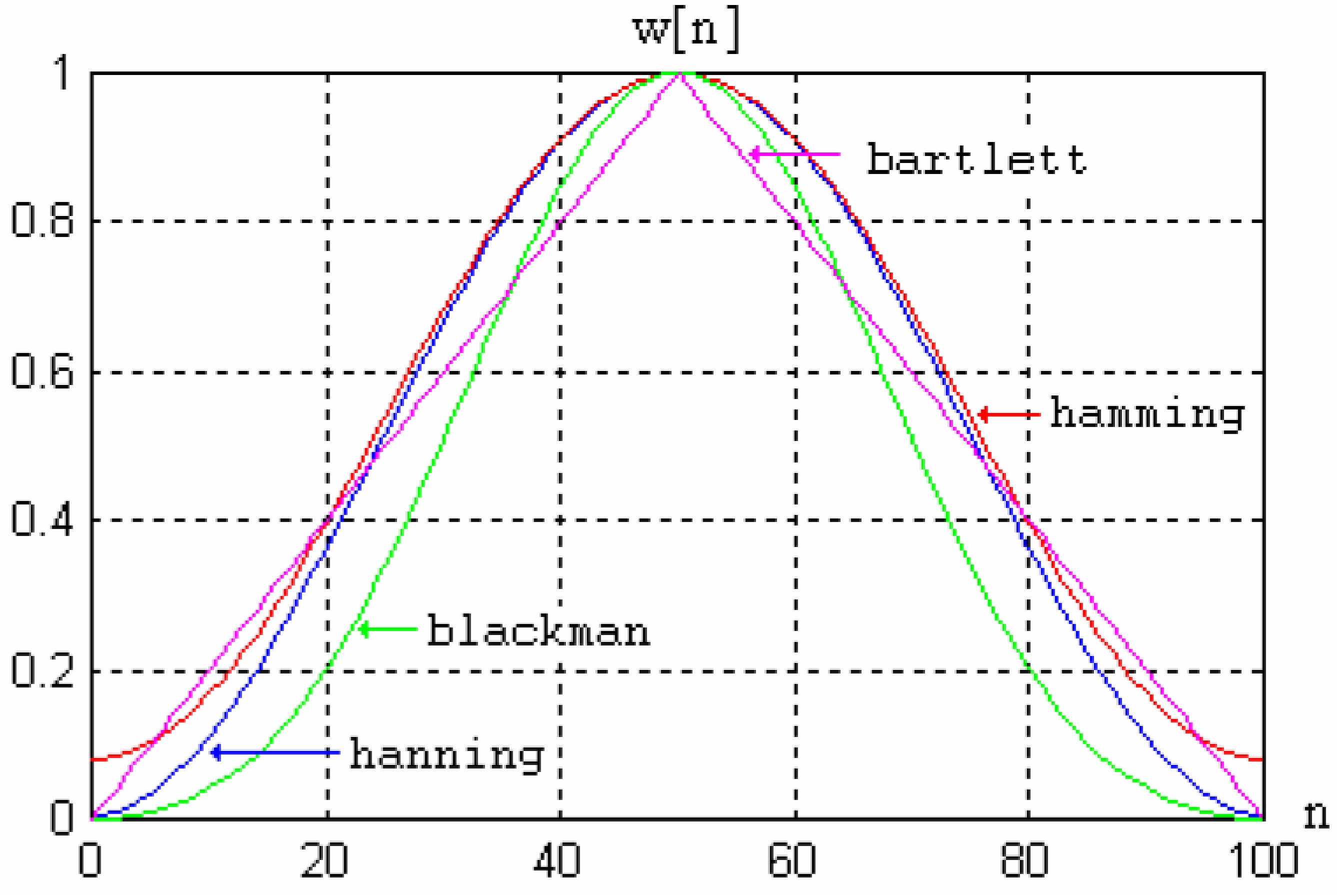
$$h[n] = \begin{cases} \dfrac{\Omega_c}{\pi} & for\ n = 0 \\ \dfrac{sin(\Omega_c n)}{n\pi} & for\ n \neq 0 \end{cases} \qquad -M \leq n \leq M$$

$$\Omega_C = \frac{2\pi F_C}{F_S}$$ = cutoff frequency in rad/sample

$$Tap = 2M + 1$$ = filter coefficient/delay pair (must be odd)

$$FilterOrder = Tap - 1$$

Common FIR Filter Window Functions,w[n]



Calculation of FIR Filter Window Function, w[n]

Window Type	Window Function $w(n)$, $-M \le n \le M$
Rectangular	1
Hanning	$0.5 + 0.5\cos\left(\frac{\pi n}{M}\right)$
Hamming	$0.54 + 0.46\cos\left(\frac{\pi n}{M}\right)$
Blackman	$0.42 + 0.5\cos\left(\frac{n\pi}{M}\right) + 0.08\cos\left(\frac{2n\pi}{M}\right)$

$$Tap = 2M + 1$$

$$FilterOrder = Tap - 1$$

Steps in the design of FIR Filter

I. Convert the cutoff frequency, if given in Hz, to radian/sample

II. Calculate M

III. Calculate the impulse response, h[n]

IV. Calculate the window function, w[n]

V. Write the windowed impulse response, (product of III and IV) $h_W[n]=h[n]w[n]$

VI. Delay the $h_W[n]$ by M samples. The result is called the filter coefficient b_n

Example #1

a) Design a 3-tap FIR lowpass filter with cutoff frequency of 800Hz and a sampling rate of 8000Hz using the Hamming window function.

b) Determine the transfer function and difference equation of the designed FIR system

Solution to Example #1

a) Design a 3-tap FIR lowpass filter with cutoff frequency of 800Hz and a sampling rate of 8000Hz using the Hamming window function.

b) Determine the transfer function and difference equation of the designed FIR system

I. Convert the cutoff frequency, if given in Hz, to radian/sample

II. Calculate M

III. Calculate the impulse response, h[n]

IV. Calculate the window function, w[n]

V. Write the windowed impulse response, $h_W[n]=h[n]\, w[n]$

VI. Delay the $h_W[n]$ by M samples. The result is called the filter coefficient b_K

SOLUTION:

$-1 \le n \le 1$

$$\Omega_C = \frac{2\pi F_C}{F_S} = \frac{2\pi(800Hz)}{8000 samples/\sec} = 0.2\pi \, \frac{rad}{sample}$$

$$Tap = 2M + 1$$
$$M = (Tap-1)/2 = (3-1)/2 = 1$$
$$FilterOrder = Tap - 1 = 3 - 1 = 2$$

$h[0] = \frac{0.2\pi}{\pi} = 0.2$

$h[1] = \frac{\sin[0.2\pi(1)]}{(1)(\pi)} = 0.187$

$h[-1] = \frac{\sin[0.2\pi(-1)]}{(-1)\pi} = 0.187$

$$h[n] = \begin{cases} \frac{\Omega_c}{\pi} & for\ n = 0 \\ \frac{\sin(\Omega_c n)}{n\pi} & for\ n \neq 0 \end{cases} \qquad -M \le n \le M$$

$$h[n] = \{0.187, \underset{\uparrow}{0.2}, 0.187\}$$

$$\text{Hamming} \quad w[n] = 0.54 + 0.46\cos\left(\frac{\pi n}{M}\right)$$

$$w[n] = \{0.08, \underset{\uparrow}{1}, 0.08\}$$

$$h_W[n] = w[n]\, h[n] = \{0.01496, \underset{\uparrow}{0.2}, 0.01496\}$$

$$\boxed{b_n = h_W[n-1] = \{0.01496, 0.2, 0.01496\}}$$

These are the filter coefficients

Continuation of Solution to Example#1

$b_n = \{0.01496, 0.2, 0.01496\}$

b) Determine the transfer function and difference equation of the designed FIR system

$$\boxed{H(z) = 0.01496 + 0.2z^{-1} + 0.01496z^{-2}}$$

Transfer Function

Since $H(z) = \frac{Y(z)}{X(z)}$

Then, $Y(z) = H(z)X(z) = 0.01496X(z) + 0.2z^{-1}X(z) + 0.01496z^{-2}X(z)$

Therefore, the difference equation of the designed FIR Low Pass Filter is:

$$\boxed{y[n] = 0.01496x[n] + 0.2x[n-1] + 0.01496x[n-2]}$$

Matlab Implementation of Example #1 using FIR1

Based on our computation, the filter coefficients for Problem #1 are:

b_n = {0.01496, 0.200, 0.01496}

Implementing this using fir1 command in matlab,

```
>> h=fir1(2,0.2,'low')
h =
   0.0651    0.8698    0.0651
```

note: fir1(order, cutoff, filter type);

order=tap-1 ; the pi on cutoff is already included in the fir1 function

Notice that the values are very much different. But if we will use a higher order... Say, 24 (tap is 25)

Matlab Implementation of Example #1 using FIR1.

- Using the formula, the filter coefficients for 25-tap are…

0.0020 0.0016 -0.0000 -0.0045 -0.0117 -0.0182 -0.0168 0.0000
0.0360 0.0873 0.1420 0.1842 0.2000 0.1842 0.1420 0.0873
0.0360 0.0000 -0.0168 -0.0182 -0.0117 -0.0045 -0.0000 0.0016
0.0020

- Using MATLAB fir1 function...

```
>> h=fir1(24,0.2,'low')

h =

 0.0020    0.0016   -0.0000   -0.0044   -0.0117   -0.0181   -0.0168    0.0000
 0.0359    0.0870    0.1415    0.1835    0.1992    0.1835    0.1415    0.0870
 0.0359    0.0000   -0.0168   -0.0181   -0.0117

-0.0044   -0.0000    0.0016    0.0020
```

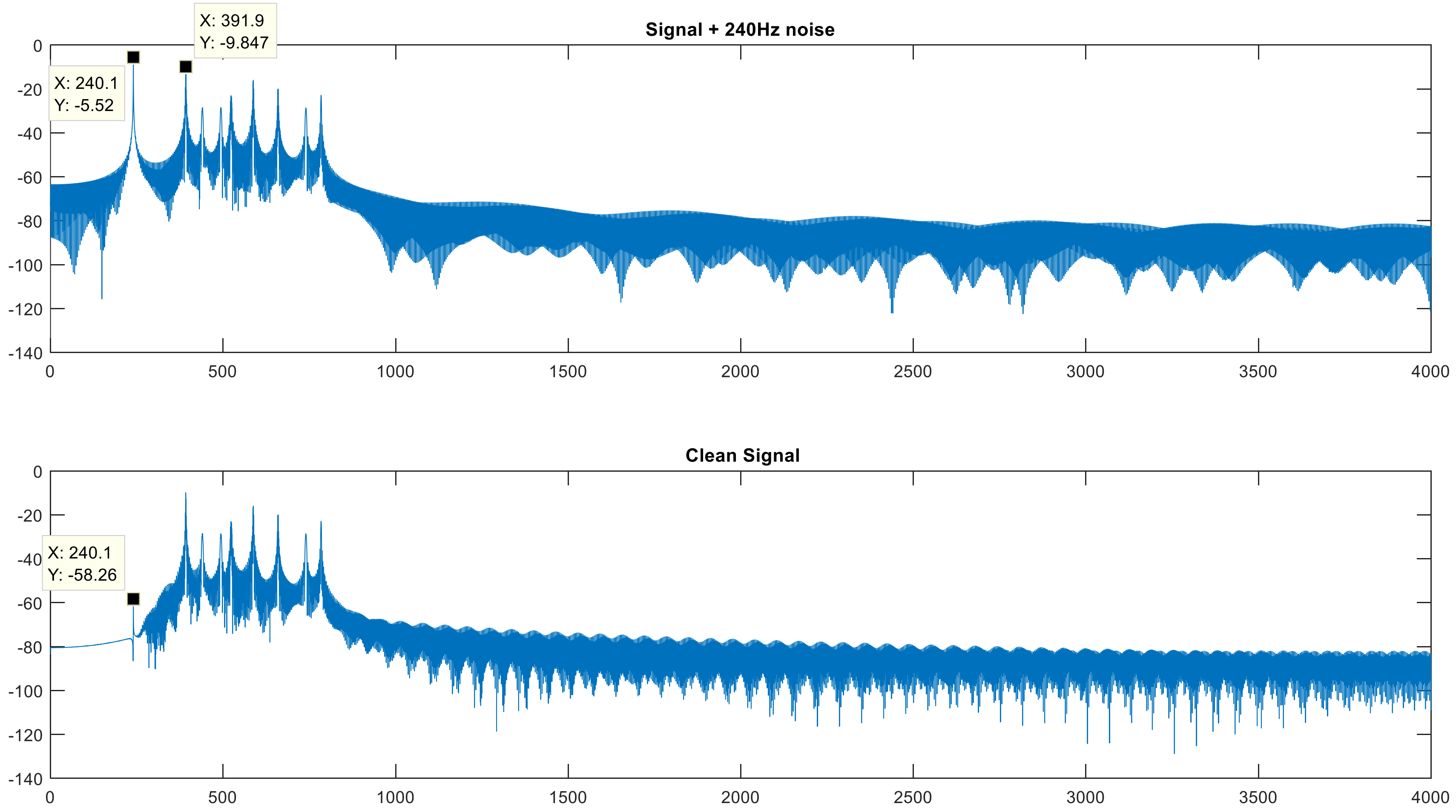
From this we can conclude that <fir1> can give us accurate results when using more taps.

Matlab Implementation of Example #1 using FIR1.

```
clc;
clear all;
close all;

[x1,Fs]=audioread('menuethum.wav');
%Create HPF, Order 250, Length 251, Wn=2*300/8000
%300Hz is the cutoff and 8000Hz is the Fs. pi is internal to <fir1>
h=fir1(250,0.075,'high');
y=filter(h,1,x1);     % num=h and den=1
%y=conv(h,x1);       %you can also use conv instead of filter
 sound(x1,Fs);
 pause(7)
 sound(y,Fs);
 pause(7)
 %PSD Plots
figure(1), subplot(2,1,1);
[Px,Fx] = periodogram(x1,[],length(x1),Fs);  %Original signal with 240Hz noise
plot(Fx,10*log10(Px));
title('Signal + 240Hz noise');
 subplot(2,1,2);
[Py,Fy] = periodogram(y,[],length(y),Fs);  %Filtered Signal
plot(Fy,10*log10(Py));
title(' Clean Signal')
 [H,W]=freqz(h);
dB=mag2db(abs(H));
figure(2),plot(W/pi,dB);
```

Matlab Implementation of Example #1 using FIR1.



Example #2

Given below is the calculated filter impulse response

$$h[n]=\{0.07503, 0.15915, 0.22508, \underset{\uparrow}{0.25}, 0.22508, 0.15915, 0.07503\}$$

a) Determine the windowed coefficient $h_W[n]$ (or windowed impulse response). Use Hamming window function.

b) Determine the FIR filter coefficients.

SOLUTION:

$$Tap = 7 = 2M+1$$

$$M = \frac{7-1}{2} = 3$$

a) $$w[n] = 0.54 + 0.46\cos\left(\frac{\pi n}{M}\right) \quad -3 \leq n \leq 3$$

substitute the values of n

$$w[n] = \{0.08, 0.31, 0.77, \underset{\uparrow}{1}, 0.77, 0.31, 0.08\}$$

$$hw[n] = h[n] \cdot w[n]$$

$$hw[n] = \{0.006, 0.04934, 0.17331, \underset{\uparrow}{0.25}, 0.17331, 0.04934, 0.006\}$$

b) $$b_k = hw[n-3] = \{0.006, 0.04934, 0.17331, 0.25, 0.17331, 0.04934, 0.006\}$$

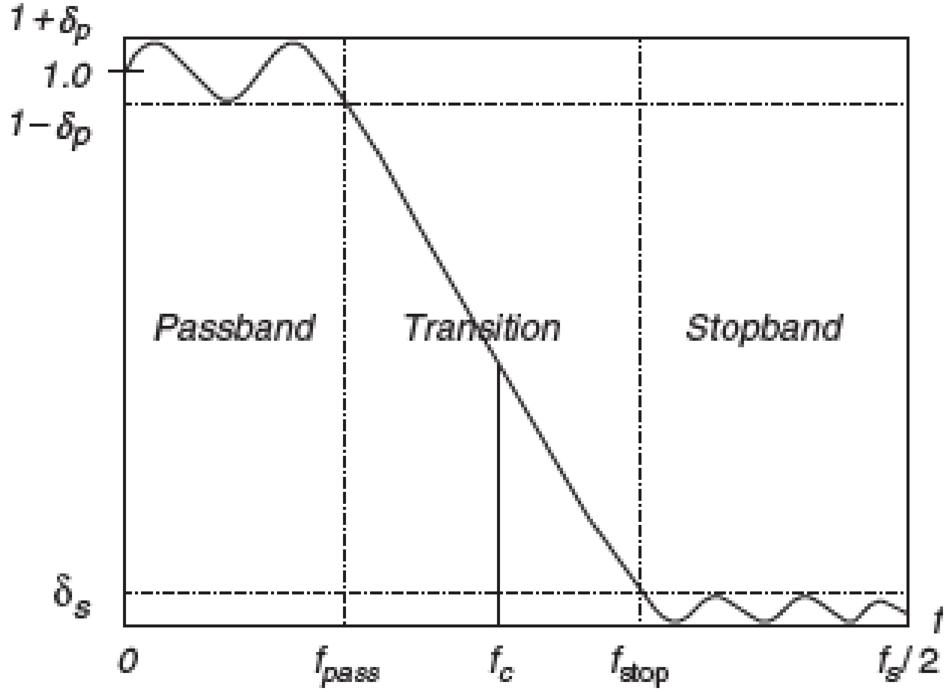
FIR filter length, N, estimation using Window Functions

$$\Delta f = \frac{|f_{stop} - f_{pass}|}{F_S}$$

Δf=normalized transition width

$$\delta_p \; dB = 20 \cdot \log_{10}(1 + \delta_p)$$

$$\delta_s \; dB = -20\log_{10}(\delta_s)$$



FIR filter length estimation using window functions (normalized transition width $\Delta f = |f_{stop} - f_{pass}|/f_s$).

Window Type	Window Function $w(n)$, $-M \le n \le M$	Window Length, N	Passband Ripple (dB)	Stopband Attenuation (dB)
Rectangular	1	$N = 0.9/\Delta f$	0.7416	21
Hanning	$0.5 + 0.5\cos\left(\frac{\pi n}{M}\right)$	$N = 3.1/\Delta f$	0.0546	44
Hamming	$0.54 + 0.46\cos\left(\frac{\pi n}{M}\right)$	$N = 3.3/\Delta f$	0.0194	53
Blackman	$0.42 + 0.5\cos\left(\frac{n\pi}{M}\right) + 0.08\cos\left(\frac{2n\pi}{M}\right)$	$N = 5.5/\Delta f$	0.0017	74

Example:

A lowpass filter using rectangular window has the following specifications:

Passband: 0-1850Hz

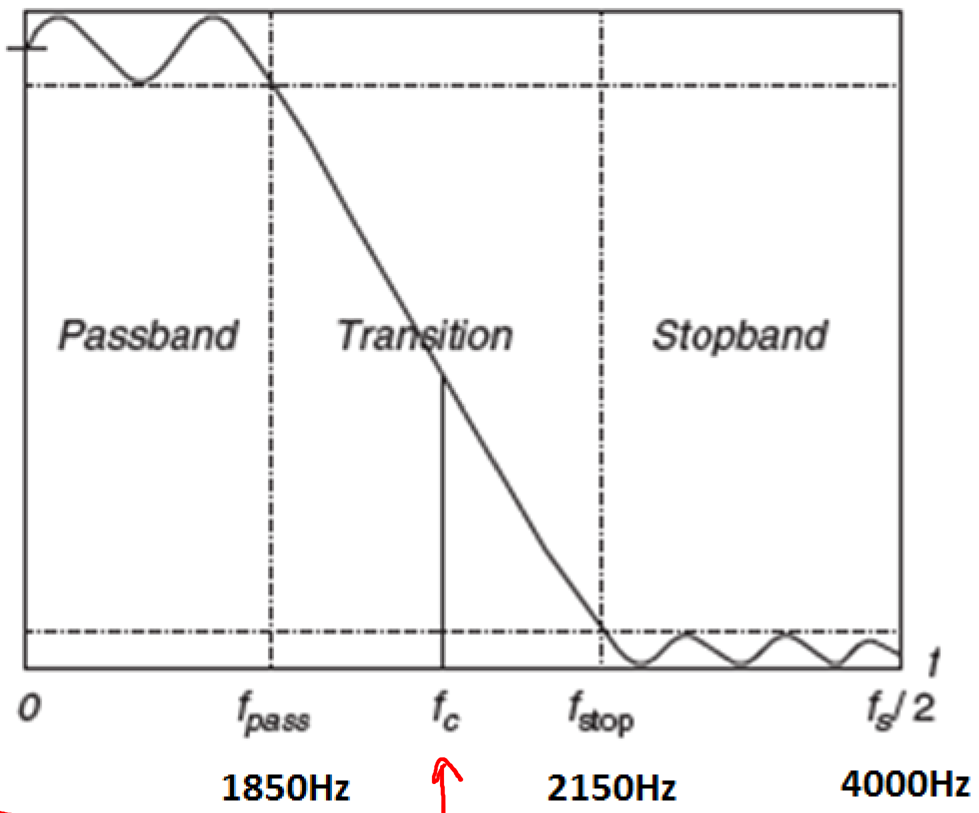
Stopband: 2150-4000Hz

Sampling Rate: 8000Hz

Required:

a)Estimate the filter length. (Taps in the filter length and is always odd)

b)Calculate the cutoff freq.



Solution:

a) $\Delta f = \dfrac{|f_{stop} - f_{pass}|}{f_s} = \dfrac{|2150Hz - 1850Hz|}{8000\ Hz}$

$= 0.0375$

From the table on the previous slide,

$N = 0.9/\Delta f = 0.9/0.0375 = 24$ use 25 Taps

b) $f_c = (1850Hz + 2150Hz)/2 = 2kHz$

Assignment:

For #s 1-2:Estimate the filter length and calculate the cutoff freq in Hz.

1) A highpass filter using Hamming Window has the following specifications:

Stopband: 0-1500Hz   Passband: 2500-4000Hz   Sampling Rate: 8000Hz

2) A Bandpass filter using Hamming window has the following specifications:

Lower Stopband: 0-500Hz   Upper Stopband: 3500-4000Hz

Passband: 1600-2300Hz   Sampling Rate: 8000Hz

3) Design a 5-tap FIR band reject (also called band stop) filter with lower cutoff frequency of 2kHz and upper cutoff frequency of 2400Hz, and a sampling rate of 8000Hz using the Hamming window method. Determine the transfer function and difference equation.

Answers to Assignment:

1) $\Delta f=0.125$; $N= 26.4$ use 27 Taps ; $f_C = 2kHz$
2) Use 25 Taps (solution is show below) ; $f_{CLower} = 1050Hz$

$$\Delta f_1 = \frac{|500-1600|}{8000} = 0.1375 \qquad N_1 = \frac{3.3}{0.1375} = 24$$

$$\Delta f_2 = \frac{|3500-2300|}{8000} = 0.15 \qquad N_2 = \frac{3.3}{0.15} = 22$$

$f_{CHigher} = 2900Hz$

3) $b_n = \{0.00748, 0.00841, 0.9, 0.00841, 0.00748\}$

$H(z)=0.00748+0.00841\, z^{-1} +0.9\, z^{-2} +0.00841\, z^{-3} +0.00748z^{-4}$

$y(n)=0.00748\, x[n]+0.00841\, x[n-1] +0.9\, x[n-2] +0.00841\, x[n-3] +0.00748x[n-4]$