

# TOWARDS SAFER AND MORE EXPRESSIVE GENETIC PROGRAMMING THROUGH DEPENDENT TYPES

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## STATEMENT OF ORIGINALITY

*This work has not previously been submitted for a degree or diploma in any university. To the best of my knowledge and belief, the dissertation contains no material previously published or written by another person except where due reference is made in the dissertation itself.*

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## ABSTRACT

There is tension between expressive program representations, and verification with respect to functional correctness, in pure functional, strongly typed genetic programs. Well-typed embedded representations are difficult to manipulate, manually typed expressive representations require proofs of progress and preservation for verification. I examine existing approaches to the traversal and manipulation of generalised algebraic data type embedded languages and show that existing dependently typed zipper specifications do not hold when the generalised algebraic data type (GADT) is indexed over a universe of types. I define a new dependently typed zipper-like structure to traverse and manipulate GADT-embedded higher order abstract syntax trees. I also use this new structure to define a type-preserving crossover operation between these trees, and providing a working proof of concept in the Idris programming language. This allows for an approach to genetic programming in pure functional languages (that support dependent types) that combines the desirable traits of expressivity and functional verification.

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— Donovan J. Crichton

# 1 | INTRODUCTION

Genetic Programming (GP) is a form of supervised learning that results in an expression, program, or function that may then be executed by a hosting programming language. Genetic Programming offers a human-readable solution to supervised learning problems that offer insight and explanation in a way that may be desirable to sensitive domains. Functional Programming (FP) is a programming paradigm that is concerned with producing programs that are easier to formally reason about than their imperative or object oriented counterparts. There is an infamous anecdote that functional programmers often use to illustrate concepts of Monadic IO, the dangers of side effects, and the desirability of type safety. This anecdote is usually given in the form of the following imperative function.

```
void do_something(){
    // do something useful
    fire_missiles(); // accidentally inserted by a tired programmer
    // return useful result
}
```

In short, functional programs (in languages that support referential transparency) won't fire the missiles.

We can imagine then a situation where the tired programmer is replaced with a genetic program. We would like similar guarantees that this genetic program will not also fire the missiles accidentally. This is *particularly* important due to the manner in which genetic programs perform supervised learning. A GP will generate a candidate program and then test that program *by executing it*. If such a program contains a call to fire the missiles, the missiles will then be fired. We concern ourselves here with ways to achieve genetic programming so that no missiles may be fired, namely through pure functional genetic programs.

In particular we investigate well-typed language representations and their manipulations through the use of dependent types. We seek to perform a type preserving crossover operation, one of the fundamental operators in the genetic programming process, such that any modifications to our programs will never result in “missiles firing”, or ill-typed expressions occurring.

Research in the field is limited. Pure functional genetic programmers not only need a knowledge of two distinct fields of computer science, but in many popular functional languages they require a knowledge of type theory as well. This is a result of an aggressive erasure of type information at run-time by such languages. Dependently typed languages have more flexible type erasure conditions and allow us to leverage the existing type systems for our program representations.

There is a tension in existing pure functional genetic programming research. Genetic Programs with expressive type systems are often hand-crafted and lacking proofs of progress and preservation, or implemented in imperative languages. Genetic Programs that are formally verified are rare, and tend to have simple type systems.

By combining the best of both approaches we aim to advance the current state of the art in formally verified genetic programming. Such systems could then be used in domains where type correctness and verification is of vital importance. For example, in defence, security, health services, or finance.

## 1.1 CONTRIBUTIONS

In particular I make the following contributions in this thesis:

- A counter example of the Hamana and Fiore’s zipper over simple inductive families that cannot be type-checked in common dependently typed languages.
- An alternative zipper-like data structure that allows traversal and manipulation of generalised algebraic data type (GADT) embedded higher order abstract syntax (HOAS) trees.
- A type correct and type preserving crossover operation for GADT-embedded HOAS expressions.

These contributions allow progress towards a complete functionally verified genetic program.

## 1.2 THESIS STRUCTURE

**BACKGROUND AND LITERATURE REVIEW** In [Chapter 2](#) we cover the necessary background in dependent types, well-typed language embeddings and the current state of pure functional genetic programs. We also highlight the areas for improvement in the existing literature.

**TRAVERSAL OVER HOAS TREES** In [Chapter 3](#) we review the difficulties in traversing and manipulating GADT-embedded HOAS trees. We then go on to examine existing techniques for alleviating these difficulties. In particular we examine the specification for a zipper over inductive families. We find that such a specification does not hold in our given use case, and go on to define a structure that allows us to traverse and manipulate our expression trees.

**VERIFIED CROSSOVER OVER PHOAS TREES** In [Chapter 4](#) we show how we may use the solution given for tree traversal in [Chapter 3](#) to define a type correct crossover

operation on GADT-embedded HOAS trees. We also provide a proof of concept implementation in the Idris programming language.

**CONCLUSION** We summarise the main contributions of the thesis and discuss areas for further work.

# 2 | BACKGROUND AND LITERATURE REVIEW

## 2.1 DEPENDENT TYPES

### 2.1.1 Algebraic Data Types

We say that types are *inhabited* by their values. While there is not one universally agreed upon definition of an algebraic data type (ADT), the term *algebraic* usually refers to an algebra which forms based on the number of inhabitants of a type. We go a step further here and define algebraic data types in a manner enabling relatively straightforward proofs of inequality between different constructor tags. First we require a concept for a type that has no inhabitants, which we denote by  $\perp$ . If we consider types as sets of values, then the logical value associated with the  $\perp$  type is  $\emptyset$ . We now give definition 1 for algebraic data types.

**Definition 1** (A general definition of an algebraic data type).

Let  $X$  denote a set of symbols (or “tags”) of cardinality  $n \geq 1$ .

Let  $Y$  denote a set of typed values.

Then an algebraic data type  $\mathcal{T}$  may be formed from the following:

$$\mathcal{T} = \forall x.(x, y) | x \in X \wedge y \in Y$$

We give the name *constructors*, sometimes *data constructors* or *value constructors* to individual elements of  $\mathcal{T}$ . As seen above each  $t \in \mathcal{T}$  is an ordered pair with the first element containing a tag from  $X$  and the second element containing a typed value from  $Y$ . The  $\forall$  quantification in the definition of  $\mathcal{T}$  shows that we must pair one of every tag from  $X$  with some element of  $Y$ . Definition 2 shows us a concrete example for an algebraic data type representing Boolean values. The type has two values as shown.

**Definition 2** (A definition of the **Bool** ADT).

$$\begin{aligned} X &= \{\text{True}, \text{False}\} \\ Y &= \{\emptyset : \perp\} \\ \mathbb{B} &= \{(\text{True}, \emptyset : \perp), (\text{False}, \emptyset : \perp)\} \end{aligned}$$

Here we see that the type of Boolean values is a set of two symbols (True and False), where each symbol is associated with no additional information. By the definition of  $\mathcal{T}$  we see that a specific tag may only appear once in an algebraic data type, regardless



of what information it may be associated with. This allows us to prove by definition that  $\text{False} \neq \text{True}$  for type  $\mathbb{B}$ .

Throughout this document we define data types through type rules. These are inference-like rules that can be read in a similar way. Informally a rule contains a number of judgements of the form  $\Gamma \vdash \mu$ . Judgements above the line are premises of the rule, with a single conclusion below the line. If all the premises hold, then the conclusion must hold. We may read the expression  $\Gamma \vdash \mu$  as “If we may derive some expression  $\mu$  from a global (typing) context  $\Gamma$  then the following (below the line) must hold”. We split the rule into two parts, once for the definition of the type itself, and then a rule for every value that we may construct. We see in definition 3 that we may always derive the `Bool` type regardless of the existing typing context. We then see that we have two values which also may be derived regardless of context: `True`, which results in a value of `Bool` and `False` which also results in the same. Informally, the type `Bool` does not depend on any other types existing in the global context for its definition, and likewise the two values of `Bool` do not depend on any other types or values for their definitions.

**Definition 3** (A definition of boolean values using typing rules).

$$\frac{}{\Gamma \vdash \text{Bool} : \text{Type}}$$

Values:

$$\frac{}{\Gamma \vdash \text{True} : \text{Bool}}$$

$$\frac{}{\Gamma \vdash \text{False} : \text{Bool}}$$

We can also define inductive algebraic data types. Consider the inductive definition of natural numbers. Here in definition 4 we have a base case `Z` and an inductive case `S(k)`. Once again `Nat` does not require any types or values derivable from our context in order to be defined as a type. The base case `Z` also requires no types or values to exist in context  $\Gamma$ . However, the inductive case requires that we may derive an existing value `k` of type `Nat`.

**Definition 4** (A definition of natural numbers using typing rules).

$$\frac{}{\Gamma \vdash \text{Nat} : \text{Type}}$$

Values:

$$\frac{}{\Gamma \vdash \text{Z} : \text{Nat}}$$

$$\frac{\Gamma \vdash k : \text{Nat}}{\Gamma \vdash \text{S}(k) : \text{Nat}}$$

Algebraic data types carry a restriction with respect to induction. All inductive terms present in the definition of the type must be parameterised by the variables that exist in the definition of the *type* (not the definition of the values). Consider definition 5 of the type of singly-linked-lists. We cannot ask that the **Cons** constructor depends on a  $xs : \text{List } B$  as a premise when we have only introduced the variable  $A$  in the type. We also cannot specify that the **Cons** case requires a concretely typed list, for example:  $xs : \text{List Nat}$ .

**Definition 5** (The inductive singly-linked list type).

$$\frac{\Gamma \vdash A : \text{Type}}{\Gamma \vdash \text{List } A : \text{Type}}$$

Values:

$$\frac{\Gamma \vdash A : \text{Type}}{\Gamma \vdash \text{Nil} : \text{List } A} \quad \frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash x : A \quad \Gamma \vdash xs : \text{List } A}{\Gamma \vdash \text{Cons}(x, xs) : \text{List } A}$$

Along with  $\perp$ , we also define our universe of types as **Type**. This universe forms an infinite tower of types, such that  $\text{Type} : \text{Type}_1$  and  $\text{Type}_1 : \text{Type}_2$  and so on. We give the type rules for common types used in this document in definitions 6, 7, and 8. It is the sum and product types (so named for the number of inhabitants) particularly that give rise to the concept of an algebra of types.

**Definition 6** (The product type).

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash B : \text{Type}}{\Gamma \vdash A \times B : \text{Type}}$$

Values:

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash B : \text{Type} \quad \Gamma \vdash x : A \quad \Gamma \vdash y : B}{\Gamma \vdash (x, y) : A \times B}$$

**Definition 7** (The sum type).

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash B : \text{Type}}{\Gamma \vdash A + B : \text{Type}}$$

Values:

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash B : \text{Type} \quad \Gamma \vdash x : A}{\Gamma \vdash \text{Left}(x) : A + B}$$

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash B : \text{Type} \quad \Gamma \vdash y : B}{\Gamma \vdash \text{Right}(y) : A + B}$$

**Definition 8** (The function type).

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash B : \text{Type}}{\Gamma \vdash A \rightarrow B : \text{Type}}$$

Values:

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash B : \text{Type} \quad \Gamma, x : A \vdash y : B}{\Gamma \vdash \lambda x : A. y : A \rightarrow B}$$

### 2.1.2 Polymorphism

We give Wadler’s definition of polymorphism as interpreted from [Strachey \(2000\)](#).

“Ad-hoc polymorphism occurs when a function is defined over several different types, acting in a different way for each type. A typical example is overloaded multiplication: the same symbol may be used to denote multiplication of integers (as in  $3 \times 3$ ) and multiplication of floating point values (as in  $3.14 \times 3.14$ ).

Parametric polymorphism occurs when a function is defined over a range of types, acting in the same manner for each type. A typical example is the **length** function, which acts in the same way on a list of integers and a list of floating point numbers.” ([Wadler and Blott, 1989](#), p. 1)

We denote an ad-hoc polymorphic constraint on a type with a side constraint on type rules. In function definitions we denote this ad-hoc polymorphic constraint through the use of the  $\Rightarrow$  symbol. Consider examples 1 and 2 respectively.

**Example 1** (The type rule for ad-hoc polymorphic addition).

$$\frac{\Gamma \vdash A : \text{Type}}{\Gamma \vdash + : A \rightarrow A \rightarrow A} \quad \Gamma \vdash \text{Num } A$$

**Example 2** (The function definition for ad-hoc polymorphic addition).

$$\begin{aligned} (+) &: \text{Num } A \Rightarrow A \rightarrow A \rightarrow A \\ x+y &= x + y \end{aligned}$$

### 2.1.3 Generalised Algebraic Data Types

Guarded recursive data types ([Xi et al., 2003](#)), first-class phantom types ([Cheney and Hinze, 2003](#)), dependent data types ([Brady, 2017](#)), inductive dependent data types ([Pierce et al., 2010](#)), and generalised algebraic data types ([Peyton Jones et al., 2004](#)). While there are some minor differences between these terms (some are specific to certain

programming languages, and carry positivity restrictions), they all denote algebraic data types with the restriction on induction relaxed. In this document we will use the term most familiar to Haskell users - Generalised Algebraic Data Type (GADT), to refer to these structures.

If ADTs may be viewed as a set of values, then GADTs may be viewed as an indexed family of sets. A canonical example of a GADT is the `Fin` type. The type of finite sets (Bove and Dybjer, 2008). `Fin` is indexed by a natural number representing a  $k$ -ary number of elements strictly less than some upper bound  $n$ . Note the type index in definition 9, we now have a family of types indexed by a natural number *value*. So `Fin 1` is an entirely separate type from `Fin 2` and so on, for some  $k$ .

**Definition 9** (*Fin*, the type of finite sets).

$$\frac{\Gamma \vdash \text{Nat} : \text{Type}}{\Gamma \vdash \text{Fin Nat} : \text{Type}}$$

Values:

$$\frac{\Gamma \vdash n : \text{Nat}}{\Gamma \vdash \text{FZ} : \text{Fin } S(n)} \qquad \frac{\Gamma \vdash n : \text{Nat} \quad \Gamma \vdash k : \text{Fin } n}{\Gamma \vdash \text{FS}(k) : \text{Fin } S(k)}$$

#### 2.1.4 Indices, Parameters and Higher Kinds

We saw in the definition of `List` that an ADT may have a type variable as a parameter. This is subject to the restriction that all inductive occurrences of `List` appearing as a premise in `List` values, are parameterised by at least one of the same variables appearing in the premise of the `List` type. We may say then that the type `List` is *higher-kinded* where kind is used in some languages (notably Haskell) to denote the type of types. We will denote higher kinded types where we are not concerned with their parameter(s) as a function between types as per example 3.

**Example 3** (Higher kinded representation of the *List* type).

$$\frac{\Gamma \vdash \text{List} : \text{Type} \rightarrow \text{Type}}{\quad}$$

GADTs may have both type parameters and indices. We distinguish between the two by whether or not they adhere to the inductive restrictions for ordinary ADTs. Consider the type of length-indexed vectors in definition 10, here the `A` index can be considered an ordinary type parameter as all inductive `Vect` terms appearing as a premise in the values are indexed by the same `A`. However the vector is *indexed* by the `Nat` type, as the `Nat` terms appearing in the premises of the values are different to the premise in the type.

**Definition 10** (The type of length indexed vectors).

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash \text{Nat} : \text{Type} \quad \Gamma \vdash n : \text{Nat}}{\Gamma \vdash \text{Vect } n \ A : \text{Type}}$$

Values:

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash \text{Nat} : \text{Type}}{\Gamma \vdash \text{Nil} : \text{Vect } Z \ A}$$

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash \text{Nat} : \text{Type} \quad \Gamma \vdash k : \text{Nat} \quad \Gamma \vdash x : A \quad \Gamma \vdash xs : \text{Vect } k \ A}{\Gamma \vdash \text{Cons}(x, xs) : \text{Vect } S(k) \ A}$$

### 2.1.5 $\Pi$ Types and Function Families

We define the concept of a function family as a function from some specific type, to our universe of types. This allows us to define a function where the co-domain changes depending on the specific *value* given from the domain. We illustrated this through example 4. We denote function families with capital first letters to distinguish from ordinary functions.

**Example 4** (A function family resulting in natural number or string codomains).

```
NatOrString : Bool → Type
NatOrString(True) = Nat
NatOrString(False) = String
```

This in-effect is a type calculation function, where the return type is calculated from the value of the input. We define the  $\Pi$  in type definition 11 and give an example of its use on our function family `NatOrString` in example 5.

**Definition 11** (The  $\Pi$  type).

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash A \rightarrow \text{Type} : \text{Type} \quad \Gamma \vdash B : A \rightarrow \text{Type} \quad \Gamma \vdash x : A}{\Gamma \vdash \Pi x : A. B : \text{Type}}$$

**Example 5** (A  $\Pi$  type mapping bools to *NatOrString*).

```
natOrString :  $\Pi x : \text{Bool}. \text{NatOrString}$ 
natOrString(True) = 2
natOrString(False) = "Test"
```

### 2.1.6 $\Sigma$ Types

The  $\Sigma$  type denotes the dependent pair, where the type of the second element depends on the value of the first element. We give the type definition in 12 and examples in 6.

**Definition 12** (The  $\Sigma$  type).

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma, x : A \vdash B : \text{Type}}{\Gamma \vdash \Sigma A : B. a : A}$$

**Example 6** (Various  $\Sigma$  of *List* values).

```
f :  $\Sigma x : \text{Type} . \text{List } x$ 
f = (Nat, Cons(1, Cons(2, Cons(3, Nil))))

g :  $\Sigma x : \text{Type} . \text{List } x$ 
g = (Bool, Cons(True, Cons(False, Cons(True, Nil))))

z :  $\Sigma x : \text{Type} . \text{List } x$ 
z = (Type, Cons(Bool, Cons(Nat, Cons(String, Nil))))
```

### 2.1.7 Propositions as Types, Proofs as Programs

Wadler (2015) gives a well-written introduction to the notion of propositions as types. In short there exists a correspondence between logic and programming such that types in dependently typed programming languages may be interpreted as propositions in first order logic according to the following table.

Logic Term	Logic Symbol	Type
Implication	$p \Rightarrow q$	$p \rightarrow q$
Conjunction	$p \wedge q$	$p \times q$
Disjunction	$p \vee q$	$p + q$
Negation	$\neg p$	$p \rightarrow \perp$
IFF/Eq	$p \equiv q, p \Leftrightarrow q$	$(q \rightarrow p) \times (p \rightarrow q)$
Universal	$\forall x. P \ x$	$\Pi x : \text{Type} . P$
Existential	$\exists x. P \ x$	$\Sigma x : \text{Type} . P$

Types and terms share the same language in dependently typed programming languages, all terms may also be types. This includes functions. In order to avoid non-termination during the type checking process, these languages often introduce a totality and positivity clause. A function is total if it is defined on all inputs and is guaranteed to terminate in some finite time. Additionally any GADTS must be *strictly positive*.

That is the constructor values must not require a value of the type under definition (an inductive step) to the right of a function arrow.

## 2.2 HIGHER ORDER ABSTRACT SYNTAX EMBEDDING

(Pfenning and Elliott, 1988) give us *higher order abstract syntax* (HOAS), a technique for representing programs in an object language (the language we construct) where variable binding and reference management are taken care of by the meta language (the language in which we program). HOAS is well known for being attractive and simple to use, but awkward to manipulate (Washburn and Weirich, 2003; Meijer et al., 1991; Atkey et al., 2009). This is particularly true when we embed our HOAS object language within a generalised algebraic data type (a so-called deep embedding). Consider example 7, the simply typed lambda calculus represented as GADT-embedded HOAS.

**Example 7** (A HOAS embedding of STLC in Haskell).

```
{-# LANGUAGE GADTs #-}
{-# LANGUAGE KindSignatures #-}
import Data.Kind (Type)
data STLC :: Type -> Type where
  Var  :: a -> STLC a
  Abs  :: (STLC a -> STLC b) -> STLC (a -> b)
  App  :: STLC (a -> b) -> STLC a -> STLC b
```

A HOAS embedding ensures that our object language is typed checked by our meta language. Ill typed terms simply will not compile. This, coupled with the notion of propositions as types, gives us a powerful ability to verify our programs with respect to functional correctness.

## 2.3 GENETIC PROGRAMMING

Genetic programming (GP) is a class of evolutionary algorithm from the broader field of evolutionary computation. This field examines meta-heuristic optimisation algorithms that have been inspired by ideas from biological evolution. Pioneered by Koza (Koza, 1992), genetic programming adapts the techniques used to simulate evolution in genetic algorithms to work on representations of executable functions. A genetic program, like other evolutionary algorithms, maintains a pool of candidate solutions (the *population*). Each member of this pool is evaluated and scored against a user-provided function (the *fitness function*) measuring the distance (in solution space) a particular candidate is from an ideal solution. Once each candidate has been evaluated, high-performing candidates are chosen to produce new offspring by exchanging feature information

(called the *crossover* operation). The new offspring also have a small chance that a feature is randomly altered (the *mutation* operation). Ideally each successive generation should consist of individual candidate solutions that, when evaluated, give a closer, or more accurate solution to the problem than the previous generation. This process is repeated until a solution is found, or until a user-specified termination criteria is met.

### Function Representation

Functions in genetic programming are commonly represented in parse tree form. This allowed for easy parsing and execution compared to the binary string representation commonly found in genetic algorithms (Koza, 1992, p. 63-77). An example of a parse tree can be found in figure 1.

$$f(x, y) = \neg x \vee (x \wedge y)$$

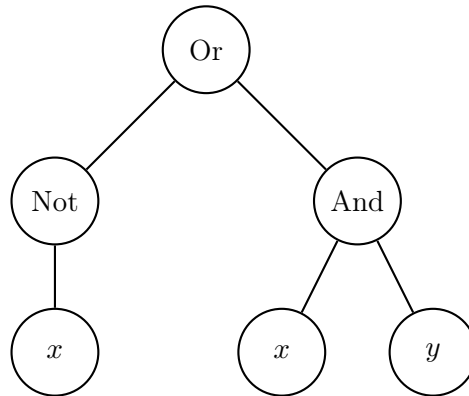


Figure 1: A parse tree

#### 2.3.1 The Crossover Operation

The crossover operator takes two high performing trees and exchanges their sub-trees at a randomly determined point (Koza, 1992, p. 101-105) (Engelbrecht, 2007, p. 180). This facilitates the exchange of features between solution candidates. Crossover is the primary operator for most genetic program implementations. Figure 2 shows two trees  $f(x, y) = \neg x \vee (x \vee y)$  and  $g(x, y) = (x \vee \neg y) \vee (\neg y \wedge \neg x)$  (where red marks the sub-trees



for which the roots were chosen at a random position for each function). The crossover function when applied to  $f$  and  $g$  yields a pair of new trees  $f'$  and  $g'$ .

## 2.4 FUNCTIONAL GENETIC PROGRAMS

The literature on functional genetic programming is sparse, perhaps understandably so given the union of two distinct fields of computer science. It also has a disproportionate preoccupation with type systems. This is also understandable when popular pure functional languages such as Haskell erase types after compile time, leaving meta-programmers to come up with creative solutions to allow run-time reasoning over types.

Yu and Clack (1998) give us *PolyGP* a polymorphic genetic programming system in Haskell. This work is notable for being the first modern pure functional genetic programming system in the literature, along with the first to implement a manual type system to perform strongly typed genetic programming (only well-typed candidates are accepted into the population (Montana, 1995)). Yu and Clack give their abstract type syntax as per example 8.

**Example 8** (Abstract type syntax for PolyGP).

```

 $\gamma$  ::  $\tau$            -- built-in type
    |  $\nu$            -- type variable
    |  $\sigma_1 \rightarrow \sigma_2$  -- function type
    |  $[\sigma_1]$       -- lists parameterised by  $\sigma_1$ 
    |  $(\sigma_1 \rightarrow \sigma_2)$  -- bracketed function type

 $\tau$  :: Int | String | Bool | Generici
 $\nu$  :: Dummyi | Temporaryi

```

The interesting cases here are clearly the **Generic** in  $\tau$ , and the entirety of  $\nu$ . Yu and clack define a generic as a type variable which forms part of the signature of the desired function to be learned. If we wish to learn the function `length :: [a] -> Int` we represent it as `length :: [Generic1] -> Int`. This is distinct from occurrences of polymorphism inside the function and terminal set (the sets of functions and values that our genetic program has to work with). Here the type system will instantiate **Dummy** variables to concrete types it may infer from the return type of a function. If a concrete type cannot be inferred the **Dummy** will be instantiated as a **Temporary** type (for delayed binding or application). Yu and Clack show here how we may represent parametric polymorphism in PolyGP, but give no mention of ad-hoc polymorphism. Yu and Clack also give no proofs of preservation or progress (Pierce and Benjamin, 2002, p. 95) for their type system.

(Diehl, 2011) shows a dependently typed crossover operation for a strongly typed stack-based genetic program (Perkis, 1994). Diehl's work is notable for being the first formally verified (with respect to functional correctness) genetic program via dependent types. Diehl first defines a language for the stack-based GP that we give in example 9.

**Example 9** (The object language of Diehl’s verified stack-based GP).

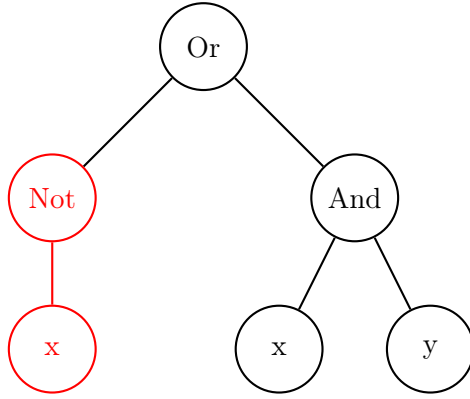
```
data Word : Set where
  true not and : Word
```

Diehl then defines **Term** as an inductive structure indexed by two natural numbers, one for the length of the stack being consumed, and the other for the length of the stack being returned. We also note Diehl’s use of the  $\Sigma$  type to represent operations where the exact value of one of the natural number indices cannot be known, including a type safe crossover that maintains the appropriate stack size invariants. Diehl’s work is type safe and verified, however the object language is trivial and practically untyped. We also note that crossover on a stack of untyped terms is significantly closer to the definition of a genetic *algorithm* (Holland et al., 1992), rather than Koza’s vision of a genetic program.

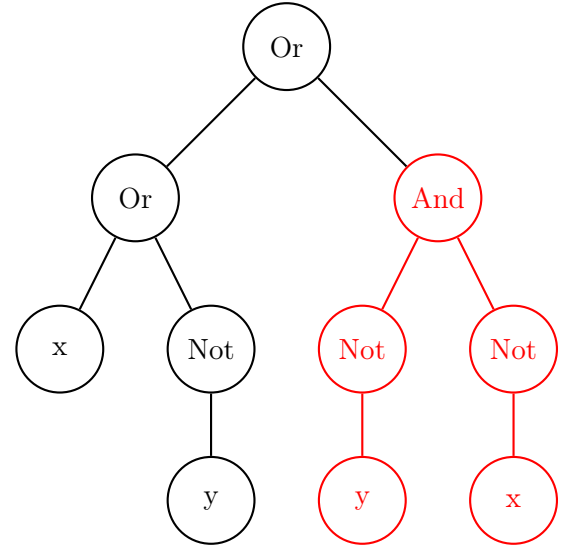
Our goal then is to take the most desirable traits from each of these approaches. We would like the expressiveness and parametric polymorphism shown by Yu and Clack, and the type safe crossover and verification properties that Diehl shows via dependent types.

We consider one of the most important properties of genetic programs over other forms of supervised machine learning, to be that the learned model is *human readable*. We note here other related examples of functional genetic programs that we exclude from consideration due to the impracticality of their object language, meta language, or both. (Briggs and O’Neill, 2008) shows how we may use SKI combinators for strongly typed genetic programming to sidestep some of the issues involved when representing variables. (Binard and Felty, 2007) give us a specification for a genetic program that learns types as well as values in the system F. However Binard and Felty represent types and terms in church-encoding and give an implementation in the C language, so there is a question of whether their abstraction-based genetic programming system is a functional genetic program. (Křen and Neruda, 2014) are concerned with parameterising the genetic program over different search strategies and show improvements with an object language of simply typed lambda calculus.

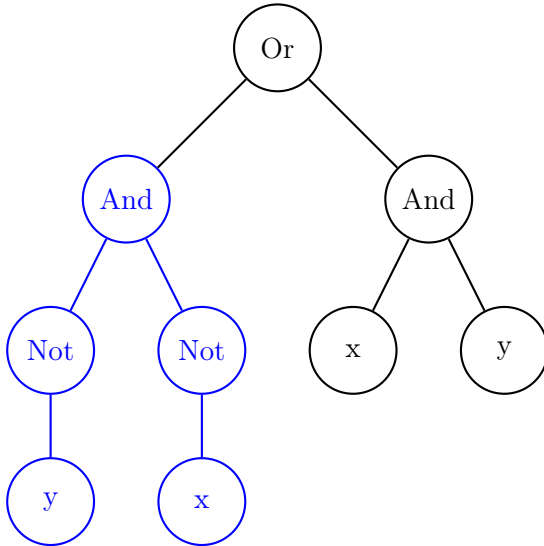
$$f(x, y) = \neg x \vee (x \wedge y)$$



$$g(x, y) = (x \vee \neg y) \vee (\neg y \wedge \neg x)$$



$$f'(x, y) = (\neg y \wedge \neg x) \vee (x \wedge y)$$



$$g'(x, y) = (x \vee \neg y) \vee \neg x$$

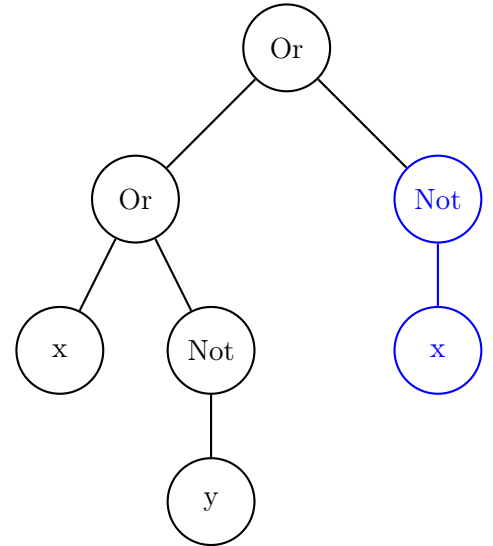


Figure 2: Crossover on expression trees.

# 3 | TRAVERSAL OVER GADT-EMBEDDED HOAS TREES

We may ultimately view a crossover operation as a substitution. We would like to substitute some particular expression in our syntax tree, with some other particular expression. In order to perform a crossover operation on two GADT embedded higher order abstract syntax trees we first need to examine ways in which we may traverse the tree structure. We also must preserve the properties of HOAS that meet our use case requirements, namely offloading variable binding and type-checking to our meta language. This chapter examines the traditional techniques functional programmers have to traverse ordinary inductive algebraic data types and why these techniques are not suited to a crossover operation on generalised algebraic data types. We then examine techniques for tree manipulation of ordinary ADTs and present a solution for traversal and expression substitution for GADT embedded trees.

## 3.1 FOLDS AND MAPS – TRAVERSAL OVER ADTS

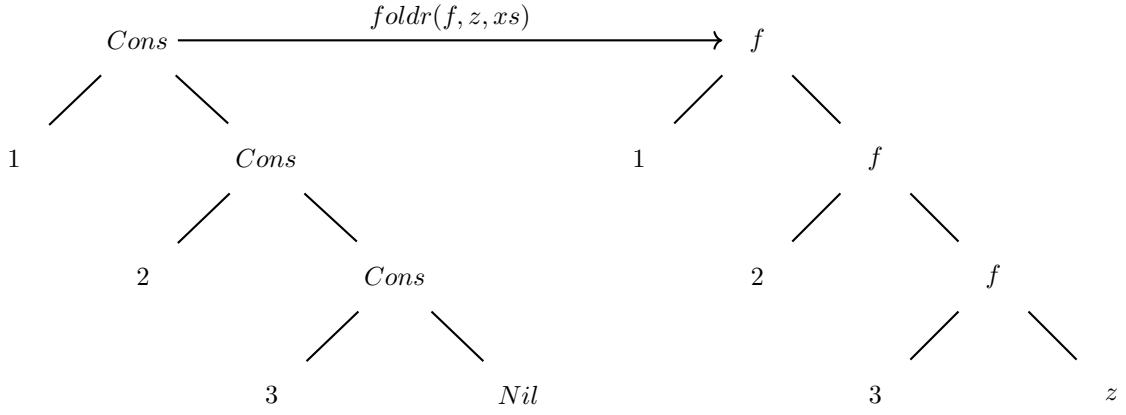
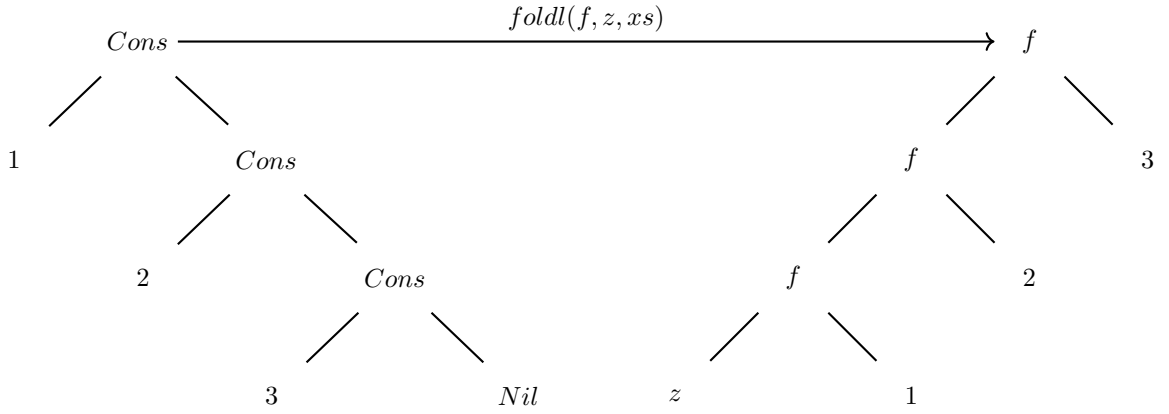
Before we explore ways to traverse well-typed GADT-embedded structures, it will be beneficial to have an understanding of common methods of traversing inductive algebraic data types. Fold and map are well known abstractions that traverse and transform these types, and are frequently introduced in beginning texts on functional programming (Bird, 2014; Allen and Moronuki, 2017; Lipovaca, 2011, for example). We cover them again here as a refresher and to show that we cannot implement these directly over generalised algebraic data type embedded higher order abstract syntax trees.

**Definition 13** (Typing rules for *foldr* and *foldl*).

$$\frac{\Gamma \vdash T : \text{Type} \rightarrow \text{Type} \quad \Gamma \vdash A : \text{Type} \quad \Gamma \vdash B : \text{Type}}{\Gamma \vdash \text{foldr} : (A \rightarrow B \rightarrow B) \rightarrow B \rightarrow T A \rightarrow B} \Gamma \vdash \text{Foldable } T$$

$$\frac{\Gamma \vdash T : \text{Type} \rightarrow \text{Type} \quad \Gamma \vdash A : \text{Type} \quad \Gamma \vdash B : \text{Type}}{\Gamma \vdash \text{foldl} : (B \rightarrow A \rightarrow B) \rightarrow B \rightarrow T A \rightarrow B} \Gamma \vdash \text{Foldable } T$$

The fold abstraction (often called reduce in imperative languages) traverses the inductive data type and yields up elements of underlying structure, applying them to the given function and accumulating value. This usually causes a reduction in the structure (hence the common name in imperative programming). The fold abstraction is expressed in modern pure functional languages as the pair of the *foldr* and *foldl* functions (see definition 13). These two functions differ in the order in which they

Figure 1: A graphical representation of *foldr* on lists.Figure 2: A graphical representation of *foldl* on lists.

accumulate the intermediate values of the function, as we illustrate in figures 1 and 2. In addition to the well-known cons-list structure, we may also fold over abstract syntax trees and accumulate the underlying values (Meijer and Jeuring, 1995). If a structure is fold-able it is customary to declare it a member of a **Foldable** interface in languages that support ad-hoc polymorphism (recall Chapter 2 (2.1.3)). Usually a definition of **foldr** is enough to declare membership of this interface as we may define **foldl** solely in terms of **foldr**.

**Definition 14** (The typing rule for *map*).

$$\frac{\Gamma \vdash F : \text{Type} \rightarrow \text{Type} \quad \Gamma \vdash A : \text{Type} \quad \Gamma \vdash B : \text{Type}}{\Gamma \vdash \text{map} : (A \rightarrow B) \rightarrow F A \rightarrow F B} \quad \Gamma \vdash \text{Functor } F$$

A simpler abstraction than fold, **map** (see definition 14) transforms every underlying element in a structure, by applying a given function to that element. Figure 3 shows

this graphically. Folds and maps are used to traverse and transform a wide variety of data types and are common abstractions that functional programmers reach for. Johann and Ghani (2008) show us that semantically GADTs are not members of the functor interface or type class. Type indexes are not necessarily polymorphic in the same way that type parameters are, this is precisely the restriction that GADTs relax. Instead GADTs rely on pattern matching each value to refine the appropriate type index. Consider example 10, a GADT provided by Johann and Ghani to represent terms:

**Example 10** (A GADT to represent terms from (Johann and Ghani, 2008)).

$$\frac{\Gamma \vdash A : \text{Type}}{\Gamma \vdash \text{Term } A : \text{Type}}$$

Values:

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash x : A}{\Gamma \vdash \text{Const}(x) : \text{Term } A}$$

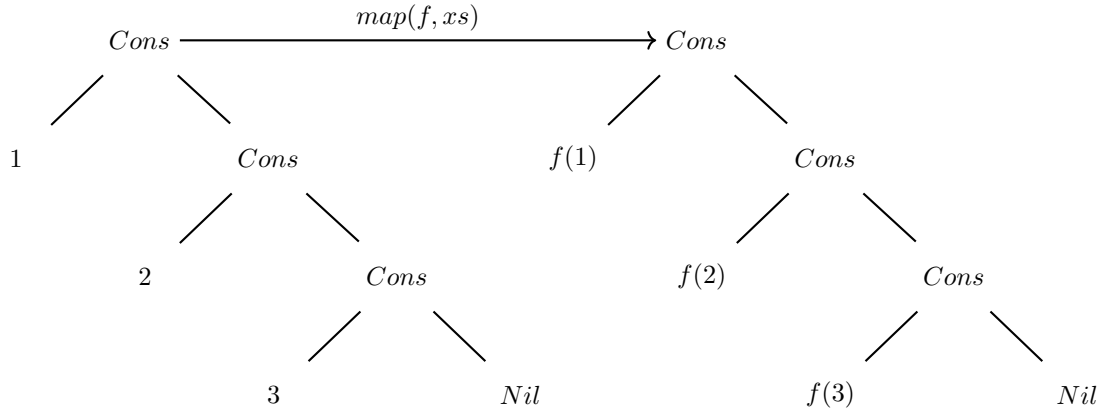
$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash B : \text{Type} \quad \Gamma \vdash x : A \quad \Gamma \vdash y : B}{\Gamma \vdash \text{Pair}(x, y) : \text{Term } (A \times B)}$$

$$\frac{\begin{array}{l} \Gamma \vdash A : \text{Type} \\ \Gamma \vdash B : \text{Type} \end{array} \quad \Gamma \vdash A \rightarrow B : \text{Type} \quad \Gamma \vdash f : \text{Term } A \rightarrow B \quad \Gamma \vdash x : \text{Term } A}{\Gamma \vdash \text{App}(f, x) : \text{Term } B}$$

Johann and Ghani consider the example of trying to `map` over the values of `Pair(x, y)`. We would like to express something of the form `map(f, Pair(x, y)) = Pair(u, v)` for some `u` and `v`. There is some semantic ambiguity here, we promised in the type of `map` that `f : A → B`, so in the case of pairs we have `f : A × B → U × V` but as `map` would iterate through the structure of the `Term`, the types of the sub expressions change. A function that maps pairs will fail to type check when given the term `Const(2)` as an argument. Yet a pair of such terms, say `Pair(Const(2), Const(2))`, is a completely valid value of type `Term`. As the type indices of the sub expressions change we cannot iterate through them with only one particular function without further restrictions on the type of the sub-expressions (for example, forcing them all to be ad-hoc polymorphic in the same manner).

Johann and Ghani continue, showing how we may derive a higher order functors and folds to structurally traverse basic GADTs. Yakushev et al. (2009) also show how we may use recursion schemes and fixed points to traverse and structurally evaluate<sup>1</sup> generalised algebraic data types. We leave the details of this approach to the interested reader however, as there is still semantic ambiguity around substitution of sub-expressions *by*

<sup>1</sup> <http://www.timphilipwilliams.com/posts/2013-01-16-fixing-gadts.html> Accessed on 07/10/19.



**Figure 3:** A graphical representation of *map* on lists.

*position*. In order for a fold or map, whether first-order or higher-order, to transform an inductive type, it may only examine the structure of its argument one node at a time. This suggests we require a structure that handles positions.

## 3.2 THE ZIPPER

Huet (1997) gives us the Zipper data structure as a means to walk down trees in a non-destructive manner. The zipper is usually represented as the pairing of a focused sub-expression, and a type capturing one-hole contexts such that the original tree may be recovered by substituting the focus into the hole. We present a similar example to Lipovaca (2011, chapter 14) by way of introduction. We first start by defining a parameterised algebraic data type to represent binary trees in definition 15.

**Definition 15** (The binary tree type).

$$\frac{\Gamma \vdash A : \text{Type}}{\Gamma \vdash \text{Tree } A : \text{Type}}$$

Values:

$$\frac{\Gamma \vdash A : \text{Type}}{\Gamma \vdash \text{Empty} : \text{Tree } A}$$

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash \text{Tree } A : \text{Type} \quad \Gamma \vdash x : A \quad \Gamma \vdash l : \text{Tree } A \quad \Gamma \vdash r : \text{Tree } A}{\Gamma \vdash \text{Node}(x, l, r) : \text{Tree } A}$$

Here we see that a tree is either empty, or contains a value and two sub-trees (although these sub-trees may in turn be empty). We define the type of one-hole contexts as a product of the parent value  $p$  and a sibling tree  $s$  as we see in definition 16.

**Definition 16** (The type of one-hole contexts for binary trees). Types:

$$\frac{\Gamma \vdash A : \text{Type}}{\Gamma \vdash \text{Context } A : \text{Type}}$$

Values:

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash \text{Tree } A : \text{Type} \quad \Gamma \vdash p : A \quad \Gamma \vdash s : \text{Tree } A}{\Gamma \vdash \text{LeftContext}(p, s) : \text{Context } A}$$

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash \text{Tree } A : \text{Type} \quad \Gamma \vdash p : A \quad \Gamma \vdash s : \text{Tree } A}{\Gamma \vdash \text{RightContext}(p, s) : \text{Context } A}$$

Once we have the context type we can represent a zipper as the product of some focus value, and a list of one-hole contexts that represent the path taken through the tree so far. Each step (definition 17) down the tree causes the list to grow, tracking the path taken from the root. Each step up the tree shrinks the list and rebuilds the new focus to include the value of the previous focus as a sub tree. This ensures that any modifications made to the focus value will propagate back up the tree.

**Definition 17** (Functions to walk along a binary tree).

$$\begin{aligned} \text{left} &: \text{Tree } A \times \text{List } (\text{Context } A) \rightarrow \text{Tree } A \times \text{List } (\text{Context } A) \\ \text{left}(\text{Empty}, \text{ctx}) &= (\text{Empty}, \text{ctx}) \\ \text{left}(\text{Node}(x, l, r), \text{ctx}) &= (l, \text{Cons}(\text{LeftContext}(x, r), \text{ctx})) \end{aligned}$$

$$\begin{aligned} \text{right} &: \text{Tree } A \times \text{List } (\text{Context } A) \rightarrow \text{Tree } A \times \text{List } (\text{Context } A) \\ \text{right}(\text{Empty}, \text{ctx}) &= (\text{Empty}, \text{ctx}) \\ \text{right}(\text{Node}(x, l, r), \text{ctx}) &= (r, \text{Cons}(\text{RightContext}(x, l), \text{ctx})) \end{aligned}$$

$$\begin{aligned} \text{up} &: \text{Tree } A \times \text{List } (\text{Context } A) \rightarrow \text{Tree } A \times \text{List } (\text{Context } A) \\ \text{up}(x, \text{Cons}(\text{LeftContext}(p, r), \text{ctx})) &= (\text{Node}(p, x, r), \text{ctx}) \\ \text{up}(x, \text{Cons}(\text{RightContext}(p, l), \text{ctx})) &= (\text{Node}(p, x, l), \text{ctx}) \end{aligned}$$

Given that the  $\text{up}$  function will substitute the focus back into the tree, we can perform a crossover operation by taking a pair of trees and exchanging their foci as per definition 18. Subsequent applications of the function  $\text{up}$  will then rebuild each tree from which we may then project out the underlying focus from the zipper.



**Definition 18** (A crossover operation (arbitrarily called  $f$ ) for zippers on binary trees).

Let  $\text{Zipper } A$  denote  $\text{Tree } A \times \text{List } (\text{Context } A)$ .

$f : \text{Zipper } A \times \text{Zipper } A \rightarrow \text{Zipper } A \times \text{Zipper } A$

$f((\text{focus}_1, \text{ctx}_1), (\text{focus}_2, \text{ctx}_2)) = ((\text{focus}_2, \text{ctx}_1), (\text{focus}_1, \text{ctx}_2))$

The zipper then is a structure to handle operations on nodes of tree-shaped inductive data types when we are concerned with a nodes position. The zipper may extract and substitute the node regardless of its particular structure, and allows manipulation of a particular focus in constant time.

```

data D : I → Set where
  K : (j : J) → (e : E) → Q[j, e] → D(d1[j, e]) → ... → D(dk[j, e]) → D(c[j])

data Ctx : I → Set where
  [] : {m : I} → Ctx(m)
  { Ki : {j : J} → {e : E} → Q[j, e] →
    D(d1[j, e]) → ... → D(dk[j, e]) → Ctx(c[j]) → Ctx(di[j, e]) }
    { with only D(di[j, e]) missing } }i ∈ [k]

data Zipper : I → Set where
  _▷_ : {m : I} → D(m) → Ctx(m) → Zipper(m)
  failure : {m : I} → Zipper(m)

downi : {j : J} → {e : E} → Zipper(c[j]) → Zipper(di[j, e])
downi(K q a1 ... ai-1 ai ai+1 ... ak ▷ C) = (ai ▷ Ki q a1 ... ai-1 ai+1 ... ak C)

upfromi : {j : J} → {e : E} → Zipper(di[j, e]) → Zipper(c[j])
upfromi(ai ▷ Ki q a1 ... ai-1 ai+1 ... ak C) = (K q a1 ... ai-1 ai ai+1 ... ak ▷ C)

```

Figure 4: A zipper over a simple inductive family  $D$  (Hamana and Fiore, 2011).

### 3.3 A TYPE DEPENDENT ZIPPER

(Hamana and Fiore, 2011) give us a general specification for a zipper over simple inductive families in their category-theoretic treatment of generalised algebraic data types and inductive families. They show this in an Agda-like syntax (Norell, 2008) as in figure 4, subject to the following definitions:

- $D$  denotes a simple inductive family.
- $Q$  denotes a constant (with respect to the type under definition) type, optionally indexed by values from  $J$  and  $E$ .
- $K$  denotes some value constructor tag of  $D$ .
- $d_k[j, e]$  represents up-to  $k$  values of  $D$  optionally indexed by values of types  $J$  and  $E$ .
- $c[j]$  represents some number of values from  $J$ .
- $i$  denotes the  $i$ th hole for the context. If  $K$  is a binary value constructor then  $i = 2$ , where  $i = 1$  may denote the left inductive  $D$  value, and  $i = 2$  may denote the right inductive  $D$  value.

The specification given in figure 4 does not hold when  $I = \mathbf{Set}$  (the type of types in Agda). Consider this zipper applied to our **Term** GADT (from our earlier example 10), as we show in example 11.

**Example 11** (A Hamana and Fiore Context type on the Term GADT shown by Johann and Ghani).

$$\frac{\Gamma \vdash A : \text{Type}}{\Gamma \vdash \text{Ctx}_{\text{hf}} A : \text{Type}}$$

Values:

$$\frac{\Gamma \vdash A : \text{Type}}{\Gamma \vdash [\ ]_{\text{hf}} : \text{Ctx}_{\text{hf}} A}$$

$$\frac{\begin{array}{l} \Gamma \vdash A, B : \text{Type} \quad \Gamma \vdash A \times B : \text{Type} \quad \Gamma \vdash \text{Term } B : \text{Type} \\ \Gamma \vdash \text{Ctx}_{\text{hf}} (A \times B) : \text{Type} \quad \Gamma \vdash r : \text{Term } B \quad \Gamma \vdash c : \text{Ctx}_{\text{hf}} (A \times B) \end{array}}{\Gamma \vdash \text{PairLeft}_{\text{hf}}(r, c) : \text{Ctx}_{\text{hf}} A}$$

$$\frac{\begin{array}{l} \Gamma \vdash A, B : \text{Type} \quad \Gamma \vdash A \times B : \text{Type} \quad \Gamma \vdash \text{Term } A : \text{Type} \\ \Gamma \vdash \text{Ctx}_{\text{hf}} (A \times B) : \text{Type} \quad \Gamma \vdash l : \text{Term } A \quad \Gamma \vdash c : \text{Ctx}_{\text{hf}} (A \times B) \end{array}}{\Gamma \vdash \text{PairRight}_{\text{hf}}(l, c) : \text{Ctx}_{\text{hf}} B}$$

$$\frac{\begin{array}{l} \Gamma \vdash A, B : \text{Type} \quad \Gamma \vdash A \rightarrow B : \text{Type} \\ \Gamma \vdash \text{Term } A : \text{Type} \quad \Gamma \vdash \text{Ctx}_{\text{hf}} B : \text{Type} \quad \Gamma \vdash r : \text{Term } A \quad \Gamma \vdash c : \text{Ctx}_{\text{hf}} B \end{array}}{\Gamma \vdash \text{AppLeft}_{\text{hf}}(r, c) : \text{Ctx}_{\text{hf}} (A \rightarrow B)}$$

$$\frac{\begin{array}{l} \Gamma \vdash A, B : \text{Type} \quad \Gamma \vdash A \rightarrow B : \text{Type} \\ \Gamma \vdash \text{Term } A : \text{Type} \quad \Gamma \vdash \text{Ctx}_{\text{hf}} B : \text{Type} \quad \Gamma \vdash r : \text{Term } A \rightarrow B \quad \Gamma \vdash c : \text{Ctx}_{\text{hf}} B \end{array}}{\Gamma \vdash \text{AppRight}_{\text{hf}}(l, c) : \text{Ctx}_{\text{hf}} A}$$

**Definition 19** (A Hamana and Fiore zipper on *Term*).

$$\frac{\Gamma \vdash A : \text{Type}}{\Gamma \vdash \text{Zipper}_{\text{hf}} A : \text{Type}}$$

Values:

$$\frac{\begin{array}{l} \Gamma \vdash A : \text{Type} \\ \Gamma \vdash \text{Term } A : \text{Type} \quad \Gamma \vdash \text{Ctx}_{\text{hf}} A : \text{Type} \quad \Gamma \vdash e : \text{Term } A \quad \Gamma \vdash c : \text{Ctx}_{\eta} A \end{array}}{\Gamma \vdash \text{Zip}_{\text{hf}}(e, c) : \text{Zipper}_{\text{hf}} A}$$

$$\frac{\Gamma \vdash A : \text{Type}}{\Gamma \vdash \text{Fail}_{\text{hf}} : \text{Zipper}_{\text{hf}} A}$$

We must now define a **down<sub>i</sub>** for each non-root value in our  $\text{Ctx}_{\text{hf}}$  type. To illustrate why the specification given in figure 4 does not hold, consider counterexample

12. Common dependent type systems (e.g those used in Agda, Idris and Coq) cannot unify the  $A$  type used in the index of the return type  $\text{Zipper}_{\text{hf}} (A \rightarrow B)$  with the  $A$  used to construct the input parameter in  $\text{App}(f, x)$ . In order to correctly represent this we would need to return a  $\Sigma$  type to capture the fact that we do not know precisely what the index of the return zipper may be. The other drawback of Hamana and Fiore’s specification for the zipper over inductive families is that we must change what function we call to traverse the zipper depending on the structure of the current node. We avoided pursuing higher order functors and folds for similar reasons, we would like to express “walking left” or “walking right” down a gadt-embedded syntax tree regardless of the current focus structure.

**Example 12** (Traversal function that will not type check for a  $\text{Zipper}_{\text{hf}}$  on the  $\text{Term}$  type).

```

downhf–AppLeft : Zipperhf B → Zipperhf (A → B)
downhf–AppLeft (Ziphf(App(f, x), c)) = Ziphf(f, AppLefthf(x, c))
downhf–AppLeft (Ziphf(e, c)) = Failhf
downhf–AppLeft (Failhf) = Failhf

```

We will then adapt Hanama and Fiore’s description of a dependently typed zipper to allow us to traverse GADT-embedded higher order abstract syntax trees. We first start by defining expressions in a small language  $\zeta$ . This language is just complex enough to force us to handle ad-hoc polymorphism, along with sub-expressions of different type to a parent expression. As we are using a deep embedding for  $\zeta$ , the meta language (the dependently typed language within which we chose to implement the zipper) will handle any type checking and variable binding (see Chapter 2 (2.1.3), and Chapter 2 (2.2)).

As we see in definition 20 we index an expression in  $\zeta$  by any type that we may derive from a global typing context  $\Gamma$ . We also see that there are only three expressions in this language:

- $\text{Lit}_{\zeta}$  allows us to embed a value from our meta language into  $\zeta$  and return an  $\text{Expr}_{\zeta}$  indexed by the type of this value.
- $\text{Add}_{\zeta}$  allows us to embed two values of some ad-hoc polymorphic type  $\Gamma \vdash \text{Num } A$  which requires an implementation of at least addition for type  $A$ . We then return an  $\text{Expr}_{\zeta}$  indexed by the same ad-hoc polymorphic type.
- $\text{Const}_{\zeta}$  allows us to embed two values, the first of a parametrically polymorphic type  $A$ , the second of a (possibly identical) parametrically polymorphic type  $B$ . We then return an  $\text{Expr}_{\zeta}$  indexed by the first type  $A$ .

**Definition 20** (A small embedded language with the potential for differing type indices in sub expressions along with values of ad-hoc polymorphic types).

$$\frac{\Gamma \vdash A : \text{Type}}{\Gamma \vdash \text{Expr}_{\zeta} A : \text{Type}}$$

Values:

$$\begin{array}{c}
\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash a : A}{\Gamma \vdash \text{Lit}_\zeta(a) : \text{Expr}_\zeta A} \quad (\Gamma \vdash \text{Show } A) \\
\\
\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash x : A \quad \Gamma \vdash y : A}{\Gamma \vdash \text{Add}_\zeta(x, y) : \text{Expr}_\zeta A} \quad (\Gamma \vdash \text{Num } A) \\
\\
\frac{\Gamma \vdash A, B : \text{Type} \quad \Gamma \vdash x : A \quad \Gamma \vdash y : B}{\Gamma \vdash \text{Const}_\zeta(x, y) : \text{Expr}_\zeta A}
\end{array}$$

One advantage of a generalised algebraic data type embedding is the ease with which we may define an evaluation function for our language. We simply map each term in our embedded language to a function in our meta language. The function argument and return types must match the indices of the sub-expressions and the parent expression respectively, as we see in definition 21. Here we give the types of the ad-hoc polymorphic function  $+$  as  $+: \text{Num } a \Rightarrow a \rightarrow a \rightarrow a$  and the parametrically polymorphic function  $\text{const}$  as  $\text{const} : a \rightarrow b \rightarrow a$  respectively.

**Definition 21** (An evaluation function for  $\text{Expr}_\zeta$ ). .

$$\begin{aligned}
&\text{eval} : \text{Expr}_\zeta A \rightarrow A \\
&\text{eval}(\text{Lit}_\zeta(x)) = x \\
&\text{eval}(\text{Add}_\zeta(x, y)) = \text{eval}(x) + \text{eval}(y) \\
&\text{eval}(\text{Const}_\zeta(x, y)) = \text{const}(\text{eval}(x), \text{eval}(y)) \text{ where } \text{const}(x, y) = x
\end{aligned}$$

Following from definition 20 we saw that we may index any expression in  $\zeta$  by any type that we may derive in our meta language. This includes the uninhabited type  $\perp$ . In order to allow us to express effectively total functions going forward we must discharge a proof obligation to say that while  $\text{Expr}_\zeta \perp$  is a valid type, it cannot be inhabited by any values (and therefore is effectively the  $\perp$  type). We use our definition of  $\text{eval}$  to prove this in theorem 1.

**Theorem 1** ( $\text{Expr}_\zeta \rightarrow \perp$ ).  $\text{Expr}_\zeta \perp$  is uninhabited.

*Proof.*

$$\begin{array}{ll}
e : \text{Expr}_\zeta \perp & \text{Assume } \text{Expr}_\zeta \perp \text{ is inhabited.} \quad (1) \\
\text{eval}(e) : \perp. & \text{By definition 21.} \quad (2) \\
\text{ex falso quodlibet} & \text{By line 2.} \quad (3)
\end{array}$$

□

Given our well-typed embedding  $\text{Expr}_\zeta$  we show from definition 22 that we may *calculate* the appropriate type when “walking” down the abstract syntax tree. We define

an indexed family of types where the index is a value of  $\text{Expr}_\zeta A$  and the elements are values of our universe of discourse type:  $\text{Type}$ . One indexed family is defined for each direction that we would like to walk down. For invalid directions, (going right or left on the unary value  $\text{Lit}_\zeta$  for example) we return the  $\perp$  type. For the valid directions we return the appropriate type of the sub-expression, given the direction and parent expression.

In  $\zeta$  the maximum arity of an expression is two and the only unary expression  $\text{Lit}_\zeta$  is a terminating node (the base case in our inductive definition of  $\text{Expr}_\zeta$ ). Only two directions are sufficient for walking down an expression tree in  $\zeta$ : walking down the left sub-expression, or walking down the right sub-expression. In particular we see in the  $\text{Const}_\zeta$  case that we return the type of the right sub-expression  $B$ , from definition 20.

**Definition 22** (indexed families representing left and right movement down an  $\text{Expr}_\zeta$  tree).

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash \text{Expr}_\zeta A : \text{Type}}{\Gamma \vdash \text{Left}_\zeta : \text{Expr}_\zeta A \rightarrow \text{Type}} \quad \frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash \text{Expr}_\zeta A : \text{Type}}{\Gamma \vdash \text{Right}_\zeta : \text{Expr}_\zeta A \rightarrow \text{Type}}$$

$\text{Left}_\zeta : \text{Expr}_\zeta A \rightarrow \text{Type}$

$\text{Left}_\zeta (\text{Lit}_\zeta(x)) = \perp$

$\text{Left}_\zeta (\text{Add}_\zeta(x, y)) = A$

$\text{Left}_\zeta (\text{Const}_\zeta(x, y)) = A$

$\text{Right}_\zeta : \text{Expr}_\zeta A \rightarrow \text{Type}$

$\text{Right}_\zeta (\text{Lit}_\zeta(x)) = \perp$

$\text{Right}_\zeta (\text{Add}_\zeta(x, y)) = A$

$\text{Right}_\zeta (\text{Const}_\zeta(x, y)) = B$

Recall the zipper as a pairing of a focus and a context from section 3.2. The context must contain enough information to rebuild the parent node given a focus. In particular the context must specify which direction was taken, from which parent, and the context under which that parent was focused. We define a new type-indexed inductive family  $\text{Ctx}_\zeta$  to capture this information.

The typing rule for our  $\text{Ctx}_\zeta$  is straightforward. We then define three possible values:

- $\text{Root}_\zeta$  represents the root, or top, of an expression tree.
- $\text{L}_\zeta(e, \text{ctx})$  represents a context state from moving left on a parent expression  $e$  where that parent had a context of  $\text{ctx}$ . The type index of this context is calculated by using  $\text{Left}_\zeta(e)$ .
- $\text{R}_\zeta(e, \text{ctx})$  represents a context state from moving right. Otherwise this value is identical to the left context except for the type index, which is now  $\text{Right}_\zeta(e)$ .

**Definition 23** (A context type to rebuild  $Expr_\zeta$  trees).

$$\frac{\Gamma \vdash A : \text{Type}}{\Gamma \vdash \text{Ctx}_\zeta A : \text{Type}}$$

Values:

$$\frac{\Gamma \vdash A : \text{Type}}{\Gamma \vdash \text{Root}_\zeta : \text{Ctx}_\zeta A}$$

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash x : \text{Expr}_\zeta A \quad \Gamma \vdash \text{ctx} : \text{Ctx}_\zeta A \quad \Gamma \vdash \text{Right}_\zeta : \text{Expr}_\zeta A \rightarrow \text{Type}}{\Gamma \vdash R_\zeta(x, \text{ctx}) : \text{Ctx}_\zeta \text{Right}_\zeta(x)}$$

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash x : \text{Expr}_\zeta A \quad \Gamma \vdash \text{ctx} : \text{Ctx}_\zeta A \quad \Gamma \vdash \text{Left}_\zeta : \text{Expr}_\zeta A \rightarrow \text{Type}}{\Gamma \vdash L_\zeta(x, \text{ctx}) : \text{Ctx}_\zeta \text{Left}_\zeta(x)}$$

From the definition of  $\text{Ctx}_\zeta$  alone, it is not immediately apparent why indexing the left and right values by the respective indexed families is necessary. This becomes clear once we define a type to represent our pairing of a focus and a context. We define a type indexed  $\text{Zipper}_\zeta$  as effectively a type-indexed product. Any given  $\text{Zipper}_\zeta A$  must have been defined using an  $\text{Expr}_\zeta A$  and a  $\text{Ctx}_\zeta A$  as per definition 24 below. From definition 23 we saw that either the type index will unify with any type (in the case of  $\text{Root}_\zeta$ ), or that the index must arise as a result of  $\text{Left}_\zeta$  or  $\text{Right}_\zeta$ . Note that all type parameters for each argument value in the zipper must be the same. This ensures that any attempt to describe an incorrect zipper state through the pairing of a context and an invalid focus type is automatically ill-typed and will be rejected as such by the compiler of our dependently typed meta language. This gives us a measure of type safety when implementing this Zipper type.

**Definition 24** (A type indexed Zipper over  $\text{Expr}_\zeta$  and  $\text{Ctx}_\zeta$ ).

$$\frac{\Gamma \vdash A : \text{Type}}{\Gamma \vdash \text{Zipper}_\zeta A : \text{Type}}$$

Values:

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash x : \text{Expr}_\zeta A \quad \Gamma \vdash \text{ctx} : \text{Ctx}_\zeta A}{\Gamma \vdash \text{Zip}_\zeta(x, \text{ctx}) : \text{Zipper}_\zeta A}$$

We now have enough information to “walk” down our zipped expression trees. We define a function for each direction from a  $\Sigma$  type to a  $\Sigma$  type. The second element of the dependent pair is the type indexed  $\text{Zipper}_\zeta a$  and the first element is the specific type index of the second element ( $a$ ). Recall from Chapter 2 (2.1.6) that a  $\Sigma$  may also be read as a first order logic proposition as per the Curry-Howard isomorphism. In this

instance we claim that a given direction function below has the following interpretation:  $\exists a, \text{Zipper}(a) \rightarrow \exists b, \text{Zipper}(b)$  where  $a, b \in \text{Type}$ . Informally we may think of this as a function that maps zippers to zippers, but where we may not be absolutely certain of their type index. There are two general cases to consider in each direction function:

- If the input zipper  $\text{pf}$  has a focus of  $\text{Lit}_\zeta(x)$  then we return the input zipper unchanged. We could also add an error-case in our zipper, or return a value of type  $\text{Maybe } \Sigma a : \text{Type} . \text{Zipper}_\zeta a$  if we wanted to explicitly fail on this case.
- If the input zipper  $\text{pf}$  has a focus of binary expression  $e$  with some left and right sub expressions of  $x$  and  $y$  respectively. Here we return a new dependent pair. The second element is a zipper with the appropriate sub expression as the focus ( $x$  if walking right,  $y$  if walking left). The context is the appropriate direction value ( $L_\zeta$  or  $R_\zeta$ ) of the previous focus  $e$  and the context under which that previous focus was derived  $\text{ctx}$ . The first element then contains the type index of this zipper.

**Definition 25** (Left and right direction functions over a  $\text{Zipper}_\zeta$ ).

$\text{left}_\zeta : \Sigma a : \text{Type} . \text{Zipper}_\zeta a \rightarrow \Sigma b : \text{Type} . \text{Zipper}_\zeta b$

$$\text{left}_\zeta(t, \text{pf}) = \begin{cases} (t, \text{pf}), & \text{if } \text{pf} = \text{Zip}_\zeta(e, \text{ctx}) \text{ where } e = \text{Lit}_\zeta(x). \\ (\text{Left}_\zeta(e), \text{Zip}_\zeta(x, L_\zeta(e, \text{ctx}))), & \text{if } \text{pf} = \text{Zip}_\zeta(e, \text{ctx}) \text{ where } e = \text{Add}_\zeta(x, y). \\ (\text{Left}_\zeta(e), \text{Zip}_\zeta(x, L_\zeta(e, \text{ctx}))), & \text{if } \text{pf} = \text{Zip}_\zeta(e, \text{ctx}) \text{ where } e = \text{Const}_\zeta(x, y) \end{cases}$$

$\text{right}_\zeta : \Sigma a : \text{Type} . \text{Zipper}_\zeta a \rightarrow \Sigma b : \text{Type} . \text{Zipper}_\zeta b$

$$\text{right}_\zeta(t, \text{pf}) = \begin{cases} (t, \text{pf}), & \text{if } \text{pf} = \text{Zip}_\zeta(e, \text{ctx}) \text{ where } e = \text{Lit}_\zeta(x). \\ (\text{Right}_\zeta(e), \text{Zip}_\zeta(y, R_\zeta(e, \text{ctx}))), & \text{if } \text{pf} = \text{Zip}_\zeta(e, \text{ctx}) \text{ where } e = \text{Add}_\zeta(x, y). \\ (\text{Right}_\zeta(e), \text{Zip}_\zeta(y, R_\zeta(e, \text{ctx}))), & \text{if } \text{pf} = \text{Zip}_\zeta(e, \text{ctx}) \text{ where } e = \text{Const}_\zeta(x, y). \end{cases}$$

We also require a way to move back up the expression tree. Recall that our goal is to use this structure to facilitate a type preserving crossover operator. We may think of this as having to substitute in an expression tree at a given node. Given our zipper structure, we would ideally like to ensure that any such substitution operation propagates through the parent contexts when moving towards the root. We define the function  $\text{up}_\zeta$  in definition 26. We see from the definition that there are three general contexts: the input focus is at the root, the focus arose from going left on a binary expression, or the focus arose from going right on a binary expression.

If the context represents the root of the tree ( $\text{Root}_\zeta$ ), we return the given input and  $\text{up}_\zeta$  effectively acts as the identity function. If the context represents a focus obtained by going left or right on binary expression then we return a new zipper, with the parent expression as the focus, and the parent context as the context. However, we substitute the appropriate sub-expression of the parent, with the focus from the input zipper.



This will do nothing in the case of an ordinary movement, but allows us to propagate any substitutions through the context.

Recall from [Chapter 2 \(2.1.7\)](#) that functions using dependent types must be total. Definition 26 is clearly partial. We claim that  $\text{up}_\zeta$  is *effectively* total however, if we are able to prove that all the missing cases are uninhabited and therefore cannot arise. The missing definitions are for zippers with contexts that attempt to walk left or right on the unary function  $\text{Lit}_\zeta$ . We show that these values are impossible by proving that they are uninhabited below in [theorem 2](#) and [theorem 3](#).

**Definition 26** (The  $\text{up}_\zeta$  function rebuilds a  $\text{Zipper}_\zeta$ ).

$$\text{up}_\zeta : \Sigma a : \text{Type} . \text{Zipper}_\zeta a \rightarrow \Sigma b : \text{Type} . \text{Zipper}_\zeta b$$

$$\text{up}_\zeta(t, \text{pf}) = \begin{cases} (t, \text{pf}), & \text{if } \text{pf} = \text{Zip}_\zeta(x, \text{Root}_\zeta). \\ (\text{Left}_\zeta(c), \text{Zip}_\zeta(\text{Add}_\zeta(e, y), \text{ctx})), & \text{if } \text{pf} = \text{Zip}_\zeta(e, \text{L}_\zeta(c, \text{ctx})) \text{ where} \\ & c = \text{Add}_\zeta(x, y). \\ (\text{Left}_\zeta(c), \text{Zip}_\zeta(\text{Const}_\zeta(e, y), \text{ctx})), & \text{if } \text{pf} = \text{Zip}_\zeta(e, \text{L}_\zeta(c, \text{ctx})) \text{ where} \\ & c = \text{Const}_\zeta(x, y). \\ (\text{Right}_\zeta(c), \text{Zip}_\zeta(\text{Add}_\zeta(x, e), \text{ctx})), & \text{if } \text{pf} = \text{Zip}_\zeta(e, \text{R}_\zeta(c, \text{ctx})) \text{ where} \\ & c = \text{Add}_\zeta(x, y). \\ (\text{Right}_\zeta(c), \text{Zip}_\zeta(\text{Const}_\zeta(x, e), \text{ctx})), & \text{if } \text{pf} = \text{Zip}_\zeta(e, \text{R}_\zeta(c, \text{ctx})) \text{ where} \\ & c = \text{Const}_\zeta(x, y). \end{cases}$$

**Theorem 2** ( $\text{L}_\zeta(\text{Lit}_\zeta(x), \text{ctx}) \rightarrow \perp$ ). Context  $\text{L}_\zeta(\text{Lit}_\zeta(x), \text{ctx})$  is uninhabited.

*Proof.*

$$\text{L}_\zeta(\text{Lit}_\zeta(x), \text{ctx}) : \text{Ctx}_\zeta \text{Left}_\zeta(\text{Lit}_\zeta(x)) \quad \text{By definition 23.} \quad (4)$$

$$\text{Left}_\zeta(\text{Lit}_\zeta(x)) : \perp \quad \text{By definition 22.} \quad (5)$$

$$\text{L}_\zeta(\text{Lit}_\zeta(x), \text{ctx}) : \text{Ctx}_\zeta \perp \quad \text{By lines 4 and 5.} \quad (6)$$

$$\text{Zip}_\zeta(e, \text{L}_\zeta(\text{Lit}_\zeta(x), \text{ctx})) : \text{Zipper}_\zeta \perp \quad \text{By definition 24 and line 6.} \quad (7)$$

$$e : \text{Expr}_\zeta \perp \quad \text{By definition 24 and line 7.} \quad (8)$$

$$\text{ex falso quodlibet} \quad \text{By theorem 1 and line 8.} \quad (9)$$

□

**Theorem 3** ( $\text{R}_\zeta(\text{Lit}_\zeta(x), \text{ctx}) \rightarrow \perp$ ). Context  $\text{R}_\zeta(\text{Lit}_\zeta(x), \text{ctx})$  is uninhabited.

*Proof.*

$$R_{\zeta}(\text{Lit}_{\zeta}(x), \text{ctx}) : \text{Ctx}_{\zeta} \text{ Right}_{\zeta}(\text{Lit}_{\zeta}(x)) \quad \text{By definition 23.} \quad (10)$$

$$\text{Right}_{\zeta}(\text{Lit}_{\zeta}(x)) : \perp \quad \text{By definition 22.} \quad (11)$$

$$R_{\zeta}(\text{Lit}_{\zeta}(x), \text{ctx}) : \text{Ctx}_{\zeta} \perp \quad \text{By lines 10 and 11.} \quad (12)$$

$$\text{Zip}_{\zeta}(e, R_{\zeta}(\text{Lit}_{\zeta}(x), \text{ctx})) : \text{Zipper}_{\zeta} \perp \quad \text{By definition 24 and line 12.} \quad (13)$$

$$e : \text{Expr}_{\zeta} \perp \quad \text{By definition 24 and line 13.} \quad (14)$$

$$\text{ex falso quodlibet} \quad \text{By theorem 1 and line 14.} \quad (15)$$

□

We will also use  $\text{up}_{\zeta}$  to define a convenience function for moving back to the root node and propagating any focus substitutions. It is important to stress that the definition given below is not strictly total due to the recursive call of  $\text{top}_{\zeta}$ . We should provide a proof of well-founded recursion to show that  $\text{top}_{\zeta}$  operates on structurally smaller objects. However, we will leave such a proof as an exercise to the reader as the definition of  $\text{up}_{\zeta}$  straightforwardly returns a context one level higher in the expression tree. Interested readers will find well-documented resources on well-founded recursion proofs for Coq<sup>2</sup>, Agda<sup>3</sup>, and Idris<sup>4</sup>. Readers may also refer to the work of [Tomé Cortiñas and Swierstra \(2018\)](#) for a well-founded recursion proof over well-typed expression trees wrapped in similar object to our zipper.

**Definition 27** ( $\text{top}_{\zeta}$  rebuilds a  $\text{Zipper}_{\zeta}$  to the root).

$$\begin{aligned} \text{top}_{\zeta} : \Sigma a : \text{Type} . \text{Zipper}_{\zeta} a &\rightarrow \Sigma b : \text{Type} . \text{Zipper}_{\zeta} b \\ \text{top}_{\zeta}(t, \text{pf}) &= \begin{cases} (t, \text{pf}) & \text{if } \text{pf} = \text{Zip}_{\zeta}(e, \text{Root}_{\zeta}). \\ \text{top}_{\zeta}(\text{up}_{\zeta}(t, \text{pf})) & \text{otherwise.} \end{cases} \end{aligned}$$

We define a substitution function,  $\text{subst}_{\zeta}$ , to substitute a focus from a zipper of a given type index, with a new focus expression of the same type index. This is achieved by using the dependent pair projection function  $\text{fst}$  on the supplied  $\Sigma$  type  $xs$  to calculate the type index of the supplied  $\text{Expr}_{\zeta}$  (recall [Chapter 2 \(2.1.6\)](#)).

**Definition 28** ( $\text{subst}_{\zeta}$  substitutes a new  $\text{Expr}_{\zeta}$  in the focus on a  $\text{Zipper}_{\zeta}$ ).

$$\begin{aligned} \text{subst}_{\zeta} : x : (\Sigma a : \text{Type} . \text{Zipper}_{\zeta} a) &\rightarrow \text{Expr}_{\zeta} \text{ fst}(x) \rightarrow \Sigma a : \text{Type} . \text{Zipper}_{\zeta} a \\ \text{subst}_{\zeta}((t, \text{Zip}_{\zeta}(e, c)), x) &= (t, \text{Zip}_{\zeta}(x, c)) \end{aligned}$$

<sup>2</sup> <https://coq.inria.fr/library/Coq.Init.Wf.html> Accessed on 05/10/19.

<sup>3</sup> <https://github.com/agda/agda-stdlib/blob/master/src/Induction/WellFounded.agda> Accessed on 05/10/19.

<sup>4</sup> [https://www.idris-lang.org/docs/current/base\\_doc/docs/Prelude.WellFounded.html](https://www.idris-lang.org/docs/current/base_doc/docs/Prelude.WellFounded.html) Accessed on 05/10/19.

Lastly, definition 29 shows how we may recover the expression tree under the zipper focus for future evaluation or substitution.

**Definition 29** ( $extract_\zeta$  extracts an  $Expr_\zeta$  under the focus of a  $Zipper_\zeta$ ).

$$\begin{aligned} extract_\zeta : x : (\Sigma a : Type . Zipper_\zeta a) &\rightarrow Expr_\zeta \text{fst}(x) \\ extract_\zeta(t, Zip_\zeta(e, ctx)) &= e \end{aligned}$$

### 3.4 DEPENDENT ZIPPER EXAMPLES

In this section we give a series of example functions and figures that show how we perform type preserving substitution and movement along well-typed expression trees. We start with example 13 defining an expression that adds the natural number 2, to the result of the function  $const(2, Test)$  where we recall that  $const$  returns the value of the first argument given. So effectively example 13 is adding 2 to 3. What makes this expression interesting is that while the second argument to  $const$  never is evaluated, it never-the-less forms part of the expression, and we should be able to substitute in a new value for the string  $Test$  in a manner that guarantees that the expression is still well-typed.

**Example 13** (An  $Expr_\zeta Nat$  containing a sub-expression of  $Expr_\zeta String$ ).

$$\begin{aligned} ex_1 &: Expr_\zeta Nat \\ ex_1 &= Add_\zeta(Lit_\zeta(2), Const_\zeta(Lit_\zeta(3), Lit_\zeta(Test))) \end{aligned}$$

Once we have defined an expression, we define a zipper with the root (or entire expression) as the focus as per example 14.

**Example 14** (A  $Zipper_\zeta Nat$  from  $ex_1 : Expr_\zeta Nat$ ).

$$\begin{aligned} ex_2 &: Zipper_\zeta Nat \\ ex_2 &= Zip_\zeta(ex_1, Root_\zeta) \end{aligned}$$

Recall that our direction functions for zippers operates on a  $\Sigma$  type as its parameter, pairing a type with a zipper indexed by that type. We wrap the zipper from example 14 with a  $\Sigma$  type by giving the type index of the zipper as the first element of the dependent pair, and value for  $ex_1$  as the second element. We see this in example 15.

**Example 15** (A  $\Sigma$  type of  $ex_2 : Zipper_\zeta Nat$ ).

$$\begin{aligned} ex_3 &: \Sigma a : Type . Zipper_\zeta a \\ ex_3 &= (Nat, ex_2) \end{aligned}$$

We then observe from example 16 that moving left gives us a dependent pair with the type of the left sub expression as the first element, and a zipper for the second

element. This zipper contains the *value* of the left sub expression as the focus, and a context that describes how we arrived at the focus (by going “ $L_\zeta$ ” on the parent expression and parent context).

**Example 16** (Moving the focus  $left_\zeta$  on  $ex_3$ ).

$$\begin{aligned} left_\zeta(ex_3) &: \Sigma a : \text{Type} . \text{Zipper}_\zeta a \\ left_\zeta(ex_3) &= (\text{Nat}, \text{Zip}_\zeta(\text{Lit}_\zeta(2), L_\zeta(ex_1, \text{Root}_\zeta))) \end{aligned}$$

Similarly, moving right from the root node gives us the appropriate dependent pair value with the return type of *const* (a natural number in this case), and a zipper with the right sub expression as the focus. We see this in example 17.

**Example 17** (Moving the focus  $right_\zeta$  on  $ex_3$ ).

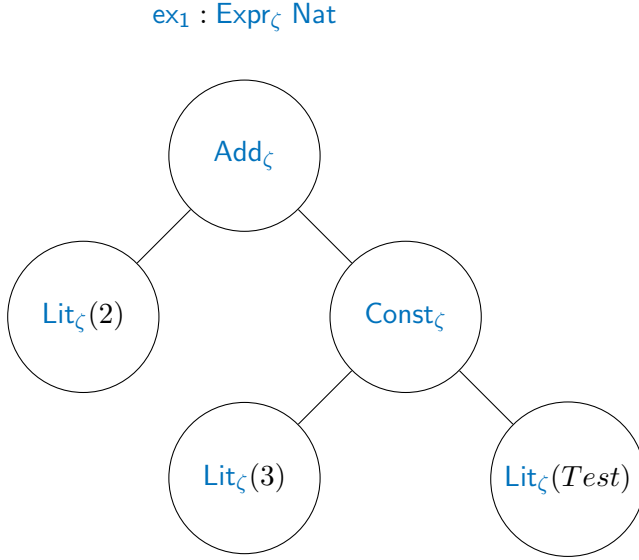
$$\begin{aligned} right_\zeta(ex_3) &: \Sigma a : \text{Type} . \text{Zipper}_\zeta a \\ right_\zeta(ex_3) &= (\text{Nat}, \text{Zip}_\zeta(\text{Const}_\zeta(\text{Lit}_\zeta(3), \text{Lit}_\zeta(\text{Test})), R_\zeta(ex_1, \text{Root}_\zeta))) \end{aligned}$$

Example 18 shows us concretely why it is necessary to give the type of the direction functions as  $\Sigma a : \text{Type} . \text{Zipper}_\zeta a \rightarrow \Sigma a : \text{Type} . \text{Zipper}_\zeta a$ . Firstly, we compose direction functions with other direction functions to form a *path*, regardless of the particular type index of a given node. Secondly we may project out the type of a focus to perform type calculations, an example of which we shall see momentarily. It is important to note that the first element of the dependent pair has changed to the correct type given the new type index of the corresponding second element.

**Example 18** (Moving the focus  $right_{\zeta\eta}$  on  $right_\zeta(ex_3)$ ).

$$\begin{aligned} right_\zeta(right_\zeta(ex_3)) &: \Sigma a : \text{Type} . \text{Zipper}_\zeta a \\ right_\zeta(right_\zeta(ex_3)) &= (\text{String}, \text{Zip}_\zeta(\text{Lit}_\zeta(\text{Test}), R_\zeta(\text{Const}_\zeta(\text{Lit}_\zeta(3), \text{Lit}_\zeta(\text{Test})), R_\zeta(ex_1, \text{Root}_\zeta)))) \end{aligned}$$

Recall that  $\text{subst}_\zeta$  has the type  $x : (\Sigma a : \text{Type} . \text{Zipper}_\zeta a) \rightarrow \text{Expr}_\zeta \text{fst}(x) \rightarrow \Sigma a : \text{Type} . \text{Zipper}_\zeta a$ . The expression given as the second argument to the function must have the same type index as the first *element* of the dependent pair that we give  $\text{subst}_\zeta$  as the first argument. Effectively  $\text{fst}$  calculates the type index of the second parameter given the first parameter. From our previous definition of zippers (definition 24) the type index of a given zipper must be the same as the type index of its focus, we now see type preserving substitution in example 19.



**Figure 5:** A graphical representation of example 13.

**Example 19** (Substituting a new focus on  $right_{\zeta}(right_{\zeta}(ex_3))$ ).

$z : \Sigma a : Type . Zipper_{\zeta} a$

$z = right_{\zeta}(right_{\zeta}(ex_3))$

$subst_{\zeta}(z, Lit_{\zeta}(Hello)) : \Sigma a : Type . Zipper_{\zeta} a$

$subst_{\zeta}(z, Lit_{\zeta}(Hello)) = (String, Zip_{\zeta}(Lit_{\zeta}(Hello), R_{\zeta}(Const_{\zeta}(Lit_{\zeta}(3), Lit_{\zeta}(Test)), R_{\zeta}(ex_1, Root_{\zeta}))))$

Lastly, we may then recover the original (modulo the substituted node) zipped expression through successive calls to  $up_{\zeta}$  as we see in example 20. Our expression is still well-typed and guaranteed to be so through the type checker in our dependently typed meta language.

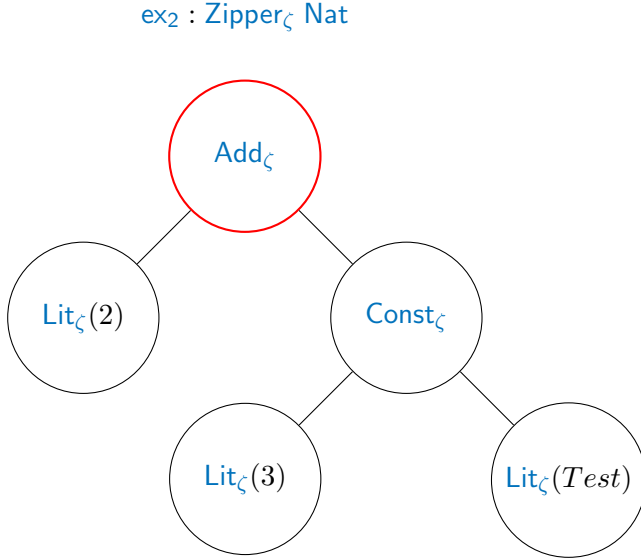
**Example 20** (Rebuilding  $subst_{\zeta}(z, Lit_{\zeta}(Hello))$ ).

$up_{\zeta}(subst_{\zeta}(z, Lit_{\zeta}(Hello))) : \Sigma a : Type . Zipper_{\zeta} a$

$up_{\zeta}(subst_{\zeta}(z, Lit_{\zeta}(Hello))) = (Nat, Zip_{\zeta}(Const_{\zeta}(Lit_{\zeta}(3), Lit_{\zeta}(Hello)), R_{\zeta}(ex_1, Root_{\zeta})))$

$up_{\zeta}(up_{\zeta}(subst_{\zeta}(z, Lit_{\zeta}(Hello)))) : \Sigma a : Type . Zipper_{\zeta} a$

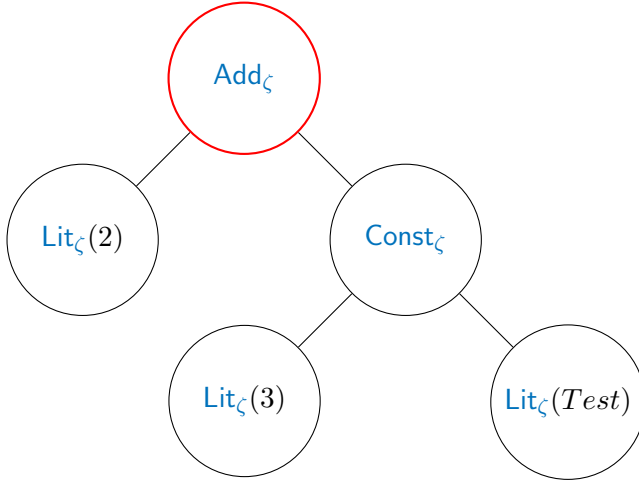
$up_{\zeta}(up_{\zeta}(subst_{\zeta}(z, Lit_{\zeta}(Hello)))) = (Nat, Zip_{\zeta}(Add_{\zeta}(Lit_{\zeta}(2), Const_{\zeta}(Lit_{\zeta}(3), Lit_{\zeta}(Hello))), Root_{\zeta}))$



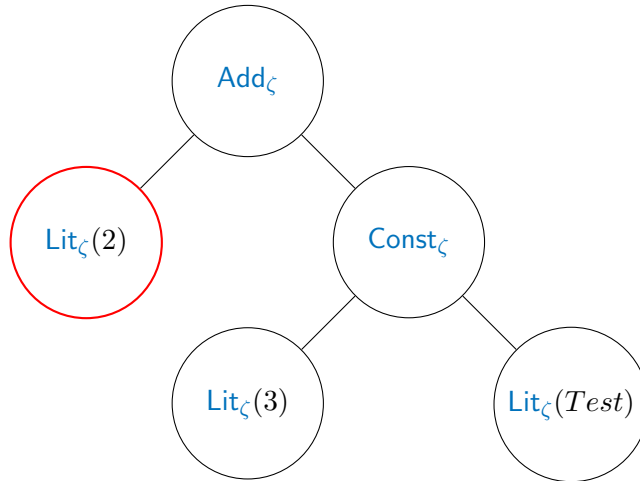
**Figure 6:** A graphical representation of a zipper on the expression seen in example 14.

### 3.5 A ZIPPER-LIKE STRUCTURE

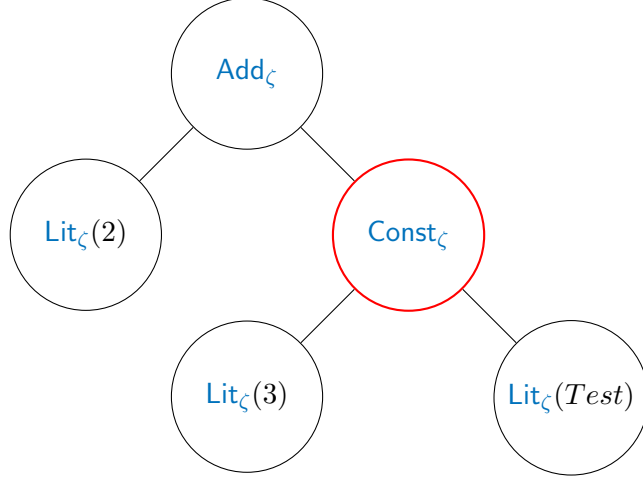
It is important to mention that the  $\text{Zipper}_\zeta$  structure we have defined here is not equivalent to a zipper from the algebra of types perspective. McBride (2001) shows that the derivative of a type is the type of one-hole context. The  $\text{Ctx}_\zeta$  (definition 23) we have defined has no “hole”, or missing type parameter that matches the focus. This concept of a hole is clearly visible in Hamana and Fiore’s specification of  $\text{Ctx}_{\text{hf}}$  (example 11). The return type of each non-empty (or non-root) context represents the missing sub-expression, which the  $\text{Zipper}_{\text{hf}}$  then pairs up with the context as the focus. In contrast our  $\text{Ctx}_\zeta$  carries around the entire parent expression, so the sub-expression is not missing, and is duplicated in the zipper twice. Once in the parent expression inside the context, and one again as the focus. This is the price we pay for the ability to call a generic  $\text{right}_\zeta$  or  $\text{left}_\zeta$  function irrespective of the current focus. The context type is indexed by a function family, and that family requires an expression. Our  $\text{Zipper}_\zeta$  type is still *semantically* a zipper however, and will satisfy our use case of enabling a type preserving crossover operation.

$$(\text{Nat}, \text{Zip}_\zeta(\text{ex}_1, \text{Root}_\zeta)) : \Sigma a : \text{Type} . \text{Zipper}_\zeta a$$


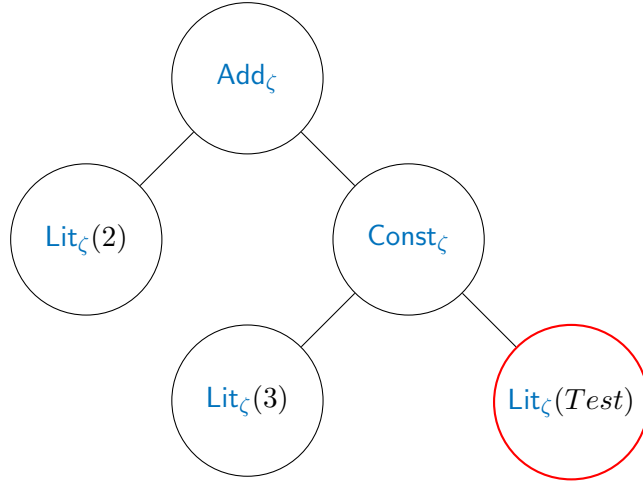
**Figure 7:** A  $\Sigma$  type that wraps the zipper in example 14.

$$(\text{Nat}, \text{Zip}_\zeta(\text{Lit}_\zeta(2), \text{L}_\zeta(\text{ex}_1, \text{Root}_\zeta))) : \Sigma a : \text{Type} . \text{Zipper}_\zeta a$$


**Figure 8:** Moving the focus left from the root node.

$$(\text{Nat}, \text{Zip}_\zeta(\text{Const}_\zeta(\text{Lit}_\zeta(3), \text{Lit}_\zeta(\text{Test})), \text{R}_\zeta(\text{ex}_1, \text{Root}_\zeta))) : \Sigma a : \text{Type} . \text{Zipper}_\zeta a$$


**Figure 9:** Moving the focus right from the root node.

$$(\text{String}, \text{Zip}_\zeta(\text{Lit}_\zeta(\text{Test}), \text{R}_\zeta(\text{Const}_\zeta(\text{Lit}_\zeta(3), \text{Lit}_\zeta(\text{Test})), \text{R}_\zeta(\text{ex}_1, \text{Root}_\zeta)))) : \Sigma a : \text{Type} . \text{Zipper}_\zeta a$$


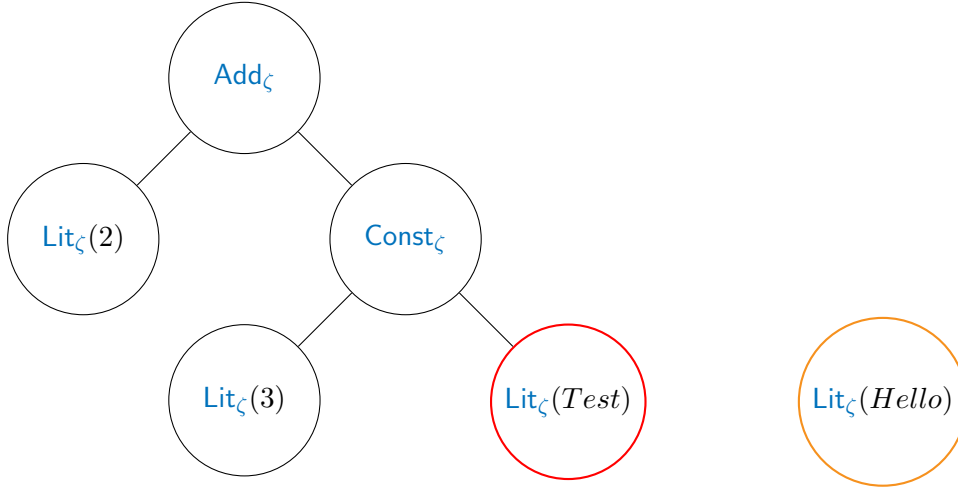
**Figure 10:** Moving the focus right twice from the root node.



Let  $z = \text{Zip}_\zeta(\text{Lit}_\zeta(\text{Test}), \text{R}_\zeta(\text{Const}_\zeta(\text{Lit}_\zeta(3), \text{Lit}_\zeta(\text{Test})), \text{R}_\zeta(\text{ex}_1, \text{Root}_\zeta))) : \text{Zipper}_\zeta \text{ String}$

$(\text{String}, z) : \Sigma a : \text{Type} . \text{Zipper}_\zeta a$

$\text{Lit}_\zeta(\text{Hello}) : \text{Expr}_\zeta \text{ fst}(\text{String}, z)$

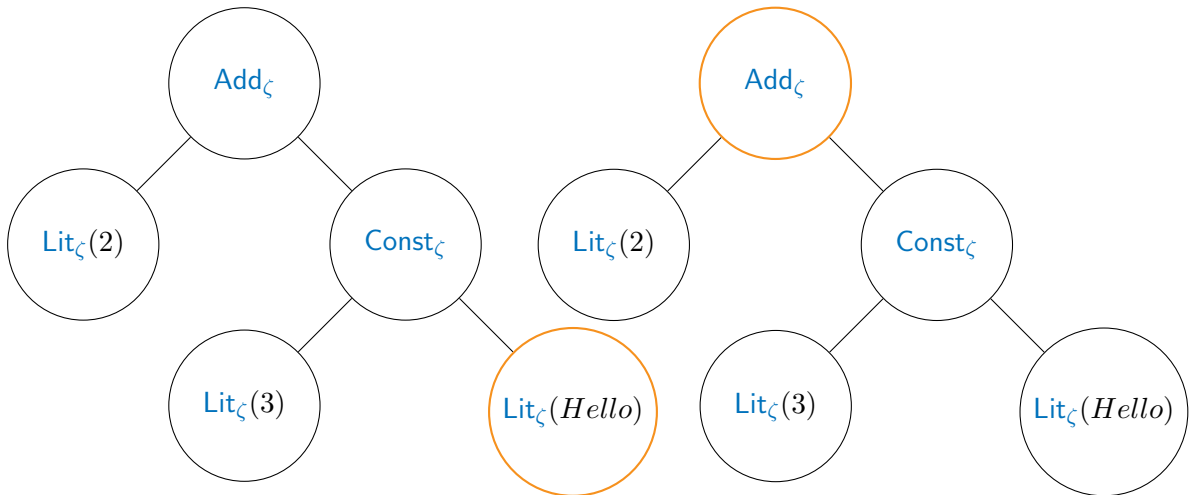


Let  $x = \text{Zip}_\zeta(\text{Lit}_\zeta(\text{Hello}), \text{R}_\zeta(\text{Const}_\zeta(\text{Lit}_\zeta(3), \text{Lit}_\zeta(\text{Test})), \text{R}_\zeta(\text{ex}_1, \text{Root}_\zeta))) : \text{Zipper}_\zeta \text{ String}$

Let  $y = \text{Zip}_\zeta(\text{Add}_\zeta(\text{Lit}_\zeta(2), \text{Const}_\zeta(\text{Lit}_\zeta(3), \text{Lit}_\zeta(\text{Hello}))), \text{Root}_\zeta) : \text{Zipper}_\zeta \text{ Nat}$

$(\text{String}, x) : \Sigma a : \text{Type} . \text{Zipper}_\zeta a$

$\text{top}_\zeta(\text{Nat}, y) : \Sigma a : \text{Type} . \text{Zipper}_\zeta a$



**Figure 11:** Substituting a new expression for the focus.  
Then rebuilding the zipper.

## 4

## VERIFIED CROSSOVER ON PHOAS TREES

In this chapter we discuss the necessary mechanics for adapting our language to the “real world”. This involves a variety of awkward solutions and work-arounds so that we may actually do something useful. We discuss the notion of parametricity and why types are usually only “almost” first class citizens of dependently typed functional languages. We then define a method in which we can bypass this restriction for a specific closed universe. We go on to discuss issues of Monadic IO and the necessity of being able to print information. This has a large flow-on effect leading to a complete redefinition of our types so far. We then discuss the necessary mechanics for a random well-typed crossover, and give the definitions and types to achieve this, discussing any proofs or language specific issues along the way. Finally, we present a proof of concept program in the Idris programming language.

## 4.1 PATTERN MATCHING ON TYPES AND PARAMETRICITY

A type preserving crossover operation must (by definition) have some manner of determining whether or not two types are equal. However, in common dependently typed functional programming languages (e.g Idris, Agda) pattern matching on type is often prohibited as it would break parametricity<sup>1</sup>. Wadler (1989) explains parametricity as a result allowing theorems to be derived from universally quantified parametrically polymorphic (recall Chapter 2 (2.1.2)) types (many functional programming languages implicitly quantify polymorphic type variables, e.g Idris, Haskell, etc). To give an example, consider the function  $f : \forall a. a \rightarrow a$ . Wadler shows that there are two theorems about the possible definitions for  $f$  under these conditions:  $f(x) = x$  or  $f(x) = f(x)$ . In dependently typed languages we gain access to the universe of discourse (the type of types: type, set, prop, \* - depending on the language). The above theorems now will not hold if we can match on the type of  $a$  instead. We could say  $f(\text{Bool}) = \text{Int}$  or  $f(\mathbb{N}) = \mathbb{N} \rightarrow \mathbb{N}$ . In order to avoid this, pattern matching on types is often disallowed in dependently typed functional languages.

In order to match on types then it is usually necessary to define a universe of types on which we would like to match at run time. (Peyton Jones et al., 2016) shows how we may use an open (easily extensible and maintainable) approach to defining universes, however for the purposes of simplicity here, we define a closed universe of types for

<sup>1</sup> Brady indicates the restriction on parametricity will be relaxed in Idris 2.

See <https://twitter.com/edwinbrady/status/1088811837376876545> Accessed on 16/10/19.

matching in definition 30. We express this through a type indexed GADT where each constructor tag returns the GADT, with the index specified to the type in question.

**Definition 30** (*TypeRep*: A data type that allows pattern matching on a type at run time).

$$\frac{\Gamma \vdash A : \text{Type}}{\Gamma \vdash \text{TypeRep}_\eta : \text{Type}}$$

Values:

$$\begin{array}{c} \frac{\Gamma \vdash \text{Bool} : \text{Type}}{\Gamma \vdash \text{TBool} : \text{TypeRep}_\eta \text{ Bool}} \quad \frac{\Gamma \vdash \text{Nat} : \text{Type}}{\Gamma \vdash \text{TNat} : \text{TypeRep}_\eta \text{ Nat}} \quad \frac{\Gamma \vdash \text{String} : \text{Type}}{\Gamma \vdash \text{TString} : \text{TypeRep}_\eta \text{ String}} \\[10pt] \frac{\Gamma \vdash B : \text{Type} \quad \Gamma \vdash A : \text{Type} \quad \Gamma \vdash x : \text{TypeRep}_\eta A \quad \Gamma \vdash y : \text{TypeRep}_\eta B}{\Gamma \vdash \text{TFunc}_\eta(x, y) : \text{TypeRep}_\eta A \rightarrow B} \\[10pt] \frac{\Gamma \vdash B : \text{Type} \quad \Gamma \vdash A : \text{Type} \quad \Gamma \vdash x : \text{TypeRep}_\eta A \quad \Gamma \vdash y : \text{TypeRep}_\eta B}{\Gamma \vdash \text{TPair}_\eta(x, y) : \text{TypeRep}_\eta A \times B} \\[10pt] \frac{\Gamma \vdash B : \text{Type} \quad \Gamma \vdash A : \text{Type} \quad \Gamma \vdash x : \text{TypeRep}_\eta A \quad \Gamma \vdash y : \text{TypeRep}_\eta B}{\Gamma \vdash \text{TSum}_\eta(x, y) : \text{TypeRep}_\eta A + B} \end{array}$$

## 4.2 RANDOM NUMBERS, IO MONADS AND PRETTY PRINTING

The crossover operation selects nodes for crossover uniformly at random (Koza, 1992, p. 101). This usually implies an external source of randomness, which in turn suggests that our programs will likely need to be run inside the IO Monad (by reading from /dev/urandom, or the system clock time, etc) (Peyton Jones, 2010). We then require a way to display these randomly generated values back to the user. This presents a problem if our expression language contains a term for lambda abstraction. Consider the following extension to our language  $\zeta$  from Chapter 3 (Definition 20):

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash B : \text{Type} \quad \Gamma \vdash A \rightarrow B : \text{Type} \quad \Gamma \vdash f : A \rightarrow \text{Expr}_\zeta B}{\Gamma \vdash \text{Lam}_\zeta(f) : \text{Expr}_\zeta (A \rightarrow B)}$$

We can begin by defining a pretty printing function over our newly extended  $\text{Expr}_\zeta$  GADT.

```

prettyζ : Exprζ A → String
prettyζ(Litζ(x)) = show(x)
prettyζ(Addζ(x, y)) = "Add(" ++ prettyζ(x) ++ "," ++ prettyζ(y) ++ ")"
prettyζ(Constζ(x, y)) = "Const(" ++ prettyζ(x) ++ "," ++ prettyζ(y) ++ ")"
prettyζ(Lamζ(f)) = λx.prettyζ(f(x)) this will not type check.

```

In the final case here we would like to come up with some  $a$  value so that we may apply it to the  $f$  parameter in our `Lamζ` constructor. However as we are working in a type system with parametricity there is no value that can satisfy the polymorphic type  $\forall a.a$ . We cannot “reach under” the lambda binder to print the lambda abstraction expression case. In languages like Haskell that do not enforce a positivity restriction, the lambda abstraction case is usually defined as `Lam : (Expr a -> Expr b) -> Expr (a -> b)`, which we can then combine with an additional dummy constructor in our expression language: `Print : String -> Expr a`. We need a stronger solution when working with dependent types.

### 4.3 FROM HOAS TO PHOAS

(Chlipala, 2008) gives us a method to solve the problem of reaching under lambda abstraction constructors in a GADT-embedded language through parametric higher order abstract syntax (PHOAS) (Chlipala, 2013, chapter 17). We add a polymorphic function family ( $\mathbf{V}$ ) to our expression indices. We then represent the types in our associated constructors as a the result of applying the appropriate type to the family  $\mathbf{V}$ . We define a more expressive language than  $\zeta$ , providing terms for lambda abstraction, function application, product types and sum types. We refer to this richer language as  $\eta$  as in definition 31.

**Definition 31** (A GADT-embedded parameteric higher order abstract syntax language  $Expr_\eta$ ).

$$\frac{\Gamma \vdash \mathbf{V} : \text{Type} \rightarrow \text{Type} \quad \Gamma \vdash A : \text{Type}}{\Gamma \vdash Expr_\eta \mathbf{V} A : \text{Type}}$$

Values:

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash V : \text{Type} \rightarrow \text{Type} \quad \Gamma \vdash t_1 : \text{TypeRep}_\eta A \quad \Gamma \vdash x : A}{\Gamma \vdash \text{Lit}_\eta(t_1, x) : \text{Expr}_\eta V A} (\Gamma \vdash \text{Show } A)$$

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash V : \text{Type} \rightarrow \text{Type} \quad \Gamma \vdash t_1 : \text{TypeRep}_\eta A \quad \Gamma \vdash x : V(A)}{\Gamma \vdash \text{Var}_\eta(t_1, x) : \text{Expr}_\eta V A}$$

$$\frac{\begin{array}{c} \Gamma \vdash A : \text{Type} \quad \Gamma \vdash B : \text{Type} \quad \Gamma \vdash V : \text{Type} \rightarrow \text{Type} \\ \Gamma \vdash t_1 : \text{TypeRep}_\eta A \quad \Gamma \vdash t_2 : \text{TypeRep}_\eta B \quad \Gamma \vdash f : V(A) \rightarrow \text{Expr}_\eta V A \end{array}}{\Gamma \vdash \text{Lam}_\eta(t_1, t_2, f) : \text{Expr}_\eta V (A \rightarrow B)}$$

$$\frac{\begin{array}{c} \Gamma \vdash A : \text{Type} \quad \Gamma \vdash B : \text{Type} \quad \Gamma \vdash V : \text{Type} \rightarrow \text{Type} \\ \Gamma \vdash t_1 : \text{TypeRep}_\eta A \quad \Gamma \vdash t_2 : \text{TypeRep}_\eta B \quad \Gamma \vdash x : \text{Expr}_\eta V A \quad \Gamma \vdash f : \text{Expr}_\eta V A \rightarrow B \end{array}}{\Gamma \vdash \text{App}_\eta(t_1, t_2, f, x) : \text{Expr}_\eta V B}$$

$$\frac{\begin{array}{c} \Gamma \vdash A : \text{Type} \quad \Gamma \vdash V : \text{Type} \rightarrow \text{Type} \\ \Gamma \vdash t_1 : \text{TypeRep}_\eta A \quad \Gamma \vdash x : \text{Expr}_\eta V A \quad \Gamma \vdash y : \text{Expr}_\eta V A \end{array}}{\Gamma \vdash \text{Add}_\eta(t_1, x, y) : \text{Expr}_\eta V A} (\Gamma \vdash \text{Num } A)$$

$$\frac{\begin{array}{c} \Gamma \vdash A : \text{Type} \quad \Gamma \vdash V : \text{Type} \rightarrow \text{Type} \\ \Gamma \vdash t_1 : \text{TypeRep}_\eta A \quad \Gamma \vdash t_2 : \text{TypeRep}_\eta B \quad \Gamma \vdash x : \text{Expr}_\eta V A \quad \Gamma \vdash y : \text{Expr}_\eta V B \end{array}}{\Gamma \vdash \text{Const}_\eta(t_1, t_2, x, y) : \text{Expr}_\eta V A}$$

$$\frac{\begin{array}{c} \Gamma \vdash A : \text{Type} \quad \Gamma \vdash V : \text{Type} \rightarrow \text{Type} \\ \Gamma \vdash t_1 : \text{TypeRep}_\eta A \quad \Gamma \vdash t_2 : \text{TypeRep}_\eta B \quad \Gamma \vdash x : \text{Expr}_\eta V A \quad \Gamma \vdash y : \text{Expr}_\eta V B \end{array}}{\Gamma \vdash \text{Pair}_\eta(t_1, t_2, x, y) : \text{Expr}_\eta V (A \times B)}$$

$$\frac{\begin{array}{c} \Gamma \vdash A : \text{Type} \\ \Gamma \vdash V : \text{Type} \rightarrow \text{Type} \quad \Gamma \vdash t_1 : \text{TypeRep}_\eta A \quad \Gamma \vdash t_2 : \text{TypeRep}_\eta B \quad \Gamma \vdash x : \text{Expr}_\eta V A \end{array}}{\Gamma \vdash \text{ELeft}_\eta(t_1, t_2, x) : \text{Expr}_\eta V (A + B)}$$

$$\frac{\begin{array}{c} \Gamma \vdash A : \text{Type} \\ \Gamma \vdash V : \text{Type} \rightarrow \text{Type} \quad \Gamma \vdash t_1 : \text{TypeRep}_\eta A \quad \Gamma \vdash t_2 : \text{TypeRep}_\eta B \quad \Gamma \vdash x : \text{Expr}_\eta V B \end{array}}{\Gamma \vdash \text{ERight}_\eta(t_1, t_2, x) : \text{Expr}_\eta V (A + B)}$$

With the addition of the typed lambda calculi terms  $\text{Lam}_\eta$  and  $\text{App}_\eta$  we require a method to partially apply constructor functions in  $\text{Expr}_\eta$  if we would like to preserve partial function application. Definition 32 shows how we may transform a function from expressions to a weak-HOAS variant that satisfies the positivity requirement of dependently typed languages. We also define a smart constructor  $\text{lam}_\eta$  to more conveniently weaken any partial applications we might find. consider the expression  $\text{Add}_\eta(t_1, \text{Lit}_\eta(t_1, 2)) : \text{Num } A \Rightarrow \text{Expr}_\eta A \rightarrow \text{Expr}_\eta A$ . We may wish to transform this into a type  $\text{Expr}_\eta A \rightarrow A$  for evaluation or use in higher order functions

such as `map`. We can then extend it using `lamη` from definition 33 for the result `lamη(t1, t1, Addη(t1, Litη(2, ))) : Num A ⇒ Exprη A → A`.

**Definition 32** (*weaken<sub>η</sub>* transforms a HOAS style lambda abstraction to a weak HOAS style abstraction).

$$\begin{aligned} \text{weaken}_\eta &: \text{TypeRep}_\eta A \rightarrow (\text{Expr}_\eta V A \rightarrow \text{Expr}_\eta V B) \rightarrow (V(A) \rightarrow \text{Expr}_\eta V B) \\ \text{weaken}_\eta(t_1, f) &= \lambda x : V(A). f(\text{Var}_\eta(t_1, x)) \end{aligned}$$

**Definition 33** (*lam<sub>η</sub>* is a convenience function for weakening HOAS-style functions to avoid direct calls to *weaken<sub>η</sub>*).

$$\begin{aligned} \text{lam}_\eta &: \text{TypeRep}_\eta A \rightarrow \text{TypeRep}_\eta B \rightarrow (\text{Expr}_\eta V A \rightarrow \text{Expr}_\eta V B) \rightarrow \text{Expr}_\eta V (A \rightarrow B) \\ \text{lam}_\eta(t_1, t_2, f) &= \text{Lam}_\eta(t_1, t_2, \text{weaken}_\eta(t_1, f)) \end{aligned}$$

The most notable change in our evaluation function for expressions is in the new `V : Type → Type` parameter. To recover the fully polymorphic evaluation we used in Chapter 3 (21) we must give `V` as the identity function on types in definition 34.

**Definition 34** (An evaluation function for *Expr<sub>η</sub>*).

$$\begin{aligned} \text{eval}_\eta &: \text{Expr}_\eta (\lambda x : \text{Type}. x) A \rightarrow A \\ \text{eval}_\eta(\text{Lit}_\eta(t_1, x)) &= x \\ \text{eval}_\eta(\text{Var}_\eta(t_1, x)) &= x \\ \text{eval}_\eta(\text{Lam}_\eta(t_1, t_2, f)) &= \lambda x : A. \text{eval}_\eta(f(x)) \\ \text{eval}_\eta(\text{App}_\eta(t_1, t_2, f, x)) &= (\text{eval}_\eta(f))(\text{eval}_\eta(x)) \\ \text{eval}_\eta(\text{Add}_\eta(t_1, x, y)) &= \text{eval}_\eta(x) + \text{eval}_\eta(y) \\ \text{eval}_\eta(\text{Const}_\eta(t_1, t_2, x, y)) &= \text{const}(\text{eval}_\eta(x), \text{eval}_\eta(y)) \\ \text{eval}_\eta(\text{Pair}_\eta(t_1, t_2, x, y)) &= (\text{eval}_\eta(x), \text{eval}_\eta(y)) \\ \text{eval}_\eta(\text{ELeft}_\eta(t_1, t_2, x)) &= \text{Left}(\text{eval}_\eta(x)) \\ \text{eval}_\eta(\text{ERight}_\eta(t_1, t_2, y)) &= \text{Right}(\text{eval}_\eta(y)) \end{aligned}$$

The additional flexibility we gain from our extension to PHOAS is readily apparent in our pretty printing function for `Exprη` (definition 35). Here we see that by forcing the `V` parameter to always return a string, we fool the type checker into believing that the variables occurring are strings or functions of strings. This is particularly interesting in our `Lamη` case. We can define a de-Brujin style index through the natural number parameter to the function, and then print nested lambda abstractions. We can now “reach under” the argument `f` because our PHOAS parameter causes `f` to have the type `String` for its first argument. Once we pass a string to `f` we can then manipulate the return expression as needed.

**Definition 35** (A pretty printer for  $Expr_\eta$ ).

```

prettyη : Nat → Exprη (λ x : Type . String) A → String
prettyη(k, Litη(t1, x))      = show(x)
prettyη(k, Varη(t1, x))      = "(Var " ++ x ++ ")"
prettyη(k, Lamη(t1, t2, f))   =
  Let x = "x" ++ show(k)
  in "(Lam " ++ x ++ " ⇒ " ++ prettyη(S(k), f(x)) ++ ")"
prettyη(k, Appη(t1, t2, f, x)) = "(App " ++ prettyη(k, f) ++ " " ++ prettyη(k, x) ++ ")"
prettyη(k, Addη(t1, x, y))    = "(" ++ prettyη(k, f) ++ " + " ++ prettyη(k, x) ++ ")"
prettyη(k, Constη(t1, t2, x, y)) = "(Const " ++ prettyη(k, x) ++ " " ++ prettyη(k, y) ++ ")"
prettyη(k, Pairη(t1, t2, x, y)) = "(" ++ prettyη(k, x) ++ "," ++ prettyη(k, y) ++ ")"
prettyη(k, ELeftη(t1, t2, x)) = "(Left " ++ prettyη(k, x) ++ ")"
prettyη(k, ERightη(t1, t2, y)) = "(Right " ++ prettyη(k, y) ++ ")"

```

While we can reach under the lambda binders for a specific type with this PHOAS implementation, it carries with it a cost. We must now “carry around” the  $\mathbb{V}$  parameter anywhere throughout our program that we suspect we may wish to eventually print or evaluate an expression. This includes inside zippers and lists. We will redefine these types slightly differently to ease our proof obligations while carrying around the PHOAS parameter where necessary.

## 4.4 ZIPPER CORRECTNESS BY CONSTRUCTION

We begin our extended definitions of zippers to accommodate the PHOAS parameter by defining improved indexed families for traversal along the tree. We say these are improved as the index has changed from  $\text{Type}$  to  $\text{Maybe Type}$ . This changes the burden of proof from one of proving that expression values are uninhabited, to a simpler one of proving that constructor tag values are not equal. We can see these changes in definition 36

**Definition 36** (Traversal families for movement along a  $Zipper_\eta$ ).

$$\begin{aligned}
 &\text{Right}_\eta : \text{Expr}_\eta \mathbb{V} A \rightarrow \text{Maybe Type} \\
 &\text{Right}_\eta(e) = \begin{cases} \text{Just}(B), & \text{if } e = \text{Pair}_\eta(a, b, x, y). \\ \text{Just}(A), & \text{if } e = \text{App}_\eta(a, b, f, x). \\ \text{Just}(A), & \text{if } e = \text{Add}_\eta(a, x, y). \\ \text{Just}(B), & \text{if } e = \text{Const}_\eta(a, b, x, y). \\ \text{Nothing}, & \text{otherwise.} \end{cases}
 \end{aligned}$$

$$\text{Left}_\eta : \text{Expr}_\eta \vee A \rightarrow \text{Maybe Type}$$

$$\text{Left}_\eta(e) = \begin{cases} \text{Just}(A), & \text{if } e = \text{Pair}_\eta(a, b, x, y). \\ \text{Just}(A \rightarrow B), & \text{if } e = \text{App}_\eta(a, b, f, x). \\ \text{Just}(A), & \text{if } e = \text{Add}_\eta(a, x, y). \\ \text{Just}(A), & \text{if } e = \text{Const}_\eta(a, b, x, y). \\ \text{Nothing}, & \text{otherwise.} \end{cases}$$

$$\text{Down}_\eta : \text{Expr}_\eta \vee A \rightarrow \text{Maybe Type}$$

$$\text{Down}_\eta(e) = \begin{cases} \text{Just}(A), & \text{if } e = \text{ELeft}_\eta(a, b, x). \\ \text{Just}(B), & \text{if } e = \text{ERight}_\eta(a, b, x). \\ \text{Nothing}, & \text{otherwise.} \end{cases}$$

We alter the  $\text{Ctx}_\eta$  to index by values of  $\text{Just}(\text{Type})$ , along with indexing by the PHOAS parameter. This further reduces the chance of programmer error from writing an invalid context. The cases where the appropriate function family returns  $\text{Nothing}$  cannot arise.

**Definition 37** (A context type,  $\text{Ctx}_\eta$  that is correct by construction). Types:

$$\frac{\Gamma \vdash V : \text{Type} \rightarrow \text{Type} \quad \Gamma \vdash A : \text{Type} \quad \Gamma \vdash \text{Maybe } A : \text{Type}}{\Gamma \vdash \text{Ctx}_\eta V (\text{Maybe } A) : \text{Type}}$$

Values:

$$\frac{\Gamma \vdash V : \text{Type} \rightarrow \text{Type} \quad \Gamma \vdash A : \text{Type} \quad \Gamma \vdash \text{Maybe } A : \text{Type}}{\Gamma \vdash \text{Root}_\eta : \text{Ctx}_\eta V \text{Just}(A)}$$

$$\frac{\begin{array}{l} \Gamma \vdash V : \text{Type} \rightarrow \text{Type} \\ \Gamma \vdash A : \text{Type} \quad \Gamma \vdash x : \text{Expr}_\eta \vee A \quad \Gamma \vdash \text{ctx} : \text{Ctx}_\eta V \text{Just}(A) \end{array}}{\Gamma \vdash \text{L}_\eta(x, \text{ctx}) : \text{Ctx}_\eta V \text{Left}_\eta(A)}$$

$$\frac{\begin{array}{l} \Gamma \vdash V : \text{Type} \rightarrow \text{Type} \\ \Gamma \vdash A : \text{Type} \quad \Gamma \vdash x : \text{Expr}_\eta \vee A \quad \Gamma \vdash \text{ctx} : \text{Ctx}_\eta V \text{Just}(A) \end{array}}{\Gamma \vdash \text{R}_\eta(x, \text{ctx}) : \text{Ctx}_\eta V \text{Right}_\eta(A)}$$

$$\frac{\begin{array}{l} \Gamma \vdash V : \text{Type} \rightarrow \text{Type} \\ \Gamma \vdash A : \text{Type} \quad \Gamma \vdash x : \text{Expr}_\eta \vee A \quad \Gamma \vdash \text{ctx} : \text{Ctx}_\eta V \text{Just}(A) \end{array}}{\Gamma \vdash \text{D}_\eta(x, \text{ctx}) : \text{Ctx}_\eta V \text{Down}_\eta(A)}$$



Our definition of the zipper is straightforward, only extending the type with the additional index for the PHOAS parameter. We also allow our  $\text{Zip}_\eta$  function to take a value of  $\text{TypeRep}_\eta$  to simplify some operations going forward.

**Definition 38** ( $\text{Zipper}_\eta$  over  $\text{Expr}_\eta$  and  $\text{Ctx}_\eta$  types). Types:

$$\frac{\Gamma \vdash V : \text{Type} \rightarrow \text{Type} \quad \Gamma \vdash A : \text{Type}}{\Gamma \vdash \text{Zipper}_\eta V A : \text{Type}}$$

Values:

$$\frac{\begin{array}{l} \Gamma \vdash V : \text{Type} \rightarrow \text{Type} \\ \Gamma \vdash A : \text{Type} \quad \Gamma \vdash t : \text{TypeRep}_\eta A \quad \Gamma \vdash x : \text{Expr}_\eta A \quad \Gamma \vdash \text{ctx} : \text{Ctx}_\eta V \text{Just}(A) \end{array}}{\Gamma \vdash \text{Zip}_\eta(t, x, \text{ctx}) : \text{Zipper}_\eta V A}$$

## 4.5 RANDOM WELL-TYPED CROSSOVER POSITIONS

Now that we have a zipper type carrying a PHOAS index we may turn our attention towards the mechanics of implementing a type safe crossover. We saw in [Chapter 2 \(2.3.1\)](#) that crossover selects randomly the node in each tree. It follows then that a type preserving crossover will select randomly from those nodes that happen to be of a particular type. We may represent all possible nodes of a given type as a list of zippers that all contain a focus of that particular type. Once again we must index this otherwise ordinary list data type by the PHOAS index as we see in [definition 39](#).

**Definition 39** (A list data type with an additional PHOAS index).

$$\frac{\Gamma \vdash V : \text{Type} \rightarrow \text{Type} \quad \Gamma \vdash A : \text{Type}}{\Gamma \vdash \text{PList}_\eta V A : \text{Type}}$$

Values:

$$\frac{\Gamma \vdash V : \text{Type} \rightarrow \text{Type} \quad \Gamma \vdash A : \text{Type}}{\Gamma \vdash \text{PNil}_\eta : \text{PList}_\eta V A}$$

$$\frac{\Gamma \vdash V : \text{Type} \rightarrow \text{Type} \quad \Gamma \vdash A : \text{Type} \quad \Gamma \vdash x : A \quad \Gamma \vdash xs : \text{PList}_\eta V A}{\Gamma \vdash \text{PCons}_\eta(x, xs) : \text{PList}_\eta V A}$$

We must also describe then common list processing functions and proof data types in order to work with a PHOAS indexed list. We first define a type that gives us a proof by construction. We saw from [Chapter 2 \(2.1.7\)](#) that types may be represented as propositions, which are proven (or witnessed) by corresponding programs (or values). We now give a concrete example of this in the  $\text{InPList}_\eta$  type given in [definition 40](#), which

represents a proof that some natural number is a valid index into the list.  $\text{InPList}_\eta$  is indexed by a  $\text{Nat}$  and our  $\text{PList}_\eta$ . It is easy enough to envisage a proof by cases approach where we consider a natural number  $n$  as a valid index into the list  $xs$ .

- $n$  is 0 and  $xs$  is empty: This is false as zero is not a valid index into the empty list.
- $n$  is the successor of  $k$  and  $xs$  is empty: Also false.
- $n$  is 0 and the list is non-empty: True, zero is always a valid index in a non-empty list.
- $n$  is the successor  $k$  and  $xs$  is non-empty: This may only be true by induction on the length of  $xs$ .

The “trick” then is to disallow the false cases, and only have values in our  $\text{InPList}_\eta$  type that represent the true cases. In both of these cases the list is non-empty. Here  $\text{InNow}_\eta$  returns an  $\text{InPList}_\eta$  indexed by zero and a non-empty list, and  $\text{InAfter}_\eta$  expects another proof type where the indexing natural number has increased by one.

**Definition 40** ( $\text{InPList}_\eta$  represents a proof by construction of valid  $\text{PList}_\eta$  indices).  
 $xs$ .

Types:

$$\frac{\Gamma \vdash V : \text{Type} \rightarrow \text{Type} \quad \Gamma \vdash A : \text{Type} \quad \Gamma \vdash k : \text{Nat} \quad \Gamma \vdash xs : \text{PList}_\eta \vee A}{\Gamma \vdash \text{InPList}_\eta k xs : \text{Type}}$$

Values:

$$\frac{\Gamma \vdash V : \text{Type} \rightarrow \text{Type} \quad \Gamma \vdash A : \text{Type} \quad \Gamma \vdash x : A \quad \Gamma \vdash xs : \text{PList}_\eta \vee A}{\Gamma \vdash \text{InNow}_\eta : \text{InPList}_\eta Z \text{PCons}_\eta(x, xs)}$$

$$\frac{\Gamma \vdash V : \text{Type} \rightarrow \text{Type} \quad \Gamma \vdash A : \text{Type} \quad \Gamma \vdash k : \text{Nat} \quad \Gamma \vdash x : A \quad \Gamma \vdash xs : \text{PList}_\eta \vee A}{\Gamma \vdash \text{InAfter}_\eta(xs) : \text{InPList}_\eta S(k) \text{PCons}_\eta(x, xs)}$$

Here in definition 41 we see a similar proof structure that captures the proposition that a list is non-empty.

**Definition 41** (A constructive proof that a  $\text{PList}_\eta$  is not empty).

$$\frac{\Gamma \vdash V : \text{Type} \rightarrow \text{Type} \quad \Gamma \vdash A : \text{Type} \quad \Gamma \vdash xs : \text{PList}_\eta \vee A}{\Gamma \vdash \text{NonEmpty}_\eta xs : \text{Type}}$$

Values:

$$\frac{\Gamma \vdash V : \text{Type} \rightarrow \text{Type} \quad \Gamma \vdash A : \text{Type} \quad \Gamma \vdash x : A \quad \Gamma \vdash xs : \text{PList}_\eta \vee A}{\Gamma \vdash \text{IsNonEmpty}_\eta : \text{NonEmpty}_\eta \text{PCons}_\eta(x, xs)}$$

We can define the append function on lists as normal (definition 42).

**Definition 42** (The append function for  $PList_\eta$ ).

$$\begin{aligned} ++_\eta &: PList_\eta A \rightarrow PList_\eta A \rightarrow PList_\eta A \\ PNil_\eta ++_\eta ys &= ys \\ PCons_\eta(x, xs) ++_\eta ys &= PCons_\eta(x, xs ++_\eta ys) \end{aligned}$$

From our earlier definition 40, we now define an indexing function into a  $PList_\eta$ . We say that this function is *verified* with respect to functional correctness as it demands that a proof be satisfied that the index is valid.

**Definition 43** (A verified index function for  $PList_\eta$ ).

$$\begin{aligned} index_\eta &: Nat \rightarrow (xs : PList_\eta V A) \rightarrow NonEmpty_\eta xs \rightarrow A \\ index_\eta(Z, PCons_\eta(x, xs), pf) &= x \\ index_\eta(S(k), PCons_\eta(x, xs), InAfter_\eta(p)) &= index_\eta(k, xs, p) \end{aligned}$$

Now that we have defined common list operations over our  $PList_\eta$  type we define a function to flatten our zipped  $Expr_\eta$  trees into a  $PList_\eta$  of zippers, with one element for each focus.  $flatten_\eta$  (definition 44) considers  $Lit_\eta$ ,  $Var_\eta$ , and  $Lam_\eta$  as the leaves or tips of the tree and capital types ( $A, B, U, A \times b$  etc) are the types of the corresponding type index of the zipper. We flatten a zipper into a list of  $\Sigma$  types where the first element of the dependent pair is the specific type index of the zipper, and the second element is a pair of the  $TypeRep_\eta$  and the associated zipper. This allows us to easily filter such a list based on its run-time type representation values. Once again we omit the proofs for well-founded recursion for flatten. Each recursive call to the left and right sub-trees on binary expressions are clearly structurally smaller.

**Definition 44** (Flatten a  $Zipper_\eta$  into a  $PList_\eta$  of  $\Sigma$  types).

Let  $S$  represent the set of all binary terms in language  $\eta$ .

$$S = \{App_\eta, Add_\eta, Const_\eta, Pair_\eta\}$$

$\forall s \in S$ , Let  $s_l$  denote the left sub-expression in  $s$  and let  $s_r$  denote the right sub-expression in  $s$ .

Let  $t$  denote some  $TypeRep_\eta A$ .

Let  $zr$  represent a zipper of  $s_r$  by focusing right on  $s$  in some context  $ctx$ .

Let  $zl$  represent a zipper of  $s_l$  by focusing left on  $s$  in some context  $ctx$ .

$$zr = Zip_\eta(t, s_r, R_\eta(s, ctx))$$

$$zl = Zip_\eta(t, s_l, L_\eta(s, ctx))$$

$$\begin{aligned}
& \text{flatten}_\eta : \text{Zipper}_\eta \vee A \rightarrow \text{PList}_\eta \vee (\Sigma a : \text{Type} . \text{TypeRep}_\eta a \times \text{Zipper}_\eta \vee a) \\
& \text{flatten}_\eta(\text{Zip}_\eta(t, e, c)) = \begin{cases} \text{PCons}_\eta((A, (t, \text{Zip}_\eta(t, e, c))), \text{PNil}_\eta) & \text{if } e = \text{Lit}_\eta(u, x) \\ \text{PCons}_\eta((A, (t, \text{Zip}_\eta(t, e, c))), \text{PNil}_\eta) & \text{if } e = \text{Var}_\eta(u, x) \\ \text{PCons}_\eta((A \rightarrow B, (t, \text{Zip}_\eta(t, e, c))), \text{PNil}_\eta) & \text{if } e = \text{Lam}_\eta(u, v, f) \\ \text{zd} ++_\eta \text{PCons}_\eta((A + B, (t, \text{Zip}_\eta(u, e, c))), \text{PNil}_\eta) & \text{if } e = \text{ELeft}_\eta(t, u, x) \\ \text{where } \text{zd} = \text{Zip}_\eta(t, x, D_\eta(e, c)) \\ \text{zd} ++_\eta \text{PCons}_\eta((A + B, (t, \text{Zip}_\eta(u, e, c))), \text{PNil}_\eta) & \text{if } e = \text{ERight}_\eta(t, u, x) \\ \text{where } \text{zd} = \text{Zip}_\eta(u, x, D_\eta(e, c)) \\ \text{zl} ++_\eta \text{zr} ++_\eta \text{PCons}_\eta((U, (u, \text{Zip}_\eta(t, e, c))), \text{PNil}_\eta) & \text{if } e \in S. \\ \text{where } U \text{ is the type index of } u. \end{cases}
\end{aligned}$$

## 4.6 VERIFIED CROSSOVER

We may now define the functions necessary for achieving a type-preserving crossover. We first define `selectNodeη` that guarantees a valid index into any non-empty `PListη` given any natural number, and returns that index. We may achieve a valid index by taking the modulo of natural number  $n$  by the length of the list. Depending on which particular language this function may be implemented in, the burden of proof to show that the modulo operation always results in a valid index (`pf2`) may be quite complex. We refer interested readers to appendix A.4 and appendix A.6 for the re-implementation of the modulo function using finite sets (Bove and Dybjer, 2008), and accompanying proofs which were necessary to implement this function in the Idris programming language.

**Definition 45** (`selectNodeη` selects a pair of zippers from a `PListη` given any natural number  $n$ ).

$$\begin{aligned}
& \text{selectNode}_\eta : \text{Nat} \rightarrow (\text{xs} : \text{PList}_\eta \vee) \rightarrow \text{NonEmpty}_\eta \text{xs} \rightarrow \Sigma a : \text{Type} . (\text{Zipper}_\eta \vee a \times \text{Zipper}_\eta \vee a) \\
& \text{selectNode}_\eta(n, \text{xs}, \text{pf}) = \text{index}_\eta(\text{mod}(n, \text{length}(\text{xs})), \text{xs}, \text{pf}_2)
\end{aligned}$$

Our `flattenη` function gives us the *Sigma* type dependent pair where the second element is a product of a zipper and it's type representation. To compare `TypeRepη` values for equality we can take the Cartesian product of two flattened zippers to give us a `PListη ∨ (Σ y : Type . (TypeRepη y × Zipperη y) × Σ z : Type . (TypeRepη z × Zipperη z))`. This will ensure that every possible point of comparison between the two trees exists in the list. We may then define a function that returns only those elements in the list where the type representation matches. It is important to note that the return type ensures that the two zippers now have the same type index. Without this crossover would not be possible. We call our function `typeRepEqη` in definition 46. Here `f` is a

function that includes the pair in the list if the type representations are equal. We refer interested readers to appendix A.7 for our implementation of `f` in Idris. This implementation is complex and involves dependent pattern matching and a number of proofs. Other dependently typed languages may be able to express `f` more succinctly.

**Definition 46** (*typeRepEq<sub>η</sub>* returns a list of valid crossover points).

```
typeRepEqη : PListη V (Σ y : Type . (TypeRepη y × Zipperη V y) × Σ z : Type . (TypeRepη z × Zipperη V z)) →
  PListη V Σ a : Type . (Zipperη V a × Zipperη V A)
typeRepEqη(xs) = foldr(f, PNilη, xs)
```

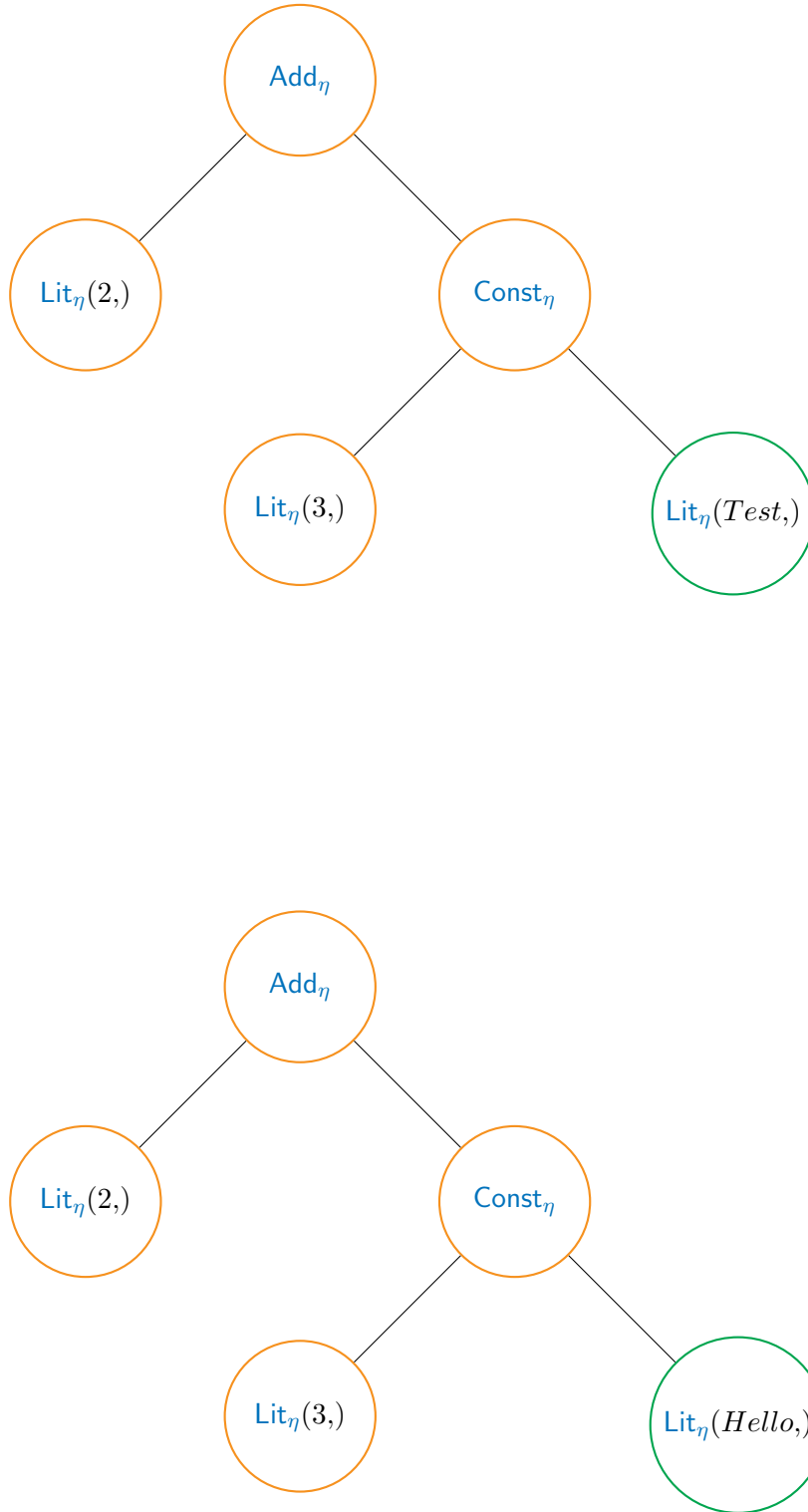
The definition of a type preserving crossover is now trivial. Given a  $\Sigma$  type of a pair of zippers indexed by the first element, we may then return a pair of zippers indexed by that element. We then only ensure that we exchange the foci between the two zippers as in definition 47.

**Definition 47** (*xover<sub>η</sub>*, A type preserving crossover).

```
xoverη : (z : Σ a : Type . (Zipperη V A × Zipperη V A)) → (Zipperη V fst(z) × Zipperη V fst(z))
xoverη(t1, (Zipη(t1, e1, ctx1), Zipη(t2, e2, ctx2))) = (Zipη(t1, e2, ctx1), Zipη(t2, e1, ctx2)))
```

## 4.7 PROOF OF CONCEPT

We provide a proof of concept of a type-preserving crossover operation implemented in the Idris programming language (version 1.3.2) (Brady, 2017). We chose Idris due to the syntactic similarity it shares with Haskell, the support it offers for programming with dependent types, and to put a fairly recently developed language through its paces. The source code for our implementation can be found in appendix A. Figure 12 demonstrates the possible crossover positions for one of the examples given in section A.8, appendix A. There are 17 possible total crossover positions,  $4 \times 4$  possible exchanges for natural numbers, and 1 possible exchange for strings.



**Figure 12:** The well-typed crossover points between two expressions.  
 The orange nodes for natural numbers.  
 The green nodes for strings.

# 5 | CONCLUSION

We have shown that Hamana and Fiore’s specification for a zipper over simple inductive families, such as the GADT of well-typed terms does not hold by way of a counter example that will not type check. We provide an alternative structure to Hamana and Fiore’s zipper that allows us to traverse well-typed GADT-embedded expression trees and perform type correct substitution. We also have given a well-typed expressive object language  $\eta$  and defined a verified type-preserving crossover operation using our zipper-like structure. We have the expressivity of parametrically polymorphic and higher-kinded terms that Yu and Clack show in PolyGP, along with the verification and type-safety guarantees from Diehl’s stack-based genetic program. Finally we have provided a working proof of concept of our type-preserving crossover operation in the Idris programming language, along with any necessary proofs required for the implementation.

## 5.1 FURTHER WORK

For further interesting research there are a plethora of approaches remaining. Readers interested in embedded syntax and dependent types may examine an implementation of our zipper structure over other forms of well-typed object languages representations such as the well-typed interpreter (Augustsson and Carlsson, 1999). Readers who are interested in genetic programming may define and implement a mutation operation over our higher order or parametrically higher order abstract syntax trees. Readers interested in general functional programming may investigate an open world approach to our run-time type representations (Peyton Jones et al., 2016). Interested readers may also investigate the type-theoretic and category-theoretic properties of our zipper-like structure, particularly with regard to the derivatives of one-hole contexts. Finally readers with an interest in more experimental computer science may wish to apply our well-typed crossover operation to supervised learning problems and give experimental results.

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## SOURCE CODE

## A.1 RUN-TIME TYPE REPRESENTATIONS: TYPEABLE.IDR

```

1  %default total
2
3  ||| A run-time representation for a type.
4  public export
5  data TypeRep : Type -> Type where
6    TBool : TypeRep Bool
7    TNat : TypeRep Nat
8    TInteger : TypeRep Integer
9    TString : TypeRep String
10   TFunc : TypeRep a -> TypeRep b -> TypeRep (a -> b)
11   TPair : TypeRep a -> TypeRep b -> TypeRep (a, b)
12   TSum : TypeRep a -> TypeRep b -> TypeRep (Either a b)
13
14   ||| A Pi Type to extract the underlying type of the run-time representation.
15   ||| TODO: Is this used anywhere?
16   public export
17   ExtractTy : (TypeRep a) -> Type
18   ExtractTy {a} t = a
19
20   ||| Boolean equality between type representations
21   ||| Unfortunately Idris is not smart enough to correctly match
22   ||| all the false cases by default.
23   public export
24   beqType : TypeRep a -> TypeRep b -> Bool
25   beqType TBool TBool = True
26   beqType TBool TNat = False
27   beqType TBool TInteger = False
28   beqType TBool TString = False
29   beqType TBool (TFunc x y) = False
30   beqType TBool (TPair x y) = False
31   beqType TBool (TSum x y) = False
32   beqType TNat TBool = False
33   beqType TNat TNat = True
34   beqType TNat TInteger = False

```

```

35 beqType TNat TString = False
36 beqType TNat (TFunc x y) = False
37 beqType TNat (TPair x y) = False
38 beqType TNat (TSum x y) = False
39 beqType TInteger TBool = False
40 beqType TInteger TNat = False
41 beqType TInteger TInteger = True
42 beqType TInteger TString = False
43 beqType TInteger (TFunc x y) = False
44 beqType TInteger (TPair x y) = False
45 beqType TInteger (TSum x y) = False
46 beqType TString TBool = False
47 beqType TString TNat = False
48 beqType TString TInteger = False
49 beqType TString TString = True
50 beqType TString (TFunc x y) = False
51 beqType TString (TPair x y) = False
52 beqType TString (TSum x y) = False
53 beqType (TFunc x y) TNat = False
54 beqType (TFunc x y) TBool = False
55 beqType (TFunc x y) TInteger = False
56 beqType (TFunc x y) TString = False
57 beqType (TFunc x y) (TFunc a b) = beqType x a && beqType y b
58 beqType (TFunc x y) (TPair a b) = False
59 beqType (TFunc x y) (TSum a b) = False
60 beqType (TPair x y) TNat = False
61 beqType (TPair x y) TBool = False
62 beqType (TPair x y) TInteger = False
63 beqType (TPair x y) TString = False
64 beqType (TPair x y) (TFunc a b) = False
65 beqType (TPair x y) (TPair a b) = beqType x a && beqType y b
66 beqType (TPair x y) (TSum a b) = False
67 beqType (TSum x y) TNat = False
68 beqType (TSum x y) TBool = False
69 beqType (TSum x y) TInteger = False
70 beqType (TSum x y) TString = False
71 beqType (TSum x y) (TFunc a b) = False
72 beqType (TSum x y) (TPair a b) = False
73 beqType (TSum x y) (TSum a b) = beqType x a && beqType y b
74
75 ||| Pretty printing for runtime representations of Types.
76 public export
77 prettyTy : TypeRep a -> String

```

```

78 prettyTy TBool = "Bool"
79 prettyTy TNat = "Nat"
80 prettyTy TInteger = "Integer"
81 prettyTy TString = "String"
82 prettyTy (TFunc x y) = "(" ++ prettyTy x ++ " -> " ++ prettyTy y ++ ")"
83 prettyTy (TPair x y) = "(" ++ prettyTy x ++ ", " ++ prettyTy y ++ ")"
84 prettyTy (TSum x y) = "(Either " ++ prettyTy x ++ " " ++ prettyTy y ++ ")"

```

## A.2 WELL-TYPED EXPRESSION TREES: EXPR.IDR

```

1  import Typeable.Typeable
2
3  %default total
4
5  ||| PHOAS Expression Language
6  ||| see 'Typeable' for TypeRep explanation.
7  public export
8  data Expr : (v : Type -> Type) -> (a : Type) -> Type where
9    Lit : Show a => {t1 : TypeRep a} -> a -> Expr v a
10   Var : {t1 : TypeRep a} -> v a -> Expr v a
11   Lam : {t1 : TypeRep a} -> {t2 : TypeRep b} ->
12         (v a -> Expr v b) -> Expr v (a -> b)
13   App : {t1 : TypeRep a} -> {t2 : TypeRep b} -> Expr v (a -> b) ->
14         Expr v a -> Expr v b
15   Add : Num a => Expr v a -> {t1 : TypeRep a} -> Expr v a -> Expr v a
16   Cnst : {t1 : TypeRep a} -> {t2 : TypeRep b} -> Expr v a -> Expr v b ->
17         Expr v a
18   EPair : {t1 : TypeRep a} -> {t2 : TypeRep b} -> Expr v a -> Expr v b ->
19         Expr v (a, b)
20   ELeft : {t1 : TypeRep a} -> {t2 : TypeRep b} -> Expr v a ->
21         Expr v (Either a b)
22   ERight : {t1 : TypeRep a} -> {t2 : TypeRep b} -> Expr v b ->
23         Expr v (Either a b)
24
25  ||| Weaken a non-positive function into a strictly positive one.
26  public export
27  weaken : {t1 : TypeRep a} -> (Expr v a -> Expr v b) ->
28         (v a -> Expr v b)
29  weaken {t1} f = \x => f (Var {t1} x)
30
31  ||| Smart constructor for the 'Lam' case so weaken doesn't need to be
32  ||| explicitly called.

```

```

33 public export
34 lam : {t1 : TypeRep a} -> {t2 : TypeRep b} -> (Expr v a -> Expr v b) ->
35     Expr v (a -> b)
36 lam {t1}{t2} f = Lam {t1} {t2} (weaken {t1} f)
37
38 ||| Polymorphic evaluator.
39 public export
40 eval : Expr (\x => x) a -> a
41 eval (Lit x) = x
42 eval (Var x) = x
43 eval (Lam f) = \x => eval (f x)
44 eval (App f x) = (eval f) (eval x)
45 eval (Add x y) = (eval x) + (eval y)
46 eval (Cnst x y) = const (eval x) (eval y)
47 eval (EPair x y) = (eval x, eval y)
48 eval (ELeft {b} x) = Left (eval x)
49 eval (ERight {a} x) = Right (eval x)
50
51 ||| A HOAS encoding of `Expr a -> String` is too polymorphic
52 ||| to reach under the lambda binding to access the body.
53 ||| So here we pretty print with PHOAS.
54 ||| (Needed to identify expressions/typereps in the IO monad)
55 public export
56 prettyE : Nat -> Expr (\_ => String) a -> String
57 prettyE k (Lit x) = show x
58 prettyE k (Var x) = "(Var " ++ x ++ ")"
59 prettyE k (Lam f) =
60     let x = "x" ++ (show k)
61     in "(Lam " ++ x ++ " => " ++ prettyE (succ k) (f x) ++ ")"
62 prettyE k (App f x) = "(App " ++ prettyE k f ++ " " ++ prettyE k x ++ ")"
63 prettyE k (Add x y) = "(" ++ prettyE k x ++ " + " ++ prettyE k y ++ ")"
64 prettyE k (Cnst x y) = "(Const " ++ prettyE k x ++ " " ++ prettyE k y ++ ")"
65 prettyE k (EPair x y) = "(Pair " ++ prettyE k x ++ " " ++ prettyE k y ++ ")"
66 prettyE k (ELeft x) = "(Left " ++ prettyE k x ++ ")"
67 prettyE k (ERight x) = "(Right " ++ prettyE k x ++ ")"

```

### A.3 ZIPPER-LIKE TRAVERSAL OF EXPRESSIONS: ZIPPER.IDR

```

1 import Expr.Expr
2 import Typeable.Typeable
3
4 %default total

```

```

5
6   ||| The indexed family calculating the type when walking left on an Expr.
7   public export
8   GoLeft : Expr v a -> Maybe Type
9   GoLeft (Lit x) = Nothing
10  GoLeft (Var x) = Nothing
11  GoLeft (Lam f) = Nothing
12  GoLeft (EPair {a} x y) = Just a
13  GoLeft (ELeft x) = Nothing
14  GoLeft (ERight x) = Nothing
15  GoLeft (App {a}{b} f x) = Just (a -> b)
16  GoLeft (Add {a} x y) = Just a
17  GoLeft (Cnst {a} x y) = Just a
18
19  ||| The indexed family calculating the type when walking right on an Expr.
20  public export
21  GoRight : Expr v a -> Maybe Type
22  GoRight (Lit x) = Nothing
23  GoRight (Var x) = Nothing
24  GoRight (Lam f) = Nothing
25  GoRight (EPair {b} x y) = Just b
26  GoRight (ELeft x) = Nothing
27  GoRight (ERight x) = Nothing
28  GoRight (App f {a} x) = Just a
29  GoRight (Add x {a} y) = Just a
30  GoRight (Cnst x {b} y) = Just b
31
32  ||| The indexed family calculating the type when walking down on an Expr.
33  public export
34  GoDown : Expr v a -> Maybe Type
35  GoDown (Lit x) = Nothing
36  GoDown (Var x) = Nothing
37  GoDown (Lam {a}{b} f) = Nothing
38  GoDown (App f x) = Nothing
39  GoDown (Add x y) = Nothing
40  GoDown (Cnst x y) = Nothing
41  GoDown (EPair x y) = Nothing
42  GoDown (ELeft {a}{b} x) = Just a
43  GoDown (ERight {b}{a} x) = Just b
44
45  ||| An ordinary Cons list but indexed by the PHOAS parameter so I don't
46  ||| need to carry it around inside Sigma types everywhere.
47  public export

```

```

48 data PHOASList : (v : Type -> Type) -> (a : Type) -> Type where
49   Nil : PHOASList v a
50   (::) : a -> PHOASList v a -> PHOASList v a
51
52   ||| Append for PHOASList.
53 public export
54   (++) : PHOASList v a -> PHOASList v a -> PHOASList v a
55   (++) Nil xs = xs
56   (++) (x :: xs) ys = x :: xs ++ ys
57
58   ||| InBounds for PHOASLists
59 public export
60 data InPList : (k : Nat) -> (xs : PHOASList v a) -> Type where
61   InNow : InPList Z (x :: xs)
62   InAfter : InPList k xs -> InPList (S k) (x :: xs)
63
64   ||| Index for PHOASLists
65 public export
66   index : (n : Nat) -> (xs : PHOASList v a) -> {auto prf: InPList n xs} -> a
67   index Z (x :: xs) = x
68   index (S k) (x :: xs) {prf = InAfter p} = index k xs
69
70   ||| The context contains all information necessary to rebuild the expression
71   ||| tree.
72 public export
73 data Context : (Type -> Type) -> Maybe Type -> Type where
74   Root : Context v (Just a)
75   L : (x : Expr v a) -> Context v (Just a) -> Context v (GoLeft x)
76   R : (x : Expr v a) -> Context v (Just a) -> Context v (GoRight x)
77   D : (x : Expr v a) -> Context v (Just a) -> Context v (GoDown x)
78
79   ||| A modified Hamana-Fiore style dependent-zipper.
80 public export
81 data Zipper : (Type -> Type) -> Type -> Type where
82   Zip : {z1 : TypeRep a} -> Expr v a -> Context v (Just a) -> Zipper v a
83
84   ||| extract the focus from a Zipper.
85 public export
86   extract : Zipper v a -> Expr v a
87   extract (Zip e c) = e
88
89   ||| A convenience function to wrap a zipper up in a Sigma type.
90 public export

```



```

91 wrap : Zipper v a -> (v : (Type -> Type) ** a : Type ** Zipper v a)
92 wrap {v} {a} z = (v ** a ** z)
93
94 ||| Rebuild a tree.
95 public export
96 up : (a : Type ** Zipper v a) -> (b : Type ** Zipper v b)
97 up (t ** pf) =
98   case pf of
99     (Zip {z1} e {a} Root) => (a ** Zip e {z1} Root)
100    (Zip e (L (Lit x) c)) impossible
101    (Zip e (L (Lam f) c)) impossible
102    (Zip e (L (App {t1}{t2} f x) c)) => (_ ** (Zip {z1=t2} (App {t1}{t2} e x) c))
103    (Zip e (L (Add {t1} x y) c)) => (_ ** (Zip {z1=t1} (Add {t1} e y) c))
104    (Zip e (L (Cnst {t1}{t2} x y) c)) => (_ ** (Zip {z1=t1} (Cnst {t1}{t2} e y) c))
105    (Zip e (L (EPair {t1}{t2} {a}{b} x y) c)) =>
106      ((a,b) ** (Zip {z1=TPair t1 t2} (EPair {t1}{t2} e y) c))
107    (Zip e (L (ELeft x) c)) impossible
108    (Zip e (L (ERight x) c)) impossible
109    (Zip e (R (Lit x) c)) impossible
110    (Zip e (R (Lam f) c)) impossible
111    (Zip e (R (App {t1}{t2}{b} f x) c)) => (b ** (Zip {z1=t2} (App {t1}{t2} f e) c))
112    (Zip e (R (Add {t1} {a} x y) c)) => (a ** (Zip {z1=t1} (Add {t1} x e) c))
113    (Zip e (R (Cnst {t1}{t2}{a} x y) c)) =>
114      (a ** (Zip {z1=t1} (Cnst {t1}{t2} x e) c))
115    (Zip e (R (EPair {t1}{t2} {a}{b} x y) c)) =>
116      ((a,b) ** (Zip {z1=TPair t1 t2} (EPair {t1}{t2} x e) c))
117    (Zip e (R (ELeft x) c)) impossible
118    (Zip e (R (ERight x) c)) impossible
119    (Zip e (D (Lit x) c)) impossible
120    (Zip e (D (Lam f) c)) impossible
121    (Zip e (D (App f x) c)) impossible
122    (Zip e (D (Add x y) c)) impossible
123    (Zip e (D (Cnst x y) c)) impossible
124    (Zip e (D (EPair x y) c)) impossible
125    (Zip e (D (ELeft {t1}{t2} {a} {b} x) c)) =>
126      (Either a b ** (Zip {z1=TSum t1 t2} (ELeft {t1}{t2} e) c))
127    (Zip e (D (ERight {t1}{t2} {a} {b} x) c)) =>
128      (Either a b ** (Zip {z1=TSum t1 t2} (ERight {t1} {t2} e) c))
129
130 ||| A convenience function to move the focus all the way to the root
131 public export
132 top : (a ** Zipper v a) -> (b ** Zipper v b)
133 top (a ** (Zip {z1} e Root)) = (a ** (Zip {z1} e Root))

```

```

134 top (a ** (Zip {z1} e c)) =
135   let s = assert_smaller (a ** (Zip e {z1} c)) (up (a ** (Zip {z1} e c)))
136   in top s
137
138 ||| Flatten a zipper into a list of all possible foci
139 public export
140 flatten : Zipper v a ->
141   PHOASList v (x : Type ** (TypeRep x, Zipper v x))
142 flatten (Zip {z1} e@(Lit {t1} {a} x) c) = [(a ** (z1, (Zip {z1} e c)))]
143 flatten (Zip {z1} e@(Var {t1} {a} x) c) = [(a ** (z1, (Zip {z1} e c)))]
144 flatten (Zip {z1} e@(Lam {t1}{t2}{a}{b} f) c) = [(a -> b ** (z1, (Zip {z1} e c)))]
145 flatten (Zip {z1} e@(App {t1}{t2} {a}{b} f x) c) =
146   let z1 = assert_total $ flatten (Zip {z1 = TFunc t1 t2} f (L e c))
147   zr = assert_total $ flatten (Zip {z1=t1} x (R e c))
148   in z1 ++ zr ++ [(b ** (z1, (Zip {z1} e c)))]
149 flatten (Zip {z1} e@(Add {t1}{a} x y) c) =
150   let z1 = assert_total $ flatten (Zip {z1=t1} x (L e c))
151   zr = assert_total $ flatten (Zip {z1=t1} y (R e c))
152   in z1 ++ zr ++ [(a ** (z1, (Zip {z1} e c)))]
153 flatten (Zip {z1} e@(Cnst {t1}{t2} x y) c) =
154   let z1 = assert_total $ flatten (Zip {z1=t1} x (L e c))
155   zr = assert_total $ flatten (Zip {z1=t2} y (R e c))
156   in z1 ++ zr ++ [(a ** (z1, (Zip {z1} e c)))]
157 flatten (Zip {z1} e@(EPair {t1}{t2}{a}{b} x y) c) =
158   let z1 = assert_total $ flatten (Zip {z1=t1} x (L e c))
159   zr = assert_total $ flatten (Zip {z1=t2} y (R e c))
160   in z1 ++ zr ++ [((a, b) ** (z1, (Zip {z1} e c)))]
161 flatten (Zip {z1} e@(ELeft {t1}{t2} {a}{b} x) c) =
162   let zd = assert_total $ flatten (Zip {z1=t1} x (D e c))
163   in zd ++ [(Either a b ** (z1, (Zip {z1} e c)))]
164 flatten (Zip {z1} e@(ERight {t1}{t2}{a}{b} x) c) =
165   let zd = assert_total $ flatten (Zip {z1=t2} x (D e c))
166   in zd ++ [(Either a b ** (z1, (Zip {z1} e c)))]
167
168 ||| substitute in a new focus expression into a zipper
169 public export
170 subst : (x : (v : (Type -> Type) ** a : Type ** Zipper v a)) ->
171   Expr (fst x) (fst (snd x)) -> Zipper (fst x) (fst (snd x))
172 subst (v ** x ** (Zip {z1} e' c)) e = Zip {z1} e c
173
174 ||| Pretty printer for contexts
175 public export
176 prettyC : Context (\_ => String) a -> String

```

```

177 prettyC Root = " Root"
178 prettyC (L e c) = "(L " ++ prettyE 0 e ++ prettyC c ++ ")"
179 prettyC (R e c) = "(R " ++ prettyE 0 e ++ prettyC c ++ ")"
180 prettyC (D e c) = "(D " ++ prettyE 0 e ++ prettyC c ++ ")"
181
182 ||| pretty printer for zippers
183 public export
184 prettyZ : Zipper (\_ => String) a -> String
185 prettyZ (Zip e c) = "Zip " ++ prettyE 0 e ++ " [C: " ++ prettyC c ++ "]"

```

## A.4 A REIMPLEMENTATION OF MOD FOR SIMPLER PROOFS: MOD.IDR

```

1 import Data.Fin
2 %default total
3
4 ||| helper function on finite sets rather than natural numbers
5 public export
6 modf : (x, y : Nat) -> {auto prf : GT y Z} -> Fin y
7 modf Z (S k) = FZ
8 modf (S k) (S j) with (strengthen (modf k (S j)))
9   modf (S k) (S j) | Left bounds = FZ
10   modf (S k) (S j) | Right rem = FS rem
11
12 ||| mod on natural numbers using finite sets
13 public export
14 modfin : (x, y : Nat) -> {auto prf : GT y Z} -> Nat
15 modfin x y = finToNat (modf x y)

```

## A.5 A LIST DATA TYPE WITH A PHOAS PARAMETER: PHOASLIST.IDR

```

1 %default total
2
3 ||| The ordinary list data type, carrying around an extra PHOAS parameter
4 public export
5 data PHOASList : (v : Type -> Type) -> (a : Type) -> Type where
6   Nil : PHOASList v a
7   (::) : a -> PHOASList v a -> PHOASList v a
8
9 ||| Append on PHOASLists

```

```

10 public export
11 (++) : PHOASList v a -> PHOASList v a -> PHOASList v a
12 (++) Nil xs = xs
13 (++) (x :: xs) ys = x :: xs ++ ys
14
15 ||| A proof that a given index is a valid index into a PHOASList
16 public export
17 data InPList : (k : Nat) -> (xs : PHOASList v a) -> Type where
18   InNow : InPList Z (x :: xs)
19   InAfter : InPList k xs -> InPList (S k) (x :: xs)
20
21 ||| An index function into a PHOASList
22 public export
23 index : (n : Nat) -> (xs : PHOASList v a) -> {auto prf: InPList n xs} -> a
24 index Z (x :: xs) = x
25 index (S k) (x :: xs) {prf = InAfter p} = index k xs
26
27 ||| Length defined for PHOASList
28 public export
29 length : PHOASList v a -> Nat
30 length [] = Z
31 length (x :: xs) = S (length xs)
32
33 ||| A non-empty proof for PHOASLists
34 public export
35 data NonEmptyPL : (xs : PHOASList v a) -> Type where
36   IsNonEmptyPL : NonEmptyPL (x :: xs)
37
38 ||| A proof that an empty non-empty proof is uninhabited for PHOASLists
39 public export
40 implementation Uninhabited (NonEmptyPL []) where
41   uninhabited IsNonEmptyPL impossible
42
43 ||| A Show implementation for PHOASLists
44 public export
45 implementation Show a => Show (PHOASList v a) where
46   show Nil = "[]"
47   show (x :: xs) = "[" ++ show x ++ showl xs
48   where
49     showl : Show a => PHOASList v a -> String
50     showl Nil = "]"
51     showl (y :: ys) = "," ++ show y ++ showl ys
52

```

```

53  ||| Semigroup implementation for PHOASLists
54  public export
55  implementation Semigroup (PHOASList v a) where
56    (<+>) = (++)
57
58  ||| Monoid implementation for PHOASLists
59  public export
60  implementation Monoid (PHOASList v a) where
61    neutral = []
62
63  ||| Foldable implementation for PHOASLists
64  public export
65  implementation Foldable (PHOASList v) where
66    foldr f acc [] = acc
67    foldr f acc (x :: xs) = f x (foldr f acc xs)
68
69    foldl f acc [] = acc
70    foldl f acc (x :: xs) = f (foldl f acc xs) x
71
72  ||| Functor implementation for PHOASLists
73  public export
74  implementation Functor (PHOASList v) where
75    map f [] = []
76    map f (x :: xs) = f x :: map f xs
77
78  ||| Applicative implementation for PHOASLists
79  public export
80  implementation Applicative (PHOASList v) where
81    pure x = [x]
82    fs <*> xs = concatMap (\f => map f xs) fs
83
84  ||| Monad implementation for PHOASLists
85  public export
86  implementation Monad (PHOASList v) where
87    xs >=> f = concatMap f xs

```

## A.6 NECESSARY PROOFS FOR CROSSOVER: PROOFS.IDR

```

1  import Data.Fin
2
3  import Mod.Mod
4  import Typeable.Typeable

```

```

5  import PHOASList.PHOASList
6
7  %default total
8
9  ||| A proof that if the run-time representation of two types are equal
10 ||| Then those types are constructively equal.
11 public export
12 typeRepInj : (prf : TypeRep a = TypeRep b) -> (a = b)
13 typeRepInj Refl = Refl
14
15 ||| Congruence on two equality proofs.
16 public export
17 cong2 : {f : Type -> Type -> Type} -> (a = c) -> (b = d) -> (f a b = f c d)
18 cong2 Refl Refl = Refl
19
20 ||| Specific two equality congruence proof for product types.
21 public export
22 cong2p : (a = c) -> (b = d) -> (a, b) = (c, d)
23 cong2p Refl Refl = Refl
24
25 ||| An equality of products implies a product of equalities.
26 public export
27 p2cong : (prf : (a, b) = (c, d)) -> (a = c, b = d)
28 p2cong Refl = (Refl, Refl)
29
30 ||| Specific two equality congruence proof for function types.
31 public export
32 cong2f : (a = c) -> (b = d) -> (a -> b) = (c -> d)
33 cong2f Refl Refl = Refl
34
35 ||| Proof of the injectivity of a boolean AND when True on pair.
36 public export
37 bandIsInjective : (x, y : Bool) -> (prf: (x && y) = True) -> (x = True, y = True)
38 bandIsInjective False False prf = absurd prf
39 bandIsInjective False True prf = absurd prf
40 bandIsInjective True False prf = absurd prf
41 bandIsInjective True True prf = (Refl, Refl)
42
43 ||| Proof that boolean equality between run-time type representations
44 ||| implies constructive equality between types.
45 public export
46 beqTypeReflectsEq : {a, b : Type} -> (x: TypeRep a) -> (y: TypeRep b) ->
47   (prf : beqType x y = True) -> (a = b)

```

```

48 beqTypeReflectsEq TBool TBool prf = Refl
49 beqTypeReflectsEq TBool TNat prf = absurd prf
50 beqTypeReflectsEq TBool TInteger prf = absurd prf
51 beqTypeReflectsEq TBool TString prf = absurd prf
52 beqTypeReflectsEq TBool (TFunc x y) prf = absurd prf
53 beqTypeReflectsEq TBool (TPair x y) prf = absurd prf
54 beqTypeReflectsEq TBool (TSum x y) prf = absurd prf
55 beqTypeReflectsEq TNat TBool prf = absurd prf
56 beqTypeReflectsEq TNat TNat prf = Refl
57 beqTypeReflectsEq TNat TInteger prf = absurd prf
58 beqTypeReflectsEq TNat TString prf = absurd prf
59 beqTypeReflectsEq TNat (TFunc x y) prf = absurd prf
60 beqTypeReflectsEq TNat (TPair x y) prf = absurd prf
61 beqTypeReflectsEq TNat (TSum x y) prf = absurd prf
62 beqTypeReflectsEq TInteger TBool prf = absurd prf
63 beqTypeReflectsEq TInteger TNat prf = absurd prf
64 beqTypeReflectsEq TInteger TInteger prf = Refl
65 beqTypeReflectsEq TInteger TString prf = absurd prf
66 beqTypeReflectsEq TInteger (TFunc x y) prf = absurd prf
67 beqTypeReflectsEq TInteger (TPair x y) prf = absurd prf
68 beqTypeReflectsEq TInteger (TSum x y) prf = absurd prf
69 beqTypeReflectsEq TString TBool prf = absurd prf
70 beqTypeReflectsEq TString TNat prf = absurd prf
71 beqTypeReflectsEq TString TInteger prf = absurd prf
72 beqTypeReflectsEq TString TString prf = Refl
73 beqTypeReflectsEq TString (TFunc x y) prf = absurd prf
74 beqTypeReflectsEq TString (TPair x y) prf = absurd prf
75 beqTypeReflectsEq TString (TSum x y) prf = absurd prf
76 beqTypeReflectsEq (TFunc x y) TBool prf = absurd prf
77 beqTypeReflectsEq (TFunc x y) TNat prf = absurd prf
78 beqTypeReflectsEq (TFunc x y) TInteger prf = absurd prf
79 beqTypeReflectsEq (TFunc x y) TString prf = absurd prf
80 beqTypeReflectsEq (TFunc x y) (TFunc z w) prf =
81   let p1 = bandIsInjective (beqType x z) (beqType y w) prf
82     rec1 = beqTypeReflectsEq x z (fst p1)
83     rec2 = beqTypeReflectsEq y w (snd p1)
84   in cong2f rec1 rec2
85 beqTypeReflectsEq (TFunc x y) (TPair z w) prf = absurd prf
86 beqTypeReflectsEq (TFunc x y) (TSum z w) prf = absurd prf
87 beqTypeReflectsEq (TPair x y) TBool prf = absurd prf
88 beqTypeReflectsEq (TPair x y) TNat prf = absurd prf
89 beqTypeReflectsEq (TPair x y) TInteger prf = absurd prf
90 beqTypeReflectsEq (TPair x y) TString prf = absurd prf

```

```

91 beqTypeReflectsEq (TPair x y) (TFunc z w) prf = absurd prf
92 beqTypeReflectsEq (TPair x y) (TPair z w) prf =
93   let p1 = bandIsInjective (beqType x z) (beqType y w) prf
94     rec1 = beqTypeReflectsEq x z (fst p1)
95     rec2 = beqTypeReflectsEq y w (snd p1)
96   in cong2 rec1 rec2
97 beqTypeReflectsEq (TPair x y) (TSum z w) prf = absurd prf
98 beqTypeReflectsEq (TSum x y) TBool prf = absurd prf
99 beqTypeReflectsEq (TSum x y) TNat prf = absurd prf
100 beqTypeReflectsEq (TSum x y) TInteger prf = absurd prf
101 beqTypeReflectsEq (TSum x y) TString prf = absurd prf
102 beqTypeReflectsEq (TSum x y) (TFunc z w) prf = absurd prf
103 beqTypeReflectsEq (TSum x y) (TPair z w) prf = absurd prf
104 beqTypeReflectsEq (TSum x y) (TSum z w) prf =
105   let p1 = bandIsInjective (beqType x z) (beqType y w) prf
106     rec1 = beqTypeReflectsEq x z (fst p1)
107     rec2 = beqTypeReflectsEq y w (snd p1)
108   in cong2 rec1 rec2
109
110 ||| Proof that x != 0 implies that x >= 0 forall natural numbers.
111 public export
112 notZimpliesGTZ : (x : Nat) -> (prf : Not (x = 0)) -> GT x Z
113 notZimpliesGTZ Z prf = void (prf Refl)
114 notZimpliesGTZ (S k) prf = LTESucc LTEZero
115
116 ||| Proof that x > 0 implies that x != 0 forall natural numbers.
117 public export
118 gTZimpliesZ : (x : Nat) -> (prf : GT x Z) -> Not (x = 0)
119 gTZimpliesZ Z prf = absurd prf
120 gTZimpliesZ (S k) prf = SIsNotZ
121
122 ||| Proof that for a given non-empty PHOASList, that the length
123 ||| of such a list must be greater than zero.
124 public export
125 nonEmptyImpliesGTZ : (xs : PHOASList v a) -> (prf : NonEmptyPL xs) ->
126   GT (length xs) Z
127 nonEmptyImpliesGTZ [] prf = absurd prf
128 nonEmptyImpliesGTZ (x :: xs) prf = LTESucc LTEZero
129
130
131 ||| Proof that for a given element and given PHOASList, the length of the list
132 ||| built from prepending the elemnt to the original list must be
133 ||| greater than zero.

```



```

134 consGTZ : (x : a) -> (xs : PHOASList v a) -> GT (length (x :: xs)) Z
135 consGTZ x [] = lteRefl
136 consGTZ x (y :: ys) = LTESucc LTEZero
137
138 ||| Proof that forall Natural numbers: 'y', given a list of length y: 'xs'
139 ||| and a proof that y is not zero. Then xs must not be empty.
140 lenGTZimpliesNonEmpty : (y : Nat) -> (xs : PHOASList v a) -> (prf : y = length xs) ->
141   (prf2 : GT y Z) -> NonEmptyPL xs
142 lenGTZimpliesNonEmpty Z xs prf prf2 = absurd prf2
143 lenGTZimpliesNonEmpty (S k) [] prf prf2 = absurd prf
144 lenGTZimpliesNonEmpty (S k) (x :: xs) prf prf2 = IsNonEmptyPL
145
146 ||| Proof that for all elements of a finite set 'f' with a supremum 'n'
147 ||| Any (successful) increments to the index of 'f' must be <= n.
148 public export
149 finLTBound : (f : Fin n) -> LTE (S (finToNat f)) n
150 finLTBound FZ = LTESucc LTEZero
151 finLTBound (FS x) =
152   let rec = finLTBound x
153   in LTESucc rec
154
155 ||| Proof that for all natural numbers: n given a PHOASList: xs
156 ||| and a proof that n < length xs then element at index n is
157 ||| guaranteed to be in the PHOASList.
158 public export
159 ltLenAlwaysBound : (n : Nat) -> (xs : PHOASList v a) ->
160   (prf : LTE (S n) (length xs)) -> InPList n xs
161 ltLenAlwaysBound Z [] prf = absurd prf
162 ltLenAlwaysBound Z (x :: xs) prf = InNow
163 ltLenAlwaysBound (S k) (x :: xs) prf =
164   let rec = ltLenAlwaysBound k xs
165   in InAfter (rec (fromLteSucc prf))
166
167 ||| proof that natural numbers x and y, given a proof that y > 0
168 ||| then mod x y + 1 <= y
169 public export
170 modfNlteN : (x, y : Nat) -> {auto prf : GT y Z} -> LTE (S (modfin x y)) y
171 modfNlteN x y = finLTBound (modf x y)

```

## A.7 VERIFIED TYPE-PRESERVING CROSSOVER: CROSSOVER.IDR

```

1  import Data.Fin
2
3  import Mod.Mod
4  import Typeable.Typeable
5  import Expr.Expr
6  import PHOASList.PHOASList
7  import Zipper.Zipper
8  import Proofs.Proofs
9
10 %default total
11
12 ||| Given a list of Sigma types of a pair of Zippers,
13 |||   and some (presumably random) natural number:
14 |||   select that index.
15 public export
16 selectNode : (n : Nat) -> (xs :
17     PHOASList v (a : Type ** (Zipper v a, Zipper v a))) ->
18     {auto prf : NonEmptyPL xs} ->
19     (a : Type ** (Zipper v a, Zipper v a))
20 selectNode n xs {prf} =
21     let p1 = nonEmptyImpliesGTZ xs prf
22         p2 = modfNlteN n (length xs) {prf = p1}
23         p3 = ltLenAlwaysBound (modfin n (length xs)) xs p2
24     in index (modfin n (length xs)) xs {prf=p3}
25
26 ||| typeRepEq takes a PHOASList of the cartesian product of all possible
27 ||| crossover pairs, and builds a PHOASList of only those pairs where crossover
28 ||| is possible.
29 public export
30 typeRepEq : (xs : PHOASList v
31     ((x : Type ** (TypeRep x, Zipper v x)),
32     (y : Type ** (TypeRep y, Zipper v y)))) ->
33     PHOASList v (a : Type ** (Zipper v a, Zipper v a))
34 typeRepEq xs = foldr f [] xs
35 where
36     f : {v : Type -> Type} ->
37         (xy : ((x : Type ** (TypeRep x, Zipper v x)),
38             (y : Type ** (TypeRep y, Zipper v y)))) ->
39         PHOASList v (a : Type ** (Zipper v a, Zipper v a)) ->
40         PHOASList v (a : Type ** (Zipper v a, Zipper v a))
41     f ((t1 ** (r1, z1)), (t2 ** (r2, z2))) xs with (beqType r1 r2) proof p

```

```

42     f ((t1 ** (r1, z1)), (t2 ** (r2, z2))) xs | True =
43         let p1 = beqTypeReflectsEq r1 r2 (sym p)
44             z3 = replace (sym p1) z2
45         in (_ ** (z1, z3)) :: xs
46     f ((t1 ** (r1, z1)), (t2 ** (r2, z2))) xs | False = xs
47
48     ||| A type preserving crossover function that is correct by construction
49     public export
50     xover : (zs : (a : Type ** (Zipper v a, Zipper v a))) ->
51         (Zipper v (fst zs), Zipper v (fst zs))
52     xover (t ** (Zip {z1} e1 c1, Zip e2 c2)) = (Zip {z1} e2 c1, Zip {z1} e1 c2)

```

## A.8 CROSSOVER RANDOM POSITIONS OF A GIVEN TYPE: MAIN.IDR

```

1  import Typeable.Typeable
2  import Mod.Mod
3  import Expr.Expr
4  import PHOASList.PHOASList
5  import Proofs.Proofs
6  import Zipper.Zipper
7  import Crossover.Crossover
8
9  import System
10
11  %default total
12
13  ex1 : Expr v Nat
14  ex1 = Add {t1=TNat} (Lit {t1=TNat} 2)
15         (Cnst {t1=TNat} {t2=TString}
16           (Lit {t1 = TNat} 3) (Lit {t1 = TString} "Test"))
17
18  ex2 : Expr v Nat
19  ex2 = Add {t1=TNat} (Lit {t1=TNat} 2)
20         (Cnst {t1=TNat} {t2=TString}
21           (Lit {t1 = TNat} 3) (Lit {t1 = TString} "Hello"))
22
23  zipExp1 : Zipper v Nat
24  zipExp1 = Zip {z1=TNat} ex1 Root
25
26  zipExp2 : Zipper v Nat
27  zipExp2 = Zip {z1=TNat} ex2 Root
28

```

```

29  ||| The cartesian product of possible focus pairs between zipExp1 and zipExp2
30  cartProdZipExp : PHOASList v ((x : Type ** (TypeRep x, Zipper v x)),
31                                (y : Type ** (TypeRep y, Zipper v y)))
32  cartProdZipExp = (\a, b => (a, b)) <$> (flatten zipExp1) <*> (flatten zipExp2)
33
34
35  ||| A control function that selects a crossover point and does not
36  ||| perform crossover
37  noOver : Nat -> ((a : Type ** Zipper v a), (b : Type ** Zipper v b))
38  noOver n = (top x, top y)
39  where
40      xys : (a : Type ** (Zipper v a, Zipper v a))
41      xys = selectNode n (typeRepEq cartProdZipExp)
42
43      x : (a : Type ** Zipper v a)
44      x = (DPair.fst xys ** (Basics.fst (DPair.snd xys)))
45
46      y : (a : Type ** Zipper v a)
47      y = (DPair.fst xys ** (Basics.snd (DPair.snd xys)))
48
49  ||| A function that selects the same point as 'noover' and performs the
50  ||| crossover operation
51  testOver : Nat -> ((a : Type ** Zipper v a), (b : Type ** Zipper v b))
52  testOver n =
53      let xys = xover $ selectNode n (typeRepEq cartProdZipExp)
54          x = (_ ** Basics.fst xys)
55          y = (_ ** Basics.snd xys)
56  in (top x, top y)
57
58  ||| A convenience function to read bytes from /dev/urandom.
59  getChars : File -> List (IO (Either FileError Char)) -> Nat ->
60              List (IO (Either FileError Char))
61  getChars f xs Z = fgetc f :: xs
62  getChars f xs (S k) = fgetc f :: getChars f xs k
63
64  main : IO ()
65  main = do
66      let path = "/dev/urandom"
67
68      Right (FHandle ptr) <- openFile path Read | Left err => do
69          putStrLn (show err)
70      cs <- sequence $ getChars (FHandle ptr) [] 4
71

```

```

72 Right str <- pure (sequence cs) | Left err => do
73   putStrLn (show err)
74
75 closeFile (FHandle ptr)
76 let ints = (\x => cast x {to=Int}) <$> str
77 (x :: xs) <- pure ints | [] => putStrLn "Error somehow we have an empty List"
78
79 let i = x
80 let n = cast i {to=Nat}
81 let xs' = (\x => cast x {to=Nat}) <$> ints
82 let s = show i
83 let control = noOver n {v=\x => x}
84 let xo = testOver n {v = \x => x}
85 let list = typeRepEq cartProdZipExp {v= (\x => String)}
86 let xolen = length list
87 let xomod = modfin n xolen
88 let printctrl = noOver n {v = \_ => String}
89 let printxo = testOver n {v = \_ => String}
90 let c1 = extract $ DPair.snd $ Basics.fst control
91 let p1 = extract $ DPair.snd $ Basics.fst printctrl
92 let c2 = extract $ DPair.snd $ Basics.snd control
93 let p2 = extract $ DPair.snd $ Basics.snd printctrl
94 let x1 = extract $ DPair.snd $ Basics.fst printxo
95 let x2 = extract $ DPair.snd $ Basics.snd printxo
96
97 putStrLn "-----GENERATING RANDOM NUMBER-----"
98 putStrLn ("Random number is: " ++ s)
99 putStrLn ("List of numbers: " ++ show (x :: xs))
100 putStrLn "-----PRINTING CONTROL EXPRESSIONS (No Crossover!)-----"
101 putStrLn (prettyE 0 p1)
102 putStrLn (prettyE 0 p2)
103 putStrLn "-----NUMBER OF VALID POSITIONS FOR CROSSOVER-----"
104 putStrLn (show xolen)
105 putStrLn "-----POSITION CHOSEN FROM VALID POSITIONS-----"
106 putStrLn (show xomod)
107 putStrLn "-----PRINTING CROSSED OVER EXPRESSION VALUES-----"
108 putStrLn (prettyE 0 x1)
109 putStrLn (prettyE 0 x2)
110 putStrLn "-----END PROGRAM-----"

```