Explicit Substitutions Part 1: STLC and Background

slides: https://github.com/donovancrichton/Talks

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The goal of this talk

- How to read (theory) syntax.
- STLC - Simply Typed Lambda Calculus.
- Substituions in anger - De-Bruijn Indicies.
- Explicit Subsitutions.
- Explicit Substitutions in modern type theory.

Reading Syntax: Grammar

Natural Numbers (Peano)

$$\mathbb{N} ::= Z \mid S\,\mathbb{N}$$

Untyped Lambda Caluclus

$$\begin{aligned} V &::= x, y, z, \ \ldots & \\ M, N &::= V & \text{Variable.} \\ &\mid M\,N & \text{Application.} \\ &\mid \lambda V.M & \text{Abstraction.} \end{aligned}$$

Reading Syntax: Grammar

Natural Numbers (Peano)

$$\mathbb{N} ::= \; Z \mid S\,\mathbb{N}$$

Reading Syntax: Grammar

Our <symbol> name...

Natural Numbers (Peano)

$$\boxed{\mathbb{N}} ::= Z \mid S\,\mathbb{N}$$

Reading Syntax: Grammar

...is defined in the following ways:

Natural Numbers (Peano)

$$\mathbb{N} \boxed{::=} Z \mid S\,\mathbb{N}$$

Reading Syntax: Grammar

The letter Z by itself.

Natural Numbers (Peano)

$$\mathbb{N} ::= \boxed{Z} \mid S\,\mathbb{N}$$

Reading Syntax: Grammar

...or...

Natural Numbers (Peano)

$$\mathbb{N} ::= \; Z \boxed{\,|\,} S\,\mathbb{N}$$

Reading Syntax: Grammar

The letter S followed by a space, followed by any $\mathbb{N}$.

Natural Numbers (Peano)

$$\mathbb{N} ::= \; Z \mid \boxed{S\,\mathbb{N}}$$

Reading Syntax: Grammar

Our set of expressions/terms called V...

Untyped Lambda Caluclus

$$
\begin{aligned}
\boxed{V} &::= x, y, z, \ \ldots \\
M, N &::= V && \text{Variable.} \\
&\mid M\ N && \text{Application.} \\
&\mid \lambda V.M && \text{Abstraction.}
\end{aligned}
$$

Reading Syntax: Grammar

is given, or defined by:

Untyped Lambda Caluclus

$$
\begin{aligned}
V \boxed{::=}\ & x, y, z,\ \ldots \\
M, N ::=\ & V && \text{Variable.} \\
\mid\ & M\ N && \text{Application.} \\
\mid\ & \lambda V.M && \text{Abstraction.}
\end{aligned}
$$

Reading Syntax: Grammar

'x', 'y', 'z', or *any other lower case letter* (lower case words also implied)

Untyped Lambda Caluclus

$$
\begin{aligned}
V &::= \boxed{x, y, z, \ \ldots} \\
M, N &::= V && \text{Variable.} \\
&\mid M\ N && \text{Application.} \\
&\mid \lambda V.M && \text{Abstraction.}
\end{aligned}
$$

Reading Syntax: Grammar

Our lambda terms, denoted by N or M
(other capital letters implied).

Untyped Lambda Caluclus

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V &::= x, y, z, \ \dots \\
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Reading Syntax: Grammar

are given by:

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Reading Syntax: Grammar

A V...

Untyped Lambda Caluclus

$$
\begin{aligned}
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M, N &::= \; \boxed{V} && \text{Variable.} \\
&\quad\; \mid M\ N && \text{Application.} \\
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$$

Reading Syntax: Grammar

…or,

Untyped Lambda Caluclus

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& \;\; \boxed{\mid} \; M\,N && \text{Application.} \\
& \;\; \mid \; \lambda V.M && \text{Abstraction.}
\end{aligned}
$$

Reading Syntax: Grammar

A lambda term (M), followed by a space, followed by another lambda term (N).

Untyped Lambda Caluclus

$$
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V &::= x, y, z, \ \dots \\
M, N &::= V && \text{Variable.} \\
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Reading Syntax: Grammar

Or,

Untyped Lambda Caluclus

$$
\begin{aligned}
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M, N &::= \; V && \text{Variable.} \\
&\quad \mid M \; N && \text{Application.} \\
&\quad \boxed{\mid} \; \lambda V.M && \text{Abstraction.}
\end{aligned}
$$

Reading Syntax: Grammar

The λ symbol, followed by a V element,
followed by a ".", followed by a lambda term (M)

Untyped Lambda Caluclus

$$
\begin{aligned}
V &::= x, y, z, \ \ldots & \\
M, N &::= V & \text{Variable.} \\
&\mid M\ N & \text{Application.} \\
&\mid \boxed{\lambda V.M} & \text{Abstraction.}
\end{aligned}
$$

Why does Grammar look like this?

Untyped Lambda Caluclus

$$
\begin{aligned}
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$$

- The smallest possible definition.

Why does Grammar look like this?

Untyped Lambda Caluclus

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\end{aligned}
$$

- The smallest possible definition.
- Can be used to generate arbitrary elements.

What about in programming?

Natural Numbers (Peano)

```haskell
data Nat = Z | S Nat

zero :: Nat
zero = Z

one :: Nat
one = S Z

two :: Nat
two = S (S Z)

three :: Nat
three = S two
```

What about in programming?

Untyped Lambda Caluclus (Idris)

```
V : Type
V = String

data Λ = Var V
       | App Λ Λ
       | Abs V Λ

id : Λ
id = Abs "x" (Var "x")

const : Λ
const = Abs "a" (Abs "b" (Var "a"))
```

Church Encoding - Naturals and Booleans

Church Booleans

$$\text{True} = \lambda a.\lambda b.a$$
$$\text{False} = \lambda a.\lambda b.b$$

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Church Booleans

$$\text{True} = \lambda a.\lambda b.a$$
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Church Naturals

$$0 = \lambda f.\lambda x.x$$
$$1 = \lambda f.\lambda x.f\,x$$
$$2 = \lambda f.\lambda x.f\,(f\,x)$$
$$\vdots$$

Church Encoding - Functions and Predicates

We use our definitions of True and False from earlier.

Functions and Predicates

$$\begin{aligned} \text{Succ}(n, f, x) &= \lambda n.\lambda f.\lambda x.\lambda f\,(n\ f\ x) \\ \text{Add}(m, n, f, x) &= \lambda m.\lambda n.\lambda f.\lambda x.m\ f\,(n\ f\ x) \\ \text{IsZero}(n) &= \lambda n.n\ (\lambda x.\text{False})\ \text{True} \end{aligned}$$

Church Encoding - Functions and Predicates

As homework, trace through IsZero and convince yourself that IsZero 0 returns True, and IsZero for any other number returns false.

Functions and Predicates

$$\begin{aligned}
\text{Succ}(n, f, x) &= \lambda n.\lambda f.\lambda x.\lambda f\,(n\ f\ x) \\
\text{Add}(m, n, f, x) &= \lambda m.\lambda n.\lambda f.\lambda x.m\ f\,(n\ f\ x) \\
\text{IsZero}(n) &= \lambda n.n\ (\lambda x.\lambda a.\lambda b.b)\ \lambda a.\lambda b.a
\end{aligned}$$

Reading Syntax: Typing Rules

Is this scary?

Typing Rules

$$\frac{\Gamma \vdash A \ \ \text{Type} \qquad \Gamma, x : A \vdash B \ \ \text{Type}}{\Gamma \vdash A \to B \ \ \text{Type}}(\mathit{Ty\text{-}Arrow})$$

$$\frac{\Gamma \vdash A \to B \ \text{Type} \qquad \Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda(x : A).M : A \to B}(\mathit{Func})$$

$$\frac{\Gamma \vdash A \to B \ \text{Type} \qquad \Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\, N : B}(\mathit{App})$$

The Program Context

The current scope of our program.

The Program Context

$$\Gamma, \Delta ::= \diamond \mid \Gamma, x : A$$

The Program Context

Our context called Γ or Δ, etc...

The Program Context

$$\boxed{\Gamma, \Delta} ::= \diamond \mid \Gamma, x : A$$

The Program Context

Is defined by:

The Program Context

$$\Gamma, \Delta \boxed{::=} \diamond \mid \Gamma, x : A$$

The Program Context

Empty (like the empty list) represented by $\diamond$.

The Program Context

$$\Gamma, \Delta ::= \boxed{\diamond} \mid \Gamma, x : A$$

The Program Context

Or,

The Program Context

$$\Gamma, \Delta ::= \diamond \boxed{\mid} \Gamma, x : A$$

The Program Context

Our program scope, extended with a variable 'x' of type 'A'.

The Program Context

$$\Gamma, \Delta ::= \diamond \mid \boxed{\Gamma, x : A}$$

Example Contexts

What is the context of this program?

```
-- assuming   empty context here.
-- assuming A → B : Type here.
data Bool = True | False

not : Bool -> Bool
not True  = False
not False = True
```

Example Contexts

What is the context of this program?

```
-- assuming   empty context here.
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data Bool = True | False

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```

Answer.

$$\Gamma = ???$$

Example Contexts

What is the context of this program?

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```

Answer.

$$\Gamma = \diamond, \ldots$$

Example Contexts

What is the context of this program?

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-- assuming   empty context here.
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data Bool = True | False

not : Bool -> Bool
not True  = False
not False = True
```

Answer.

$$\Gamma = \diamond, A \to B : \text{Type}, \ldots$$

Example Contexts

What is the context of this program?

```
-- assuming   empty context here.
-- assuming A → B : Type here.
data Bool = True | False

not : Bool -> Bool
not True  = False
not False = True
```

Answer.

$$\Gamma = \diamond, A \to B : \text{Type}, \text{Bool} : \text{Type}, \text{True} : \text{Bool}, \text{False} : \text{Bool}, \ldots$$

Example Contexts

What is the context of this program?

```
-- assuming   empty context here.
-- assuming A → B : Type here.
data Bool = True | False

not : Bool -> Bool
not True  = False
not False = True
```

Answer.

$$\Gamma = \diamond, A \rightarrow B : \text{Type}, \text{Bool} : \text{Type}, \text{True} : \text{Bool}, \text{False} : \text{Bool}, \text{not} : \text{Bool} \rightarrow \text{Bool}$$

The Function Type Formation Rule

If the derivations (true statements) hold above the line (premise), then the derivations hold below the line (conclusion).

The Type Formation Rule

$$\frac{\qquad}{\qquad}\,(\textit{Ty-Arrow})$$

The Function Type Formation Rule

...from an arbitrary context, Γ

The Type Formation Rule

$$\frac{\boxed{\Gamma}}{}\;(\textit{Ty-Arrow})$$

The Function Type Formation Rule

...we may derive (produce, obtain...)

The Type Formation Rule

$$\frac{\Gamma\boxed{\vdash}}{}(\textit{Ty-Arrow})$$

The Function Type Formation Rule

...an arbitrary type, called A

The Type Formation Rule

$$\frac{\Gamma \vdash \boxed{A \text{ Type}}}{\phantom{\Gamma \vdash A \text{ Type}}}\,(\mathit{Ty\text{-}Arrow})$$

The Function Type Formation Rule

AND

The Type Formation Rule

$$\frac{\Gamma \vdash A \text{ Type} \quad \boxed{}}{}\,(\textit{Ty-Arrow})$$

The Function Type Formation Rule

...From the same arbitrary context Γ, extended with a variable 'x' of type A

The Type Formation Rule

$$\frac{\Gamma \vdash A \text{ Type} \qquad \boxed{\Gamma, x : A}}{} (\textit{Ty-Arrow})$$

The Function Type Formation Rule

...we may derive

The Type Formation Rule

$$\frac{\Gamma \vdash A \text{ Type} \qquad \Gamma, x : A \boxed{\vdash}}{}\,(\mathit{Ty\text{-}Arrow})$$

The Function Type Formation Rule

some arbitrary type B.

The Type Formation Rule

$$\frac{\Gamma \vdash A \ \mathsf{Type} \qquad \Gamma, x : A \vdash \boxed{B \ \mathsf{Type}}}{}\,(\textit{Ty-Arrow})$$

The Function Type Formation Rule

Then, from that arbitrary Γ (program scope)

The Type Formation Rule

$$\frac{\Gamma \vdash A \text{ Type} \qquad \Gamma, x : A \vdash B \text{ Type}}{\boxed{\Gamma}} \, (\textit{Ty-Arrow})$$

The Function Type Formation Rule

...we may derive

The Type Formation Rule

$$\frac{\Gamma \vdash A \;\mathsf{Type} \qquad \Gamma, x : A \vdash B \;\mathsf{Type}}{\Gamma \boxed{\vdash}} (\mathit{Ty\text{-}Arrow})$$

The Function Type Formation Rule

...an arrow type between them

The Type Formation Rule

$$\frac{\Gamma \vdash A \ \mathsf{Type} \qquad \Gamma, x : A \vdash B \ \mathsf{Type}}{\Gamma \vdash \boxed{A \to B}}(\textit{Ty-Arrow})$$

The Function Type Formation Rule

...that is also a type.

The Type Formation Rule

$$\frac{\Gamma \vdash A \text{ Type} \qquad \Gamma, x : A \vdash B \text{ Type}}{\Gamma \vdash A \to B \boxed{\text{ Type}}}(\textit{Ty-Arrow})$$

The Function Term Formation Rule

If the premises hold above, so the conclusion holds below.

The Function Term Former

$$\frac{\qquad}{\qquad}(Abs)$$

The Function Term Formation Rule

From our program scope Γ we can obtain an arbitrary function type $A \to B$.

The Function Term Former

$$\frac{\boxed{\Gamma \vdash A \to B \text{ Type}}}{}\,(Abs)$$

The Function Term Formation Rule

..And..

The Function Term Former

$$\frac{\Gamma \vdash A \to B \text{ Type} \quad \boxed{}}{} (Abs)$$

The Function Term Formation Rule

From our program scope $\Gamma, x : A$ we can derive a term $M : B$.

The Function Term Former

$$\frac{\Gamma \vdash A \to B \text{ Type} \qquad \boxed{\Gamma, x : A \vdash M : B}}{}(Abs)$$

The Function Term Formation Rule

Then we may derive a lambda abstraction term, where x has type A and the body is M, where the entire abstraction has type $A \to B$.

The Function Term Former

$$\frac{\Gamma \vdash A \to B \text{ Type} \qquad \Gamma, x : A \vdash M : B}{\boxed{\Gamma \vdash \lambda(x : A).M : A \to B}}(Abs)$$

The Function Term Elimination Rule

If the premises hold above, so the conclusion holds below.

The Function Term Eliminator

$$\frac{\qquad}{\qquad}(App)$$

The Function Term Elimination Rule

From our program scope Γ we can obtain some term M with type $A \to B$.

The Function Term Eliminator

$$\frac{\boxed{\Gamma \vdash M : A \to B}}{}\,(App)$$

The Function Term Elimination Rule

And, from Γ we can also obtain a term N of type A.

The Function Term Eliminator

$$\frac{\Gamma \vdash M : A \to B \qquad \boxed{\Gamma \vdash N : A}}{}(App)$$

The Function Term Elimination Rule

Then we can produce a term $M\,N$ of type B.

The Function Term Eliminator

$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\boxed{\Gamma \vdash M\,N : B}}(App)$$

STLC: Grammar

The Simply Typed Lambda Calculus: Grammar

The Simply Typed Lambda Calculus: Grammar

$V ::= \ x, y, z, \ \dots$ Variable Set

STLC: Grammar

The Simply Typed Lambda Calculus: Grammar

$$
\begin{aligned}
V &::= \; x, y, z, \; \ldots && \text{Variable Set} \\
\alpha, \beta &::= \; * \mid \alpha \to \beta && \text{Types}
\end{aligned}
$$

STLC: Grammar

The Simply Typed Lambda Calculus: Grammar

$$V ::= \; x, y, z, \; \ldots \qquad \text{Variable Set}$$

$$\alpha, \beta ::= \; * \mid \alpha \to \beta \qquad \text{Types}$$

$$M, N ::= V \mid M\, N \mid \lambda(V : \alpha).M \qquad \text{Terms}$$

STLC: Grammar

The Simply Typed Lambda Calculus: Grammar

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\begin{aligned}
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\Gamma &::= \; \diamond \mid \Gamma, x : \alpha && \text{Contexts}
\end{aligned}
$$

STLC: Grammar

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\end{aligned}
$$

Why $x : \alpha$ in the Contexts set, and not $V : \alpha$?

STLC: Grammar

The Simply Typed Lambda Calculus: Grammar

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\begin{aligned}
V &::= x, y, z, \; ... && \text{Variable Set} \\
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M, N &::= V \mid M\,N \mid \lambda(V : \alpha).M && \text{Terms} \\
\Gamma &::= \diamond \mid \Gamma, x : \alpha && \text{Contexts}
\end{aligned}
$$

Why $x : \alpha$ in the Contexts set, and not $V : \alpha$?

V is the variable set in our *object* language, meaning it is part of the syntax of the language we are defining.
x represents a variable in our meta language - the language we use to specify and reason about our object language.

STLC: Typing Rules/Judgements

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$$\frac{\Gamma \vdash \alpha \text{ Type} \qquad \Gamma, x : \alpha \vdash \beta \text{ Type}}{\Gamma \vdash \alpha \to \beta \text{ Type}}(\to\textit{-Form})$$

STLC: Typing Rules/Judgements

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$$\frac{\Gamma \vdash \alpha \text{ Type} \qquad \Gamma, x : \alpha \vdash \beta \text{ Type}}{\Gamma \vdash \alpha \to \beta \text{ Type}} (\to\textit{-Form})$$

$$\frac{\Gamma \vdash \mathit{alpha} \to \beta \text{ Type} \qquad \Gamma, x : \alpha \vdash M : \beta}{\Gamma \vdash \lambda(x : \alpha).M : \alpha \to \beta} (\to\textit{-Intro})$$

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$$\frac{\Gamma \vdash \alpha \to \beta \ \mathsf{Type} \qquad \Gamma \vdash M : \alpha \to \beta \qquad \Gamma \vdash N : \alpha}{\Gamma \vdash M\, N : \beta} (\to\textit{-Elim})$$

Free Variables

FV

A variable is x considered *free* in a lambda term if it refers to a binding outside the *scope* of the current lambda term. Defined inductively.

Free Variables

The set *FV* of free variables.

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$$\begin{aligned} FV(V) &= V \\ FV(M\ N) &= FV(M) \cup FV(N) \\ FV(\lambda(V : \alpha).M) &= FV(M) \backslash V \end{aligned}$$

Free Variables

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\begin{aligned}
FV(V) &= V \\
FV(M\,N) &= FV(M) \cup FV(N) \\
FV(\lambda(V:\alpha).M) &= FV(M) \backslash V
\end{aligned}
$$

Examples

$$
\begin{array}{ll}
\lambda(x:\alpha \to \beta).\lambda(y:\beta).x\,y & FV = \emptyset \\
\lambda(x:\alpha \to \beta).\boxed{\lambda(y:\beta).x\,y} & \mathrm{FV} = \{x\}
\end{array}
$$

α-equivalence/conversion

$\lambda(x : \beta).x = \lambda(y : \beta).y$

Two lambda terms are said to be α-*equivalent* if we can rename bound variables and still have the same term. We may α-convert or α-rename a lambda term into another α-equivalence term.

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Examples of α-equivalence.

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Examples of α-equivalence.

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α-equivalence/conversion

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Two lambda terms are said to be α-*equivalent* if we can rename bound variables and still have the same term. We may α-convert or α-rename a lambda term into another α-equivalence term.

Examples of α-equivalence, and non-equivalence.

$$\lambda(x : \beta).x = \lambda(y : \beta).y$$
$$\lambda(x : \beta).\lambda(x : \beta).x = \lambda(y : \beta).\lambda(x : \beta).x$$
$$\lambda(x : \beta).\lambda(x : \beta).x \neq \lambda(y : \beta).\lambda(x : \beta).y$$

Substitution

$M[x := N]$ or $M[N/x]$ or $M[x \mapsto N]$

Substitution is a *meta* operation that allows us to perform computation with lambda calculus. We may replace (almost) any variable x that occurs in term M with term N.

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Example Substitutions

$$(\lambda(x : \mathbb{N}).2 + x)[3/x] = \lambda(x : \mathbb{N}).2 + 3$$

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Example Substitutions

$$(\lambda(x : \mathbb{N}).2 + x)[3/x] = \lambda(x : \mathbb{N}).2 + 3$$
$$(\lambda(x : \beta \to \beta).x)[\lambda(y : \beta).y/x] = \lambda(x : \beta \to \beta).\lambda(y : \beta).y$$

Substitution

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Substitution is a *meta* operation that allows us to perform computation with lambda calculus. We may replace (almost) any variable x that occurs in term M with term N.

Example Substitutions

$$(\lambda(x : \mathbb{N}).2 + x)[3/x] = \lambda(x : \mathbb{N}).2 + 3$$

$$(\lambda(x : \beta \to \beta).x)[\lambda(y : \beta).y/x] = \lambda(x : \beta \to \beta).\lambda(y : \beta).y$$

$$(\lambda(x : \beta).y)[x/y] \neq \lambda(x : \beta).x$$

Capture Avoiding Substituion

Rule 1. $x[T/x] = T$

When substituting into a variable, we just have the term.

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Rule 2. $y[T/x] = y$ if $x \neq y$

Substituting by a different variable does nothing.

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Rule 3. $(M\ N)[T/x] = M[T/x]\ N[T/x]$

Substitution distributes over application.

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Substituting by a different variable does nothing.

Rule 3. $(M\ N)[T/x] = M[T/x]\ N[T/x]$

Substitution distributes over application.

Rule 4. $(\lambda(x : \alpha).M)[T/x] = \lambda(x : \alpha).M$

Abstraction substitution is neutral for the binding variable.

Capture Avoiding Substituion

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Rule 2. $y[T/x] = y$ if $x \neq y$

Substituting by a different variable does nothing.

Rule 3. $(M\ N)[T/x] = M[T/x]\ N[T/x]$

Substitution distributes over application.

Rule 4. $(\lambda(x : \alpha).M)[T/x] = \lambda(x : \alpha).M$

Abstraction substitution is neutral for the binding variable.

Rule 5. $(\lambda(y : \alpha).M)[T/x] = \lambda(y : \alpha).M[T/x]$ if:
$x \neq y$ and $y \notin FV(T)$

Substitution is conditional through an abstraction body.

Computation

β-reduction: $(\lambda(x : \alpha).M)\ N \to M[N/x]$

β-reduction is the essence of computation for the STLC.
If we see an application after an abstraction, we may perform the given substitution then strip off the lambda binder.

Computation

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Example of *beta*-reduction.

$$
\begin{aligned}
(\lambda(x:\alpha).x)\ y &\to \lambda(x:\alpha).x[y/x] && \text{replace abs-app with substitution.}\\
&\to \lambda(x:\alpha).y && \text{perform the substitution.}\\
&\to y && \text{eliminate the relevant binder.}
\end{aligned}
$$

De Bruijn Indices: (1972) Lambda Calculus Notation with Nameless Dummies ...

Nicolaas Govert
De Bruijn

De-Bruijn Indices

Replacing letters with numbers (why?)

Instead of using variable names to denote variable terms, we use numbers to denote a distance from the relevant binder. Best shown by example.

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$$\lambda(x : \alpha).\lambda(y : \alpha).\lambda(z : \alpha).y\,(x\,z) = \lambda\alpha.\lambda\alpha.\lambda\alpha.2\;(3\;1)$$

$$\lambda(x : \alpha).x = \lambda\alpha.1$$

$$\lambda(y : \alpha).y = \lambda\alpha.1$$

Lift-and-shift

Substitution: Given $(\lambda\alpha.M)\ N$

1. Find the instances of the binding sites in $\lambda\alpha.M$.
2. Decrement the free variables in M to account for the removal of the outer binder.
3. Replace the binding sites with N but increment any necessary free variables so they occur under the appropriate number of binders.

(I think we need an example).

Lift-and-shift

1. Identify binding sites in the λ abstraction.

$\lambda\alpha.\lambda\beta.2\ 3))\ (\lambda\alpha.1\ 4) \rightarrow \lambda\alpha.\lambda\beta.\Box\ 3$

2. Decrement free variables that are affected by the removal of the outer binder.

$\lambda\alpha.\lambda\beta.\Box\ 3 \rightarrow \lambda\beta.\Box\ 2$

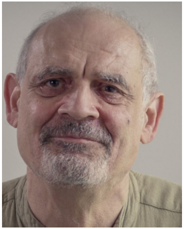
3. Increment the free variables in the applied term to match the number of binders.

$\lambda\beta.\Box\ 2 \rightarrow \lambda\beta.(\lambda\alpha.1\ \boxed{4+1})\ 2$
$\lambda\beta.\Box\ 2 \rightarrow \lambda\beta.(\lambda\alpha.1\ 5)\ 2$

Explicit Substitutions: (1991) Explicit Substitutions

Martin Abadi

Luca Cardelli



Pierre-Louis Curien

Jean-Jacques Levy

Extending the STLC with Explicit Substitutions

STLC with Explicit Substitutions: Grammar

$$\alpha, \beta ::= \ * \mid \alpha \to \beta \qquad \text{Types}$$

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Extending the STLC with Explicit Substitutions

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$$\begin{aligned} \alpha, \beta &::= \; * \mid \alpha \to \beta && \text{Types} \\ M, N &::= 1 \mid M\,N \mid \lambda\alpha.M \mid M[\gamma] && \text{Terms} \\ \gamma, \delta &::= id \mid \uparrow \mid \gamma, M : \alpha \mid \gamma \circ \delta && \text{Substitutions} \end{aligned}$$

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STLC with Explicit Substitutions: Grammar

$\alpha, \beta ::= \ * \mid \alpha \to \beta$	Types
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$\gamma, \delta ::= id \mid \uparrow \mid \gamma, M : \alpha \mid \gamma \circ \delta$	Substitutions
$\Gamma ::= \ \diamond \mid \Gamma, \alpha$	Contexts

Additional Terms

$1 \in$ Terms

1 is De Bruijn index 1, the variable pointing to the most recent binder.
This replaces our need for a variable term in our lambda calculus.

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This replaces our need for a variable term in our lambda calculus.

$M[\gamma] \in$ Terms

A substitution of γ in M because *gamma* tells us what and where.

Explicit Substitutions