Enforcing a discipline of Total Functional Programming through Dependent Types

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Preliminaries

- Slides and Examples available at:
  https://github.com/donovancrichton/Talks
- This talk: BFPG/TotalFPThroughDepTypes

About me

- PhD Candidate
- Computing Foundations
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- Software Engineer

Alex (I): A Lexer Generator Library (GHC)

Character Sets and Macros

```
-- character sets
$digit  = 0-9
$small  = a-z
$big    = A-Z
$dash   = \-
$gt     = \>
$prime  = \'
$uscore = \_
$lambda = \\
$idchar = [$small $big $digit $prime $uscore]
-- character set macros
@bigid   = big idchar*
@smallid = small idchar*
@arrow   = $dash $gt
```

Alex (II): A Lexer Generator Library (GHC)

Regex Rules and Tokens

```
-- <state>     <regex>    {<func>}
   <0>         @arrow     {tokArrow}
   <0>         @bigid     {tokBigId}
   <0>         @smallid   {tokSmallId}
   <0>         $lambda    {tokLambda}
{
  data Token = TSmallId | TBigId
    | TArrow | TLambda

  tokLambda :: input -> Alex Token
  tokLambda input = pure TLambda

  tokArrow :: input -> Alex Token
  tokArrow input = pure TArrow
}
```

Alex (III): A Lexer Generator Library (GHC)

Output Token List

```haskell
  lex "\foo -> \bar -> \baz -> Baz foo bar"
-- gives us something like:
  [TLambda, TSmallId, TArrow,
   TLambda, TSmallId, TArrow,
   TLambda, TSmallId, TArrow,
   TBigId, TSmallId, TSmallId]
```

Happy (I): A Parser Generator Library (GHC)

Token Directive: Mirrors the Lexer

```
{
  import qualified Lexer as LEX
  import qualified ParseTree as AST
}
%token
  smallIdent {LEX.TSmallId}
  bigIdent   {LEX.TBigId}
  arrow      {LEX.TArrow}
  lambda     {LEX.TLambda}
```

Happy (II): A Parser Generator Library (GHC)

Production Rules and Sadness

```
Term :: {AST.Term}
  : smallIdent                   {parseVar $1}
  | bigIdent                     {parseDataCon $1}
  | lambda smallIdent
      arrow Term                 {parseLambda $2 $4}
  | Term Term                    {parseApp $1 $2}
{
  parseVar :: LEX.Token -> AST.Term
  parseVar (TSmallId) = MkSmallRef
  parseVar _          = ?whatgoeshere

  parseDataCon :: LEX.Token -> AST.Term
  parseDataCon (TBigId) = MkDataCon
  parseDataCon _        = ?whatabouthere?
}
```

Awkward Pattern-Matches, A type is too large.

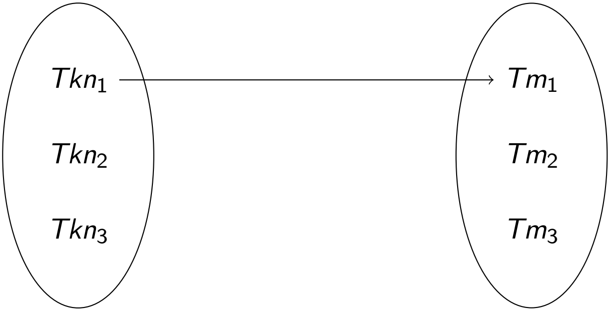
A quick note on the Algebra of Types.

Types

$Unit = \{*\}$
$Bool = \{True, False\}$
$Pair(A, B) = A \times B$
$Either(A, B) = A \sqcup B$
$Maybe(A) = \{*\} \sqcup A$
$A \to B = A \mapsto B$

Cardinality

$|Unit| = 1$
$|Bool| = 2$
$|Pair(A, B)| = |A| \times |B|$
$|Either(A, B)| = |A| + |B|$
$|Maybe(A)| = 1 + |A|$
$|A \to B| = |B|^{|A|}$

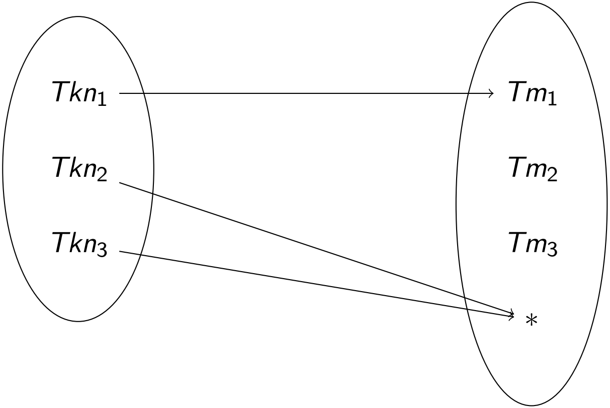


Overview of Totality

Totality

A function is total if it is *defined over all inputs*, and guaranteed to terminate.

```haskell
f :: Token -> Maybe Term
```



Why do we want totality?

Compositionality

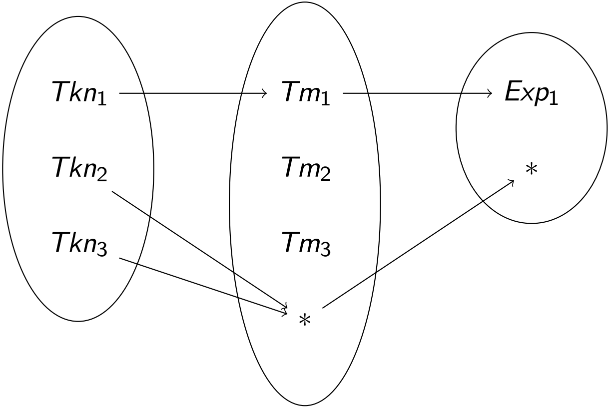
John Hughes (1978) "Why Functional Programming Matters": Argues that compositionality is the backbone of modular programming.
Partial functions do not compose.

Larger co-domains

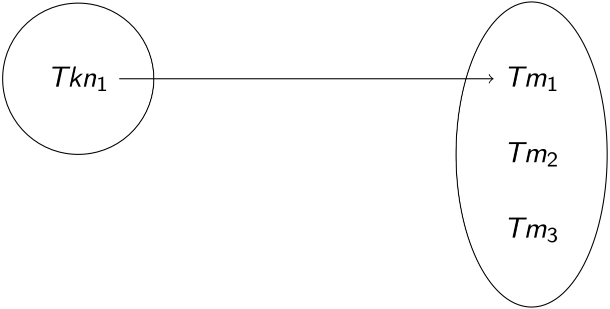
We can regain totality by increasing the size of our co-domains, i.e with a Maybe type, but now we can only compose with functions that take Maybe types. This leads to a lot of unnecessary extending of domains.

An (unsatisfying) example of extending the co-domain

```
f :: Token -> Maybe Term
g :: Maybe Term -> Maybe Exp
```



Ideally, we'd like to restrict the domain.



Fortunately, we can do this via dependent types.

Indexing our valid tokens by tokens.

Token and ValidToken Types

```
data Token =
  TBigId
  | TSmallId
  | TArrow
  | TLambda
  | TNull

data ValidToken : Token -> Type where
  VTNull     : ValidToken TNull
  VTBigId    : ValidToken TBigId
  VTSmallId  : ValidToken TSmallId
  VTArrow    : ValidToken TArrow
  VTLambda   : ValidToken TLambda
```

Indexing our valid tokens by tokens.

Token and ValidToken Types

```
-- null Token to satisfy totality checker.
total
match : String -> (a : Token ** ValidToken a)
match "foo" = (_ ** VTSmallId)
match "bar" = (_ ** VTSmallId)
match "baz" = (_ ** VTSmallId)
match "Foo" = (_ ** VTBigId)
match "=>"  = (_ ** VTArrow)
match "\\"  = (_ ** VTLambda)
match _     = (_ ** VTNull)
```

An introduction to dependent pairs.

Dependant Pairs (Σ types) in code.

```
data DPair : (a : Type) -> (a -> Type) -> Type
  MkDPair : {p : a -> Type}
         -> (fst : a) -> p fst -> DPair a p

fst : DPair a p -> a
fst (MkDPair x prf) = x

snd : {p : a -> Type}
   -> (rec : DPair a p) -> p (fst rec)
snd (MkDPair x prf) = prf
```

Using dependent pairs and GADTs to restrict our input domains.

Total Parsing with no maybes or errors

```
total
parseSmallId : ValidToken TSmallId -> Term
parseSmallId VTSmallId = MkRef

total
example : Term
example = parseSmallId (DPair.snd (match "foo"))
```

What about compositionality?

Dependent Function Composition

```
cm : {a : Type}
 -> {b : a -> Type}
 -> {c : {x : a} -> b x -> Type}
 -> ({x : a} -> (y : b x) -> c y)
 -> (g : (x : a) -> b x)
 -> ((x : a) -> c (g x))
cm f g = \x => f (g x)

example2 : (x : String)
        -> ValidToken (DPair.fst (match x))
example2 = (DPair.snd {p = ValidToken} `cm` match)
```

Conclusions

Dependent Types can restrict our input domains.

We saw how, using dependant pairs and GADTs, we may restrict our input domains to be precisely the type we actually care about, avoiding the need to wrap types in superfluous Eithers and Maybes.

Still not really compositional

When types dependent on one-another it can be difficult to compose them, even with dependent composition, due to the dependency on the initial input. This suggests that dependent types are less modular than 'plain' types (according to Hughes) however there are other practical advantages to dependent types.