CSC 565: Graph Theory
Fall 2019

North Carolina State University
Computer Science

Lecture 28: Dec 2, 2019

Lecturer: Don Sheehy <drsheehy@ncsu.edu> *Scribe: Hongyi Fan, Guangyu Yu*

1 Decision problems

Decision problems: input: number output: accept/reject

Not all decision problems are decidable. Halting problem is the problem of determining, from a description

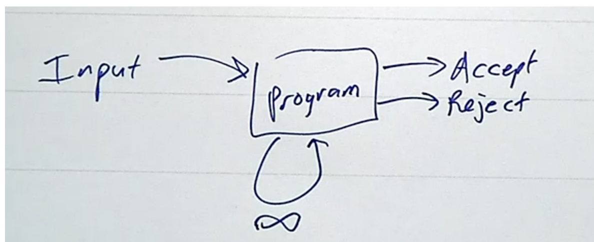

Figure 1: Decision problem

of an arbitrary computer program and an input, whether the program will finish running, or continue to run forever. Programs are stored as data, or string of bits, or just number.

Cantor's Diagonalization: a mathematical proof that there are infinite sets which cannot be put into one-to-one correspondence with the infinite set of natural numbers.

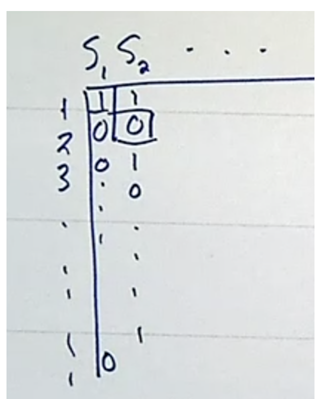

Figure 2: Cantor's Diagonalization

$S_* = \{i : i \not\subseteq S_i\}$

Suppose $S_* = S_i$ for some i
i$\subseteq$S$_*$ iff i$\not\subseteq$S$_*$ $\Rightarrow\perp$

Diagonal Language: D = {P|P(<P>) does not accept }

Theorem 1. *D is undecidable.*

Proof. Suppose D is decidable.
Let Q be a program that decides D.
Q $\in D$?
$if\ Q(<Q>)$ accepts then Q $\notin D$
$if\ Q(<Q>)$ rejects then Q $\in D$ □

Reduction
A reduces to B: A is not harder than B, so we can use a solution to B to solve A.
Suppose A is impossible then B is impossible.

Halt = $\{<P, w> | P(w)\ halts\ \}$

Theorem 2. *halt is undecidable.*

Proof. Suppose Q decides halt.
program: R(P):
Run Q(P, <P>)
if it accepts run p(<P>) and return opposite of P(<P>)
Else reject
R decides Diag. $\Rightarrow\perp$ □

2 A problem with graph

Goal: construct a space X $\subseteq$R^2
s.t. The problem $Embed_X$ = {G: G is a graph that can be embeded in X } is undecidable.

U: undecidable problem as a set of numbers.
X = $\bigcup\limits_{n\in U} geom(\mathrm{G}_n)$
Reduction: instances(a number n) to graphs(G_n: C_n + triangles on edges)
Claim: G_n embeds into X iff n $\in U$
G$_n$ $\in$Embed$_x$

Proof. $G_n \hookrightarrow \mathrm{G}_m \implies m = n$
if $m < n$ then there is no place for all n deg 4 vertices to map to.
if $m > n$ then 2 adjacent vertices in G_n are not adjacent in G_m.
if $\mathrm{G}_n \hookrightarrow X$ **then** $n \in U\ because f(\mathrm{G}_n)$ is in some connected component of X.
so f(G_n) $\subseteq \mathrm{G}_m \implies m = n$.
$cr(G)$ =min number of crossings for any embedding of G into the plane.
$m \leq 3n - 6$ for planar graphs
$cr(G) \geq m - 3n + 6$
$G = (V, E)|V| = n, |E| = m$
$H \subseteq G$,V_H is formed by removing each vertex of g with probability of 1-p
E(—V_H—)=pn E(—E_H—)=p^2m
E(cr(H)) $\leq$p^{4}cr(G) m^3=n^4
so cr(G) $\geq$n^2/64 ∎