CSC 565: Graph Theory | **North Carolina State University**
Fall 2019 | **Computer Science**

Lecture 1: Aug 21, 2019

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1 Pre-class logistics

This class is evaluated through

- Quizzes and Tests: 70% (In class quizzes x 6)
- Scribe Notes 30% (in team of 3)

2 Introduction: What is graph?

When we talk about graphs, we could first start with some of the basic building component of it:

- Number

1, 2

- Set

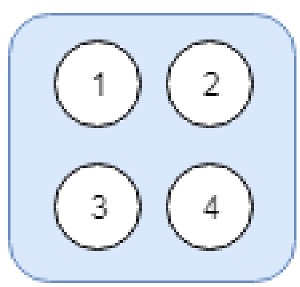

Figure 1: Set of numbers

- Graph

What are some typical graphs? Someone may list the following:

Computer Network, Circuit, Social Network, Recommending system, maps, Data Bases, Neural Networks, etc...

And since we're computer scientists, what are some of the graph based algorithms?

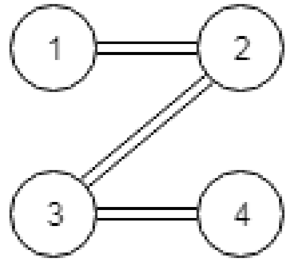

Figure 2: Picture of a graph

Dijkstra, Floyd, BFS, DFS, etc...

Graphs are the most important abstraction in computation (after numbers and sets)

- They describe binary relations (i.e. sets of pairs of things)
- As the name implies, we often draw graphs

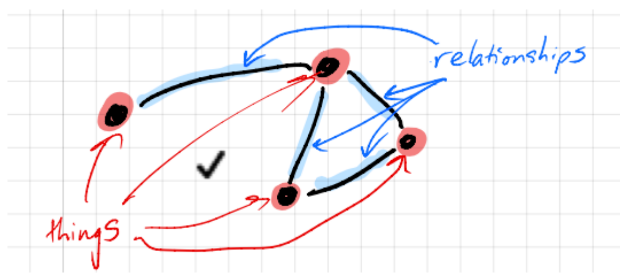

Figure 3: The drawing is not the graph. It's only a picture.

The abstractions of many of these listing could be identified as graph, but what makes a graph graph?

3 The Basics

Definition 1. Graph: A **graph** is a pair (V,E) where V is any set and E $\subseteq \binom{V}{2}$. Elements of V are called **Vertices** (singular: vertex), elements of E are called **Edges**

in this definition, we used some notations, of V and E, here we define if two vertices a and b belongs to V, then we have $a, b \in V$, and $(a, b) \in E$, sometimes people use other notations, and they are equivalent: $\{a, b\} = (a, b) = (b, a)$

As you might have discovered, we treat (a, b) and (b, a) as the same. In this semester we will be studying specifically undirected graphs, specifically we would be focusing on un-directed graphs without self loops, there could only be one edge between two given vertices.

As you could see in Figure 3, some graphs could be drawn on a 2D plane, making it easier for people to visualize, however one have to know that these visualizations are not graphs, graphs are mathematical abstractions of vertices, edges, and relationships. In this sense, we always represent graphs in this notation:

$$G = (V, E) = (\{a, b\}, \{(a, b), (b, c)\})$$

Definition 2. Adjacent and Incident: Two vertices a and b are **adjacent** if $(a, b) \in E$ (if there's an edge connecting them); A vertex v is **incident** to edge e if $v \in e$, in the same sense e is incident to v at the same time.

If we formulate incident, it would be written in this way:

$$G = (V, E)$$

$$I = (V \cup E, \{(v, e) | e \in E \& v \in V\})$$

Definition 3. Degree: The degree of a vertex $v \in V$ is the number of edges incident to v

Some fun fact about this is that, in a graph we have:

$$\sum_{v \in V} deg(v) = 2|E|$$

Because for each edge, there will always be 2 vertices on each end of the edge.

3.1 Examples

G_1**:**

$V(G_1) = \{a, b, c\}$

$E(G_1) = \{\{a, b\}, \{b, c\}\}$

G_2**:**

$V(G_2) = \{1, 2, 3\}$

$E(G_2) = \{\{1, 2\}\}$

Notation: It's easier to write (a,b) instead of $\{a,b\}$*. In this case, it's assumed (a,b) = (b,a).*

- Two vertices u,v are **adjacent** if $(u, v) \in E$
- An edge e and a vertex v are **incident** if $v \in e$ (i.e. $e = (u, v)$ for some $u \in V$
- The number of edges incident to a vertex v is called a degree of v, and is written deg(v)

4 Questions could be asked(and you should care)about a graph

1. Is it connected? (i.e. is it all one piece)
2. Does the graph have any cycles?
3. What is the shortest path form one vertex to another?
4. Can we assign a small number of colors to the vertices so that no two adjacent vertices have the same color?

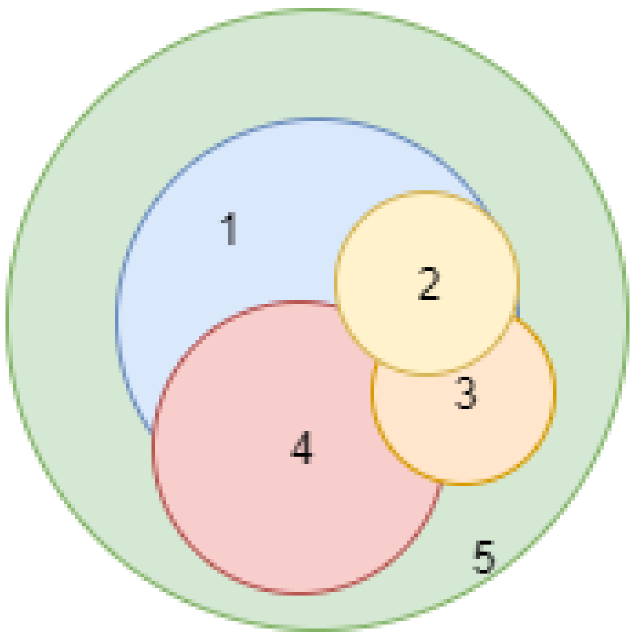

Figure 4: Is 4 color always enough to color a map?

5. Can we draw the graph so that no two edges cross?
6. Is one graph "equal" [isomorphic] to another (allowing the vertices to be relabelled)

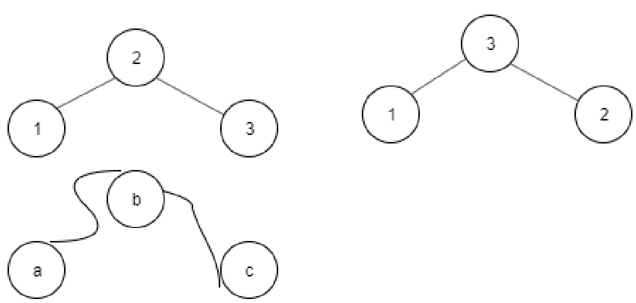

Figure 5: Graphs with different labels could still be isomorphic.

7. Does one graph contain another graph (or its equivalent)? [subgraph isomorphism is a NP complete problem]

8. How quickly will a random walk on a graph mix?
9. How many spanning trees (minimally connected subgraphs using all the vertices) does a graph contain?

In general, graph theory would research into combinatorial questions such as 1 3, topological questions such as 4 5, computational questions such as 6 7 as well as algebraic questions such as 8

5 Different Perspective on Graphs

combinatorial, Computation, Geometry, Topology, Algebra. As much as possible, we will try to represent these different perspectives as **categories** and our change of perspective as **functions**. I will tell you what these words mean.

5.1 Sets and Functions

This should all be review I will use all these concepts, definitions, and notation will reckless abandonment. The definition of a graph depends on the notion of a set.

You should know:

1. What is a set?

   Elements, Membership, Empty Set (∅), Cordiality
2. Set Relations and Elements

   $a \in S$ "a is in S" or "a is an element of S"

   $A \subset B$, $A \subseteq B$, $A = B$; subset, strict subset, and equality

   Note that if $A \subseteq B$ and $B \subseteq A$ then $A = B$

   Sometimes a set could also be written in mathematical formulations, for example:

   $$\{a \in \mathbb{Z} | a \equiv 3 \ mod \ 5\}$$
3. Set Operations

   Union: $A \cup B$

   Intersection: $A \cap B$

   Difference: $A \setminus B$

   Complement: $\hat{A}$

   Cartesian Product: AXB

   $\bigcup_{i=1}^{n} A_i$, $\bigcap_{i=1}^{n} A_i$
4. Notation $\{1, 2, 3\}$

   (Sub) Set Builder: $\{x \epsilon \mathbb{R} | x \geq 2\}$

   Predicate: $x \geq 2$
5. Functions $f : A \Rightarrow B$

   domain, range, injective, surjective, bijective, inverse, preimage, composition

5.2 The Category of Sets

- Set fuctions: $f : A \Rightarrow B$ or $A \xrightarrow{f} B$

  A is the **domain** or **source**

  B is the **range** or **target**

- Functions can be **composed**

  $A \xrightarrow{f} B \xrightarrow{g} C$

  $A \xrightarrow{g \circ f} C$

  $x \epsilon A : (g \circ f)(x) = g(f(x)) \epsilon C$

- **Inclusion** (as a function)

  If $a \subseteq B$ there exists a unique injection $f : A \Rightarrow B$ such that for all $x \epsilon A : f(x) = x$

- Identity Functions

  For any set A there is a unique fucntion $id_A : A \Rightarrow A$ such that for all $x \epsilon A \quad id_A(x) = x$

- Let A,B be sets and $f : A \Rightarrow B$

  Image $imf = \{f(x) : x \epsilon A\} \subseteq B$

- Let $S \subseteq A$

  Restriction $f_{\setminus S} : S \Rightarrow B$ $f_{\setminus S}(x) = f(x)$ (for all $x \epsilon S$)

  Image of a set $f(s) = imf_{\setminus S} = \{f(x) : x \epsilon S\}$

  (*Note: This is an abuse of notation and I'm not sorry.*)

  Preimage $T \subseteq B$ $f^{-1}(T) = \{x \epsilon A : f(x) \epsilon T\}$

  (*Another abuse of notation.* f^{-1} *could also be an inverse.*)

  Inverse If f is bijective then there is a unique function $f^{-1} : B \Rightarrow A$ such that $f \circ f^{-1} = id_B$ and $f^{-1} \circ f = id_A$