CSC 565: Graph Theory	**North Carolina State University**
Fall 2019	**Computer Science**

Lecture 29: Dec 4, 2019

Lecturer: Don Sheehy `<drsheehy@ncsu.edu>` *Scribe: Hongyi Fan, Guangyu Yu*

1 P and NP problem

Definition 1. decision problem with polynomial solutions is called **P problem**

If A is a decision problem, V_A is called **verifier** if

$$A = \{a \mid < a, b > \in V_A \text{ for some b}\}$$

Definition 2. decision problem with polynomial verifiers is called **NP problem**

2 Polynomial-time reduction

Consider A and B are decision problems, and f : Instance of A $\rightarrow$ Instance of B,

Requirement 1:

$$a \in A \text{ iff } f(a) \in B$$

Requirement 2:

f runs in polynomial time

Denotation:

$$A \leq_p B$$

Example:

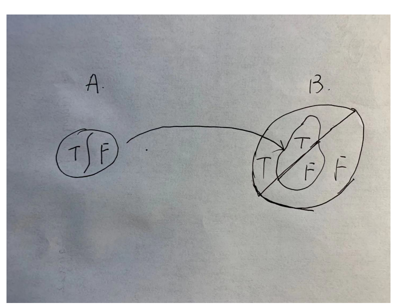

Figure 1: Polynomial time reduction

3 NP-complete problem

Definition 3. If A is a decision problem and A $\in$ NP, such that

$$\forall B \in \text{NP},\ B \leq_p A$$

then A is called **NP-complete problem**

3.1 SAT problem

SAT problem: determining if there exists an interpretation that satisfies a given boolean expression.

Example:

$$(x_1 \vee \bar{x}_2 \vee x_3 \vee x_4) \wedge (x_4 \vee x_2 \vee \bar{x}_5)$$

Theorem 1. *SAT problem is NP-complete*

3.2 VC(vetex cover) problem

Definition 4. A **vetex cover(VC)** is a subset $V^{'} \in V$, such that

$$\forall e \in E,\ e \cap V^{'} = \emptyset$$

VC problem: given a Graph G and size k, does G have a VC of size k?

3.3 3SAT $\leq_p$ VC

Transformation rules from boolean expression to graph G:

$$\forall \text{ variable } x \to x, \bar{x} \in V, (x, \bar{x}) \in E$$
$$\forall \text{ clause } (a \vee b \vee c) \to a, b, c \in V, (a, b), (a, c), (b, c) \in E$$

Consider 3SAT expression consists n variables and m clauses.

- For every triangle generated by the clauses, we need at least 2 vertices in our VC otherwise we can't cover 3 edges in this triangle, so we need at least $2m$ vertices in our VC.
- For every edge generated by the variables, we need at least 1 vertices in our VC to cover this edge, so we need at least n vertices in our VC.

In summary, the size k of VC is at least $2m + n$, that is

$$\text{If Exp} \in \text{3SAT, then}(G, 2m + n) \in VC$$

So, VC problem is NP-complete

3.4 IS(independent set) problem

Definition 5. A **independent set(IS)** is a set of v such that no two of which are adjacent.

IS problem: given a Graph G and size k, does G have a IS of size k?

Relation between VC and IS:

$$V^{'} \in V \text{ is a VC iff } V \setminus V^{'} \text{is an IS.}$$

in another word,

$$(G, k) \in \text{VC iff } (G, n-k) \in \text{IS}$$

So, IS problem is NP-complete

3.5 Clique problem

Definition 6. A **clique** is a subset of V such that every two distinct vertices in the clique are adjacent

Clique problem: given a Graph G and number k, does $K_k \in G$?

Relation between IS and clique:

$$(G, k) \in \text{IS iff } (\bar{G}, k) \in \text{clique}$$

So, clique problem is NP-complete

3.6 Subgraph Isomophism(SubGI) problem

SubGI problem: given a Graph G and H, does G have a subgraph isomorphic to k?

Relation between clique and SubGI:

$$(G, k) \in \text{clique iff } (G, K_k) \in \text{SubGI}$$

So, SubGI problem is NP-complete