CSC 565: Graph Theory **North Carolina State University**
Fall 2019 **Computer Science**

Lecture 25: Nov 18, 2019

Lecturer: Don Sheehy `<drsheehy@ncsu.edu>` *Scribe: Niranj Jyothish, Xun Jia*

1 Incidence Matrix as a Graph Representation

A graph can be represented as a matrix which represents it as mapping from edges to vertices. This form of a matrix is called an Incidence Matrix (B).

$$B = \begin{bmatrix} . & 1 & . \\ . & 0 & . \\ . & 0 & . \\ . & 1 & . \end{bmatrix}$$

Lets say, the k^{th} column has the ones at i^{th} and j^{th} row. This form represents edge, $e_k = (v_i, v_j)$

B $\in \{0,1\}^{n\times m}$ where n is number of vertices and m is edges.

B : $\{0,1\}^n - > \{0,1\}^m$

Also written as, Pow(E) $- >$ Pow(V). Set of edges to vertices.

Lets say, $u \subseteq V$ which implies $u \in \{0,1\}^n$

$$u_i = \begin{cases} 1 & if v_i \in u \\ 0 & o/w \end{cases}$$

And $X \subseteq E$ which implies $X \in \{0,1\}^m$

Note: Multigraphs can be represented using Incidence Matrix but cannot represented by Adjacency Matrix.

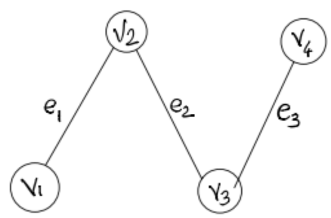
2 Kernel of a Graph

Lets think of a new operation. We know in an incidence matrix, B , $e_i = (v_j, v_k)$

$B[1]i = \begin{bmatrix} 1 \\ . \\ . \\ 1 \end{bmatrix}$ 1s at j^{th} and k^{th} row

This implies $Be_i = v_j + v_k$

Example: Figure 1



From figure 1, $B(e_1 + e_3) = Be_1 + Be_3 = v_1 + v_2 + v_3 + v_4$

$B(e_1 + e_2) = v_1 + v_2 + v_2 + v_3 = v_1 + v_3$

Lets say, there is a path from u to v. $P \subseteq E$

BP = u + v

Let C be the edges of a cycles ! What is BC? BC = 0.

Kernel, ker B = v : Bv = 0

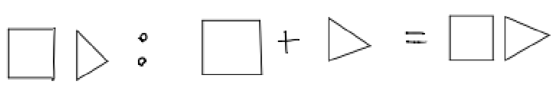
$u, v \subseteq kerB$
B(u+v) Bu + Bv = 0 + 0 = 0

Kernel is also called cycle space of G

2.1 Extending idea of kernels to cycle space

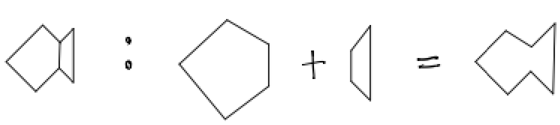
What if, we add two cycle graphs. What resultant graphs we can expect.

Case 1: When egde sets of two cycle graphs are disjoint.



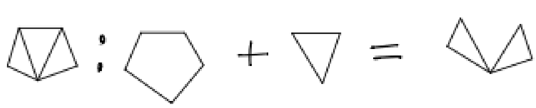
We get the same graph.

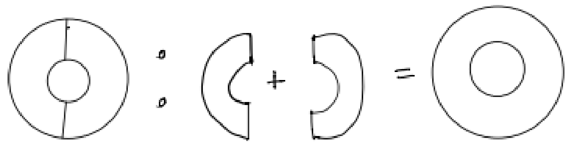
Case 2: When edges set of two cycle graphs share an edge.



We get the cycle graphs glued.

Find a case where, when two or more cycle in a graph gets added to form disjoint cycles.





Thus, the cycle space denoted by the basis of the graph not only gives an idea about the individual cycles in the graph but also, the addition of cycles.

Fact: Bounded faces of a 3-connected planar form a basis for kerB.

2.2 Using kernels to denote connectivity

Lets say we have a graph with $v_i, v_j \in V$

We say it is connected , if there is a path $P \subseteq E$ from v_i to v_j

We have seen earlier in the lecture about the behaviour of incidence matrix, B on paths. When we operate B on paths we just get the end points of a graph.Therefore,

BP = v_i + v_j

v_i = v_j + BP

Image of B: im B = { v : v = Bx for some x }

v_i , v_j are connected iff v_i = v_j + imB

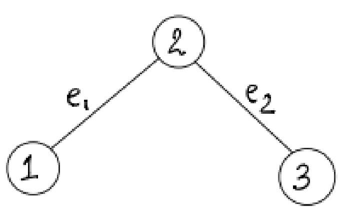
$v_i \in V$, $[v_i]$ = v_i + imB

Quotient vector space is denoted by :
V /imB $\cong$ connected components . $[v_i]$ = [v_j]

$[v_i + v_j\ ]$ = v_i + v_j + imB = $[v_i]$ + $[v_j]$

Equality in V / imB is the same as connected in G.

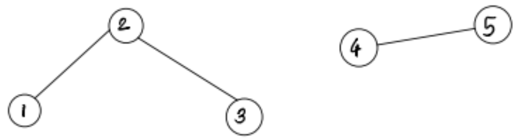
Example figure 2:



$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

imB = { ϕ , $\{1,2\}$, $\{2,3\}$, $\{1,3\}$ }

Example figure 3:



imB = { ϕ , $\{1,2\}$, $\{2,3\}$, $\{1,3\}$, $\{4,5\}$ }

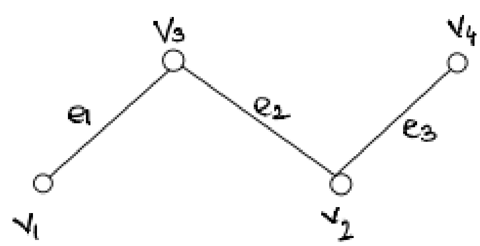
3 Another form of Incidence Matrix

$$\partial = \begin{bmatrix} . & 1 & . \\ . & 0 & . \\ . & 0 & . \\ . & -1 & . \end{bmatrix}$$

Lets say, the k^{th} column has the ones at i^{th} and j^{th} row. This form represents edge, $e_k = (v_i, v_j)$

$$\partial = \begin{cases} 1 & \text{if } e_k = (v_i, v_j) i < j \\ -1 & \text{if } e_k = (v_i, v_j) i > j \\ 0 & 0/w \end{cases}$$

Example figure 4:



$$\partial = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\partial(e_1 + e_2 + e_3) =$

$$\partial = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$= (v_1 - v_3) + (v_2 - v_3) + (v_2 - v_4) = v_1 - 2v_2 - 2v_3 - v_4$

$\partial(e_1 - e_2 + e_3) = (v_1 - v_3) - (v_2 - v_3) + (v_2 - v_4) = v_1 - v_4$

We looked at symmetric matrices and how wrangling matrices with its transpose gives symmetric matrices preserving properties and information. like rank of a matrix.

Lets take ∂ and multiply with ∂^T. $\partial\partial^T \in R^{n\times n}$

$[\partial\partial^T]_{ii} = \sum_{j=1}^{m} \partial_{ij}\partial^T_{ji} = \sum_{j=1}^{m} \partial^2_{ij} = \deg(v_i)$

$$\partial_{ij} = \begin{cases} 1 & if v_i \in u \\ 0 & o/w \end{cases}$$

$[\partial\partial^T]_{ik} = \sum_{j=1}^{m} \partial_{ij}\partial^T_{jk} = \sum_{j=1}^{m} \partial_{ij}\partial_{kj}$

$$\partial_{ij}\partial_{kj} = \begin{cases} -1 & if e_j = (v_i, v_k) \\ 0 & o/w \end{cases}$$

Therefore $\partial\partial^T$ is Laplacian Matrix.

Also, Fact : ker $(\partial\partial^T) \cong$ connected components.