

# Heterogeneous Agent Models

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# Heterogeneous Agent Models

- ▶ Motivation
- ▶ Cross-sectional Distributions
  - ▶ Kolmogorov Forward/Fokker-Planck Equations
- ▶ Parameter Estimation
- ▶ Steady State: Huggett procedure
- ▶ Nonstochastic Dynamics
- ▶ Stochastic Dynamics
  - ▶ Krusell-Smith Method
  - ▶ Perturbation Method

# Motivation

- ▶ Dynamic optimization describes behavior of single agent
  - ▶ Consumer, Firm, Household, Government
- ▶ How to describe population of these agents?
  - ▶ Derive cross-sectional distribution implied by behavior
  - ▶ Consider feedback from distribution to behavior
  - ▶ Solve for fixed point: equilibrium
- ▶ Applications
  - ▶ Distribution allows estimating model parameters
  - ▶ May care about feedback
    - ▶ Role of heterogeneity in economic environment
  - ▶ May care about distributions directly
    - ▶ Find how it evolves or changes

## Example Model: (Approximately) Huggett (1993)

- ▶ Standard consumption savings problem we have seen all class
- ▶ Assume 1 bond with return  $R$ , lower limit  $\underline{a}$  on borrowing
- ▶ Typical agent  $i$  chooses  $c_t^i$  to solve

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{(c_t^i)^{1-\gamma}}{1-\gamma}$$

$$\text{s.t. } w_{t+1}^i = R_t(w_t^i - c_t^i) + s_t^i$$

$$R_t(w_t^i - c_t^i) \geq \underline{a}, \quad \underline{a} < 0$$

- ▶ Income  $s_t^i \sim g(\cdot)$  drawn iid over time **and** across individuals
- ▶ Everyone solves same problem, *but*
  - ▶ Receives different draw of income  $s$
  - ▶ So chooses different consumption  $c$
  - ▶ Ends up with different wealth  $w$
- ▶ Heterogeneity encoded in random variables
  - ▶ Model provides complete statistical description of data

# Huggett Model: Goals

- ▶ Tasks
  1. Derive joint distribution of consumption, income, wealth  $(c_t^i, s_t^i, w_t^i)$
  2. Allow feedback: let interest rate  $R$  be endogenous
  3. Add “aggregate” variability, so distribution also random
- ▶ Distribution of  $s^i$  is given  $g(s)$
- ▶ Policy function  $c(w)$ , law of motion for  $w$  solved for by standard (dynamic programming) methods
- ▶ Use above objects to find *distribution*  $m_t(w)$  of  $w_t^i$ 
  - ▶ This is Kolmogorov Forward/Fokker Planck equation
- ▶ Feedback is feature added to model: use general equilibrium
  - ▶ Simplest: Market for bonds clears with 0 net supply

$$\int (w_t^i - c_t^i) di = 0$$

- ▶ For now, assume steady state:
  - ▶ Individual income and choices dynamic, stochastic
  - ▶ Cross-section distribution (and so  $R_t$ ) fixed

# Kolmogorov Forward Equation

- ▶ Turns policy rule into density rule
- ▶ Intuition simpler in finite state case
  - ▶ Let  $[H]_{ij} = Pr(x' = j | x = i)$
- ▶ Then, given population with  $Pr(x = i) = [m]_i$ , next period distribution is

$$m' = H^T m$$

- ▶ Rule is transpose (“adjoint”) of Markov transition matrix
  - ▶ Hence called “adjoint Markov operator”
- ▶ Continuous state case same: switch integrand in conditional expectation
  - ▶  $H[\phi] := E[\phi(x') | x] = \int \phi(x') f(x' | x) dx'$
  - ▶  $H^*[m] = \int m(x) f(x' | x) dx$
- ▶ Obtain next density from conditional density, current density
- ▶ Use initial (or terminal) conditions or steady state requirement  $m = m'$  to find implied path
- ▶ Model doesn't directly give conditional density, but can derive it

## Deriving KFE From Policy Rule

- ▶ Assume rule  $x' = h(x, \epsilon)$ , where  $\epsilon \sim G(\epsilon)$
- ▶ Population follows above rule, with independent draws of  $\epsilon$
- ▶ Assume current distribution of  $x$  in this population is  $\mu(x)$
- ▶ “Adding up” rule: expectation of any function of  $x'$  tomorrow is expectation of that function over distribution of  $x, \epsilon$
- ▶ For all  $\phi(\cdot)$ , distribution  $\mu'$  satisfies

$$\int \phi(x') d\mu'(x') = \int \int \phi(h(x, \epsilon)) d\mu(x) dG(\epsilon)$$

- ▶ Simplifies considerably if  $h(x, \epsilon)$  has inverse  $h_x^{-1}(x')$  in  $\epsilon$  for every  $x$ , and  $\mu, G$  have densities  $m, g$
- ▶ (Apply above to  $1\{x' \leq z\}$  to get law for CDF, differentiate to get PDFs, and apply change of variables)

$$m(x') = \int g(h_x^{-1}(x')) \left| \det \frac{dh_x^{-1}(x')}{dx'} \right| m(x) dx$$

- ▶ This expresses  $m(x') = \int f(x'|x) m(x) dx$  using conditional density implied by rule

# Continuous time case: Fokker-Planck Equation

- ▶ In continuous time case,  $x$  follows diffusion equation

$$dx = \mu(x, t)dt + \sigma(x, t)dW$$

- ▶ Evolution of conditional expectation described by “infinitesimal generator”  $A[\phi] = \frac{dE[\phi(x_{t+\Delta})|x_t]}{d\Delta}$
- ▶ Apply Itô's lemma to obtain  $A[\phi] = \phi'(x)\mu(x) + \frac{\sigma(x)^2}{2}\phi''(x)$
- ▶ Evolution of distribution described by its  $(L^2)$  adjoint  $A^*$
- ▶ Density  $m(x, t)$ , it evolves as  $\frac{\partial}{\partial t}m(x, t) =$

$$A^*[m(x, t)] := -\frac{\partial}{\partial x}(m(x, t)\mu(x, t)) + \frac{\partial^2}{\partial x^2}\left(\frac{\sigma(x, t)^2}{2}m(x, t)\right)$$

- ▶ Steady state: Assume  $\mu, \sigma$  depend only on  $x$ , set LHS to 0
  - ▶ Second order PDE in  $m$
- ▶ Can extend to jump processes, multiple dimensions



# Solving Kolmogorov Forward Equation

- ▶ Solve for policy using standard dynamic programming method
- ▶ KFE is a functional equation
  - ▶ Represent density  $m(x)$  by function representation
  - ▶ Linear in coefficients if  $m(x) = \sum_{i=1}^k a_i \phi_i(x)$
- ▶ If discrete or discretized, KFE is a Markov matrix  $H^*$ 
  - ▶ Transpose of discretized differential operator in HJB equation
- ▶ May need to impose that density nonnegative, sums to 1
  - ▶ Divide each column by sum of entries
  - ▶ Or add additional equation
- ▶ If solving for steady state, this is fixed point  $m = H^* m$ 
  - ▶ Can use fixed point iteration
- ▶ Linear fixed point is eigenproblem
  - ▶ Use eigensolver to find eigenvector with eigenvalue closest to 1
    - ▶ 0 in continuous time
  - ▶ Guaranteed to exist if normalized (Frobenius-Perron thm)
  - ▶ May need to multiply by  $-1$  to ensure sign correct
- ▶ Can “solve” by simulating many agents forward
  - ▶ Error bad, but no need to derive formula

# Application 1: Estimation

- ▶ Above results enough to perform estimation if all agents iid
- ▶ Given parameters  $\theta$ , solve optimization for law of motion
- ▶ Solve KFE for steady state density  $\tilde{m}(x, \theta)$
- ▶ Given data  $\{x_i\}_{i=1}^n$ , find MLE or Bayes
  - ▶  $\hat{\theta}_{MLE} = \arg \max_{\theta} \sum_{i=1}^n \log \tilde{m}(x_i, \theta)$
  - ▶ Given prior  $\pi(\theta)$ , run MCMC to sample from  $\pi(\theta|X) \propto \tilde{m}(x_i, \theta)\pi(\theta)$
- ▶ Note: need numerical error in likelihood of  $o(\frac{1}{\sqrt{n}})$  to be able to ignore approximation error in asymptotic results<sup>1</sup>
- ▶ With panel data, can use conditional densities directly
  - ▶ Derive these from Fokker-Planck in continuous time
- ▶ When some individual variables unobserved, need to marginalize
  - ▶ Use filtering methods (Kalman if linear, Particle o/w)
  - ▶ or just simulate and match moments

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<sup>1</sup>And other conditions: see Kristensen & Salanié (2017)

# Adding Feedback

- ▶ Optimization and KFE describe population behavior in given environment
- ▶ Close model by adding response of environment to behavior
  - ▶ E.g. market clearing. Price matches supply and demand
  - ▶ Huggett Model: interest rate  $R$  solves
$$\int (w - c(w, R))m(w)dw = 0$$
- ▶ More generally: parameter  $P$  of decision problem depends on distribution through some “equilibrium condition”
- ▶ For some problems “ $P$ ” is itself a function
  - ▶ Nash equilibrium: respond directly to others’ behavior
  - ▶ Spatial/network models: response at every value of argument
- ▶ Steady state equilibrium is
  - ▶ Decision rule  $h(\cdot)$ , distribution  $\mu$ , parameter  $P$ , st.
    1. Given  $P$ ,  $h(\cdot)$  satisfies optimality condition
    2.  $\mu$  satisfies steady state KFE derived from  $h(\cdot)$
    3.  $P$  satisfies equilibrium condition given  $h(\cdot)$ ,  $\mu$

## Solution Procedure: “Huggett Algorithm”

- ▶ Write function  $J(P)$  for equilibrium
  1. Given  $P$ , solve optimization problem for policy rule  $h(\cdot)$
  2. Use  $h(\cdot)$  to solve KFE for steady state distribution  $\mu$
  3. Output value of equilibrium condition equation given  $h, \mu, P$
- ▶ Solve  $J(P) = 0$  by, e.g., bisection
- ▶ Step 1 by Value/Policy iteration or projection for Euler equation
- ▶ Step 2 by fixed point iteration or eigensolver
- ▶ Step 3 usually involves numerical integration
- ▶ If  $P$  a vector, use nonlinear solver
  - ▶ Keep tolerance large, since interior loops approximate
- ▶ If  $P$  is function, use functional equation methods (projection, etc) to solve  $J(P) = 0$
- ▶ Nested structure convenient because interior problems often contraction *conditionally* on value of other arguments

## Example: Huggett Model

- ▶ Reduce optimization problem to Bellman or Euler equation
  - ▶ Bellman easier, more robust, Euler gets rid of max
- ▶ Fix  $R$ , solve Euler for consumption rule

$$\ell(w) = \mathbb{E}_t \beta R \int g(w' - R(w - c(w, \ell(w), R))) c(w', \ell(w'), R)^{-\gamma} dw'$$

- ▶ where  $c(w, \ell, R) = \min\{\ell^{1/\gamma}, w - \frac{a}{R}\}$
- ▶ “Parameterized Expectations” transform:  $\ell(w) = \beta \cdot$  expected m.u. of consumption (Christiano & Fisher 2000)
  - ▶ Turns Euler equation into fixed point problem
  - ▶ Gets rid of Lagrange multipliers in equation
  - ▶ Represent  $\ell(w)$  by basis functions, run fixed point iteration
- ▶ Solve KFE for  $m(w)$

$$m(w') = \int g(w' - R(w - c(w, \ell(w), R))) m(w) dw$$

- ▶ Represent  $m(w)$  by basis functions, solve by eigensolver
- ▶ Bisection in  $R$  to solve  $\int (w - c(w, \ell(w), R)) m(w) dw = 0$

# Results: Equilibrium Policy Function and State Distribution

## Steady State Policy and Wealth Distribution

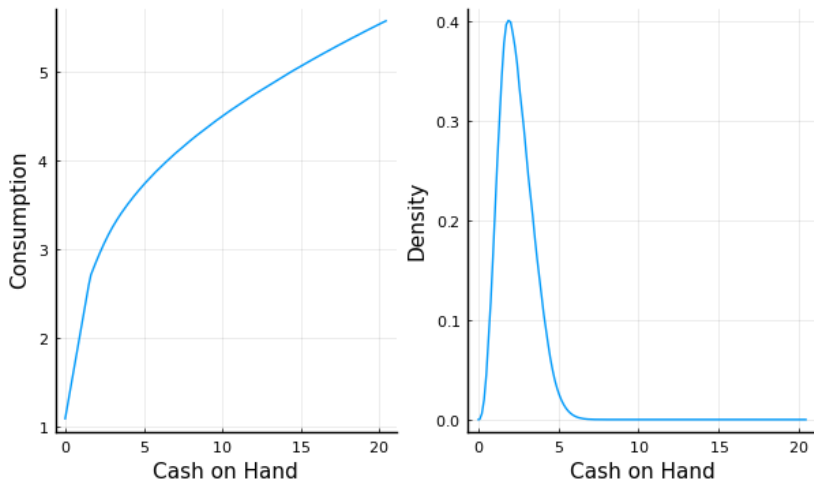


Figure 1: Huggett Model Results

## Extension: Nonstochastic dynamics

- ▶ Suppose initial condition  $t = 0$  not a steady state
- ▶ Nonstochastic dynamics can be solved by shooting method
  - ▶ Use Huggett algorithm to solve for steady state
  - ▶ Choose large  $T$  for time economy returns to steady state
  - ▶ Conjecture **sequence**  $\{P_t\}_{t=0}^T$ ,  $P_t = P^*$  for  $t > T$
  - ▶ Solve optimization problem given sequence (eg by iterating value functions backward)
  - ▶ Solve KFE forward from  $\mu_0$  using sequence of policies
  - ▶ Check *sequence* of equilibrium conditions
  - ▶ Iterate over prices until equilibrium conditions satisfied
- ▶ Describes path response to unanticipated economic changes
  - ▶ “MIT Shock”: non-equilibrium parameter change
    - ▶ Start at a steady state
    - ▶ Change a parameter of economic environment
    - ▶ Solve for path back to new steady state

# Stochastic Dynamics

- ▶ Consider adding random variables  $Z_t$  that affect all agents
  - ▶ Business cycles: income distribution fluctuates
  - ▶ Aggregate policy (fiscal/monetary/regulatory)
- ▶ Creates major computational problem
  - ▶ Whole distribution shifts in response to shocks
  - ▶ Stochastic law of endogenous distribution enters decision problem
  - ▶ Value and policy functions take distribution as argument
- ▶ This is a functional equation “squared”:
  - ▶ Functions to solve for are functions of functions: “Operators”
  - ▶ E.g. map from income distribution to spending
- ▶ Try to solve by functional equation approach
  - ▶ Projection
  - ▶ Perturbation
- ▶ How to represent infinite dimensional function?
  - ▶ Curse of dimensionality
  - ▶ Need approximation which reduces dimension



## Example: Huggett with aggregate shocks

- ▶ Take original Huggett model but
- ▶ instead of income  $s_t^i \stackrel{iid}{\sim} g(\cdot)$  face income  $s_t^i + z_t$ 
  - ▶  $s_t^i \stackrel{iid}{\sim} g(\cdot | \sigma_t)$ ,  $\sigma_t, z_t$  shared for all agents
- ▶  $Z_t = (\sigma_t, z_t)$  are aggregate shocks to mean and variance of income risk
- ▶ Direct effect: 2 more variables in decision problem
- ▶ Indirect effects
  1. since shared, whole wealth distribution depends on them
  2. Interest rate now stochastic
  3. to keep track of interest rate, need to know state of shocks **and** distribution
- ▶ Decision rule  $c(w, Z, m(\cdot))$  takes additional arguments
  - ▶ One of which is a distribution

# Dimension reduction method 1: Krusell Smith method

- ▶ Projection represents decision/value function by basis functions over arguments
- ▶ Polynomials or discretization over  $\infty$  arguments not representable
- ▶ So cut down # of arguments
- ▶ Maybe you don't really need full distribution  $m(w)$
- ▶ Approximate by small number of summary statistics  $\zeta$ 
  - ▶ Mean  $\zeta_1 = \int w m(w) dw$ , Variance  $\zeta_2 = \int w^2 m(w) dw$ , etc
  - ▶ Include enough info to describe components of state that matter for agents: eg, prices
  - ▶  $c(w, Z, \zeta)$  takes (small) finite number of arguments
- ▶ Idea: “approximate aggregation”
  - ▶ Only care about distribution as way to forecast  $R$
  - ▶ Can get good forecast accuracy with just moments
  - ▶ Use this forecast in place of true law in decision rule
  - ▶ Solve for fixed point in which implied decision rule generates law consistent with forecast

# Krusell Smith Method: Steps

1. Conjecture a simple parameterized law of motion for  $\zeta$ 
    - ▶ Typically linear conditional on shocks: e.g.  $\zeta' = \beta_0(Z) + \beta_1(Z)\zeta$
  2. Solve optimization problem with conditional expectation over distribution given by above law
  3. Simulate shocks  $Z$ , population of agents following policy, and prices each period
  4. Re-estimate parameters  $\beta$  for moments using simulated data
  5. Repeat until coefs from simulated moments match conjectured coefficients
- ▶ Works well if you get good forecasts with just moments
    - ▶ Not a feature of all models: can fail, especially if many shocks
    - ▶ Interpret as behavioral: agents discard info to make forecast

# Perturbation Approach

- ▶ Linearization method from last class can be applied directly
  - ▶ Instead of scalar variables, allow **functions** to be variables
  - ▶ Instead of scalar derivatives, take **functional** derivatives
- ▶ Huggett model:
  - ▶ Variables are  $\ell_t(\cdot), m_t(\cdot), \sigma_t, z_t, R_t$
  - ▶ Equations are Euler, KFE, market clearing, and law of shocks
- ▶ Functional derivative of a nonlinear operator  $F(\cdot)$  is a linear map between functions
  - ▶  $\int k(x, y)[\cdot]dx$  is linear map taking  $f(x)$  to  $f(y)$
- ▶ Still  $\infty$  dimensional but can approximate by Galerkin approach
  - ▶ Function valued variables replaced by vector of  $k$  coefficients
  - ▶ Derivatives of equilibrium conditions just  $k \times k$  matrices
- ▶ As  $k \rightarrow \infty$ , approximate derivatives converge
- ▶ Apply standard RE solver to derivative matrices
  - ▶ Approximates first order Taylor approx of policy operator

## Perturbation method: Details

- ▶ To use linearization, need nonstochastic steady state values
  - ▶ Only shut down *aggregate* shocks (e.g,  $\sigma_t, z_t$ , *not*  $s_t^i$ )
  - ▶ Exactly result of Huggett algorithm!
- ▶ Need system of equations differentiable w.r.t. functions
  - ▶ Usually means using FOC approach
- ▶ How to take functional derivatives?
  - ▶ Derive symbolically, then project to get coefs (Childers 2018)
- ▶ Alternate approach (Reiter (2009), Winberry (2018), Childers (2022))
  1. Represent functions by vector of coefficients or discretized values
  2. Take scalar derivatives wrt points/coefs
- ▶ Discretization works, but inefficient for smooth functions
- ▶ Better method: take derivatives wrt points, get coefficients by interpolation

## Modified Perturbation: Steps

1. Represent Model as system of functional equations

$$\mathbb{E}\mathcal{F}(x, y, x', y', \sigma) = 0$$

2. Solve for function values at nonstochastic steady state

$$\mathcal{F}(x^*, y^*, x^*, y^*, 0) = 0$$

3. Represent functions by their values at a set of grid points, and integral equations by weighted sums

$$\{\tilde{\mathcal{F}}(\{x(s_j)\}_{j=1}^n, \{y(s_j)\}_{j=1}^n, \{x'(s'_j)\}_{j=1}^n, \{y'(s'_j)\}_{j=1}^n, \sigma)(t_i)\}_{i=1}^n = 0$$

4. Differentiate w.r.t. function values around steady state

$$(\tilde{\mathcal{F}}_x, \tilde{\mathcal{F}}_y), (\tilde{\mathcal{F}}_{x'}, \tilde{\mathcal{F}}_{y'})$$

5. Apply matrix transforms to get coefficient representation

$$(\tilde{A}, \tilde{B}) = (M(\tilde{\mathcal{F}}_x, \tilde{\mathcal{F}}_y)M^\top, M(\tilde{\mathcal{F}}_{x'}, \tilde{\mathcal{F}}_{y'})M^\top)$$

6. Apply standard linear ratex solution code to matrices

# Perturbation Application: Huggett with Aggregate Shocks

- ▶ Solve for steady state setting  $z, \sigma$  shocks to 0
- ▶ Represent equilibrium conditions with functions  $\ell(w), m(w)$  replaced by values at grid points
  - ▶  $\{m(w_i)\}_{i=1}^{nw} \{ \ell(w_i) \}_{i=1}^{nw}$
  - ▶ Integrals replaced by weighted  $w$ / quadrature weights  $\pi^i$
- ▶ E.g., Kolmogorov equation becomes  $0 = \mathcal{F}^2(.) (w'_j) =$

$$m_{t+1}(w'_j) - \sum \pi^i g(w'_j - R_t(w_i + z_t - c(w_i + z_t, \ell_t(w_i), R_t)); \sigma_t) m_t(w_i)$$

- ▶ Euler, market clearing equations replaced similarly
- ▶ Apply automatic differentiation to get derivatives
- ▶ Map matrix on function values to matrix on coefficients
  - ▶ E.g., by Wavelet or just Histogram (piecewise constant) transform  $M$
- ▶ Plug into RE algorithm to get policy functions
  - ▶  $\tilde{h}_x$  maps  $m(.), z, \sigma$  to  $m'(.), z', \sigma'$
  - ▶  $\tilde{g}_x$  maps  $m(.), z, \sigma$  to  $\ell(.), R$

# IRF: Response of wealth distribution to $z$

cash dist. response to aggregate income shock

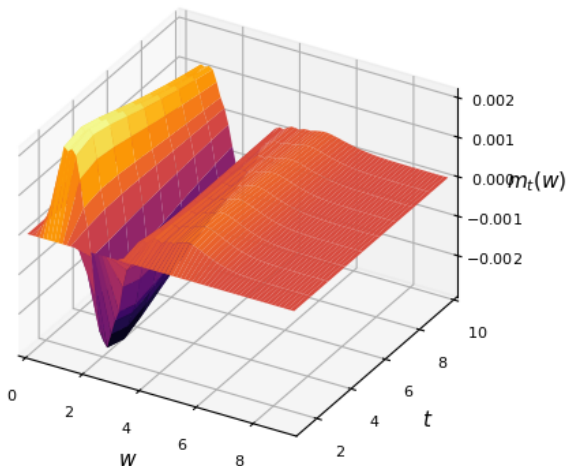


Figure 2: Deviation from Steady State Density



## IRF: Response of consumption function to $z$

consumption response to aggregate income shock

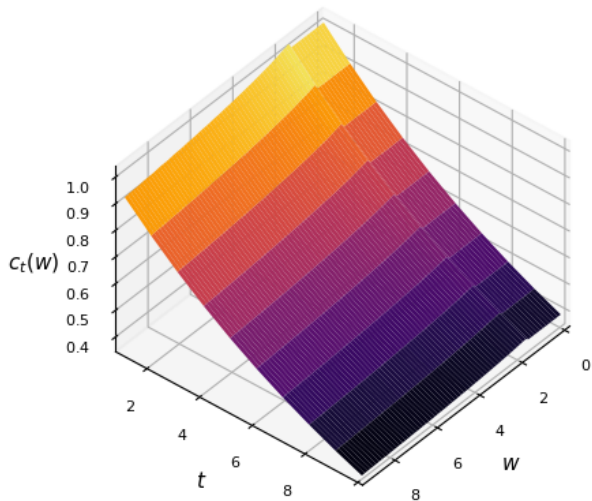


Figure 3: Deviation from Steady State Policy Function

## Simulation: Exogenous Shock Process $z_t$

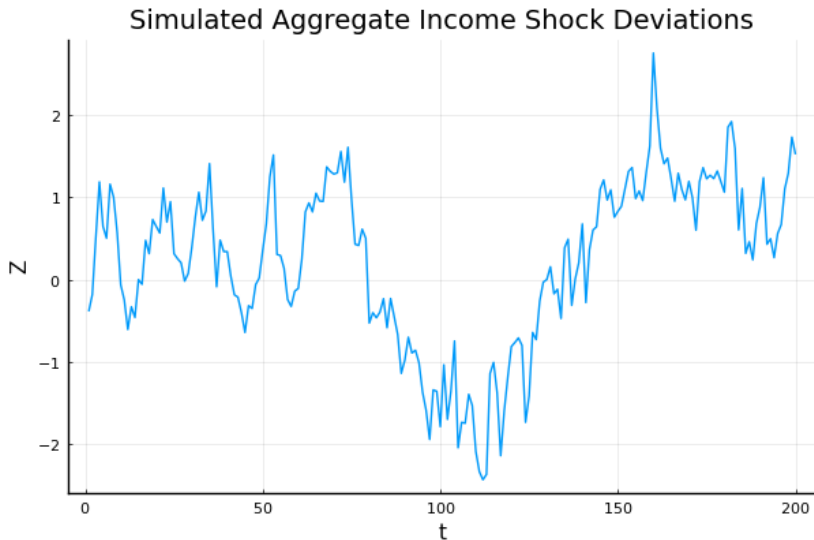


Figure 4: Deviation of  $z$  from Steady State

## Simulation: Exogenous Shock Process $\sigma_t$

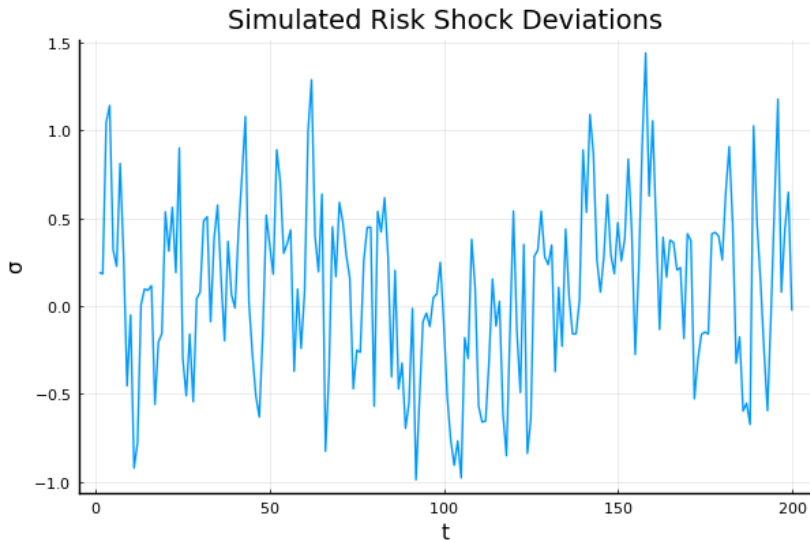


Figure 5: Deviation of  $\sigma$  from Steady State

# Simulation: Endogenous Wealth Distribution

## Simulated Cash Distribution Deviations

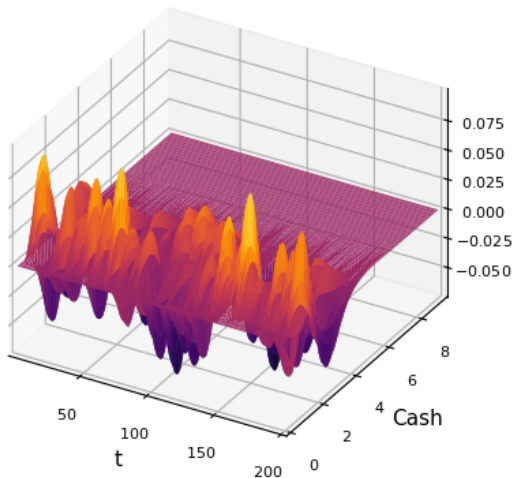


Figure 6: Deviation of  $m(w)$  from Steady State

# Simulation: Endogenous Consumption Function

## Simulated Consumption Function Deviations

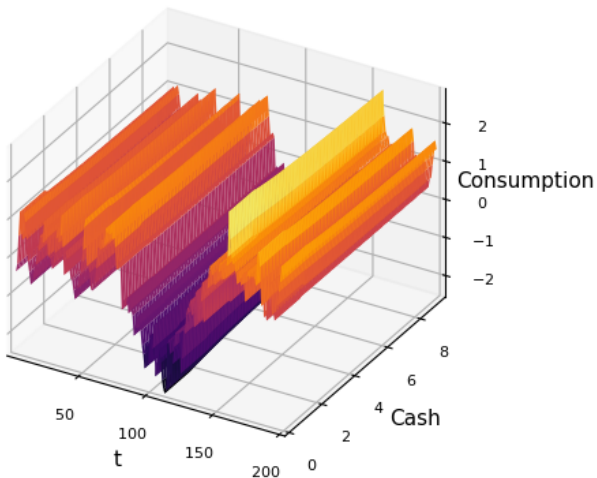


Figure 7: Deviation of  $c(w)$  from Steady State

## Simulation: Endogenous Interest Rate

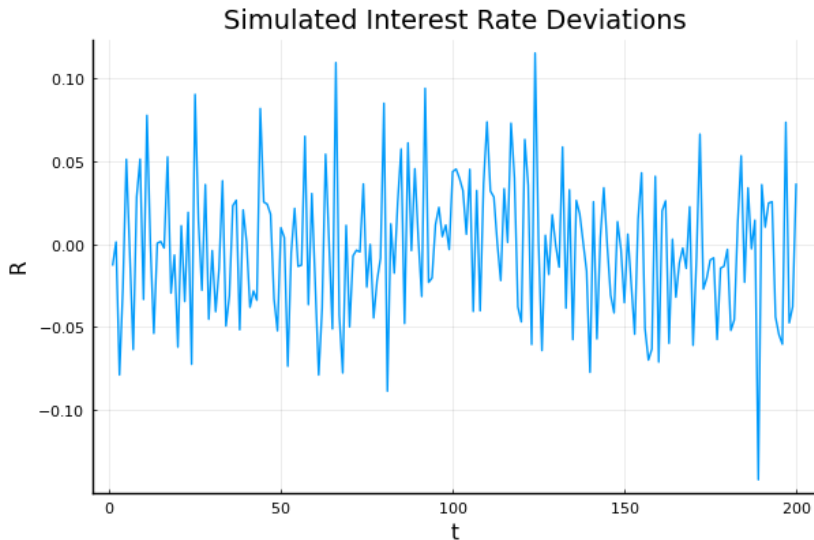


Figure 8: Deviation from Steady State  $R$

# Conclusions

- ▶ Dynamic models can incorporate heterogeneity and (mediated) interactions by deriving distribution
- ▶ Steady state needs 1 more functional equation (Kolmogorov) and a single loop
- ▶ Dynamics feasible by incorporating distribution as a state for projection/perturbation
- ▶ Krusell-Smith: Standard projection methods workable by massive dimensionality reduction
  - ▶ Preserves global, nonlinear effect of aggregates
  - ▶ No accuracy guarantees
- ▶ Perturbation: take functional derivatives, apply to projected distributions
  - ▶ Preserves nonlinearity in individual decisions, not in aggregates
  - ▶ Applicable even with many or infinite shocks
    - ▶ Whole function can be a stochastic process
  - ▶ Strong guarantees for first order

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