Heterogeneous Agent Models

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Heterogeneous Agent Models

- Motivation
- Cross-sectional Distributions
  - Kolmogorov Forward/Fokker-Planck Equations
- Parameter Estimation
- Steady State: Huggett procedure
- Nonstochastic Dynamics
- Stochastic Dynamics
  - Krusell-Smith Method
  - Perturbation Method

Motivation

- Dynamic optimization describes behavior of single agent
  - Consumer, Firm, Household, Government
- How to describe population of these agents?
  - Derive cross-sectional distribution implied by behavior
  - Consider feedback from distribution to behavior
  - Solve for fixed point: equilibrium
- Applications
  - Distribution allows estimating model parameters
  - May care about feedback
    - Role of heterogeneity in economic environment
  - May care about distributions directly
    - Find how it evolves or changes

Example Model: (Approximately) Huggett (1993)

- Standard consumption savings problem we have seen all class
- Assume 1 bond with return R, lower limit $\underline{a}$ on borrowing
- Typical agent i chooses c_t^i to solve

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{(c_t^i)^{1-\gamma}}{1-\gamma}$$

$$\text{s.t. } w_{t+1}^i = R_t(w_t^i - c_t^i) + s_t^i$$

$$R_t(w_t^i - c_t^i) \geq \underline{a}, \ \underline{a} < 0$$

- Income $s_t^i \sim g(.)$ drawn iid over time **and** across individuals
- Everyone solves same problem, *but*
  - Receives different draw of income s
  - So chooses different consumption c
  - Ends up with different wealth w
- Heterogeneity encoded in random variables
  - Model provides complete statistical description of data

Huggett Model: Goals

- Tasks
  1. Derive joint distribution of consumption, income, wealth (c_t^i, s_t^i, w_t^i)
  2. Allow feedback: let interest rate R be endogenous
  3. Add "aggregate" variability, so distribution also random
- Distribution of s^i is given $g(s)$
- Policy function $c(w)$, law of motion for w solved for by standard (dynamic programming) methods
- Use above objects to find *distribution* $m_t(w)$ of w_t^i
  - This is Kolmogorov Forward/Fokker Planck equation
- Feedback is feature added to model: use general equilibrium
  - Simplest: Market for bonds clears with 0 net supply

$$\int (w_t^i - c_t^i) \mathrm{d}i = 0$$

- For now, assume steady state:
  - Individual income and choices dynamic, stochastic
  - Cross-section distribution (and so R_t) fixed

Kolmogorov Forward Equation

- Turns policy rule into density rule
- Intuition simpler in finite state case
  - Let $[H]_{ij} = Pr(x' = j|x = i)$
- Then, given population with $Pr(x = i) = [m]_i$, next period distribution is

$$m' = H^T m$$

- Rule is transpose ("adjoint") of Markov transition matrix
  - Hence called "adjoint Markov operator"
- Continuous state case same: switch integrand in conditional expectation
  - $H[\phi] := E[\phi(x')|x] = \int \phi(x')f(x'|x)dx'$
  - $H^*[m] = \int m(x)f(x'|x)dx$
- Obtain next density from conditional density, current density
- Use initial (or terminal) conditions or steady state requirement $m = m'$ to find implied path
- Model doesn't directly give conditional density, but can derive it

Deriving KFE From Policy Rule

- Assume rule $x' = h(x, \epsilon)$, where $\epsilon \sim G(\epsilon)$
- Population follows above rule, with independent draws of ϵ
- Assume current distribution of x in this population is $\mu(x)$
- "Adding up" rule: expectation of any function of x' tomorrow is expectation of that function over distribution of x, ϵ
- For all $\phi(.)$, distribution μ' satisfies

$$\int \phi(x')d\mu'(x') = \int \int \phi(h(x,\epsilon))d\mu(x)dG(\epsilon)$$

- Simplifies considerably if $h(x, \epsilon)$ has inverse $h_x^{-1}(x')$ in ϵ for every x, and μ, G have densities m, g
- (Apply above to $1\{x' \leq z\}$ to get law for CDF, differentiate to get PDFs, and apply change of variables)

$$m(x') = \int g(h_x^{-1}(x')) \left| \det \frac{dh_x^{-1}(x')}{dx'} \right| m(x)dx$$

- This expresses $m(x') = \int f(x'|x)m(x)dx$ using conditional density implied by rule

Continuous time case: Fokker-Planck Equation

- In continuous time case, x follows diffusion equation

$$dx = \mu(x,t)dt + \sigma(x,t)dW$$

- Evolution of conditional expectation described by "infinitesimal generator" $A[\phi] = \frac{dE[\phi(x_{t+\Delta})|x_t]}{dt}$
- Apply Itō's lemma to obtain $A[\phi] = \phi'(x)\mu(x) + \frac{\sigma(x)^2}{2}\phi''(x)$
- Evolution of distribution described by its (L^2) adjoint A^*
- Density $m(x,t)$, it evolves as $\frac{\partial}{\partial t}m(x,t) =$

$$A^*[m(x,t)] := -\frac{\partial}{\partial x}(m(x,t)\mu(x,t)) + \frac{\partial^2}{\partial x^2}(\frac{\sigma(x,t)^2}{2}m(x,t))$$

- Steady state: Assume μ, σ depend only on x, set LHS to 0
  - Second order PDE in m
- Can extend to jump processes, multiple dimensions

Solving Kolmogorov Forward Equation

- Solve for policy using standard dynamic programming method
- KFE is a functional equation
  - Represent density $m(x)$ by function representation
  - Linear in coefficients if $m(x) = \sum_{i=1}^{k} a_i \phi_i(x)$
- If discrete or discretized, KFE is a Markov matrix H^*
  - Transpose of discretized differential operator in HJB equation
- May need to impose that density nonnegative, sums to 1
  - Divide each column by sum of entries
  - Or add additional equation
- If solving for steady state, this is fixed point $m = H^* m$
  - Can use fixed point iteration
- Linear fixed point is eigenproblem
  - Use eigensolver to find eigenvector with eigenvalue closest to 1
    - 0 in continuous time
  - Guaranteed to exist if normalized (Frobenius-Perron thm)
  - May need to multiply by -1 to ensure sign correct
- Can “solve” by simulating many agents forward
  - Error bad, but no need to derive formula

Application 1: Estimation

- Above results enough to perform estimation if all agents iid
- Given parameters θ, solve optimization for law of motion
- Solve KFE for steady state density $\tilde{m}(x, \theta)$
- Given data $\{x_i\}_{i=1}^n$, find MLE or Bayes
  - $\hat{\theta}_{MLE} = \arg\max_\theta \sum_{i=1}^n \log \tilde{m}(x_i, \theta)$
  - Given prior $\pi(\theta)$, run MCMC to sample from $\pi(\theta|X) \propto \tilde{m}(x_i, \theta)\pi(\theta)$
- Note: need numerical error in likelihood of $o(\frac{1}{\sqrt{n}})$ to be able to ignore approximation error in asymptotic results[1]
- With panel data, can use conditional densities directly
  - Derive these from Fokker-Planck in continuous time
- When some individual variables unobserved, need to marginalize
  - Use filtering methods (Kalman if linear, Particle o/w)
  - or just simulate and match moments

[1] And other conditions: see Kristensen & Salanié (2017)

Adding Feedback

- Optimization and KFE describe population behavior in given environment
- Close model by adding response of environment to behavior
  - E.g. market clearing. Price matches supply and demand
  - Huggett Model: interest rate R solves $\int (w - c(w, R))m(w)\mathrm{d}w = 0$
- More generally: parameter P of decision problem depends on distribution through some "equilibrium condition"
- For some problems "P" is itself a function
  - Nash equilibrium: respond directly to others' behavior
  - Spatial/network models: response at every value of argument
- Steady state equilibrium is
  - Decision rule $h(.)$, distribution μ, parameter P, st.

  1. Given P, $h(.)$ satisfies optimality condition
  2. μ satisfies steady state KFE derived from $h(.)$
  3. P satisfies equilibrium condition given $h(.), \mu$

Solution Procedure: "Huggett Algorithm"

- Write function $J(P)$ for equilibrium
  1. Given P, solve optimization problem for policy rule $h(.)$
  2. Use $h(.)$ to solve KFE for steady state distribution μ
  3. Output value of equilibrium condition equation given h, μ, P
- Solve $J(P) = 0$ by, e.g., bisection
- Step 1 by Value/Policy iteration or projection for Euler equation
- Step 2 by fixed point iteration or eigensolver
- Step 3 usually involves numerical integration
- If P a vector, use nonlinear solver
  - Keep tolerance large, since interior loops approximate
- If P is function, use functional equation methods (projection, etc) to solve $J(P) = 0$
- Nested structure convenient because interior problems often contraction *conditionally* on value of other arguments

Example: Huggett Model

- Reduce optimization problem to Bellman or Euler equation
  - Bellman easier, more robust, Euler gets rid of max
- Fix R, solve Euler for consumption rule

$$\ell(w) = \mathbb{E}_t \beta R \int g(w' - R(w - c(w, \ell(w), R))) c(w', \ell(w'), R)^{-\gamma} \mathrm{d}w'$$

  - where $c(w, \ell, R) = \min\{\ell^{1/\gamma}, w - \frac{\mathsf{a}}{R}\}$
- "Parameterized Expectations" transform: $\ell(w) = \beta \cdot$ expected m.u. of consumption (Christiano & Fisher 2000)
  - Turns Euler equation into fixed point problem
  - Gets rid of Lagrange multipliers in equation
  - Represent $\ell(w)$ by basis functions, run fixed point iteration
- Solve KFE for $m(w)$

$$m(w') = \int g(w' - R(w - c(w, \ell(w), R))) m(w) \mathrm{d}w$$

  - Represent $m(w)$ by basis functions, solve by eigensolver
- Bisection in R to solve $\int (w - c(w, \ell(w), R)) m(w) \mathrm{d}w = 0$

Results: Equilibrium Policy Function and State Distribution

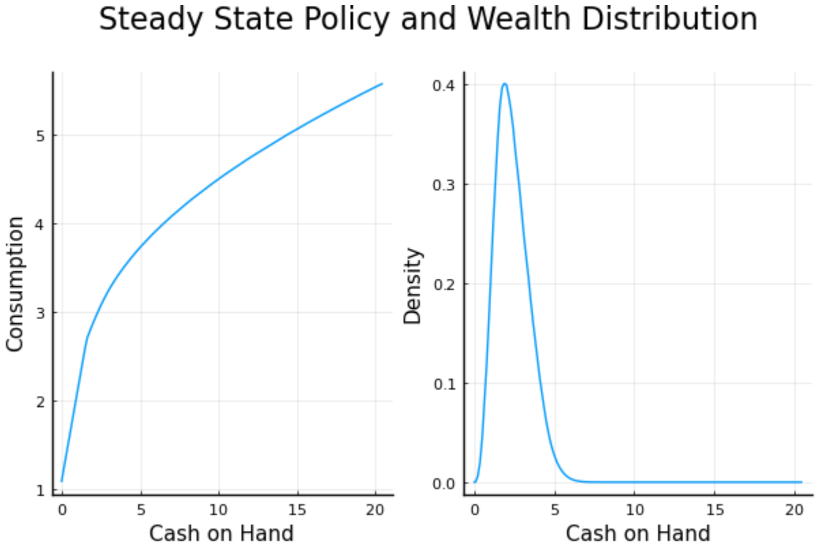

Figure 1: Huggett Model Results

Extension: Nonstochastic dynamics

- Suppose initial condition $t = 0$ not a steady state
- Nonstochastic dynamics can be solved by shooting method
  - Use Huggett algorithm to solve for steady state
  - Choose large T for time economy returns to steady state
  - Conjecture **sequence** $\{P_t\}_{t=0}^{T}$, $P_t = P^*$ for $t > T$
  - Solve optimization problem given sequence (eg by iterating value functions backward)
  - Solve KFE forward from μ_0 using sequence of policies
  - Check *sequence* of equilibrium conditions
  - Iterate over prices until equilibrium conditions satisfied
- Describes path response to unanticipated economic changes
  - "MIT Shock": non-equilibrium parameter change
    - Start at a steady state
    - Change a parameter of economic environment
    - Solve for path back to new steady state

Stochastic Dynamics

- Consider adding random variables Z_t that affect all agents
  - Business cycles: income distribution fluctuates
  - Aggregate policy (fiscal/monetary/regulatory)
- Creates major computational problem
  - Whole distribution shifts in response to shocks
  - Stochastic law of endogenous distribution enters decision problem
  - Value and policy functions take distribution as argument
- This is a functional equation "squared":
  - Functions to solve for are functions of functions: "Operators"
  - E.g. map from income distribution to spending
- Try to solve by functional equation approach
  - Projection
  - Perturbation
- How to represent infinite dimensional function?
  - Curse of dimensionality
  - Need approximation which reduces dimension

Example: Huggett with aggregate shocks

- Take original Huggett model but
- instead of income $s_t^i \stackrel{iid}{\sim} g(.)$ face income $s_t^i + z_t$
  - $s_t^i \stackrel{iid}{\sim} g(.|\sigma_t)$, σ_t, z_t shared for all agents
- $Z_t = (\sigma_t, z_t)$ are aggregate shocks to mean and variance of income risk
- Direct effect: 2 more variables in decision problem
- Indirect effects
  1. since shared, whole wealth distribution depends on them
  2. Interest rate now stochastic
  3. to keep track of interest rate, need to know state of shocks **and** distribution
- Decision rule $c(w, Z, m(.))$ takes additional arguments
  - One of which is a distribution

Dimension reduction method 1: Krusell Smith method

- Projection represents decision/value function by basis functions over arguments
- Polynomials or discretization over ∞ arguments not representable
- So cut down # of arguments
- Maybe you don't really need full distribution $m(w)$
- Approximate by small number of summary statistics ζ
  - Mean $\zeta_1 = \int wm(w)dw$, Variance $\zeta_2 = \int w^2 m(w)dw$, etc
  - Include enough info to describe components of state that matter for agents: eg, prices
  - $c(w, Z, \zeta)$ takes (small) finite number of arguments
- Idea: "approximate aggregation"
  - Only care about distribution as way to forecast R
  - Can get good forecast accuracy with just moments
  - Use this forecast in place of true law in decision rule
  - Solve for fixed point in which implied decision rule generates law consistent with forecast

Krusell Smith Method: Steps

1. Conjecture a simple parameterized law of motion for ζ
   - Typically linear conditional on shocks: e.g. $\zeta' = \beta_0(Z) + \beta_1(Z)\zeta$
2. Solve optimization problem with conditional expectation over distribution given by above law
3. Simulate shocks Z, population of agents following policy, and prices each period
4. Re-estimate parameters β for moments using simulated data
5. Repeat until coefs from simulated moments match conjectured coefficients

- Works well if you get good forecasts with just moments
  - Not a feature of all models: can fail, especially if many shocks
  - Interpret as behavioral: agents discard info to make forecast

Perturbation Approach

- Linearization method from last class can be applied directly
  - Instead of scalar variables, allow **functions** to be variables
  - Instead of scalar derivatives, take **functional** derivatives
- Huggett model:
  - Variables are $\ell_t(.), m_t(.), \sigma_t, z_t, R_t$
  - Equations are Euler, KFE, market clearing, and law of shocks
- Functional derivative of a nonlinear operator $F(.)$ is a linear map between functions
  - $\int k(x, y)[.]dx$ is linear map taking $f(x)$ to $f(y)$
- Still ∞ dimensional but can approximate by Galerkin approach
  - Function valued variables replaced by vector of k coefficients
  - Derivatives of equilibrium conditions just $k \times k$ matrices
- As $k \to \infty$, approximate derivatives converge
- Apply standard RE solver to derivative matrices
  - Approximates first order Taylor approx of policy operator

Perturbation method: Details

- To use linearization, need nonstochastic steady state values
  - Only shut down *aggregate* shocks (e.g, σ_t, z_t, *not* s_t^i)
  - Exactly result of Huggett algorithm!
- Need system of equations differentiable w.r.t. functions
  - Usually means using FOC approach
- How to take functional derivatives?
  - Derive symbolically, then project to get coefs (Childers 2018)
- Alternate approach (Reiter (2009), Winberry (2018), Childers (2022))
  1. Represent functions by vector of coefficients or discretized values
  2. Take scalar derivatives wrt points/coefs
- Discretization works, but inefficient for smooth functions
- Better method: take derivatives wrt points, get coefficients by interpolation

Modified Perturbation: Steps

1. Represent Model as system of functional equations

$$\mathbb{E}\mathcal{F}(x, y, x', y', \sigma) = 0$$

2. Solve for function values at nonstochastic steady state

$$\mathcal{F}(x^*, y^*, x^*, y^*, 0) = 0$$

3. Represent functions by their values at a set of grid points, and integral equations by weighted sums

$$\{\tilde{\mathcal{F}}(\{x(s_j)\}_{j=1}^n, \{y(s_j)\}_{j=1}^n, \{x'(s_j')\}_{j=1}^n, \{y'(s_j')\}_{j=1}^n, \sigma)(t_i)\}_{i=1}^n = 0$$

4. Differentiate w.r.t. function values around steady state

$$(\tilde{\mathcal{F}}_x, \tilde{\mathcal{F}}_y), (\tilde{\mathcal{F}}_{x'}, \tilde{\mathcal{F}}_{y'})$$

5. Apply matrix transforms to get coefficient representation

$$(\tilde{A}, \tilde{B}) = (M(\tilde{\mathcal{F}}_x, \tilde{\mathcal{F}}_y)M^\intercal, M(\tilde{\mathcal{F}}_{x'}, \tilde{\mathcal{F}}_{y'})M^\intercal)$$

6. Apply standard linear ratex solution code to matrices

Perturbation Application: Huggett with Aggregate Shocks

- Solve for steady state setting z, σ shocks to 0
- Represent equilibrium conditions with functions $\ell(w), m(w)$ replaced by values at grid points
  - $\{m(w_i)\}_{i=1}^{nw}$ $\{\ell(w_i)\}_{i=1}^{nw}$
  - Integrals replaced by weighted w/ quadrature weights π^i
- E.g., Kolmogorov equation becomes $0 = \mathcal{F}^2(.)(w_j') =$

$$m_{t+1}(w_j') - \sum \pi^i g(w_j' - R_t(w_i + z_t - c(w_i + z_t, \ell_t(w_i), R_t)); \sigma_t) m_t(w_i)$$

- Euler, market clearing equations replaced similarly
- Apply automatic differentiation to get derivatives
- Map matrix on function values to matrix on coefficients
  - E.g., by Wavelet or just Histogram (piecewise constant) transform M
- Plug into RE algorithm to get policy functions
  - $\tilde{h}_x$ maps $m(.), z, \sigma$ to $m'(.), z', \sigma'$
  - $\tilde{g}_x$ maps $m(.), z, \sigma$ to $\ell(.), R$

IRF: Response of wealth distribution to z

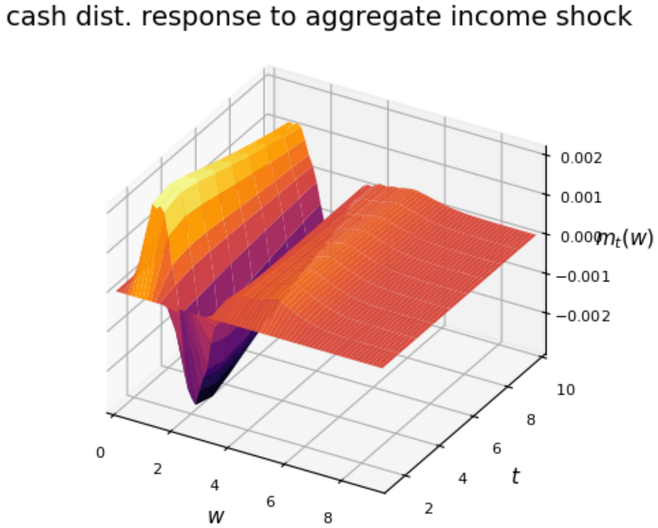

Figure 2: Deviation from Steady State Density

IRF: Response of consumption function to z

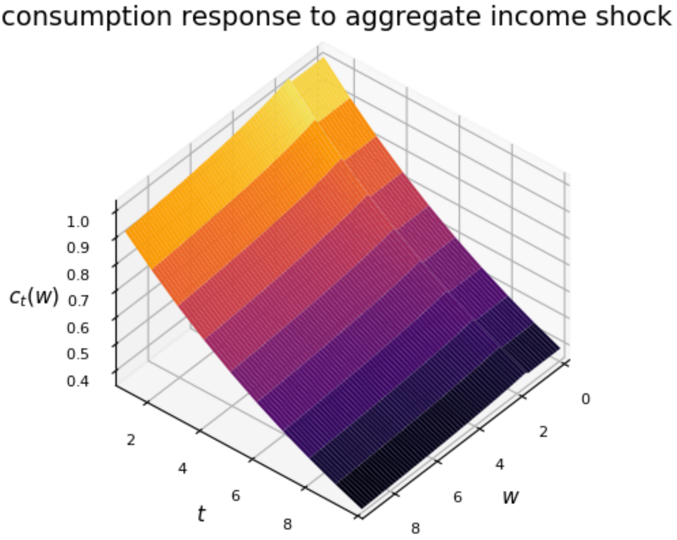

Figure 3: Deviation from Steady State Policy Function

Simulation: Exogenous Shock Process z_t

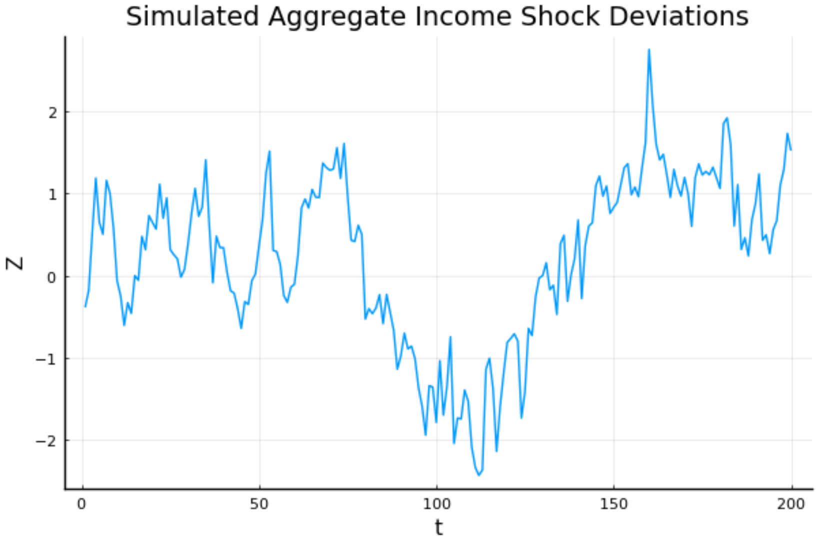

Figure 4: Deviation of z from Steady State

Simulation: Exogenous Shock Process σ_t

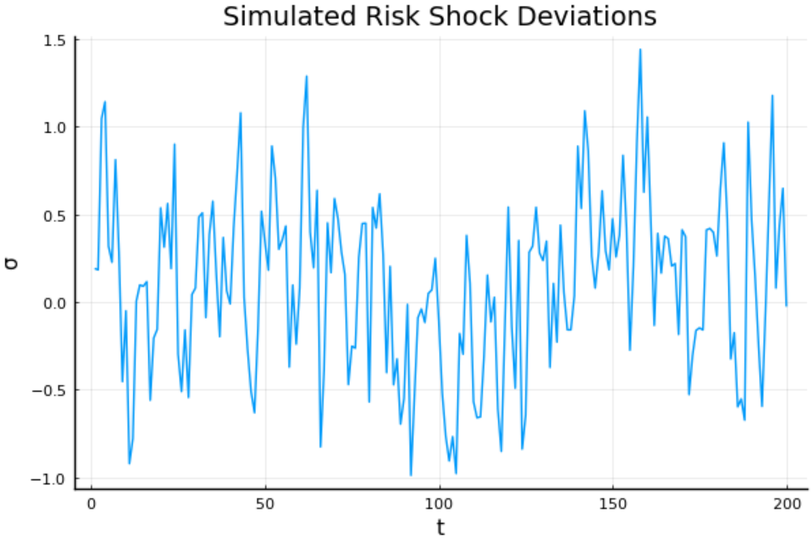

Figure 5: Deviation of σ from Steady State

Simulation: Endogenous Wealth Distribution

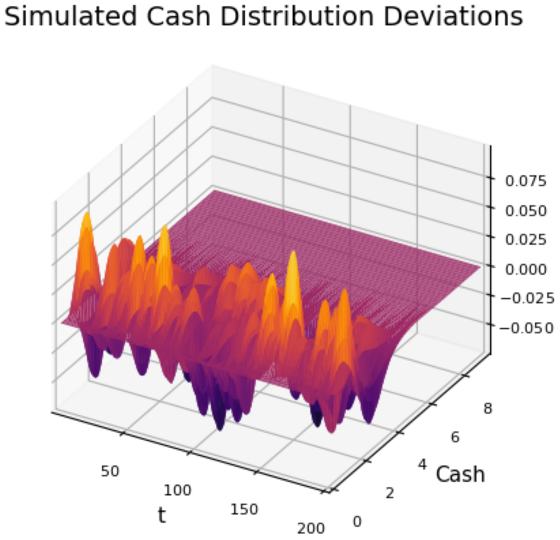

Figure 6: Deviation of $m(w)$ from Steady State

Simulation: Endogenous Consumption Function

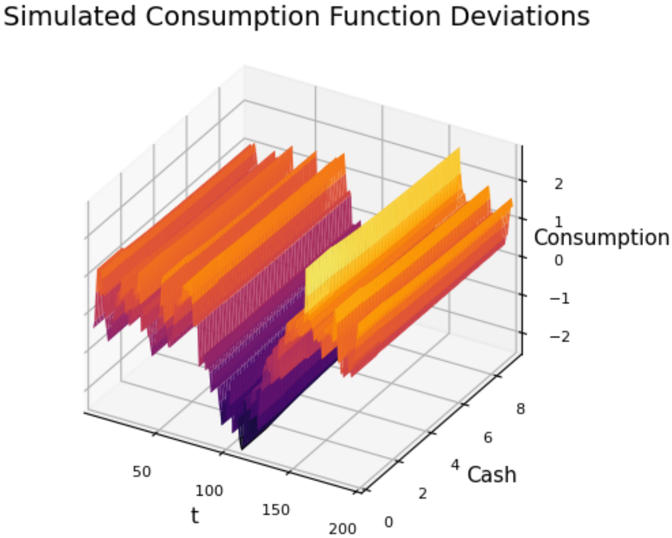

Figure 7: Deviation of $c(w)$ from Steady State

Simulation: Endogenous Interest Rate

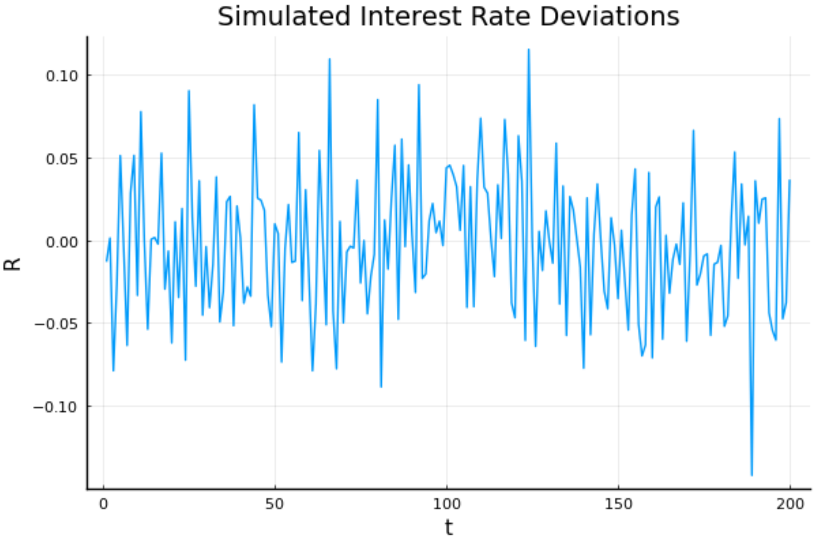

Figure 8: Deviation from Steady State R

Conclusions

- Dynamic models can incorporate heterogeneity and (mediated) interactions by deriving distribution
- Steady state needs 1 more functional equation (Kolmogorov) and a single loop
- Dynamics feasible by incorporating distribution as a state for projection/perturbation
- Krusell-Smith: Standard projection methods workable by massive dimensionality reduction
  - Preserves global, nonlinear effect of aggregates
  - No accuracy guarantees
- Perturbation: take functional derivatives, apply to projected distributions
  - Preserves nonlinearity in individual decisions, not in aggregates
  - Applicable even with many or infinite shocks
    - Whole function can be a stochastic process
  - Strong guarantees for first order

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