Optimal Control approach to dynamic models

Judd Sections 10.6-10.7

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The Plan

- Dynamic Programming approach:
  - Rewrite the model as Bellman equation
  - Solve for policy and value functions
- **Optimal control approach**:
  - Hamiltonian and Pontryagin optimality conditions
  - Derive ODE for state and control (policy), as functions of time
  - Then derive ODE for policy function
- Today:
  - Finite horizon; Example: lifetime savings
  - Infinite horizon; Example: optimal growth
  - Examples are cont. time, cont. state, determ. transition
  - Bonus: Stochastic case: "Forward Backward SDE"

Optimal Control problem with finite horizon

$$\begin{array}{rl} \max_u & \int_0^T e^{-\rho t}\pi(x,u,t)dt + g(x(T)) \\ \text{subject to:} & \dot{x} = f(x,u,t), \quad x(0) = x_0, \end{array}$$

where

- t is time, $\rho > 0$ is the discount rate
- $x \in \mathbb{R}^n$ is a vector of state variables;
- $u \in \mathbb{R}^m$ is a vector of control variables;
- $\pi : \mathbb{R}^{n+m+1} \to \mathbb{R}$ is the payoff flow;
- $g : \mathbb{R}^n \to \mathbb{R}$ is the terminal payoff; and
- $f : \mathbb{R}^{n+m+1} \to \mathbb{R}^n$ is the law of motion (state transition process).

Solution: Pontryagin conditions

Hamiltonian = "Lagrangian for functions":

$$H(x,u,\lambda,t) = \pi(x,u,t) + \lambda^\top f(x,u,t),$$

where $\lambda \in R^n$ is a vector of costate variables ("multipliers")

1. The optimality condition: $u = \arg\max H(x,u,\lambda,t)$

   $$\text{FOC: } 0 = \frac{\partial H}{\partial u} = \pi_u(x,u,t) + \lambda^\top f_u(x,u,t).$$

2. Law of motion: $\dot{x} = \frac{\partial H}{\partial \lambda} \Rightarrow \dot{x} = f(x,u,t)$.
3. Costate equation: $\dot{\lambda} = \rho\lambda - \frac{\partial H}{\partial x} =$
   $= \rho\lambda - \pi_x(x,u,t) - \lambda' f_x(x,u,t)$
4. Initial condition: $x(0) = x_0$.
5. Transversality condition (TVC): $\lambda(T) = g'(x(T))$;
   1. In BVP: $g(x(T)) \equiv 0$; impose $x(T) = x_T$ instead $\Rightarrow$ terminal condition on $x(T)$ replaces the TVC.

Example: Life-cycle consumption

Let $A(t)$ be assets that consumer holds at time t:

$$\begin{array}{rl} \max_c & \int_0^T e^{-\rho t} u(c) dt \\ \text{subject to} & \dot{A} = f(A) + w - c, \\ & A(0) = A(T) = 0 \end{array}$$

1. Hamiltonian: $H = u(c) + \lambda \left[f(A) + w - c \right]$
2. The optimality condition $H_c = u'(c) - \lambda = 0$
    1. Differentiate w.r.t. time: $\dot{\lambda} = u''(c)\dot{c}$.
3. Costate: $\dot{\lambda} = \rho\lambda - H_A = \rho\lambda - \lambda f'(A)$

Use 2 & 2.a to eliminate λ from 3

Example continued

- Combine optimality and costate eq-ns:

$$\dot{c} = \frac{u'(c)}{u''(c)}\left(\rho - f'(A)\right), \tag{1}$$

- Law of motion and boundary conditions

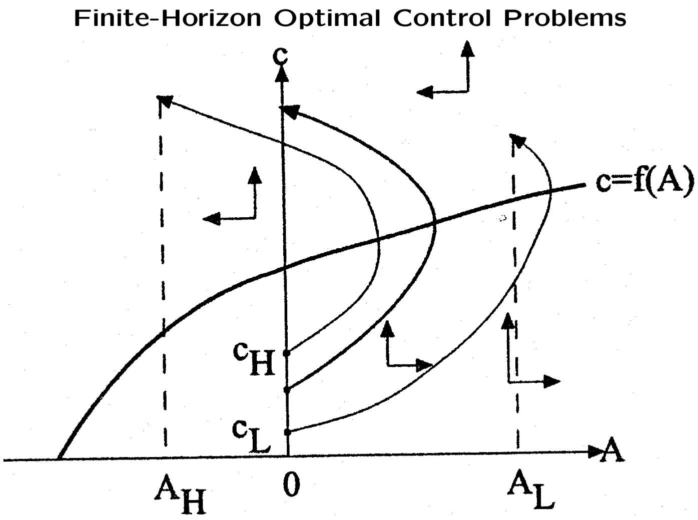
$$\begin{aligned} \dot{A} &= f(A) + w - c, & (2) \\ A(0) &= A(T) = 0 & (3) \end{aligned}$$

- (1)-(3) form a BVP
- Shooting: pick c_0 to ensure $A(T) = 0$

Phase diagram (Figure 10.2)

- Axes are current level of consumption and capital
- Short arrows represent signs of $\dot{c} = c'(t)$ and $\dot{A}$
- $c = f(A)$ line is really $c = f(A) + w$; (2) tells us that:
  - $\dot{A} = 0$ on the line
  - $\dot{A} > 0$ (arrows right) below the line, as $f(A) + w > c$
  - $\dot{A} < 0$ (arrows left) above the line, as $f(A) + w < c$
- Assume $f'(A) > \rho$. Then (1) implies $\dot{c} > 0$,
  so there are only up arrows, and no down ones.
- Curved arrows = *trajectories* $\{A(t), c(t)\}_{t \in [0,T]}$ for different c_0
  - Their direction is governed by short arrows
  - Tip of the arrow corresponds to $A(T), c(T)$,
  - We reach $A(T) = 0$ by adjusting c_0 (c_L, c_H, etc.)

Fig 10.2: Phase diagram in life cycle model



Shooting in a life-cycle consumption problem. Source: Judd, K. (1998), Figure 10.2.

Aside on modeling choices

- Finite horizon should be justified
  - Truncating an infinite-horizon problem creates terminal effects
- In most cases, "death" is random
  - Adjustment to discount factor: $\tilde{\beta} = \beta \Pr\{survival\}$
  - Poisson arival of death in cont. time
- Attempt at justifying Lifecyle problem:
  - $T =$ retirement age: require $A(T) = \bar{A} > 0$
- Infinite-horizon models should have steady state
  - Or stable (limiting) distribution of states
  - It should be inside the modeled interval
  - Otherwise, value function is driven by value at the edge of the interval, which is undetermined
  - If known to *asymptote* to steady state, can pretend it arrives at large finite T: justify by a turnpike theorem
  - Solution to infinite growth: rescale state (e.g. K per capita)

Optimal control with Infinite horizon

No more terminal period T

$$\begin{array}{rl} \max_u & \int_0^\infty e^{-\rho t}\pi(x,u,t)dt \\ \text{subject to:} & \dot{x} = f(x,u,t), \quad x(0) = x_0, \end{array}$$

The system of ODEs consists of:

- Optimality condition: $\frac{\partial H}{\partial u} = 0$
- Law of motion: $\dot{x} = \frac{\partial H}{\partial \lambda} = f(x, U(x,\lambda,t), t)$.
- Initial condition: $x(0) = x_0$.
- Costate equation: $\dot{\lambda} = \rho\lambda - \frac{\partial H}{\partial x}$
- **TVC**: $\lim_{t\to\infty} e^{-\rho t}|\lambda(t)^T x(t)| < \infty$
  - Satisfied if variables converge to a **steady state**: $(u^*, x^*) : \dot{u} = \dot{x} = 0$

Example: Optimal growth

- Recall the optimal growth problem

$$\begin{array}{rl} \max_c & \int_0^\infty e^{-\rho t} u(c) dt \\ \text{subject to:} & \dot{k} = f(k) - c, \qquad k(0) = k_0 \end{array}$$

- The optimality condition $0 = u'(c) - \lambda$ implies $\dot{\lambda} = u''(c)\dot{c}$.
- Thus, the system of ODEs is

$$\dot{c} = \frac{u'(c)}{u''(c)} \left(\rho - f'(k)\right)$$
$$\dot{k} = f(k) - c$$
$$k(0) = k_0$$
$$\lim_{t \to \infty} e^{-\rho t} |\lambda(t) k(t)| < \infty$$

- TVC is satisfied if $\forall \epsilon > 0 \; \exists T$ s.t. $\forall t \geq T$, $\left|\dot{k}(t)\right|, |\dot{c}(t)| < \epsilon$

Steady state & Phase diagram

- The steady state (k^*, c^*) satisfies $\dot{k} = \dot{c} = 0$
- Implies

$$\begin{aligned} f(k^*) - c^* &= 0 \\ \rho - f'(k^*) &= 0 \end{aligned}$$

- Algorithm 10.3: Shoot for $c(T) = c^*$, where T is the first time such that $\dot{k}(T) \leq 0$ or $\dot{c}(T) \leq 0$.
- Phase diagram (Figure 10.3, assume $k_0 < k^*$):
  - If c_0 is too large, then the path crosses the $\dot{k} = 0$ line, so that $k(t)$ keeps falling and $c(t)$ keeps rising.
  - If c_0 is too small, the path crosses the $\dot{c} = 0$ line, so that $k(t)$ keeps rising and $c(t)$ keeps falling.

Reverse shooting

- Problem: $k(T)$ too sensitive to $c(0)$ when T is large.
- Solution – reverse shooting (from T to 0):
  Initial state is not very sensitive to terminal state
- Reverse the direction of time ($s = -t$), conditions change sign:

$$\dot{c} = -\frac{u'(c)}{u''(c)}\left(\rho - f'(k)\right),$$
$$\dot{k} = -\left(f(k) - c\right),$$

- $\Rightarrow$ Phase diagram in figure 10.4
- The unstable manifold of the new system
  is the stable manifold of the old system.
- So shooting succeeds

Shooting and Reverse Shooting

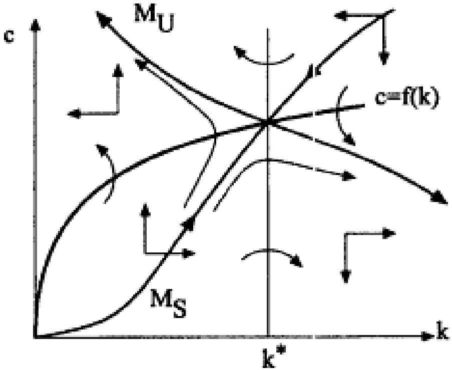


Figure 10.3
Shooting in a saddle-point problem

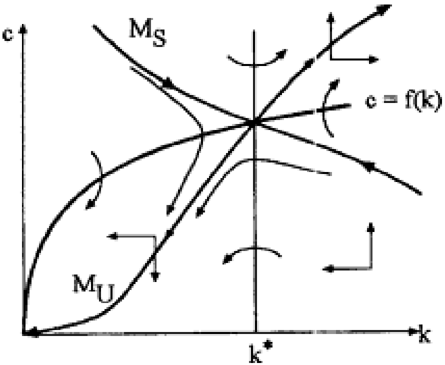


Figure 10.4
Reverse shooting in a saddle-point problem

Stochastic Case: Forward Backward SDE

$$\begin{array}{ll} \max_u & \int_0^T \pi(x,u,t)dt + g(x(T)) \\ \text{subject to:} & dx = f(x,u,t)dt + \sigma(x,u,t)dW, \quad x(0) = x_0, \end{array}$$

- Generalized Hamiltonian

$$H(x,u,\lambda,z,t) = \pi(x,u,t) + \lambda^T f(x,u,t) + Tr(\sigma^T(x,u,t)z)$$

1. Optimality: $\hat{u}_t = \arg\max H(x_t,u,\lambda,z_t,t)$
2. Law of motion: $dx_t = f(x_t,\hat{u}_t,t)dt + \sigma(x_t,\hat{u}_t,t)dW_t$.
3. Costate equation: $d\lambda_t = -\nabla_x H(x_t,\hat{u}_t,\lambda_t,z_t,t)dt + z_t dW_t$
4. Initial condition: $x(0) = x_0$
5. Transversality condition (TVC): $\lambda(T) = g'(x(T))$

- Solve SDE system for (x_t,u_t,λ_t,z_t): 1 extra costate
- where $\lambda_t = \nabla_x V(x_t,t)$, $z_t = \sigma(x_t,\hat{u}_t,t)^T \text{Hess}_x V(x_t,t)$

Opt. control vs. Dyn.programming

The Good:

- No need to solve for value function, or depend on covergence
- Provides timepaths $(c(t), k(t))$ directly
- Works well with finite-horizon models

The Bad:

- Stochastic transition case difficult: "Forward Backward SDE"
  - Additional costate z_t is stochastic
  - It has to be specified as function of transition process

The Ugly:

- Derivation-intensive, can be numerically unstable

References

- Judd Ch. 10
- Morton Kamien and Nancy Schwartz (2012) *Dynamic optimization: the calculus of variations and optimal control in economics and management*
  - Classic with proofs and derivations
- Hu and Laurière "Recent Developments in Machine Learning Methods for Stochastic Control and Games"
  - Survey with numerical methods including for FBSDE approach