Modelling with Cosserat Theory: Octopus Edition

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Introduction

Definition: A *rod* is a material object in three-dimensional space characterised by having one of its dimensions significantly larger than the other two (i.e. longer than it is thick), also called quasi-one dimensional.
We can model an elastic rod (one that can return to its original shape after deformation) as a Cosserat rod. Our aim will be to model an octopus arm as such. Named after the Cosserat brothers in 1909, the Cosserat model is a generalised version of the Kirchhoff model that allows for axial stretching and shearing. So in order to understand Cosserat's model, we must first understand Kirchhoff's model. The following introduction to Elastic Rod Theory is an adaptation of the one given in [2].

Elastic Rod Theory (Kirchhoff's Version)

The geometric shape of our rod is governed by a central curve,

$$\mathbf{r}(s,t) : [0,L] \times [0,t] \longrightarrow \mathbb{R}^3 \times \mathbb{R}$$

which we call the *centreline* of the rod. s is the arc-length of $\mathbf{r}$ and t is time.

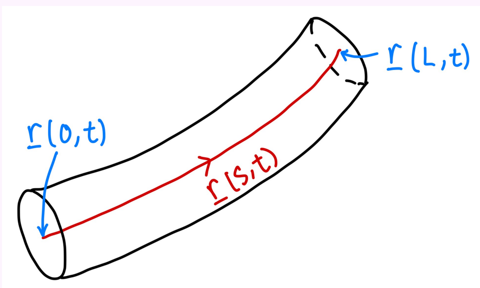

Figure 1. rod with centreline (red); start and end points (blue) at $\mathbf{r}(0,t)$ and $\mathbf{r}(L,t)$, respectively

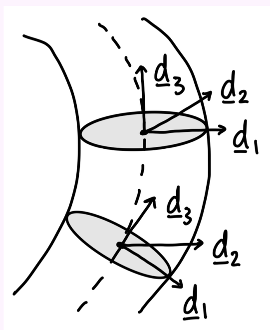

Figure 2. local basis along the centreline at two different cross sections

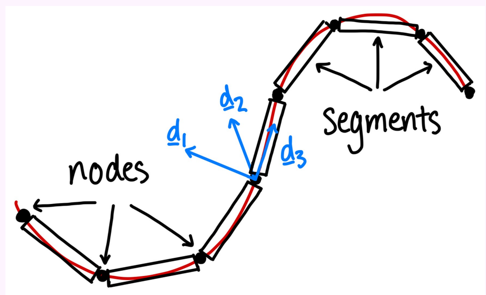

Figure 3. assembly of segments (and nodes) making up an overall bent rod

To quantify the changing shape of the rod, we label an orthonormal basis $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ in $\mathbb{R}^3$ such that $\mathbf{d}_3$ points in the direction of the tangent, $\mathbf{r}_s = \frac{\partial \mathbf{r}}{\partial s}$, to the centreline, and $\mathbf{d}_1$ and $\mathbf{d}_2$ spans the material cross section of the rod (Figure 2). We call this the *local frame* of $\mathbf{r}$. Given a vector in the laboratory (Eulerian) frame, $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, we can find the corresponding vector in the rod's local (Lagrangian) frame, $\mathbf{x}_{\mathcal{L}} = a\mathbf{d}_1 + b\mathbf{d}_2 + c\mathbf{d}_3$, by putting it through the rotational matrix $\mathbf{Q}(s,t) \in SO(3)$. Like so:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \mathbf{d}_3 \end{pmatrix} \qquad \mathbf{x}_{\mathcal{L}} = \mathbf{Q}\mathbf{x}\,, \qquad \mathbf{x} = \mathbf{Q}^{\top}\mathbf{x}_{\mathcal{L}}\,, \qquad \mathbf{Q}^{\top}\mathbf{Q} = \mathbf{1}$$

The third equivalence comes from the definition of an orthogonal matrix, $\mathbf{1}$ is the identity matrix.

Kirchhoff's model pictures the rod as an assembly of short, *undeformable* segments that are connected at nodes (Figure 3), allowing them to twist and bend relative to each other. Available movements include:

- bending in the $\mathbf{d}_1$ direction (so in the $\{\mathbf{d}_1, \mathbf{d}_3\}$ plane), represented by κ_1
- bending in the $\mathbf{d}_2$ direction ($\{\mathbf{d}_2, \mathbf{d}_3\}$ plane), represented by κ_2
- twisting about $\mathbf{d}_3$, represented by κ_3

We define the following to track how our rod changes with respect to time and space:

$$\frac{\partial \mathbf{d}_i}{\partial t} = \boldsymbol{\omega} \times \mathbf{d}_i = (\mathbf{Q}^{\top}\boldsymbol{\omega}_{\mathcal{L}}) \times \mathbf{d}_i \qquad i = 1,2,3 \qquad (1)$$

$$\frac{\partial \mathbf{d}_i}{\partial s} = \boldsymbol{\kappa} \times \mathbf{d}_i = (\mathbf{Q}^{\top}\boldsymbol{\kappa}_{\mathcal{L}}) \times \mathbf{d}_i \qquad i = 1,2,3 \qquad (2)$$

where $\boldsymbol{\omega}$ is the *angular velocity*, and $\boldsymbol{\kappa}$ is the *generalised curvature*. The remaining equations will be introduced in their generalised form in Cosserat's column.

Elastic Rod Theory (Cosserat's Version)

Adding in 3 more degrees of freedom at each cross section, we now come onto the Cosserat model.

Consider stretching, changing the centreline means changing the arc-length associated with our rod. We set the unstretched state (with arc-length s) to be the *rest configuration*. Say that the stretched state or *current configuration* has arc-length $\hat{s}$, then the factor of stretching is $e = \frac{d\hat{s}}{ds}$, and we have $e \cdot \mathbf{r}_s$ as the tangent vector of the current rod. In the event of shearing, the $\mathbf{d}_3$ component points away from the tangent vector. We represent this with the shearing term, $\boldsymbol{\sigma} = e \cdot \mathbf{r}_s - \mathbf{d}_3$. In the case that $e = 1$ and $\mathbf{d}_3 = \mathbf{t}$, we get Kirchhoff's model for inextensible and unshearable rods.

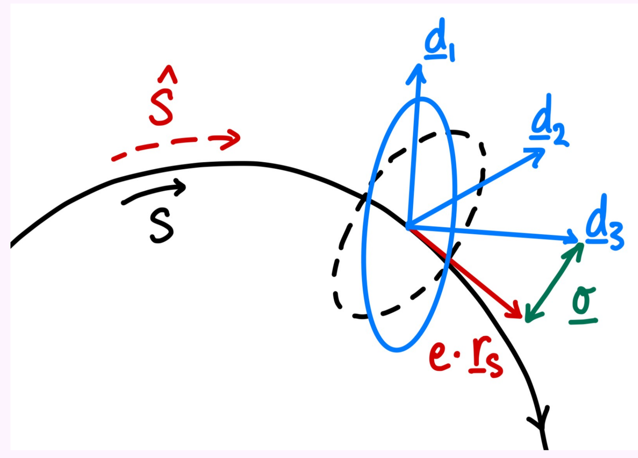

Figure 4. rest configuration arc-length and cross section (black); stretched arc-length and tangent vector (red); cross section of sheared rod (blue); shearing term (green)

Obviously, any stretching and or shearing will affect more than just the $\mathbf{d}_3$ component. For example, assuming our rod is *volume preserving*, any stretching along the axis will cause it to thin out radially, shrinking the *cross sectional area*, A. This lowers its *second moment of inertia*, I, which in turn reduces the rod's bending/twisting stiffness. A thinner cross section also results in a lower shearing/stretching stiffness, making the rod easier to shear/stretch. So we must scale our model accordingly.

Finally, defining *velocity* to be

$$\frac{\partial \mathbf{r}}{\partial t} = \mathbf{v} \qquad (3)$$

we can use Newton's second law to get the following equations for the conservation of momentum:

Conservation of linear momentum:

$$\frac{\partial(\rho A \mathbf{v})}{\partial t} = \overbrace{\frac{\partial}{\partial s}\left(\frac{\mathbf{Q}^{\top}\mathbf{S}\boldsymbol{\sigma}_{\mathcal{L}}}{e}\right)}^{\text{internal shear/stretch force}} + e\mathbf{f} \qquad (4)$$

Where ρ is the constant *material density*, $\mathbf{S}$ is the diagonal matrix containing the *shearing/stretching stiffness coefficients*, and $e\mathbf{f}$ is the scaled *external force*.

Conservation of angular momentum:

$$\frac{\partial}{\partial t}\left(\frac{\rho \mathbf{I}\boldsymbol{\omega}_{\mathcal{L}}}{e}\right) = \overbrace{\frac{\partial}{\partial s}\left(\frac{\mathbf{B}\boldsymbol{\kappa}_{\mathcal{L}}}{e^3}\right) + \frac{\boldsymbol{\kappa}_{\mathcal{L}} \times \mathbf{B}\boldsymbol{\kappa}_{\mathcal{L}}}{e^3}}^{\text{bend/twist couples}} + \overbrace{\left(\mathbf{Q}\frac{\partial \mathbf{r}}{\partial s} \times \mathbf{S}\boldsymbol{\sigma}_{\mathcal{L}}\right)}^{\text{shear/stretch couples}} + \underbrace{\left(\rho \mathbf{I} \cdot \frac{\boldsymbol{\omega}_{\mathcal{L}}}{e}\right) \times \boldsymbol{\omega}_{\mathcal{L}}}_{\text{transport term}} + \underbrace{\frac{\rho \mathbf{I}\boldsymbol{\omega}_{\mathcal{L}}}{e^2} \cdot \frac{\partial e}{\partial t}}_{\text{unsteady dilation}} + e\mathbf{c}_{\mathcal{L}} \qquad (5)$$

Where $\mathbf{B}$ is the diagonal matrix containing the *bending/twisting stiffness coefficients*, and $e\mathbf{c}_{\mathcal{L}}$ is the scaled *external couple*.

(1)-(5) make up the governing equations of the Cosserat model (for full derivation, see *Forward and inverse problems in the mechanics of soft filaments*[2] or my report).

References

[1] PyElastica - Release 0.2.3 Gazzola Lab - Cosserat Rods, 19 May 2022.

[2] M. Gazzola, L. H Dudte, A. G. McCormick, and L. Mahadevan. Forward and inverse problems in the mechanics of soft filaments. *R. Soc. open sci.* **5**:171628, 2018.

[3] E.B.L. Kenedy, K.C. Buresch, P. Boiapally, et al. Octopus arms exhibit exceptional flexibility. *Scientific Reports*, 2020.

[4] W. M. Kier and M. P. Stella. The arrangement and function of octopus arm musculature and connective tissue. *Journal of Morphology, 268:831-843*, 2007.

Application: Modelling an Octopus Arm as an Elastic Rod

My arm has finite degrees of freedom (7 to be exact: 3 at the shoulder, 3 at the wrist, and 1 at the elbow), this is due to the fact that I only have 3 points (joints) between my arm bones that allow for motion. Unlike mine, an octopus's arm, devoid of bones, is free to bend and twist at any point along itself. Meaning they have essentially infinite degrees of freedom. This makes them a very useful template for robot design.

Kirchhoff or Cosserat

Octopus arms, like many other cephalopod limbs, are *muscular hydrostats*, meaning they are composed of a densely packed arrangement of muscle fibres[4]. This soft muscular structure allows them to extend and contract, therefore Cosserat's rod would be the one best suited for the model we're trying to achieve.

Fixed Endpoint Boundary Condition

In this project, we will focus on the modelling of one arm. Octopus arms are generally connected to an octopus body. To represent this in our model, we will be fixing one end of our Cosserat rod to a point. We make the assumption that the rest of the body (including its other 7 arms) will be stationary and will have negligible effects on the modelled arm.

Accounting for Changing in Thickness

As shown in Figure 5, the shape of an octopus's arm is not cylindrical. The proximal base of the arm has a wider radius than the distal tip. In other words, it tapers out as it distances from the main body. Article [3] tells us that this tapering is uniform and [4] indicates that the proportion of each type of muscle fibre at every cross section along the arm is relatively constant. Naturally, we can assume the thicker the arm, the more resistant it will be to bending and twisting. So to account for that, we will impose a decreasing bending stiffness as we travel from base to tip. Another point to note is that octopuses have structures called *papillae* in their dermal layer which protrude and retract, giving them the ability to change their skin texture to match the environment they wish to blend into. This could be something we consider later on in the project, but for now, we will assume smooth skin all over our octopus. To simplify things further, we will also not be considering the effects of the suckers.

Figure 5. ©Joel Satore/Photo Ark. A common octopus

Simulating with PyElastica

For this project, we will be solving our equations and simulating our elastic rod (octopus arm) using the software PyElastica.

Starting with an example (for modelling snakes) from Gazzola lab[1], I made modifications based on the points above to achieve the result shown in Figure 6.

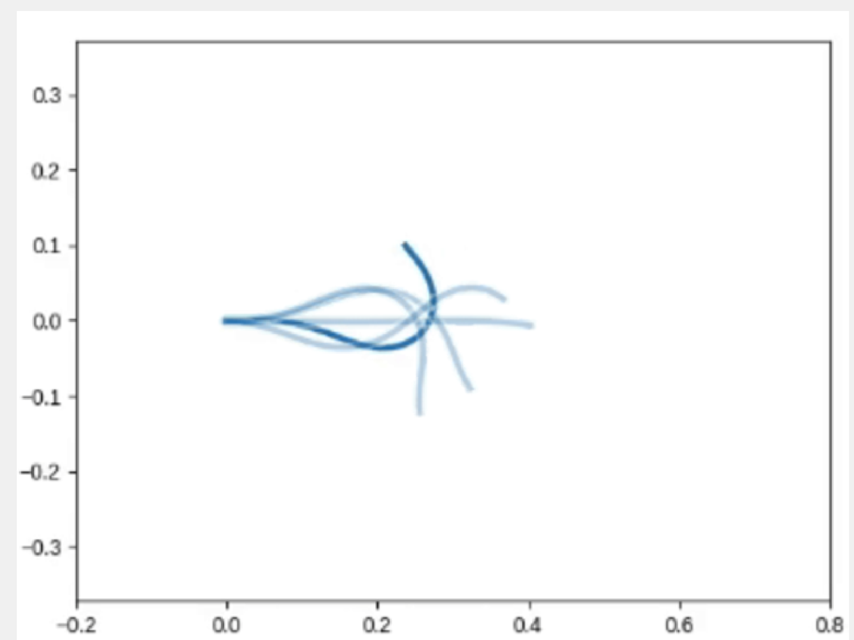

Figure 6. planar view of the movement of an octopus arm modelled as Cosserat rod

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