Problems on Probability and Random Variables

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Questions marked with (*) are from *Probability Models* by Sheldon Ross

1 Probability

1. (*) A box contains three marbles: one red, one green, and one blue. Consider an experiment that consists of taking one marble from the box then replacing it in the box and drawing a second marble from the box. What is the sample space? If, at all times, each marble in the box is equally likely to be selected, what is the probability of each point in the sample space?

2. (*) Repeat Question 1 when the second marble is drawn without replacing the first marble.

3. (*) An individual uses the following gambling system at Las Vegas. She bets \$1 that the roulette wheel will come up red. If she wins, she quits. If she loses then she makes the same bet a second time only this time she bets \$2; and then regardless of the outcome, quits. Assuming (unrealistically!) that she has a probability of 1/2 of winning each bet, what is the probability that she goes home a winner? Why is this system not used by everyone?

4. (*) If $P(E) = 0.9$ and $P(F) = 0.8$, show that $P(E \cap F) \geq 0.7$. In general, show that: $P(E \cap F) \geq P(E) + P(F) - 1$. This is known as Bonferroni's inequality.

5. Bayes theorem states that $P(E|F) = \frac{P(E)P(F|E)}{P(F)}$ whenever $P(F) > 0$. Prove that Bayes theorem is correct.

6. (*) Suppose that 5 percent of men and 0.25 percent of women are colour-blind. A colour-blind person is chosen at random. What is the probability of this person being male? Assume that there are an equal number of males and females.

7. (*) A die is rolled twice. What is the conditional probability that the first die is six given that the sum of the dice is seven?

2 Random variables

1. Let X be a discrete random variable with the following distribution: $P(X = 1) = 0.2$, $P(X = 2) = 0.4$, $P(X = 5) = 0.4$. Compute the mean and variance of X.

2. Recall that $E(X)$ denotes the mean (expected value of) X. Show that $E(aX + b) = aE(X) + b$ for any scalars a and b. Prove this for both discrete and continuous random variables.

3. Show that, for any two random variables X and Y, $E(X+Y) = E(X) + E(Y)$.

4. Find two random variables X and Y such that $E(XY) \neq E(X)E(Y)$.

5. Show that, for any random variable, and any scalar a we have $\mathrm{Var}(aX) = a^2\mathrm{Var}(X)$.

6. Show that $\mathrm{cov}(X,Y) = E(XY) - E(X)E(Y)$.

7. Show that if X and Y are independent then $E(XY) = E(X)E(Y)$.

8. Show that if X and Y are independent then $\mathrm{cov}(X,Y) = 0$.