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# Subset solving algorithm: A novel sensitivity-based method for damage detection of structures

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ABSTRACT

We present an efficient method for structural damage detection using natural frequencies. The method, which is based on sensitivity analysis of the structure, consists of two main stages. In the first stage, the structural elements are ordered based on their damage probability into a vector referred to as elements damage probability ordering vector (EDPOV). In the second stage, a rather small subset of EDPOV elements are judiciously selected to form a nonlinear system of equations, which are subsequently solved to detect potential damages. In the second stage, the procedure of subset selecting and solving iterates until a feasible solution is achieved. In order to assess the merits of the proposed method some illustrative cases are studied. Numerical results demonstrate the high efficiency of the proposed algorithm compared to those found in the literature.

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## 1. Introduction

The need of industry to health monitoring and curing makes damage detection techniques as one of the most active fields of research in the recent years. These techniques, in which the damages are identified through a non-destructive test (NDT) instead of visual or locally experimental techniques, have been successfully applied to many practical problems [1,2].

The process of damage detection can be performed using many structural responses. However, among them the use of dynamic response of structures such as natural frequencies, complete or incomplete mode shapes and frequency response functions are the most utilized ones. Carden [3] has been provided a comprehensive literature review regarding recent efforts on the vibration-based damage detection method. However, in practice, it has been recognized, mode shapes are more difficult to estimate than natural frequencies [4]. Therefore, in this study, the changes of natural frequencies are used as a criterion for detecting structural damages. In this work the damage is represented by a reduction in the Young modulus.

Damage detection methods can be generally divided into two branches: optimization-based methods and direct methods. In the first one, damaged elements and damage extends are searched through an optimization process until the response of a hypothesized damaged structure equals those of a real damaged structure. Since in general damage detection the number, location and extent of the damage(s) are all unknowns, therefore their corresponding optimization method is computationally expensive. Many efforts have been devoted to reduce the size of search space in the optimization-based problems and also to introduce the direct methods [5,6] that can be found very fast in comparison with the optimization-based methods.

During the last years, many useful approaches based on the mentioned methods have been reported in the literature. A correlation-based damage detection approach referred to as multiple damage location assurance criterion (MDLAC), has

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been presented by Messina et al. [7] for localizing multiple damages. Further, for estimating the size of defects, the first and second order approximations are applied. In a study, Koh and Dyke [8] have employed genetic algorithm as the optimizer to maximize correlation of measured and hypothetically damaged structure, namely MDLAC index, to detect damages of a cable stayed bridge. A new study has also been made by Lee [9] to identify the multiple beams' cracks which are modeled by hinges and rotational springs, in a beam structure. The sensitivity of responses with respect to both cracks depths and locations has been derived using the finite difference method. The established sensitivity matrix is further used in Newton–Raphson procedure for equalizing the measured and analytically synthesized natural frequencies to obtain location and depths of the crack(s). A comparison between the sensitivity-based methods with the optimization-based methods for damage detection using natural frequencies has been made by Gomes and Silva [10]. In that study, a modified MDLAC has been used as the objective function for their optimization-based method. While both methods were good for detecting the damage sites but not so good for predicting the damage extents. Au et al. [11] used the incomplete mode shapes and natural frequencies for damage detection. To limit the search space, the possible damaged element sites have been approximately identified using the elemental energy quotient difference ratio (EEQDR). Afterward, a genetic algorithm with the small number of initial population, referred to as micro-genetic algorithm, has been employed for damage detection. An application of the micro-genetic algorithm to detect the single crack in a real cracked beam has also been employed by Vakil-Baghmisheh et al. [12]. Recently, Naseralavi et al. by addition of two new operates, specialized the usual genetic algorithm for damage detection. Their method was applied to damage detection using both static and dynamic responses, and its efficiency was concluded [13].

The current research intends to overcome the inaccuracy and high computational effort of the previous works. In this study, an efficient algorithm having two simple stages is proposed. In the first stage, the total element numbers with respect to their damage probability are arranged into a vector. The vector is named here as the elements damage probability ordering vector (EDPOV) and it is generated using the sensitivity matrix which is the first order derivatives of eigenvalues with respect to elemental Young modulus of the undamaged structure. In the second stage of the algorithm, using a sub-algorithm named subset selection unit (SSU) a subset of most potentially damaged elements is selected. After that a set of nonlinear equations for the selected subset is produced and solved by means of the nonlinear equation solving methods. If the obtained solution is feasible, it will be considered as the true damage solution. Otherwise, the SSU is called again to select another subset for solving. This procedure will be repeated until the convergence criteria are satisfied. In addition, in each iteration of the second stage, the probability of other subsets to have all damaged elements has been mathematically derived.

This paper is organized as follows: The application of the sensitivity analysis for the damage detection is presented in Section 2. The proposed algorithm is presented in Section 3, Probability of a subset having all damaged elements and its corresponding theories are discussed in Section 4 and then five illustrative test examples are considered in Section 5. Finally, Section 6 concludes the study.

## 2. Damage detection using sensitivity-based analysis

The problem of structural damage detection is formally equivalent to finding a set of damage variables in a way that the analytical responses of the structure match the measured ones in an optimal way. The mathematical expression of the problem considering the natural frequencies as valuable responses can be defined as:

$$\mathbf{F}_d = \mathbf{F}(\mathbf{X}) \Rightarrow \mathbf{X} = ?, \tag{1}$$

where $\mathbf{X} = \{x_1, x_2, \ldots, x_n\}^T$ is a damage variable vector having $n$ components equal to the total number of structural elements in which $x_i$ is the damage ratio (the ratio of damaged element stiffness to the healthy one) of the $i$th element, $\mathbf{F}_d = \{f_{d1}, f_{d2}, \ldots, f_{dm}\}^T$ is the vector of the first $m$ frequencies of an existing damaged structure and $\mathbf{F}(\mathbf{X}) = \{f_1(\mathbf{X}), f_2(\mathbf{X}), \ldots, f_m(\mathbf{X})\}^T$ is the vector of the first $m$ natural frequencies of a hypothetically damaged structure that can be evaluated from the analytical model.

Using the first order method to approximate the vector of natural frequencies of a structure, expression (1) can be expressed as follows:

$$\mathbf{F}_d = \mathbf{F}(\mathbf{X}) = \mathbf{F}_h + \frac{\partial \mathbf{F}}{\partial \mathbf{X}} \Delta \mathbf{X} + \cdots \Rightarrow \mathbf{F}_d - \mathbf{F}_h = \Delta \mathbf{F} \simeq \mathbf{S} \Delta \mathbf{X}, \tag{2}$$

where $\mathbf{F}_h$ is the frequency vector of the healthy structure, $\Delta \mathbf{X}$ is the vector of damage variable change and $\mathbf{S} = \partial \mathbf{F}/\partial \mathbf{X}$ is referred to as the sensitivity matrix.

Noting $\lambda_i = 4\pi^2 f_i^2$ in which $\lambda_i$ and $f_i$ are the $i$th eigenvalue and frequency of the structure, respectively, expression (2) can also be rewritten with respect to the eigenvalue vector as:

$$\Delta \boldsymbol{\lambda} \simeq \hat{\mathbf{S}} \Delta \mathbf{X}, \tag{3}$$

where $\hat{\mathbf{S}} = [\hat{s}_{ij} = \partial \lambda_i / \partial x_j]$ is the sensitivity matrix of eigenvalues with respect to the damage variables.

Practically, the number of measured frequencies $m$ is usually less than the unknown variables $n$. Hence, this is an indeterminate problem in mathematics and has infinite solutions. Normally, the result with minimum Euclidean norm is taken as a primitive approximation for damaged elements. Therefore,

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$$\Delta \mathbf{X} \simeq \hat{\mathbf{S}}^{+} \Delta\lambda, \tag{4}$$

where $\hat{\mathbf{S}}^{+}$ is the pseudo-inverse of $\hat{\mathbf{S}}$ which can be found by singular value decomposition.

Xia and Hao [14] have used the expression (4) for damage detection in a noisy measurement response of structures. In two other studies, Gue [15] and Gue and Li [16] have also used expression (4) as one of the sources of information for identifying the damages.

A fast approach for calculating the derivatives of eigenvalues with respect to an arbitrary variable using the stiffness and mass matrices of structural elements and the eigenvectors have been presented in Refs. [17,18]. This approach can be simply employed here to determine each arbitrary component $\hat{s}_{ij}$ of the required sensitivity matrix in an index notation form as:

$$\hat{s}_{ij} = \frac{\partial \lambda^{(i)}}{\partial x_j} = \frac{\partial \mathbf{K}_{rs}}{\partial x_j} \phi_r^{(i)} \phi_s^{(i)} - \lambda^{(i)} \frac{\partial \mathbf{M}_{rs}}{\partial x_j} \phi_r^{(i)} \phi_s^{(i)}, \quad i = 1, \ldots, m, \; j = 1, \ldots, n, \tag{5}$$

where $\mathbf{K}$ is the stiffness matrix, $\mathbf{M}$ is the mass matrix of the original system, and $\phi^{(i)}$ denotes the $i$th eigenvector. The suffixes $r$ and $s$ must be summed over the number of the rows and columns of the stiffness and mass matrices of the particular element under consideration. In this study, since the damage variables are considered as a reduction in elasticity modules of an individual element, they will not affect the mass matrix, thus, the second term of the second side of Eq. (5) will be vanished.

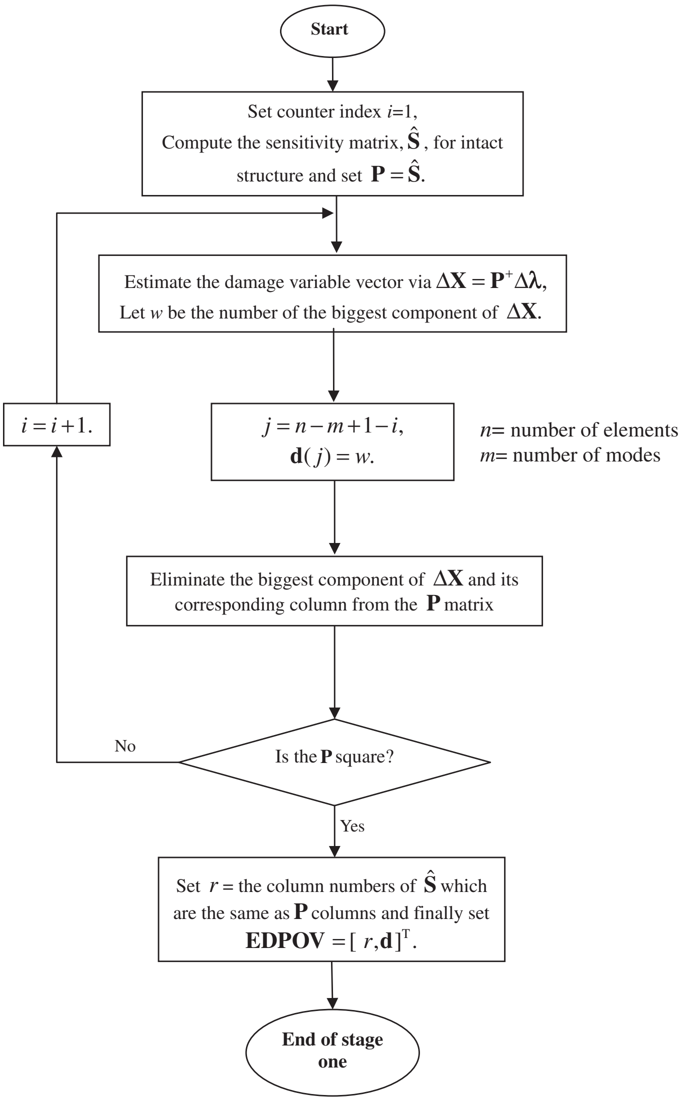



**Fig. 1a.** The flowchart of the first stage of SSA

## 3. The proposed method: subset solving algorithm (SSA)

A powerful algorithm is proposed to identify the multiple structural damages using the natural frequency changes. Here, it is assumed that the number of damaged elements is lower than or equal to the number of considered frequencies. The algorithm has two simple stages having different roles. In the first stage, all of the structural elements are ordered into a vector according to their damage probability. The vector is named here as the element damage probability ordering vector (EDPOV) and produced in a procedure using the sensitivity matrix of eigenvalues with respect to the damage variables, $x_i$, in the undamaged structure state. This EDPOV has been further used in the second stage of the algorithm that will be declared

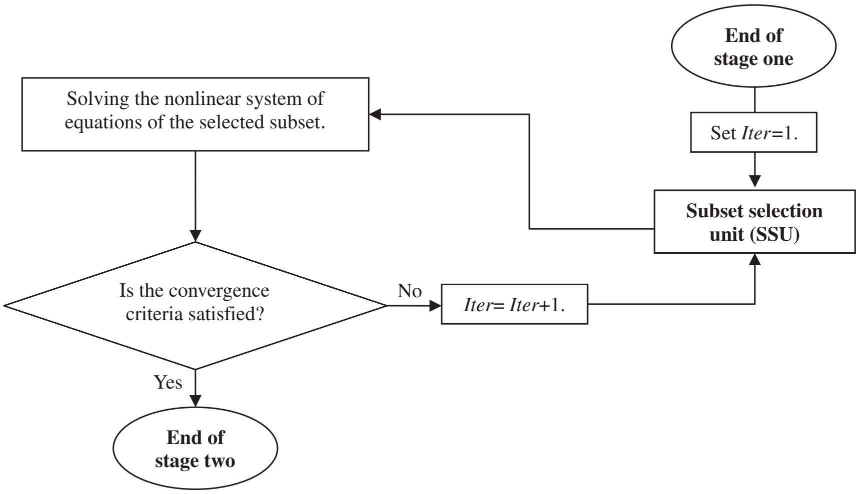



**Fig. 1b.** The flowchart of the second stage of the SSA

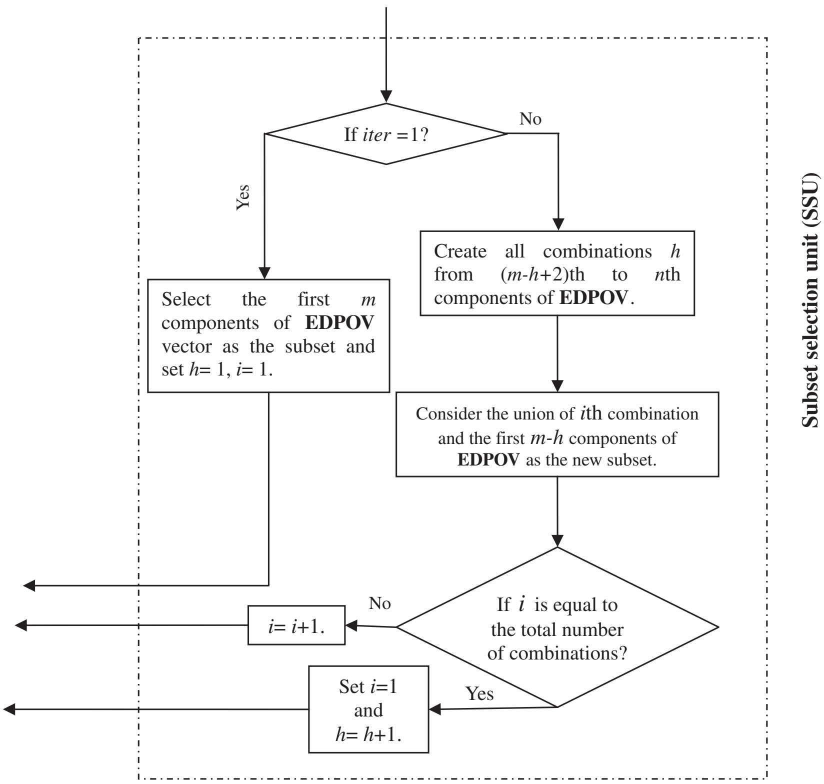



**Fig. 1c.** The flowchart of subset selection unit (SSU).

in detail at follows. In this study, a subset of elements that includes all damaged elements is called correct subset, otherwise it is called false subset.

As denoted in pervious section, by using Eq. (4), the damaged elements and the undamaged ones can be somewhat distinguished. However, our numerical experience for finding a correct subset demonstrates, it is more trustable rather than using Eq. (4), one iteratively find the most potentially undamaged element to be eliminated from the choice. Consequently, one could order the elements with respect to their damage probability based on their elimination ordering. Therefore, the first stage of the algorithm is designed as follows:

1. Set counter index $i = 1$ and pick the initial parameters of the problem.
2. Compute the sensitivity matrix of the original structure, $\hat{\mathbf{S}}$, using Eq. (5). The total number of rows and columns of the sensitivity matrix are equal to the total number of considered eigenvalues $m$ and structural elements $n$, respectively.
3. Compute the pseudo-inverse of sensitivity matrix, $\hat{\mathbf{S}}^{+}$, and obtain an initial guess for the change of damage variable vector by using the matrix equation $\Delta\mathbf{X} = \hat{\mathbf{S}}^{+}\Delta\lambda$.
4. Store the position of the biggest component of $\Delta\mathbf{X}$ referred to the less potentially damaged element number as the $n - i + 1$th component of EDPOV vector. Eliminate the biggest component of $\Delta\mathbf{X}$ and its corresponding column from the sensitivity matrix and increase $i$ one unit.
5. If the size of $\Delta\mathbf{X}$ is not equal to the number of utilized frequencies for damage detection go to step 3, otherwise go to next step.
6. Fill the first $m$ components of EDPOV vector by the element numbers which the current $\Delta\mathbf{X}$ entries refer to them and then go to the stage two.

In the second stage, a subset of elements having the number of members equal to the considered natural frequencies is selected. Next, the system of nonlinear equations $\mathbf{F}_d = \mathbf{F}(\mathbf{X})$, where $\mathbf{X}$ is the vector of damage ratio in which the entries associated with the selected elements are considered as unknowns and the rest are kept zeros. Therefore, in such system of equations the number equations are equal to the number of unknowns. If the real damaged elements are present in the selected elements subset, the system of nonlinear equations will converge to the real damage solution; otherwise, it will diverge or will converge to an unfeasible solution. Unfeasible solutions are the solutions having negative damage extent(s) (or damage ratios lower then a certain limit, e.g. −0.01, for noisy data cases) or damage extent(s) more than unity. When the problem does not converge to a feasible solution, it indicates that the selected subset of elements does not have all the damaged elements, so another set of elements should be selected. For choosing a subset of elements having high probability to be correct, a sub-algorithm named subset selection unit (SSU) using EDPOV vector is utilized. The process of subset selecting and solving should be followed until a feasible solution achieved. In this study, the Newton–Raphson and arc-length methods [19] as the most popular nonlinear equation solvers are employed.

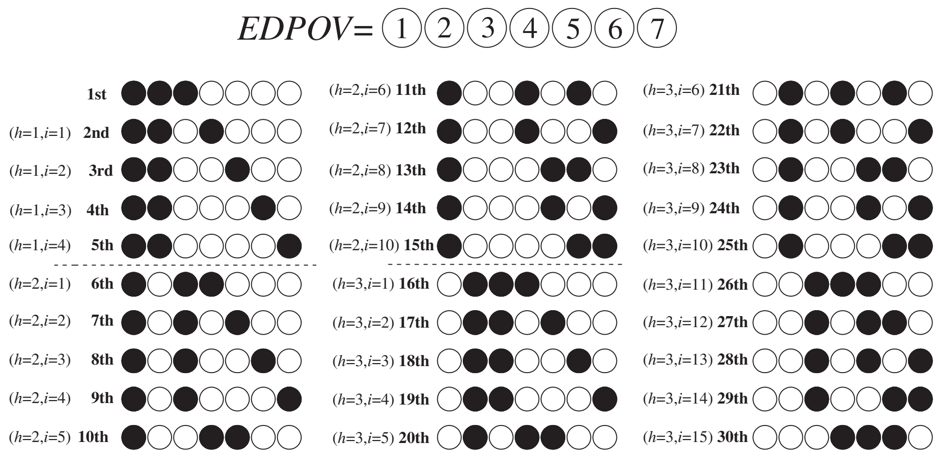



**Fig. 2.** Subset selection ordering of SSU, for a structure with seven members, and damage detection using the first three frequencies.

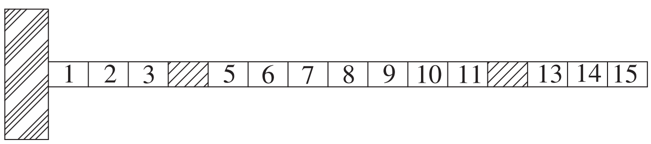



**Fig. 3.** Cantilever beam having 15-elements, in which elements 4 and 12 are damaged.

The second stage of the proposed algorithm may be summarized as the following steps:

1. Select a subset of $m$ potentially damaged elements by calling the SSU.
2. Solve the simultaneous nonlinear equations $\mathbf{F}(\mathbf{X}) = \mathbf{F}_d$, where only the entries of $\mathbf{X}$ associated with the selected $m$ elements are considered as unknowns and the left are kept zeros. Solution procedure could be followed by using the well-known Newton–Raphson or arc-length methods.
3. If the convergence is not achieved to a feasible solution, go to step 1; otherwise, the obtained solution is considered as the true damage solution.

When a subset is considered individually, the probability of the subset to be a correct one only depends on its elements' position in the EDPOV vector. However, while two subsets with many similar members are highly probable to be correct, and then, one of them is proven to be false, the correctness probability of the other is reduced. Therefore, in the iterative process of the stage 2, the probability of a subset to be a correct one, not only depend on its elements' situation in the EDPOV vector but also depends on the subset's intersections with the previously selected subsets that are proven to be false ones (see Section 4). This concept has been considered in the design of SSU. Therefore, the SSU part conducts the procedure of subset selection in an efficient way to facilitate extracting a correct subset. The SSU consists of the following steps:

1. For the first time calling of SSU, the first $m$ components of EDPOV are selected as the subset and also counter indexes $h$ and $i$ are set to unity. For the other times the steps 2 and 3 mentioned below should be followed.
2. Establish all different combinations of selecting $h$ components from the $(m - h + 2)$th to the $n$th components of EDPOV vector. Consider the union of $i$th combination and the first $m - h$ components of EDPOV vector as the new subset. For more description, combinations of selecting three members from $A = [x_1, x_2, x_3, x_4, x_5]$ is ordered as follows:

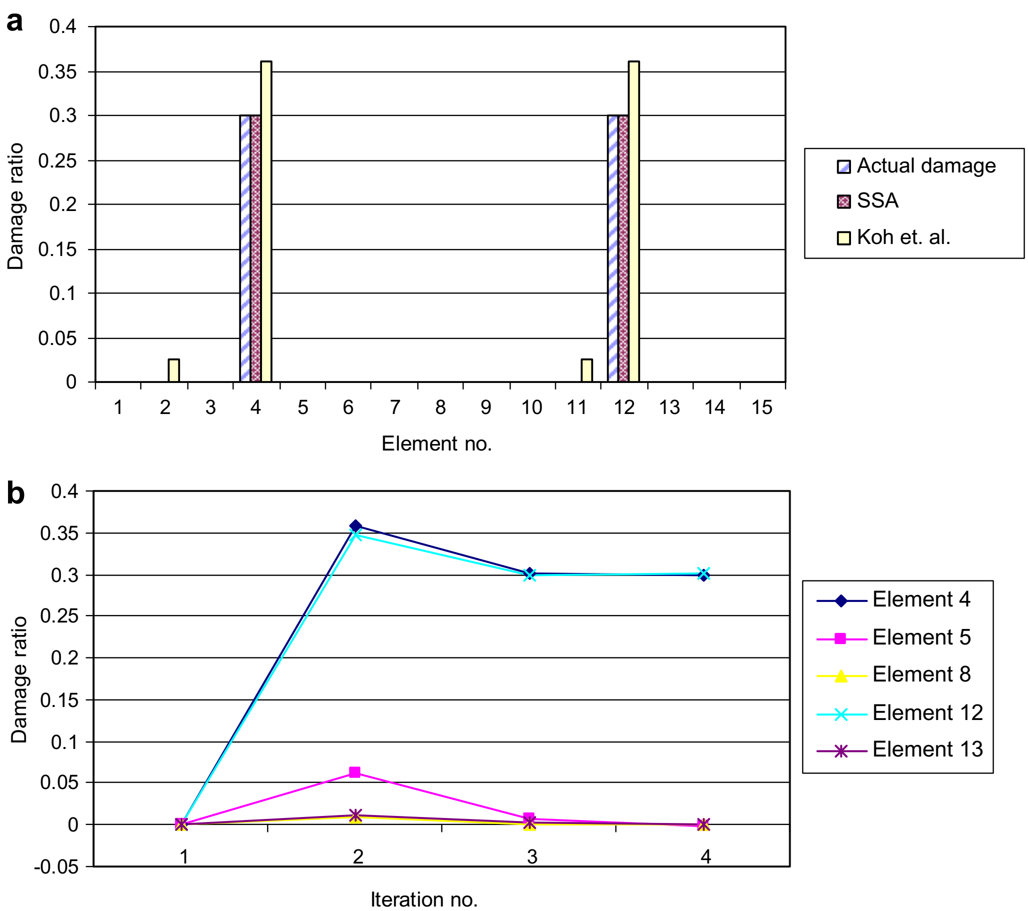
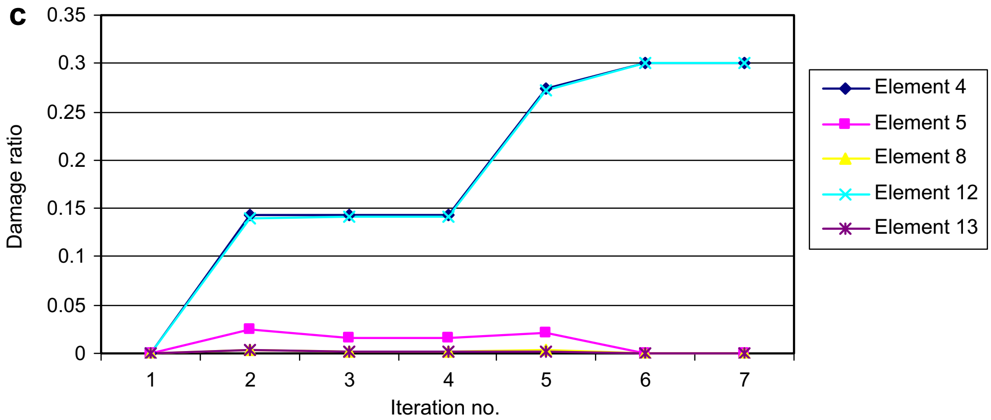



**Fig. 4.** Solution results for 15 element cantilever beam: (a) Damage identification result; (b) damage ratio changes of potentially flawed elements during the Newton–Raphson process; (c) damage ratio changes of potentially flawed elements during the arc-length process and (d) norm of residual during the iterations of nonlinear solution procedure.



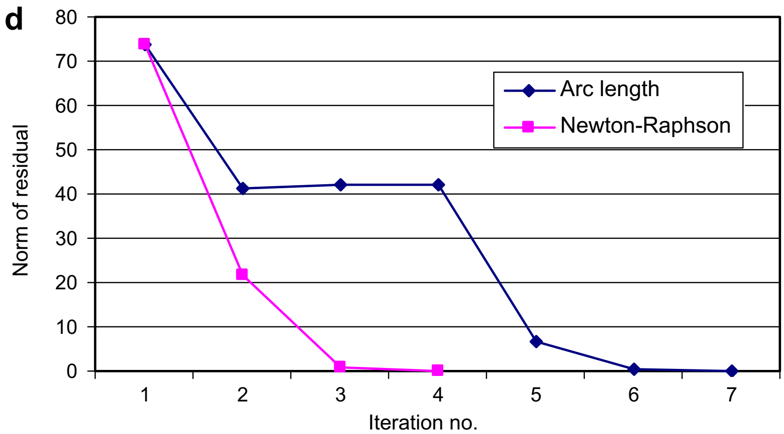




**Fig. 4** (*continued*)

$$\textit{ordering of combinations}: \{1,2,3\},\{1,2,4\},\underbrace{\{1,2,5\}}_{3rd\ combination},\{1,3,4\},\underbrace{\{1,3,5\}}_{5th\ combination}.$$

3. This combinations ordering for subscripts can be directly achieved using combntns function of MATLAB software.
4. If $i$ is the last combination, then set $i = 1$ and $h = h + 1$; otherwise, $i = i + 1$.

For more details, the step by step procedure of proposed damage detection algorithm is shown through Figs. 1a–c, respectively.

For more description about the method, as an example, a 7-element structure in which the damages are to be detected using the first three frequencies is considered. The ordering of subset selections by SSU sub-algorithm is given in Fig. 2. In such structure, if for instance we have EDPOV as $[x_2, x_5, x_1, x_6, x_7, x_3, x_4]^T$, so the corresponding equations for the 1st, 7th and 10th subsets that in the second stage should to be considered are $\mathbf{F}(x_2, x_5, x_1) = \mathbf{F}_d$, $\mathbf{F}(x_2, x_1, x_7) = \mathbf{F}_d$, and $\mathbf{F}(x_2, x_6, x_7) = \mathbf{F}_d$, respectively.

As it can be seen from Fig. 2, the way SSU selects the subsets is predetermined and harmonic, thus easy to follow. If the elements ordered with respect to their damage probabilities by any other methods rather than EDPOV, SSU may also be adopted there.

## 4. The likelihood of selecting all damaged elements

By considering the present algorithm, SSA, a question may be arisen that is; if EDPOV vector order the elements with respect to damage potentiality, how beneficial will be the way that SSU selects the elements? In other words; if one iteratively select the subsets by using SSU, and in each iteration the selected subset be proven to be false; is the next subset, more probable to be correct rather than the other followings? The debates of this section are to derive a mathematical formulation for investigating it further (see Section 5.1).

A damaged structure with at least one damaged element with $n$ elements is considered. The elements and their damaged probabilities are denoted by $\ell$ and $p_\ell$, respectively. The $p_\ell$'s are assumed to be predetermined values which could be reasonably assigned from the position of the elements in EDPOV vector (like what in done in Section 5.1) or any extra information, if there is. For simplicity, it is assumed that damaging of separate elements is independent.

Here, the subsets of $S = \{1, 2, \ldots, n\}$ are denoted by $A_i$'s. The statement that "$A_i$ is a correct subset" is denoted by $\hat{A}_i$ and the symbol $\neg$ states negation of a statement. The symbols $\wedge$ and $\vee$ are, respectively, the logic and, and the logic or of the Boolean algebra.

The event $E$ presents "all elements are undamaged". The probability of the first selected subset (i.e. $A_1$) being a correct one is obtained from the following conditional probability:

$$P(\hat{A}_1|\neg E) = \frac{P(\hat{A}_1 \wedge \neg E)}{P(\neg E)} = \frac{P(\hat{A}_1)\cdot P(\neg E)}{P(\neg E)} = \frac{\overbrace{\prod_{\ell \in S-A_1}(1-p_\ell)}^{P(\hat{A}_1)} \times \overbrace{\left[1-\prod_{\ell \in S}(1-p_\ell)\right]}^{P(\neg E)}}{1-\prod\limits_{\ell \in S}(1-p_\ell)}. \tag{6}$$

The process of subset selection proceeds until a correct subset is found. Assuming that the first $k$ subsets are not correct subsets, one can find the probability of $k+1$th being a correct subset by using Bayes theorem as:

$$P\left(\hat{A}_{k+1}|\left(\wedge_{i=1}^{k}\neg\hat{A}_i\right)\wedge\neg E\right) = \frac{P\left(\left(\wedge_{i=1}^{k}\neg\hat{A}_i\right)\wedge\neg E|\hat{A}_{k+1}\right)P(\hat{A}_{k+1})}{P\left(\left(\wedge_{i=1}^{k}\neg\hat{A}_i\right)\wedge\neg E\right)}. \tag{7}$$

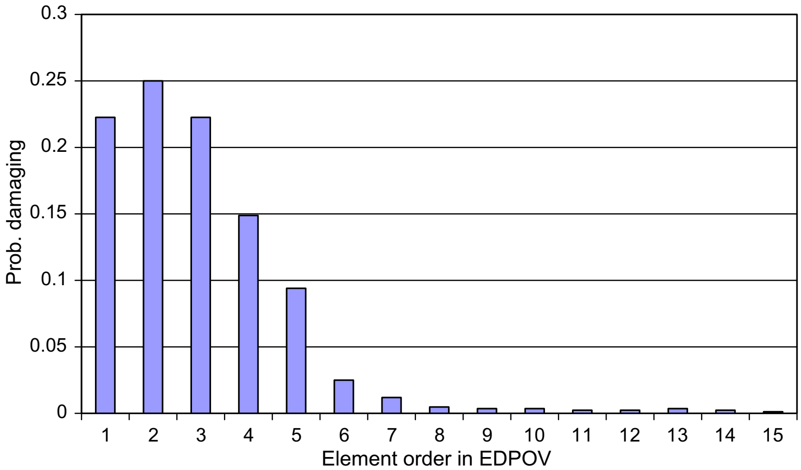



**Fig. 5.** Damage probability of structural elements with respect to their ordering in EDPOV vector.

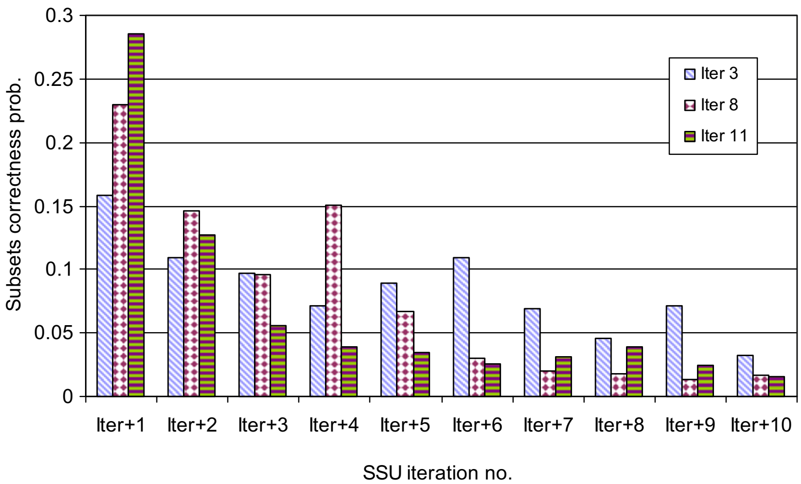



**Fig. 6.** The probability of correctness of 10 next following subsets from 3rd, 8th and 11th iterations of SSU.

The denominator of Eq. (7) can be determined as follows:

$$P\left(\left(\wedge_{i=1}^{k}\neg\hat{A}_i\right)\wedge\neg E\right)=P\left(\left(\wedge_{i=1}^{k}\neg\hat{A}_i\right)\right)=P\left(\neg\left(\vee_{i=1}^{k}\hat{A}_i\right)\right)=1-P\left(\vee_{i=1}^{k}\hat{A}_i\right). \tag{8}$$

The value $P\left(\vee_{i=1}^{k}\hat{A}_i\right)$ can be calculated from inclusion–exclusion principle as follows:

$$P\left(\vee_{i=1}^{k}\hat{A}_i\right)=\sum_{r=1}^{k}\left\{(-1)^{r-1}\sum_{\substack{I\subset\{1,2,\ldots,k\}\\ N(I)=r}}P(\hat{A}_I)\right\}, \tag{9}$$

where $A_I=\bigcap_{i\in I}A_i$ and $N(\cdot)$ represents number of elements of the corresponding subset.

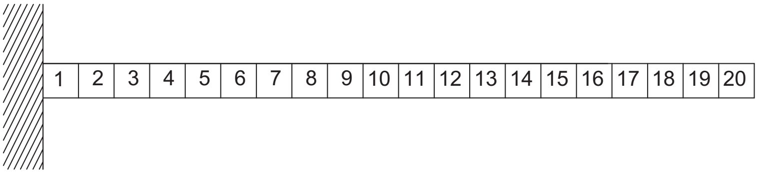



**Fig. 7.** Twenty-element beam with element numbering.

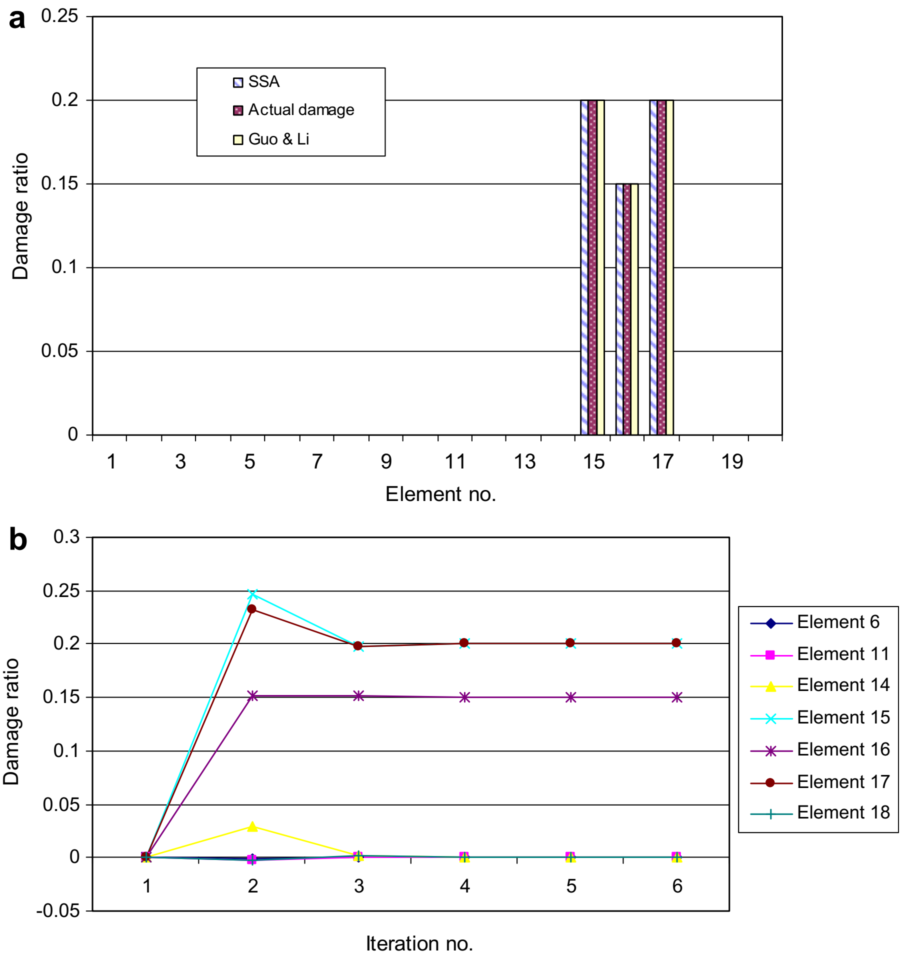



**Fig. 8.** Solution results for case 1 of 20 element cantilever beam: (a) Damage identification result; (b) damage ratio changes of potentially flawed elements during the Newton–Raphson process; (c) damage ratio changes of potentially flawed elements during the arc-length process and (d) norm of residual during the iterations of nonlinear solution procedure.

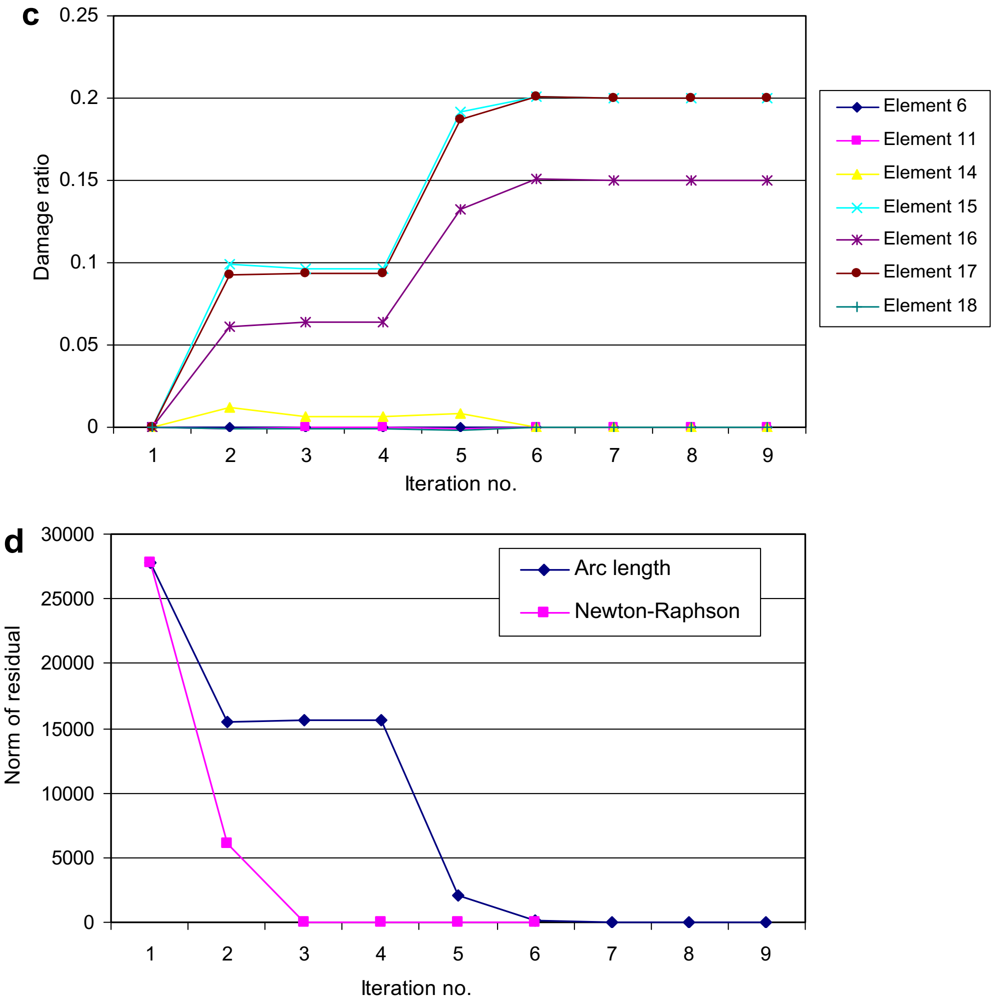



**Fig. 8** (*continued*)

In general, the conditional probability $P(\cdot|\hat{A}_i)$ can be calculated alike $P(\cdot)$ just by changing damage probability of elements as follows:

$$p_\ell^{(i)} = \begin{cases} p_\ell, & \ell \in A_i, \\ 0, & \ell \in S - A_i. \end{cases} \tag{10}$$

For simplicity $P(\cdot|\hat{A}_i)$ is presented by $\overline{P}_{(i)}(\cdot)$ in which $\overline{P}_{(i)}$ indicates such changing in damage probability of elements. Therefore, probability $P\left(\left(\wedge_{i=1}^{k}\neg\hat{A}_i\right)\wedge\neg E|\hat{A}_{k+1}\right)$ which is in the numerator of Eq. (7) is equal to $\overline{P}_{(k+1)}\left(\left(\wedge_{i=1}^{k}\neg\hat{A}_i\right)\wedge\neg E\right)$. Similar to Eq. (8), it can be written as:

$$\overline{P}_{(k+1)}\left(\left(\wedge_{i=1}^{k}\neg\hat{A}_i\right)\wedge\neg E\right) = 1 - \overline{P}_{(k+1)}\left(\vee_{i=1}^{k}\hat{A}_i\right). \tag{11}$$

Finally, considering $P(\hat{A}_i) = \prod_{\ell\in S-A_i}(1-p_\ell)$ and above discussion, Eq. (7) is determined as follows:

$$P\left(\hat{A}_{k+1}|\left(\wedge_{i=1}^{k}\neg\hat{A}_i\right)\wedge\neg E\right) = \frac{\prod_{\ell\in S-A_{k+1}}(1-P_\ell)\times\left\{1-\sum_{r=1}^{k}\left[(-1)^{r-1}\sum_{\substack{I\subset\{1,2,\ldots,k\}\\ N(I)=r}}\overline{P}_{(k+1)}(\hat{A}_I)\right]\right\}}{\left\{1-\sum_{r=1}^{k}\left[(-1)^{r-1}\sum_{\substack{I\subset\{1,2,\ldots,k\}\\ N(I)=r}}P(\hat{A}_I)\right]\right\}}. \tag{12}$$

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## 5. Case studies

The efficiency of the proposed method is verified by five test examples explained bellow. In all examples, the damage is simulated by using a reduction in the Young modulus of damaged elements and noise is neglected. For all examples, the initial load factor for the arc-length method is selected as $\gamma = 0.40$.

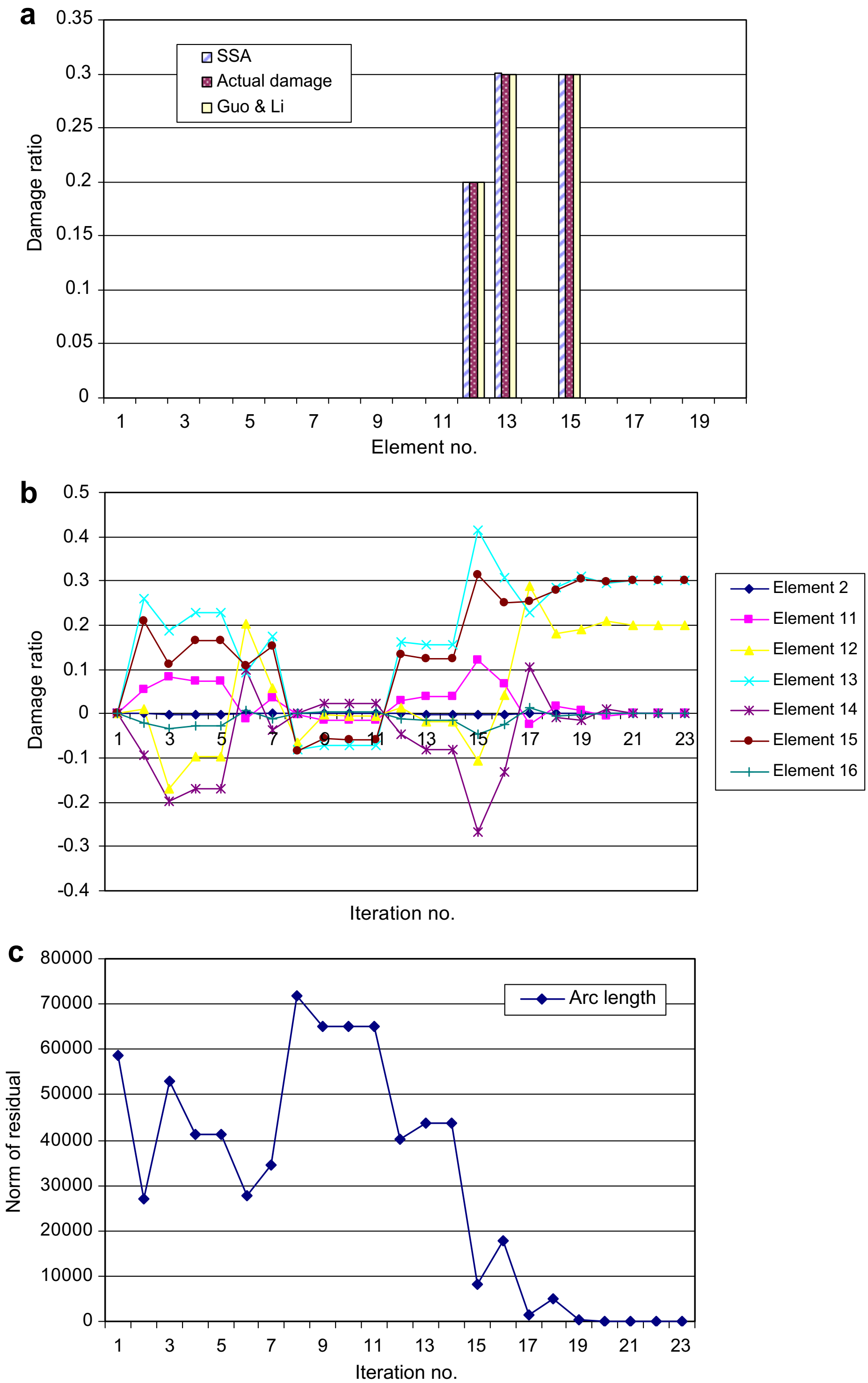


**Fig. 9.** Solution results for damage case 2 of 20-element cantilever beam: (a) Damage identification result; (b) damage ratio changes of potentially flawed elements during the arc-length process and (c) norm of residual during the iterations of nonlinear solution procedure.

### 5.1. A 15-element cantilever beam

A 15-element cantilever beam which was previously studied by Koh and Dyke [8] is considered to assess the efficiency of the proposed algorithm. The length, thickness and width of the beam are 2.74, 0.00635 and 0.0760 m, respectively, and the

**Table 1**
A comparison of runtimes of SSA with those of Gue and Li.

| Device properties | Gue and Li [13] 3.0 GHz processor and 512 MB memory | | The present work: SSA 1.8 GHz processor and 1 GB memory | |
|---|---|---|---|---|
| Cases | Case 1 | Case 2 | Case 1 | Case 2 |
| Runtime (s) | 289.2 | 348.9 | 0.19 | 0.28 |

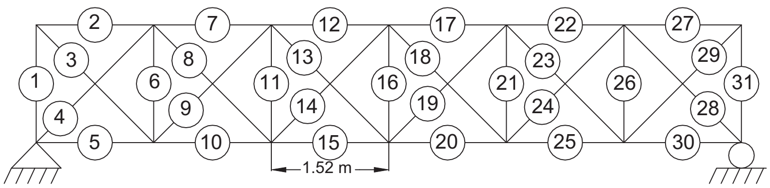


**Fig. 10.** Thirty-one-element planar truss.

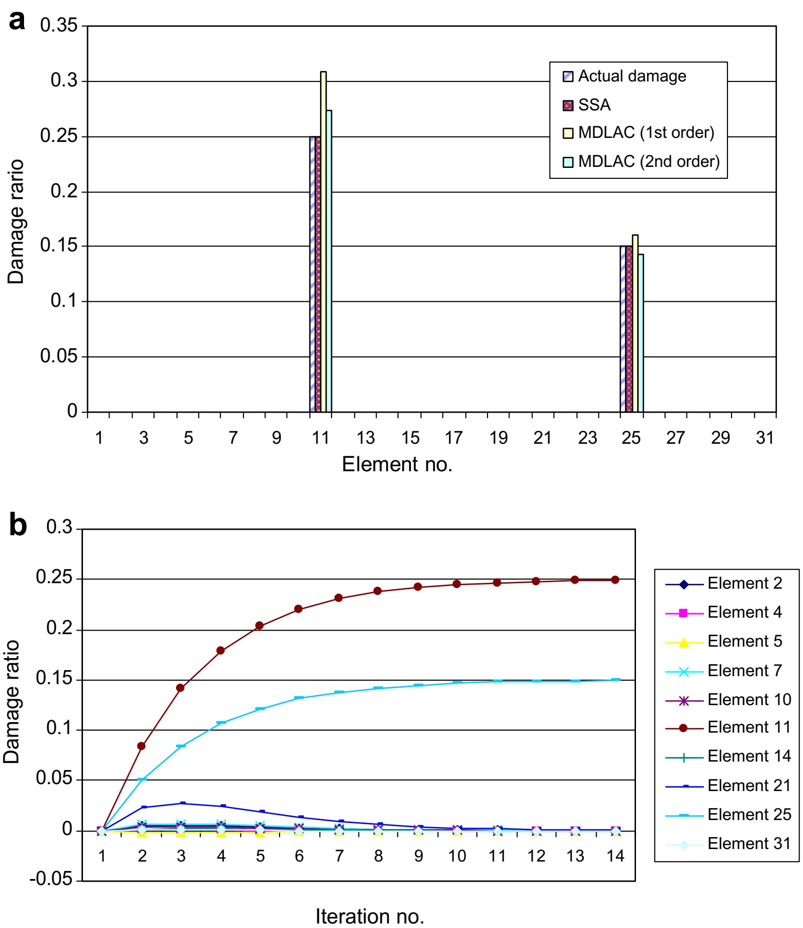


**Fig. 11.** Solution results for the damage case 1 of planar truss: (a) Damage identification result; (b) damage ratio variations of potentially damaged elements using the Newton–Raphson method; (c) damage ratio variations of potentially damaged elements using the arc-length method and (d) norm of residual during the iterations of nonlinear solution procedure.

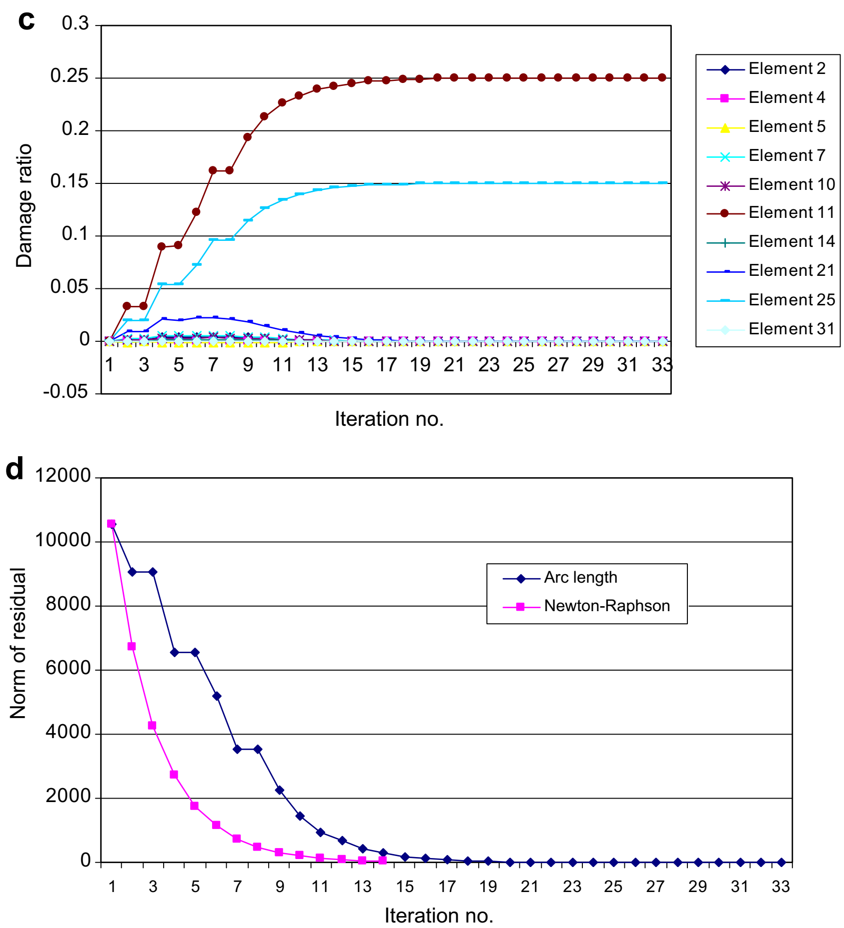


**Fig. 11** (*continued*)

elements are numbered from the fixed end as shown in Fig. 3. In Ref. [8] the elements 4 and 12 have been assumed to be damaged by the extent of 30% and the process of damage detection has been proceed using the first five frequencies with three different ways from which the solution of genetic algorithm with the MDLAC criterion has been found as the best result. Here, the best result of Koh and Dyke is selected to be compared with those obtained by the proposed algorithm as what is shown in Fig. 4a.

For this example, the elements 4, 5, 8, 12 and 13 are the first five elements of EDPOV vector. As indicated, these elements which are considered as the first subset include the real damaged elements. The corresponding nonlinear system of equations has been solved with Newton–Raphson and arc-length method and both methods converge to the exact damage ratios. The changing of damage ratios of potentially flawed elements during the Newton–Raphson and arc-length procedure can be seen in Fig. 4b and c, respectively. The norm of residual of eigenvalue vector in the iterative procedure of the Newton–Raphson and arc-length methods is also depicted in Fig. 4d. All of the figures demonstrate that the proposed algorithm can detect the actual locations and extents of damaged elements.

In this case study, an investigation on the performance of SSU has been done either. In a random model, the elements of the structure are simulated to be damaged with probability of 1/15. The damage extent for damaged elements is considered randomly in the interval [0.1, 0.5] with a uniform probability distribution. This model is generated 10,000 times and in each run the EDPOV vector, and the damaged element positions in EDPOV is found. And further, elemental damage probability with respect to their position in EDPOV is obtained, as what is shown in Fig. 5.

Here, while the first $k$ subsets selected by SSU are failed to be correct, the correctness probability of 10 next following subsets (i.e. $k+1$th, $k+2$th, ..., $k+10$th) is investigated. For this, by assuming the elemental damage probability as discussed presented in Fig. 5 and by using Eq. (12) the correctness probabilities of 10 following subsets from 3rd, 8th and 11th iterations of SSU are derived and illustrated in Fig. 6.

As it can be seen from Fig. 6, the probability of correctness of right next subset proposed by SSU, which is presented in the first column (the Iter + 1th subset), is higher than the nine others. This means that, through the iterations, SSU choose a highly probable correct subset to be solved by nonlinear system of equations' solvers.

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### 5.2. A 20-element cantilever beam

Here, another cantilever beam which has been studied previously by Gue and Li [16] is presented. The beam has the length of 1, the height of 0.01 and the width of 0.01 m. Also, Young's modulus and mass density are 210 GPa and 7860 kg/m$^3$, respectively. The finite element model of the beam including 20 elements leading to 40 degrees of freedom is shown in Fig. 7. Two damage cases are induced at the beam as follows:

Case 1: the elements 15, 16 and 17 are damaged by the extent of 20%, 15% and 20%, respectively.
Case 2: the elements 12, 13 and 15 are damaged by the extent of 20%, 30% and 30%, respectively.

Gue and Li [16] have used the first 10 frequencies and two mode shapes to detect the structural damages but, in this study only the first seven frequencies are used. Damage identification results for cases 1 and 2 are shown in Figs. 8 and 9, respectively. As can be seen in Figs. 8a and 9a the sites and extents of two damage cases induced are correctly detected alike Gue and Li study. However, by paying attention to Table 1, it is revealed that our consuming-times are much less than those presented by Gue and Li. It should be noted that the presented runtimes of the proposed algorithm is related to the case performing via the arc-length method as the nonlinear equation solver.

For damage case 1, the first seven entries of EDPOV vector are 6, 11, 14, 15, 16, 17 and 18 which consist of real damage elements 15, 16 and 17. The variations of unknowns for case 1 during solving the nonlinear equations using the Newton–Raphson and arc-length methods are shown in Figs. 8b and 9c, respectively. The variations of norm of residual of eigenvalue vector in the iterative solutions of Newton–Raphson and arc-length methods are shown in Fig. 8d. In damage case 2, the Newton–Raphson method cannot converge, so for this case only the arc-length method results are provided as shown in Fig. 9a–c, respectively.

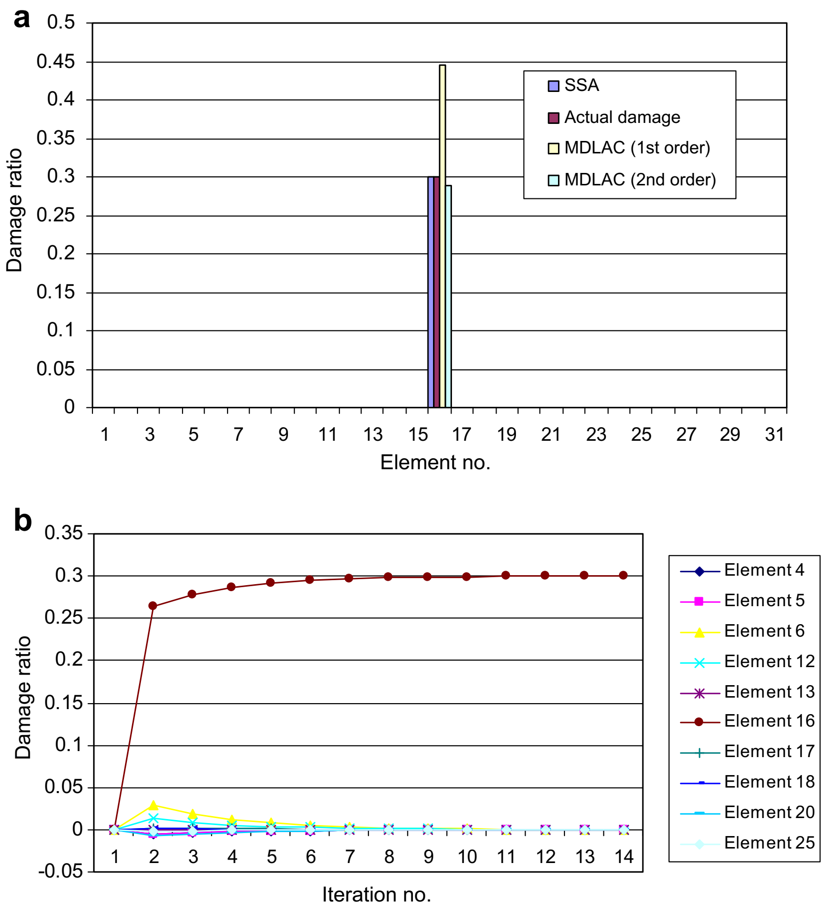



**Fig. 12.** Solution results for the damage case 2 of planar truss: (a) Damage identification result; (b) damage ratio variations of potentially damaged elements using the Newton–Raphson method; (c) damage ratio variations of potentially damaged elements using the arc-length method and (d) norm of residual during the iterations of nonlinear solution procedure.

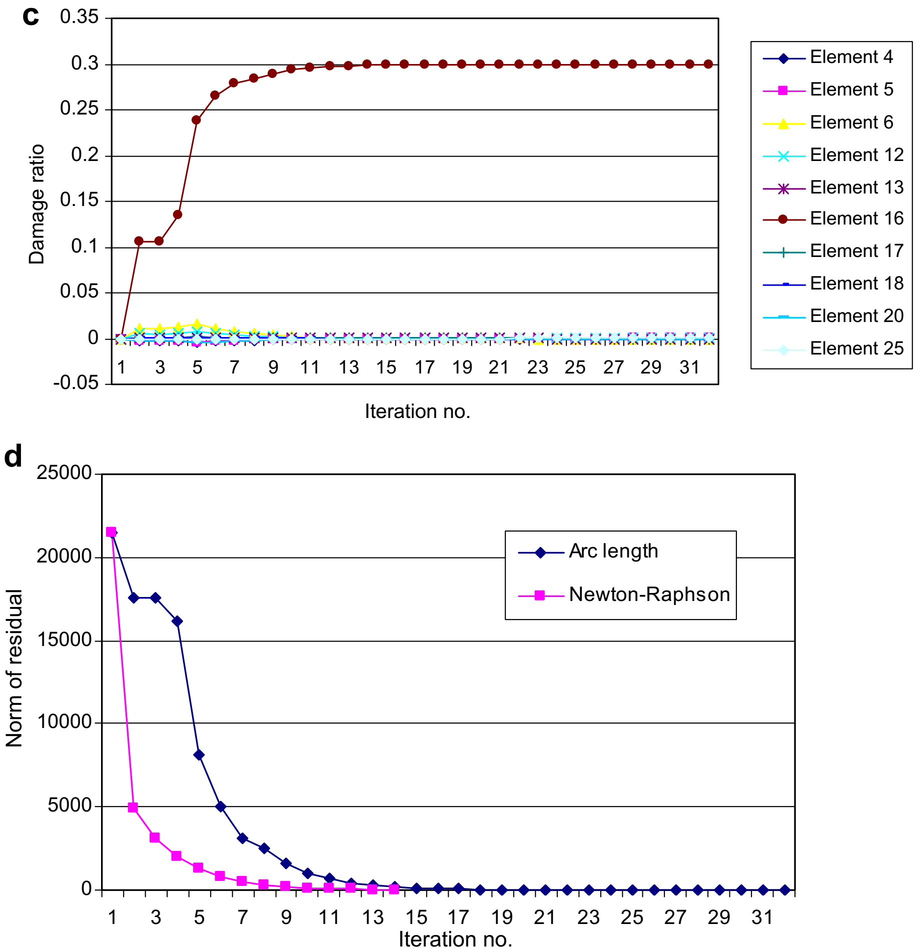



**Fig. 12** (*continued*)

### *5.3. A 31-element planar truss*

In order to show the capability of the proposed algorithm for structural damage detection, a 31-bar planar truss as shown in Fig. 10 is selected and the results of the algorithm are compared with those of first and second order approximation methods reported by Messina et al. [7].

Two damage cases are considered here; in case 1, damages are induced in elements 11 and 25 with the extents of 25% and 15%, respectively, while in case 2, a damage is induced only on element 16 with the extent of 30%. The damages are detected using the first 10 natural frequencies. The damages identified via the proposed algorithm for damage cases 1 and 2 are shown in Figs. 11a and 12a, respectively. It can be seen that the proposed algorithm can achieve the actual sizes and locations of damages comparing to the methods reported in Ref. [7].

The variations of damage ratios of potentially flawed elements for cases 1 and 2 during the Newton–Raphson and arc-length procedures can be seen in Figs. 11b, c and 12b and c, respectively. The norm of residual of eigenvalue vector in iterative procedure of the Newton–Raphson and arc-length methods is also depicted in Figs. 11d and 12d, respectively. The figures demonstrate that the convergence speed of two methods is very high. Moreover, it can be observed that the convergence rate of the Newton–Raphson method is higher than the arc-length method.

### *5.4. A gravity concrete dam*

A plane strain model of a concrete gravity dam having 24 triangular elements as shown in Fig. 13 is considered as the forth example. The dam has the height of 50 m, Young's modulus of 20 GPa, Poisson's ratio of 0.2 and mass density of 2402 kg/m$^3$. For this example, the elements 3 and 16 are assumed to be damaged with the extents of 15% and 30%, respectively. The damages are detected using the first eight frequencies. The results are shown in Fig. 14.

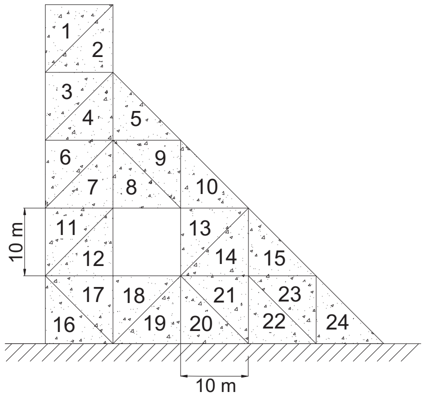



**Fig. 13.** Plan strain model of a gravity dam.

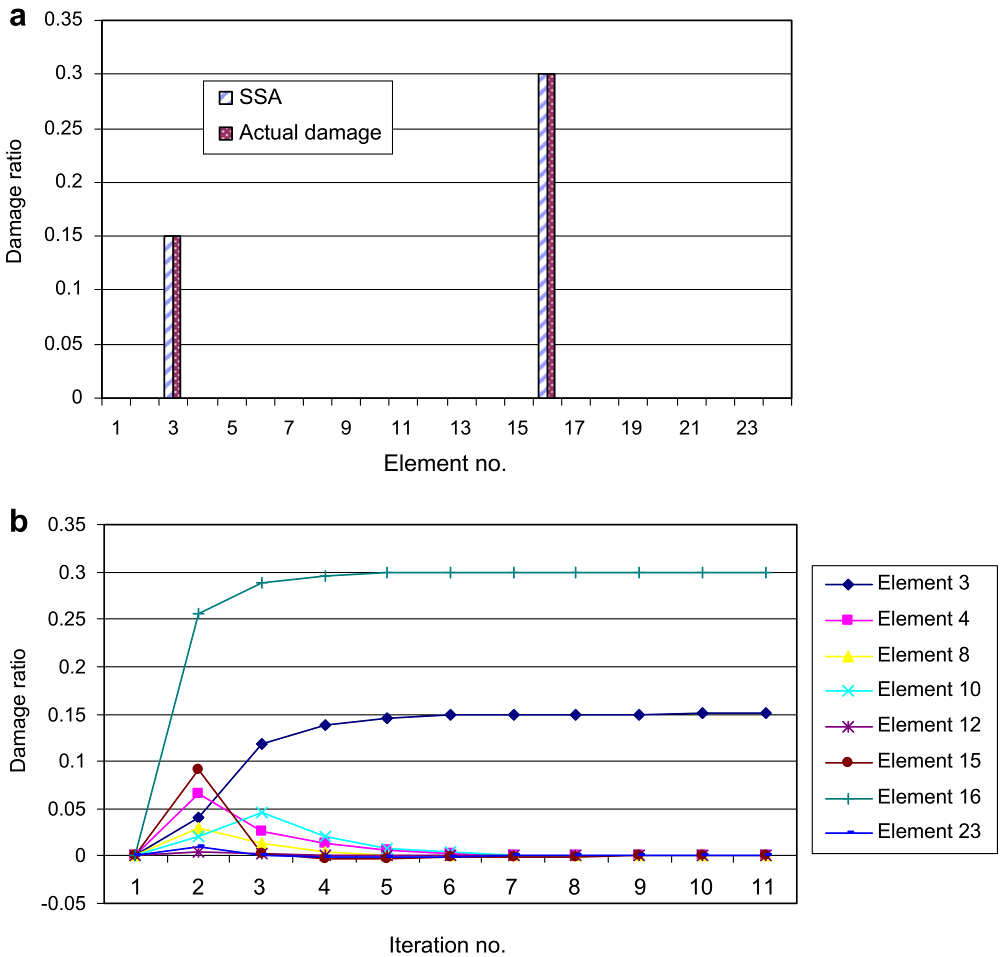
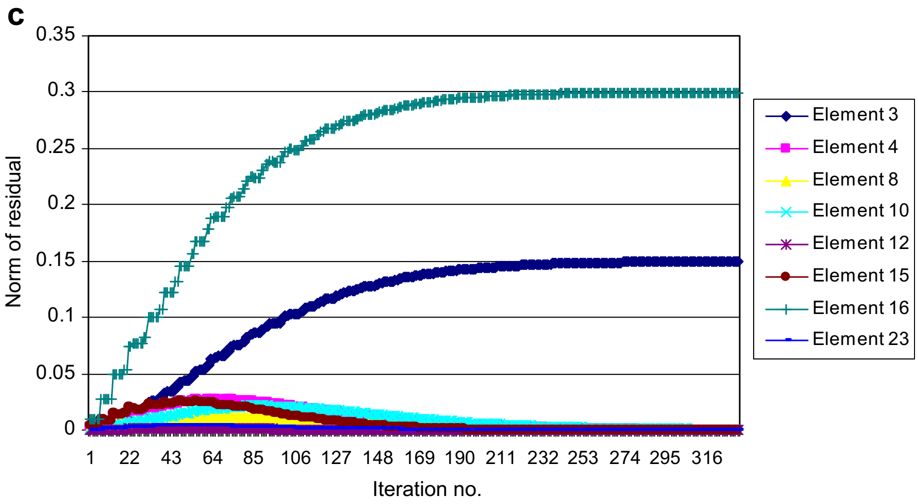



**Fig. 14.** Solution results for the gravity dam: (a) Damage identification result; (b) damage ratio variations of potentially damaged elements using the Newton–Raphson method; (c) damage ratio variations of potentially damaged elements using the arc-length method and (d) norm of residual during the iterations of nonlinear solution procedure.

*5.5. A 35-element bending plate*

A bending plate having 35 square elements with one fixed end and a simple point support as shown in Fig. 15 is considered as the last example. The plate has the length of 0.35 m, the width of 0.25 m, the thickness of 0.01 m, Young's modulus of



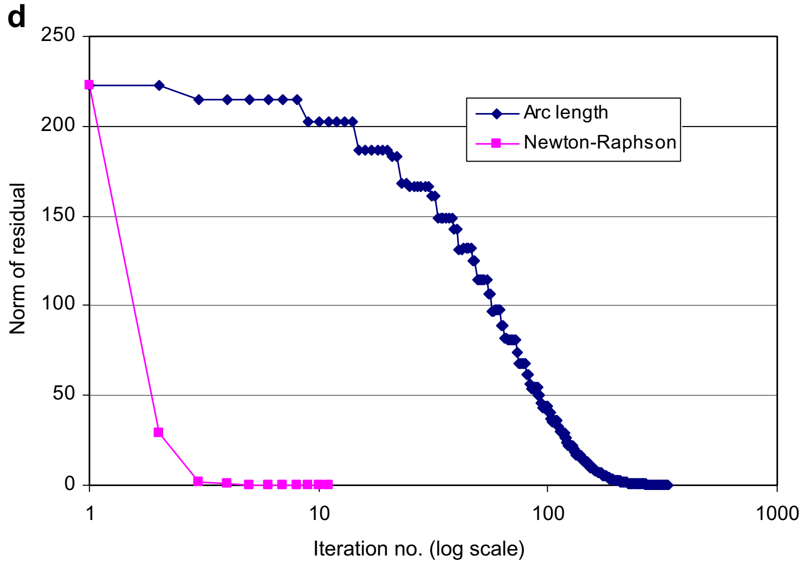




**Fig. 14** (*continued*)

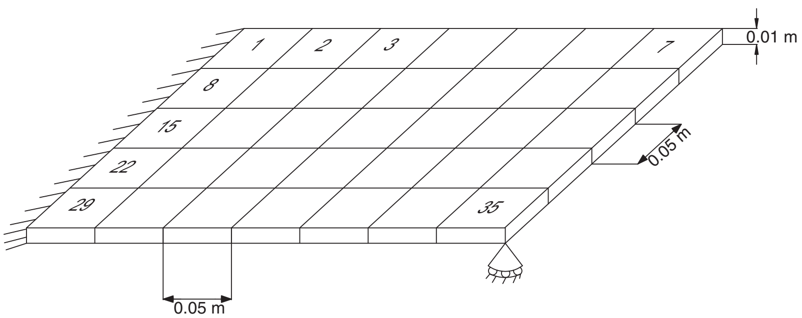



**Fig. 15.** Bending plate with the elements numbers.

200 GPa, Poisson's ratio of 0.3, and mass density of 7860 kg/m$^3$. The first 10 natural frequencies of the structure are considered to identify the damages. The damages are induced in elements 1, 9, 10, 26 by the extents of 20%, 30%, 25% and 15%,

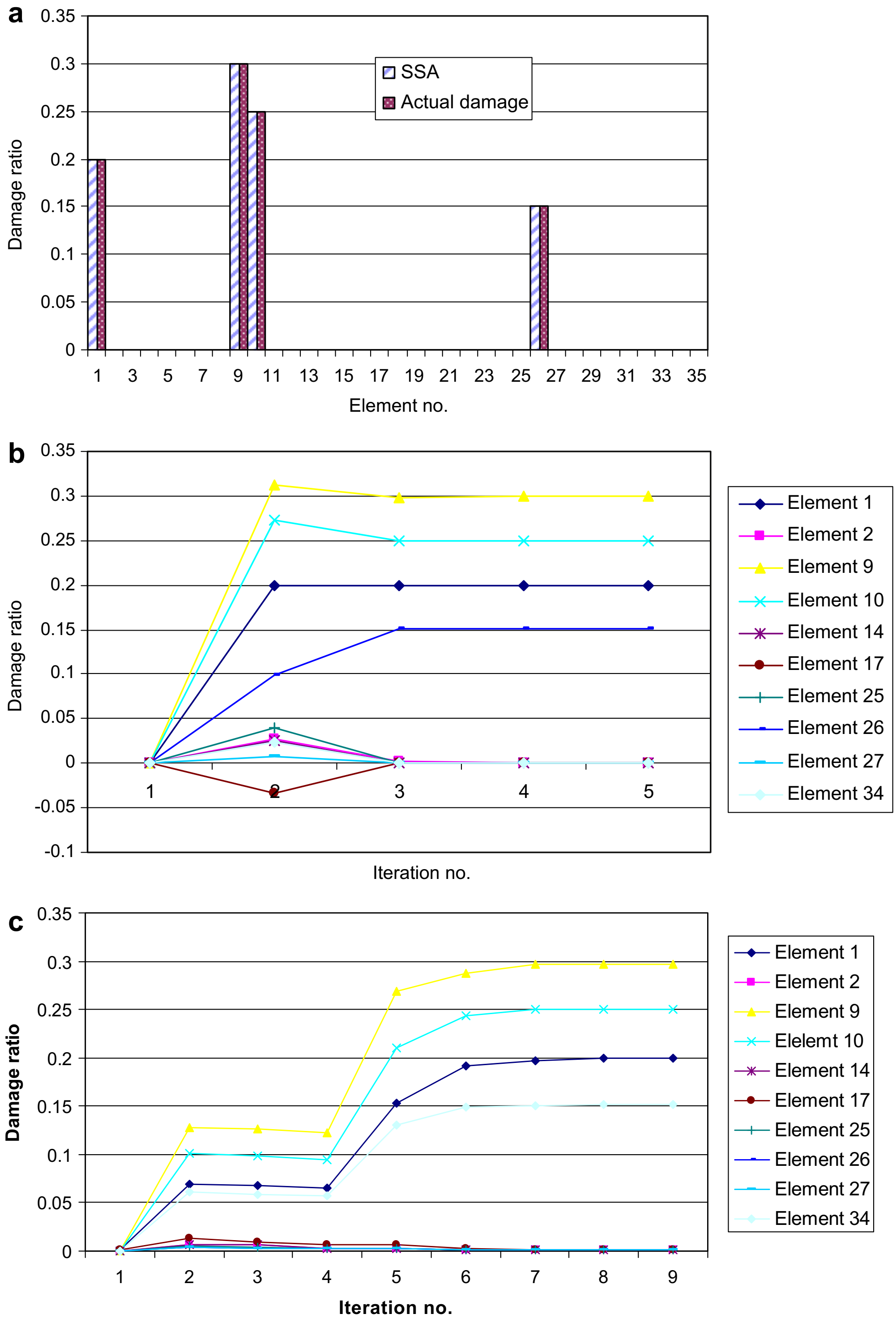



**Fig. 16.** Solution results for the plate structure: (a) Damage identification result; (b) damage ratio variations of potentially damaged elements using the Newton–Raphson method and (c) damage ratio variations of potentially damaged elements using the arc-length method.

respectively. The proposed algorithm is applied to the plate and the results of damage detection are depicted in Fig. 16. The figure demonstrates the high performance of the algorithm for accurately identifying the damages induced.

However, in practice 10 frequencies is hard to estimate and also noise effects is also exist. Therefore, for a scenario in which the elements 1, 21, 33 are damaged by the extent of 20%, 30% and 30%, respectively, the different conditions for

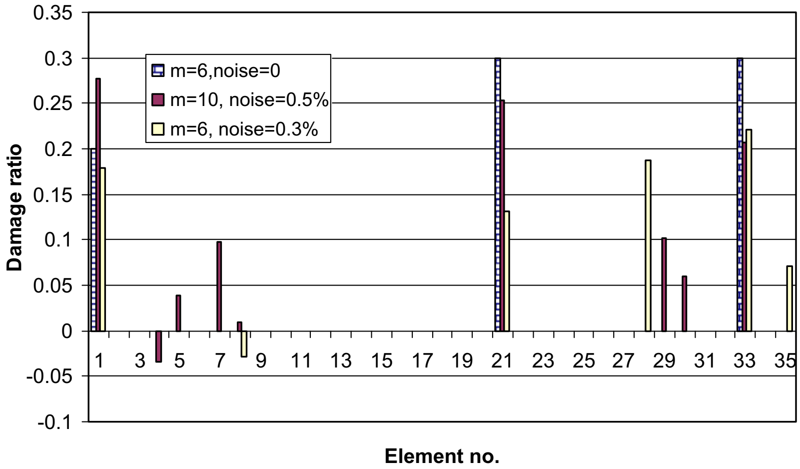



**Fig. 17.** Comparison between different conditions for damage detection of plate ($m$ is no. of frequencies).

**Table 2**
The runtime of SSA for both states using Newton–Raphson and arc-length.

| Runtime (s) | 15-Element beam | Truss | | Dam | Plate |
|---|---|---|---|---|---|
| | | Case 1 | Case 2 | | |
| SSA (Newton–Raphson) | 0.15 | 0.16 | 0.25 | 0.282 | 0.531 |
| SSA (arc-length) | 0.17 | 0.29 | 0.37 | 1.704 | .66 |

noise levels and number of frequencies are investigated and the results are concisely given in Fig. 17. The noisy $i$th frequency, $\bar{f}_i$, is simulated using $\bar{f}_i = f_i(1 + r \cdot \varepsilon)$ where $r$ is a uniformly distributed random number between $-1$ and 1 and $\varepsilon$ is the noise level.

The runtimes of the algorithm for all the structures, in the cases of employing the Newton–Raphson and arc-length method are given in Table 2. It can be seen that for most examples the runtime is less than 1 s and it is also revealed that the Newton–Raphson is faster than the arc-length method.

### 5.6. The performance of subset selection unit

Here, the computational efficiency of selecting the elements using the subset selection unit (SSU) is investigated. For this aim, for some predetermined number of damages, the sites and extents of damages are randomly generated 10,000 times using the Monte Carlo simulation. Subsequently, for each simulated damaged structure, the total number of required iterations in the proposed algorithm to achieve the correct subset is calculated. The results for the previously studied structures for the different cases with 2, 3 and 4 damaged elements are presented in Fig. 18a–c, respectively. As it can be seen, the selection order of subsets is in such way that the most probable correct subsets are verified first. It can also be observed that the first iteration has a high probability to be a correct subset with respect to other iterations. Therefore, this kind of subset selection causes the expectation of required iterations to reach the correct subset to be much less than randomly subset selection. In Table 3, an illustrative comparison between the expectation of required iterations for the cases with and without using SSU is provided. In the randomly subset selection case (i.e. the case without SSU), double selection of a false subset is prevented. For randomly subset selection, the expectation of required iterations (ERI) to reach a correct subset can be computed from the following equation:

$$ERI = \frac{\binom{n-d}{m-d}}{\binom{n}{M}} + \sum_{i=2}^{j_{max}} \left\{ i \times \frac{\binom{n-d}{m-d}}{\binom{n}{m} - i + 1} \times \prod_{j=1}^{i-1} \frac{\binom{n}{m} - \binom{n-d}{m-d} - j + 1}{\binom{n}{m} - j + 1} \right\}, \tag{13}$$

where $m$, $n$ and $d$ are the number of frequencies, elements and damaged elements, respectively, and $j_{\max}$ is the maximum feasible value that $i$ can take.

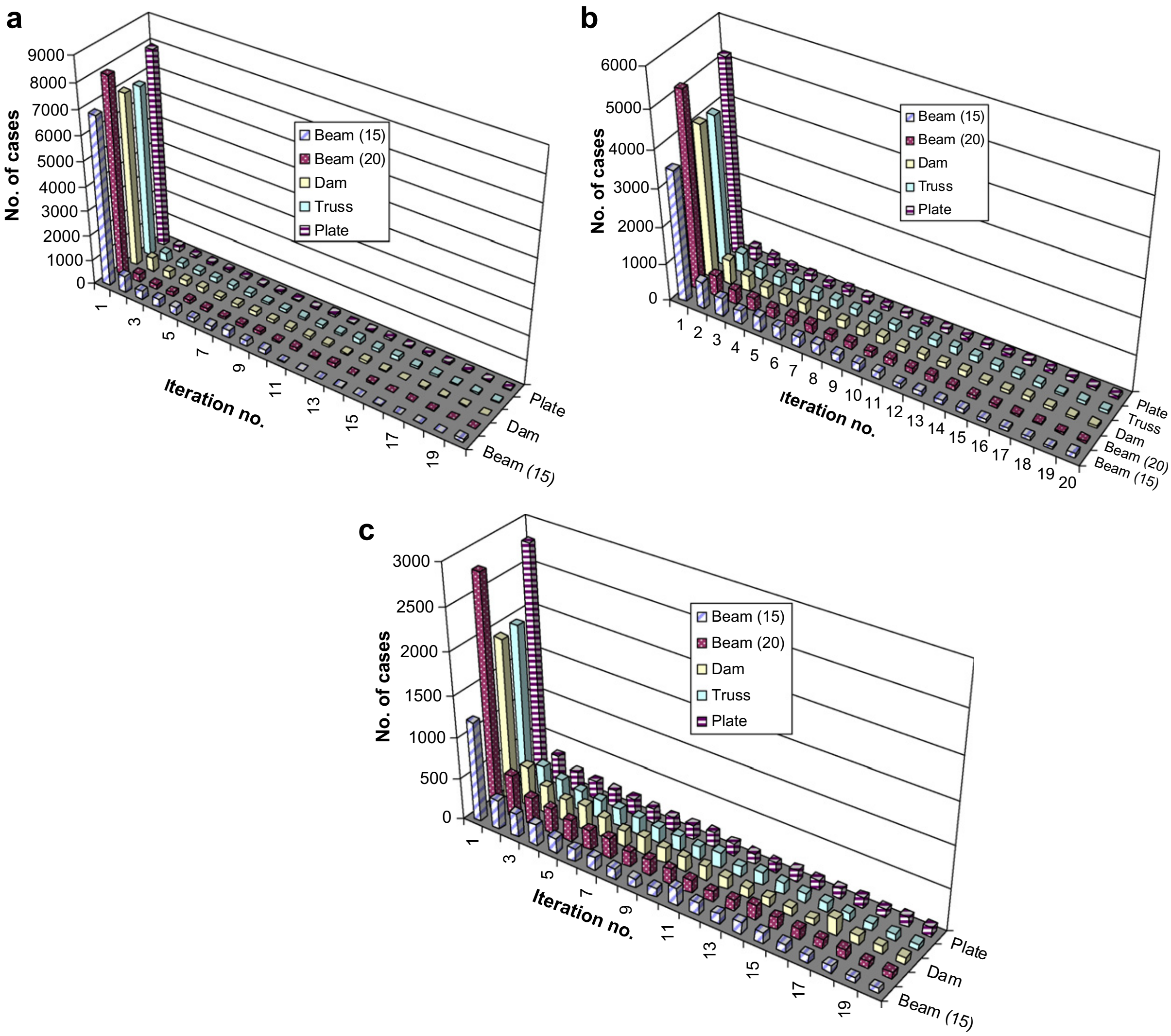



**Fig. 18.** The needed iterations to reach a correct subset for 10,000 randomly damaged structures for: (a) two damaged elements; (b) three damaged elements and (c) four damaged elements.

**Table 3**
Expectation of requited iterations in the cases with and without SSU for 2 and 3 damages.

| ERI | | Beam (15) | Beam (20) | Truss | Plate | Dam |
|---|---|---|---|---|---|---|
| Two damages | With SSU | 3.372 | 2.840 | 3.478 | 4.980 | 3.567 |
| | Without SSU | 10.467 | 9.047 | 10.162 | 12.852 | 9.744 |
| Three damages | With SSU | 18.462 | 12.733 | 19.811 | 37.660 | 25.021 |
| | Without SSU | 44.836 | 32.558 | 37.804 | 54.000 | 35.491 |

## 6. Conclusions

A novel sensitivity-based algorithm is proposed to accurately identify the locations and extents of multiple structural damages. The suggested algorithm includes two stages having different actions. In the first stage, all structural elements are ordered according to their damage probability into a vector named here as the element damage probability ordering vector (EDPOV). In the second stage, a subset of elements is selected using the subset selection unit (SSU) by employing the EDPOV and then the corresponding set of nonlinear equations is solved. The process of the second stage is iteratively continued until the convergence to a feasible solution is achieved. The method is applied to five illustrative examples. The first three of them have been previously studied by the other researchers. It is impressive to note that for all the examples the presented algorithm successfully achieved to the actual damages. By comparison the results of the presented algorithm with those available in the literature, the accuracy and speed of the method are also concluded. It is also observed that for solving the set of nonlinear equations, the Newton–Raphson method converges in less iterations than the arc-length method, however, arc-length method shows a more powerful convergence in highly nonlinear paths. Finally, it is revealed that the procedure of subset selection for establishing the set of nonlinear equations is an appropriate way for a fast achievement to a correct subset.

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