Greedy Algorithms: Fractional Knapsack

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Algorithmic Toolbox
Data Structures and Algorithms

Outline

1. Long Hike
2. Fractional Knapsack
3. Pseudocode and Running Time

Long Hike

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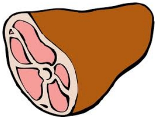

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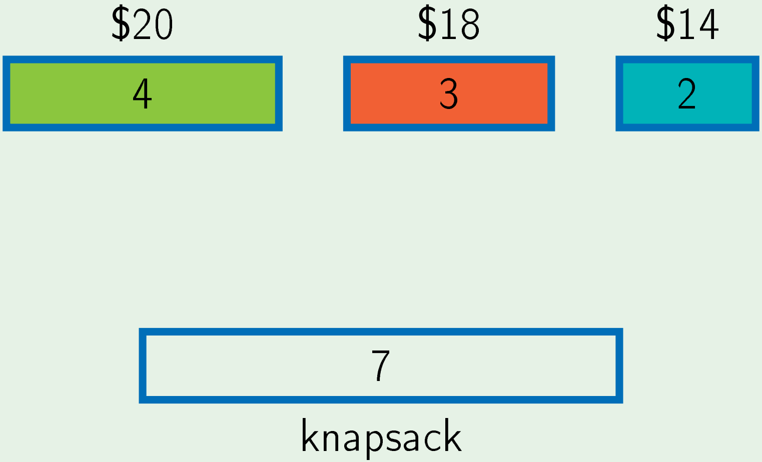
Fractional knapsack

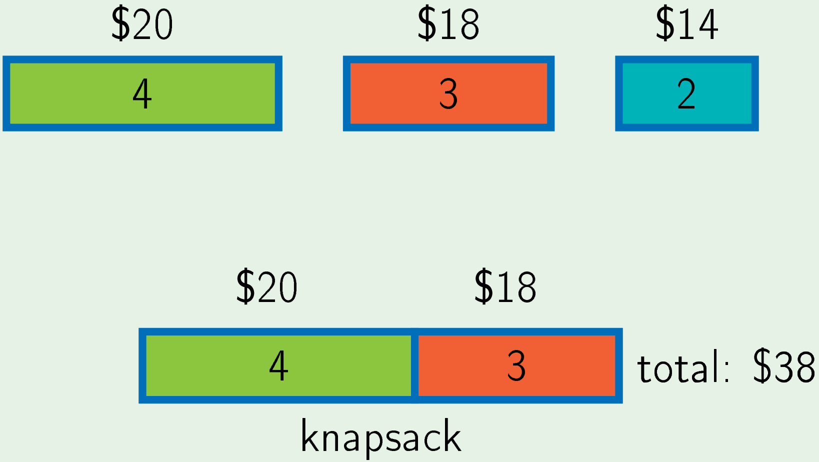
Input: Weights $w_1, \dots, w_n$ and values $v_1, \dots, v_n$ of n items; capacity W.

Output: The maximum total value of fractions of items that fit into a bag of capacity W.

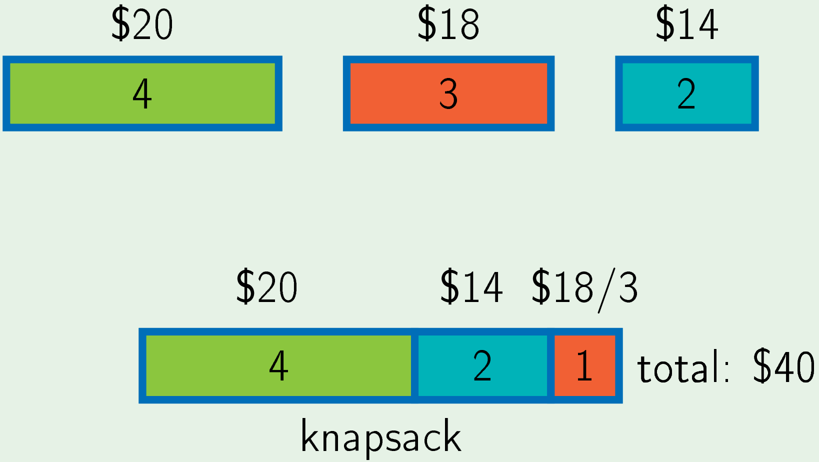
Example



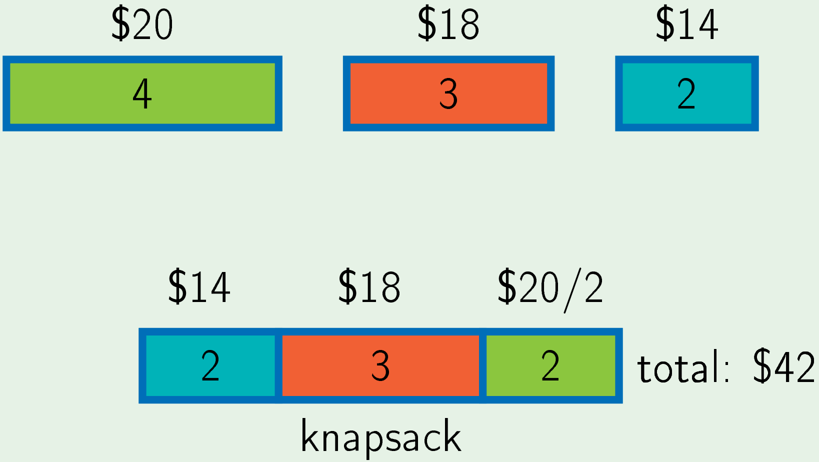
Example



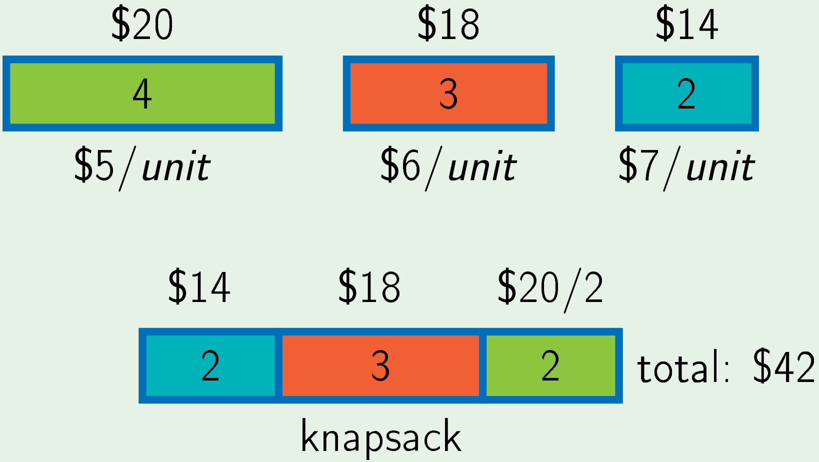
Example



Example



Example

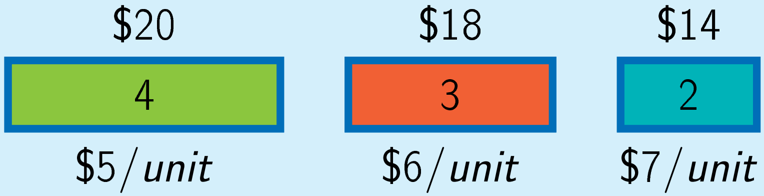


Safe move

Lemma

There exists an optimal solution that uses as much as possible of an item with the maximal value per unit of weight.

Proof



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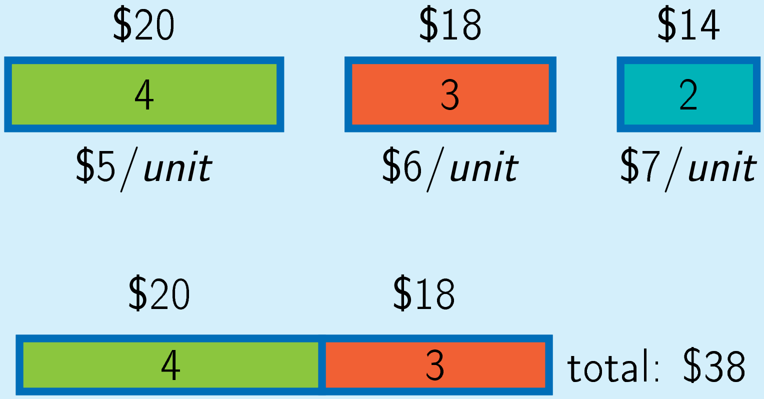
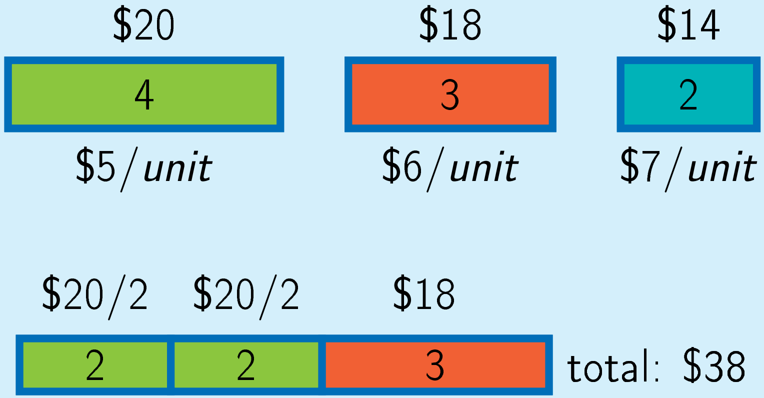
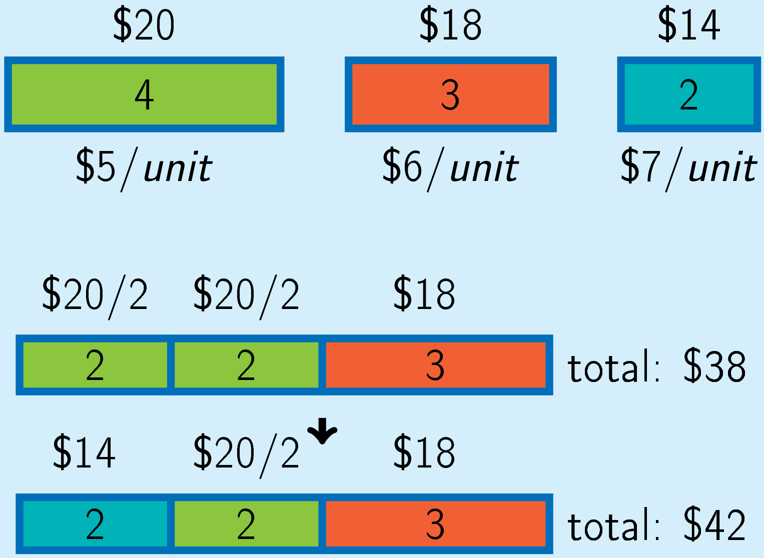


Proof



Proof


$20
4
$5/unit
$18
3
$6/unit
$14
2
$7/unit
$20/2
2
$20/2
2
$18
3
total: $38
$14
2
$20/2
2
$18
3
total: $42

Greedy Algorithm

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Knapsack$(W, w_1, v_1, \dots, w_n, v_n)$

```
$A \leftarrow [0, 0, \dots, 0]$, $V \leftarrow 0$
repeat $n$ times:
    if $W = 0$:
        return $(V, A)$
    select $i$ with $w_i > 0$ and max $\frac{v_i}{w_i}$
    $a \leftarrow \min(w_i, W)$
    $V \leftarrow V + a\frac{v_i}{w_i}$
    $w_i \leftarrow w_i - a$, $A[i] \leftarrow A[i] + a$, $W \leftarrow W - a$
return $(V, A)$
```

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The running time of `Knapsack` is $O(n^2)$.

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- Main loop is executed n times
- Overall, $O(n^2)$ $\square$

Optimization

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- First, sort items by decreasing $\frac{v}{w}$

Assume $\frac{v_1}{w_1} \geq \frac{v_2}{w_2} \geq \cdots \geq \frac{v_n}{w_n}$

```
Knapsack($W, w_1, v_1, \ldots, w_n, v_n$)

$A \leftarrow [0, 0, \ldots, 0]$, $V \leftarrow 0$
for $i$ from 1 to $n$:
   if $W = 0$:
      return $(V, A)$
   $a \leftarrow \min(w_i, W)$
   $V \leftarrow V + a\frac{v_i}{w_i}$
   $w_i \leftarrow w_i - a$, $A[i] \leftarrow A[i] + a$, $W \leftarrow W - a$
return $(V, A)$
```

Asymptotics

- Now each iteration is $O(1)$
- `Knapsack` after sorting is $O(n)$
- Sort + `Knapsack` is $O(n \log n)$