CMPS/MATH 2170 Discrete Mathematics – Spring 24

3/12/24

Midterm

Name: ______________________________

Student ID: ______________________________

- Put your name on the exam.
- This exam is closed-book, closed-notes, and closed-calculators. You are allowed to use a helper sheet (one single-sided letter page).
- If you have a question, **stay seated** and raise your hand.
- Please try to write legibly – if we cannot read it you may not get credit.
- **Do not waste time** – if you cannot solve a question immediately, skip it and return to it later.

1) Logic		15
2) Rules of Inference		10
3) Proof by Contradiction		15
4) Counterexample		5
5) Sets		10
6) Functions		15
7) Sequences		10
8) Induction		10
Bonus		10
Total		100

1 Logic (15 points)

1. **Equivalence (10 points)**
   Show that $(p \to r) \lor (q \to r)$ and $(p \land q) \to r$ are logically equivalent by using the laws of propositional logic (Zybooks table 1.5.1). Show each step.

$$
\begin{aligned}
(p \to r) \lor (q \to r) &\equiv (\neg p \lor r) \lor (\neg q \lor r) && \text{Conditional Identities} \\
&\equiv (\neg p \lor r) \lor (r \lor \neg q) && \text{Commutative Law} \\
&\equiv \neg p \lor ((r \lor r) \lor \neg q) && \text{Associative Law} \\
&\equiv \neg p \lor (r \lor \neg q) && \text{Idempotent Laws} \\
&\equiv \neg p \lor (\neg q \lor r) && \text{Commutative Laws} \\
&\equiv (\neg p \lor \neg q) \lor r && \text{Associative Laws} \\
&\equiv \neg(p \land q) \lor r && \text{De Morgan's Laws} \\
&\equiv (p \land q) \to r && \text{Conditional Identity}
\end{aligned}
$$

2. **Translation (5 points)**
   Let the domain be the members of a chess club. The predicate $B(x, y)$ means that person x has beaten person y at some point in time. Give a logical expression equivalent to the following English statement.

   "Everyone has won at least one game."

   $\forall x \exists y (x \neq y \land B(x, y))$.

2 Rules of Inference (10 points)

Consider the following collection of premises:

> "I ate dinner or I ate lunch."
> "If I ate lunch, then I neither ate dinner nor breakfast."
> "If I ate dinner, then I ate breakfast."
> "I ate breakfast."

Use rules of inference to infer "I ate dinner." from these premises.
You may use the rules in Zybooks table 3.8 and table 1.5.1.
Let $D =$"I ate dinner", $L =$"I ate lunch" and $B =$"I ate breakfast."

B	Hypothesis	(1)
$B \vee D$	Addition from 1	(2)
$\neg(\neg B \wedge \neg D)$	De Morgan's Law from 2	(3)
$\neg(\neg D \wedge \neg B)$	Commutative Law from 3	(4)
$L \to (\neg D \wedge \neg B)$	Hypothesis	(5)
$\neg L$	Modus Tollens from 4 and 5	(6)
$D \vee L$	Hypothesis	(7)
D	Disjunctive Syllogism from 7 and 6	(8)

3 Proof by Contradiction (15 points)

Use a proof by contradiction to show that for any five real numbers $a_1, a_2, ..., a_5$, at least one of them is greater than or equal to the average of these numbers. Recall that the average of the five numbers is given by $(a_1 + a_2 + \cdots + a_5)/5$.

Proof. Assume to the contrary that there exists a set of 5 numbers $a_1, a_2, \ldots, a_5$ that are all less than their average. Denote their average by $A = \frac{a_1+a_2+a_3+a_4+a_5}{5}$. By assumption, we have the inequalities $a_1 < A$, $a_2 < A$, $a_3 < A$, $a_4 < A$ and $a_5 < A$. We add these five inequalities together to find

$$
\begin{aligned}
a_1 + a_2 + a_3 + a_4 + a_5 &< 5A && \text{Adding the 5 inequalities} \\
a_1 + a_2 + a_3 + a_4 + a_5 &< 5\frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} && \text{Definition of } A \\
a_1 + a_2 + a_3 + a_4 + a_5 &< a_1 + a_2 + a_3 + a_4 + a_5 && \text{Cancelling } \frac{5}{5} = 1.
\end{aligned}
$$

The last line is a contradiction, because it assets that the real number $a_1 + a_2 + a_3 + a_4 + a_5$ is less than itself. From the contradiction we learn that our initial assumption was false, and there are no real numbers a_1, a_2, a_3, a_4, a_5 that are each less than their average. In other words, every set of 5 real numbers has at least one that is at least as great as their average. □

4 Counterexample (5 points)

Disprove the following statement:

$$\forall x, y \in \mathbb{R}^+, x^2 + y^2 \geq 2\min(x, y)$$

Let $x = y = 0.5$. This is a counterexample because

$$0.5^2 + 0.5^2 = 0.5 \not\geq 2\min(0.5, 0.5) = 1.$$

5 Sets (10 points)

Let $A = \{1, 3, 5, 7\}$, $B = \{5, 6, 7\}$, and $C = \{1, 2, 3, 4, 5, 6, 7\}$. Please fill in the blanks below. (2 points each).

$A - C =$ $\emptyset$

$B \cap C =$ {5,6,7}

$A \cup B =$ {1,3,5,6,7}

$\{1\} \times A = \{(1, 1), (1, 3), (1, 5), (1, 7)\}$

$|P(A \cap B)| = 4$

6 Functions (15 points)

Let $f : (1, \infty) \to (0, \infty)$ be defined by $f(x) = \sqrt{x-1}$.

1. (10 points) Use the definitions of surjective, injective, and bijective to prove that f is bijective on this domain and codomain.

   - Surjective: The function $f : (1, \infty) \to (0, \infty)$ is surjective if $\forall y \in (0, \infty), \exists x \in (1, \infty)$ such that $f(x) = y$. To check that this is satisfied, let $y \in (0, \infty)$. Define x by $x = y^2 + 1$. Note that $x > 1$, since $y^2 > 0$, so $x \in (1, \infty)$. Observe that

     $$f(x) = \sqrt{x-1} = \sqrt{(y^2+1)-1} = \sqrt{y^2} = y.$$

     The last equality uses that $y > 0$. We have found a particular $x \in \mathbb{R}^+$ such that $f(x) = y$. Since y was arbitrary from the set $(0, \infty)$, this argument shows that every element $y \in (0, \infty)$ has a preimage in $(1, \infty)$, so f is surjective.

   - Injective: The function $f : (1, \infty) \to (0, \infty)$ is injective if $\forall x_1, x_2 \in (1, \infty), f(x_1) = f(x_2) \to x_1 = x_2$. Suppose $x_1, x_2 \in (1, \infty)$ are such that $f(x_1) = f(x_2)$. Then

     $$\begin{aligned} \sqrt{x_1 - 1} &= \sqrt{x_2 - 1} && \text{Definition of } f \\ x_2 - 1 &= x_1 - 1 && \text{Squaring both sides} \\ x_2 &= x_1 && \text{Adding 1 to both sides} \end{aligned}$$

   - Bijective: f is bijective if it is both surjective and injective. We have proven that f is surjective and injective, so we have proven that f is bjective.

2. (5 points) Give a domain and a codomain for which f is injective but not surjective. Justify your answer shortly.

   $f : \emptyset \to (0, \infty)$ is injective but not surjective on the domain $\emptyset$ and codomain $(0, \infty)$. It is trivial that f is injective, because $\forall x_1, x_2 \in \emptyset, f(x_1) = f(x_2) \to x_1 = x_2$ holds because of the universal quantification over the emptyset. To see that f is not surjective, take $y = 2 \in (1, \infty)$. There is no $x \in \emptyset$ such that $f(x) = y$ because there is no $x \in \emptyset$ at all.

7 Sequences (10 points)

1. (4 points) Give an example of a geometric sequence that is neither non-increasing nor non-decreasing.

   The sequence $1, -1, 1, -1, \ldots$ is geometric because the common ratio between successive terms is constant, -1. It is not non-decreasing because the even numbered terms are all less than their predecessor. It is not non-increasing because the odd numbered terms are all greater than their predecessor.

2. (2 point) Express $5 + 15 + 25 + 35 + 45 + \cdots + 95 + 105$ in summation notation
   $\sum_{i=0}^{10} 5 + 10i$

3. (4 points) Calculate the value of the expression from (2).
   $\sum_{i=0}^{10} 5 + 10i = 5 \cdot 11 + \frac{10 \cdot 10 \cdot 11}{2} = 605$

8 Induction (10 points)

Show by mathematical induction that for all $n \in \mathbb{N}$ satisfying $n \geq 3$, the following holds.

$$n^2 \geq 2n + 1.$$

Proof.

1. First, we check the base case. When $n = 3$, we verify that $9 \geq 2 \cdot 3 + 1 = 7$.
2. We assume as the inductive assumption that $k \in \mathbb{N}$ satisfies $k^2 \geq 2k + 1$ for some $k \geq 3$.
3. For the inductive step, we would like to show that $(k+1)^2 \geq 2(k+1) + 1$.

$$\begin{aligned}
(k+1)^2 &= k^2 + 2k + 1 && \text{FOIL (Distributive property)} \\
&\geq 2k + 1 + 2k + 1 && \text{Inductive hypothesis} \\
&\geq 2k + 8 && k \geq 3 \\
&\geq 2k + 3 && 8 \geq 3 \\
&= 2(k+1) + 1 && \text{Distributive property}
\end{aligned}$$

The calculation verifies the inductive step and completes the inductive proof that $n^2 \geq 2n + 1$ for all $n \in \mathbb{N}$ satisfying $n \geq 3$.

□