Discrete Math Counting Introduction

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  - How many pairs of twin primes (numbers p and $p+2$ that are both prime) are there? (unknown)

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- There are 4 factors of 46.

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- In this solution, why did we multiply the number of choices for each n_i'?

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- Mathematically, the product rule states that $|A \times B| = |A||B|$.

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- The total number of moves is $8 + 14 = 22$.

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- When we count "one two three..." we are establishing a bijection with $\{1, 2, 3, \ldots\}$.

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- An elaborate theory justifies the slogan "Some infinities are bigger than others."
- In Discrete math, we should stick to the finite, or at least countably finite sets.

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- There are $\frac{10 \times 9}{2}$ handshakes.

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- Example: How many ways can 10 people sit if there are 11 chairs? (Just 1: everybody sits down in a chair.)