CMPS/MATH 2170 Discrete Mathematics – Fall 23

10/3/23

Midterm

Name:

- Put your name on the exam.
- This exam is closed-book, closed-notes, and closed-calculators. You are allowed to use a helper sheet (one single-sided letter page).
- If you have a question, **stay seated** and raise your hand.
- Please try to write legibly – if we cannot read it you may not get credit.
- **Do not waste time** – if you cannot solve a question immediately, skip it and return to it later.

1) Logic		15
2) Rules of Inference		10
3) Functions		15
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6) Sets		10
7) Functional Completeness		10
8) Summations		10
Bonus		10
Total		100

1 Logic (15 points)

1. **Equivalence (5 points)**
Let p and q be propositions. Use a truth table to determine whether
$(p \vee q) \rightarrow p$ is equivalent to $q \rightarrow p$.

p	q	$p \vee q$	$(p \vee q) \rightarrow p$	$q \rightarrow p$	
T	T	T	T	T	
T	F	T	T	T	
F	T	T	F	F	
F	F	F	T	T	

Are $(p \vee q) \rightarrow p$ and $q \rightarrow p$ equivalent? Justify your answer.

Answer:

Yes. The two columns corresponding to these propositions are identical in the table above.

2. **Negation (5 points)**
Simplify the following Boolean formula such that it contains no negation operators $\neg$, but possibly $\leq$ symbols.

$$\neg\left((\exists p \in \mathbb{N})\ (\forall q \in \mathbb{Z}) :\ p > q\right)$$

(Hint: Use $a \leq b \equiv \neg(a > b)$.)

Answer:

$$(\forall p \in \mathbb{N})(\exists q \in \mathbb{Z}) : p \leq q$$

3. **Translation (5 points)**
Express the following sentence as a logic formula with quantifiers:

"For every positive integer, there is a different integer with the same square."

Answer:

$(\forall n \in \mathbb{N} - \{0\})(\exists m \in \mathbb{Z}) : m \neq n \wedge m^2 = n^2$

2 Rules of Inference (10 points)

Consider the following collection of premises:

> "If Dracula is out, then it is raining or it is night or the full moon shines."
> "Dracula is out."
> "It is not raining."
> "Whenever the full moon shines, it is night."

Use rules of inference to infer "It is night." from these premises.
You may use the rules in Zybooks table 3.8 and table 1.5.1.

Proposition	Symbol	
Dracula is out	D	
It is raining	R	
The full moon shines	M	
It is night	N	

Statement number	Statement	Justification
1	$D \to (R \vee N \vee M)$	Premise
2	D	Premise
3	$\neg R$	Premise
4	$M \to N$	Premise
5	$R \vee N \vee M$	Modus Ponens 1,2
6	$R \vee (N \vee M)$	Associative law 5
7	$N \vee M$	Disjunctive Syllogism, $3, 6$
8	$\neg M \vee N$	Conditional identity 4
9	$M \vee N$	Commutative identity 7
10	$N \vee N$	Resolution $8, 9$
11	N	Idempotent law 10

3 Functions (15 points)

Let $f(x) = 7x - \frac{1}{2}$.

1. (10 points) Give a domain and a codomain for which f is bijective. Use the definitions of surjective, injective, and bijective to prove that f is bijective.

   (Solution 1) Let $A = B = \emptyset$. Then $f : A \to B$ is a bijection. Let us show that it is an injection. By definition, we must show that $\forall x, y \in A, f(x) = f(y) \implies x = y$. This holds trivially, since there are no $x, y \in A$.

   Now let us show that f is a surjection. By definition, we must show that $(\forall y \in B)(\exists x \in A) f(x) = y$. Again, since $B = \emptyset$, this is true trivially.

   (Alternatively, one could pick $A = \{0\}, B = \{-\frac{1}{2}\}$.)

   (Solution 2) Let $A = B = \mathbb{Q}$. The $f : A \to B$ is a bijection. First, we show that f is an injection. Let $x, y \in A$. Assume that $f(x) = f(y)$. Then $fx - \frac{1}{2} = 7y - \frac{1}{2}$, so $7x = 7y$, and after dividing by 7, we find $x = y$. This shows that f is injective.

   To show that f is a surjection, let $z \in B$. Let $w = \frac{1}{7}(z + \frac{1}{2})$. Because $\mathbb{Q}$ is closed under addition and multiplication, $w \in \mathbb{Q} = A$. We defined w so that it satisfies $f(w) = 7w - \frac{1}{2} = 7(\frac{1}{7}(z + \frac{1}{2})) - \frac{1}{2} = z$. Since z was arbitrary, this shows (by universal generalization) that $(\forall z \in B)(\exists w \in A) f(w) = z$, which is the definition of f being surjective.

2. (5 points) Give a domain and a codomain for which f is injective but not surjective. Justify your answer shortly.

   (Solution 1)Let the domain be $A = \emptyset$ and let the codomain be $B = \mathbb{R}$. Then f is injective but not surjective. f is injective trivially, because $(\forall x, y \in \emptyset) : f(x) = f(y)$.

   The function f is not surjective because, for example, the element $10 \in \mathbb{R}$ has no preimage under f.

   (Solution 2) Let $A = \mathbb{Q}$ and let $B = \mathbb{R}$. The function f is injective because $f : A \to \mathbb{Q}$ is injective by problem 1 (solution 2), and because $\mathbb{Q} \subset \mathbb{R}$. The function f cannot be surjective because $|\mathbb{Q}| < |\mathbb{R}|$.

4 Proof By Contradiction (15 points)

Consider the theorem below.

Theorem: If x is irrational then $\frac{x}{2}$ is irrational.

Prove the theorem using a proof by contradiction.

Assume for the sake of contradiction that $x \in \mathbb{R} - \mathbb{Q}$ yet $\frac{x}{2} \in \mathbb{Q}$. Then $\exists a, b \in \mathbb{Z}$ with $b \neq 0$ such that $\frac{x}{2} = \frac{a}{b}$. Multiplying both sides of this equation by 2 results in the equation $x = \frac{2a}{b}$. Since $2a \in \mathbb{Z}$ and $b \neq 0$, we have shown that $x \in \mathbb{Q}$, but this contradicts that $x \in \mathbb{R} - \mathbb{Q}$.

From the contradiction, we learn that our initial assumption was wrong, so there is no $x \in \mathbb{R} - \mathbb{Q}$ such that $\frac{x}{2} \in \mathbb{Q}$. In other words, if x is irrational, then $\frac{x}{2}$ is irrational.

5 Counterexample (5 points)

Disprove the following statement:

$$(\forall x, y \in \mathbb{Q}) : x^2 + y^2 + xy \geq x + y$$

Set $x = y = \frac{1}{2}$. Then we find that

$$\frac{3}{4} = (\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{2} + \frac{1}{2}) < \frac{1}{2} + \frac{1}{2} = 1,$$

and this is ca counterexample to the claim.

6 Sets (10 points)

Let $A = \{1, 3, 5, 7\}$, $B = \{1, 2, 3, 4\}$ and $C = \{1, 2, 3, 4, 5, 6, 7\}$. Please fill in the blanks below. (2 points each).

$A - C = \emptyset$

$B \cap C = \{1, 2, 3, 4\}$

$A \cap B = \{1, 3\}$

$|A \times B| = 16$

$|2^{A \cap B}| = 4$

(Hint: In order to find $|2^{A \cap B}|$ and $|A \times B|$ you do not need to specify the power set.)

7 Functional Completeness (10 points)

Recall that the nor operator $\downarrow$ is defined by $p \downarrow q = \neg(p \vee q)$. Show that $\downarrow$ is functionally complete by expressing each operation in the set $\{\wedge, \neg\}$ in terms of $\downarrow$.

Let p and q be propositions. Then $\neg p = \neg(p \vee p) = p \downarrow p$. This expresses $\neg$ in terms of $\downarrow$.

Since $p \downarrow q = \neg(p \vee q) = \neg p \wedge \neg q$, it follows that $(\neg p) \downarrow (\neg q) = \neg p \wedge \neg q$. Using that $\neg p = p \downarrow p$ and $\neg q = q \downarrow q$, we find that $p \wedge q = (p \downarrow p) \downarrow (q \downarrow q)$. This expresses $\wedge$ in terms of $\downarrow$.

8 Summations (10 points)

1. (1 point) Express $5 + 15 + 25 + 35 + 45 + \cdots + 95 + 105$ in summation notation.

   $\sum_{k=0}^{10} 5 + 10k$

2. (8 points) Calculate the value of the expression from part 1.

   $\sum_{k=0}^{10} 5 + 10k = \sum_{k=0}^{10} 5 + 10 \sum_{k=0}^{10} k = 55 + 10\frac{10 \cdot 11}{2} = 55 + 550 = 605$

3. (1 point) How many terms are in the series? (For example, the series $1 + 2 + 3$ has 3 terms.)

   There are 11 terms.