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A characterization of Delsarte's linear programming bound as a ratio bound

Carlos J. Luz *

Escola Superior de Tecnologia de Setúbal/Instituto Politécnico de Setúbal, Campus do IPS, Estefanilha, 2914-508 Setúbal, Portugal

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Abstract

It is well known that the ratio bound is an upper bound on the stability number $\alpha(G)$ of a regular graph G. In this note it is proved that, if G is a graph whose edge is a union of classes of a symmetric association scheme, the Delsarte's linear programming bound can alternatively be stated as the minimum of a set of ratio bounds. This result follows from a recently established relationship between a set of convex quadratic bounds on $\alpha(G)$ and the number $\vartheta'(G)$, a well known variant of the Lovász theta number, which was introduced independently by Schrijver [A. Schrijver, A comparison of the Delsarte and Lovász bounds, IEEE Trans. Infor. Theory 25 (1979) 425–429] and McEliece et al. [R.J. McEliece, E.R. Rodemich, H.C. Rumsey Jr., The Lovász bound and some generalizations, J. Combin. Inform. System Sci. 3 (1978) 134–152].

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1. Introduction

Let $G = (V, E)$ be a simple undirected graph where $V = \{1, \dots, n\}$ denotes the vertex set and E is the edge set. It will be supposed that G has at least one edge, i.e., E is not empty. We will write $ij \in E$ to denote the edge linking nodes i and j of V. The adjacency matrix of G will

* Tel.: +351 265 790018; fax: +351 265 721869.
E-mail address: cluz@est.ips.pt

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be denoted by A_G (this is a symmetric matrix of order n whose entries (i, j) are equal to 1 if $ij \in E$ and 0 otherwise). If we substitute some or all of the ones of A_G by any real numbers and the resulting matrix is non-null and symmetric, we obtain a so called weighted adjacency matrix of G. Additionally, if in a weighted adjacency matrix of G we substitute some or all entries corresponding to the non-edges $ij \notin E$ by negative real numbers, and the resulting matrix is non-null and symmetric, we obtain a matrix that will be referred to as an *extended weighted adjacency matrix* of G.

A stable set (independent set) of G is a subset of nodes of V whose elements are pairwise nonadjacent. The stability number (or independence number) of G is defined as the cardinality of a largest stable set and is usually denoted by $\alpha(G)$. A maximum stable set of G is a stable set with $\alpha(G)$ nodes. Finding $\alpha(G)$ is a NP-hard problem which has originated a great amount of research works in the literature (see, for example, [1,6,10,13,16] and [2] for a survey).

It is well known that the stability number $\alpha(G)$ of a regular graph G of order n satisfies

$$\alpha(G) \leqslant \frac{-\lambda_{\min}(A_G) \times n}{\lambda_{\max}(A_G) - \lambda_{\min}(A_G)}.$$

(Throughout this note, $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ will denote respectively the smallest and the greatest eigenvalue of a matrix M.)

As proved by Delsarte [8], the above inequality (whose right-hand side is known as the *ratio bound*) is valid for strongly regular graphs. In an unpublished work, Hoffman extended the result of Delsarte proving that the ratio bound is an upper bound on the stability number of any regular graph (the referred work appears mentioned in [5]; this monograph also presents a more general result that implies Hoffman's result, which was proved in [3]; for related informations, see [4]). In [12], Lovász showed that for regular graphs the ratio bound is also an upper bound on his theta number. In [14], it is proved that for regular graphs the ratio bound coincides with the following convex quadratic upper bound $\upsilon(G)$ on $\alpha(G)$:

$$\upsilon(G) = \max\{2e^{\mathrm{T}}x - x^{\mathrm{T}}(H + I)x : x \geqslant 0\}, \tag{1}$$

where e is the $n \times 1$ all ones vector, I is the identity matrix of order n and $H = A_G/(-\lambda_{\min}(A_G))$.

In this note it is proved that, if G is a graph whose edge set is a union of classes of an association scheme, Delsarte's linear programming bound on $\alpha(G)$ (which will be denoted by $\mathrm{Del}(G)$) can alternatively be stated as the minimum of a set of ratio bounds, i.e.,

$$\mathrm{Del}(G) = \min_{C} \frac{-\lambda_{\min}(C) \times n}{\lambda_{\max}(C) - \lambda_{\min}(C)},$$

where C is a extended weighted adjacency matrix of G such that the all-ones vector belongs to the eigenspace of $\lambda_{\max}(C)$. This result follows from a relationship that can be established between a set of convex quadratic bounds on $\alpha(G)$ that generalize $\upsilon(G)$ and the number $\vartheta'(G)$, a well known variant of the Lovász theta number, which was introduced independently by Schrijver [18] and McEliece et al. [17].

We first briefly review Delsarte's linear programming bound and the $\vartheta'(G)$ bound. Then we relate this bound with a class of convex quadratic bounds on $\alpha(G)$ and prove the main result.

2. Delsarte's linear programming bound

We first recall the definition of symmetric association scheme as a necessary background for reviewing Delsarte's linear programming bound (we essentially followed Refs. [8,18,19]).

A symmetric association scheme is a pair $(X, \mathscr{R})$ where X is a finite set with m elements and $\mathscr{R} = (R_0, R_1, \ldots, R_n)$ is a family of binary relations of X (i.e., a subset family of the cartesian product $X \times X$) that satisfy the following conditions:

A.1 $R_0 = \{(x, x) : x \in X\}$;
A.2 $R_k^{-1} = \{(y, x) : (x, y) \in R_k\} = R_k$, for $k = 0, 1, \ldots, n$;
A.3 $\mathscr{R}$ partitions $X \times X$;
A.4 For $i, j, k = 0, \ldots, n$ and $(x, y) \in R_k$, there exists a non-negative integer $p_{i,j}^k$ such that, for all $(x, y) \in R_k$,

$$|\{z \in X : (x, z) \in R_i \wedge (z, y) \in R_j\}| = p_{i,j}^k.$$

The numbers $p_{i,j}^k$ are the scheme parameters and from A.2 it follows that $p_{i,j}^k = p_{j,i}^k$. Each pair (X, R_i), $i = 1, \ldots, n$, can be considered as a regular graph of valence $v_i = p_{ii}^0$. We have $v_0 = p_{0,0}^0 = 1$, $p_{i,j}^0 = \delta_{i,j} v_i$ and $v_0 + v_1 + \cdots + v_n = m$.

Let $D_0 = I$, where I is the identity matrix of order m, and, for $i = 1, \ldots, n$, let D_i be the adjacency matrix of (X, R_i). Having in mind the above definition, these matrices satisfy:

D.1 $\sum_{i=0}^n D_i = ee^{\mathrm{T}}$, where e is now the all ones vector or order m;
D.2 $D_i D_j = \sum_{t=0}^n p_{j,i}^t D_t, \quad \forall i, j = 0, 1, \ldots, n.$

As the D_i are 0–1 matrices, condition D.1 implies that $D_0, D_1, \ldots, D_n$ are linearly independent. Condition D.2 asserts that the product of any two of these matrices belongs to the subspace spanned by $\{D_0, D_1, \ldots, D_n\}$ and so this set is a basis of a commutative algebra, known as the Bose–Mesner algebra. From a spectral result of linear algebra [11, Theorem 1.3.19] it follows that $D_0, D_1, \ldots, D_n$ (as well as all matrices of the Bose–Mesner algebra) are simultaneously diagonalizable (i.e., there exists an orthogonal matrix U such that $U^{\mathrm{T}} D_0 U, U^{\mathrm{T}} D_1 U, \ldots, U^{\mathrm{T}} D_n U$ are diagonal matrices) and there exists a matrix $P = [P_{ji}]_{i,j=0}^n$ such that $P_{0i}, P_{1i}, \ldots, P_{ni}$ are the eigenvalues of D_i that form the ith column of $P (i = 0, \ldots, n)$. Besides, $P_{j0}, P_{j1}, \ldots, P_{jn}$ form the jth row of P and are the eigenvalues associated with a common eigenvector u of $D_0, D_1, \ldots, D_n$, respectively (i.e., $D_0 u = P_{j0} u, \ldots, D_n u = P_{jn} u$).

The set of columns of U is the union of $n + 1$ subsets of eigenvectors, being the jth subset $(j = 0, 1, \ldots, n)$ formed by m_j eigenvectors which constitute a basis of the common eigenspace to $D_0, D_1, \ldots, D_n$ associated with $P_{j0}, P_{j1}, \ldots, P_{jn}$, respectively. Hence m_j is the dimension of this common eigenspace and $m_0 + m_1 + \cdots + m_n = m$.

By D.1, U diagonalizes ee^{T}. Since this matrix has eigenvalues m and 0 with multiplicities 1 and $m - 1$, respectively, one column of U is precisely the eigenvector $e/\sqrt{m}$ of ee^{T} corresponding to the eigenvalue m. Usually one considers $e/\sqrt{m}$ as the first column of U and sets $m_0 = 1$ and $P_{00} = 1$. As $D_1, \ldots, D_n$ are adjacency matrices of regular graphs, the rest of the first row (row 0) of P includes the valences of these graphs, i.e.,

$$P_{01} = v_1, \ldots, P_{0n} = v_n.$$

Let $Q = [Q_{ji}]_{i,j=0}^n$ be the matrix such that

$$Q_{ij} = \frac{m_j}{v_i} P_{ji}. \tag{2}$$

Thus the first row (row 0) of Q is formed by the geometric multiplicities $m_0, m_1, \ldots, m_n$ as, by (2),

$$Q_{0j} = \frac{m_j}{v_0} P_{j0} = m_j, \quad j = 0, 1, \ldots, n,$$

since 1 is the unique eigenvalue of D_0 (i.e., $P_{j0} = 1 \forall j$) and $v_0 = 1$. Additionally, it can be proved that

$$\sum_{j=0}^{n} P_{jr} Q_{sj} = m\delta_{rs} \quad \text{and} \quad \sum_{i=0}^{n} P_{ri} Q_{is} = m\delta_{rs}, \tag{3}$$

i.e., P and Q/m are inverse matrices. The matrices P and Q are called respectively the first and the second eigenvalues matrices of the scheme.

Let $E_0 = \frac{1}{m} ee^{\mathrm{T}}$ the orthogonal projection matrix over the eigenspace spanned by the first column of U. For $j = 1, \ldots, n$, let $u_1^{(j)}, \ldots, u_{m_j}^{(j)}$ be the columns of U spanning the eigenspace of dimension m_j common to $D_0, D_1, \ldots, D_n$ and associated with $P_{j0}, P_{j1}, \ldots, P_{jn}$, respectively. Setting

$$E_j = \sum_{t=1}^{m_j} u_t^{(j)} u_t^{(j)\mathrm{T}}$$

we have that the matrices $E_0, E_1, \ldots, E_n$ are symmetric and satisfy:

E.1 $E_i E_j = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$
E.2 $\sum_{i=0}^{n} E_i = I$;
E.3 $\{E_0, E_1, \ldots, E_n\}$ is a basis of the Bose–Mesner algebra;
E.4 $D_i = \sum_{j=0}^{n} P_{ji} E_j$, $\forall i = 0, \ldots, n$.

A classic example of association scheme is now presented. Let n and q be natural numbers and let X be the set of vectors of length n, with entries in $\{0, \ldots, q-1\}$. For $k = 0, 1, \ldots, n$ let

$$R_k = \{(x, y) \in X \times X : d_{\mathrm{H}}(x, y) = k\},$$

where $d_{\mathrm{H}}(x, y)$ denotes the Hamming distance between the vectors x and y, i.e., the number of coordinate places in which x and y differ. Setting $\mathscr{R} = (R_0, R_1, \ldots, R_n)$, it can be seen that $(X, \mathscr{R})$ is a symmetric association scheme, called a Hamming scheme. Usually it is represented by $H(n, q)$ and has the following values of v_i, m_j and P_{ji}:

$$v_i = \binom{n}{i} (q-1)^i, \quad m_j = \binom{n}{j} (q-1)^j$$

and

$$P_{ji} = K_i(j) = \sum_{t=0}^{i} (-q)^t (q-1)^{i-t} \binom{n-t}{i-t} \binom{j}{t}$$

for $i, j = 0, 1, \ldots, n$ ($K_i(j)$ is a Krawtchouk polynomial of degree i in the variable j). In these scheme the matrices P and Q are equal as can be easily checked.

The Hamming schemes are very important in coding theory (for this theme see for example [15]). In a Hamming scheme $(X, \mathscr{R} = (R_0, R_1, \ldots, R_n))$ a code is a subset Y of X. One of the main coding problems is to estimate the maximum cardinality of a code Y such that no two elements in Y have Hamming distance less or equal to a given value d. To formulate this problem in the language of association schemes it is necessary to introduce the following notion: given $M \subset \{0, 1, \ldots, n\}$ with $0 \in M$, a subset Y of X is a M-clique if $(x, y) \in \cup_{i \in M} R_i$, for all

$x, y \in Y$. Therefore, the above coding problem consists on determining the maximum cardinality of $\{0, d, d+1, \ldots, n\}$-cliques in the Hamming scheme. It should be noted that determining a M-clique in a association scheme is equivalent to obtain a stable set of vertices in the graph $G = (X, E)$, where $E = \cup_{i \notin M} R_i$. Hence, to determine the maximum cardinality of a M-clique is the same as computing $\alpha(G)$.

To give an upper bound on the maximum cardinality of a M-clique in a symmetric association scheme $(X, \mathscr{R})$ with $\mathscr{R} = (R_0, R_1, \ldots, R_n)$, Delsarte [8] defined, for each $Y \subseteq X$, the inner distribution vector $a = (a_0, \ldots, a_n)^{\mathrm{T}}$ of Y:

$$a_i = \frac{|R_i \cap (Y \times Y)|}{|Y|}, \quad i = 0, 1, \ldots, n.$$

Thus, $a_0 = 1$, $\sum_{i=0}^{n} a_i = |Y|$ and, if Y is a M-clique, $a_i = 0$ for $i \notin M$. Delsarte proved that the inner distribution vector a of any $Y \subseteq X$ verifies $\sum_{j=0}^{n} Q_{ij} a_j \geqslant 0$, for all i (i.e., $Q^{\mathrm{T}} a \geqslant 0$). In addition he also proved that, for any M-clique Y, its cardinal $|Y|$ is less or equal to the optimal value $\mathrm{Del}(G)$ of the following primal-dual pair of linear programing problems, known as Delsarte's linear programming bound:

$$\begin{aligned}
|Y| &\leqslant \mathrm{Del}(G) \\
&= \max\left\{\sum_{j=0}^{n} a_j : a_0, \ldots, a_n \geqslant 0,\ a_0 = 1 \text{ and } a_j = 0 \text{ for } j \notin M,\ \sum_{j=0}^{n} Q_{ij} a_j \geqslant 0,\ \forall i\right\} \\
&= \min\left\{\sum_{i=0}^{n} b_i : b_0, \ldots, b_n \geqslant 0,\ b_0 = 1 \text{ and } \sum_{i=0}^{n} P_{ij} b_i \leqslant 0 \text{ for } j \in M\backslash\{0\}\right\}.
\end{aligned} \tag{4}$$

As a M-clique corresponds to an stable set in the graph $G = (X, E)$ defined above, we conclude that $\mathrm{Del}(G)$ can be viewed as an upper bound on $\alpha(G)$.

3. The $\vartheta'(G)$ bound

The Lovász $\vartheta(G)$ number was introduced in [12] and is probably the most famous upper bound on $\alpha(G)$. It can be computed in polynomial time as proved by Grötschel et al. [9] and verifies

$$\alpha(G) \leqslant \vartheta(G) \leqslant \bar{\chi}(G),$$

a fact known as the *Lovász sandwich theorem.* ($\bar{\chi}(G)$ denotes the minimum number of cliques of G that cover V, a clique being any subset of V such that the induced subgraph is complete.)

Schrijver [18] and McEliece et al. [17] gave, independently, a bound $\vartheta'(G)$ on the stability number $\alpha(G)$ which is generally sharper then $\vartheta(G)$, i.e., such that

$$\alpha(G) \leqslant \vartheta'(G) \leqslant \vartheta(G)$$

for each graph G. Schrijver [18] characterized the bound $\vartheta'(G)$ as follows:

$$\vartheta'(G) = \min_{A} \lambda_{\max}(A), \tag{5}$$

where $\lambda_{\max}(A)$ denotes the largest eigenvalue of A, and the minimum is taken over the set of all symmetric matrices $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ such that $a_{ij} = 1$ if $i = j$ and $a_{ij} \geqslant 1$ if $ij \notin E$. Since we are assuming that G has at least one edge, we can eliminate the matrix ee^{T} from this set. In fact, if $\vartheta'(G) = \lambda_{\max}(ee^{\mathrm{T}}) = n$, then $\bar{\chi}(G) = n$ (recall the sandwich theorem) and thus G would have no edge.

So, if $A \neq ee^{\mathrm{T}}$ is one of the above symmetric matrices, we have that $C = ee^{\mathrm{T}} - A \neq 0$ is an extended weighted adjacency matrix of G. Consequently, setting $A = ee^{\mathrm{T}} - C$, $\vartheta'(G)$ can be formulated as

$$\vartheta'(G) = \min_{C} \lambda_{\max}(ee^{\mathrm{T}} - C), \tag{6}$$

where C is an extended weighted adjacency matrix of G.

The papers [17,18] assert that $\vartheta'(G)$ coincides with Delsarte's linear programming bound [8] when the edge set of G is a union of some classes of a symmetric association scheme $(X, \mathscr{R})$. In fact, Theorem 3 of [18] states: "$\vartheta'(G)$ is equal to the linear programming bound for M-cliques in $(X, \mathscr{R})$". In the proof of this theorem it is considered the matrix

$$A = \vartheta'(G)I - \sum_{i,j=0}^{n} \frac{b_j}{m_j} Q_{ij} D_i + ee^{\mathrm{T}},$$

where $b_0, b_1, \ldots, b_n$ are the optimal solutions of the minimization problem (4) and I is the identity matrix of order $m = |X|$. The matrix A is symmetric, satisfies $a_{ij} = 1$ if $i = j$, $a_{ij} \geqslant 1$ if $ij \notin E$ and its largest eigenvalue coincides with $\vartheta'(G)$. By (6), the matrix to be used below

$$C = \sum_{i,j=0}^{n} \frac{b_j}{m_j} Q_{ij} D_i - \vartheta'(G)I,$$

is an extended weighted adjacency matrix of G.

4. Relating $\vartheta'(G)$ with convex quadratic bounds on $\alpha(G)$

Let $C = [c_{ij}] \in \mathbb{R}^{n\times n}$ be an extended weighted adjacency matrix of a graph $G = (V, E)$, i.e., a non-null real symmetric matrix such that $c_{ij} = 0$ if $i = j$ and $c_{ij} \leqslant 0$ if $ij \notin E$. Associated with C define the following quadratic programming problem:

$$(P_G(C)), \quad \upsilon(G, C) = \max\left\{2e^{\mathrm{T}}x - x^{\mathrm{T}}(H_C + I)x : x \geqslant 0\right\},$$

where $H_C = C/(-\lambda_{\min}(C))$. Note that, like the Hessian of quadratic problem (1), the matrix H_C is indefinite since its trace is null and not all c_{ij} entries are null. Consequently, $\lambda_{\min}(H_C) = -1$ and the problem $(P_G(C))$ is convex.

We show first that $\upsilon(G, C)$ is an upper bound on the stability number of a graph $\alpha(G)$. It should be noted that $\upsilon(G, C)$ generalizes the upper bound (1), since $\upsilon(G) = \upsilon(G, A_G)$.

Proposition 1. *For any extended weighted adjacency matrices C of a graph $G = (V, E)$, the number $\upsilon(G, C)$ is an upper bound on $\alpha(G)$.*

Proof. As $\lambda_{\min}(H_C) = -1$, the problem $(P_G(C))$ is convex as stated above. To see that $\upsilon(G, C)$ is an upper bound on $\alpha(G)$ for all matrices C, let x be a characteristic vector of any maximum independent set S of G (defined by $x_i = 1$ if $i \in S$ and $x_i = 0$ otherwise). Since the vector x is a feasible solution of $(P_G(C))$, we have

$$\upsilon(G, C) \geqslant 2e^{\mathrm{T}}x - x^{\mathrm{T}}x - x^{\mathrm{T}}H_C x = 2\alpha(G) - \alpha(G) - \frac{1}{-\lambda_{\min}(C)} \sum_{i,j} c_{ij} x_i x_j$$

$$= \alpha(G) - \frac{1}{-\lambda_{\min}(C)}\left(\sum_{i\in V} c_{ii}x_i^2 + 2\sum_{ij\in E} c_{ij}x_ix_j + 2\sum_{ij\notin E} c_{ij}x_ix_j\right).$$

As $\lambda_{\min}(C) < 0$, $c_{ii} = 0$ for all $i \in V$, $x_ix_j = 0$ if $ij \in E$ and $c_{ij} \leqslant 0$ if $ij \notin E$, the inequality $\upsilon(G, C) \geqslant \alpha(G)$ is true for all extended weighted adjacency matrices C of G. $\square$

When $e \in \operatorname{Ker}\{C - \lambda_{\max}(C)I\}$, $\upsilon(G, C)$ can be given as a ratio bound.

Proposition 2. *Let G be a graph of order n with at least one edge. If C is an extended weighted adjacency matrix of G such that $e \in \operatorname{Ker}\{C - \lambda_{\max}(C)I\}$, then*

$$\upsilon(G, C) = \frac{-\lambda_{\min}(C) \times n}{\lambda_{\max}(C) - \lambda_{\min}(C)}.$$

Proof. Let $x = \frac{-\lambda_{\min}(C)}{\lambda_{\max}(C)-\lambda_{\min}(C)}e$. As $x \geqslant 0$ and $(H_C + I)x = e$, the Karush–Kuhn–Tucker conditions associated with $(P_G(C))$ are satisfied. Consequently, x is an optimal solution of $(P_G(C))$ and hence $\upsilon(G, C) = e^{\mathrm{T}}x = \frac{-\lambda_{\min}(C)\times n}{\lambda_{\max}(C)-\lambda_{\min}(C)}$. $\square$

We now relate $\vartheta'(G)$ with the convex quadratic upper bounds $\upsilon(G, C)$.

Theorem 1. *Let G be a graph with at least one edge. Then for any extended weighted adjacency matrix C of graph G, we have $\vartheta'(G) \leqslant \upsilon(G, C)$.*

Proof. Let C be an extended weighted adjacency matrix of a graph G of order n and suppose that $(P_G(C))$ is not unbounded for otherwise the theorem is true.

Let x be an optimal solution of $(P_G(C))$. The Karush–Kuhn–Tucker conditions applied to this problem guarantee that the following conditions are true:

$$x \geqslant 0, \quad (H_C + I)x \geqslant e \quad \text{and} \quad x^{\mathrm{T}}(H_C + I)x = e^{\mathrm{T}}x = \upsilon(G, C). \tag{7}$$

As $H_C + I$ is positive semidefinite we can write $H_C + I = U^{\mathrm{T}}U$. Denoting the columns of U by $u_1, \ldots, u_n$, define a matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ such that

$$a_{ij} = 1 - \frac{u_i^{\mathrm{T}}u_j}{(c^{\mathrm{T}}u_i)(c^{\mathrm{T}}u_j)} \quad \text{if} \quad i \neq j,$$
$$a_{ii} = 1,$$

where $c = \upsilon^{-1/2}Ux$ (we use υ to abbreviate $\upsilon(G, C)$). By (7), we have $U^{\mathrm{T}}c = \upsilon^{-1/2}U^{\mathrm{T}}Ux \geqslant \upsilon^{-1/2}e$, hence $a_{ij} \geqslant 1$ if $ij \notin E$ (since $u_i^{\mathrm{T}}u_j \leqslant 0$ if $ij \notin E$) and $\frac{1}{(c^{\mathrm{T}}u_i)^2} \leqslant \upsilon$, for all i. Conditions (7) also imply $c^{\mathrm{T}}c = \upsilon^{-1}x^{\mathrm{T}}(H_C + I)x = 1$ and thus we can write

$$-a_{ij} = \left(c - \frac{u_i}{c^{\mathrm{T}}u_i}\right)^{\mathrm{T}}\left(c - \frac{u_j}{c^{\mathrm{T}}u_j}\right)$$

and

$$\upsilon - a_{ii} = \left(c - \frac{u_i}{c^{\mathrm{T}}u_i}\right)^2 + \upsilon - \frac{1}{(c^{\mathrm{T}}u_i)^2}.$$

These equations imply that $\upsilon I - A$ is positive semidefinite and hence $\lambda_{\max}(A) \leqslant \upsilon$. Finally, by (5), we conclude $\vartheta'(G) \leqslant \upsilon(G, C)$ as desired. □

Combining this Theorem with Proposition 2 we conclude immediately:

Corollary 2. *Let G be a graph of order n with at least one edge. Then for any extended weighted adjacency matrix C of G such that $e \in \mathrm{Ker}\{C - \lambda_{\max}(C)I\}$, we have*

$$\vartheta'(G) \leqslant \frac{-\lambda_{\min}(C)}{\lambda_{\max}(C) - \lambda_{\min}(C)} n.$$

5. A new characterization of Delsarte's bound

Using the above results, Delsarte's linear programming bound can be stated as follows.

Theorem 3. *Let $(X, \mathscr{R})$ be a symmetric association scheme where $\mathscr{R} = (R_0, R_1, \ldots, R_n)$ and $M \subset \{0, 1, \ldots, n\}$ with $0 \in M$. Let $G = (X, E)$ be the graph whose edge set is $E = \cup_{i \notin M} R_i \neq \emptyset$. Then,*

$$\mathrm{Del}(G) = \min_{C} \frac{-\lambda_{\min}(C)}{\lambda_{\max}(C) - \lambda_{\min}(C)} |X|, \tag{8}$$

where C is an extended weighted adjacency matrix of G such that $e \in \mathrm{Ker}\{C - \lambda_{\max}(C)I\}$.

Proof. From Section 3, we know that, if $b_0, b_1, \ldots, b_n$ are optimal solutions of the minimization problem (4)

$$C = \sum_{i,j=0}^{n} \frac{b_j}{m_j} Q_{ij} D_i - \vartheta'(G)I,$$

is an extended weighted adjacency matrix of G. As $D_i = \sum_{k=0}^{n} P_{ki} E_k$ (recall the E.4 condition in Section 1), we have

$$C = \sum_{k=0}^{n} \left\{ \sum_{i=0}^{n} \left(\sum_{j=0}^{n} \frac{b_j}{m_j} Q_{ij} \right) P_{ki} \right\} E_k - \vartheta'(G)I. \tag{9}$$

Hence, taking into account (3), the eigenvalues of C are

$$\begin{aligned} \sum_{i=0}^{n} \left(\sum_{j=0}^{n} \frac{b_j}{m_j} Q_{ij} \right) P_{ki} - \vartheta'(G) &= \sum_{j=0}^{n} \frac{b_j}{m_j} \left(\sum_{i=0}^{n} Q_{ij} P_{ki} \right) - \vartheta'(G) \\ &= m \sum_{j=0}^{n} \frac{b_j}{m_j} \delta_{jk} - \vartheta'(G) \\ &= m \frac{b_k}{m_k} - \vartheta'(G) \end{aligned} \tag{10}$$

for $k = 0, 1, \ldots, n$. Since there exists $k \in \{1, \ldots, n\}$ such that $b_k = 0$, the smallest eigenvalue of C is $-\vartheta'(G)$. (In fact, if for all $k \in \{1, \ldots, n\}$, $b_k > 0$, the complementary conditions of linear programming would imply that $Q^{\mathrm{T}}a = (s, 0, \ldots, 0)^{\mathrm{T}}$ with $s \geqslant 0$. As $(Q^{\mathrm{T}})^{-1} = P^{\mathrm{T}}/m$ and $a_0 = 1$, we would have $s = m$, $a = (v_0, v_1, \ldots, v_n)^{\mathrm{T}}$ and $\vartheta'(G) = \sum_{i=0}^{n} a_i = m$. Since $\vartheta'(G) \leqslant \vartheta(G) \leqslant \bar{\chi}(G) \leqslant m$, $\bar{\chi}(G) = m$ and hence $E = \emptyset$, a contradiction.)

On the other hand, from Lemma 3.6 of Delsarte's work [8], it immediately follows that the optimal solutions $b_0, b_1, \ldots, b_n$ of problem (4) satisfy $b_k \leqslant m_k$, for all $k = 0, 1, \ldots, n$. Thus the greatest eigenvalue of C is less or equal than $m - \vartheta'(G)$. From (9) and (10) we can write

$$C = m \sum_{k=0}^{n} \frac{b_k}{m_k} E_k - \vartheta'(G) I \tag{11}$$

and, as $E_0 = \frac{1}{m} e e^{\mathrm{T}}$, $Ce = (m - \vartheta'(G))e$. Hence $\lambda_{\max}(C) = m - \vartheta'(G)$ and $e \in \mathrm{Ker}\{C - \lambda_{\max}(C) I\}$.

Therefore,

$$\vartheta'(G) = \frac{\vartheta'(G) \times m}{(m - \vartheta'(G)) + \vartheta'(G)} = \frac{-\lambda_{\min}(C)}{\lambda_{\max}(C) - \lambda_{\min}(C)} |X|,$$

and, since $\mathrm{Del}(G) = \vartheta'(G)$, the result follows from Corollary 2. $\square$

In many cases the minimum in (8) is attained when C is the adjacency matrix of the graph. For example, Delsarte [8] has proved that this true for the strongly regular graphs. Recently, De Klerk and Pasechnik [7] proved that the same is also valid for the orthogonality graph $\Omega(n)$, a graph whose vertices correspond to the vectors $\{0, 1\}^n$, two vertices being adjacent if and only if their Hamming distance is equal to $n/2$.

Finally, we observe an example in the opposite direction. Let G be the graph given in [18] whose vertices correspond to the vectors $\{0, 1\}^6$, two vertices being adjacent if and only if their Hamming distance is at most 3. This is the graph whose edge set is the union of classes R_1, R_2 and R_3 of the Hamming scheme $H(6, 2)$, for which $\vartheta'(G) = 4$, $\vartheta(G) = 16/3$ and

$$\upsilon(G) = \frac{-\lambda_{\min}(A_G)}{\lambda_{\max}(A_G) - \lambda_{\min}(A_G)} \times 2^6 \approx 13,54.$$

As $\mathrm{Del}(G) = \vartheta'(G) = 4$, the minimum in (8) is not attained for the matrix A_G. Using (11) a matrix C whose spectrum is $\{[-4]^{51}, [-3.2695]^6, [27.2695]^6, [60]^1\}$ can be obtained. Hence the minimum in (8) is attained for this matrix since

$$\frac{-\lambda_{\min}(C)}{\lambda_{\max}(C) - \lambda_{\min}(C)} \times 2^6 = 4.$$

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