Information Principles and the Quantum Set

Victor Bankston

April 18, 2019

Models for Distributed Computation

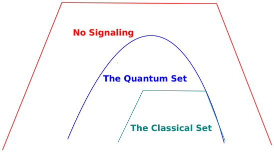
- Message passing:
  - Limited by the speed of light
  - No obvious way to coordinate processes otherwise
- Non-Locality
  - Features of the universe can depend on other distant "non-local" parts of the universe.
  - Counterintuitive in light of relativity
  - Can be achieved via entanglement
  - A potential means of coordination without communication

A metaphor for non-locality

- We have Boxes, which abstract physics
- Alice (Bob) chooses an input $\{0, 1\}$ and receives probabilistically an output $\{0, 1\}$.
  - A priori, Alice's output may depend on not only her but also Bob's input and output, as well as random variables.
  - This allows for correllations between their outputs.
- We will impose various constraints on these boxes.
  - Most generally, they cannot transfer information.

Box Distributions

- Boxes are characterized by their joint probability distribution: $Pr\left(a_1 a_2 a_3 \mid x_1 x_2 x_3\right)$.
- Which joint probability distributions can be achieved?
  - By choosing randomly between two distributions, we can always form convex combinations.



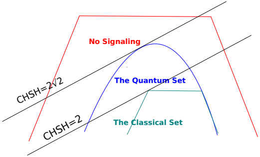
- We would like an information principle which characterizes the quantum set.
  - Helpful for identifying information applications.

Bell Inequalities

- A mathematical definition of correllation.
- Mathematical Motiff: Study a space by studying functions on that space.
  - These bodies may be studied by optimizing linear functions over them.
  - A convex set is defined by the set of hyperplanes tangent to it.
  - Each hyperplane represents an inequality which that set satisfies.

Bell Inequalities

- A mathematical definition of correllation.
- Mathematical Motiff: Study a space by studying functions on that space.
  - These bodies may be studied by optimizing linear functions over them.
  - A convex set is defined by the set of hyperplanes tangent to it.
  - Each hyperplane represents an inequality which that set satisfies.



- Example: For 2 parties, the CHSH inequality:
  $Pr(00 \mid 00) + Pr(11 \mid 00) + Pr(00 \mid 01) + Pr(11 \mid 01) + Pr(00 \mid 10) + Pr(11 \mid 10) - Pr(00 \mid 11) - Pr(11 \mid 11)$
  - Classically, maximized at 2.
  - Can acheive $2\sqrt{2}$ using the quantum set (Tsirelson's bound)
  - $\therefore$ the quantum set is strictly larger than the classical set.

Bell Inequalities

- A mathematical definition of correllation.
- Mathematical Motiff: Study a space by studying functions on that space.
  - These bodies may be studied by optimizing linear functions over them.
  - A convex set is defined by the set of hyperplanes tangent to it.
  - Each hyperplane represents an inequality which that set satisfies.
- Example: For 2 parties, the CHSH inequality:
  $Pr(00 \mid 00) + Pr(11 \mid 00) + Pr(00 \mid 01) + Pr(11 \mid 01) + Pr(00 \mid 10) + Pr(11 \mid 10) - Pr(00 \mid 11) - Pr(11 \mid 11)$
  - Classically, maximized at 2.
  - Can acheive $2\sqrt{2}$ using the quantum set (Tsirelson's bound)
  - $\therefore$ the quantum set is strictly larger than the classical set.
- Example: For 3 parties, Guess Your Neighbor's Input

  $$Pr(000 \mid 000) + Pr(110 \mid 011) + Pr(011 \mid 101) + Pr(101 \mid 110) \leq 1$$

  - Same bound for both quantum and classical correlations.

Some Information Principles

- No Signalling
  - The statistics of a person's outcome should only depend on his input.
  - (But the outcome itself may can depend on non-local events!)
  - $\forall \widetilde{x}_2 \in \{0,1\} Pr\left(a_1 \mid x_1\right) = \sum_{a_2 \in \{0,1\}} Pr\left(a_1 a_2 \mid x_1 \widetilde{x}_2\right)$
    - (i.e. both summations on the right have the same value, so $Pr\left(a_1 \mid x_1\right)$ is well-defined.)
  - Defines a larger set of correlations than then Quantum Set.

Some Information Principles

- No Signalling
  - The statistics of a person's outcome should only depend on his input.
  - (But the outcome itself may can depend on non-local events!)
  - $\forall \widetilde{x}_2 \in \{0,1\} Pr\left(a_1 \mid x_1\right) = \sum_{a_2 \in \{0,1\}} Pr\left(a_1 a_2 \mid x_1 \widetilde{x}_2\right)$
    - (i.e. both summations on the right have the same value, so $Pr\left(a_1 \mid x_1\right)$ is well-defined.)
  - Defines a larger set of correlations than then Quantum Set.
- Information Causality
  - If Alice sends Bob m bits, then Bob cannot learn more than m bits about Alice, even using his correlations.
  - When $m = 0$, we recover no-signalling.
  - Compatible with Quantum Mechanics.
  - Has been proposed as a candidate to characterize the quantum set.
  - Naturally stated as a bipartite principle.

Main Argument:

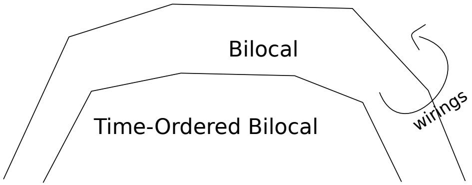
- To create multipartite principles from bipartite ones, apply the bipartite principle over every bipartition.
- However, there is a tripartite distribution which is classical over any bipartition, but is not in the Quantum Set!
- Since it is classical over any bipartition, it will satisfy any information principle based on bipartite concepts.
- To characterize the quantum set, we will need an inherently tripartite information principle.

Classical Probabilities

- Classically, we assume the collection of Boxes is in some state, λ, which determines the probability distributions of all the boxes for each input. (There may be additional local sources of randomness)
  - The classical assumption is that for any state, the Boxes behave like pairwise independent random variables.
  - $Pr(a_1 a_2 a_3 \mid x_1 x_2 x_3 \lambda) = \sum_\lambda p(\lambda) Pr(a_1 \mid x_1 \lambda) Pr(a_2 \mid x_2 \lambda) Pr(a_3 \mid x_3 \lambda)$
- We would like to define what it means to be classical over some bipartition of the parties, such as $1 - 23$.

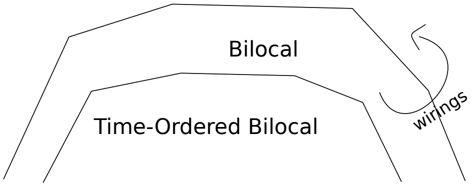
“Classical over a bipartition”

- Naive Definition (Bilocal distributions):
  - $P(a_1 a_2 a_3 \mid x_1 x_2 x_3) = \sum_\lambda p(\lambda) P_1(a_1 \mid x_1 \lambda) P_{23}(a_2 a_3 \mid x_2 x_3 \lambda)$
  - Allows arbitrary correlations between 2 and 3
    - Even those where a_2 depends on x_3 and a_3 depends on x_2.
    - In reality, one of 2 and 3 must measure first.
  - If 2 feeds his output into 3's input, the resulting probability distribution does not factor.
  - We can create non-locality with 1 through local (between 2 and 3) means!
  - “Classical over a bipartition” should be preserved under any protocol 2 and 3 implement.



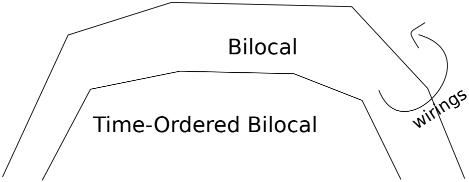
"Classical over a bipartition"

- Naive Definition (Bilocal distributions):
  - $P(a_1 a_2 a_3 \mid x_1 x_2 x_3) = \sum_\lambda p(\lambda) P_1(a_1 \mid x_1 \lambda) P_{23}(a_2 a_3 \mid x_2 x_3 \lambda)$
  - Allows arbitrary correlations between 2 and 3
    - Even those where a_2 depends on x_3 and a_3 depends on x_2.
    - In reality, one of 2 and 3 must measure first.
  - If 2 feeds his output into 3's input, the resulting probability distribution does not factor.
  - We can create non-locality with 1 through local (between 2 and 3) means!
  - "Classical over a bipartition" should be preserved under any protocol 2 and 3 implement.



Classical over a bipartition

- Updated Definition (Time Ordered Bilocal Distributions):
- $P(a_1 a_2 a_3 \mid x_1 x_2 x_3) =$
  $\sum_\lambda p(\lambda) P(a_1 \mid x_1) P_{2\to 3}(a_2 a_3 \mid x_2 x_3, \lambda) =$
  $\sum_\lambda p(\lambda) P(a_1 \mid x_1) P_{2\leftarrow 3}(a_2 a_3 \mid x_2 x_3, \lambda)$
  - where $P_{2\to 3}$ satisfies
    $P_{2\to 3}(a_2 \mid x_2, \lambda) = \sum_{a_3} P_{2\to 3}(a_2 a_3 \mid x_2 x_3, \lambda)$ and
  - $P_{2\leftarrow 3}$ satisfies $P_{2\leftarrow 3}(a_2 \mid x_2, \lambda) = \sum_{a_3} P_{2\leftarrow 3}(a_2 a_3 \mid x_2 x_3, \lambda)$
  - 2 or 3 must be able to measure first.
- Being able to think of 2 or 3 as measuring first gives a local model for any protocol 2 and 3 enact.
  - It is not claimed that TOBL is closed under wirings.



The Counterexample:

	000	001	101	011	100	101	110	111
000	$\frac{2}{3}$	0	0	0	0	0	0	$\frac{1}{3}$
001	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$
101	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	$\frac{1}{6}$	0	$\frac{1}{6}$
011	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{6}$	0
100	$\frac{1}{3}$	0	0	$\frac{1}{6}$	$\frac{1}{3}$	0	0	$\frac{1}{6}$
101	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0
110	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0
111	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0
- Invariant with respect to renaming parties
- Guess Your Neighbor's Input is $\frac{7}{6} > 1$: It's not in the quantum set!
- Found via linear program maximizing GYNI over TOBL

Decompositions

- Decomposition: $\sum_\lambda p(\lambda)\, P(a \mid x_1)\, P_{2\to 3}(bc \mid x_2 x_3, \lambda)$
  - So c can depend on x_2.

λ	p_λ	a_0	a_1	b_0	b_1	c_{00}	c_{01}	c_{10}	c_{11}
1	$\frac{1}{12}$	0	0	0	1	0	1	0	1
2	$\frac{1}{12}$	0	0	0	0	0	1	0	1
3	$\frac{1}{12}$	0	0	0	0	0	0	0	1
4	$\frac{1}{12}$	0	0	0	1	0	0	0	1
5	$\frac{1}{12}$	0	1	0	1	0	0	0	0
6	$\frac{1}{12}$	0	1	0	0	0	1	0	0
7	$\frac{1}{12}$	0	1	0	0	0	0	0	0
8	$\frac{1}{12}$	0	1	0	1	0	1	0	0
9	$\frac{1}{6}$	1	0	1	1	1	1	1	0
10	$\frac{1}{6}$	1	1	1	0	1	0	1	1

Decompositions

- Decomposition
  $p\left(abc \mid x_1 x_2 x_3\right) = \sum_\lambda p\left(\lambda\right) P\left(a \mid x_1\right) P_{2\leftarrow 3}\left(bc \mid x_2 x_3, \lambda\right)$

λ	p_λ	a_0	a_1	b_{00}	b_{01}	b_{10}	b_{11}	c_0	c_1
1	$\frac{1}{12}$	0	0	0	0	0	1	0	0
2	$\frac{1}{12}$	0	0	0	0	0	1	0	1
3	$\frac{1}{12}$	0	0	0	0	1	1	0	0
4	$\frac{1}{12}$	0	0	0	0	1	1	0	1
5	$\frac{1}{12}$	0	1	0	0	0	0	0	0
6	$\frac{1}{12}$	0	1	0	0	0	0	0	1
7	$\frac{1}{12}$	0	1	0	0	1	0	0	0
8	$\frac{1}{12}$	0	1	0	0	1	0	0	1
9	$\frac{1}{6}$	1	0	1	1	1	0	1	1
10	$\frac{1}{6}$	1	1	1	1	0	1	1	0

Conclusion:

- Information Causality cannot characterize the quantum set.
- Information Causality for multiple parties is defined based on a bipartite principle
- The counter-example distribution will obey information causality,
  - The distribution behaves classically over any bipartite set.
- The distribution is not in the quantum set.
  - Its value for GYNI is $\frac{7}{6} > 1$
- We will need a multipartite principle to characterize the quantum set.
  - Or we need to reconsider how to extend a bipartite principle to a multipartite one.