NOTES ON LOVASZ THETA

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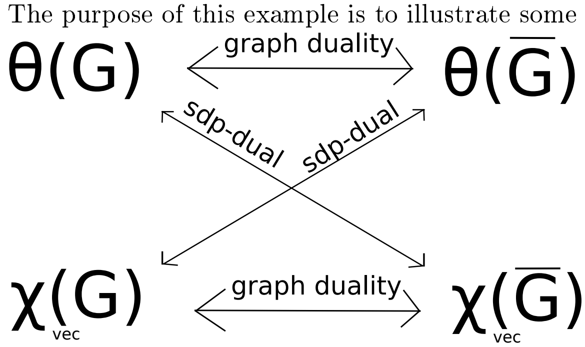
The purpose of this example is to illustrate some of the structure to $\vartheta(G)$.



Though the quantities above are scalars, they arise from arrangements of vectors. I will write down the vectors associated with each quantity above when G is the Petersen Graph.

Defining ϑ

First, we need a series of definitions to define ϑ:

Definition 1. Given a graph G, an orthonormal representation of G is a mapping $r : V(G) \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ (for some $n \in \mathbb{N}$) such that if $i \neq j \in V(G)$, with $i \not\sim j$, then $r(i) \perp r(j)$. (Be careful: a vertex is not adjacent to itself).

Note that each graph has at least one representation, where v maps the verticies each to its own orthonormal vector.

Definition 2. A valuation of an orthonormal representation $val(r)$ is

$$\min_{\psi} \max_{v \in V(G)} \frac{1}{\left(\psi^T r(v)\right)^2}$$

where ψ ranges over all unit vectors (of the target space of r).

Given an orthonormal representation, its valuation is how tightly it can be embedded into a cone around some vector (ψ).

Definition 3. Define $\vartheta(G)$ to be the minimum valuation over all orthonormal representations of G.

We can show that this minimum is actually attained. We will use Bolzano-Weirstrass. To see this, fix n, and consider the orthonormal representations of the form: $v : V(G) \to \mathbb{S}^{n-1} \subset \mathbb{R}^n$. Observe that $\vartheta(G)$ remains unchanged if we require that $\psi = (1, 0, 0, 0 \dots)$: These valuations are defined by an inner product, which will not change if we apply a fixed unitary U to every vector. Choose U to send $\psi \mapsto (1, 0, 0, 0 \dots)$.

Fixing ψ, take a sequence of orthonormal representations whose values converge to $\vartheta(G)$. Observe that these orthonormal representations themselves can be considered as bounded vectors of dimension $n \cdot V(G)$, by concatenating all $V(G)$ vectors of dimension n. By the Bolzano-Weirstrass theorem, these have a convergent subsequence, so there is an accumulation point, r_∞, which we must show is an orthonormal representation.

Our convergent subsequence of orthonormal representations gives rise to $V(G)$ convergent sequences of vectors. We must show that each sequence of vectors goes to a unit vector, and that when $i \not\sim j$, with $i \neq j$, we have $r_\infty(i)^T r_\infty(j) = 0$. Both of these are consequences of the fact that dot products are continuous: $0 = \lim_{n\to\infty} r_n(i)^T r_n(j) = (\lim_{n\to\infty} r_n(i))^T (\lim_{n\to\infty} r_n(j))$.

There is no claim that such optimal representations are unique.

Graphs

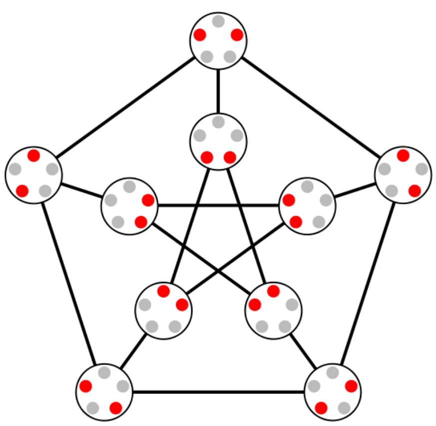
Definition 4. Define the Kneser Graph $k(n,r)$ to have $\binom{n}{r}$ verticies labeled by r-element subsets from a universe of size n. Two verticies are adjacent if their corresponding sets are disjoint. We assume that $n \geq 2r$

Kneser graphs are vertex and edge transitive. Given any two verticies (edges), there is an automorphism which sends one to the other.

Theorem 5. *If G is vertex and edge transitive, then $\vartheta(G)\,\vartheta(\overline{G}) = n$, and $\vartheta(G) = \frac{-n\lambda_n}{\lambda_1 - \lambda_n}$*

This powerful theorem was originally used to find $\vartheta(k(n,r))$. The proof of the theorem builds on the relations in the diagram.

Definition 6. The Petersen Graph, P, is the Kneser graph, $k(5,2)$.



We start with some graph properties.

Claim 7. The clique number of the kneser graph $\omega(k(n,r)) = \lfloor \frac{n}{r} \rfloor$, so $\omega(P) = 2$

A clique corresponds to a collection of disjoint sets.

Claim 8. The coloring number $\chi(k(n,r)) = n - 2r + 2$, so $\chi(P) = 3$

This was a big open problem for many years. The optimal coloring is the following: Order the elements of the universe $u_1, \ldots, u_n$, and divide them into 3 pieces with sizes $n-2r$, r and r. Let x be an r-set. If it intersects the first piece, color x with the color i, where $i=\min\{i \mid u_i \in x\}$. Otherwise, x is contained entirely in the last two pieces. These remaining verticies form a subgraph, where each vertex has a unique neighbor, and these can be colored with two colors.

Claim 9. The independence number of the kneser graph is $\alpha(k(n,r)) = \binom{n-1}{r-1}$, so $\alpha(P)=4$

The collection of r-subsets which each contain u_1 is a set of this size. It isn't hard to show this is optimal.

Claim 10. The clique covering number is $q(k(n,r)) = \left\lceil \frac{\binom{n}{k}}{\left\lfloor \frac{n}{k} \right\rfloor} \right\rceil$, $q(P)=5$

(!) (Claim found on Wolfram Mathworld.)

Claim 11. $\vartheta(k(n,r)) = \binom{n-1}{r-1}$, and $\vartheta(k(n,r)) = \frac{n}{r}$, so $\vartheta(P)=4, \vartheta\left(\overline{P}\right) = \frac{5}{2}$

This is proven by Theorem 5 and some tricky algebra, but this avoids (or at least obscures) creating explicit orthonormal representations, which is the point of this example.

Relations between Graph Constants

Theorem 12. $\alpha(G)\chi(G) \geq \mid V(G) \mid$

Each color is an independent set, and a proper coloring colors every vertex.

Theorem 13. *For any graph G, $\alpha(G) \leq \vartheta(G) \leq \chi\left(\overline{G}\right) = q(G)$*

In an orthonormal representation, an independent set, α, of G must be sent to a collection of pairwise independent vectors. For such vectors, it is easy to see that $\max_{v_i \in \alpha} \frac{1}{(\psi^T r(v_i))^2}$ is minimized when $\psi = \frac{\sum_{v_i \in \alpha} r(v_i)}{\sqrt{|V(G)|}}$ (when ψ is between all the vectors.) In this case, $\frac{1}{(\psi^T r(v_i))^2} = \mid \alpha \mid$, and this lower bound holds for all orthonormal representations. This shows $\alpha(G) \leq \vartheta(G)$.

Suppose we have clique cover of size $q(G)$. Define an orthonormal representation by choosing $q(G)$ pairwise orthonormal vectors. Send each clique to one of these vectors. This provides an explicit orthonormal representation with valuation $q(G)$. The minimum over all orthonormal representations may be less.

Orthonormal Representations

We start with the graph $\overline{k(n,r)}$ and construct an optimal orthonormal representation in dimension n, with orthonormal basis $e_1, \ldots, e_n$ The choice is obvious: disjoint sets need to go to orthonormal vectors. Set $e_i^T r(v_j) = \frac{1}{\sqrt{r}}$ if $u_i \in v_j$, and 0 otherwise. Set $\psi = \frac{1}{\sqrt{n}}(1,1,\ldots,1)$. It is immediate that this is an orthonormal representation with valuation $\frac{1}{(\psi^T r(v_i))^2} = \frac{r \cdot n}{r^2} = \frac{n}{r}$.

Definition 14. Given an orthonormal representation, we can define the cost of a vertex to be $c(v) = \left(\psi_1^T (r_1(v_i))\right)^2$. This corresponds to the quantum-mechanical probability of measuring $r_1(v_1)$ when measuring from state ψ.

Theorem 15. *(Certification of Orthonormal Representations): if we have two orthonormal representations r_1, r_2 of G and $\overline{G}$ and for all $i \in V(G)$ we have $c_1(v_i) = \frac{1}{\vartheta}$, and we also have $\sum_i c_1(v_i) c_2(v_i) = 1$, then $\vartheta = \vartheta(G)$*

Proof. We have the explicit orthonormal representation r_1, so $\vartheta(G) \le \vartheta$. For the other direction, we use an alternate definition ϑ, $\vartheta(G) = \max_{Rep(\overline{G})} \sum_i c(v_i)$.

$$\vartheta = \sum_i \vartheta c_1(v_i) c_2(v_i) = \sum_i c_2 v_i \le \max_{r \in Rep(\overline{G})} \sum_i c(v_i) = \vartheta(G)$$

□

The argument above also shows that certificates always exist.

Definition 16. Given a non-empty closed convex set $P \subset \mathbb{R}_P^n$ with the property that $x \in P$ and $0 \le x' \le x$ then $x' \in P$, the antiblocker of P is

$$AB(P) = \left\{x \in \mathbb{R}_+^n : y^T x \le 1 \text{ for all } y \in P\right\}$$

Definition 17. $TH(G) = \left\{(c(v_i), v_i \in V(G)) \in \mathbb{R}_+^{V(G)}\right\}$. These are assignable probabilities.

Theorem 18. $AB(TH(G)) = TH(\overline{G})$

(!) These concepts provide a geometric description of the ϑ. ϑ is the maximal of a linear functional over $TH(G)$. This linear functional has hyperplanes as its level sets, and the optimal value corresponds to a level set which lies tangent to $TH(G)$. Thus, $\vartheta(G)$ is the smallest simplex S such that $TH(G) \subset S$. If we take the antiblocker of this picture, we seek the largest cube C such that $C \subset AB(TH(G)) = TH(\overline{G})$. This explains the formula $\vartheta(G) = \min_{O.R.} \max_i \frac{1}{c(v_i)} = \max_{O.R.} \min_i c(v_i)$.

Next, we give an orthonormal representation of $\vartheta(P)$, which will certify the optimality of the orthonormal representation given at the begining of this section.

Assume a basis of size 10, $\{e_{s_1}, e_{s_2}, \ldots, e_{s_{10}}\}$ labeled by the $\binom{5}{2}$ subsets of the graph. Let $\psi = \frac{1}{\sqrt{10}}(1, 1, \ldots, 1)$. Finally, assume that we will have $e_{s_i}^T r(v_j) = x_{|s_i \cap v_j|}$. This is a plausible assumption, because it will result in vectors whose orthongonality relations are invariant with respect to the automorphism group of P.

The fact that intersection sets must be sent to orthonormal vectors translates into the contstraint

$$x_0^1 + 3x_1^2 + 4x_0x_1 + 2x_1x_2 = 0$$

At the same time, we would like to minimize $\frac{10 \cdot \left(x_2^2 + 6x_1^2 + 3x_0^2\right)}{(x_2 + 6x_1 + 3x_0)^2}$. According to Wolfram Alpha the minimum is 4, when $(x_0, x_1, x_2) = \left(1, -4 - \sqrt{15}, 6 + \sqrt{15}\right)$, or when $(x_0, x_1, x_2) = \left(1, \sqrt{15} - 4, 6 - \sqrt{15}\right)$.

Vector Colorings of Graphs

Definition 19. Given a graph G, we assign a unit vector to each vertex. This time, we would like adjacent verticies to be sent to vectors whose dot product is as negative as possible. If $\chi(G) = k$, then we can associate each color with a vector in the regular k-simplex in $\mathbb{R}^{k+1}$. Such vectors have inner product $\frac{-1}{k-1}$. In light of this, we define $\chi_{vec}(G) = \min\left\{k \mid v_i^T v_j \le \frac{-1}{k-1}\right\}$

Theorem 20. $\chi_{vec}(G) = \vartheta(\overline{G})$

This is proven by the fact that the two problems can be expressed as semidefinite-programming duals of one another. Alternatively, there is a concrete way to move between optimal representations of the coloring problem and optimal representations for ϑ. Namely, if x_i is an optimal vector-coloring for $\vartheta\left(\overline{G}\right)$ and ψ is some unit vector orthogonal to each x_i, then $u_i = \frac{1}{\sqrt{\vartheta}}\left(\psi + \sqrt{\vartheta - 1}x_i\right)$ (!)

Next, we will provide an optimal vector coloring of P. Assume a basis of size 5, and that we will map $\star\star\circ\circ\circ \mapsto (a, a, b, b, b)$, and extend this map by permutations of S_5. If x, y are two vector representations of intersecting sets, we would like to minimize

$$min_{a,b}\frac{x^T y}{\|x\|\,\|y\|} = \frac{4ab + b^2}{2a^2 + 3b^2}$$

The minumum occurs at $a = -3, b = 2$, and gives

$$\frac{x^T y}{\|x\|\,\|y\|} = \frac{-24 + 4}{18 + 12} = -\frac{2}{3} = \frac{-1}{\frac{5}{2} - 1}$$

Finally, an optimal vector coloring of $\overline{P}$ can be found by assuming a basis of size 10 (the same basis we used for $\vartheta\left(P\right)$) and three variables, x_0, x_1, x_2. This gives us the optimization problem:

$$min_{x_0,x_1,x_2}\frac{x^T y}{\|x\|\,\|y\|} = \frac{x_0^2 + 3x_1^2 + 4x_0x_1 + 2x_1x_2}{3x_0^2 + 6x_1^2 + x_2^2}$$

The minimum (according to Wolfram) is found at $\left(-\frac{1}{\sqrt{18}}, \frac{1}{\sqrt{18}}, -\frac{1}{\sqrt{2}}\right)$ and gives

$$\frac{x^T y}{\|x\|\,\|y\|} = \frac{\frac{1}{18} + \frac{3}{18} - 4\frac{1}{18} - 2\frac{1}{6}}{3\frac{1}{18} + 6\frac{1}{18} + \frac{1}{2}} = \frac{-6}{18} = \frac{-1}{4 - 1}$$