Phase Points as Ontic States in Finite-Dimensional Systems

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1 Introduction

Phase space is an important tool in classical mechanics. In classical mechanics, states are determined by the position and momentum of every particle. In quantum mechanics, the uncertainty principle asserts that position and momentum of particles cannot be defined simultaneously.

[1] describes the quantum phase-space construction for odd degrees of freedom. In [2], it is shown that negative probabilities in this construction can be associated with Bell Inequalities.

The arguments in [2] can be interpreted as showing that phase points are the classical states which underlie compatible outcomes to all stabilizer measurements with an ancilla.

We will argue that phase points are the classical states in a stronger sense. Namely, any set of compatible outcomes for any subset of separable measurements and all entangled measurements will have separable outcomes which are supported on at a common point in phase space. We can then regard this common point as the underlying 'classical state.'

2 Background

2.1 Phase Space Geometry

Let $d = p^n$ be an odd power of a prime. We start by defining the (single particle) finite phase space over powers of odd primes. This is made easier by the fact that finite fields of order d, $\mathbb{F}_d$, exist.

Definition 1. *The affine space* $Aff(\mathbb{F}_d)$ *is a triple,* (P, L, I)*, where* $P = \mathbb{F}_d \times \mathbb{F}_d$ *is the set of points,* $L = \bigcup L_{a,b,c}$ *with* $L_{a,b,c} = \{(x, y) \in P \mid (ax + by = c)\}$ *is the set of lines, and* $I \subset P \times L$ *is the incidence relation, defined by* $(P, L) \in I$ *if* $P \in L$*.*

We will use subscripts to denote components of the triple. For example, $Aff(\mathbb{F}_d)_p$ is the set of points of $Aff(\mathbb{F}_d)$.

Proposition 1. *The lines of $Aff(\mathbb{F}_d)$ form $d+1$ parallel classes, each of size d. These parallel classes are called directions.*

Definition 2. *The canonical coefficients of a line L, are (a,b,c) where $L = L_{1,b,c}$ or $L = L_{0,1,c}$.*

Definition 3. *In an Affine space, a triangle is a set of 3 lines which pairwise intersect but which do not all 3 intersect.*

Definition 4. *In an Affine space, a star at a point p is the set of lines which pass through p.*

2.2 Wigner Function

Let $\mathcal{D}$ be the set of $d \times d$ density matrices. The Wigner Function defined in [1] is a function $W : \mathcal{D} \times Aff(\mathbb{F}_d) \to \mathbb{R}$. Its main property is that it marginalizes over sets of parallel lines:

Definition 5. *Given a density matrix ρ and a direction r, we define the marginal distribution of ρ over r, $Mar(\rho, r) : r \to \mathbb{R}$, defined by*

$$Mar(\rho, l) = \sum_{p \in L} W_\rho(L)$$

In [1], it is shown that the marginals are indeed probability distributions.

The secondary property of the Wigner Function is its covariance relation.

Proposition 2. *Let $\rho \in \mathcal{D}$, and let $S \in SL_2(\mathbb{F}_d)$, and $a \in Aff(\mathbb{F}_d)_p$. Let $\rho' := U\rho U^\dagger$. Then*

$$W_\rho(p) = W'_\rho(Sp + a) \tag{1}$$

That is, affine transformations of phase space can be realized by Clifford Unitaries, and every Clifford Unitary is of this form.

See [1] Theorem 7 and Theorem 3.

2.3 Stabilizer States

todo:0 Geometry 1 Wigner Function 2 lines/stabilizer states 3 outcomes, vectors, 3.5 Graphs, orthogonalities 4 No Detection Events 5 Review [2]'s constructions Let $\mathcal{M} = \mathcal{M}_s \cup \mathcal{M}_e$ be the measurements in the construction.

Definition 6. *A set of vertices $I \subset G(\mathcal{M})$ is completely classical if it intersects the clique of each measurement non-trivially.*

Definition 7. *A collection of vertices $V \subset G(\mathcal{M})$ is compatibly extendable if V is independent, and is contained in a completely classical set of vertices*

Definition 8. *The main-qudit lines of a set of vertices $V \subset G(\mathcal{M})$ are the supports of the main-qudit part of the separable outcomes in phase space.*

By abuse of notation, we can also describe vectors in this way.

3 Main Arguments

Let's see how [2] can be interpreted as asserting that the classical states in phase space are phase points.

Theorem 1. *The main qudit-lines of a completely classical set of vertices are stabilizer states whose Wigner-Function supports are a star in phase space.*

Conversely, the $d + 1$ lines a star correspond to a compatibly extendable collection of vertices.

Proof. Theorem 6 shows that a completely classical set of vertices will have main qudit lines which cover every phase-space point.

If three lines form a triangle, the $d + 1$ lines will only cover at most $p^2 - 1$ points. Since the main qudit lines are non-parallel and have no triangles, they must meet at a common point and form a star.

Conversely, the common point of the main qudit lines, combined with the common outcome for the ancilla provide points in the 2-particle phase space where all separable outcomes are supported. We can choose the entangled outcomes so they also contain this point in their support. □

This means that forcing the entangled measurements to provide complete, non-contradictory outcomes ensures that phase points can be considered the deterministic states of phase space in the Abramsky/Brandenburger model where each measurement should have exactly one outcome.

We would also like phase-space points to be the classical states in the no-detection-events setting. This means that we allow partial assignments for the separable measurements, so some directions may be left undetermined. The classical sets are (not necessarily maximal) independent sets, and our claim is that the main lines of any independent set meet at a common point.

Theorem 2. *Let l_1, l_2, l_3 form a triangle in p^n-dimensional phase space. The set of outcomes $\{O(l_1), O(l_2), O(l_3)\}$ is incompatible with any set of outcomes for the entangled measurements.*

Proof. Suppose these outcomes can be extended compatibly. Then, we can choose new coordinates for the separable spaces. This change of coordinates will be Clifford, and act separably between the main qudit and ancilla. We will be able to consider them as changes in coordinates of the measurement outcomes. Note that these transformations will map stabilizer states to stabilizer states and the entangled outcomes to other entangled outcomes.

In the new coordinates, we will show that the main qudit lines must pass through a single point, so they must still do so in the original coordinates.

Suppose our 3 lines form a triangle $\{a, b, c\}$. We can find a translation which transfers a to the origin. Then, we can find an element of $SL(2, d)$ which moves b to the point $(0, 1)$. Then, the set of affine maps which fix a and b but can affect c are of the form $\begin{bmatrix} 1, 0 \\ r, 1 \end{bmatrix}$

If c has coordinates (x_c, y_c), then choose r such that $rx_c + y_c = -1$. This choice can be made because $x_c \neq 0$, since $\{a, b, c\}$ form a triangle (so their images under affine transformations do as well.)

To be more specific about the change in coordinates, we are mapping by the affine transformation $p' = S(p - a) = Sp - Sa$, where

$$S = \begin{bmatrix} 1 & 0 \\ \frac{-1 - \frac{y_c - y_a}{y_b - y_a}}{(y_b - y_a)(x_c - x_a) - (x_b - x_a)(y_c - y_a)} & 1 \end{bmatrix} \begin{bmatrix} y_b - y_a & -(x_b - x_a) \\ 0 & (y_b - y_a)^{-1} \end{bmatrix} \tag{2}$$

as long as $y_b - y_a \neq 0$. Let l_1 be the line from a to b, l_2 be the line from b to c, and l_3 be the line from c to a. (We unbind the names a, b and c at this point in the proof.)

Let $l_i = \{(k_{b_i} + b_i y, y) \mid y \in \mathbf{F}_d\}$. Observe that this notation agrees with the proof of Theorem 6 in [2], since

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k \\ y \end{bmatrix} = \begin{bmatrix} k + by \\ y \end{bmatrix} \tag{3}$$

Moreover, our transformations chosen so that $b_3 - b_2 = b_2 - b_1$.

We adopt notation from [2], and choose coordinates on the ancilla so that $l = 0$.

Observe that equations (39) continues to hold for all b and (42) continue to hold, as long as $b \in \{b_1, b_2, b_3\}$.

Therefore, equation (44) holds for $b = b_i$ and $c = b_{i+1} - b_i$.

$$k_{b_{i+1}} - k_{b_i} = (b_{i+1} - b_i)(2z_{2,\gamma'} - z_{1,\gamma}) \tag{4}$$

Let $Y_{i,\gamma} = (2z_{2,\gamma'} - z_{1,\gamma})$ and $\tilde{l}_i \coloneqq l_{i+1} - l_i$. Then $Y_{1,\gamma} = Y_{2,\gamma}$

$$\tilde{l}_1(Y_{1,\gamma}) = 0 \tag{5}$$

$$\tilde{l}_2(Y_{1,\gamma}) = 0 \tag{6}$$

$$\tilde{l}_3(Y_{3,\gamma}) = 0 \tag{7}$$

Adding (3) and (4) gives

$$-\frac{l_1 - l_3}{2}(Y_{i,\gamma}) = 0 \implies \tilde{l}_3(Y_1, \gamma) = 0 \tag{8}$$

Since $\tilde{l}_3$ is the difference of two lines with different slopes, it cannot be horizontal, so $Y \coloneqq Y_{3,\gamma} = Y_{1,\gamma}$, and we can conclude that the pairwise differences of our lines meet at a point, $(0, Y)$.

But this means that all 3 lines meet at Y.

$\square$

The above theorem lends some support to the assertion that we can treat phase points as classical states.

It would be nice if this theorem could be generalized to multiple particles.

Definition 9. *When ϕ ranges over Clifford unitaries on n qudits, call the outcomes $\sum_j |\phi(k), k\rangle$ the fully entangled outcomes. Parameterizing $\phi = w\mu(s)$ where w is Weyl and s is symplectic then partitions the fully entangled outcomes into measurements, indexed by s.*

Conjecture 1. *We can replace the set of fully entangled measurements with an exponentially smaller set and retain the result that any measurement outcomes will be incompatible with a triangle on the main qudit.*

We have a possible strategy in mind for the exponentially smaller set of measurements which ban triangles.

Conjecture 2. *Suppose we have n d-dimensional particles and assume we have outcomes for all entangled measurements on each qudit. Then these outcomes are incompatible with a triangle on the main qudit.*

If true, then our entangled measurements on the qud^nit can be replaced with entangled measurements on all qu$d$its at once. That is, we choose to perform the same entangled measurement on each qudit. The number of outcomes of these measurements will still scale exponentially in n.

Conjecture 3. *Suppose a d^n-dimensional particle (simulated by n qudits) has 3 separable outcomes whose lines form a triangle. The lines of this triangle are d^n dimensional isotropic spaces which restrict to 3 dimensional isotropic spaces over any 3 qubits. For some triple of qudits, the 3 dimensional isotropic spaces will be incompatible with outcomes to some stabilizer measurement.*

The 3 in the last conjecture is chosen because 3 points of a triangle must have distinct coordinates.

If these conjectures hold, we will be able to enforce that the d^n dimensional particle's classical states are phase-space points, and we will use a number of entangled measurements which scales only polynomially in n.

4 Bell Inequalities and the Negativity of the Wigner Function

Numerical evidence suggests(!) the following: [TODO: define labyrinth states as fixed points of twirls of operators of SL2]

Conjecture 4. *The labyrinth states on a particle of dimension $d = p^n$ have a total negativity which grows as $O(\sqrt{d})$. The number of negative points is at least some fraction f of the total number of points, and the normalized negativity at the negative points ($dW_\rho(a)$) is bounded below by a constant.*

If true, we can use the negativity of the Wigner Function to construct Bell Inequalities.

Theorem 3. *Suppose that ρ is a state whose Wigner Function is negative at points in $N \subset Aff(\mathbb{F}_d)$ with sum-negativity $\mathcal{SN} < 0$. Further assume that the ontic states are phase point operators. Consider the stabilizer outcomes which pass through negative points, and are weighted by the number of negative points which they cover. These outcomes and their weightings define a Bell Inequality whose classical maximum (assuming phase points as classical states) is $|N|$, and whose quantum value is $|N| + d\mathcal{SN} < |N|$.*

Proof. The quantum value is obtained from the sum negativity. $\sum w(l)v(l) = \sum_{p\in N}\sum_{l\ni p} v(l) = \sum_{p\in N} 1 + dv(p) = |N| + d\mathcal{SN}$

The classical bound arises from the fact that classically, the sum-negativity is zero. □

In order to study this Bell Inequality, we should study $\mathcal{SN}$.

Proposition 3. *For a qudit system, $\mathcal{SN} \in O(\sqrt{d})$.*

Proof. For a state ρ, $\frac{1}{d^2} \leq \sum_{a\in Aff(\mathbb{F}_d)} (W_\rho(a))^2 \leq \frac{1}{d}$ due to the uncertainty principle in [4]. We may as well assume ρ is pure, so we obtain the upper bound. Thus, the L^2 norm of the Wigner Function is bounded. Observe that the L^1 norm is $1 + 2|\mathcal{SN}|$. Therefore, $\mathcal{SN}$ is maximized when the L^1 norm is maximized.

If we drop the restriction that ρ be positive and only apply the restriction on its L^2 norm, we can use the fact that the maximum L^1 norm for a fixed L^2 norm will occur when all values are equal in absolute value, so suppose this is the case and let $x = |W_\rho(a)|$ be this absolute value.

Then $d^2x^2 = \frac{1}{d} \implies x = \frac{1}{\sqrt{d^3}}$.

Since no more than half of the points may be negative, this provides an upper bound of $O(\sqrt{(d)})$. □

Proposition 4. *We can calculate $\mathcal{SN}$ for a random pure state, as defined in [5].*

Let ρ be a random pure state as defined in [5]. Then $\mathbb{E}(\mathcal{SN}_\rho) = \sum_a \mathbb{E}(S_\rho(a)))$, where $S_\rho(a) = \begin{cases} 0 & \text{if } W_\rho(a) < 0 \\ W_\rho(a) & \text{otherwise} \end{cases}$

Since the random distribution on pure state is defined so that it is invariant under application of unitaries and the phase point operators of the Wigner Function are related by unitary transformations, it follows that $\mathbb{E}(S_\rho(a)) = \mathbb{E}(S_\rho((0,0)))$ and it suffices to just find the expected value at the origin.

Recall that $W_\rho((0,0)) = Tr(A(0)\rho$, where $A(0)$ is the parity operator which has $\frac{(d+1)}{2}$ eigenvalues of $\frac{1}{d}$ and $\frac{d-1}{2}$ eigenvalues $-\frac{1}{d}$.

Thus, we must integrate

$$\int_{\sum_{i=0}^{d-1} x_i = 1} f(x)dx_i \tag{9}$$

Where $f(x) = \min\left(0, \frac{1}{d}\sum_{i=0}^{\frac{d-1}{2}} x_i - \frac{1}{d}\sum_{i=\frac{d+1}{2}}^{d-1} x_i\right)$

Changing variables and using x_0 as slack gives [3]

$$\sqrt{d+1}\int_{\sum_{i=1}^{d-1} x_i \leq 1} \tilde{f}(x)dx \tag{10}$$

where $\tilde{f}(x) = f(1-x, x)$.

Next, let $y_i = x_{i+1}$ for $i \in [1, \frac{d-1}{2} - 1]$ and $z_i = x_{i+\frac{d-1}{2}}$ for $i \in [1, \frac{d-1}{2}]$, and let $t = \sum_{i=1}^{d-1} x_i$. Summarizing $\sum_i y_i$ as y and $\sum_i z_i$ as z, our integral can be written

$$\int_{t=0}^{1}\int_{y\leq t}\int_{z\leq 1-t} \tilde{f}(t,y,z)dzdydt \tag{11}$$

Observe that we can write

$$\tilde{f}(t,y,z) = \min(0, \frac{1}{d}((1-t)+(t-y-z)+y-z)) = \min(0, \frac{1}{d}(1-2z)) \tag{12}$$

We limit the bounds of the integral to the support of $\tilde{f}$

$$\frac{1}{d}\int_{t=0}^{\frac{1}{2}}\int_{y\leq t}\int_{\frac{1}{2}\leq z\leq 1-t}(1-2z)dzdydt \tag{13}$$

Let $a = \frac{d-1}{2} - 1$ and $b = \frac{d-1}{2}$ be the number of y and z variables respectively.

The integral can be divided into two parts. The constant term can be handled as

$$\frac{1}{d}\int_{t=0}^{\frac{1}{2}}\int_{y\leq t}\int_{\frac{1}{2}\leq z\leq 1-t} 1dzdydt \tag{14}$$

$$= \frac{1}{db!}\int_{t=0}^{\frac{1}{2}}\int_{y\leq t}((1-t)^b - \frac{1}{2^b})dydt \tag{15}$$

$$= \frac{1}{da!b!}\int_{t=0}^{\frac{1}{2}}((1-t)^b - \frac{1}{2^b})t^a dt \tag{16}$$

$$= \frac{1}{da!b!}\int_{t=0}^{\frac{1}{2}}(1-t)^b t^a - \frac{t^a}{2^b}dt \tag{17}$$

$$= \frac{1}{da!b!}(\int_{t=0}^{\frac{1}{2}}(1-t)^b t^a dt - \frac{1}{(a+1)2^{d-1}}) \tag{18}$$

The linear term can be handled similarly.

$$-\frac{1}{d}\int_{t=0}^{\frac{1}{2}}\int_{y\leq t}\int_{\frac{1}{2}\leq z\leq 1-t} 2zdzdydt \tag{19}$$

$$= -\frac{2}{d}\int_{t=0}^{\frac{1}{2}}\int_{y\leq t}\frac{(1-t)^{b+1} - \frac{1}{2^{b+1}}}{(b+1)(b-1)!}dydt \tag{20}$$

$$= -\frac{2}{da!(b+1)(b-1)!}\int_{t=0}^{\frac{1}{2}}((1-t)^{b+1} - \frac{1}{2^{b+1}})t^a dt \tag{21}$$

$$= -\frac{2}{da!(b+1)(b-1)!}\int_{t=0}^{\frac{1}{2}}((1-t)^{b+1})t^a - \frac{1}{(a+1)2^d}dt \tag{22}$$

Thus, we have an explicit formula for the expected value of the sum negativity. Let

$$A \coloneqq \frac{1}{da!b!}\int_{t=0}^{\frac{1}{2}}(1-t)^b t^a dt \tag{23}$$

$$B \coloneqq \frac{-2}{da!(b+1)(b-1)!}\int_{t=0}^{\frac{1}{2}}(1-t)^{b+1}t^a dt \tag{24}$$

$$X \coloneqq \frac{2}{da!(b+1)(b-1)!}\frac{1}{(a+1)2^d} - \frac{1}{da!b!}\frac{1}{(a+1)2^{d-1}} \tag{25}$$

Then the expected sum negativity is given by $\mathcal{SN} = d^2\frac{\sqrt{d}(A+B+X)}{vol(\Delta^{(d-1)}))} = d^2(d-1)!(A+B+X)$

Since the quantities above can be expressed with the incomplete beta function, this is an explicit formula for the sum negativity. However, it is not very useful because it is unclear how this quantity grows as d increases. For this reason, we try to find asymptotic bounds.

First, we will approximate $dd!X$.

$$X = \frac{1}{da!(b-1)!(a+1)2^{d-1}}(-\frac{1}{b} + \frac{1}{b+1}) \tag{26}$$

$$dd!X = -\frac{d!}{a!(b-1)!(a+1)2^{d-1}}\frac{1}{b(b+1)} = -d\binom{d-1}{a+1}\frac{1}{2^{d-1}(b+1)} \tag{27}$$

We approximate $\binom{d-1}{a+1}$ by applying Stirling's approximation to $\binom{d-1}{\frac{d-1}{2}} \approx \sqrt{2}\frac{2^{d-1}}{\sqrt{\pi(d-1)}}$.

$$\approx -\frac{d}{2^{d-1}(b+1)}\sqrt{2}\frac{2^{d-1}}{\sqrt{\pi(d-1)}} \approx -\sqrt{\frac{1}{2\pi d}} \tag{28}$$

This quantity becomes negligible as d increases.

At this point, we depart to describe a useful integration formula: It is well known (I assume this can be proven with induction) that

$$\int_0^1 (1-t)^b t^a dt = \frac{a!b!}{(a+b+1)!} = \frac{a!b!}{(d-1)!} \tag{29}$$

We divide the interval $[0,1] = [0,\frac{1}{2}) \cup [\frac{1}{2},1]$

$$= \int_0^{\frac{1}{2}}(1-t)^b t^a dt + \int_{\frac{1}{2}}^1 (1-t)^b t^a dt \tag{30}$$

Let $t' = (1-t)$ in the second integral to obtain

$$\int_{\frac{1}{2}}^{1} (1-t)^b t^a dt = -\int_{\frac{1}{2}}^{0} t'^b (1-t')^a dt' = \int_{0}^{\frac{1}{2}} t'^b (1-t')^a dt' \tag{31}$$

To summarize, we have shown that

$$\int_{0}^{\frac{1}{2}} (1-t)^b t^a dt + \int_{0}^{\frac{1}{2}} t'^b (1-t')^a dt' = \frac{a!b!}{(a+b+1)!} \tag{32}$$

Next, we approximate B

$$B \ \approx \bar{B} \coloneqq \frac{-2}{da!b!} \int_{0}^{\frac{1}{2}} (1-t)^{b+1} t^a dt \tag{33}$$

which is justified because, $dd!(B - \bar{B}) = dd! \frac{2}{da!(b+1)!} \int_0^{\frac{1}{2}} (1-t)^{b+1} t^a dt$. We use integration by parts and 32 to conclude that

$$\int_{0}^{\frac{1}{2}} (1-t)^{b+1} t^a dt = \frac{1}{a+1} (1-t)^{b+1} t^{a+1} \Big|_{0}^{\frac{1}{2}} + \frac{b+1}{a+1} \int_{0}^{\frac{1}{2}} (1-t)^b t^{a+1} dt \tag{34}$$

$$= \frac{1}{(a+1)2^d} + \frac{1}{2} \frac{(a+1)!b!}{(d+1)!} \tag{35}$$

Substituting this,

$$dd!(B - \bar{B}) = dd! \frac{2}{da!(b+1)!} \left(\frac{1}{(a+1)2^d} + \frac{1}{2} \frac{(a+1)!b!}{(d+1)!} \right) \tag{36}$$

$$dd!(B - \bar{B}) = \frac{d!}{(a+1)!(b+1)!2^{d-1}} + \frac{(a+1)}{(d+1)(b+1)} \tag{37}$$

The second summand is $O(\frac{1}{d})$. We bound the first summand by using the central binomial coefficient formula together with Sperner's Theorem:

$$\frac{d!}{(a+1)!(b+1)!} \leq \sqrt{2} \frac{2^d}{\sqrt{\pi d}} \tag{38}$$

This bounds the first summand by

$$\sqrt{2} \frac{2^d}{\sqrt{\pi d}} \frac{1}{2^{d-1}} = \sqrt{\pi} 2^{\frac{3}{2}} \frac{1}{\sqrt{d}} \tag{39}$$

We are ready to approximate $A + \bar{B}$.

$$A + \bar{B} = \frac{1}{da!b!} (\int_{t=0}^{\frac{1}{2}} (1-t)^b t^a dt - 2 \int_{0}^{\frac{1}{2}} (1-t)^{b+1} t^a dt) \tag{40}$$

$$= \frac{1}{da!b!}\int_0^{\frac{1}{2}}(1-t)^b t^a(1-2(1-t))dt \tag{41}$$

$$= \frac{1}{da!b!}(-\int_0^{\frac{1}{2}}(1-t)^b t^a dt + 2\int_0^{\frac{1}{2}}(1-t)^b t^{a+1} dt) \tag{42}$$

The second integral can be evaluated by recognizing that $a+1=b$ and applying 32.

$$2\int_0^{\frac{1}{2}}(1-t)^b t^{a+1} dt = \frac{(a+1)!b!}{d!} \tag{43}$$

We can apply integration by parts to the second integral in 30 to obtain an analogue of 32 for the first integral.

$$\int_{\frac{1}{2}}^1 (1-t)^b t^a dt = \frac{1}{a+1} t^{a+1}(1-t)^b \bigg|_{\frac{1}{2}}^1 + \frac{b-1}{a+1}\int_{\frac{1}{2}}^1 t^{a+1}(1-t)^{b-1} dt \tag{44}$$

$$= -\frac{1}{(a+1)2^{d-1}} + \frac{b-1}{a+1}\int_{\frac{1}{2}}^1 t^{a+1}(1-t)^{b-1} dt \tag{45}$$

Next, let $t' = (1-t)$

$$\int_{\frac{1}{2}}^1 (1-t)^b t^a dt = -\frac{1}{(a+1)2^{d-1}} + \frac{b-1}{a+1}\int_0^{\frac{1}{2}} (1-t')^{a+1} t'^{b-1} dt' \tag{46}$$

since $a+1=b$, we get back our original integral. We substitute this into 30.

$$\int_0^{\frac{1}{2}}(1-t)^b t^a dt - \frac{1}{(a+1)2^{d-1}} + \frac{a}{b}\int_0^{\frac{1}{2}}(1-t)^b t^a dt = \frac{a!b!}{(d-1)!} \tag{47}$$

$$\int_0^{\frac{1}{2}}(1-t)^b t^a dt = \frac{1}{1+\frac{a}{b}}(\frac{a!b!}{(d-1)!} + \frac{1}{(a+1)2^{d-1}}) \tag{48}$$

Now we substitute 48 and 43 into 42 and multiply through by $dd!$.

$$dd!(A+\bar{B}) = \frac{dd!}{da!b!}\left(-(\frac{1}{1+\frac{a}{b}}(\frac{a!b!}{(d-1)!} + \frac{1}{(a+1)2^{d-1}})) + \frac{(a+1)!b!}{d!}\right) \tag{49}$$

$$= \frac{d!}{a!b!}\left(\frac{(a+1)!b!}{d!} - \frac{1}{1+\frac{a}{b}}\frac{a!b!}{(d-1)!}\right) - \frac{d!}{a!b!}\frac{1}{1+\frac{a}{b}}\frac{1}{(a+1)2^{d-1}} \tag{50}$$

We can simplyify the first term as

$$a+1-\frac{db}{a+b} = \frac{d-1}{2} - \frac{d\frac{d-1}{2}}{d-1} \in O(1) \tag{51}$$

Now we turn to the second term. We drop the factor of $\frac{1}{1+\frac{a}{b}}$ since it clearly goes to $\frac{1}{2}$ as d grows.

$$-\frac{dd!}{da!b!}\frac{1}{(a+1)2^{d-1}} = -d\frac{(d-1)!}{(a+1)!b!}\frac{1}{2^{d-1}} \approx -d\sqrt{2}\frac{2^{d-1}}{\sqrt{\pi(d-1)}}\frac{1}{2^{d-1}} \in O(\sqrt{d}) \tag{52}$$

Moreover, collecting constants, we can see that random pure states will have an expected value of around $-\frac{1}{2}\sqrt{\frac{2}{\pi}}\sqrt{d} \approx -0.4\sqrt{d}$ as d becomes large.

References

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