Phase Points as Ontic States in Finite-Dimensional Systems

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July 30, 2020

1 Introduction

Phase space is an important tool in classical mechanics. In classical mechanics, states are determined by the position and momentum of every particle. In quantum mechanics, the uncertainty principle asserts that position and momentum of particles cannot be defined simultaneously.

[1] describes the quantum phase-space construction for odd, finite degrees of freedom. In [2], it is shown that negative probabilities in this construction can be associated with Bell Inequalities.

The arguments in [2] can be interpreted as showing that phase points are the classical states which underlie compatible outcomes to all stabilizer measurements with an ancilla.

We will argue that phase points are the classical states in a stronger sense. Namely, any set of compatible outcomes for any subset of separable measurements and all entangled measurements will have separable outcomes which are supported on at a common point in phase space. We can then regard this common point as the underlying 'ontic state.'

2 Background

todo:1 Wigner Function 2 lines/stabilizer states 3 outcomes, vectors, 3.5 Graphs, orthogonalities 4 No Detection Events 5 Review [2]'s constructions Let $\mathcal{M} = \mathcal{M}_{\int} \cup \mathcal{M}_{\rceil}$ be the measurements in the construction.

Definition 1. *A set of vertices $I \subset G(\mathcal{M})$ is completely classical if it intersects the clique of each measurement non-trivially.*

Definition 2. *A collection of vertices $V \subset G(\mathcal{M})$ is compatibly extendable if V is independent, and is contained in a completely classical set of vertices*

Definition 3. *The main-qudit lines of a set of vertices $V \subset G(\mathcal{M})$ are the supports of the main-qudit part of the separable outcomes in phase space.*

By abuse of notation, we can also describe vectors in this way.

3 Main Arguments

Let's see how [2] can be interpreted as asserting that the classical states in phase space are phase points.

Theorem 1. *The (main qudit part of the) separable outcomes of a completely classical set of vertices are stabilizer states whose Wigner-Function supports in phase space are lines, and these lines meet at a common point.*

Conversely, any point in phase space has $d+1$ lines passing through it. These $d+1$ lines correspond to a compatibly extendable collection of vertices.

Proof. Theorem 6 shows that a completely classical set of vertices will have main qudit lines which cover every phase-space point.

If three lines form a triangle, the $d+1$ lines will only cover at most p^2-1 points. Since the main qudit lines are non-parallel and have no triangles, they must meet at a common point.

Conversely, the common point of the main qudit lines, combined with the common outcome for the ancilla provide points in the 2-particle phase space where all separable outcomes are supported. We can choose the entangled outcomes so they also contain this point in their support. □

This means that forcing the entangled measurements to provide complete, non-contradictory outcomes ensures that phase points can be considered the deterministic states of phase space in the Abramsky/Brandenburger model, where each measurement should have exactly one outcome.

We would also like phase-space points to be the classical states in the no-detection events setting. This means that we allow partial assignments for the separable measurements, so some directions may be left undetermined. The classical sets are (not necessarily maximal) independent sets, and our claim is that the main lines of any independent set meet at a common point.

Theorem 2. *Let l_1, l_2, l_3 form a triangle in p^n-dimensional phase space. The set of outcomes $\{O(l_1), O(l_2), O(l_3)\}$ is incompatible with any set of outcomes for the entangled measurements.*

Proof. Suppose these outcomes can be extended compatibly. Then, we can choose new coordinates for the separable spaces. This change of coordinates will be Clifford, and act separably between the main qudit and ancilla. We will be able to consider them as changes in coordinates of the measurement outcomes. Note that these transformations will map stabilizer states to stabilizer states and the entangled outcomes to other entangled outcomes.

In the new coordinates, we will show that the main qudit lines must pass through a single point, so they must still do so in the original coordinates.

Suppose our 3 lines form a triangle $\{a, b, c\}$. We can find a translation which transfers a to the origin. Then, we can find an element of $SL(2, d)$ which moves b to the point $(0, 1)$. Then, the set of affine maps which fix a and b but can affect c are of the form $\begin{bmatrix} 1, 0 \\ r, 1 \end{bmatrix}$

If c has coordinates (x_c, y_c), then choose r such that $rx_c + y_c = -1$. This choice can be made because $x_c \neq 0$, since $\{a, b, c\}$ form a triangle (so their images under affine transformations do as well.)

Let l_1 be the line from a to b, l_2 be the line from b to c, and l_3 be the line from c to a. (We unbind the names a, b and c at this point in the proof.)

Let $l_i = \{(k_{b_i} + b_i y, y) \mid y \in \mathbf{F}_d\}$. Observe that this notation agrees with the proof of Theorem 6 in [2], since

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k \\ y \end{bmatrix} = \begin{bmatrix} k + by \\ y \end{bmatrix} \tag{1}$$

Moreover, our transformations chosen so that $b_3 - b_2 = b_2 - b_1$.

We adopt notation from [2], and choose coordinates on the ancilla so that $l = 0$.

Observe that equations (39) continues to hold for all b and (42) continue to hold, as long as $b \in \{b_1, b_2, b_3\}$.

Therefore, equation (44) holds for $b = b_i$ and $c = b_{i+1} - b_i$.

$$k_{b_{i+1}} - k_{b_i} = (b_{i+1} - b_i)(2z_{2,\gamma'} - z_{1,\gamma}) \tag{2}$$

Let $Y_{i,\gamma} = (2z_{2,\gamma'} - z_{1,\gamma})$ and $\tilde{l}_i := l_{i+1} - l_i$. Then $Y_{1,\gamma} = Y_{2,\gamma}$

$$\tilde{l}_1(Y_{1,\gamma}) = 0 \tag{3}$$

$$\tilde{l}_2(Y_{1,\gamma}) = 0 \tag{4}$$

$$\tilde{l}_3(Y_{3,\gamma}) = 0 \tag{5}$$

Adding (3) and (4) gives

$$-\frac{l_1 - l_3}{2}(Y_{i,\gamma}) = 0 \implies \tilde{l}_3(Y_1, \gamma) = 0 \tag{6}$$

Since $\tilde{l}_3$ is the difference of two lines with different slopes, it cannot be horizontal, so $Y := Y_{3,\gamma} = Y_{1,\gamma}$, and we can conclude that the pairwise differences of our lines meet at a point, $(0, Y)$.

But this means that all 3 lines meet at Y.

$\square$

It would be nice if this theorem could be generalized to multiple particles.

Definition 4. *When ϕ is Clifford on n qudits, call the outcomes $\sum_j |\phi(k), k\rangle$ the fully entangled outcomes. Parameterizing $phi = w * \mu(s)$ where w is Weyl and s is symplectic partitions the fully entangled outcomes into measurements, indexed by s.*

Conjecture 1. *Suppose we have n p-dimensional particles and 3 outcomes to Clifford measurements on them, and 3 ancilla qubits. If these are compatible with a the fully entangled stabilizer measurements, then the main qupit spaces intersect at a common point.*

If true, we could replace the entangled measurements on a d^n-dimensional particle by an exponentially smaller set, namely the fully entangled stabilizer measurements on all sets of 3 particles, plus 3 ancilla qudits.

Definition 5. *The vector space generated by a collection Q of qudits the vector subspace of dimension $2 \mid Q \mid$ whose projection*

Conjecture 2. *Suppose a d^n-dimensional particle has 3 separable outcomes whose lines form a triangle. The lines of this triangle are d^n dimensional isotropic spaces which restrict to 3 dimensional isotropic spaces over any 3 qubits. For some particular triple, these restricted isotropic spaces will fulfill the hypothesis of Conjecture 1.*

If these conjectures hold, we will be able to enforce that the d^n dimensional particle's classical states are phase-space points, and we will use a number of entangled measurements which scales only polynomially in n.

References

[1] David Gross. Hudson's theorem for finite-dimensional quantum systems. *Journal of Mathematical Physics*, 2006.

[2] M. Howard, J. Wallman, V. Veitch, and J. Emerson. Contextuality supplies the magic for quantum computation. 2014.