Exponential Function, e^x

Compound Interest

Assume we invest €P at interest Rate, R% per annum.
Let $r = R/100$.
e.g. If interest rate is 10% per annum then $r = 0.1$.

Compounded	1	times/year, amount is	$P * (1+r)$
"	2	"	$(P * (1+\frac{r}{2})) * (1+\frac{r}{2})$ $= P * (1+\frac{r}{2})^2$
"	3	"	$P * (1+\frac{r}{3})^3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
"	n	"	$P * (1+\frac{r}{n})^n$
"	∞	"	$P * \lim_{n\to\infty}(1+\frac{r}{n})^n$

If the interest rate is compounded instantaneously, then the amount $= P * \lim_{n\to\infty}(1+\frac{r}{n})^n$.

Calculating $\lim_{n\to\infty}(1+\frac{r}{n})^n$

Recall Binomial expansion of $(1+x)^n$

$(1+x)^n = 1 + n*x + \frac{n*(n-1)}{2!}*x^2 + \frac{n*(n-1)*(n-2)}{3!}*x^3 + \cdots + x^n$

This is written as:

$$(1+x)^n = \begin{pmatrix} n \\ 0 \end{pmatrix} * x^0 + \begin{pmatrix} n \\ 1 \end{pmatrix} * x + \begin{pmatrix} n \\ 2 \end{pmatrix} * x^2 + \begin{pmatrix} n \\ 3 \end{pmatrix} * x^3 + \cdots + \begin{pmatrix} n \\ n \end{pmatrix} * x^n$$

i.e.

$$(1+x)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} x^k$$

where $x^0 = 1$, $\begin{pmatrix} n \\ 0 \end{pmatrix} = 1$ and $\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!*(n-k)!}$.

Note: $n! = n*(n-1)*\cdots*1$ and $0! = 1$.

$$(1+\frac{r}{n})^n$$

$$= 1 + n * \frac{r}{n} + \frac{n*(n-1)}{2!} * \left(\frac{r}{n}\right)^2 + \frac{n*(n-1)*(n-2)}{3!} * \left(\frac{r}{n}\right)^3 + \dots$$

$$= 1 + r + \frac{n*(n-1)}{n*n} * \frac{r^2}{2!} + \frac{n*(n-1)*(n-2)}{n*n*n} * \frac{r^3}{3!} + \dots$$

$$= 1 + r + \frac{r^2}{2!} * (1-\frac{1}{n}) + \frac{r^3}{3!} * (1-\frac{1}{n}) * (1-\frac{2}{n}) + \dots$$

$\therefore$

$$\lim_{n\to\infty}(1+\frac{r}{n})^n$$

$$= \lim_{n\to\infty}\left(1 + r + \frac{r^2}{2!} * (1-\frac{1}{n}) + \frac{r^3}{3!} * (1-\frac{1}{n}) * (1-\frac{2}{n}) + \dots\right)$$

As $n \to \infty$ then $\frac{1}{n} \to 0$ and $\frac{2}{n} \to 0$, etc. $\therefore$

$$\lim_{n\to\infty} \left(1 + r + \frac{r^2}{2!} * (1 - \frac{1}{n}) + \frac{r^3}{3!} * (1 - \frac{1}{n}) * (1 - \frac{2}{n}) + \dots\right)$$
$$= 1 + r + \frac{r^2}{2!} + \frac{r^3}{3!} + \dots$$

but it is known (and proved later) that

$e^r = 1 + r + \frac{r^2}{2!} + \frac{r^3}{3!} + \dots$ where $ln\, e = 1$
$\therefore$
$\lim_{n\to\infty}(1 + \frac{r}{n})^n = e^r$

Amount compounded instantaneously is
$= P * \lim_{n\to\infty}(1 + \frac{r}{n})^n.$
$= P * e^r$

Example $P * (1 + \frac{r}{k})^k$

If r is the interest rate and P is the principle or initial amount then $P * (1 + \frac{r}{k})^k$ gives the final amount, A, when interest is compounded k times a year.
After 2 years, final amount A will be
$A = (P * (1 + \frac{r}{k})^k) * (1 + \frac{r}{k})^k = P * (1 + \frac{r}{k})^{k*2}$
After t years final amount, A, will be $A = P * (1 + \frac{r}{k})^{k*t}$

Deposit Growth

How many years will it take a €1000 deposit to grow to €1500 if it is invested at a yearly rate of 12% compounded quarterly.

$r = .12$ and $k = 4$. Using formula $A = P * (1 + \frac{r}{k})^{k*t}$
Given $1500 = 1000 * (1 + \frac{.12}{4})^{4*t}$, find t.

Example (Cont'd)

$$1500 = 1000 * (1 + \frac{0.12}{4})^{4*t} \therefore$$

$$\frac{1500}{1000} = (1 + 0.03)^{4*t} \therefore$$

$$\frac{3}{2} = 1.03^{4*t} \therefore$$

$$ln(\frac{3}{2}) = 4 * t * ln(1.03) \therefore$$

$$ln(3) - ln(2) = 4 * t * ln(1.03)$$
$$\{\ ln(1.03) = .029,\ ln(3) = 1.098.\ \ ln(2) = .693\ \} \therefore$$

$$1.098 - .693 = 4 * t * .029 \therefore$$

$$\frac{.405}{4 * .029} = t \therefore$$

$$t \approx 3.5$$

Instantaneous Compound Interest Example

Let $P = 100$, $r = 0.1$ then
Compounded once per year (Simple interest)
$= 100 * (1 + 0.1) = 110$
Compounded instantaneously in the year $= 100 * e^{0.1}$ but
$e^{0.1} = 1.105 \therefore$
$100 * e^{0.1} = 110.5$.

Compound Interest after t years:
After 1 year compounded instantaneously, Amount $= P * e^r$
After 2 years compounded instantaneously,
Amount $= (P * e^r) * e^r = P * e^{2*r}$
...
After t years compounded instantaneously, Amount $= P * e^{t*r}$

Inverse Functions

Let $f : \mathbb{R} \to \mathbb{R}$ be a function from Reals to Reals.
Since f is a function, we can a use $f(a)$ to denote the result of applying the function, f, to the argument, a. We say "a is mapped to $f(a)$" and this can be symbolised as $a \mapsto f(a)$ 'a maps to $f(a)$'.
For example, we can define a function, *cube*, that cubes a number as:

$$cube : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto x^3$$

e.g. $cube(-2) = -8$ and $cube(2) = 8$.
The inverse of a function undoes what the function does and so the inverse of the function, *cube*, is the **cube root** function, $cube^{-1}$.
$cube^{-1}(-8) = -2$ and $cube^{-1}(8) = 2$.

Bijective Function:

A function $f : \mathbb{R} \to \mathbb{R}$ is **Bijective** if for $x, y \in \mathbb{R}$ and $x \neq y$ then $f(x) \neq f(y)$.
i.e. no two distinct inputs are mapped to the same output.
We assume that for each $y \in \mathbb{R}$ there is an $x \in \mathbb{R}$ such $f(x) = y$,
i.e. for each 'output value' there is an 'input value'.
Bijective functions have inverses. The inverse of a function f is denoted by f^{-1}.
Properties of Inverse:

- $f(f^{-1}(x)) = x$
- $f^{-1}(f(x)) = x$.

If $y = f^{-1}(x)$ then $f(y) = x$ and if $y = f(x)$ then $f^{-1}(y) = x$.
The function f^{-1} undoes what f does.

Examples

The function

$$cube : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto x^3$$

is bijective and so has an inverse. The inverse is:

$$cube\_root : \mathbb{R} \to \mathbb{R}$$

$$x \mapsto \sqrt[3]{x}$$

The function

$$square : \mathbb{R} \to \mathbb{R}$$

$$x \mapsto x^2$$

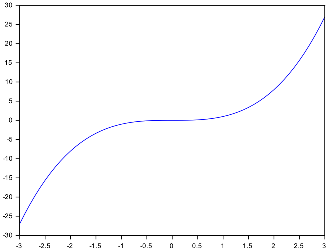
is not bijective as, e.g. $2 \neq -2$ and $square(2) = square(-2)$.

Graph view of a bijective function

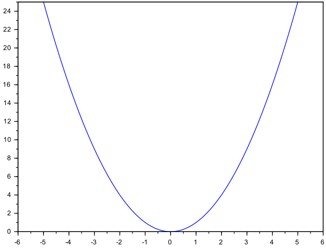
If a line parallel to the $x - axis$ cuts the graph of f in more that one place then the function f is not bijective.

The function, $cube(x) = x^3$, is bijective and so has an inverse. For a bijective function, each line parallel to the $x - axis$ cuts the curve and each line cuts the curve at just one point.

The function, $square(x) = x^2$, is not bijective and its inverse is not a function.



$$cube(x) = x^3$$



$$square(x) = x^2$$

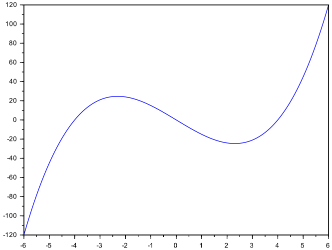
Square as a bijective function

Let $\mathbb{R}_{\geq 0} = \{x | x \in \mathbb{R} \, and \, x \geq 0\}$, the non-negative Reals.
Consider defining a function,
$Sq : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$
$s.t. \; x \mapsto x^2$.
The function, Sq, is bijective and so has an inverse.
$Sq^{-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$
$x \mapsto \sqrt{x}$
where $y = \sqrt{x} \, iff \, y^2 = x \, and \, y \geq 0$ i.e. y is the non-negative root.
i.e. for $x, y \geq 0$,

$$y = Sq^{-1}(x) \, iff \, Sq(y) = x$$

Non Bijective function

Consider the function
$f : \mathbb{R} \to \mathbb{R}$ such that $x \mapsto x^3 - 16 * x$. The graph of f is:



This function is not bijective as the line $y = 0$ cuts the graph in more than one place.
In particular, $f(-4) = f(0) = f(4) = 0$.

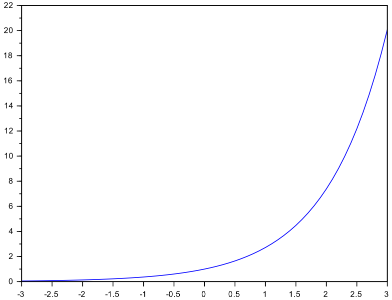
Note: $\mathbb{R}^+$ is the set of Reals > 0

Definition e^x

Recall the Natural logarithm, $ln : \mathbb{R}^+ \to \mathbb{R}$. The function, ln, is a bijective function (see graph of $ln(x)$), therefore it has an inverse, ln^{-1}. The function ln^{-1} is defined as the exponential function and denoted by exp, i.e. $exp = ln^{-1}$.
Note: $exp : \mathbb{R} \to \mathbb{R}^+$ i.e exp is defined for all Real numbers and for all $x \in \mathbb{R}$, $exp(x) > 0$.
e.g. $exp(0) = 1$ as $ln^{-1}(0) = 1$ i.e. $ln(1) = 0$.
We have definition:
$a = exp(b)$ iff $ln(a) = b$.

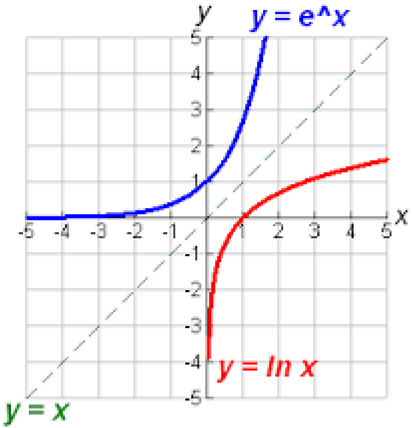
- $exp(ln(x)) = x$, for $x \in \mathbb{R}^+$
- $ln(exp(x)) = x$, for $x \in \mathbb{R}$

Graph of $exp(x)$



The function, $exp(x)$, is bijective. From graph, $exp(0) = 1$.
If (a, b) lies of the graph of exp, then (b, a) lies on the graph of ln.

Graph of ln(x) and exp(x)



$exp(x) = e^x$

Recall that $ln(x) = log_e(x)$ and $y = log_e(x)$ iff $e^y = x$.

We can define e^x as:

$$y = e^x \; iff \; ln(y) = x.$$

i.e.

$$e^{ln(x)} = x$$

Since $y = exp(x) \; iff \; ln(y) = x \; \therefore$

$$exp(x) = e^x.$$

Derivative of e^x

Let $y = e^x$ then $ln(y) = x \therefore$
$\frac{d}{dx}(ln(y)) = \frac{dx}{dx}$.
From calculus $\frac{dx}{dx} = 1$ and from properties of $ln(x)$
$\frac{d}{dx}(ln(y)) = \frac{1}{y} * \frac{dy}{dx} \therefore$
$\frac{1}{y} * \frac{dy}{dx} = 1 \therefore$
$\frac{dy}{dx} = y$
but $y = e^x \therefore$
$\frac{d}{dx}(e^x) = e^x$.

Integral of e^x

$\int e^x dx = e^x + c$

Derivative via Chain Rule: e^u

If u is a function of x, then $\frac{d}{dx}(e^u) = e^u * \frac{du}{dx}$.

Example:
Find $\frac{dy}{dx}$ when $y = e^{x^2-x}$

$\frac{dy}{dx} = e^{x^2-x} * (2 * x - 1)$

Definition a^b

Assume $a, b \in \mathbb{R}$ and $a > 0$.
From above, $e^{ln(x)} = x$.
For $a, b \in \mathbb{R}$ and $a > 0$, we define:

$$a^b = (e^{ln(a)})^b = e^{b*ln(a)}.$$

Derivative of $a^x, a > 0$

Derivative of a^x when a is a constant and $a > 0$.

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{ln(a)*x}) = e^{ln(a)*x} * ln(a) = a^x * ln(a).$$

Derivative of r^s

r and s are functions of x and we assume that $r > 0$ as $ln(r)$ will be involved. Let $t = ln(r)$ then $e^t = e^{ln(r)} = r$ $\therefore$

$r^s = (e^t)^s = e^{t*s}$

$\frac{d}{dx}(r^s) = \frac{d}{dx}(e^{t*s}) = e^{t*s} * \frac{d}{dx}(t * s)$

Recall the derivative of product $u * v$:
$\frac{d}{dx}(u * v) = u * \frac{dv}{dx} + v * \frac{du}{dx}$

Derivative of r^s,

$$\frac{d}{dx}(r^s) = e^{t*s} * \frac{d}{dx}(t*s) \text{ where } t = ln(r)$$

$$= e^{t*s} * \left(t * \frac{ds}{dx} + s * \frac{dt}{dx}\right)$$

$$= (e^{ln(r)})^s * \left(ln(r) * \frac{ds}{dx} + s * \frac{d}{dx}(ln(r)\right) \text{ as } t = ln(r),\ r = e^t$$

$$= r^s * \left(ln(r) * \frac{ds}{dx} + s * \frac{1}{r} * \frac{dr}{dx}\right)$$

$$= r^s * \left(ln(r) * \frac{ds}{dx} + \frac{s}{r} * \frac{dr}{dx}\right)$$

Example, Find $\frac{d}{dx}(\cos x)^{Sin\, x})$

Since $\frac{d}{dx}(r^s) = r^s * \left(ln(r) * \frac{ds}{dx} + \frac{s}{r} * \frac{dr}{dx}\right)$

$\frac{d}{dx}(\cos x)^{\sin x}$

$= (\cos x)^{\sin x} * \left(ln(\cos x) * \frac{d}{dx}(\sin x) + \frac{\sin x}{\cos x} * \frac{d}{dx}(\cos x)\right)$

$= (\cos x)^{\sin x} * \left(ln(\cos x) * (\cos x) - \frac{\sin x}{\cos x} * (\sin x)\right)$

$= (\cos x)^{\sin x} * (ln(\cos x) * (\cos x) - (\tan x) * (\sin x))$

Exercise

Find $\frac{d}{dx}\left(x^x\right)$.

Note: $x^x = \left(e^{ln(x)}\right)^x = e^{x*ln(x)}$.