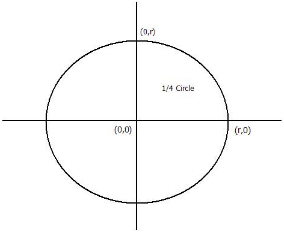
Area of Circle using Integration

Area of Circle

From Archimedes, it is known that the area of circle is proportional to the square of the radius i.e.
Area of circle with radius r is πr^2.
Archimedes proved this result using the Method of Exhaustion which had been invented by Eudoxus.
He also proved that the length of the circumference of a circle with radius r is $2\pi r$.
He considered the area of circle with radius, r, as equal to the area of a right angled triangle who height is r and whose base is $2\pi r$, the length of the the circumference.
Area of such a triangle $= \frac{1}{2} base \times height$ i.e. $\frac{1}{2}(2\pi r) \times r$ i.e. πr^2.

The equation of circle with radius, r, and centre at the origin, is $x^2 + y^2 = r^2$. Expressing this as a function, we get $y = \sqrt{r^2 - x^2}$.



To find the area of a $\frac{1}{4}$circle we find $\int_0^r \sqrt{r^2 - x^2}\, dx$

Digression to Trigonometry

From Trigonometry we have:

$$cos(A + B) = (cos\, A)(cos\, B) - (sin\, A)(sin\, B) \therefore$$
$$cos\, 2A = cos^2 A - sin^2 A$$
$$\{\ cos^2\theta + sin^2\theta = 1 \therefore sin^2\theta = 1 - cos^2\theta\ \}$$
$$\therefore$$
$$cos\, 2A = cos^2 A - (1 - cos^2 A) \therefore$$
$$cos\, 2A = 2\, cos^2 A - 1 \therefore$$
$$cos^2 A = \frac{1 + cos\, 2A}{2}$$

Finding $\int_0^r \sqrt{r^2 - x^2}\, dx$

To find $\int_0^r \sqrt{r^2 - x^2}\, dx$ we use the substitution:

$x = r\, sin\, \theta \ (0 \leq \theta \leq \frac{\pi}{2}) \therefore$
$dx = r\, cos\, \theta\, d\theta$

Also, we can change the limits of integration

for limit $x = 0$, then $r\, sin\, \theta = 0$ i.e. $sin\, \theta = 0 \therefore \theta = 0$
for limit $x = r$ then $r\, sin\, \theta = r$ i.e. $sin\, \theta = 1 \therefore \theta = \frac{\pi}{2}$
$\therefore$

Finding $\int_0^r \sqrt{r^2 - x^2}\, dx$ (Cont'd)

Area of $\frac{1}{4}$circle =

$$\int_0^r \sqrt{r^2 - x^2}\, dx$$

$$\begin{aligned}
&= \int_0^{\frac{\pi}{2}} \sqrt{r^2 - r^2 sin^2\theta}\; r\, cos\,\theta\; d\theta \\
&= \int_0^{\frac{\pi}{2}} r\sqrt{1 - sin^2\theta}\; r\, cos\,\theta\; d\theta \\
&= r^2 \int_0^{\frac{\pi}{2}} (cos\,\theta)\,(cos\,\theta)\; d\theta \\
&= r^2 \int_0^{\frac{\pi}{2}} cos^2\,\theta\; d\theta \\
&= r^2 \int_0^{\frac{\pi}{2}} \frac{1 + cos\,2\theta}{2}\; d\theta
\end{aligned}$$

Finding $\int_0^r \sqrt{r^2 - x^2}\, dx$ (Cont'd)

$\therefore$ Area of $\frac{1}{4}$circle $=$

$$\begin{aligned}
\int_0^r \sqrt{r^2 - x^2}\, dx &= r^2 \int_0^{\frac{\pi}{2}} \frac{1 + cos\, 2\theta}{2}\, d\theta \\
&= \left[\frac{r^2}{2} \left(\theta + \frac{sin\, 2\theta}{2} \right) \right]_0^{\frac{\pi}{2}} \\
&= \frac{r^2}{2} \left(\frac{\pi}{2} + \frac{sin\, \pi}{2} \right) - \frac{r^2}{2} \left(0 + \frac{sin\, 0}{2} \right) \\
&= \frac{r^2}{2} \left(\frac{\pi}{2} + 0 \right) - \frac{r^2}{2} (0 + 0) \\
&= \frac{\pi r^2}{4}
\end{aligned}$$