Section II

Electric Circuit Analysis

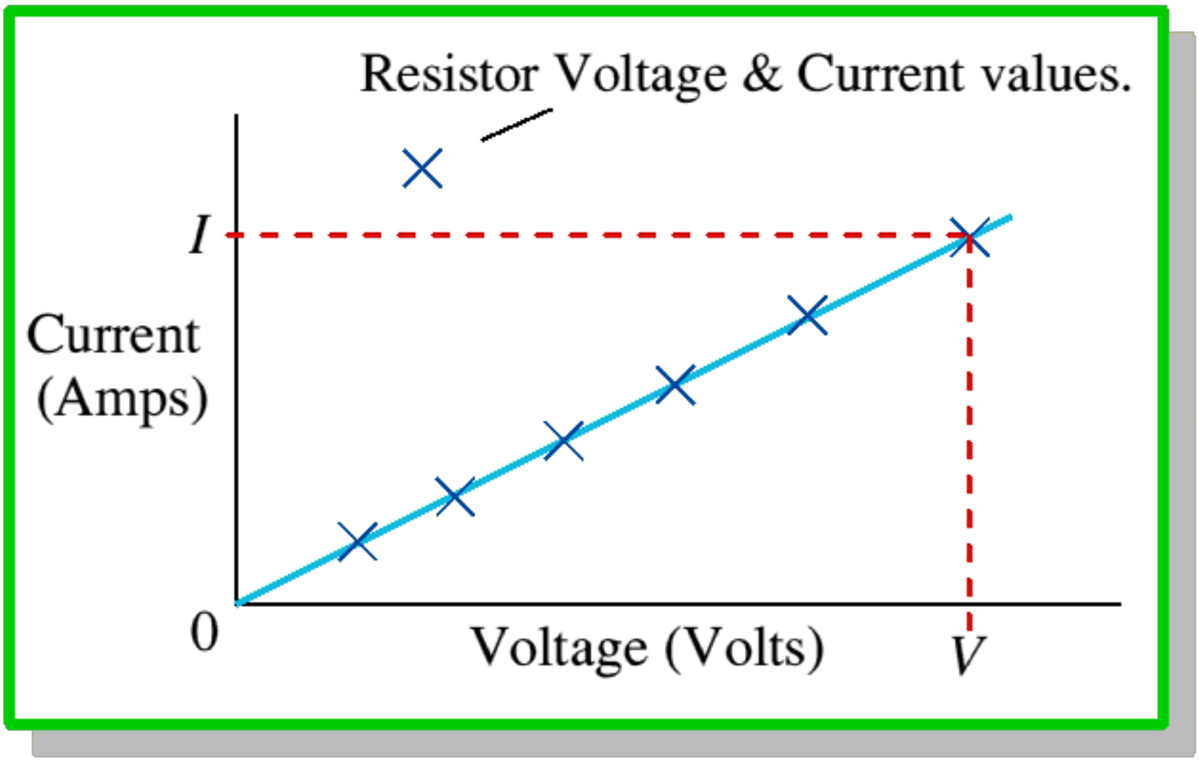
(i) The Resistor

The Resistor

The resistor is a two-terminal element which impedes the flow of electrical current (through it) converting some of the electrical energy into another form (usually heat and light).

The degree to which it impedes current flow is given numerically by its Resistance Value – measured in 'Ohms' (Ω).

Every material possesses this property (resistance) to some extent. When we refer to devices known as 'resistors', we are referring to what are known as 'Ohmic Materials' – where there is a linear relationship between the applied voltage and the current.



The relationship between the terminal voltage of a resistor and the current flowing through it is given by Ohm's Law:

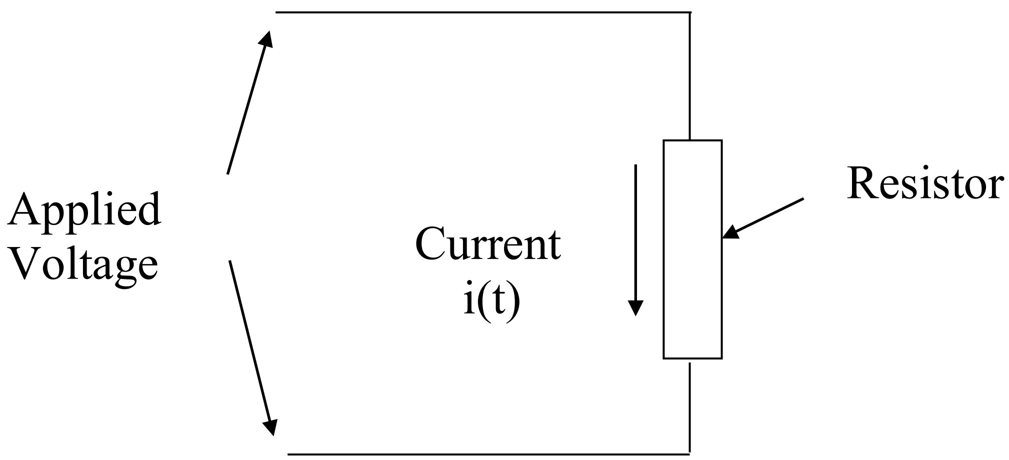
$$v(t) = Ri(t)$$

- 'R' is known as the Resistance
- 't' is time
- 'v(t)' is voltage across the terminals
- 'i(t)' is the current flowing through the device

$$[R] = \frac{V}{A} \equiv \Omega \equiv 'Ohm'$$

Electrical symbols for the resistor are:

Hence:



Since $$p(t) = v(t)i(t)$$

Then from Ohm's Law:

$$\boxed{p(t) = i^2(t)R}$$

Or:

$$\boxed{p(t) = \frac{v^2(t)}{R}}$$

- for a resistor.

Example

Consider a heater which is connected to a fixed electrical power supply – e.g. mains. Such heaters are common in many household items such as heaters, toasters, hairdryers, etc.

The power of this heater (i.e. the rate at which heat is produced) is:

$$P_{heater} = \frac{V^2(t)}{R_{heater}}$$

It is obvious from this equation that a smaller resistance will give a greater heating effect.

Note: if R $\rightarrow$0 then $P_{heater} \rightarrow \infty$ => burnout of circuit which is undesirable so R must be kept above a certain minimum.

Example

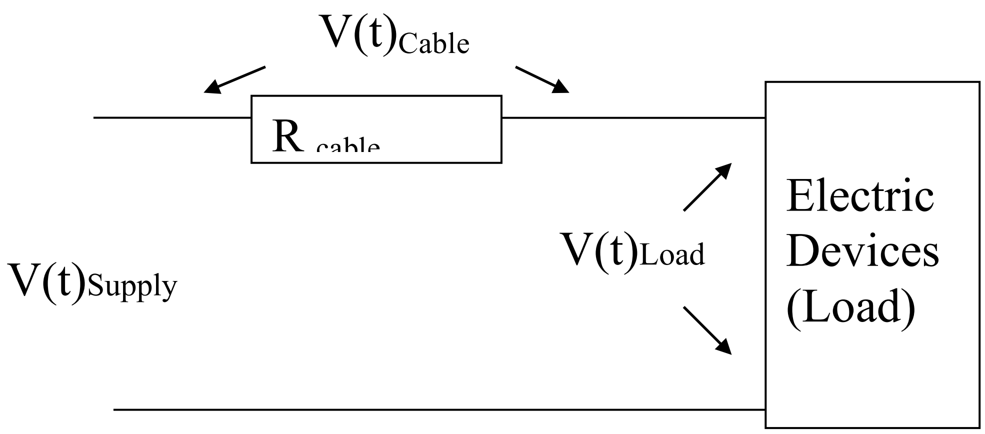
Consider the transmission of power through electrical power lines. In this case we have a fixed amount of power available for transmission.

$$P_{Transmission} = v(t)_{Supply} i(t)$$

$$P_{Loss} = i^2(t) R_{Cable}$$

Therefore to minimize power loss (i.e. maximise the transmitted power) we have two alternatives:

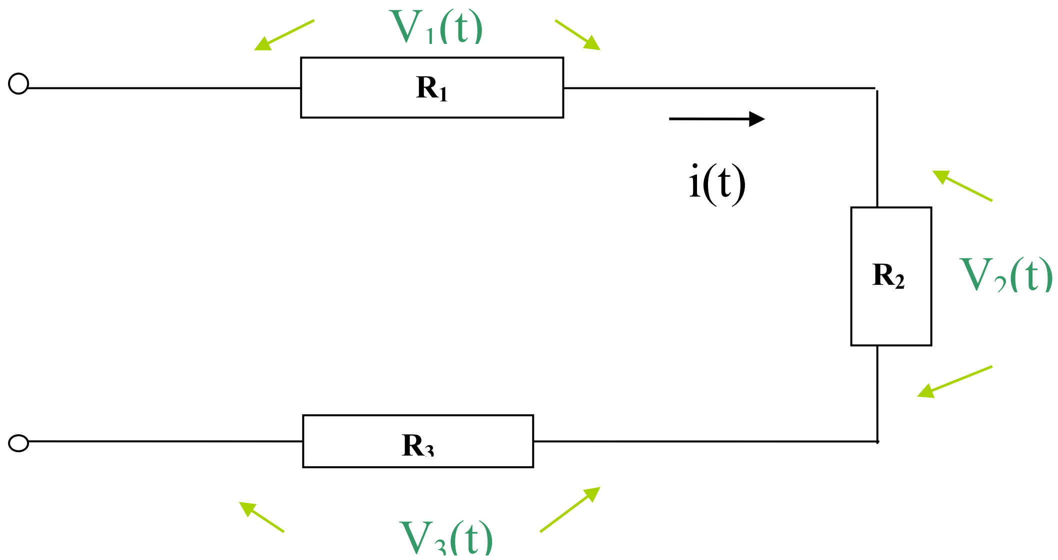
a. Reduce R_{cable}.
b. Make i(t) as small as possible.



In making i(t) small we must raise v(t) to deliver the same power. Hence high voltage transmission cables.

(ii) Resistors Connected in Series

Consider the following setup:



By Ohm's Law:

$$V_1(t) = R_1\, i(t)$$
$$V_2(t) = R_2\, i(t)$$
$$V_3(t) = R_3\, i(t)$$

But: $V(t) = V_1(t) + V_2(t) + V_3(t)$

$\Rightarrow V(t) = i(t)\,(R_1 + R_2 + R_3)$

Hence for ‘n’ resistors in series:

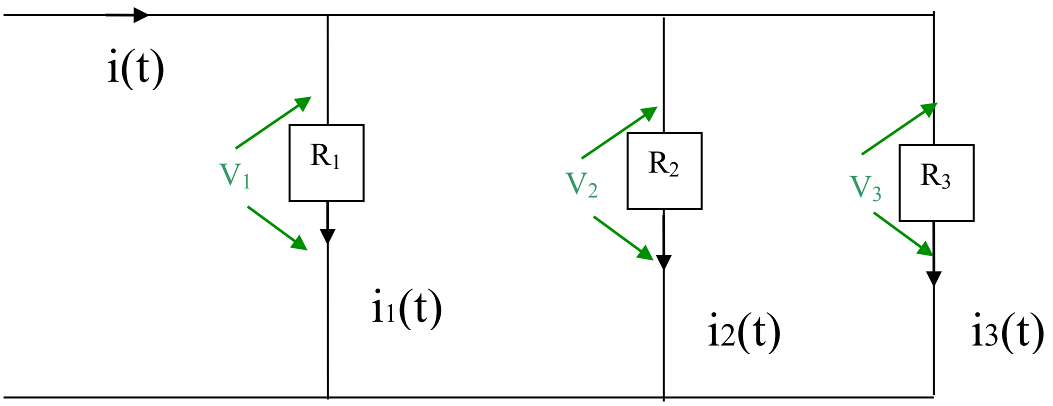
$$R_{Total} = R_1 + R_2 + R_3 + \ldots\ldots + R_n$$

and

$$v(t) = i(t)R_{Total}$$

(iii) Resistors Connected in Parallel

Consider the following setup:



Then: $i(t) = i_1(t) + i_2(t) + i_3(t)$

$$v_1(t) = R_1\, i(t)$$
$$v_2(t) = R_2\, i(t)$$
$$v_3(t) = R_3\, i(t)$$

=> $i_1(t) = \dfrac{v(t)}{R_1}$

etc.

$$\Rightarrow \quad i(t) = \frac{v(t)}{R_1} + \frac{v(t)}{R_2} + \frac{v(t)}{R_3}$$

$$\Rightarrow \quad \frac{v(t)}{R_{total}} = \frac{v(t)}{R_1} + \frac{v(t)}{R_2} + \frac{v(t)}{R_3}$$

$$\Rightarrow \quad \frac{1}{R_{total}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

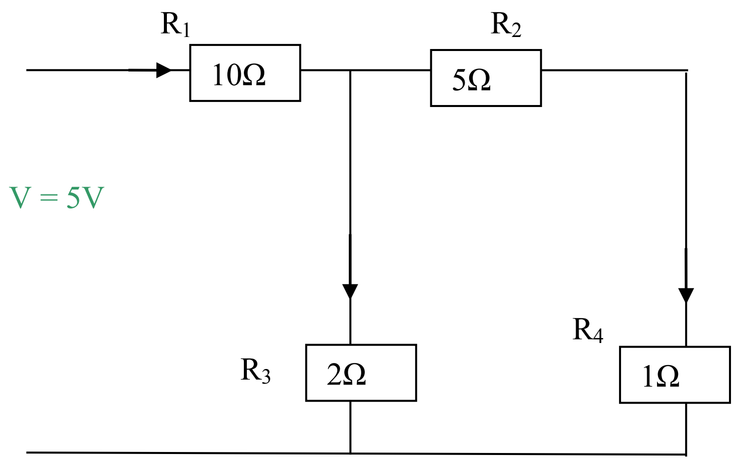
Hence for 'n' resistors in parallel:

$$\boxed{\frac{1}{R_{Total}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots\ldots.. + \frac{1}{R_n}}$$

$$\boxed{v(t) = i(t) R_{Total}}$$

Example

Consider the following circuit:



Find the total current drawn from the source.

Any purely resistive circuit can be reduced to a source in series with a single resistance!

The problem is to determine the total resistance, R_{Total}, and then, by Ohm's Law, the total current, I, drawn from the source.

So:

R_2 is in series with R_4 forming R_5.

- i.e. $R_5 = R_2 + R_4$

R_3 is in parallel with R_5, forming R_6.

- i.e. 1/R6 = 1/R3 + 1/R5

R_{Total} is then $R_1 + R_6$

$$\Rightarrow I_{Total} = \frac{V_{Supply}}{R_{Total}}$$

Calculations:

$R_5 = R_2 + R_4 = 5 + 1 = 6\Omega$
$R_6 = R_3 \,||\, R_5 = (1/6 + 1/2)^{-1} = 1.5\Omega$
$R_{Total} = R_1 + R_6 = 10 + 1.5 = 11.5\Omega$

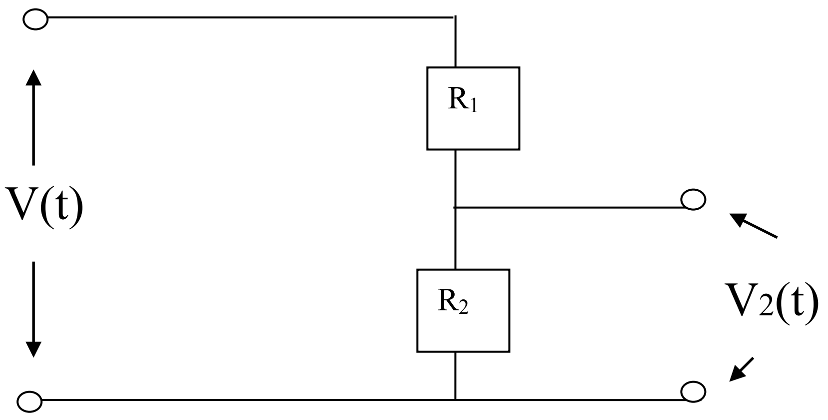
Hence:

$$I_{Total} = \frac{5}{11.5} = 0.43\text{ A}$$

(iv) The Potential Divider

An important application of resistors is the Potential Divider. It is used to obtain a lower p.d. from a higher one.

i.e.



Clearly: $v(t) = i(t)\,(R_1 + R_2)$

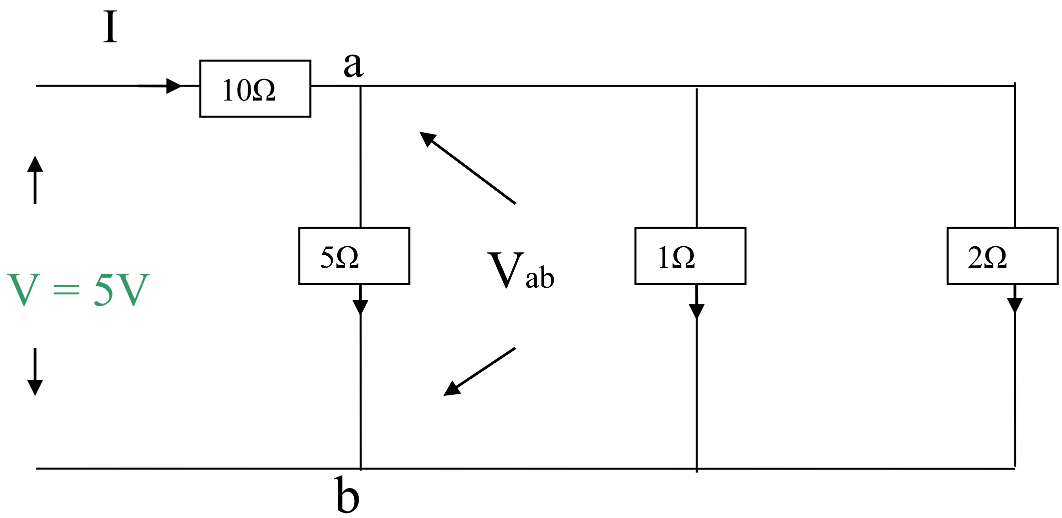
But: $v_2(t) = R_2\, i(t)$

Hence:

$$v_2(t) = v(t)\left(\frac{R_2}{R_1 + R_2}\right)$$

Example

Find the potential difference between points 'a' and 'b' in the following circuit:



Three of the resistors are in parallel which implies:

$$R_{Total} = \left(\frac{1}{5} + \frac{1}{1} + \frac{1}{2} \right)^{-1} = 0.59\ \Omega$$

These three resistors can then be treated as a single resistor in series with the 10Ω resistor which implies:

$$R_{Total} = 10 + 0.59 = 10.59\ \Omega$$

$$I = 5 / 10.59 = 0.47\ A$$

Hence:

V_{ab}= 0.47 * 0.59 = 0.27 V

Or:

Using the Potential Divider Rule

$$V_{ab} = 5\left(\frac{0.59}{10+0.59}\right)V = 0.27V$$

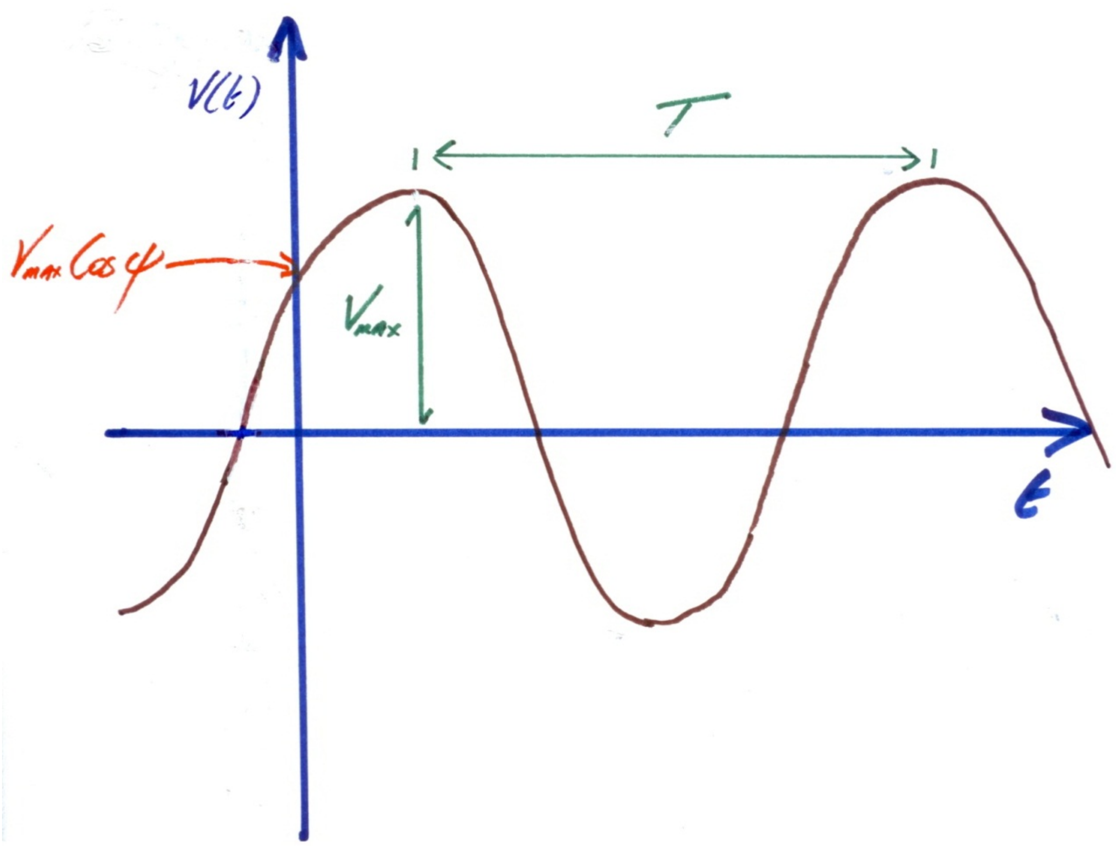
Let's say the source was sinusoidal – that is the p.d. between its terminals varies in a sinusoidal fashion:

Then the supply voltage is ac and is described by the general equation:

$$\boxed{v(t) = V_{\max} cos(\omega t + \varphi)}$$

- Vmax is the peak voltage
- ω is the frequency in radians per second (= 2πf = 2π/T).
- Ψ is the phase shift.

Diagram:



Let's say the peak voltage is 5V and the frequency is 50Hz. We assume the phase shift is zero unless otherwise specified.

Then:

$$v(t) = 5\cos(100\pi t)V$$

By repeating the above example with this supply we get:

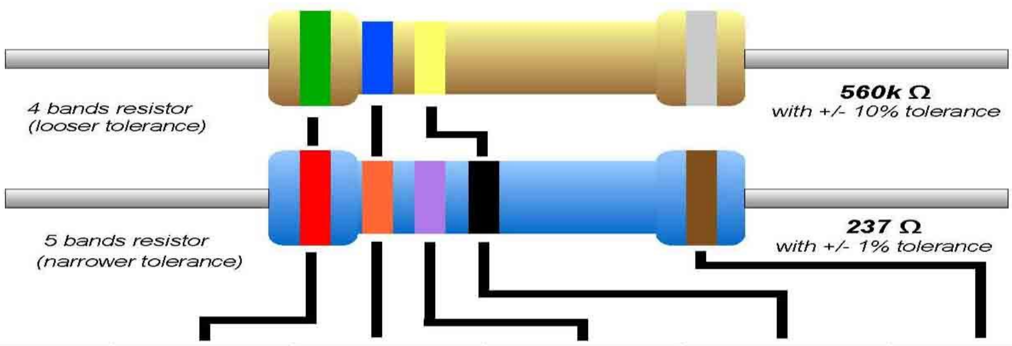
$$v_{ab}(t) = 0.27\cos(100\pi t)V$$

Reading Work: ‘Sinusoidal Alternating Waveforms’ - Introductory Circuit Analysis – Boylestad

Resistor Colour Codes:

Resistors do not have their numerical Ohmic value printed on them. Rather a colour code is used where each colour band corresponds to a numerical value. The colour code is given here:

Resistor Color Code

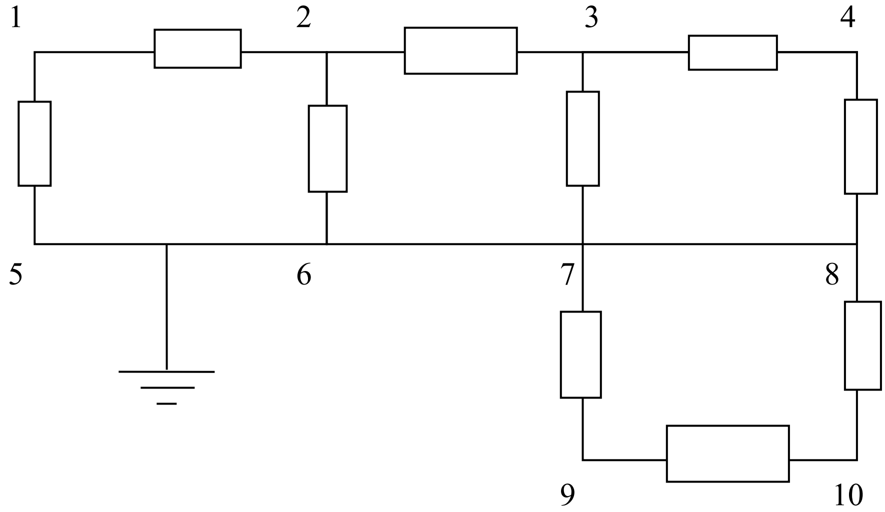


Color	1st Band	2nd Band	3rd Band	Multiplier	Tolerance
Black	0	0	0	x 1 Ω	
Brown	1	1	1	x 10 Ω	+/- 1%
Red	2	2	2	x 100 Ω	+/- 2%
Orange	3	3	3	x 1K Ω	
Yellow	4	4	4	x 10K Ω	
Green	5	5	5	x 100K Ω	+/- .5%
Blue	6	6	6	x 1M Ω	+/- .25%
Violet	7	7	7	x 10M Ω	+/- .1%
Grey	8	8	8		+/- .05%
White	9	9	9		
Gold				x .1 Ω	+/- 5%
Silver				x .01 Ω	+/- 10%

(v) Electric Circuit Analysis

Terminology

Consider the following circuit (or electrical n/w):



The boxes denotes circuit elements which may be sources or components.

Nodes

Points 1 though 10 are referred to as nodes.

Branches

Sections [1,2], [2,3],[3,7],etc are referred to as branches.

Loops

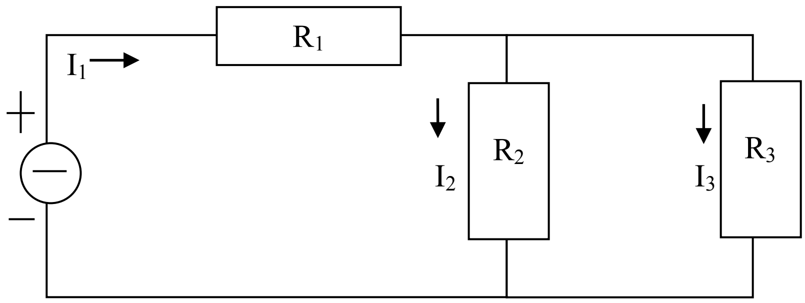
[1,2,6,5],[2,3,6,7],[1,4,8,5],etc are referred to as loops.

Kirchoffs Laws

Kirchoff's Current Law (KCL)

- states that the algebraic sum of electrical currents at any node in an electrical circuit is equal to zero at every instant in time.

In other words, the total current entering a node must equal the total current leaving the node. Current entering the node is given a positive sign and that leaving is given a negative sign such that the algebraic sum at any node equals zero.



$$I_1 = I_2 + I_3$$

i.e.

$$I_1 + (-I_2) + (-I_3) = 0$$

i.e.

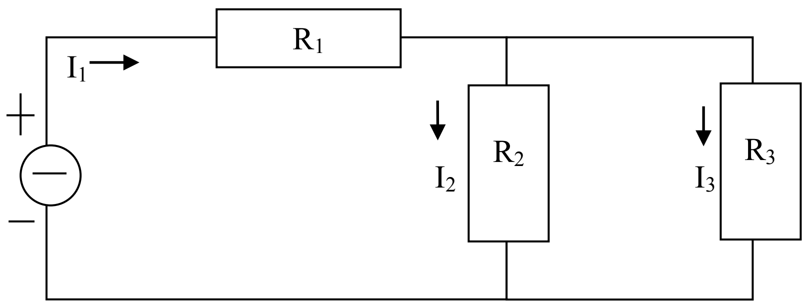
$$\boxed{\sum I_{Node} = 0}$$

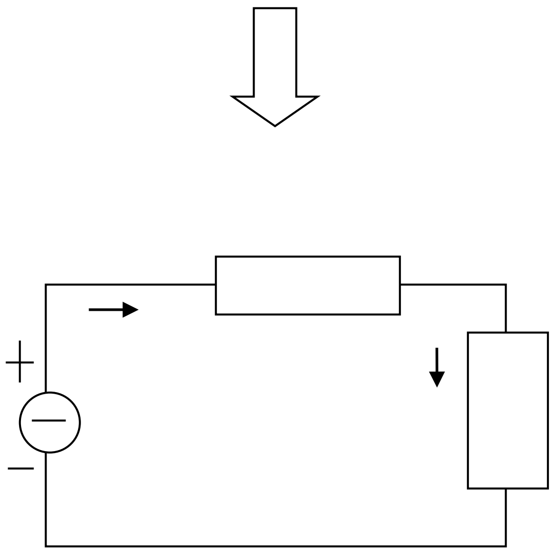
Kirchoff's Voltage Law (KVL)

- *states that the algebraic sum of branch voltages around any loop in an electrical circuit is equal to zero at every instant in time.*

i.e.

$$\sum_{loop} V_{branch} = 0$$





$$R_4 = R_2 \parallel R_3$$
$$E = V_1 + V_4$$

i.e.

$$E + (-V_1) + (-V_4) = 0$$

This is simply a statement of the principle of conservation of energy.

The Principle of Superposition

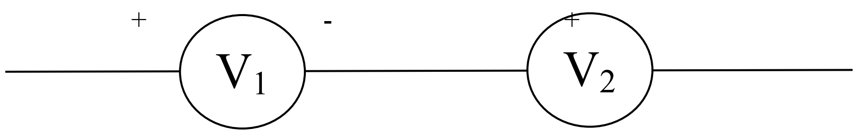
- states that *the current through (or voltage across) a circuit across a circuit element in a literal bilateral electrical n/w is equal to the algebraic sum of currents or voltages produced independently by each source*.

Terminology

<u>Linear</u> – ideal sources and circuit elements

Example:

Given sources:



$$V = I(R_1 + R_2 + \ldots + R_n)$$

⇨ linear circuit

Bilateral – no change in circuit behaviour if voltage or current is reversed. - e.g. a circuit containing diodes is not a bilateral circuit.

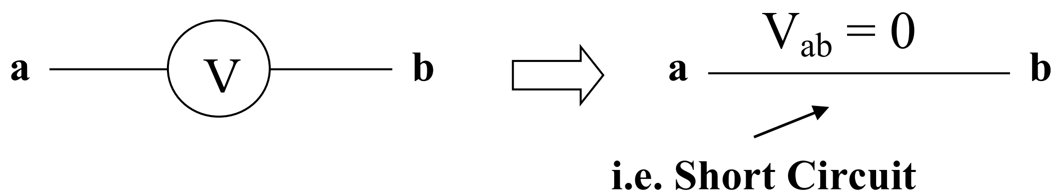
Summary

In short, if we are analysing a circuit with many sources (voltage and/or circuit), we can do this in a linear bilateral circuit by considering the effect produced independently by each source.

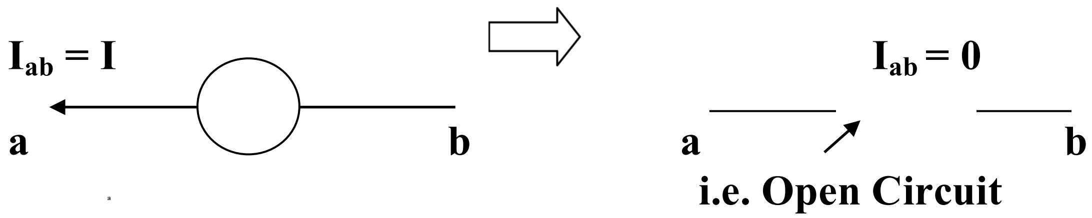
This is done by performing the analysis separately for each source setting the other sources to zero and then summing the results.

Setting Sources to Zero

Voltage Sources



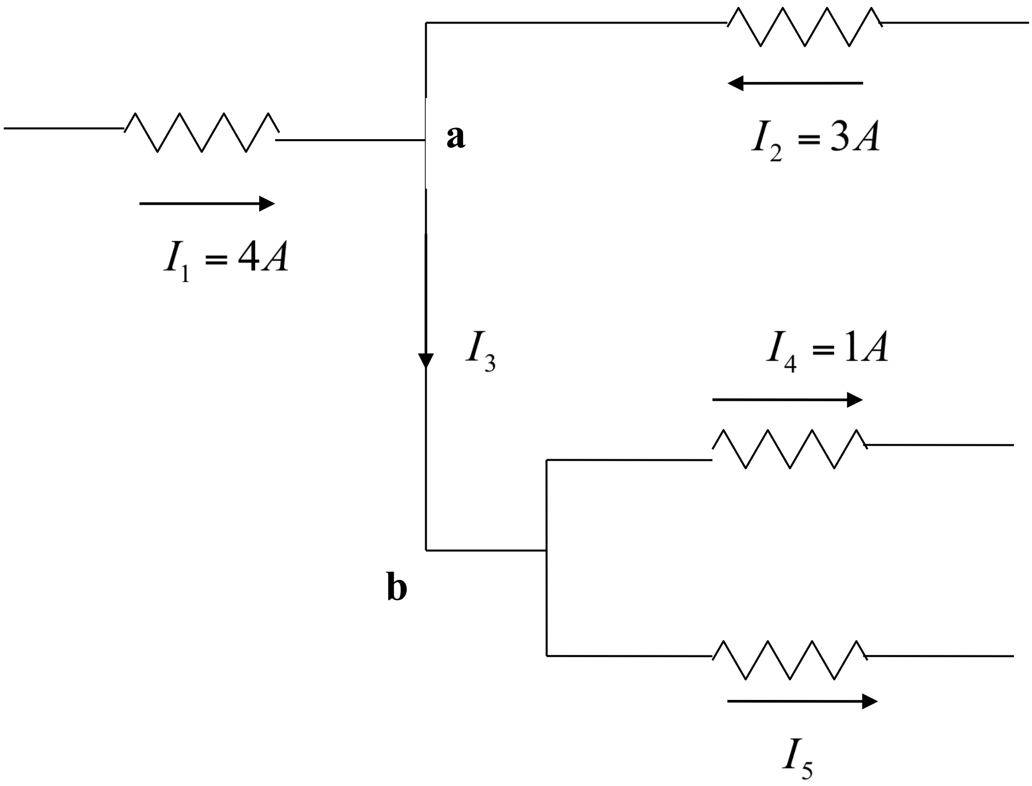
Current Sources



Example

KCL:

(Boylestad 9th Ed. Ch. 6, Example 6.15, p179.)



Determine the currents I_3 and I_5 in the above circuit using KCL.

At Node ‘a’:

$$I_1 + I_2 = I_3 \Rightarrow$$

$$I_3 = 4A + 3A = 7A$$

At Node ‘b’:

$$I_3 = I_4 + I_5 \Rightarrow$$

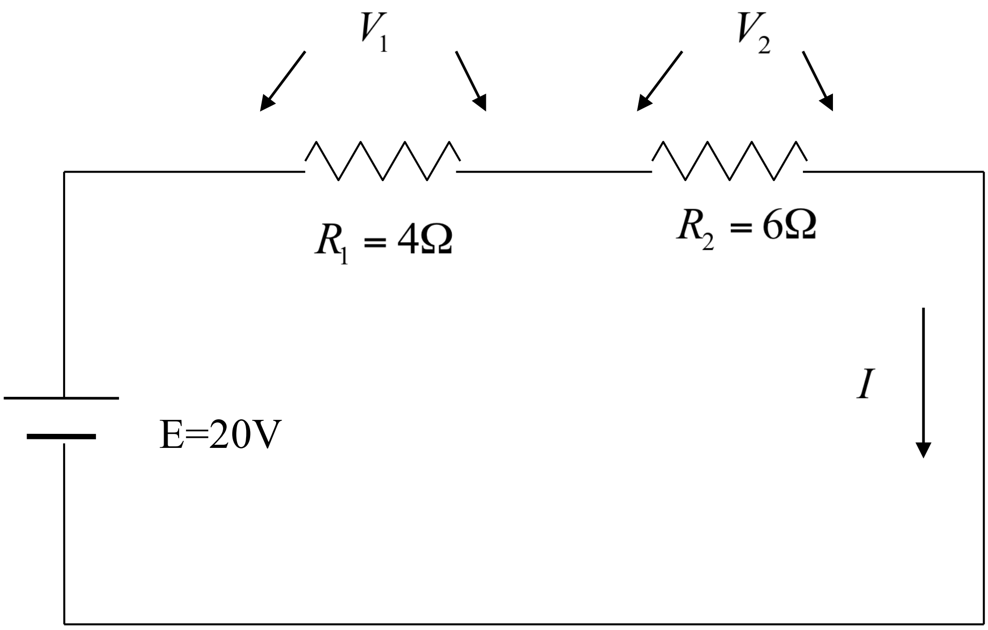
$$I_5 = 7A - 1A = 6A$$

- see related examples

Example

For the following circuit find:

1) R_T
2) I
3) V_1, V_2
4) The power dissipated by the 4Ω and 6Ω resistors.
5) The power delivered by the battery to the circuit.
6) Verify KVL for the circuit.



$$1)R_T = 4\Omega + 6\Omega = 10\Omega$$

$$2)I = \frac{E}{R_T} = 2A$$

$$3)V_1 = IR_1 = 8V, V_2 = IR_2 = 12V$$

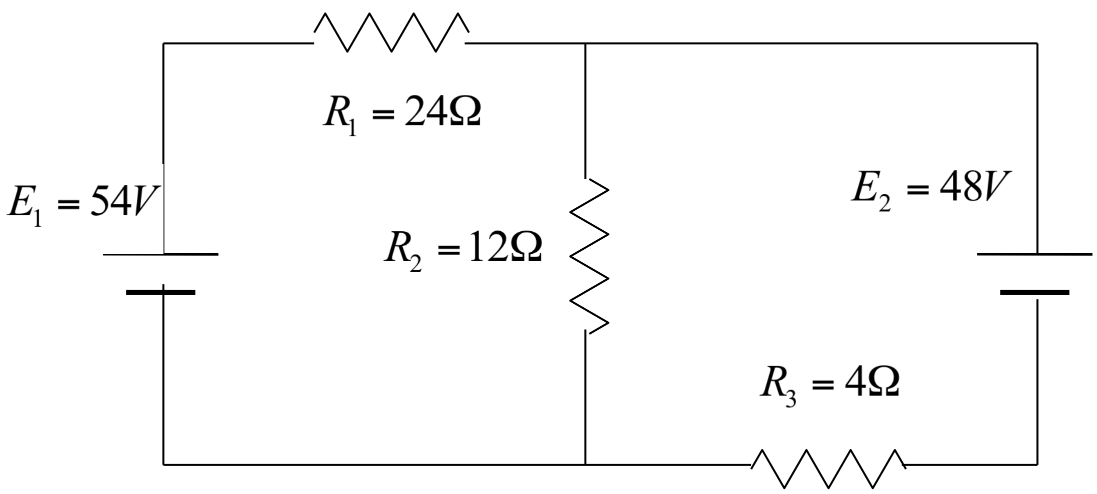
$$4)P_{4\Omega} = \frac{V_1^2}{R_1} = 16W, P_{6\Omega} = \frac{V_2^2}{R_2} = 24W$$

$$5)P_E = EI = 40W(= P_{4\Omega} + P_{6\Omega})$$

$$6)KVL \Rightarrow E - V_1 - V_2 = 0 \Rightarrow 20V = 8V + 12V - Check$$

Example:

Determine the current through the 4Ω resistor in the following circuit:



Shorting the $48V$ source gives:

$$R_{T(a)} = R_1 + R_2 \,||\, R_3 = 27\Omega \Rightarrow$$

$$I_{T(a)} = \frac{E_1}{R_{T(a)}} = 2A \Rightarrow$$

$$I_{R_3(a)} = \frac{E_1 - I_{T(a)}R_1}{R_3} = 1.5A$$

Note the following:

1) The above does not make use of the ‘Current Divider Rule’ – for which there is no need.
2) $I_{R_3(a)}$ flows from left to right

Shorting the 54V source gives:

$$R_{T(b)} = R_3 + R_1 \,||\, R_2 = 12\Omega \Rightarrow$$

$$I_{T(b)} = \frac{E_2}{R_{T(b)}} = 4A \Rightarrow$$

$$I_{3(b)} = -4A$$

Note:

$I_{R_3(b)}$ flows from right to left

Then:

$$I_{R_3} = I_{R_3(a)} + I_{R_3(b)} = 1.5A - 4A = -2.5A$$

Since we chose ‘left to right’ as meaning the ‘postive direction’ it follows that since I_{R_3} turns out negative that it flows from ‘right to left’. Alternatively we could write $I_{R_3} = 2.5A$ where we specify that I_{R_3} flows from ‘right to left’.

Exercise:

Suppose the sources in the above example were non-ideal, each with an internal resistance of 2Ω, what would the resulting current I_{R_3} be?

(vi) AC Power

Let P_{AC} be the <u>average</u> ac power over an interval.

Since $p(t) = v(t).i(t)$ then:

$$p_{AC} = \frac{1}{T}\int_0^T v(t)i(t)dt$$

Hence for resistors we use:

$$\boxed{p_{AC} = \frac{R}{T}\int_0^T i^2(t)dt}$$

or

$$\boxed{p_{AC} = \frac{1}{RT}\int_0^T v^2(t)dt}$$

Root Mean Squared (rms)Current and voltage

By definition, the root mean square (rms) value of a periodic (with period T) time variant function y(t) is:

$$Y_{rms} = \sqrt{\frac{1}{T}\int_0^T y^2(t)\,dt}$$

Hence:

$$\boxed{I_{rms} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\,dt}}$$

and

$$\boxed{V_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt}}$$

Hence:

$$\boxed{P_{ac} = RI_{rms}^2}$$

$$\boxed{P_{ac} = \frac{V_{rms}^2}{R}}$$

and

$$\boxed{P_{ac} = V_{rms} I_{rms}}$$

The root mean square value of an ac quantity gives the ‘effective’ dc value which would effect the same power.

Peak Value Vs RMS Value:

There is an important relationship between the peak values of sinusoidal currents and voltages and their rms values. To see this consider:

$$i(t) = I_{pk} Cos(\omega t + \phi) \Rightarrow$$

$$i^2(t) = I_{pk}^2 Cos^2(\omega t + \phi) \Rightarrow$$

$$i^2(t) = \frac{I_{pk}^2}{2}\left(1 + Cos(2\omega t + 2\phi)\right)$$

$$I_{rms} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\,dt} = \sqrt{\int_0^T \frac{I_{pk}^2}{2}\left(1 + Cos(2\omega t + 2\phi)\right)}$$

Hence:

$$\boxed{I_{rms} = \frac{I_{pk}}{\sqrt{2}}}$$

The same can be proven for V_{rms}. Hence:

$$V_{rms} = \frac{V_{pk}}{\sqrt{2}}$$

AC currents and voltages are often expressed in terms of their rms values. Take for example the mains voltage which is 230V. It is implicit that this is an rms value since we know the mains is not dc. The frequency of the mains is 50Hz. Hence the mathematical expression for the mains voltage is:

$$v_{mains}(t) = 230\sqrt{2}Cos(100\pi)V$$

So the peak voltage for the mains is about 325V.

Example:

A 120V dc source delivers 3.6W to a load. Determine the peak value of the applied current and voltage of an ac source delivering the same power to the load.

$$P = VI \Rightarrow$$

$$I = \frac{P}{V} = \frac{3.6}{120} = 30mA = I_{rms} \Rightarrow$$

$$I_{pk} = \sqrt{2}I = 42.42mA$$

$$V_{pk} = \sqrt{2}V = 169.68V$$

Question: Is this answer unique? Why?

- see related examples