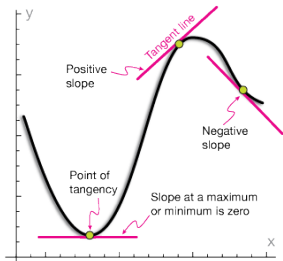


Differentiation and Integration

Differentiation and Integration

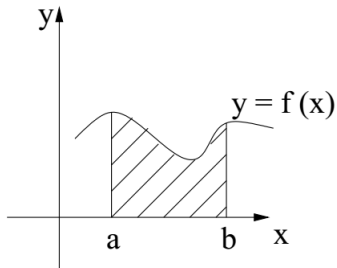
Geometric View

Differentiation



Slope of curve at point p
= slope of tangent at point p

Integration



Area under the curve between (a, b)
$$= \int_a^b f(x) dx$$

Derivative Notation

Notation

Let $y = f(x)$ then

$$f'(x) = \frac{dy}{dx}$$

i.e.

$$f'(x) = \frac{d(f(x))}{dx}$$

e.g.

let $f(x) = \sin x$ then

$$f'(x) = \frac{d(\sin x)}{dx} = \cos x$$

Slope at point (x, y) .

With $y = f(x)$ then

$f'(x)$ is the slope of the curve of f at the point (x, y)

Anti-Derivative = Integration

Anti-Derivative = Integration

Show $f(x) = \int f'(x) dx$ i.e. $f(x) = \int \frac{d}{dx}(f(x)) dx$

Let $y = f(x)$ then

$$\frac{dy}{dx} = f'(x) \therefore$$

$$dy = f'(x) dx \therefore$$

$$\int dy = \int f'(x) dx \therefore$$

$$y = \int f'(x) dx$$

$$f(x) = \int f'(x) dx$$

Example

$$\frac{d(\sin x)}{dx} = \cos x \therefore$$

$$d(\sin x) = (\cos x) dx \therefore$$

$$\int d(\sin x) = \int (\cos x) dx$$

$$\sin x = \int (\cos x) dx \text{ i.e.}$$

$$\text{if } \frac{d(\sin x)}{dx} = \cos x \text{ then } \sin x = \int (\cos x) dx$$

Integration reverses the action of differentiation i.e.

- Integrating the entry in the left column gives the entry in the right column and
- differentiating the entry in the right column gives the entry in the left column.

$f(x)$	$\xrightarrow{\text{integrate}} \xleftarrow{\text{differentiate}}$	$\int f(x)dx$
$x^n, n \neq -1$		$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$ i.e. x^{-1}		$\ln x$
$\cos x$		$\sin x$
$\sin x$		$-\cos x$
$\tan x$		$\ln(\sec x)$
$\frac{1}{1+x^2}$		$\tan^{-1}x$

Note: $y = \tan^{-1} x$ iff $\tan y = x$

also, $\ln x = \log_e x$, the natural logarithm

Definite and Indefinite Integration

Indefinite Integral

An indefinite integral, $\int f(x)dx$ is an anti derivate and results in a function.

e.g $\int (\cos x)dx = \sin x$

- **Constant of Integration**

Since the derivative of a constant, C , is 0, the anti derivative can involve a constant so that:

$$\int (\cos x)dx = \sin x + C \quad \text{as} \quad \frac{d(\sin x + C)}{dx} = \cos x$$

Definite Integral

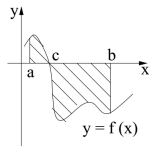
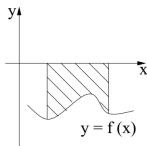
Definite Integral

The Definite Integral is motivated from the 'area under a curve' view of integration.

$\int_a^b f(x)dx$ is a number which can correspond to the 'area under a curve'.

- 'Negative Area'

A negative area corresponds to area below the x - axis



$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Definite Integral (Cont'd)

Given $F(x) = \int f(x)dx$ i.e. $\frac{d(F(x))}{dx} = f(x)$ i.e. $F'(x) = f(x)$
we evaluate the definite integral

$$\begin{aligned}\int_a^b f(x)dx &= F(x)|_a^b \\ &= F(b) - F(a)\end{aligned}$$

Examples

- $\int_1^2 x \, dx$

$$\begin{aligned}\int_1^2 x \, dx &= \left. \frac{x^2}{2} \right|_1^2 \\ &= \frac{2^2}{2} - \frac{1^2}{2} \\ &= \frac{3}{2}\end{aligned}$$

$$\int_1^2 x \, dx$$

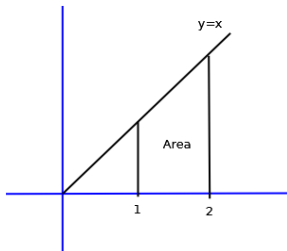
Area

$$= (\text{Area of } \triangle \text{ from 0 to 2}) - (\text{Area of } \triangle \text{ from 0 to 1})$$

$$= \left(\frac{1}{2} * 2\right) * 2 - \left(\frac{1}{2} * 1\right) * 1$$

$$= 2 - \frac{1}{2}$$

$$= \frac{3}{2}$$



- $\int_1^2 \frac{1}{x} dx$

$$\begin{aligned}\int_1^2 \frac{1}{x} dx &= \ln x \Big|_1^2 \\ &= \ln 2 - \ln 1 \\ &= \ln 2 \text{ as } \ln 1 = 0\end{aligned}$$

- $\int_1^2 \frac{1}{x^2} dx$

$$\begin{aligned}\int \frac{1}{x^2} dx &= \int x^{-2} dx \\ &= \frac{x^{-2+1}}{-2+1} \\ &= \frac{x^{-1}}{-1} \\ &= \frac{-1}{x}\end{aligned}$$

$$\begin{aligned}\int_1^2 \frac{1}{x^2} dx &= \left. \frac{-1}{x} \right|_1^2 \\ &= \frac{-1}{2} - \frac{-1}{1} \\ &= \frac{-1}{2} + 1 \\ &= \frac{1}{2}\end{aligned}$$

Some Properties of Integration

- Integral of a sum = sum of the integrals

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$$

- Factor a constant outside the integral; let c be a constant

$$\int (c * f(x))dx = c * \int f(x)dx$$

Example

- $\int \left(3x^{\frac{1}{2}} - \frac{2}{x} + 4\sin x \right) dx$

$$\int \left(3x^{\frac{1}{2}} - \frac{2}{x} + 4\sin x \right) dx$$

$$= \int 3x^{\frac{1}{2}} dx - \int \frac{2}{x} dx + \int 4\sin x dx$$

$$= 3 \int x^{\frac{1}{2}} dx - 2 \int \frac{1}{x} dx + 4 \int \sin x dx$$

$$= 3 \left(\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right) - 2 \ln x + 4(-\cos x)$$

$$= 2x^{\frac{3}{2}} - 2 \ln x - 4 \cos x$$

Integration by Substitution

Consider differentiating e^{x^3} : i.e. find $\frac{dy}{dx}$ when

$$y = e^{x^3}$$

Use Chain Rule: Let $u = x^3$ then

$$\left. \begin{array}{l} y = e^u \\ \frac{dy}{du} = e^u \end{array} \right| \quad \left| \begin{array}{l} u = x^3 \\ \frac{du}{dx} = 3x^2 \end{array} \right.$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} * \frac{du}{dx} \\ &= e^u * 3x^2 \\ &= 3x^2 e^{x^3} \end{aligned}$$

Since integration is anti-derivative

$$\int 3x^2 e^{x^3} dx = e^{x^3}$$

This is the basis of the Integration technique:

Integration by Substitution

Integration by Substitution

Consider the problem of evaluating

$$\int 3x^2 e^{x^3} dx$$

We use a **Substitution**, we use the substitution, $u = x^3$.

Since $u = x^3$, $\frac{du}{dx} = 3x^2$ i.e. using differentials, $du = 3x^2 dx$

We can rewrite $3x^2 e^{x^3}$ as $e^{x^3} 3x^2$

Using the substitution

$$\begin{aligned}\int e^{x^3} 3x^2 dx &= \int e^u du \\ &= e^u \\ &= e^{x^3}\end{aligned}$$

i.e.

$$\int 3x^2 e^{x^3} dx = e^{x^3}$$

Integrals of the form: $g(f(x))f'(x)$

Integrals of the form: $g(f(x))f'(x)$

$e^{x^3} 3x^2$ is of the form $g(f(x))f'(x)$ (a constant item may be involved) where f' is the derivative of f and $g(u) = e^u$.

Since multiplication is commutative, $g(f(x))f'(x) = f'(x)g(f(x))$.

Evaluate $\int g(f(x))f'(x)dx$

Let $u = f(x)$ then $du = f'(x)dx$

$$\int g(f(x))f'(x)dx = \int g(u)du$$

Evaluating $\int g(f(x))f'(x)dx$ has been reduced to evaluating $\int g(u)du$.

- $\int x^3 \sqrt{16 + x^4} \, dx$

Consider evaluating $\int x^3 \sqrt{16 + x^4} \, dx$

$x^3 \sqrt{16 + x^4}$ is of form $f'(x)g(f(x))$ where $g(u) = \sqrt{u}$.

$f(x) = 16 + x^4 \therefore f'(x) = 4x^3$ (a constant term is involved)

Let $u = 16 + x^4 \therefore \frac{du}{dx} = 4x^3 \therefore du = 4x^3 dx$ i.e. $x^3 dx = \frac{1}{4} du$

Example

$$\int x^3 \sqrt{16 + x^4} \, dx$$

Let $u = 16 + x^4 \therefore \frac{du}{dx} = 4x^3 \therefore du = 4x^3 dx$ i.e. $x^3 dx = \frac{1}{4} du$

$$\begin{aligned} \int x^3 \sqrt{16 + x^4} \, dx &= \frac{1}{4} \int \sqrt{u} \, du \\ &= \frac{1}{4} \int u^{\frac{1}{2}} \, du \\ &= \frac{1}{4} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C \\ &= \frac{1}{6} (16 + x^4)^{\frac{3}{2}} + C \end{aligned}$$

Note: Differentiating $\frac{1}{6}(16 + x^4)^{\frac{3}{2}} + C$ returns $x^3 \sqrt{16 + x^4}$

Consider integrals of the form:

- $\int \frac{f'(x)}{f(x)} dx$ i.e. of the form $g(f(x))f'(x)$ where $g(u) = \frac{1}{u}$

Example

$$\int \frac{x^4}{x^5+2} dx$$

let $u = x^5 + 2$, then $\frac{du}{dx} = 5x^4 \therefore \frac{du}{5} = x^4 dx$

$$\begin{aligned} \int \frac{x^4}{x^5+2} dx &= \frac{1}{5} \int \frac{1}{u} du \\ &= \frac{1}{5} (\ln u) + C \\ &= \frac{1}{5} \ln(x^5 + 2) + C \end{aligned}$$

Example

$$\int x^3(x^4 + 4)^5 dx$$

let $u = x^4 + 4$, then $\frac{du}{dx} = 4x^3 \therefore \frac{du}{4} = x^3 dx$

$$\begin{aligned}\int x^3(x^4 + 4)^5 dx &= \int (x^4 + 4)^5 x^3 dx \\ &= \frac{1}{4} \int u^5 du \\ &= \frac{1}{4} \frac{u^6}{6} + C \\ &= \frac{(x^4 + 4)^6}{24} + C\end{aligned}$$

Trigonometry Functions

$f'(x)$	$\xleftarrow{\text{Differentiate}}$ $\xrightarrow{\text{Integrate}}$	$f(x)$
$\cos x$		$\sin x$
$\sin x$		$-\cos x$
$\tan x$		$\ln(\sec x) = -\ln(\cos x)$
		Note: $\sec x = \frac{1}{\cos x}$

Example

$$\int \sin 6x \, dx$$

$$\text{Let } u = 6x, \, du = 6dx$$

$$\begin{aligned}\int \sin 6x \, dx &= \frac{1}{6} \int \sin u \, dx \\ &= \frac{1}{6} * (-\cos u) \\ &= -\frac{1}{6} \cos 6x\end{aligned}$$

- $\int \frac{\sin x}{2 + \cos x} dx$

This is of the form $g(f(x))f'(x)$ where $g(u) = \frac{1}{u}$ and $f(x) = 2 + \cos x$

Let $u = 2 + \cos x$, then $\frac{du}{dx} = -\sin x$, $\therefore du = -\sin x dx$

$$\begin{aligned}\int \frac{\sin x}{2 + \cos x} dx &= \int \frac{\sin x dx}{u} \\ &= -\int \frac{du}{u} \\ &= -\ln u + C \\ &= -\ln(2 + \cos x) + C\end{aligned}$$

- $\int (\sin x)(\cos^2 x) dx$

Let $u = \cos x \therefore du = -\sin x dx$

$$\begin{aligned}\int (\sin x)(\cos^2 x) dx &= \int (\cos^2 x) \sin x dx \\ &= - \int u^2 du \\ &= -\frac{u^3}{3} + C \\ &= -\frac{1}{3} \cos^3 x + C\end{aligned}$$

Prove that $\int \tan x \, dx = -\ln(\cos x) + C$

- **Method 1.** Differentiate Right Hand Side.

Let $y = -\ln(\cos x)$.

Substitution: let $u = \cos x$ then $y = -\ln u \therefore$

$$\frac{dy}{du} = -\frac{1}{u} \text{ and } \frac{du}{dx} = -\sin x$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\ &= \left(-\frac{1}{u}\right) (-\sin x) \\ &= \frac{\sin x}{\cos x} \\ &= \tan x\end{aligned}$$

- **Method 2.** Evaluate $\int \tan x \, dx$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\text{Let } u = \cos x \therefore du = -\sin x \, dx$$

$$\begin{aligned}\int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx \\&= - \int \frac{1}{\cos x} (-\sin x) \, dx \\&= - \int \frac{1}{u} \, du \\&= -\ln u + C \\&= -\ln(\cos x) + C\end{aligned}$$

Note: $\ln\left(\frac{1}{x}\right) = \ln(x^{-1}) = -\ln x \therefore -\ln(\cos x) = \ln\left(\frac{1}{\cos x}\right) \therefore$
 $\int \tan x \, dx = \ln\left(\frac{1}{\cos x}\right) = \ln(\sec x)$ as $\sec x = \frac{1}{\cos x}$.

$\int \sin^3 x \, dx$

- $\int \sin^3 x \, dx$

$$\sin^3 x = (\sin^2 x)(\sin x) = (1 - \cos^2 x)(\sin x) = \sin x - (\cos^2 x)(\sin x)$$

$$\begin{aligned}\int \sin^3 x \, dx &= \int (\sin x - (\cos^2 x)(\sin x)) \, dx \\&= \int \sin x \, dx - \int (\cos^2 x)(\sin x) \, dx \\&= -\cos x - \int (\cos^2 x)(\sin x) \, dx \\&\quad \{ \text{previous example} \} \\&= -\cos x + \frac{\cos^3 x}{3} + C\end{aligned}$$

- $\int \sin^2 x \, dx$

From Trigonometry: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, also
 $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$$\begin{aligned}\int \sin^2 x \, dx &= \int \frac{1}{2}(1 - \cos 2x) dx \\&= \frac{1}{2} \int dx - \frac{1}{2} \int \cos 2x \, dx \\&= \frac{x}{2} - \frac{\sin 2x}{4} + C\end{aligned}$$

Not of form $g(f(x))f'(x)$

$$\int \frac{x}{\sqrt{2x+2}} dx$$

This is not of the form $g(f(x))f'(x)$

To evaluate this integral:

let $u = 2x + 2$ then $du = 2dx$

$$\int \frac{x}{\sqrt{2x+2}} dx = \frac{1}{2} \int \frac{x}{\sqrt{u}} du$$

Since $u = 2x + 2 \therefore x = \frac{u-2}{2} \therefore$

$$\begin{aligned} \frac{1}{2} \int \frac{x}{\sqrt{u}} du &= \frac{1}{4} \int \frac{u-2}{\sqrt{u}} du \\ &= \frac{1}{4} \int \left(\sqrt{u} - \frac{2}{\sqrt{u}} \right) du \\ &= \frac{1}{4} \int \left(u^{\frac{1}{2}} - 2u^{-\frac{1}{2}} \right) du \end{aligned}$$

$$\begin{aligned}\frac{1}{4} \int (u^{\frac{1}{2}} - 2u^{-\frac{1}{2}}) du &= \frac{1}{4} \left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}} - 2 \left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right) \right) \\ &= \frac{1}{6} u^{\frac{3}{2}} - u^{\frac{1}{2}} \\ &= \frac{1}{6} (2x + 2)^{\frac{3}{2}} - (2x + 2)^{\frac{1}{2}} + C\end{aligned}$$