Matrix Inverse by Determinants

Properties of Matrix Inverse

- $(A^{-1})^{-1} = A$
- $(A * B)^{-1} = B^{-1} * A^{-1}$ as $(A * B) * (B^{-1} * A^{-1)} = A * (B * B^{-1}) * A^{-1} = A * Id * A^{-1} = Id$.
- $(A^T)^{-1} = (A^{-1})^T$ where M^T is the Transpose of M.
- $|A^{-1}| = \frac{1}{|A|}$ where $|M|$ is the determinant of M.

Inverse of 2×2 Matrices by Determinants

Recall that the inverse of a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

can be found by:

$$A^{-1} = \frac{1}{|A|} * \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Minor and Cofactor

Minor M_{ij}

Recall that the minor M_{ij} of an entry a_{ij} in a matrix , $A = [a_{ij}]$, is the **determinant** of the submatrix of A formed by deleting row i and column j of the matrix A.

Cofactor C_{ij}

When M_{ij} is a minor of of an entry a_{ij}, then $(-1)^{i+j} M_{ij}$, which can be denoted by C_{ij}, is the **cofactor** of the entry a_{ij} i.e.

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Example: The determinant of a matrix, $A = [a_{ij}]_{n \times n}$ is (expanding along row 1)

$$|A| = a_{11} * C_{11} + a_{12} * C_{12} + .. + a_{1n} * C_{1n}$$

Matrix of Cofactors

Let $A = [a_{ij}]$ then the **matrix of cofactors** is $[C_{ij}]$, where C_{ij} is the cofactor of a_{ij}.
Each cofactor, C_{ij} involves calculating the determinant M_{ij} (the minor of the entry a_{ij}) and also $(-1)^{i+j}$ as $C_{ij} = (-1)^{i+j} M_{ij}$.
For a 3×3 matrix we get the pattern for $(-1)^{i+j}$:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Matrix Inverse in terms of Cofactors

Let $A = [a_{ij}]$ and let C_{ij} be the cofactor of a_{ij} then

$$A^{-1} = \frac{1}{|A|} * [C_{ij}]^T$$

- Calculate $|A|$
- Calculate the matrix of cofactors $[C_{ij}]$
- Calculate the transpose, $[C_{ij}]^T$.

Then $A^{-1} = \frac{1}{|A|} * [C_{ij}]^T$

Example

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & -3 \\ 4 & -2 & 0 \end{bmatrix}$

Find A^{-1} by the Matrix of Cofactors method.

- Calculate $|A|$

$|A|$
$= 1 * C_{11} + 2 * C_{12} + (-1) * C_{13}$

$C_{11} = (-1)^{1+1} * \begin{vmatrix} 5 & -3 \\ -2 & 0 \end{vmatrix}$ $\quad C_{12} = (-1)^{1+2} * \begin{vmatrix} 0 & -3 \\ 4 & 0 \end{vmatrix}$ $\quad C_{13} = (-1)^{1+3} * \begin{vmatrix} 0 & 5 \\ 4 & -2 \end{vmatrix}$

$\therefore$

$C_{11} = \begin{vmatrix} 5 & -3 \\ -2 & 0 \end{vmatrix}$ $\quad C_{12} = -\begin{vmatrix} 0 & -3 \\ 4 & 0 \end{vmatrix}$ $\quad C_{13} = \begin{vmatrix} 0 & 5 \\ 4 & -2 \end{vmatrix}$

i.e.

$$C_{11} = -6 \quad C_{12} = -12 \quad C_{13} = -20$$

$$\begin{aligned} &|A| \\ &= 1 * C_{11} + 2 * C_{12} + (-1) * C_{13} \\ &= 1 * (-6) + 2 * (-12) + (-1) * (-20) \\ &= -6 - 24 + 20 \\ &= -10 \end{aligned}$$

To get A^{-1} we also need to calculate $[C_{ij}]$ and transpose, $[C_{ij}]^T$.

Matrix of Cofactors $[C_{ij}]$

$$C_{11} = \begin{vmatrix} 5 & -3 \\ -2 & 0 \end{vmatrix} \quad C_{12} = -\begin{vmatrix} 0 & -3 \\ 4 & 0 \end{vmatrix} \quad C_{13} = \begin{vmatrix} 0 & 5 \\ 4 & -2 \end{vmatrix}$$

i.e. $C_{11} = -6$ $\quad C_{12} = -12$ $\quad C_{13} = -20$

$$C_{21} = -\begin{vmatrix} 2 & -1 \\ -2 & 0 \end{vmatrix} \quad C_{22} = \begin{vmatrix} 1 & -1 \\ 4 & 0 \end{vmatrix} \quad C_{23} = -\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix}$$

i.e. $C_{21} = 2$ $\quad C_{22} = 4$ $\quad C_{23} = 10$

Cont'd

$$C_{31} = \begin{vmatrix} 2 & -1 \\ 5 & -3 \end{vmatrix} \qquad C_{32} = -\begin{vmatrix} 1 & -1 \\ 0 & -3 \end{vmatrix} \qquad C_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix}$$

i.e. $\quad C_{31} = -1 \qquad C_{32} = 3 \qquad C_{33} = 5$

$$[C_{ij}] = \begin{bmatrix} -6 & -12 & -20 \\ 2 & 4 & 10 \\ -1 & 3 & 5 \end{bmatrix} \text{ and } [C_{ij}]^T = \begin{bmatrix} -6 & 2 & -1 \\ -12 & 4 & 3 \\ -20 & 10 & 5 \end{bmatrix}$$

Matrix Inverse

$$
\begin{aligned}
A^{-1} &= \frac{1}{|A|} * [C_{ij}]^T \\
&= \frac{1}{-10} * \begin{bmatrix} -6 & 2 & -1 \\ -12 & 4 & 3 \\ -20 & 10 & 5 \end{bmatrix} \\
&= \frac{1}{10} * \begin{bmatrix} 6 & -2 & 1 \\ 12 & -4 & -3 \\ 20 & -10 & -5 \end{bmatrix}
\end{aligned}
$$