KE Deduction (KE Tableaux)

Determining that an argument is valid by KE Deduction is based on Refutation. An argument is valid if it is not possible for the premises to be True and the conclusion False. A proof by refutation attempts to find a way to make the premises True and the conclusion False; the argument is valid if the attempt leads to a contradiction. If it is possible to find a state where the premises are True and the conclusion False then the argument is not valid.

A propositional expression is a Tautology if its negation is a Contradiction. A propositional expression is said to be Contingent if it is neither a Tautology nor a Contradiction, i.e. for some states the expression is True and for some other states the expression is False. For example, $p \vee \neg p$ is a Tautology, $p \wedge \neg p$ is a Contradiction and $p \vee q$ is Contingent.

A KE Deduction Rule

One of the KE Deduction Rules is:

$$\begin{array}{ll} \text{From} & P \rightarrow Q \\ & P \\ \hline \text{Conclude} & Q \end{array}$$

i.e. $P \rightarrow Q, P \vdash Q$

Given premises $P \rightarrow Q$ and P, we can conclude Q. This KE Deduction Rule has the classical name 'modus ponens' or in modern times, 'the detachment rule'.

This rule can be shown by Truth Table:

To show $P \rightarrow Q, P \models Q$, we show $\models (P \rightarrow Q) \rightarrow (P \rightarrow Q)$.

Since $\models S \rightarrow S$ then $\models (P \rightarrow Q) \rightarrow (P \rightarrow Q)$.

$P \rightarrow Q, P \models Q$ can also be rewritten as $\models (P \rightarrow Q) \wedge P \rightarrow Q$. By Truth Table it can be shown that $\models (P \rightarrow Q) \wedge P \rightarrow Q$.

Example

Consider whether $(P \to Q) \to R \vdash (\neg R \to P)$. This can be shown by an 8 row Truth Table but consider a proof by refutation using KE Deduction. We make use of the following KE Deduction Rules:

$$\frac{\neg(P \to Q)}{\begin{array}{c} P \\ \neg Q \end{array}} \qquad \frac{\begin{array}{c} P \to Q \\ \neg Q \end{array}}{\neg P}$$

From $\dfrac{\neg(P \to Q)}{\begin{array}{c} P \\ \neg Q \end{array}}$ we also have $\dfrac{\neg(P \to Q)}{P}$ and $\dfrac{\neg(P \to Q)}{\neg Q}$

Validity of Rule

To show a rule such as $\dfrac{\neg(P \to Q) \quad P}{\neg Q}$ is logically valid, we show $\models \neg(P \to Q) \to P \wedge \neg Q$. This can be done by Truth Table.

P	Q	$\neg(P \to Q) \to P \wedge \neg Q$
F	F	T
F	T	T
T	F	T
T	T	T

$(P \to Q) \to R \vdash \neg R \to P$

Is it possible for the premise $(P \to Q) \to R$ to be true and the conclusion, $\neg R \to P$ false i.e. $\neg(\neg R \to P)$. We try to show a contradiction.

1	$(P \to Q) \to R$	
2	$\neg(\neg R \to P)$	Negated Conclusion
3	$\neg R$	from 2
4	$\neg P$	from 2
5	$\neg(P \to Q)$	from (1,3)
6	P	from 5
7	*Contradiction*	$\times$(4,6)

Comment on Proof

From line #2 $\neg(\neg R \to P)$, we can conclude both #3 $\neg R$ and #4 $\neg P$. From lines #1 $(P \to Q) \to R$ and #3 $\neg R$ we conclude #5 $\neg(P \to Q)$. From #5 we conclude #6 P. We get a contradiction from #4 and #6.
Given the premise $(P \to Q) \to R$ and assuming $\neg(\neg R \to P)$ we get a contradiction and therefore

$$(P \to Q) \to R \models \neg R \to P$$

KE Deduction Rules

For convenience and reference, the KE Deduction rules are divided into categories.

α (alpha) Rules

$$\frac{P \wedge Q}{\begin{array}{c} P \\ Q \end{array}} \qquad \frac{\neg(P \vee Q)}{\begin{array}{c} \neg P \\ \neg Q \end{array}} \qquad \frac{\neg(P \rightarrow Q)}{\begin{array}{c} P \\ \neg Q \end{array}} \qquad \frac{\neg\neg P}{P}$$

KE Deduction Rules (Cont'd)

β (beta) Rules

$$\frac{\begin{array}{c} P \vee Q \\ \neg P \end{array}}{Q} \qquad \frac{\begin{array}{c} \neg(P \wedge Q) \\ P \end{array}}{\neg Q} \qquad \frac{\begin{array}{c} P \rightarrow Q \\ P \end{array}}{Q} \qquad \frac{\begin{array}{c} P \rightarrow Q \\ \neg Q \end{array}}{\neg P}$$

Branching Rule, B :

$$\begin{array}{ccc} & \wedge & \\ \mathrm{P} & & \neg\mathrm{P} \end{array}$$

Previously, we showed the following by Truth Table and also by refutation.

$$\frac{s \vee b,\ \ b \rightarrow \neg d,\ \ c \vee d}{c \vee s}$$

i.e.

$$s \vee b,\ \ b \rightarrow \neg d,\ \ c \vee d \models c \vee s$$

We can now prove it via KE Deduction.

$S \lor B, B \rightarrow \neg D, C \lor D \vdash C \lor S$

1	$S \lor B$	
2	$B \rightarrow \neg D$	
3	$C \lor D$	
4	$\neg(C \lor S)$	Negated Conclusion
5	$\neg C$	$\alpha(4)$
6	$\neg S$	$\alpha(4)$
7	B	$\beta(1,6)$
8	$\neg D$	$\beta(2,7)$
9	D	$\beta(3,5)$
10	*Contradiction*	$\times(8,9)$

Superman Argument

Superman Argument

Consider the following argument that determines whether Superman exists.

If Superman was able and willing to prevent evil, he would do so.
If Superman was unable to prevent evil, he would be powerless.
Superman does not prevent evil and
if Superman was unwilling to prevent evil, he would be malevolent.
If Superman exists then he is neither powerless nor malevolent.

Therefore

Superman does not exist.

Abbreviating the Propositions

We can abbreviate the various propositions used in the argument. Let

A: Superman is able to prevent evil
W: Superman is willing to prevent evil
P: Superman is powerless
M: Superman is malevolent
E: Superman prevents evil
S: Superman exists

Abbreviating the Argument

Using the proposition abbreviations, we abbreviate the premises:

- If Superman was able and willing to prevent evil, he would do so.

$$P_1 : \quad A \wedge W \rightarrow E$$

- If Superman was unable to prevent evil, he would be powerless

$$P_2 : \quad \neg A \rightarrow P$$

- Superman does not prevent evil and if Superman was unwilling to prevent evil, he would be malevolent.

$$P_3 : \ \neg E \wedge (\neg W \rightarrow M)$$

- If Superman exists then he is neither powerless nor malevolent.

$$P_4 : \ S \rightarrow \neg P \wedge \neg M$$

We abbreviate the conclusion :

- Superman does not exist.

$$\neg S$$

Argument abbreviated

Show

$$\vdash P_1 \wedge P_2 \wedge P_3 \wedge P_4 \rightarrow \neg S$$

The 5 premises P_1, P_2, P_3, P_4 and the conclusion $\neg S$ involve 6 basic propositions, A, W, P, M, E, S and the validity of the argument can be shown by Truth Table using the 6 basic propositions but this would need $2^6 (= 64)$ rows in the Truth Table. It is more efficient and instructive to prove it using KE Deduction.

Superman Proof

$A \wedge W \rightarrow E \ldots 1$

$\neg A \rightarrow P \ldots 2$

$\neg E \wedge (\neg W \rightarrow M) \ldots 3$

$S \rightarrow \neg P \wedge \neg M \ldots 4$

{ Negate Conclusion: $\neg S$ }

$S \ldots 5$

|

$\beta(4{,}5)$

$\neg P \wedge \neg M \ldots 6$

$\alpha(6)$

$\neg P \ldots 7$

$\neg M \ldots 8$

$\alpha(3)$

$\neg E \ldots 9$

$\neg W \rightarrow M \ldots 10$

«Continued»

Superman Proof (Cont'd)

β(9,1)

¬(A∧W). . .11

β(7,2)

A. . .12

β(12,11)

$\neg W \ldots 13$

β(13,10)

M. . .14

$\times$(14,8)

Solution LogicPalet

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1	$A \bigwedge W \to E$	Hypothesis
2	$\neg A \to P$	Hypothesis
3	$\neg E \bigwedge (\neg W \to M)$	Hypothesis
4	$S \to \neg P \bigwedge \neg M$	Hypothesis
5	$\neg\{\neg S\}$	NegatedConclusion
6	S	PropRule(5)
7	$\neg P \bigwedge \neg M$	PropRule(4,6)
8	$\neg P$	PropRule(7)
9	$\neg M$	PropRule(7)

Solution LogicPalet (Cont'd)

10	$\neg E$	PropRule(3)
11	$\neg W \rightarrow M$	PropRule(3)
12	$\neg\{A \bigwedge W\}$	PropRule(10,1)
13	$\neg\{\neg A\}$	PropRule(2,8)
14	A	PropRule(13)
15	$\neg\{W\}$	PropRule(14,12)
16	M	PropRule(15,11)
	Contradiction	9,16

KE Deduction Rules for $\equiv$

To deal with the operator, $\equiv$, we need further rules.
Properties of $\equiv$:

- Symmetric: $P \equiv Q = Q \equiv P$
- Negation: $\neg(P \equiv Q) = \neg P \equiv Q = P \equiv \neg Q$.
- Opposite: $P \equiv Q = \neg P \equiv \neg Q$

KE Deduction Rules for $\equiv$(Cont'd)

η (eta) Rules

$$\frac{\begin{array}{c} P \equiv Q \\ P \end{array}}{Q} \qquad \frac{\begin{array}{c} P \equiv Q \\ Q \end{array}}{P} \qquad \frac{\begin{array}{c} \neg(P \equiv Q) \\ P \end{array}}{\neg Q} \qquad \frac{\begin{array}{c} \neg(P \equiv Q) \\ Q \end{array}}{\neg P}$$

$$\frac{\begin{array}{c} P \equiv Q \\ \neg P \end{array}}{\neg Q} \qquad \frac{\begin{array}{c} P \equiv Q \\ \neg Q \end{array}}{\neg P} \qquad \frac{\begin{array}{c} \neg(P \equiv Q) \\ \neg P \end{array}}{Q} \qquad \frac{\begin{array}{c} \neg(P \equiv Q) \\ \neg Q \end{array}}{P}$$

Contraposition $\models P \to Q \equiv \neg Q \to \neg P$

Previously Truth Tables were used to prove $\models P \to Q \equiv \neg Q \to \neg P$. This can be proved via KE Deduction.

$\neg$(P$\to$Q$\equiv$ $\neg$Q$\to$ $\neg$P)...1

Case

«Left»	«Right»
P$\to$Q...2	$\neg$(P$\to$Q)...7
η(1,2)	η(1,7)
$\neg$($\neg$Q$\to$ $\neg$P)...3	$\neg$Q$\to$ $\neg$P...8
α(3)	α(7)
$\neg$Q...4	P...9
P...5	$\neg$Q...10
β(2,5)	β(8,10)
Q...6	$\neg$P...11
$\times$(4,6)	$\times$(9,11)

De Morgan's Law

It is straightforward to show by Truth Tables that
$\models \neg(P \vee Q) \equiv \neg P \wedge \neg Q$.
It can also be proved by KE Deduction, i.e. assume
$\neg(\neg(P \vee Q) \equiv \neg P \wedge \neg Q)$ and show that this leads to a contradiction.
In proofs, we implicitly use the 'double negation' rule, i.e. we write $\neg\neg P$ as P.

De Morgan

$\neg(\neg(P\vee Q)\equiv \neg P\wedge\neg Q)\ldots 1$

Case

«Left»	«Right»
$P\vee Q\ldots 2$	$\neg(P\vee Q)\ldots 7$
$\eta(1,2)$	$\eta(1,7)$
$\neg P\wedge\neg Q\ldots 3$	$\neg(\neg P\wedge\neg Q)\ldots 8$
$\alpha(3)$	$\alpha(7)$
$\neg P\ldots 4$	$\neg P\ldots 9$
$\neg Q\ldots 5$	$\neg Q\ldots 10$
$\beta(2,4)$	$\beta(8,9)$
$Q\ldots 6$	$Q\ldots 11$
$\times(5,6)$	$\times(10,11)$

Comment on KE Proof

This proof makes use of a deduction rule that is similar to a De Morgan Law.
The deduction rule

$$\frac{\neg(P \vee Q)}{\begin{array}{c}\neg P \\ \neg Q\end{array}}$$

can be rewritten as $\vdash \neg(P \vee Q) \rightarrow \neg P \wedge \neg Q$

Contingent argument

John or Joyce will go to the party.
If Joyce goes to the party then, Clare will go unless Stephen goes.
Stephen will go if John does not go.
Therefore,
Clare will go to the party.

[from J. Kelly 'Essence of Logic']

We use the abbreviations:

- J : John will go to the party
- Y : Joyce will go to the party
- C : Clare will go to the party
- S : Stephen will go to the party

Formalise argument

The argument can be abbreviated as:

Note: $Q \textit{ unless } P = Q \textit{ if not } P = \neg P \rightarrow Q = P \vee Q$

$$
\begin{array}{ll}
 & J \vee Y \\
 & Y \rightarrow S \vee C \\
 & \neg J \rightarrow S \\
\therefore & C
\end{array}
$$

Argument Truth Tree

$J \vee Y \ldots 1$

$Y \rightarrow (S \vee C) \ldots 2$

$\neg J \rightarrow S \ldots 3$

$\neg C \ldots 4$

«L»	«R»
$J \ldots 5$	$\neg J$

«LL»	«LR»
$Y \ldots 6$	$\neg Y$
$\beta(2,6)$
$S \vee C \ldots 6$
$\beta(4,6)$
S
«Open»

Not a valid argument

A path in the KE Deduction (marked <<Open>>) cannot be closed hence the set of propositions $\{J \vee Y, Y \rightarrow (S \vee C), \neg J \rightarrow S, \neg C\}$ is consistent (i.e. all true). Thus it is possible for the all the premises $\{J \vee Y, Y \rightarrow (S \vee C), \neg J \rightarrow S\}$ to be True and the conclusion, C, to be False. The argument is not valid.
The set $\{J \vee Y, Y \rightarrow (S \vee C), \neg J \rightarrow S, \neg C\}$ is True in the state:

C	J	S	Y
F	T	T	T

which can be obtained by traversing the left subtrees.

Check by Truth Table

Check whether

$$(J \vee Y) \wedge (Y \to S \vee C) \wedge (\neg J \to S) \to C$$

is a Tautology.
Consider the following row of the Truth Table:
(variables in alphabetical order)

C	J	S	Y	$(J \vee Y)$	$\wedge$	$(Y \to S \vee C)$	$\wedge$	$(\neg J \to S)$	$\to$	C
$\vdots$	$\vdots$	$\vdots$	$\vdots$							
F	T	T	T			T			**F**	F
$\vdots$	$\vdots$	$\vdots$	$\vdots$							

$(J \vee Y) \wedge (Y \to S \vee C) \wedge (\neg J \to S) \to C$ is not a Tautology.
The argument
$J \vee Y, Y \to (S \vee C), \neg J \to S \models C$ is not valid.