Solutions for Tutorial Exercises Week 9

Qs. 1 (Matrix Inverse via Cayley-Hamilton Theorem)

Let the matrix

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 2 \end{bmatrix}$$

1. Show that the characteristic polynomial, $p(t)$, of A is $-t^3 + 2 * t^2 + 3 * t - 6$
2. Determine A^{-1} using the Cayley-Hamilton Theorem.

Solution Qs. 1.

$$\begin{vmatrix} -1-t & 2 & 0 \\ 1 & 1-t & 0 \\ 2 & -1 & 2-t \end{vmatrix}$$

$$= (-1-t)\begin{vmatrix} 1-t & 0 \\ -1 & 2-t \end{vmatrix} - 2 * \begin{vmatrix} 1 & 0 \\ 2 & 2-t \end{vmatrix}$$

$$= (-1-t) * ((1-t) * (2-t)) - 2 * (2-t)$$

$$= (t^2 - 1) * (2-t) - 4 + 2 * t$$

$$= -t^3 + 2 * t^2 + 3 * t - 6$$

Soln. Qs. 1 (Cont'd)

From Cayley-Hamilton Theorem:
$-A^3 + 2 * A^2 + 3 * A - 6 * Id = 0$

$(-A^2 + 2 * A + 3 * Id) * A = 6 * Id$

{multiply both sides by A^{-1} }
$-A^2 + 2 * A + 3 * Id = 6 * A^{-1}$

$A^{-1} = \frac{1}{6} * (-A^2 + 2 * A + 3 * Id)$

Soln. Qs. 1 (Cont'd)

Calculate: $\frac{1}{6} * (-A^2 + 2 * A + 3 * Id)$

$$A^2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 4 \end{bmatrix}$$

$$-A^2 + 2 * A + 3 * Id =$$

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ -1 & -1 & -4 \end{bmatrix} + \begin{bmatrix} -2 & 4 & 0 \\ 2 & 2 & 0 \\ 4 & -2 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6} * \begin{bmatrix} -2 & 4 & 0 \\ 2 & 2 & 0 \\ 3 & -3 & 3 \end{bmatrix}$$

Qs. 2 (Matrix Inverse by Matrix of Co-Factors)

Find the inverse of the following matrix, A, using the **matrix of co-factors** method.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 3 \\ 3 & 1 & -2 \end{bmatrix}$$

Recall for a Matrix $A = [a_{ij}]_{n\times n}$, the Minor M_{ij} is the determinant of the submatrix of A obtained by deleting the i^{th} row and the j^{th} column. The **matrix of co-factors** is the matrix $[C_{ij}]$ where $C_{ij} = (-1)^{i+j} M_{ij}$and M_{ij} is the minor of the matrix entry a_{ij}. The inverse, A^{-1}, is obtained by:

$$A^{-1} = \frac{1}{det(A)} * [C_{ij}]^T$$

Qs. 2
Find the inverse of the following matrix, A, using the **matrix of co-factors** method.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 3 \\ 3 & 1 & -2 \end{bmatrix}$$

Recall for a Matrix $A = [a_{ij}]_{n\times n}$, the Minor M_{ij} is the determinant of the submatrix of A obtained by deleting the i^{th} row and the j^{th} column. The **matrix of co-factors** is the matrix $\tilde{A} = [\tilde{a}_{ij}]_{n\times n}$ where $\tilde{a}_{ij} = (-1)^{i+j} M_{ij}$. The inverse, A^{-1}, is obtained by:

$$A^{-1} = \frac{1}{det(A)} * (\tilde{A})^T$$

Qs. 2. Solution

$$
\begin{aligned}
|A| &= \begin{vmatrix} 1 & 2 & 4 \\ -1 & 0 & 3 \\ 3 & 1 & -2 \end{vmatrix} \\
&= 1 * (-3) - 2 * (-1 * -2 - 3 * 3) + 4 * (-1) \\
&= -3 + 14 - 4 \\
&= 7
\end{aligned}
$$

Qs. 2. Solution (Cont'd)

Let $[C_{ij}]$ be the matrix of co-factors.

$$
\begin{aligned}
[C_{ij}] &= \begin{pmatrix} +\begin{vmatrix} 0 & 3 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} -1 & 3 \\ 3 & -2 \end{vmatrix} & +\begin{vmatrix} -1 & 0 \\ 3 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 4 \\ 1 & -2 \end{vmatrix} & +\begin{vmatrix} 1 & 4 \\ 3 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \\ +\begin{vmatrix} 2 & 4 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 4 \\ -1 & 3 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} \end{pmatrix} \\
&= \begin{pmatrix} -3 & -((-1*-2)-(3*3 & -1)) \\ -((2*-2)-4) & (1*-2)-4*3) & -(1-2*3) \\ 6 & -(3-(4*-1)) & -(2*-1) \end{pmatrix} \\
&= \begin{pmatrix} -3 & 7 & -1 \\ 8 & -14 & 5 \\ 6 & -7 & 2 \end{pmatrix}
\end{aligned}
$$

Qs. 2. Solution (Cont'd)

$$A^{-1} = \frac{1}{7} * \begin{pmatrix} -3 & 8 & 6 \\ 7 & -14 & -7 \\ -1 & 5 & 2 \end{pmatrix}$$

Check:

$$\frac{1}{7} * \begin{pmatrix} -3 & 8 & 6 \\ 7 & -14 & -7 \\ -1 & 5 & 2 \end{pmatrix} * \begin{pmatrix} 1 & 2 & 4 \\ -1 & 0 & 3 \\ 3 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$