Finding Roots by Newton Raphson Method

Newton Raphson

Recall from before we approximated the value of $f(a+h)$ by the corresponding point on the tangent line as

$f(a+h) \approx f(a) + h * f'(a)$

We can use this result as the basis for finding a root of the function, f.

To find a root of f, find r such that $f(r) = 0$.

Finding the Root

Let the value a be an approximation of the root, so that the value, a, is close to the root, $a + h$, where h is small. Since $a + h$ is a root, then $f(a + h) = 0$.
From above,

$f(a + h) \approx f(a) + h * f'(a) \therefore$
$0 \approx f(a) + h * f'(a)$
$h \approx \frac{-f(a)}{f'(a)}$, provided, $f'(a) \neq 0$

We can use $\frac{-f(a)}{f'(a)}$ as an approximation for h and so we can use $a + \frac{-f(a)}{f'(a)}$ as a better approximation than the value, a, for the root, $a + h$, of the function, f.

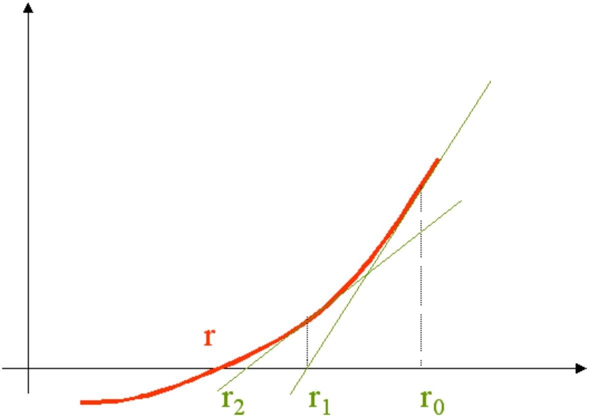
Newton Raphson Method

Let r_0 be a initial approximation of a root, r. From the initial approximation we can generate a sequence of terms $r_1, r_2, \ldots$ that give better approximations.

We get an iterative sequence of terms:

$$\begin{aligned} r_0 &= a \\ r_{n+1} &= r_n - \frac{f(r_n)}{f'(r_n)} \end{aligned}$$

Newton-Raphson method



Succesive Tangents

The technique of the Newton Raphson method is to use successive tangents to the curve and find where they cross the $x - axis$ until the root is found.
As in the diagram, above, if the initial approximation is r_0, then we find the tangent to the curve at $(r_0, f(r_0))$. This tangent crosses the $x - axis$ at $x = r_1$. We can find r_1 by finding the equation of the tangent line at $(r_0, f(r_0))$ and finding where it crosses the $x - axis$.
From before, the equation of the tangent at $(r_0, f(r_0))$ is:

$$y - f(r_0) = f'(r_0) * (x - r_0)$$

We wish to find the value of x when $y = 0$, i.e. find where the tangent line crosses the $x - axis$. Let the tangent cross the $x - axis$ at $(r_1, 0)$, find r_1.

$$0 - f(r_0) = f'(r_0) * (r_1 - r_0) \therefore$$

$$-f(r_0) = f'(r_0) * (r_1 - r_0) \therefore$$

$$r_1 - r_0 = \frac{-f(r_0)}{f'(r_0)} \therefore$$

$$r_1 = r_0 - \frac{f(r_0)}{f'(r_0)} \therefore$$

the tangent line crosses the $x - axis$ at $r_1 = r_0 - \frac{f(r_0)}{f'(r_0)}$.
In general, if we have an approximation of the root, r_n, then we let $r_{n+1} = r_n - \frac{f(r_n)}{f'(r_n)}$.
This gives an iterative sequence of successive approximations:
$r_0, r_1, r_2, \ldots r_n, \ldots$

Find Square Root

Find an approximation to $\sqrt{n}$ for $n \in \mathbb{R}_{\geq 0}$.
We find $\sqrt{n}$ by finding the (positive) root of $x^2 - n = 0$ i.e. we find r such that $r^2 - n = 0$ i.e. $r^2 = n$ i.e. $r = \sqrt{n}$.

With $f(x) = x^2 - n$ then $f'(x) = 2 * x$.
With an initial approximation as $r_0 = a$ we get an iterative sequence:

$$r_0 = a$$
$$r_{n+1} = r_n - \frac{r_n^2 - n}{2 * r_n} \therefore$$
$$r_{n+1} = \frac{1}{2} * \left(r_n + \frac{n}{r_n} \right)$$

Square Root Example

Find $\sqrt{10}$

$r_0 = 3$

$r_{n+1} = \frac{1}{2} * \left(r_n + \frac{10}{r_n} \right)$

$$r_1 = \frac{1}{2} * (3 + \frac{10}{3}) = \frac{19}{6}$$

$$r_2 = \frac{1}{2} * (\frac{19}{6} + \frac{60}{19}) = \frac{721}{228}$$

$$r_2 \approx 3.162$$

$$r_3 = \frac{1}{2} * (\frac{721}{228} + \frac{10 * 228}{721}) = \frac{721}{228}$$

$\therefore$

$\sqrt{10} \approx 3.162$

Finding Reciprocal

Finding Reciprocals

Given b, find $\frac{1}{b}$. We can use reciprocals to implement division. We can consider $\frac{a}{b}$ as $a * \frac{1}{b}$.
To find $\frac{1}{b}$ we find the root of the function, $f(x) = b - \frac{1}{x}$ i.e.
if $f(r) = 0$ then $0 = b - \frac{1}{r}$ i.e. $r = \frac{1}{b}$.

If $f(x) = b - \frac{1}{x}$ then $f'(x) = \frac{1}{x^2}$.
We can find the root by the Newton Raphson iterative process:

$$r_0 = \textit{initial value}$$
$$r_{n+1} = r_n - \frac{f(r_n)}{f'(r_n)}$$

Finding Reciprocal (Cont'd)

With $f(x) = b - \frac{1}{x}$ we generate a sequence of approximations by:

$r_0 = initial\ value$

$r_{n+1} = r_n - \frac{b - \frac{1}{r_n}}{\frac{1}{r_n^2}}$

i.e.

r_{n+1}

$= r_n - r_n^2 * (b - \frac{1}{r_n})$

$= r_n * (1 - r_n * (b - \frac{1}{r_n})$

$\therefore$

$r_{n+1} = r_n * (2 - b * r_n)$

Division by Reciprocal

Since calculating r_{n+1} in terms of r_n and b involves only multiplication and subtraction we can calculate $\frac{1}{b}$ involving only multiplication and subtraction.
In the early days of computing, the operations of addition (subtraction) and multiplication were 'hard-wired' but division was calculated in 'software' via the Newton Raphson method to calculate reciprocals i.e. $\frac{a}{b} = a * \frac{1}{b}$. The Newton Raphson method to calculate reciprocals uses just subtraction and multiplication.

Reciprocal Example

Given 1.37 find $\frac{1}{1.37}$.

Initial approximation:
Since 1.37 is close to $\frac{4}{3}$ let us use:

$$r_0 = \frac{3}{4} = 0.75$$

$$r_{n+1} = r_n * (2 - 1.37 * r_n) \therefore$$

$$r_1 = 0.75 * (2 - 1.37 * 0.75) = 0.7293$$

$$r_2 = 0.7293 * (2 - 1.37 * 0.7293) = 0.7299$$

$$\therefore$$

$$\frac{1}{1.37} \approx 0.7299$$

Find an approximation to the root of the polynomial: $x^3 - 2 * x - 5$

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Let $f(x) = x^3 - 2 * x - 5$ then $f'(x) = 3 * x^2 - 2$ and we get the Newton Raphson iterations:

Using 2 as an initial value,

$r_0 = 2,$

$r_{n+1} = r_n - \frac{r_n^3 - 2 * r_n - 5}{3 * r_n^2 - 2}$

Simiplifying, we get,

$r_{n+1} = \frac{2 * r_n^3 + 5}{3 * r_n^2 - 2}$

Finding Roots of Polynomials (Cont'd)

We get:

$r_1 = \frac{2*2^3+5}{3*2^2-2} = \frac{21}{10} = 2.1$

$r_2 = \frac{2*(2.1)^3+5}{3*(2.1)^2-2} = 2.094$

$r_3 = \frac{2*(2.094)^3+5}{3*(2.094)^2-2} = 2.094$

$\therefore$

a root of $x^3 - 2*x - 5$ is 2.094

Check:

$(2.094)^3 - 2*2.094 - 5 = -0.006$

Root of $x - cos\, x$

Find a value, x, such that $x = cos\, x$.
(A solution for $x = sin\, x$ is easy as $sin\, 0 = 0$.)
Let $f(x) = x - cos\, x$ then $f'(x) = 1 + sin\, x$.
With value 1 as the initial value we get:
$r_0 = 1$
$r_{n+1} = r_n - \frac{r_n - cos\, r_n}{1 + sin\, r_n}$
We get successive terms:

$$r_1 = 1 - \frac{1 - cos\, 1}{1 + sin\, 1} = 0.75036$$

$$r_2 = 0.75036 - \frac{0.75036 - cos\, 0.75036}{1 + sin\, 0.75036} = 0.73911$$

$$r_2 = 0.73911 - \frac{0.73911 - cos\, 0.73911}{1 + sin\, 0.73911} = 0.73908$$

Check: $cos\, 0.73908 \approx 0.73908$