Homogeneous Linear Equations

Homogeneous equations

A system of m Linear Equations in n unknowns is said to be **Homogeneous** if it is of the form

$$
\begin{aligned}
a_{11} * x_1 + a_{12} * x_2 + \cdots + a_{1n} * x_n &= 0 \\
a_{21} * x_1 + a_{22} * x_2 + \cdots + a_{2n} * x_n &= 0 \\
\vdots \quad & \quad \vdots \\
a_{m1} * x_1 + a_{n2} * x_2 + \cdots + a_{mn} * x_n &= 0
\end{aligned}
$$

Such a system of Homogeneous equations is consistent as it has the solution:
$x_1 = 0, x_2 = 0, \ldots x_n = 0$, called the trivial solution.
A system of Homogeneous equations either has:

- Only the trivial solution, or
- Infinitely many non-trivial solutions, in addition to the trivial solution.

Homogeneous equations, Example

Consider the system of Homogeneous equations:

$$\begin{aligned} 2*x_1+2*x_2-x_3+x_5 &= 0 \\ -x_1-x_2+2*x_3-3*x_4+x_5 &= 0 \\ x_1+x_2-2*x_3-x_5 &= 0 \\ x_3+x_4+x_5 &= 0 \end{aligned}$$

The Augmented Matrix is:

Example, (Cont'd)

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Reduce to Reduced Row Echelon Form:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This corresponds to the following system of linear equations:

Example, (Cont'd)

$$\begin{aligned} x_1 + x_2 + x_5 &= 0 \\ x_3 + x_5 &= 0 \\ x_4 &= 0 \end{aligned}$$

i.e.

$$\begin{aligned} x_1 &= -x_2 - x_3 \\ x_3 &= -x_5 \\ x_4 &= 0 \end{aligned}$$

Let $x_5 = t$, and $x_2 = s$ we have solutions:
$x_1 = -s - t$, $x_2 = s$, $x_3 = -t$, $x_4 = 0$, $x_5 = t$.

'Square' Homogeneous Equations

If the number of equations is the same as the number of unknowns (e.g. $m = n$ in the above system) we have:

$$\begin{aligned}
a_{11} * x_1 + a_{12} * x_2 + \cdots + a_{1n} * x_n &= 0 \\
a_{21} * x_1 + a_{22} * x_2 + \cdots + a_{2n} * x_n &= 0 \\
\vdots \quad &\quad \vdots \\
a_{n1} * x_1 + a_{n2} * x_2 + \cdots + a_{nn} * x_n &= 0
\end{aligned}$$

and the corresponding matrix of coefficients is:

$$\begin{bmatrix}
a_{11} & a_{12} & \dots & a_{1n} \\
a_{21} & a_{22} & & a_{2n} \\
\vdots & & & \vdots \\
a_{n1} & a_{n2} & \dots & a_{nn}
\end{bmatrix}$$

'Square' Homogeneous Equations (Cont'd)

If the matrix of coefficients has an inverse then the solution is unique and for Homogeneous equations, the solution is $x_1 = 0, x_2 = 0, \ldots x_n = 0$, the trivial solution.
If the matrix of coefficients has no inverse then the matrix is Singular. Recall that if a matrix, M, is singular then the determinant, $|M| = 0$ i.e.

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = 0$$

If the determinant of the matrix of coefficients is zero, then there are infinitely many non-trivial solutions, in addition to the trivial solution.