# Example: Finding Eigenvalues and Eigenvectors

Find the eigenvalues and eigenvectors of the matrix:

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

## 1 Solution:

First, the eigenvalues are found and then for each of the eigenvalues the corresponding eigenvectors are found.

### 1.1 Find the Eigenvalues

Find the values of $\lambda$ that satisfy the characteristic equation:

$$|A - \lambda * Id| = 0$$

where $Id$ is the $3 \times 3$ identity matrix.

$A - \lambda * Id$

$$= \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} - \lambda * \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{bmatrix}$$

#### 1.1.1 Calculate $|A - \lambda * Id|$

$$\begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix}$$

$$=(1-\lambda) * \begin{vmatrix} -5-\lambda & 3 \\ -6 & 4-\lambda \end{vmatrix} - (-3) * \begin{vmatrix} 3 & 3 \\ 6 & 4-\lambda \end{vmatrix} + 3 * \begin{vmatrix} 3 & -5-\lambda \\ 6 & -6 \end{vmatrix}$$

$= (1-\lambda)*((-5-\lambda)*(4-\lambda)$
$+3*(3*(4-\lambda)-6*3)$
$+3*(3*(-6)-6*(-5-\lambda))$

$= (1-\lambda)*((-20+5*\lambda-4*\lambda+\lambda^2)+18)+3*((12-3*\lambda)-18)+3*(-18-(-30-6*\lambda))$

$= (1-\lambda)*(-2+\lambda+\lambda^2)+3*(-6-3*\lambda)+3*(12+6*\lambda)$

$= -2+\lambda+\lambda^2+2*\lambda-\lambda^2-\lambda^3-18-9*\lambda+36+18*\lambda$

$= 16+12*\lambda-\lambda^3$

$\therefore |A-\lambda*Id| = 16+12*\lambda-\lambda^3$
the characteristic polynomial of the Matrix $A$.

#### 1.1.2 Find integer solutions for $\lambda$ in $\lambda^3-12*\lambda-16=0$

An integer solution for $\lambda^3-12*\lambda-16=0$ divides the constant 16. Possible factors of 16 are:
$\pm1, \pm2, \pm4, \pm8, \pm16$
By trial we find that $4^2-12*4-16=0$ and so $\lambda-4$ is a factor of $\lambda^3-12*\lambda-16$
Divide $\lambda^3-12*\lambda-16$ by $\lambda-4$

$$\begin{array}{ll}
 & \lambda^2+4*\lambda+4 \\ \cline{2-2}
\lambda-4 & \big) \; \lambda^3-12*\lambda-16 \\
 & \lambda^3-4*\lambda^2 \\ \cline{2-2}
\text{subtract} & 4*\lambda^2-12*\lambda-16 \\
 & 4*\lambda^2-16*\lambda \\ \cline{2-2}
\text{subtract} & 4*\lambda-16 \\
 & 4*\lambda-16 \\ \cline{2-2}
\text{subtract} & 0
\end{array}$$

$\therefore$
$\lambda^3-12*\lambda-16 = (\lambda-4)*(\lambda^2+4*\lambda+4)$

**Factors of $\lambda^2+4*\lambda+4$**

By inspection $\lambda^2+4*\lambda+4 = (\lambda+2)^2$ or by formula;
the formula for the roots of $a*\lambda^2+b*\lambda+c$

$$\begin{aligned}
\lambda &= \frac{-b\pm\sqrt{b^2-4*a*c}}{2*a} \\
&= \frac{-4\pm0}{2}
\end{aligned}$$

Factors of $\lambda^3-12*\lambda-16$ are $\lambda-4$, $\lambda+2$ and $\lambda+2$
i.e. $\lambda^3-12*\lambda-16 = (\lambda-4)*(\lambda+2)*(\lambda+2)$

The roots of $\lambda^3 - 12 * \lambda - 16$ are 4 and $-2$, where $-2$ is a repeated root.
The eigenvalues of the matrix $A$ are $\lambda_1 = 4$ and $\lambda_2 = -2$.
Recall, $\lambda^3 - 12 * \lambda - 16 = 0$ is the **characteristic equation** of the matrix $A$.

## 1.2 Finding the Eigenvectors

We find the eigenvectors corresponding to the eigenvalues.

- For each eigenvalue, $\lambda$, we have $(A - \lambda * Id) * x = 0$,
  where the vectors, $x$, are the corresponding eigenvectors for the eigenvalue, $\lambda$.

- Find the vectors, $x$, by Gauss/Jordan elimination, i.e. reduce the augmented matrix, $[A - \lambda * x \,|\, 0]$ to reduced row echelon form and solve the associated linear system.

### 1.2.1 Case when the eigenvalue is $\lambda_1 = 4$

Find vectors, $x$, which satisfy $(A - 4 * Id) * x = 0$

$$A - 4 * Id = \begin{bmatrix} 1-4 & -3 & 3 \\ 3 & -5-4 & 3 \\ 6 & -6 & 4-4 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix}$$

Augmented Matrix:

$$\begin{bmatrix} -3 & -3 & 3 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{bmatrix}$$

$R1 := \frac{R1}{-3}$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{bmatrix}$$

$R2 := R2 - 3$*R1
$R3 := R3 - 6 * R1$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -12 & 6 & 0 \\ 0 & -12 & 6 & 0 \end{bmatrix}$$

$R2 := \frac{R2}{-12}$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & -12 & 6 & 0 \end{bmatrix}$$

$R3 := R3 + 12 * R2$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R1 := R1 - R2$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rewriting this augmented matrix back as a linear system we get:

$$\begin{aligned} x_1 - \frac{1}{2} * x_3 &= 0 \\ x_2 - \frac{1}{2} * x_3 &= 0 \end{aligned}$$

$x_1$ and $x_2$ are lead variables and we let $t$ be the parameter for $x_3$.

We get the eigenvector, $x$, corresponding to the eigenvalue, $\lambda_1 = 4$ :

$$x = \begin{bmatrix} \frac{1}{2} * t \\ \frac{1}{2} * t \\ t \end{bmatrix} = t * \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

With $t = 2$ we get an eigenvector, $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, for $A$ corresponding to the eigenvalue, 4.

**Check:**

$$A*x = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1*1-3*1+3*2 \\ 3*1-5*1+3*2 \\ 6*1-6*1+4*2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} = 4* \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

#### 1.2.2 Case when the eigenvalue is $\lambda_2 = -2$

$$A + 2 * Id = \begin{bmatrix} 1+2 & -3 & 3 \\ 3 & -5+2 & 3 \\ 6 & -6 & 4+2 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix}$$

Augmented Matrix

$$\begin{bmatrix} 3 & -3 & 3 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{bmatrix}$$

$$R1 := \frac{R1}{3}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{bmatrix}$$

$$R2 := R2 - 3 * R1$$

$$R3 := R3 - 6 * R1$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rewritng the augmented matrix back as a linear system:

$$x_1 - x_2 + x_3 = 0$$

i.e.

$$x_1 = x_2 - x_3$$

$x_1$ is the lead variable and let $s$ and $t$ be parameters for $x_2$ and $x_3$.
The eigenvectors $x$, corresponding to the eigenvalue, $-2$ have the form:

$$\begin{aligned} x &= \begin{bmatrix} s-t \\ s \\ t \end{bmatrix} \\ &= s * \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t * \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

In particular, with $s = 1 = t$ we have an eigenvector corresponding to the eigenvalue, -2:

$$x = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

**Check:**

$A * x$

$$= \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1*0-3*1+3*1 \\ 3*0-5*1+3*1 \\ 6*0-6*1+4*1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}$$

$$= -2 * \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$