Natural Logarithm, $ln(x)$

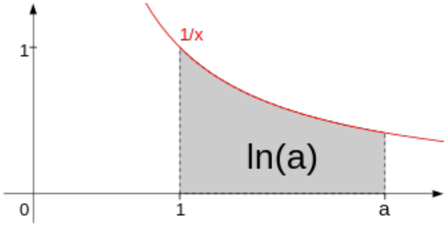
$ln(x)$

From Integral Calculus, $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ where $n \neq -1$.
Case $n = -1$, $\int x^{-1} dx = \int \frac{1}{x} dx = ln\, x$ (natural logarithm function)

Definition

For $x \geq 1$

$$ln(x) = \int_1^x \frac{1}{t}\, dt$$



(See https://en.wikipedia.org/wiki/Natural_logarithm)

$ln(x)$ (Cont'd)

For $0 < x < 1$, $ln(x) = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt$

- $ln(1) = 0$ as $\int_1^1 \frac{1}{t}\,dt = 0$
- if $x > 1$ then $ln(x) > 0$
- if $0 < x < 1$ then $ln(x) < 0$
- $ln(x)$ is not defined when $x \leq 0$.

Derivative of $ln(x)$

Since $ln(x) = \int_1^x \frac{1}{t} dt$, from Calculus theory we have

$\frac{d}{dx}(ln(x)) = \frac{d}{dx} \int_1^x \frac{1}{t} dt = \frac{1}{x}$

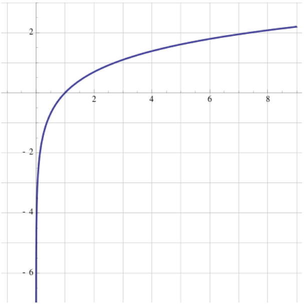
Chain Rule Derivative

If $u(x)$ is a function of x then
$\frac{d}{dx}(ln(u)) = \frac{1}{u} * \frac{du}{dx}$

e.g.

- $\frac{d}{dx}(ln(x^2)) = \frac{1}{x^2} * (2 * x) = \frac{2}{x}$
- $\frac{d}{dx}(ln(1 + x)) = \frac{1}{1+x} * \frac{d}{dx}(1 + x) = \frac{1}{1+x}$

Graph of $ln(x)$



(See https://en.wikipedia.org/wiki/Natural_logarithm)

Since the function $ln(x)$ is continuous, there is a number, x , such that $ln(x) = 1$. This number is named, e , and so $ln(e) = 1$.
In decimal notation $e \approx 2.718281828$ or $e \approx \frac{87}{32}$.
e is an irrational number.

Properties of $ln(x)$

In terms of 'standard' logarithms, $ln(x) = log_e(x)$

The function $ln(x)$ has the properties of logarithms.
$a > 0, b > 0$

- $ln(a^n) = n * ln(a)$
- $ln(a * b) = ln(a) + ln(b)$
- $ln(a/b) = ln(a) - (ln\, b)$

Calculating $ln(x)$

In order to calculate $ln\, x$, we first find a converging series for calculating $ln(1+x)$ when $|x| < 1$ i.e. when $-1 < x < 1$. When $-1 < x < 1$, then $1 + x > 0$ and so $ln(1+x)$ is defined.

Note: For $x = 1$
$ln\, 1 = 0$

Series $ln(1 + x)$

From above: $\frac{d}{dx}(ln(1 + x)) = \frac{1}{1+x} \therefore$

$d\,(ln(1 + x)) = \frac{1}{1+x}\,dx \therefore$

$\int d(ln(1 + x)) = \int \frac{1}{1+x} dx$ i.e.

$ln(1 + x) = \int \frac{1}{1+x} dx$

Evaluate $\frac{1}{1+x}$ as a series:

$$
\begin{array}{r|l}
 & 1 - x \quad +x^2 \quad \dots \\
\hline
1 + x & 1 \\
 & \underline{1 + x} \\
 & -x \\
 & \underline{-x - x^2} \\
 & x^2 \\
 & \underline{x^2 + x^3} \\
 & \dots
\end{array}
$$

Check:

If $\frac{1}{1+x} = 1 - x + x^2 - x^3 \dots$ then $1 = (1 + x) * (1 - x + x^2 - x^3 \dots)$

Cont'd

$$(1+x)*(1-x+x^2-x^3\dots)$$

$$= 1*(1-x+x^2-x^3\dots)$$
$$+x*(1-x+x^2-x^3\dots)$$

$$= 1-x+x^2-x^3\dots$$
$$+x-x^2+x^3\dots$$

$$= 1$$

Geometric Series: for $|r| < 1$

$a + a * r + a * r^2 + a * r^3 + \cdots = \frac{a}{a-r}$
Consider

$$\begin{array}{rcll} S & = & a + & a * r + a * r^2 + a * r^3 + \dots \\ r * S & = & & a * r + a * r^2 + a * r^3 + \dots \end{array}$$

$\therefore \mathit{subtracting},\ S - r * S = a$
i.e. $S * (1 - r) = a$
$\therefore S = \frac{a}{1-r}$, for $|r| < 1$
For $|r| < 1$,

$$a + a * r + a * r^2 + a * r^3 + \cdots = \frac{a}{1-r} \ ,$$

let $a = 1$ and $r = -x$ we get

$$1 - x + x^2 - x^3 \cdots = \frac{1}{1 + x}$$

Calculate $ln(1+x)$

$$
\begin{aligned}
ln(1+x) &= \int \frac{1}{1+x} dx \\
&= \int (1 - x + x^2 - x^3 \dots) dx \\
&= \int 1 dx - \int x dx + \int x^2 dx - \dots \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots
\end{aligned}
$$

For $|x| < 1$ i.e. $-1 < x < 1$ the series converges $\therefore$

$$ln(1+x) = x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \tfrac{x^4}{4} \dots$$

To find $ln(t)$ for any $0 < t$, we consider two cases:

1. $0 < t < 2$
2. $2 < t$

Note:$ln\, 1 = 0$

if $0 < x < 1$ then $ln\, x < 0$

if $1 < x$ then $ln\, x > 0$.

Case $0 < t < 2$

If $0 < t < 2$ then $-1 < t - 1 < 1$.

Let $x = t - 1$ then $t = (1 + t - 1) = (1 + x)$

$ln(t) = ln(1 + x)$ where $x = t - 1$

Since $0 < t < 2$ then $-1 < x < 1$

To calculate $ln(t)$ when $0 < t < 2$

let $x = t - 1$ and calculate $ln(1 + x)$ where

$$ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots$$

Calculate $ln(t)$, for $0 < t < 2$

Note: $ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots$

Example

e.g. $t = \frac{1}{3}$ and so $0 < \frac{1}{3} < 1 < 2$
Let $x = \frac{1}{3} - 1 \therefore x = -\frac{2}{3}$ and so $-1 < x < 1$

$$\begin{aligned} ln\left(\frac{1}{3}\right) &= ln(\left(1 + \left(-\frac{2}{3}\right)\right)) \\ &= \left(-\frac{2}{3}\right) - \frac{1}{2} * \left(-\frac{2}{3}\right)^2 + \frac{1}{3} * \left(-\frac{2}{3}\right)^3 - \dots \\ &= -\frac{2}{3} - \frac{2}{9} - \frac{8}{81} \dots \\ &\approx -\frac{80}{81} \approx -0.988 \end{aligned}$$

Example

$t = \frac{4}{3} \therefore 1 < t < 2$ and so $0 < t < 2$

Let $x = \frac{4}{3} - 1 \therefore x = \frac{1}{3}$ and so $-1 < x < 1$

$$
\begin{aligned}
ln\left(\frac{4}{3}\right) &= ln\left(1 + \frac{1}{3}\right) \\
&= \left(\frac{1}{3}\right) - \frac{1}{2} * \left(\frac{1}{3}\right)^2 + \frac{1}{3} * \left(\frac{1}{3}\right)^3 - \dots \\
&= \frac{1}{3} - \frac{1}{18} + \frac{1}{81} \\
&= \frac{47}{162}
\end{aligned}
$$

Calculate $ln(t)$, for $2 < t$

Case $2 < t$: e.g. $t = 3$
If $1 < x$ then $\frac{1}{x} < 1$; also $0 < \frac{1}{x} \therefore 0 < \frac{1}{x} < 1$

From the properties of ln,
$ln(x^k) = k * ln(x) \therefore$
$ln(\frac{1}{x}) = ln(x^{-1}) = -ln(x)$
To calculate $ln(x)$ when $2 < x$, calculate $-ln(\frac{1}{x})$
e..g. to calculate $ln(3)$, calculate $-ln(\frac{1}{3})$ using the above.
Note: 2 ways of calculating $ln(t)$ when $1 < t < 2$

- Use series for $ln(1 + x)$ with $x = t - 1$ or
- Via $-ln\left(\frac{1}{t}\right)$ i.e. $-ln(1 + x)$ with $x = \frac{1}{t} - 1 = \frac{1-t}{t}$
  but $-ln\left(1 + \frac{1-t}{t}\right) = -ln\left(\frac{t+1-t}{t}\right) = -ln\left(\frac{1}{t}\right) = ln(t)$

$\therefore$ to calculate $ln(t)$ with $1 < t < 2$ use series $ln(1 + x)$ with $x = t - 1$.

Series for $\ln(\frac{1+x}{1-x})$

From the properties of $ln(x)$

$\ln(\frac{1+x}{1-x}) = ln(1+x) - ln(1-x)$

$ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots$, also,
$ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \dots$,
Subtract:
$ln(1+x) - ln(1-x) = 2*(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} \dots) \therefore$

$$\ln(\frac{1+x}{1-x}) = 2*(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} \dots), \text{ for } |x| < 1 .$$

This series converges 'faster' than the series for $ln(1+x)$.

Calculate $ln(t)$ using $\ln\left(\frac{1+x}{1-x}\right)$

We can use $\ln(\frac{1+x}{1-x})$ to calculate $ln(t)$, for $t > 1$.
(**Note**: for $0 < t < 1, ln\, t = -ln\,(t^{-1}) = -ln(\frac{1}{t})$ where $\frac{1}{t} > 1$)
For $t > 1$, let
$t = \frac{1+x}{1-x} \therefore$
$(1 - x) * t = 1 + x \therefore$
$t - x * t = 1 + x \therefore$
$x * (t + 1) = t - 1 \therefore$
$x = \frac{t-1}{t+1}$
Since $t > 1$, then $|x| < 1 \therefore$ for $t > 1$

$$ln(t) = \ln(\frac{1+x}{1-x}), \text{ where } x = \frac{t-1}{t+1} \text{ and } |x| < 1$$

Example

Example

Calculate $ln(3)$ using series for $\ln(\frac{1+x}{1-x})$

For $|x| < 1$, use series for

$$\ln(\frac{1+x}{1-x}) = 2 * (x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} \dots)$$

to calculate $ln(3)$.

With $t = 3$, $x = \frac{3-1}{3+1} = \frac{1}{2}$ and so $|x| < 1$,

i.e. $ln(3) = \ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) = ln\left(\frac{\frac{3}{2}}{\frac{1}{2}}\right) = ln(3)$

Cont'd

$$
\begin{aligned}
ln(3) &= ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) \\
&= 2 * \left(\frac{1}{2} + \frac{\left(\frac{1}{2}\right)^3}{3} + \frac{\left(\frac{1}{2}\right)^5}{5} \dots\right) \\
&\approx 2 * \left(\frac{1}{2} + \frac{1}{24} + \frac{1}{160}\right) \\
&= \frac{263}{240} = 1.0958
\end{aligned}
$$

Note:
This approximation for $ln(3)$ is better than the approximation using the series for $ln(1+x)$.

Series for Evaluating π

From Trigonometry, $tan\left(\frac{\pi}{4}\right) = 1 \therefore$
$\frac{\pi}{4} = tan^{-1}(1)$
Sometimes tan^{-1} is referred to as *arctan*.
From integral calculus:
$tan^{-1}(x) = \int \frac{1}{1+x^2}dx$
From geometric series: $\frac{a}{1-r} = a + a * r + a * r^2 + \dots$,
let $a = 1,\ r = -x^2$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

Digression $tan^{-1}x$

Show $\frac{d}{dx} tan^{-1}x = \frac{1}{1+x^2} \therefore \int \frac{1}{1+x^2} dx = tan^{-1}(x)$
let $y = tan^{-1}x \therefore tan\, y = x$. Using chain rule:
$\frac{d(tan\, y)}{dy} \frac{dy}{dx} = \frac{dx}{dx} = 1$
From differential calculus, $\frac{d(tan\, y)}{dy} = sec^2 y$
$\therefore \frac{dy}{dx} = 1/\left(sec^2 y\right)$
But $cos^2 x + sin^2 x = 1 \therefore 1 + \frac{sin^2 x}{cos^2 x} = \frac{1}{cos^2 x} \therefore$
$sec^2 y = 1 + tan^2 y$ i.e.
$sec^2 y = 1 + x^2$
From above
$\frac{dy}{dx} = 1/\left(sec^2 y\right)$
$\frac{dy}{dx} = \frac{1}{1+x^2}$

Digression Cont'd

Evaluate $\int \frac{1}{1+x^2} dx$
Let $x = tan\,\theta \therefore dx = sec^2\theta\, d\theta$,
also, $1 + x^2 = 1 + tan^2\theta = sec^2\theta$

$$\begin{aligned}
\int \frac{1}{1+x^2} dx &= \int \frac{1}{sec^2\theta} sec^2\theta\, d\theta \\
&= \int 1 d\theta \\
&= \theta \\
&= tan^{-1} x \\
&\quad as\, x = tan\,\theta
\end{aligned}$$

Series for Evaluating π

$tan^{-1}(x)$

$= \int \frac{1}{1+x^2} dx$

$= \int (1 - x^2 + x^4 - x^6 + \dots) dx$

$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \dots$

With $x = 1$, $tan^{-1} 1 = \frac{\pi}{4}$

$$\frac{\pi}{4} = tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$