1 Sets Continued

1.1 Set Properties

Set Properties

Set Properties

Sets have properties similar but not the same as Arithmetic.
Let U be the Universal set of elements of interest.
Let $X, Y, Z \subseteq U$ The basic operators on sets are:

- Complement: $\overline{X}$
- Intersection: $X \cap$Y
- Union $X \cup Y$

Set Props. Cont'd

Fundamental Properties of Set Theory Operators

Identity

$$X \cap U = X \qquad X \cup \{\} = X$$

Anihilation

$$X \cap \{\} = \{\} \qquad X \cup U = U$$

Complement

$$X \cap \overline{X} = \{\} \qquad X \cup \overline{X} = U$$

Idempotent

$$X \cap X = X \qquad X \cup X = X$$

Commutativity

$$X \cap Y = Y \cap X \qquad X \cup Y = Y \cup X$$

Set Props Cont'd

Associativity

$(X \cap Y) \cap Z = X \cap (Y \cap Z) \quad (X \cup Y) \cup Z = X \cup (Y \cup Z)$

Distributivity: $\cap$ over $\cup$

$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$

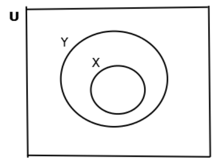
Distributivity: $\cup$ over $\cap$

$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$

Elementary Properties of Sets

- $\overline{\overline{X}} = X$
- $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$

- $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$
- $X \subseteq Y \equiv \overline{Y} \subseteq \overline{X}$



- Also $Y \subseteq X \equiv \overline{X} \subseteq \overline{Y}$

Elementary Properties (Cont'd)

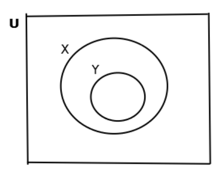
- $X = Y \equiv \overline{X} = \overline{Y}$

  Proof:
  $X = Y \equiv X \subseteq Y \, and \, Y \subseteq X$
  $\equiv \overline{Y} \subseteq \overline{X} \, and \, \overline{X} \subseteq \overline{Y}$
  $\equiv \overline{X} = \overline{Y}$

Set Theory Theorems

Set Theory Theorems

- $Y \subseteq X \quad \equiv \quad X \cup Y = X$
- $Y \subseteq X \quad \equiv \quad X \cap Y = Y$



$Y \subseteq X \equiv X \cup Y = X$

Show $Y \subseteq X \equiv X \cup Y = X$

1. $Y \subseteq X \rightarrow X \cup Y = X$
2. $X \cup Y = X \rightarrow Y \subseteq X$

Proof. (1.)
Assume $Y \subseteq X$,
show $X \cup Y = X$ i.e. $X \cup Y \subseteq X$ and $X \subseteq X \cup Y$
Show $X \cup Y \subseteq X$
let $z \in X \cup Y$
$\therefore z \in X$ or $z \in Y$ $\square$

Cont'd

Proof. Case $z \in X$

$\therefore z \in X$

Case $z \in Y$

$\{$assuming $Y \subseteq X$ $\}$

$\therefore z \in X$.

Show $X \subseteq X \cup Y$

True, from properties of $\cup$. $\square$

$Y \subseteq X \equiv X \cup Y = X$ **(Cont'd)**

Show(2.) $X \cup Y = X \rightarrow$ **Y⊆X**

Proof. (2.)

Assume $X \cup Y = X$, show $Y \subseteq X$

let $z \in Y$,

$\therefore z \in X \cup Y$

$\{$assuming $X \cup Y = X$ $\}$

$\therefore z \in X$ $\square$

$Y \subseteq X \equiv X \cap Y = Y$

Show $Y \subseteq X \equiv X \cap Y = Y$

i.e. Show

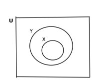
1. $Y \subseteq X \rightarrow X \cap Y = Y$
2. $X \cap Y = Y \rightarrow Y \subseteq X$

Proof. Exercise $\square$

$X \subseteq Y \equiv X \cap \overline{Y} = \{\}$

Theorem 1. $X \subseteq Y \equiv X \cap \overline{Y} = \{\}$

Diagram:



Show $X \subseteq Y \rightarrow X \cap \overline{Y} = \{\}$

Assume $X \subseteq Y$ i.e. from above (swapping X and Y): $X \cap Y = X$ $\therefore$ $X \cap \overline{Y}$ $= (X \cap Y) \cap \overline{Y} = X \cap \left(Y \cap \overline{Y}\right) = X \cap \{\} = \{\}$

Show $X \cap \overline{Y} = \{\} \to X \subseteq Y$

Show $X \cap \overline{Y} = \{\} \to X \subseteq Y$

Assume $X \cap \overline{Y} = \{\}$ As $X \subseteq Y \equiv X \cap Y = X$, show $X = X \cap Y$

$$\begin{aligned} X &= X \cap U \\ &= X \cap \left(Y \cup \overline{Y}\right) \\ &= (X \cap Y) \cup \left(X \cap \overline{Y}\right) \\ &= (X \cap Y) \cup \{\} \\ &= X \cap Y \end{aligned}$$

De Morgan's Laws

De Morgan's Laws

1. $\overline{(X \cap Y)} = \overline{X} \cup \overline{Y}$ – De Morgan 1
2. $\overline{(X \cup Y)} = \overline{X} \cap \overline{Y}$ – De Morgan 2

De Morgan 1 Veitch Diagram

$$X \cap Y \quad = \quad \begin{array}{c|c|c|} & \multicolumn{2}{c}{\text{Y}} \\ \cline{2-3} & 0 & 0 \\ \cline{2-3} X & 0 & 1 \\ \cline{2-3} \end{array} \quad \therefore \quad \overline{X \cap Y} \quad = \quad \begin{array}{c|c|c|} & \multicolumn{2}{c}{\text{Y}} \\ \cline{2-3} & 1 & 1 \\ \cline{2-3} X & 1 & 0 \\ \cline{2-3} \end{array}$$

$$\overline{X} \quad = \quad \begin{array}{c|c|c|} & \multicolumn{2}{c}{\text{Y}} \\ \cline{2-3} & 1 & 1 \\ \cline{2-3} X & 0 & 0 \\ \cline{2-3} \end{array} \qquad \overline{Y} \quad = \quad \begin{array}{c|c|c|} & \multicolumn{2}{c}{\text{Y}} \\ \cline{2-3} & 1 & 0 \\ \cline{2-3} X & 1 & 0 \\ \cline{2-3} \end{array}$$

$\therefore$

$$\overline{X} \cup \overline{Y} \quad = \quad \begin{array}{c|c|c|} & \multicolumn{2}{c}{\text{Y}} \\ \cline{2-3} & 1 & 1 \\ \cline{2-3} X & 1 & 0 \\ \cline{2-3} \end{array} \quad = \quad \overline{X \cap Y}$$

Proof of De Morgan's Law 1

Proof of De Morgan 1 $\overline{X \cap Y} = \overline{X} \cup \overline{Y}$

1. Show $\overline{X} \cup \overline{Y} \subseteq \overline{X \cap Y}$
2. Show $\overline{X \cap Y} \subseteq \overline{X} \cup \overline{Y}$

Show 1. $\overline{X} \cup \overline{Y} \subseteq \overline{X \cap Y}$

Theorem 2. $\overline{X} \cap \overline{X \cap Y} = \overline{X}$

Proof.

$X \cap Y \subseteq X$

$\{ A \subseteq B \equiv \overline{B} \subseteq \overline{A} \}$

$\equiv \overline{X} \subseteq \overline{X \cap Y}$

$\{ A \subseteq B \equiv A \cap B = A \}$

$\equiv \overline{X} \cap \overline{X \cap Y} = \overline{X}$ □

Corollary 3. $\overline{Y} \cap \overline{X \cap Y} = \overline{Y}$

Show 1. $\overline{X} \cup \overline{Y} \subseteq \overline{X \cap Y}$ **(Cont'd)**

Theorem 4. $\overline{X} \cup \overline{Y} \subseteq \overline{X \cap Y}$

Recall: $A \subseteq B \equiv A \cap B = A$

Proof.

$\overline{X} \cup \overline{Y} \subseteq \overline{X \cap Y}$

$\equiv \left(\overline{X} \cup \overline{Y}\right) \cap \overline{X \cap Y} = \overline{X} \cup \overline{Y}$

$\{ \cap$ Distributes over $\cup \}$

$\equiv \left(\overline{X} \cap \overline{X \cap Y}\right) \cup \left(\overline{Y} \cap \overline{X \cap Y}\right) = \overline{X} \cup \overline{Y}$

$\{$ by Thms. $\overline{X} \cap \overline{X \cap Y} = \overline{X}$ and $\overline{Y} \cap \overline{X \cap Y} = \overline{Y}$ $\}$

$\equiv \overline{X} \cup \overline{Y} = \overline{X} \cup \overline{Y}$

$\equiv True$ □

Show 2. $\overline{X \cap Y} \subseteq \overline{X} \cup \overline{Y}$

Theorem 5. $\overline{\overline{X} \cup \overline{Y}} \cup X = X$

Proof.

$\overline{X} \subseteq \overline{X} \cup \overline{Y}$

$\equiv \overline{\overline{X} \cup \overline{Y}} \subseteq \overline{\overline{X}}$

$\equiv \overline{\overline{X} \cup \overline{Y}} \subseteq X$

$\equiv \overline{\overline{X} \cup \overline{Y}} \cup X = X$ □

Corollary 6. $\overline{\overline{X} \cup \overline{Y}} \cup Y = Y$

Show 2. $\overline{X \cap Y} \subseteq \overline{X} \cup \overline{Y}$ **(Cont'd)**

Theorem 7. $\overline{X \cap Y} \subseteq \overline{X} \cup \overline{Y}$

Proof.

$\overline{X \cap Y} \subseteq \overline{X} \cup \overline{Y}$

$\{ A \subseteq B \equiv \overline{B} \subseteq \overline{A} \}$

$\equiv \overline{\overline{X} \cup \overline{Y}} \subseteq \overline{\overline{X \cap Y}}$

$\equiv \overline{\overline{X} \cup \overline{Y}} \subseteq X \cap Y$

$\equiv \overline{\overline{X} \cup \overline{Y}} \cup (X \cap Y) = X \cap Y$

$\equiv \left(\overline{\overline{X} \cup \overline{Y}} \cup X\right) \cap \left(\overline{\overline{X} \cup \overline{Y}} \cup Y\right) = X \cap Y$

$\equiv X \cap Y = X \cap Y$

$\equiv True$ □

Prove De Morgan 2 $\overline{X \cup Y} = \overline{X} \cap \overline{Y}$

Theorem 8. $\overline{X \cup Y} = \overline{X} \cap \overline{Y}$

Proof.
$\overline{X \cup Y} = \overline{X} \cap \overline{Y}$
$\{\ A = B \equiv \overline{A} = \overline{B}\ \}$
$\equiv \overline{\overline{X \cup Y}} = \overline{\overline{X} \cap \overline{Y}}$
$\equiv X \cup Y = \overline{\overline{X} \cap \overline{Y}}$
{ De Morgan 1 }
$\equiv X \cup Y = \overline{\overline{X}} \cup \overline{\overline{Y}}$
$\equiv X \cup Y = X \cup Y$
$\equiv True$ ∎

1.2 Cardinality of Sets

Cardinality of Sets

Disjoint Sets

Sets X and Y are disjoint iff $X \cap Y = \{\}$.
We define $|X|$ as the size of set X,
i.e. $|X|$ is the number of elements in X .
Sometimes $\#X$ is used instead of $|X|$.
With $A = \{2, 3, 5, 7, 11, 13, 17, 19\}$, $|A| = 8$.

Lemma 1

$$B \subseteq A \rightarrow |A - B| = |A| - |B|$$

Lemma 2

$$A \cap B = \{\} \rightarrow |A \cup B| = |A| + |B|$$

Cardinalty Cont'd

Theorem $|A \cup B| = |A| + |B| - |A \cap B|$

We can split $A \cup B$ into disjoint sets:
i.e. $A \cup B = (A - (A \cap B)) \cup (B - (A \cap B)) \cup (A \cap B)$

Proof.

$$\begin{array}{rcl} |A \cup B| & = & |(A - (A \cap B)) \cup (B - (A \cap B)) \cup (A \cap B)| \\ & & \{all\, these\, disjoint\} \\ & = & |A - (A \cap B)| + |B - (A \cap B)| + |A \cap B| \\ & & \{A \cap B \subseteq A\, and\, A \cap B \subseteq B\} \\ & = & |A| - |A \cap B| + |B| - |A \cap B| + |A \cap B| \\ & = & |A| + |B| - |A \cap B| \end{array}$$

∎

Cardinality of Sets

Cardinality $A \cup B \cup C$

$$\begin{aligned}
|A \cup B \cup C| &= |(A \cup B) \cup C| \\
&= |A \cup B| + |C| - |(A \cup B) \cap C| \\
&= \{Set\,Theory\,distributive\,law\} \\
&\quad\; |A \cup B| + |C| - |(A \cap C) \cup (B \cap C)| \\
&= |A| + |B| - |A \cap B| + |C| \\
&\quad\; -(|A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)| \\
&= |A| + |B| + |C| \\
&\quad\; -(|A \cap B| + |A \cap C| + |B \cap C|) \\
&\quad\; +|A \cap B \cap C|
\end{aligned}$$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Example

Students pass the year if they pass all 3 exams A, B, C.
For a particular year it was found that

- 3% failed all 3 papers
- 9% failed papers B and C
- 10% failed papers A and C
- 12% failed papers A and B
- 32% failed paper A
- 30% failed paper B
- 46% failed paper C

1. What percentage of students passed the year
2. What percentage failed exactly one paper.

Solution

Solution:

			B	B
	20	30	6	12
A	13	7	3	9
		C	C	

1.3 Power Set

Power Set

The Power Set, $P(S)$, of a set S, is the set of subsets of S,
i.e. $x \in P(S) \equiv x \subseteq S$.
If $|S| = n$ then $|P(S)| = 2^n$.

Example 9. $S = \{a, b, c\}$
$P(S) = \{\, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \,\}$
where $\emptyset$ is the empty set, i.e. $\emptyset = \{\}$.

In forming the subsets of S e.g.$S = \{0, 1, 2, 3, \ldots, n-1\}$, we have 2 choices for each element; to include it or exclude it.

2 choices for 0, 2 choices for 1, 2 choices for 2 etc.

Total #choices $= 2 * 2 * \cdots * 2$ (n times) $= 2^n$.

There is a natural correspondence between the subsets of $0, 1, 2, 3, \ldots, n-1$ and binary numbers.

Subsets and Binary

subset	$n-1$	$\ldots$	k	$\ldots$	3	2	1	0
$\{\}$	0	$\ldots$	0	$\ldots$	0	0	0	0
$\{0\}$	0	$\ldots$	0	$\ldots$	0	0	0	1
$\{1\}$	0	$\ldots$	0	$\ldots$	0	0	1	0
$\{0, 1\}$	0	$\ldots$	0	$\ldots$	0	0	1	1
$\vdots$								
$\{\ldots, k, \ldots\}$			1					
$\vdots$								
$\{0, 1, 2, \ldots k, \ldots n\}$	1	$\ldots$	1	$\ldots$	1	1	1	1

- 0 in column, k, indicates that k is not in the subset
- 1 in column, k, indicates that k is in the subset.

Binary and Decimal

Binary	decimal
$0 \ldots 0$	$0 = 0 * 2^{n-1} + \cdots + 0 * 2^2 + 0 * 2^1 + 0 * 2^0$
$0 \ldots 1$	$1 = 0 * 2^{n-1} + \cdots + 0 * 2^2 + 0 * 2^1 + 1 * 2^0$
$0 \ldots 10$	$2 = 0 * 2^{n-1} + \cdots + 0 * 2^2 + 1 * 2^1 + 0 * 2^0$
$0 \ldots 11$	$3 = 0 * 2^{n-1} + \cdots + 0 * 2^2 + 1 * 2^1 + 1 * 2^0$
	$\vdots$
$1 \ldots 1$	$2^n - 1 = 1 * 2^{n-1} + \cdots + 1 * 2^2 + 1 * 2^1 + 1 * 2^0$

$|P(S)| = 2^{|S|}$ Proof by Induction

$|P(S)| = 2^{|S|}$

Let $|S| = n$. Proof by induction on n.

Base Case:

$n = 0$

If $|S| = 0$ then $S = \emptyset \therefore P(S) = \{\emptyset\}$.

$|\{\emptyset\}| = 1$ tf $|P(S)| = 1 = 2^0 = 2^{|S|}$.

Induction Step:

Assume true for n, show true for $n + 1$.

i.e. Assume (if $|A| = n$ then $|P(A)| = 2^n$),

show (if $|S| = n + 1$ then $|P(S)| = 2^{n+1}$).

Induction Step

Assume $|S| = n + 1$.

Consider an element, x, of S, i.e. $x \in S$.

Discard x , then we have $S - \{x\}$ and $\therefore |S - \{x\}| = n$.

By induction, $|P(S - \{x\})| = 2^n$.

The original subsets of S consist of

- those that do not have the element, x, i.e. the subsets of $S - \{x\}$. and $|P(S - \{x\})| = 2^n$.
- those that do have the element, x, which are the subsets of of $S - \{x\}$ with the element, x, added in, giving 2^n subsets.

$\therefore |P(S)| = 2^n + 2^n = 2^{n+1}$.

Cantor's Theorem, $|\mathbb{N}| \neq |P(\mathbb{N})|$

CardinalityofSets

Let $S = \{0, 1, 2\}$ and $P(S) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$, $\therefore$

$|S| = 3$ and $|P(S)| = 8$ and in this case $|S| \neq |P(S)|$. For a finite set, S, $|S| \neq |P(S)|$.

Sets with same Cardinality

Two sets have the same cardinality iff there is a one to one (1-1) correspondence between both sets.

Let $A = \{a, b, c, d, e, \ldots, x, y, z\}$and $B = \{1, 2, 3, \ldots 26\}$ then $|A| = |B|$ as we have the 1-1 correspondence

a	b	c	$\ldots$	y	z
1	2	3	$\ldots$	25	26

$|\mathbb{N}| = |Even|$

$|\mathbb{N}| = |Even|$

Consider infinite sets:

Infinite sets S_1 and S_2 have the same cardinality if there is a one to one (1-1) correspondence between both sets.

Let $Even$ be the set of even natural numbers then $|\mathbb{N}| = |Even|$ as:

$Even$	0	2	4	6	$\dots$	$2*n$	$\dots$
$\mathbb{N}$	0	1	2	3	$\dots$	n	$\dots$

There is a (1-1) correspondence between the two sets $\mathbb{N}$ and $Even$. The sets $\mathbb{N}$ and $Even$ have the same cardinality i.e. $|\mathbb{N}| = |Even|$, even though $Even \subseteq \mathbb{N}$ and $Even \neq \mathbb{N}$.

$|\mathbb{N}| = |\mathbb{Z}|$

$|\mathbb{N}| = |\mathbb{Z}|$

Consider a 1-1 correspondence between $\mathbb{N}$ and $\mathbb{Z}$,

$\mathbb{N}$	$\dots$	$-(2*n+1)$	$\dots$	3	1	0	2	4	$\dots$	$2*n$	$\dots$
$\mathbb{Z}$	$\dots$	$-n$	$\dots$	-2	-1	0	1	2	$\dots$	n	$\dots$

The negative integers are in (1-1) correspondence with the odd natural numbers and the positive integers are in (1-1) correspondence with the even natural numbers.

Proof of Cantor's Theorem $|\mathbb{N}| \neq |P(\mathbb{N})|$

Cantor's Theorem

$|\mathbb{N}| \neq |P(\mathbb{N})|$

Proof is by contradiction.

Assume $|\mathbb{N}| = |P(\mathbb{N})|$, $\therefore$ there is a (1-1) correspondence between $\mathbb{N}$ and $P(\mathbb{N})$.

$\mathbb{N}$	0	1	$\dots$	n	$\dots$
$P(\mathbb{N})$	$sub(0)$	$sub(1)$	$\dots$	$sub(n)$	$\dots$

where $sub(n)$ is the subset corresponding to n.

Also, for each subset, S, of $\mathbb{N}$ there is a matching element in $\mathbb{N}$,

i.e. for each element $S \in P(\mathbb{N})$, there is an element, $k \in \mathbb{N}$, such that $sub(k) = S$. **Note**: $S \in P(\mathbb{N})$ iff $S \subseteq \mathbb{N}$.

Cantor's Thm. (Cont'd)

For each subset, $sub(n)$, of $\mathbb{N}$, either $n \in sub(n)$ or $n \notin sub(n)$.

Define a subset D of $\mathbb{N}$, such that

$$D = \{k \in \mathbb{N} : \ k \notin sub(k)\}$$

i.e. for $k \in \mathbb{N}$,

$$k \in D \equiv k \notin sub(k)$$

Note similarity with Russell Set, R, where
$R = \{x \mid x \notin x\}$
i.e. $x \in R \equiv x \notin x$.

Cantor's Thm. (Cont'd)

Since $D \subseteq \mathbb{N}$, i.e. $D \in P(\mathbb{N})$,
there is an element, $d \in \mathbb{N}$, such that $sub(d) = D$, $\therefore$

$$d \in sub(d) \equiv d \in D$$

but from the definition of D,

$$d \in D \equiv d \notin sub(d)$$

and so $d \in sub(d) \equiv d \notin sub(d)$, a contradiction. This contradiction arose due to assuming that $|\mathbb{N}| = |P(\mathbb{N})|$ $\therefore$ $|\mathbb{N}| \neq |P(\mathbb{N})$.