1 Set Theory

Set Theory

1. Preliminaries
2. Russell's Paradox
3. Set Operations
4. Set Properties
5. Cardinality of Sets
6. Power Set
7. Cantor's Theorem on Infinite Sets.

1.1 Preliminaries

Defining Sets

Sets can be defined by extension i.e. listing the elements or by a property.

- By Extension i.e. listing the elements: e.g. $A = \{2, 3, 5, 7, 11, 13, 17, 19\}$
- By a Property (or a Predicate): e.g. $B = \{n \mid n < 20 \;\; and \;\; n \, is \, prime\}$
  - $\{\}$ is the empty set; it has no elements. e.g. $\{\} = \{x \mid x \neq x\}$.

Notation: $\in$
Read $x \in S$ as "x is an element of the set S" or "x is in S".
Query:
Does the set $E = \{x \mid x = x\}$, the set of everything, exist.
i.e. $x \in E \equiv x \notin \{\}$

Subset $\subseteq$

$X \subseteq Y$ iff every element of X is an element of Y.
i.e. there is no element, z, such that $z \in X$ and $z \notin Y$.

Theorem 1. *$\{\} \subseteq X$, for all sets X.*

Since there are no elements in the empty set, $\{\}$, there is no element, z, in the empty set $\{\}$ such $z \in \{\}$ and $z \notin X$ hence $\{\} \subseteq X$.

Equality of Sets

Equality of Sets

Two sets X and Y are equal if they have the same elements, i.e. all the elements in X are in Y and all the elements in Y are in X.

$$X = Y \text{ iff } X \subseteq Y \text{ and } Y \subseteq X$$

Example 2. Sets
$A = \{2, 3, 5, 7, 11, 13, 17, 19\}$ and
$B = \{n | n < 20 \;\; and \;\; n \, is \, prime\}$
are equal.

Example: Equal Sets

Example 3. Let $S = \{x \in \mathbb{R} \mid x^2 < x\}$ and $T = \{x \in \mathbb{R} \mid 0 < x < 1\}$
Show $S = T$.
Show i) $S \subseteq T$ and ii) $T \subseteq S$

Proof. i) $S \subseteq T$
Let $x \in S$ then
$x^2 < x$
$\equiv x^2 - x < 0$
$\equiv x(x-1) < 0$

x and $x - 1$ have opposite signs or parity; □

Proof cont'd

Proof. Case: $x > 0$ and $x - 1 < 0$
If $x > 0$ and $x - 1 < 0$ i.e. $x < 1$ then $0 < x < 1$.
$\therefore$ (therefore) $x \in T$.
Case: $x < 0$ and $x - 1 > 0$
Impossible, as $x - 1 < x$. □

Proof Cont'd

Proof. ii) $T \subseteq S$
Let $x \in T$ then $0 < x < 1$
$\therefore x < 1$,
$\{$multiply both sides by x and since $0 < x\}$
$\therefore x^2 < x$,
$\therefore x \in S$. □

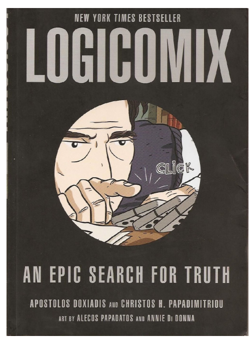
Number Sets

- $\mathbb{N}$, Natural Numbers {0,1,2,...}
- $\mathbb{Z}$, Integers {...-2, -1, 0, 1, 2, ...}
- $\mathbb{Q}$, Rationals (Fractions) $\{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$
- $\mathbb{R}$, Real numbers e.g. all the points on the number line.

1.2 Russell's Paradox

Bertrand Russell

Logicomix Book



https://www.logicomix.com/en/index.html https://www.youtube.com/watch?v=XebglmXrgEc

Bertrand Russell
Bertrand Russell (1872–1970) was a British philosopher, logician, essayist and social critic best known for his work in mathematical logic and analytic philosophy.

Over the course of a long career, Russell also made significant contributions to a broad range of other subjects, including ethics, politics, educational theory, the history of ideas and religious studies, cheerfully ignoring Hooke's admonition to the Royal Society against "meddling with Divinity, Metaphysics, Moralls, Politicks, Grammar, Rhetorick, or Logick" (Kreisel 1973, 24). In addition, generations of general readers have benefited from his many popular writings on a wide variety of topics in both the humanities and the natural sciences. Like Voltaire, to whom he has been compared (Times of London 1970, 12)), he wrote with style and wit and had enormous influence.

(https://plato.stanford.edu/entries/russell/)

Russell's Paradox

Example 4. Library Reference Books

A library has reference books, some of which refer to themselves and some which do not. The Librarian creates a new reference book, R, with the entries of all those (and only those) reference books that do not reference themselves. The book, R, is a reference book and the librarian now considers whether they should include a reference to R in the book R.

Russell's Paradox Cont'd

Should R include a reference to itself

- Case: Include a reference to R in R: By definition of R, it should contain references to books that do not reference themselves. Since by assumption a reference to R is included in R then, by definition, R should not include a reference to R.
- Case: Exclude a reference to R in R: By assumption, R does not contain a reference to R hence by definition of R, the book R should include a reference to R.

In either case, we get a contradiction.

Simpler Version

Consider a library with just 25 reference books, labeled A to Y. Some of these reference themselves, some don't. Consider a new reference book, Z, which is made of just the entries of the books A to Y which do not reference themselves. If book A does not list itself then book Z includes an entry for A. Should Z reference itself or not?

We may need to be more precise. We could include a time limit. Book Z is to include all books that do not reference themselves by 12noon. Therefore, since book Z is not in the library at 12noon, it will not be mentioned in his own entries.

Simpler Version Cont'd

Continuing with time limit then books A to Y exist at 12 noon but book Z does not. Book Z is created by 1pm containing entries to all books in A to Y that do not reference themselves by 12noon and so book Z does not contain a reference to itself. If time continues and we consider all reference books in library at 1pm then we update Z to include a reference to Z. At 2pm we update book Z to exclude a reference to book Z and we continue on forever. Book Z is never completed and so is more of a project than a book in the library.

Normal Sets

A set is called *normal* (or ordinary) if it does not contain itself. The set of numbers $s = \{1, 2, 3\}$ is normal as $s \notin \{1, 2, 3\}$, i.e. $s \notin s$. It is not easy to think of a set that is not normal but a possibility would be the set of everything i.e. E above.

Consider a set, N, the set of all normal sets: i.e. For any x, $x \in N$ iff x is a normal set.

$N = \{x \mid x \notin x\}$

$x \in N \equiv x \notin x$

Substituting N for x we get

$N \in N \equiv N \notin N$

This is a contradiction. So there is a problem in defining a set using the property, $x \notin x$.

(Local) Universal Set

Also there is likewise a problem in defining a set $E = \{x \mid x = x\}$ and so in standard set theory, the set of all sets is not regarded as a set.

If one is to use set thery, one begins by first determining the set of elements of interest. That is, one defines a local universal set, U. For example, in mathematics, the set $\mathbb{R}$ (real numbers) is frequently used as a local universal set, U.

1.3 Set Operations

Set Operations

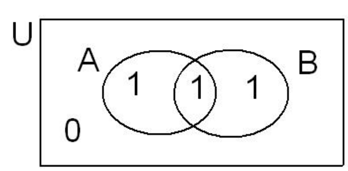
Union, intersection, difference, complement.

- Union: $X \cup Y = \{z \mid z \in X \text{ or } z \in Y\}$ $z \in X \cup Y$ iff either $z \in X$ or $z \in Y$ or z is in both X and Y **Note**: $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$
- Intersection: $X \cap Y = \{z \mid z \in X \text{ and } z \in Y\}$ z is in both X and Y **Note**: $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$
- Difference: $X - Y = \{z \mid z \in X \text{ and } z \notin Y\}$ $z \in X$ but z is not in Y.
- Complement: $\overline{X} = U - X$ where U is a universal set of elements of interest. In particular, $\overline{\{\}} = U$ and $\overline{U} = \{\}$

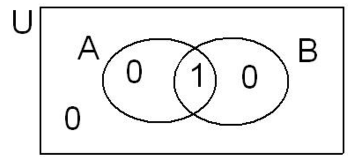
Venn Diagrams

Venn Diagrams

$A \cup B$

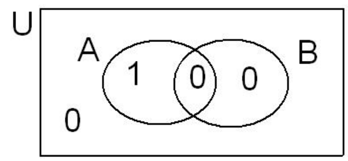


$A \cap B$

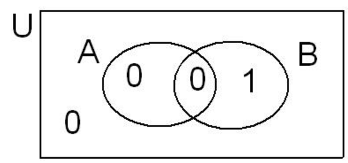


Venn Diagrams (Cont'd)

$A - B$

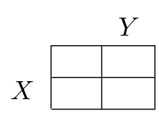


$B - A$



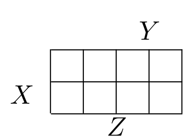
Veitch Diagram/Karnaugh Map

Instead of Circles or Ellipses representing subsets of U, one can use Squares or Rectangles. The various subsets are represented as equal area divisions of U. If we are concerned with 2 subsets X and Y of U then in a Veitch Diagram/Karnaugh Map, the subset X is respresented as the bottom half division and the subset Y as the right half division.



Veitch Diagram/Karnaugh Map (Cont'd)

For 3 subsets X, Y and Z of U, we have the Veitch Diagram/Karnaugh Map



X is the 'bottom half'. Y the 'right half' and Z the 'middle half' of the Universal set, U.

Veitch .v. Karnaugh

An ancestor of the Veitch Diagram/Karnaugh Map was proposed by Marquand in 1881. Veitch rediscovered the Marquand diagram in his 1952 article

"A Chart Method for Simplifying Truth Functions",

The 'Veitch/Marquand Chart' was modified by Karnaugh in 1953 and even though we use the Karnaugh map variation we abbreviate the 'Veitch Diagram/Karnaugh Map' to 'Veitch Diagram'. We call Veitch Diagrams what others may call Karnaugh Maps.

Veitch Diagram: Union and Intersection

Veitch Diagram: Union and Intersection

e.g. $X \cup Y$

$$X \cup Y = \quad \begin{array}{c|c|c|} & & Y \\ \cline{2-3} & 0 & 1 \\ \cline{2-3} X & 1 & 1 \\ \cline{2-3} \end{array}$$

e.g. $X \cap Y$

$$X \cap Y = \quad \begin{array}{c|c|c|} & & Y \\ \cline{2-3} & 0 & 0 \\ \cline{2-3} X & 0 & 1 \\ \cline{2-3} \end{array}$$

Veitch Diagram

Each of the 'cells' in the Veitch Diagram represent a subset of the whole 'box' which represents the Universal set, U.

$$\text{Universal Set} \quad \begin{array}{c|c|c|} & \overline{Y} & Y \\ \hline \overline{X} & \overline{X} \cap \overline{Y} & \overline{X} \cap Y \\ \hline X & X \cap \overline{Y} & X \cap Y \\ \hline \end{array}$$

From above, $X \cup Y$ is the union of disjoint subsets of U.

$$X \cup Y = (X \cap \overline{Y}) \cup (X \cap Y) \cup (\overline{X} \cap Y)$$

One Variable Sets

Set expression with one variable:

e.g. set X is represented as

$$X = \quad \begin{array}{c|c|} \cline{2-2} & 0 \\ \cline{2-2} X & 1 \\ \cline{2-2} \end{array}$$

e.g. $\overline{X}$, the complement of X

$$\overline{X} = \begin{array}{c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{2-2} & 1 \\ \cline{2-2} X & 0 \\ \cline{2-2} \end{array}$$

Note: $\overline{\overline{X}} = X$.

Set Difference

e.g. $X - Y$

$$X - Y = \begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{2}{c}{Y} \\ \cline{2-3} & 0 & 0 \\ \cline{2-3} X & 1 & 0 \\ \cline{2-3} \end{array}$$

Note: $X - Y = X \cap \overline{Y}$

e.g. $Y - X$

$$Y - X = \begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{2}{c}{Y} \\ \cline{2-3} & 0 & 1 \\ \cline{2-3} X & 0 & 0 \\ \cline{2-3} \end{array}$$

Note: $Y - X = Y \cap \overline{X}$

Defining Set Operations

Example 5. Define a set operator, $\oplus$, such that $X \oplus Y = (X \cap \overline{Y}) \cup (Y \cap \overline{X})$

$$\begin{array}{c|c|c|} \multicolumn{1}{c}{X \cap \overline{Y}} & \multicolumn{2}{c}{Y} \\ \cline{2-3} & 0 & 0 \\ \cline{2-3} X & 1 & 0 \\ \cline{2-3} \end{array} \quad \cup \quad \begin{array}{c|c|c|} \multicolumn{1}{c}{Y \cap \overline{X}} & \multicolumn{2}{c}{Y} \\ \cline{2-3} & 0 & 1 \\ \cline{2-3} X & 0 & 0 \\ \cline{2-3} \end{array}$$

'Union' these to get

$$X \oplus Y \quad = \quad \begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{2}{c}{Y} \\ \cline{2-3} & 0 & 1 \\ \cline{2-3} X & 1 & 0 \\ \cline{2-3} \end{array}$$

Example Exercise

Exercise: Determine whether $(X - Y) \cup (Y - X) = (X \cup Y) - (X \cap Y)$

LHS $(X - Y) \cup (Y - X)$

$$\begin{array}{c|c|c|} \multicolumn{1}{c}{X - Y} & \multicolumn{2}{c}{Y} \\ \cline{2-3} & 0 & 0 \\ \cline{2-3} X & 1 & 0 \\ \cline{2-3} \end{array} \quad \cup \quad \begin{array}{c|c|c|} \multicolumn{1}{c}{Y - X} & \multicolumn{2}{c}{Y} \\ \cline{2-3} & 0 & 1 \\ \cline{2-3} X & 0 & 0 \\ \cline{2-3} \end{array}$$

'Union' these to get

$$(X - Y) \cup (Y - X) \quad = \quad \begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{2}{c}{Y} \\ \cline{2-3} & 0 & 1 \\ \cline{2-3} X & 1 & 0 \\ \cline{2-3} \end{array}$$

Exercise Cont'd

RHS $(X \cup Y) - (X \cap Y)$

$$
\begin{array}{cc} & \text{Y} \\ X \cup Y & \\ & \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 1 \\ \hline \end{array} \\ \text{X} & \end{array}
\quad - \quad
\begin{array}{cc} & Y \\ X \cap Y & \\ & \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \\ X & \end{array}
$$

'Difference' these to get

$$
(X \cup Y) - (X \cap Y) \quad = \quad \begin{array}{cc} & Y \\ X & \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} \end{array}
$$

Therefore

$X \oplus Y$

$= (X \cap \overline{Y}) \cup (Y \cap \overline{X})$

$= (X - Y) \cup (Y - X)$

$= (X \cup Y) - (X \cap Y)$

3-Variable Set Expressions

Set expression with 3 variables

Example 6. $X \cup \overline{Y} \cup Z$

$$
\begin{array}{cc} & Y \\ X & \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 0 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} \\ & Z \end{array}
$$

Cont'd

Example 7. Determine $X \cup (X \cap Z) \cup (Y \cap Z)$

$$
\begin{array}{cc} & Y \\ X & \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} \\ & Z \end{array}
\cup
\begin{array}{cc} X \cap Z & Y \\ X & \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ \hline \end{array} \\ & Z \end{array}
$$

$$
\cup
\begin{array}{cc} Y \cap Z & Y \\ X & \begin{array}{|c|c|c|c|} \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline \end{array} \\ & Z \end{array}
$$

Cont'd

'Union' these together to get:

$$
\begin{array}{cc} & Y \\ X & \begin{array}{|c|c|c|c|} \hline 0 & 0 & 1 & 0 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} \\ & Z \end{array}
$$

$(X \oplus Y) \oplus Z$

Example 8. **Example_A**: Determine $(X \oplus Y) \oplus Z$

Let $A = X \oplus Y$ i.e. $A = (X \cap \overline{Y}) \cup (Y \cap \overline{X}))$ $\therefore$

$$\text{A} \quad = \quad \begin{array}{r|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{4}{c}{Y} \\ \cline{2-5} & 0 & 0 & 1 & 1 \\ \cline{2-5} X & 1 & 1 & 0 & 0 \\ \cline{2-5} \multicolumn{1}{c}{} & \multicolumn{4}{c}{Z} \end{array}$$

$(X \oplus Y) \oplus Z$ **Cont'd**

$A \oplus Z = (A \cap \overline{Z}) \cup (Z \cap \overline{A})$

$$\begin{array}{r|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{4}{c}{Y} \\ \cline{2-5} & 0 & 0 & 1 & 1 \\ \cline{2-5} X & 1 & 1 & 0 & 0 \\ \cline{2-5} \multicolumn{1}{c}{} & \multicolumn{4}{c}{Z} \end{array} \quad \cap \quad \begin{array}{r|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{4}{c}{Y} \\ \cline{2-5} & 1 & 0 & 0 & 1 \\ \cline{2-5} X & 1 & 0 & 0 & 1 \\ \cline{2-5} \multicolumn{1}{c}{} & \multicolumn{4}{c}{Z} \end{array}$$

$$A \quad \cap \quad \overline{Z}$$

$$\cup$$

$$\overline{A} \quad \cap \quad Z$$

$$\begin{array}{r|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{4}{c}{Y} \\ \cline{2-5} & 1 & 1 & 0 & 0 \\ \cline{2-5} X & 0 & 0 & 1 & 1 \\ \cline{2-5} \multicolumn{1}{c}{} & \multicolumn{4}{c}{Z} \end{array} \quad \cap \quad \begin{array}{r|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{4}{c}{Y} \\ \cline{2-5} & 0 & 1 & 1 & 0 \\ \cline{2-5} X & 0 & 1 & 1 & 0 \\ \cline{2-5} \multicolumn{1}{c}{} & \multicolumn{4}{c}{Z} \end{array}$$

$(X \oplus Y) \oplus Z$ **Cont'd**

i.e.

$$\begin{array}{r|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{4}{c}{Y} \\ \cline{2-5} & 0 & 0 & 0 & 1 \\ \cline{2-5} X & 1 & 0 & 0 & 0 \\ \cline{2-5} \multicolumn{1}{c}{} & \multicolumn{4}{c}{Z} \end{array}$$

$$\cup$$

$$\begin{array}{r|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{4}{c}{Y} \\ \cline{2-5} & 0 & 1 & 0 & 0 \\ \cline{2-5} X & 0 & 0 & 1 & 0 \\ \cline{2-5} \multicolumn{1}{c}{} & \multicolumn{4}{c}{Z} \end{array}$$

i.e.

$$(X \oplus Y) \oplus Z \quad = \quad \begin{array}{r|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{4}{c}{Y} \\ \cline{2-5} & 0 & 1 & 0 & 1 \\ \cline{2-5} X & 1 & 0 & 1 & 0 \\ \cline{2-5} \multicolumn{1}{c}{} & \multicolumn{4}{c}{Z} \end{array}$$

$X \oplus (Y \oplus Z)$

Example 9. **Example_B** Determine $X \oplus (Y \oplus Z)$

Consider $\oplus$ as 'addition mod 2'

Let $B = Y \oplus Z \therefore$ by adding individual cells 'mod 2' we get:

$$
\begin{array}{rcl}
B & = & \begin{array}{c|c|c|c|c|} \multicolumn{5}{r}{Y} \\ \cline{2-5} & 0 & 0 & 1 & 1 \\ \cline{2-5} X & 0 & 0 & 1 & 1 \\ \cline{2-5} \multicolumn{5}{c}{Z} \end{array} \quad \oplus \quad \begin{array}{c|c|c|c|c|} \multicolumn{5}{r}{Y} \\ \cline{2-5} & 0 & 1 & 1 & 0 \\ \cline{2-5} X & 0 & 1 & 1 & 0 \\ \cline{2-5} \multicolumn{5}{c}{Z} \end{array} \\
& = & \begin{array}{c|c|c|c|c|} \multicolumn{5}{r}{Y} \\ \cline{2-5} & 0 & 1 & 0 & 1 \\ \cline{2-5} X & 0 & 1 & 0 & 1 \\ \cline{2-5} \multicolumn{5}{c}{Z} \end{array} \\
& = & Y \quad \oplus \quad Z
\end{array}
$$

$X \oplus (Y \oplus Z)$ **Cont'd**

$\therefore X \oplus (Y \oplus Z) = X \oplus B$

$$
\begin{array}{ccc}
\begin{array}{c|c|c|c|c|} \multicolumn{5}{r}{Y} \\ \cline{2-5} & 0 & 0 & 0 & 0 \\ \cline{2-5} X & 1 & 1 & 1 & 1 \\ \cline{2-5} \multicolumn{5}{c}{Z} \end{array} & \oplus & \begin{array}{c|c|c|c|c|} \multicolumn{5}{r}{Y} \\ \cline{2-5} & 0 & 1 & 0 & 1 \\ \cline{2-5} X & 0 & 1 & 0 & 1 \\ \cline{2-5} \multicolumn{5}{c}{Z} \end{array} \\
X & \oplus & B
\end{array}
$$

$$
\begin{array}{rl}
= & \begin{array}{c|c|c|c|c|} \multicolumn{5}{r}{Y} \\ \cline{2-5} & 0 & 1 & 0 & 1 \\ \cline{2-5} X & 1 & 0 & 1 & 0 \\ \cline{2-5} \multicolumn{5}{c}{Z} \end{array} \\
= & X \quad \oplus \quad (Y \quad \oplus \quad Z)
\end{array}
$$

Disjoint Union, Symmetric Difference

From **Example_A** and **Example_B**

$$(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z)$$

Since $\oplus$ is associate we can insert brackets to suit. From above the $X \oplus Y \oplus Z$ consists of 'cells' i.e.

$$X \oplus Y \oplus Z = (X \cap \overline{Y} \cap Z) \cup (\overline{X} \cap \overline{Y} \cap Z) \cup (X \cap Y \cap Z) \cup (\overline{X} \cap Y \cap \overline{Z})$$

The set operator, $\oplus$ is referred to as 'Disjoint Union' or 'Symmetric Difference'. From above:

$$
\begin{array}{rcl}
X \oplus Y & = & (X \cap \overline{Y}) \cup (Y \cap \overline{X}) \\
& = & (X - Y) \cup (Y - X) \\
& = & (X \cup Y) - (X \cap Y)
\end{array}
$$

Labelling Cells in Veitch Diagram

We can lable the 'cells' in a Veitch Diagram using binary numbers.

$$\begin{array}{cc|c|c|c|c|}
 & & & & y & \\
 & & 00 & 01 & 11 & 10 \\
\cline{3-6}
 & 0 & 000 & 001 & 011 & 010 \\
\cline{3-6}
x & 1 & 100 & 101 & 111 & 110 \\
\cline{3-6}
 & & & z & & \\
\end{array}$$

Note: Moving from one cell to a neighbour is a one 'bit' change.

Using Decimal numbers we can label the cells as:

$$\begin{array}{cc|c|c|c|c|}
 & & & & y & \\
 & & 00 & 01 & 11 & 10 \\
\cline{3-6}
 & 0 & 0 & 1 & 3 & 2 \\
\cline{3-6}
x & 1 & 4 & 5 & 7 & 6 \\
\cline{3-6}
 & & & z & & \\
\end{array}$$

Binary representation of the sets {} and U

Consider the set $B = \{0, 1\}$.

We can represent the empty set, $\{\}$ by 0 and the Universal set, U, by 1. If x and y are elements of $\{0, 1\}$ or using the representatation,x and y are elements of the set $\{\{\}, U\}$ then

$x \oplus y$	0	1
0	**0**	**1**
1	**1**	**0**

$x \cup y$	0	1
0	**0**	**1**
1	**1**	**1**

$x \cap y$	0	1
0	**0**	**0**
1	**0**	**1**

$x - y$	0	1
0	**0**	**0**
1	**1**	**0**

Also, $\overline{0} = 1$ and $\overline{1} = 0$ as $\overline{\{\}} = U$ and $\overline{U} = \{\}$.

In some textbooks, the symbol, 0, is used for the empty set, $\{\}$, and the symbol, 1, is used for the Universal set, U .

$\cap$, $\cup$, in terms of $*_2$, $+_2$

Let $x, y \in B$ where $B = \{0, 1\}$.

Abbreviate $(x + y \, mod \, 2)$ as $(x +_2 y)$ and $(x * y \, mod \, 2)$ as $(x *_2 y)$.

We can define $x \cap y$, $x \cup y$, $\overline{x}$, $x - y$ and $x \oplus y$ in terms of $+_2$ and $*_2$.

$$\begin{aligned}
x \cap y &= x *_2 y \\
x \cup y &= x +_2 y +_2 x *_2 y \\
\overline{x} &= x +_2 1 \\
x - y &= x *_2 (y +_2 1) \\
x \oplus y &= x +_2 y
\end{aligned}$$