Inverse of a (2×2) Matrix

Identity Matrix

The identity matrix for Matrix multiplication is the matrix, Id, such that for a matrix, M, $M * Id = Id * M = M$.

For 2×2 matrices the Identity matrix is: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

as for a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} a*1+b*0 & a*0+b*1 \\ c*1+d*0 & c*0+d*1 \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{aligned}$$

Simlarly, $Id * M = M$

Also, for a vector, $\begin{bmatrix} x \\ y \end{bmatrix}$, we have $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

Inverse of 2x2 Matrix

The inverse of a Matrix, M, is a Matrix, X, such that:

$$M * X = X * M = Id$$

The inverse, if it exists, of a Matrix, M, is denoted by M^{-1}.
In arithmetic, the inverse of a number, n , can be written as n^{-1}, as $n * n^{-1} = 1$.

Finding Inverse of 2x2 Matrix

Given a matrix, $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

find a matrix, $M^{-1} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e.

$$\begin{bmatrix} a*x+b*z & a*y+b*w \\ c*x+d*z & c*y+d*w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Cont'd

Since Matrices are equal if corresponding entries are equal, the above gives rise to the following two systems of simultaneous equations.

I	II
$a * x + b * z = 1$	$a * y + b * w = 0$
$c * x + d * z = 0$	$c * y + d * w = 1$

Given the values a, b, c and d find the values of x, y, z and w.

Using 2×2 Determinants

If M is a matrix, let $|M|$ be the Determinant of M.

From above, $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Using 2×2 Determinants we can find the solutions to the simultaneous equations.

Equations I:

$$x = \frac{\begin{vmatrix} 1 & b \\ 0 & d \end{vmatrix}}{|M|} = \frac{d}{|M|} \qquad z = \frac{\begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix}}{|M|} = \frac{-c}{|M|}$$

Equations II:

$$y = \frac{\begin{vmatrix} 0 & b \\ 1 & d \end{vmatrix}}{|M|} = \frac{-b}{|M|} \qquad w = \frac{\begin{vmatrix} a & 0 \\ c & 1 \end{vmatrix}}{|M|} = \frac{a}{|M|}$$

Matrix Inverse, 2×2

The Inverse of $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\begin{aligned} M^{-1} &= \begin{bmatrix} \frac{d}{|M|} & \frac{-b}{|M|} \\ \frac{-c}{|M|} & \frac{a}{|M|} \end{bmatrix} \\ &= \frac{1}{|M|} * \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \end{aligned}$$

Check Matrix Inverse, 2x2

The Inverse of $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{|M|} * \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Check:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \left(\frac{1}{|M|} * \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)$$

$$= \frac{1}{|M|} * \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)$$

$$= \frac{1}{|M|} * \begin{bmatrix} a*d - b*c & -a*b + b*a \\ c*d - d*c & -c*b + d*a \end{bmatrix}$$

$$= \frac{1}{a*d - b*c} * \begin{bmatrix} a*d - b*c & 0 \\ 0 & a*d - b*c \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Inverse Example

Let $M = \begin{bmatrix} 5 & 8 \\ 3 & 5 \end{bmatrix}$.

The Determinant, $|M| = \begin{vmatrix} 5 & 8 \\ 3 & 5 \end{vmatrix} = 5 * 5 - 3 * 8 = 1$.

$$M^{-1} = \frac{1}{1} * \begin{bmatrix} 5 & -8 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 5 & -8 \\ -3 & 5 \end{bmatrix}$$

Check:

$$\begin{bmatrix} 5 & 8 \\ 3 & 5 \end{bmatrix} * \begin{bmatrix} 5 & -8 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 5*5+8*(-3) & 5*(-8)+8*5 \\ 3*5+5*(-3) & 3*(-8)+5*5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Also, $\begin{bmatrix} 5 & -8 \\ -3 & 5 \end{bmatrix} * \begin{bmatrix} 5 & 8 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Matrix Inverse 2×2 (Cont'd)

Let $M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

- Exchange the elements on the (down) Diagonal
- Change the signs of the items on the (up) Diagonal
- Multiply by $\frac{1}{|M|}$

to get

$$M^{-1} = \frac{1}{|M|} * \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Note: $|M|$ is also written as $det(M)$.

Matrix Method for Solving Simultaneous Equations

Recall that to solve the simultaneous equation

$$\begin{aligned} 5 * x + 8 * y &= 18 \\ 3 * x + 5 * y &= 11 \end{aligned}$$

in terms of Matrices and Vectors, we find a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ such that

$$\begin{bmatrix} 5 & 8 \\ 3 & 5 \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18 \\ 11 \end{bmatrix}$$

Multiply both sides by the inverse of $\begin{bmatrix} 5 & 8 \\ 3 & 5 \end{bmatrix}$ which from above is $\begin{bmatrix} 5 & -8 \\ -3 & 5 \end{bmatrix}$ i.e.

$$\begin{bmatrix} 5 & 8 \\ 3 & 5 \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -8 \\ -3 & 5 \end{bmatrix} * \begin{bmatrix} 5 & 8 \\ 3 & 5 \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & -8 \\ -3 & 5 \end{bmatrix} * \begin{bmatrix} 18 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 * 18 + (-8) * 11 \\ (-3) * 18 + 5 * 11 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

i.e. $x = 2$ and $y = 1$

Cont'd

In general, for a Matrix M and vector, v and constant vector, k, if

$$M * v = k$$

then

$$v = M^{-1} * k$$

provided, M^{-1} exists.

Matrix Inverse by Gaussian Approach

Matrix Inverse by Gaussian Approach

Consider the matrix, $\begin{bmatrix} 5 & 8 \\ 3 & 5 \end{bmatrix}$. We create an 'Augmented Matrix' by attaching the Identity Matrix, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ to get

$$\left[\begin{array}{cc|cc} 5 & 8 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right]$$

From previous Gaussian elimination, the allowed row operations are:

- Interchange rows R_i and R_j
- Multiply row, R, by a number n to give $R := n * R$
- Add a multiple of one row to another: for $i \neq j$ $R_i := R_i + n * R_j$

Cont'd

$$\left[\begin{array}{cc|cc} 5 & 8 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right]$$

$R1 := R1/5$

$$\left[\begin{array}{cc|cc} 1 & \frac{8}{5} & \frac{1}{5} & 0 \\ 3 & 5 & 0 & 1 \end{array}\right]$$

$R2 := R2 - 3 * R1$

$$\left[\begin{array}{cc|cc} 1 & \frac{8}{5} & \frac{1}{5} & 0 \\ 0 & 5 - \frac{24}{5} & \frac{-3}{5} & 1 \end{array}\right]$$

Simplify

$$\left[\begin{array}{cc|cc} 1 & \frac{8}{5} & \frac{1}{5} & 0 \\ 0 & \frac{1}{5} & \frac{-3}{5} & 1 \end{array}\right]$$

$R2 := R2 * 5$

$$\left[\begin{array}{cc|cc} 1 & \frac{8}{5} & \frac{1}{5} & 0 \\ 0 & 1 & -3 & 5 \end{array}\right]$$

$R1 := R1 - \frac{8}{5} * R2$

$$\left[\begin{array}{cc|cc} 1 & \frac{8}{5} - \frac{8}{5} & \frac{1}{5} - \frac{-3*8}{5} & 0 - \frac{8}{5} * 5 \\ 0 & 1 & -3 & 5 \end{array}\right]$$

Simplify

$$\left[\begin{array}{cc|cc} 1 & 0 & 5 & -8 \\ 0 & 1 & -3 & 5 \end{array}\right]$$

$\therefore$

inverse of $\begin{bmatrix} 5 & 8 \\ 3 & 5 \end{bmatrix}$ is $\begin{bmatrix} 5 & -8 \\ -3 & 5 \end{bmatrix}$ i.e.

if $M = \begin{bmatrix} 5 & 8 \\ 3 & 5 \end{bmatrix}$ then

$$M^{-1} = \begin{bmatrix} 5 & -8 \\ -3 & 5 \end{bmatrix}$$