Contradiction

A *contradiction* is always false. What is proven is always true and since a contradiction is always false it cannot be proven.
Central to mathematics is the concept of a proof, i.e. given a set of premises, how can one infer or deduce the conclusion. A conclusion is deducible from a set of premises if it is not possible for all the premises to be true and the conclusion false.
For example, the conclusion, Q, can be deduced from the set of premises $\{P, P \rightarrow Q\}$ i.e. it is not possible for both P and $P \rightarrow Q$ to be true and the sentence Q to be false.
If P and $P \rightarrow Q$ are both true then Q must also be true.
From the Truth Table of the operator, $\rightarrow$, the only situation where $P \rightarrow Q$ is false is when P is true and Q is false. Therefore, if we suppose Q to be false and we have assumed that P is true, then $P \rightarrow Q$ is false contradicting our assumption that $P \rightarrow Q$ is true.

Tautology

The sentences in Propositonal Logic that are always true are called **Tautologies**, e.g. $P \vee \neg P$ is a Tautology.
If a propositional sentence P is a contradiction i.e. always false (e.g. $P \wedge \neg P$) then its negation is a Tautology i.e. always true.
Since $P \wedge \neg P$ is a contradiction, then $\neg(P \wedge \neg P)$ is a Tautology.
By De Morgan's Law, $\neg(P \wedge \neg P) = \neg P \vee P$.

Tautology (Cont'd)

Definition

Tautology

If a propositional sentence, P, evaluates to T in all states (i.e. P is always true) then P is a **Tautology**. In particular, the constant, *True*, is a Tautology.

Notation $\models$

We abbreviate "P is a Tautology" to "$\models P$". The symbol $\models$ is pronounced 'double turnstyle' as there is also a 'single turnstyle' symbol $\vdash$.
We can read "$\models P$" as "P is a Tautology".

Tautology (Cont'd)

If P is a Tautology then it is always true, i.e. P is true no matter what values are assigned to the variables in P. We can check whether a sentence, P, is a Tautology by using a Truth Table.

Example: $\models p \vee \neg p$ as

p	$p \vee \neg p$
F	T
T	T

$\models p \to q \equiv \neg q \to \neg p$

Example: $\models p \to q \equiv \neg q \to \neg p$ as

p	q	$p \to q$	$\equiv$	$\neg$	q	$\to$	$\neg$	p
F	F	**T**	**T**	T		**T**	T	
F	T	**T**	**T**	F		**T**	T	
T	F	**F**	**T**	T		**F**	F	
T	T	**T**	**T**	F		**T**	F	

In Propositional Logic, Tautologies correspond to Logical Truths.

$\models p \wedge q \equiv p \equiv q \equiv p \vee q$

Example: Show $\models p \wedge q \equiv p \equiv q \equiv p \vee q$
Restoring brackets we get: $((((p \wedge q) \equiv p) \equiv q) \equiv (p \vee q))$

p	q	$(((p \wedge q)$	$\equiv$	$p)$	$\equiv$	$q)$	$\equiv$	$(p \vee q)$
F	F	F	T		**F**		**T**	**F**
F	T	F	T		**T**		**T**	**T**
T	F	F	F		**T**		**T**	**T**
T	T	T	T		**T**		**T**	**T**

It can be shown that the operator, $\equiv$, is associative, i.e.

$$(p \equiv q) \equiv r \;=\; p \equiv (q \equiv r)$$

As a consequence we can rewrite $p \wedge q \equiv p \equiv q \equiv p \vee q$ for example, as $(p \wedge q \equiv p) \equiv (q \equiv p \vee q)$ or as $p \wedge q \equiv (p \equiv q \equiv p \vee q)$.

Logical Implication

Logical Implication vs Conditional

There is a close connection between the conditional operator, $\rightarrow$, and mathematical/logical implication.
In mathematics, 'implies' normally means 'logically implies'

For propositional sentences, P and Q,

P **Logically Implies** Q iff $P \rightarrow Q$ is a Tautology.

We use "$P \models Q$ " for "P Logically Implies Q", $\therefore$

$$P \models Q \quad \mathit{iff} \quad \models P \rightarrow Q$$

Logical Implication (Cont'd)

More generally, if Ps is a set of propositional sentences and P and Q are propositional sentences then

$$Ps \cup \{P\} \models Q \quad \textit{iff} \quad Ps \models P \rightarrow Q$$

Some logicians abbreviate $Ps \cup \{P\}$ as $Ps,\ P$ so that

$$Ps,\ P \models Q \quad \textit{iff} \quad Ps \models P \rightarrow Q$$

In particular, if Ps is empty in $Ps \models Q$, i.e. $\{\} \models Q$ then

$$\{\} \models Q \quad \textit{iff} \quad \models Q$$

i.e. if Q is deducible or derivable from the empty set of premises then Q is a Tautology.

Premises and Conclusion

Let the set of propositional sentences $Ps = \{P_1, P, \ldots, P_n\}$ then we can write $Ps \models Q$ as

$$P_1,\ P_2, \ldots,\ P_n \models Q$$

$P_1,\ P_2, \ldots,\ P_n$ are the Premises and Q is the Conclusion.
An argument with premises, $P_1,\ P_2, \ldots,\ P_n$ and a conclusion, Q, is valid iff $P_1,\ P_2, \ldots,\ P_n \models Q$
i.e. an argument is valid if the premises entail the conclusion i.e. if the conclusion is deducible or derivable from the premises.

Premises and Conclusion (Cont'd)

Since

$$P_1,\ P_2, \ldots,\ P_n \models Q$$

is the same as

$$P_1,\ P_2, \ldots,\ P_{n-1} \models P_n \rightarrow Q$$

we get by continuing,

$$P_1,\ P_2, \ldots,\ P_n \models Q \text{ iff } \models P_1 \rightarrow P_2 \rightarrow \cdots \rightarrow P_n \rightarrow Q$$

Recall that the operator $\rightarrow$ is right associative.

Alternative Notation: For $P_1,\ P_2, \ldots,\ P_n \models Q$ we can use

$$\frac{P_1,\ P_2, \ldots,\ P_n}{Q}$$

Transitivity of $\rightarrow$

For any propositional expressions, P, Q and R the argument with

premises, $P \rightarrow Q,\ Q \rightarrow R$ and conclusion $P \rightarrow R$

is valid i.e.
$P \rightarrow Q,\ Q \rightarrow R \models P \rightarrow R$ i.e.

$$\frac{P \rightarrow Q,\ Q \rightarrow R}{P \rightarrow R}$$

as it is not possible for both $P \rightarrow Q$ and $Q \rightarrow R$ to be true and $P \rightarrow R$ to be false.

Transitivity of $\rightarrow$(Cont'd)

Show $P \rightarrow Q,\ Q \rightarrow R \models\ P \rightarrow R$
i.e. show $P \rightarrow Q \models (Q \rightarrow R) \rightarrow (P \rightarrow R)$
i.e. show $\models (P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R))$

P	Q	R	$(P \rightarrow Q)$	$\rightarrow$	$((Q \rightarrow R)$	$\rightarrow$	$(P \rightarrow R))$
F	F	F	**T**	**T**	T	**T**	T
F	F	T	**T**	**T**	T	**T**	T
F	T	F	**T**	**T**	F	**T**	T
F	T	T	**T**	**T**	T	**T**	T
T	F	F	**F**	**T**	T	**F**	F
T	F	T	**F**	**T**	T	**T**	T
T	T	F	**T**	**T**	F	**T**	F
T	T	T	**T**	**T**	T	**T**	T

Logically Equivalent

Logically Equivalent

P is **logically equivalent** to Q iff $\models P \equiv Q$

We may use "$P = Q$" for "P is logically equivalent to Q" $\therefore$

$$P = Q \quad iff \quad \models P \equiv Q$$

In particular,

$$P = Q \quad iff \quad P \models Q \text{ and } Q \models P$$

De Morgan's Laws

De Morgan's Laws

Show $\neg(P \wedge Q) = \neg P \vee \neg Q$ i.e. show $\models \neg(P \wedge Q) \equiv \neg P \vee \neg Q$

P	Q	$\neg$	$(P$	$\wedge$	$Q)$	$\equiv$	$\neg P$	$\vee$	$\neg Q$
F	F	**T**		F		**T**	T	**T**	T
F	T	**T**		F		**T**	T	**T**	F
T	F	**T**		F		**T**	F	**T**	T
T	T	**F**		T		**T**	F	**F**	F

Also,

- $\neg(P \vee Q) = \neg P \wedge \neg Q$

$P \to Q \to R = P \wedge Q \to R$

Show $P \to Q \to R = P \wedge Q \to R$
i.e. show $\models P \to (Q \to R) \equiv P \wedge Q \to R$

P	Q	R	P	$\to$	$(Q \to R)$	$\equiv$	$P \wedge Q$	$\to$	R
F	F	F	F	*T*	T	**T**	F	*T*	F
F	F	T	F	*T*	T	**T**	F	*T*	T
F	T	F	F	*T*	F	**T**	F	*T*	F
F	T	T	F	*T*	T	**T**	F	*T*	T
T	F	F	T	*T*	T	**T**	F	*T*	F
T	F	T	T	*T*	T	**T**	F	*T*	T
T	T	F	T	*F*	F	**T**	T	*F*	F
T	T	T	T	*T*	T	**T**	T	*T*	T

Cont'd

Since $P_1, P_2 \models Q$ is the same as $\models P_1 \rightarrow P_2 \rightarrow Q$ and $P_1 \rightarrow P_2 \rightarrow Q = P_1 \wedge P_2 \rightarrow Q$ then $P_1, P_2 \models Q$ is the same as $\models P_1 \wedge P_2 \rightarrow Q$.
But, $\models P_1 \wedge P_2 \rightarrow Q$ is the same as $P_1 \wedge P_2 \models Q$ $\therefore$

$$P_1, P_2 \models Q \text{ is the same as } P_1 \wedge P_2 \models Q\ .$$

More generally,

$$P_1,\ P_2, \ldots,\ P_n \models Q \text{ is the same as } P_1 \wedge P_2, \cdots \wedge P_n \models Q$$

Cont'd

To show that an argument with **Premises** $P_1, P_2, \ldots, P_n$ and **Conclusion** Q is valid, we show that if the premises $P_1, P_2, \ldots, P_n$ are (all) true then the conclusion, Q, is also true. That is, an argument is valid if it is not possible for all the premises to be true and the conclusion false.

Example: $\neg(P \wedge Q), P \models \neg Q$ **i.e.** $\models \neg(P \wedge Q) \wedge P \rightarrow \neg Q$

P	Q	$(\neg$	$(P$	$\wedge$	$Q)$	$\wedge$	$P)$	$\rightarrow$	$\neg Q$
F	F	T		F		**F**		**T**	**T**
F	T	T		F		**F**		**T**	**F**
T	F	T		F		**T**		**T**	**T**
T	T	F		T		**F**		**T**	**F**

Proving an Argument

Is the following argument valid:

Either the vicar or the butler shot the earl.
If the butler shot the earl then the butler was not drunk at 9pm.
Either the vicar is a liar or the butler was drunk at 9pm

therefore

either the vicar is a liar or he shot the earl

Let us abbreviate the propositions in the above argument.

- s: the vicar shot the earl
- b: the butler shot the earl
- d: the butler was drunk at 9pm
- c: the vicar is a liar

Argument

We need to show the following:

$$\frac{s \vee b,\ \ b \to \neg d,\ \ c \vee d}{c \vee s}$$

i.e. show

$$s \vee b,\ \ b \to \neg d,\ \ c \vee d \models c \vee s$$

To show this we show

$$\models s \vee b \to (b \to \neg d) \to c \vee d \to c \vee s$$

Show by Truth Table

To show $\models s \vee b \to (b \to \neg d) \to c \vee d \to c \vee s$, we use a truth table and show that in all states it is true. Since the sentence has 4 variables, 16 rows are needed in the Truth Table.
(Restore brackets for clarity; recall $\to$ is right associative)

s	b	d	c	$(s \vee b)$	$\to$	$((b \to \neg d) \to ((c \vee d) \to (c \vee s)))$
F	F	F	F		T	
					T	
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	
					T	
T	T	T	T		T	

From the truth table, we conclude that

$$\models s \vee b \to (b \to \neg d) \to c \vee d \to c \vee s$$

Therefore the original argument is valid.

Alternative

From above, since $P \to Q \to R = P \wedge Q \to R$ we can rewrite

$$\models s \vee b \to (b \to \neg d) \to c \vee d \to c \vee s$$

as

$$\models (s \vee b) \wedge (b \to \neg d) \wedge (c \vee d) \to c \vee s$$

Using Short Cuts to create Truth Table

Using Short Cuts for Evaluating sentences

Consider the Truth Table for $(s \vee b) \wedge (b \rightarrow \neg d) \wedge (c \vee d) \rightarrow c \vee s$.

If any of $(s \vee b), (b \rightarrow \neg d), (c \vee d)$ is F, then $(s \vee b) \wedge (b \rightarrow \neg d) \wedge (c \vee d)$ is F.

From the Truth Table for the operator, $\rightarrow$, if P is F then $P \rightarrow Q$ is T and also if Q is T then $P \rightarrow Q$ is T.

From the Truth Table for the operator, $\vee$, if either of P or Q is T then so is $P \vee Q$.

Truth Table with Short Cuts

First 8 rows in Truth Table

s	b	d	c	$((s \vee b)$	$\wedge$	$(b \rightarrow \neg d)$	$\wedge$	$(c \vee d))$	$\rightarrow$	$(c \vee s)$
F	F	F	F	F					**T**	
F	F	F	T	F					**T**	
F	F	T	F	F					**T**	
F	F	T	T	F					**T**	
F	T	F	F					F	**T**	
F	T	F	T						**T**	T
F	T	T	F			F			**T**	
F	T	T	T			F			**T**	

Cont'd

Next 8 rows in Truth Table

Since in the following 8 rows, the variable, s, is T the so is $c \vee s$.

s	b	d	c	$((s \vee b)$	$\wedge$	$(b \to \neg d)$	$\wedge$	$(c \vee d))$	$\to$	$(c \vee s)$
T	F	F	F						**T**	T
T	F	F	T						**T**	T
T	F	T	F						**T**	T
T	F	T	T						**T**	T
T	T	F	F						**T**	T
T	T	F	T						**T**	T
T	T	T	F						**T**	T
T	T	T	T						**T**	T

Cont'd

In general, we can write

$$P,\ P_2, \ldots,\ P_n \models Q$$

as

$$P_1 \wedge P_2, \cdots \wedge P_n \models Q$$

i.e.

$$\models P_1 \wedge P_2, \cdots \wedge P_n \rightarrow Q$$

To show:

$$P,\ P_2, \ldots,\ P_n \text{ logically implies } Q$$

we show

$$P_1 \wedge P_2, \cdots \wedge P_n \rightarrow Q \text{ is a Tautology}$$

Refutation Approach

Refutation Approach

The Rufutation Approach may be more efficient that using Truth Tables.
To show
$\models (s \vee b) \wedge (b \rightarrow \neg d) \wedge (c \vee d) \rightarrow c \vee s$
we show that there is no state in which
$((s \vee b) \wedge (b \rightarrow \neg d) \wedge (c \vee d)) \rightarrow c \vee s$ is F.
That is, we attempt to refute $(s \vee b) \wedge (b \rightarrow \neg d) \wedge (c \vee d) \rightarrow c \vee s$
i.e. we attempt to make it false.
Attempted Refutation:
Suppose $(s \vee b) \wedge (b \rightarrow \neg d) \wedge (c \vee d) \rightarrow c \vee s$ is F
then
$(s \vee b) \wedge (b \rightarrow \neg d) \wedge (c \vee d)$ is T and
$c \vee s$ is F.

Refutation (Cont'd)

i.e.

$$\begin{array}{cc} ((s \lor b) \land (b \to \neg d) \land (c \lor d)) & c \lor s \\ T & F \end{array}$$

$\therefore$ from the properties of $\land$ and $\lor$,

$$\begin{array}{ccccc} (s \lor b) & (b \to \neg d) & (c \lor d) & c & s \\ T & T & T & F & F \end{array}$$

1. From $c = F$ we get that $d = T$ since $(c \lor d) = T$.
2. From $s = F$ and $(s \lor b) = T$, we get that $b = T$
3. Since $b = T$ and $(b \to \neg d) = T$, we get that $\neg d = T$, i.e. $d = F$.
4. From 1. we deduced that $d = T$ and from 3. that $d = F$ and so we have a conflict i.e. a contradiction.
5. tf. the attempt at refutation failed.

Cont'd

It is not possible to make $((s \vee b) \wedge (b \to \neg d) \wedge (c \vee d)) \to c \vee s$ false.

The argument

Either the vicar or the butler shot the earl.
If the butler shot the earl then the butler was not drunk at 9pm.
Either the vicar is a liar or the butler was drunk at 9pm

therefore

either the vicar is a liar or he shot the earl

is valid as it is not possible to make all the premises $(s \vee b), (b \to \neg d), (c \vee d)$ true and the conclusion, $c \vee s$, false.