Simultaneous Equations

Old Way!

Consider the equations:

$$\begin{aligned} 5*x+8*y &= 18 \\ 3*x+5*y &= 11 \end{aligned}$$

$$\begin{array}{ll} R1 := (-3)*R1 & -15*x-24*y = -54 \\ R2 := 5*R2 & 15*x+25*y = 55 \\ \hline \quad \text{Add} & \qquad\qquad\qquad y = 1 \end{array}$$

i.e. $y = 1$

Back substitute to get the value of the variable x :

$$5*x+8*1 = 18$$

$\therefore x = 2$

General Form

Consider

$$
\begin{aligned}
a_1 * x + a_2 * y &= k_1 \\
b_1 * x + b_2 * y &= k_2
\end{aligned}
$$

$$
\begin{array}{lrcr}
R1 := (-b_1) * R1 & -b_1 * a_1 * x - b_1 * a_2 * y & = & -b_1 * k_1 \\
R2 := a_1 * R2 & a_1 * b_1 * x + a_1 * b_2 * y & = & a_1 * k_2 \\
\text{Add} & (a_1 * b_2 - b_1 * a_2) * y & = & a_1 * k_2 - b_1 * k_1
\end{array}
$$

i.e.

$$y = \frac{a_1 * k_2 - b_1 * k_1}{a_1 * b_2 - b_1 * a_2}$$

Cont'd

Similarly, to get the value of x:

$$
\begin{array}{lrcr}
R1 := R1 * b_2 & b_2 * a_1 * x + b_2 * a_2 * y & = & b_2 * k_1 \\
R2 := R2 * (-a_2) & -a_2 * b_1 * x - a_2 * b_2 * y & = & -a_2 * k_2 \\
\text{Add} & (a_1 * b_2 - b_1 * a_2) * x & = & k_1 * b_2 - k_2 * a_2
\end{array}
$$

i.e.

$$x = \frac{k_1 * b_2 - k_2 * a_2}{a_1 * b_2 - b_1 * a_2}$$

No Solution?

Solution to Simultaneous equation depends on $a_1 * b_2 - b_1 * a_2 \neq 0$.
There is no (unique) solution
when $a_1 * b_2 - b_1 * a_2 = 0$ i.e. $a_1 * b_2 = b_1 * a_2$ i.e. $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

No (unique) solution

No (unique) solution when $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

Geometric View

In Cartesian (Co-Ordinate) Geometry, a line is defined by the set of points (x, y) satisfying the equation:

$$y = m * x + c$$

where m is the slope of the line. i.e. $m = \frac{rise}{run}$.
We can rewrite the line $a_1 * x + a_2 * y = k_1$
as

$$y = -\frac{a_1}{a_2} * x_1 + \frac{k_1}{a_2}$$

Also, the line $b_1 * x + b_2 * y = k_2$ can be rewritten as:

$$y = -\frac{b_1}{b_2} * x_1 + \frac{k_1}{b_2}$$

$\therefore$ if $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ then the two lines have the same slope.

Cont'd

Two lines have the same slope when

- The lines are parallel
- The lines are coincident, i.e. they are the same line.

A solution to the simultaneous equations:

$$a_1 * x + a_2 * y = k_1$$

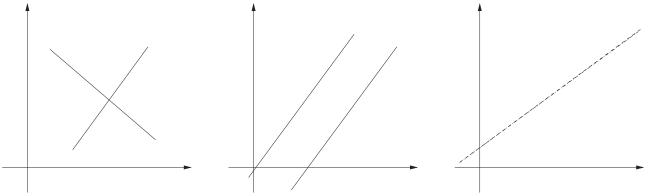
$$b_1 * x + b_2 * y = k_2$$

is where the lines intersect (cross over) at one point; the solution.

Linear Systems in Two Unknowns I

- One equation: $ax + by = c$ represents a line in $\mathbb{R}^2$. There are infinitely many solutions of this equation (one for each point on the line). (Here a, b and c are real numbers, and a and b are not both zero).
- Two equations: Solutions of linear equations in two variables correspond to intersection points of lines.

Three ways in which lines can intersect:



Intersect at unique point   Parallel lines - no intersection point   Lines coincide

Linear Systems

A function from a set A to a set B written as $f : A \to B$ is a relation or rule such that for each element, x, in A there is a most one result, $f(x)$, in B.
A function $f : \mathbb{R} \to \mathbb{R}$ is **linear** if the variable in the expression for f has a first degree i.e. a power of 1. e.g. $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 3 * x + 4$.
A function,$g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = x^2 - 4$ is not linear due to the power of 2 on the variable x.

Root of a Function

A root of a function, f, is the value that makes the function result zero. For example, a root of the function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = 3 * x + 4$ is $-\frac{4}{3}$ as $3 * (-\frac{4}{3}) + 4 = 0$.

Equation

A equation is an expression of the form $f(x) = 0$. e.g.
$3 * x + 4 = 0$.

A solution to an equation is a value that makes the equation true.
Example:
The number $-\frac{4}{3}$ is a solution to the equation $3 * x + 4 = 0$.

Calculating Solution to Simultaneous Equations

From above we can solve the simulataneous equation:

$$\begin{aligned} a_1 * x + a_2 * y &= k_1 \\ b_1 * x + b_2 * y &= k_2 \end{aligned}$$

by

$$x = \frac{k_1 * b_2 - k_2 * a_2}{a_1 * b_2 - b_1 * a_2} \quad y = \frac{a_1 * k_2 - b_1 * k_1}{a_1 * b_2 - b_1 * a_2}$$

Calculating Solution to Simultaneous Equations Cont'd

Example: Solve

$$\begin{aligned} 5 * x + 8 * y &= 18 \\ 3 * x + 5 * y &= 11 \end{aligned}$$

Using general form:

$$a_1 * x + a_2 * y = k_1$$
$$b_1 * x + b_2 * y = k_2$$

$x = \frac{k_1 * b_2 - k_2 * a_2}{a_1 * b_2 - b_1 * a_2}$ i.e. $x = \frac{18*5-11*8}{5*5-3*8} = \frac{90-88}{25-24} = 2$

$y = \frac{a_1 * k_2 - b_1 * k_1}{a_1 * b_2 - b_1 * a_2}$ i.e. $y = \frac{5*11-3*18}{5*5-3*8} = \frac{55-54}{25-24} = 1$

How would this calculating approach be exended to 3x3 Simultaneous Equations?

Gaussian Elimination:

We consider a more algorithmic approach , named "Gaussian Elimination", after the mathematican, Carl Friedrich Gauss. We can write

$$\begin{aligned} 5*x+8*y &= 18 \\ 3*x+5*y &= 11 \end{aligned}$$

in an abbreviated 'matrix' way by regarding the variables as implicit.

$$\begin{array}{lcc|c} R1 & 5 & 8 & 18 \\ R2 & 3 & 5 & 11 \end{array}$$

$$\begin{array}{lcc|c} R1 := \frac{R1}{5} & 1 & \frac{8}{5} & \frac{18}{5} \\ R2 & 3 & 5 & 11 \end{array}$$

Cont'd

$$\begin{array}{lcc|c} R1 & 1 & \frac{8}{5} & \frac{18}{5} \\ R2 := R2 - 3 * R1 & 0 & 5 - \frac{24}{5} & 11 - \frac{54}{5} \end{array}$$

Converting back to using the variables x and y

$$\begin{aligned} x + \frac{8}{5} * y &= \frac{18}{5} \\ \frac{25-24}{5} * y &= \frac{55-54}{5} \end{aligned}$$

i.e. $y = 1$

By back substitution, as above, we find $x = 2$.

Avoiding Back Substitution

We can avoid back substitution by the following
From above:

$$\begin{array}{lcc|c} R1 & 1 & \frac{8}{5} & \frac{18}{5} \\ R2 & 0 & \frac{1}{5} & \frac{1}{5} \end{array}$$

$$\begin{array}{lcc|c} R1 & 1 & \frac{8}{5} & \frac{18}{5} \\ R2 := R2 * 5 & 0 & 1 & 1 \end{array}$$

$$\begin{array}{lcc|c} R1 := R1 - \frac{8}{5} * R2 & (1 - \frac{8}{5} * 0) & \frac{8}{5} - \frac{8}{5} & \frac{18}{5} - \frac{8}{5} \\ R2 & 0 & 1 & 1 \end{array}$$

Simplification

Simplifying the above:

$$\begin{array}{lcc|c} R1 & 1 & 0 & 2 \\ R2 & 0 & 1 & 1 \end{array}$$

Translating back to using variables x and y

$$\begin{aligned} 1 * x + 0 * y &= 2 \\ 0 * x + 1 * y &= 1 \end{aligned}$$

i.e. $x = 2$ and $y = 1$

Allowed Operations on Rows

The follows operations on rows are allowed which do not change the solutions:

- Interchange rows R_i and R_j
- Multiply row, R, by a number n to give $R := n * R$
- Add a multiple of one row to another: for $i \neq j$
  $R_i := R_i + n * R_j$

Gauss-Jordan Elimination in Operation VIII

Historical context:

- The method of Gaussian elimination is mentioned in the Chinese mathematical text “Jiuzhang suanshu“ or “The Nine Chapters on the Mathematical Art“. It dates to between 150BC and 179AD.
- In Europe it appears in the notes of Isaac Newton(1642 -1727) in 1670, where he commented that all the algebra books he knew of lacked a lesson for solving simultaneous equations – which he then supplied.
- Cambridge University published his notes as “Arithmetica Universalis” in 1707.
- By the end of the 18th century the method was a standard lesson in algebra textbooks.
- In 1810 Carl Friedrich Gauss devised a notation for symmetric elimination that was adopted in the 19th century by professional hand computers to solve the normal equations of least-squares problems (a methodology used for curve fitting to a set of points) .
- The algorithm we know was named after Gauss only in the 1950s as a result of confusion over the history of the subject

Reading materials

Linear algebra

- Linear Algebra
  J. Hefferon, Online textbook: http://joshua.smcvt.edu/linearalgebra/
- Elementary Linear Algebra
  K. R. Matthews, chapter 1, Online textbook: http://www.numbertheory.org/book/
- Elementary Linear Algebra
  Howard Anton, Chris Rorres, Wiley (any edition will do)

Non-course specific reading material

- How Not to Be Wrong: The Power of Mathematical Thinking Jordan Ellenberg, Penguin, 2014
- How to Read and Do Proofs Daniel Solow, Wiley, 2010.
- How to Solve It George Pólya, Penguin.