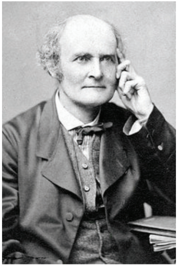
Linear Equations
Consistent and Inconsistent

How old are matrices?

- Matrices were invented by the British mathematician Arthur Cayley (1821-1895).
- Cayley presented a paper giving the rule for matrix operations and the conditions under which a matrix has an inverse to the Royal Society in 1858.
- Cayley's friend, James Joseph Sylvester (1814-1897), was the person who first used the term "matrix" in 1850.



Arthur Cayley (1821-1895)

James Joseph Sylvester (1814-1897)

m Linear Equations in n unknowns

m Linear Equations in n unknowns

A system of m Linear Equations in n unknowns

$$
\begin{aligned}
a_{11} * x_1 + a_{12} * x_2 + \cdots + a_{1n} * x_n &= c_1 \\
a_{21} * x_1 + a_{22} * x_2 + \cdots + a_{2n} * x_n &= c_2 \\
\vdots \qquad &\quad \vdots \\
a_{m1} * x_1 + a_{m2} * x_2 + \cdots + a_{mn} * x_n &= c_m
\end{aligned}
$$

is:

- **Consistent**, if it has a least one solution
- **Inconsistent**, if it has no solution

The i^{th} equation of the m equations above may be rewritten as:

$$\sum_{j=1}^{n} a_{ij} x_j = c_i$$

Linear Equations in Matrix Form

The above system of m linear equations in n unknowns can be written in matrix form as:

$$A * x = c$$

i.e.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$$

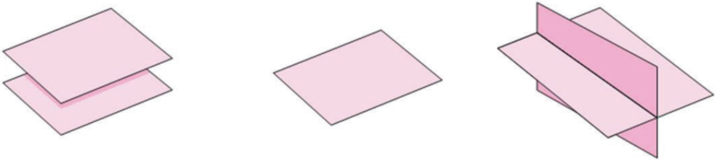
Note:
m may not be same as n .
The vector x of unknowns has length n while the vector c of constants has length m.
When row i 'dives' into the column vector, x, we get the sum: $\sum_{j=1}^{n} a_{ij} x_j$.

With $n = 3$, the equations have 3 unknowns. We could write an equation in 3 unknowns, x, y and z as:

$$a * x + b * y + c * z = d.$$

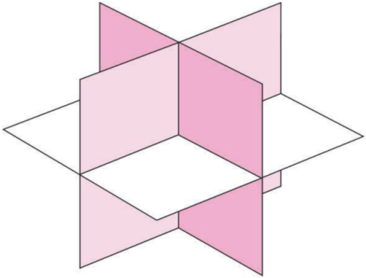
(a, b and c are not zero constants)
This equation represents a plane in $\mathbb{R}^3$. With one equation, there are infinite solutions.

With 2 linear equations in 3 unknowns:
No solution or infinite solutions.



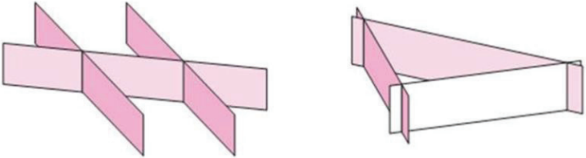
Geometric view (Cont'd)

With 3 linear equations in 3 unknowns:
Unique solution:

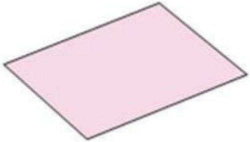


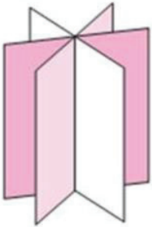
Geometric view (Cont'd)

With 3 linear equations in 3 unknowns:
No solution:



With 3 linear equations in 3 unknowns:
Infinite solutions:
when the planes intersect either in overlapping planes or intersect in a line.





Augmented Matrix

From the system of m Linear Equations in n unknowns

$$\begin{aligned}
a_{11} * x_1 + a_{12} * x_2 + \cdots + a_{1n} * x_n &= c_1 \\
a_{21} * x_1 + a_{22} * x_2 + \cdots + a_{2n} * x_n &= c_2 \\
\vdots \qquad &\qquad \vdots \\
a_{m1} * x_1 + a_{n2} * x_2 + \cdots + a_{mn} * x_n &= c_m
\end{aligned}$$

we have the augmented matrix of the system as:

Augmented Matrix (Cont'd)

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & c_1 \\ a_{21} & a_{22} & \dots & a_{2n} & c_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & c_m \end{array}\right]$$

or dropping the 'dividing line'

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & c_1 \\ a_{21} & a_{22} & \dots & a_{2n} & c_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & c_m \end{bmatrix}$$

Gauss/Jordan Elimination Method

The Gauss/Jordan method uses elementary row operations to reduce an augmented matrix to a simpler form so that the solution set of the orginal system of linear equations can be found, if there is a solution.
The elementary row operations are:
(let R_k name the k^{th} row)

Allowed row operations

1. Interchange (swap) rows R_i and R_j
2. Multiply a row by a non-zero scalar (number):
   $R_i := k * R_i$.
3. Add a multiple of one row to another:
   $R_i := R_i + k * R_j$ where $i \neq j$.

Row Equivalent Matrices

Matrices A and B are row equivalent if one is obtained from the other by elementary row operations.
If the augmented matrices A and B are row equivalent then they have the same solution set.
Given an initial augumented matrix A for a system of linear equations and the matrix A is reduced by row operations to a row equivalent form B which is simpler then B can be used to find the solution set for the initial matrix A.

Row Equivalent Matrices (Cont'd)

The Gauss/Jordan method reduces an initial augmented matrix A to a row equivalent simpler form B using elementary row operations.

The Gauss/Jordan method reduces the matrix to **Reduced Row Echelon Form**.

This is done in two stages:

1. Reduce matrix A to **Row Echelon Form**
   then
2. Reduce this result further to **Reduced Row Echelon Form**

Row Equivalent Matrices (Cont'd)

The matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & & c_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & c_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & c_m \end{bmatrix}$$

is first converted to the form

$$\begin{bmatrix} 1 & a_{12}^* & \cdots & a_{1n}^* & c_1^* \\ 0 & 1 & \cdots & a_{2n}^* & c_2^* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c_m^* \end{bmatrix}$$

where the 'starred' entries are the final values in **row echelon form**. Then the matrix is further converted to **Reduced Row Echelon Form** i.e.

Row Equivalent Matrices (Cont'd)

$$\begin{bmatrix} 1 & a_{12}^{*} & \cdots & a_{1n}^{*} & c_1^{*} \\ 0 & 1 & & a_{2n}^{*} & c_2^{*} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{m2} & \cdots & 1 & c_m^{*} \end{bmatrix}$$

is further reduced to the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & c_1^{**} \\ 0 & 1 & \cdots & 0 & c_2^{**} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c_m^{**} \end{bmatrix}$$

From the **Reduced Row Echelon Form**, the soluton set for the initial system of linear equations can be found.

(From Dictionary: ***Echelon***
a formation of troops, ships, aircraft, or vehicles in parallel rows with the end of each row projecting further than the one in front.)

Row Echelon Form of a Matrix

1. All zero rows (if any) are at the bottom of the matrix
2. In a non-zero row, the first non zero entry is 1 (the leading 1)
3. Each leading 1 in a row is to the right of the leading 1 in the row above it.
   As a consequence, all entries, if any, in the column below the leading 1 are zeros.

Example: Row-Echelon Form

The Row Echelon Form of the 3×6 matrix:

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Reduced Row-Echelon Form

Reduced Row-Echelon Form

A Matrix is in Reduced Row-Echelon Form if:

1. it is Row-Echelon Form
2. Each column that has a leading 1 has zeros elsewhere in the column.

Reduced Row-Echelon Form (Cont'd)

The **Reduced Row Echelon Form** of

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

It is in Row Echelon Form and each column that has a leading 1 in a row has zeros elsewhere.

Algorithm for Row Echelon Form

Given an $m \times n$ matrix, A. (Using $:=$ for assignment)

```
i := 1;
while ( A is not in Row Echelon Form )
{
    Locate the leftmost non-zero column;
    If top of column is zero, swap with another row
    to bring an entry, k ≠ 0, to the top;
    If ( k ≠ 1 )
        R_i := R_i / k; // row i has leading 1
    Use suitable multiples of R_i to add to other rows
    below so that entries below the leading 1 are zeros;
    // Ignore Row i
    i := i+1;
    // Consider remaining submatrix
}
```

Example Row Echelon Form

Reduce

$$A = \begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

to Row Echelon Form.
As $A_{11} = 0$, swap $R1$ and $R2$

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Example (Cont'd)

$$R1 := \frac{R1}{2}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

$$R3 := R3 - 2 * R1$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

Ignore $R1$, and process submatrix

$$R2 := \frac{R2}{-2}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & \frac{-7}{2} & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

$$R3 := R3 - 5 * R2$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & \frac{-7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

Ignore $R2$, process submatrix

Example (Cont'd)

$$R3 := 2 * R3$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & \frac{-7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Matrix is in Row Echelon Form

Algorithm: Reduced Row Echelon Form

First convert a matrix to Row Echelon Form:

```
Begin with  last nonzero row;
Work upwards;
Add suitable multiples of each row to the rows
above to introduce zeros above the leading 1;
```

From above we have the Row Echelon Form:

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & \frac{-7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Starting with the 3^{rd} row and 5^{th} column, work upwards:

Example (Cont'd)

$$R2 := R2 + \frac{7}{2} * R3$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$R1 := R1 - 6 * R3$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$R1 := R1 + 5 * R2$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Matrix is now in

Reduced Row Echelon Form

The initial augmented matrix:

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

represents the linear system:

$$\begin{aligned} -2 * x_3 + 7 * x_5 &= 12 \\ 2 * x_1 + 4 * x_2 - 10 * x_3 + 6 * x_4 + 12 * x_5 &= 28 \\ 2 * x_1 + 4 * x_2 - 5 * x_3 + 6 * x_4 + -5 * x_5 &= -1 \end{aligned}$$

Cont'd

The Reduced Row Echelon Form:

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

represents the linear system:

$$\begin{aligned} x_1 + 2 * x_2 + 7 * x_4 &= 7 \\ x_3 &= 1 \\ x_5 &= 2 \end{aligned}$$

No Solution Example

Consider the augmented matrix:

$$\begin{bmatrix} 0 & 0 & 4 & 0 \\ 5 & 5 & -1 & 5 \\ 2 & 2 & -2 & 5 \end{bmatrix}$$

$$\text{swap } R1 \text{ and } R3$$
$$\begin{bmatrix} 2 & 2 & -2 & 5 \\ 5 & 5 & -1 & 5 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

Cont'd

$$R1 := \frac{R1}{2}$$
$$\begin{bmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 5 & 5 & -1 & 5 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

$$R2 := \frac{R2}{4}$$
$$\begin{bmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 1 & \frac{-15}{8} \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

$$R2 := R2 - 5 * R1$$
$$\begin{bmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 4 & \frac{-15}{2} \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

$$R3 := R3 - 4 * R2$$
$$\begin{bmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 1 & \frac{-15}{8} \\ 0 & 0 & 0 & \frac{15}{2} \end{bmatrix}$$

Linear System with No Solution

$$R3 := \frac{2}{15} * R3$$

$$\begin{bmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 1 & \frac{-15}{8} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix is now in Row Echelon Form.

To covert to Reduced Row Echelon Form, continue with

Cont'd

$R2 := R2 + \frac{15}{8} * R3$

$$\begin{bmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R1 := R1 - \frac{5}{2} * R3$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R1 := R1 + R2$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix is now in Reduced Row Echelon Form

Cont'd

The initial augmented matrix

$$\begin{bmatrix} 0 & 0 & 4 & 0 \\ 5 & 5 & -1 & 5 \\ 2 & 2 & -2 & 5 \end{bmatrix}$$

represents the system of Linear Equations

$$\begin{aligned} 4 * x_3 &= 0 \\ 5 * x_1 + 5 * x_2 - x_3 &= 5 \\ 2 * x_1 + 2 * x_2 - 2 * x_3 &= 5 \end{aligned}$$

Cont'd

The Reduced Row Echelon Form matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

represents the system of Linear Equations

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_3 &= 0 \\ 0 &= 1 \end{aligned}$$

From which we can conclude that the initial system of Linear Equations is inconsistent as $0 \neq 1$ i.e. the initial system has no solution.