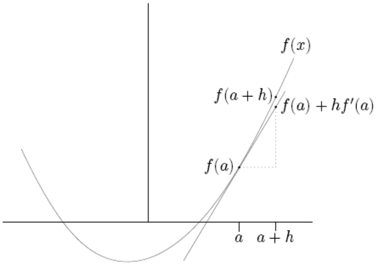
Taylor and Maclaurin Series

Slope of a Curve

The slope of a curve $y = f(x)$ at a point, $(a, f(a))$, is the slope of the tangent to the curve at point $(a, f(a))$. From Calculus, the slope of the curve at point $(a, f(a))$ is $f'(a)$, the derivative at point, a.
We can use the tangent line a point, a, on a curve $y = f(x)$, to approximate the values of $f(x)$ close to the point, a, i.e. to approximate the values $f(a + h)$ when h is small (h close to 0).

Aproximate $f(a+h)$

Introduction to the Taylor expansion

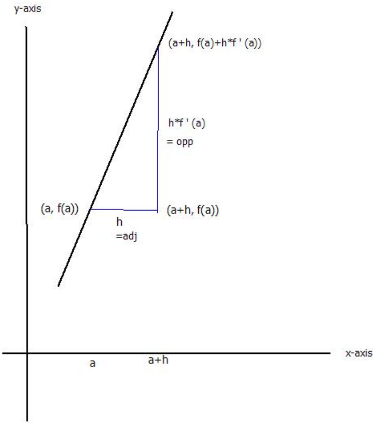


We can approximate a point on a curve at $x = a + h$ by the corresponding point on the tangent:

$$f(a+h) \approx f(a) + hf'(a)$$

For h close to 0, it is a good approximation.

Finding point on Tangent Line



Slope of tangent line $= f'(a)$
Slope of tangent line $\frac{opp}{adj} = \frac{opp}{h} = f'(a) \therefore opp = h * f'(a)$

Equation of Tangent line at $f(a)$

Recall the equation of a line: $y = m * x + c$ where m is the slope of the line.
From Calculus, the slope of the curve at point $(a, f(a))$ is $f'(a)$ $\therefore$ the equation of the tangent to the curve at $(a, f(a))$ is
$y = f'(a) * x + c$.
We can find the value of c since the point $(a, f(a))$ lies on the tangent line. We have:
$f(a) = f'(a) * a + c$ $\therefore$
$c = f(a) - a * f'(a)$.
The equation of the tangent to the curve at $(a, f(a))$ is:
$y = f'(a) * x + f(a) - a * f'(a)$.

Value of $(a + h)$ onTangent line

We approximate the value of $f(a + h)$ by the corresponding point on the tangent line
$y = f'(a) * x + f(a) - a * f'(a)$.
If $x = a + h$ then
$y = f'(a) * (a + h) + f(a) - a * f'(a)$ i.e.
$y = f'(a) * a + f'(a) * h + f(a) - a * f'(a) \therefore$
$y = f'(a) * h + f(a)$ i.e.
$(a + h,\ f(a) + h * f'(a))$ lies on tangent line, as in diagram above.

Introduction to the Taylor expansion I

- Remember that:
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- You can also think about it this way:
$$f'(a) = \frac{f(a+h) - f(a)}{h} + \text{terms which goes to zero as } h \to 0$$

- The terms which goes to zero as $h \to 0$ have to be $h \times$ something and are written $O(h)$. More generally, we note:

Definition:

$$O(h^n) \quad \equiv \quad \text{terms of the form } h^n \times \text{something}$$

Introduction to the Taylor expansion II

- So

$$f'(a) = \frac{f(a+h) - f(a)}{h} + O(h)$$

Multiplying by h, we have :

$$h \cdot f'(a) = f(a+h) - f(a) + O(h^2)$$

or we can write the following Taylor expansion of order 2:

$$f(a+h) = f(a) + h \cdot f'(a) + O(h^2)$$

When h is small (e.g. $h = 0.1$), then h^2 is even smaller ($h^2 = 0.01$).

We have an $O(h^2)$ expansion of $f(a+h)$ as
$f(a+h) = f(a) + h * f'(a) + O(h^2)$
With $a = 0$ and $h = x$, where x is small (x close to 0).
$f(x) = f(0) + x * f'(0) + O(x^2)$
This can be generalised to more terms.
Consider finding an approximation for $f(x)$ using the series:

$$f(x) = a_0 + a_1 * x + a_2 * x^2 + a_3 * x^3 + \dots$$

We have:
$f(0) = a_0 \therefore$

$$f(x) = f(0) + a_1 * x + a_2 * x^2 + a_3 * x^3 + \dots$$

Consider $f'(x)$
$f'(x) = a_1 + a_2 * 2 * x + a_3 * 3 * x^2 + \dots \therefore$
$f'(0) = a_1 \therefore$

$$f(x) = f(0) + f'(0) * x + a_2 * x^2 + a_3 * x^3 + \dots$$

Continuing: Differentiate $f'(x)$ to get
$f''(x) = a_2 * 2 + a_3 * 3 * 2 * x + \cdots \therefore$
$f''(0) = a_2 * 2$ i.e.
$a_2 = \frac{f''(0)}{2} \therefore$

$$f(x) = f(0) + f'(0) * x + \frac{f''(0)}{2} * x^2 + a_3 * x^3 + \dots$$

Maclaurin Series (Cont'd)

Assuming $f(x)$ can be differentiated successively, i.e. we can find $f'''(x)$, $f^{(4)}(x)$, $f^{(5)}(x)$ etc. then

$a_3 = \frac{f'''(0)}{3!}$, $a_4 = \frac{f^{(4)}(0)}{4!}$, $a_5 = \frac{f^{(5)}(0)}{5!}$ etc.

where $k! = k * (k-1)..2 * 1$ "k factorial"

Maclaurin Series

We get an approximation for $f(x)$ as the **Maclaurin Series**

$$f(x) = f(0) + x * f'(0) + \frac{x^2}{2!} * f''(0) + \frac{x^3}{3!} * f'''(0) + \dots$$

Maclaurin Series for e^x

Let $f(x) = e^x$ then $f(0) = e^0 = 1$

$$f'(x) = e^x \qquad \therefore f'(0) = e^0 = 1$$

$$f''(x) = e^x \qquad \therefore f''(0) = e^0 = 1$$

$$f'''(x) = e^x \qquad \therefore f'''(0) = e^0 = 1$$

$$f^{(4)}(x) = e^x \qquad \therefore f^{(4)}(0) = e^0 = 1$$

Maclaurin Series for e^x (Cont'd)

Maclaurin Series

$$f(x) = f(0) + x * f'(0) + \frac{x^2}{2!} * f''(0) + \frac{x^3}{3!} * f'''(0) + \frac{x^4}{4!} * f^{(4)}(0) + \dots$$

With $f(x) = e^x$ we get

$$\begin{aligned} e^x &= 1 + x * 1 + \frac{x^2}{2!} * 1 + \frac{x^3}{3!} * 1 + \frac{x^4}{4!} * 1 + \dots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \end{aligned}$$

This justifies the series for e^x, used previously in the Compound Interest calculation.

$\lim_{n\to\infty}(1 + \frac{r}{n})^n$
$= \sum_{k=0}^{\infty} \frac{r^k}{k!}$
$= 1 + r + \frac{r^2}{2!} + \frac{r^3}{3!} + \dots$
$= e^r$

Maclaurin Series for $sin\, x$

Let $f(x) = sin\, x$ then $f(0) = sin\, 0 = 0$

$$f'(x) = cos\, x \qquad \therefore f'(0) = cos\, 0 = 1$$

$$f''(x) = -sin\, x \qquad \therefore f''(0) = -sin\, 0 = 0$$

$$f'''(x) = -cos\, x \qquad \therefore f'''(0) = -cos\, 0 = -1$$

$$f^{(4)}(x) = sin\, x \qquad \therefore f^{(4)}(0) = sin\, 0 = 0$$

$$f^{(5)}(x) = cos\, x \qquad \therefore f^{(5)}(0) = cos\, 0 = 1$$

Maclaurin Series

$f(x) = f(0) + x * f'(0) + \frac{x^2}{2!} * f''(0) + \frac{x^3}{3!} * f'''(0) + \frac{x^4}{4!} * f^{(4)}(0) + \dots$

With $f(x) = \sin x$

$f(0) = 0$	$f'(0) = 1$	$f''(0) = 0$	$f'''(0) = -1$
$f^{(4)}(0) = 0$	$f^{(5)}(0) = 1$		

$\sin x = 0 + x * 1 + \frac{x^2}{2!} * 0 + \frac{x^3}{3!} * (-1) + \frac{x^4}{4!} * 0 + \frac{x^5}{5!} * 1 + \dots \therefore$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Differentiating both sides we get

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Digression: Euler Formula: $e^{i\theta} = cos\,\theta + i * sin\,\theta$

$i^2 = -1$, i is a complex number.

$$\begin{array}{ccl} cos\,\theta & = & 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots \\ i * sin\,\theta & = & i * \theta - \frac{i*\theta^3}{3!} + \frac{i*\theta^5}{5!} - \frac{i*\theta^7}{7!} + \ldots \\ \textbf{add} & & \textbf{add} \\ cos\,\theta + i * sin\,\theta & = & 1 + i * \theta - \frac{\theta^2}{2} - \frac{i*\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots \\ & = & e^{i\theta} \end{array}$$

i.e. $e^{i\theta} = cos\,\theta + i * sin\,\theta$
Let $\theta = \pi$ then $e^{i\pi} = cos\,\pi + i * sin\,\pi$ i.e.

$$e^{i\pi} = -1$$

i.e.

$$e^{i\pi} + 1 = 0$$

Taylor Series

The Taylor Series about a value, a, is a generalisation of the Maclaurin series.
For small x, we can approximate, $f(a+x)$

$$f(a+x) = f(a) + x * f'(a) + \frac{x^2}{2!} * f''(a) + \frac{x^3}{3!} * f'''(a) + \frac{x^4}{4!} * f^{(4)}(a) + \dots$$

When $a = 0$, we have a special case of the Taylor series (the Maclaurin series)

$$f(x) = f(0) + x * f'(0) + \frac{x^2}{2!} * f''(0) + \frac{x^3}{3!} * f'''(0) + \frac{x^4}{4!} * f^{(4)}(0) + \dots$$

In $f(a+x)$, above, let $x = (x-a)$ to get another version of the Taylor series about a:
$f(x) = f(a) + (x-a) * f'(a) + \frac{(x-a)^2}{2!} * f''(a) + \frac{(x-a)^3}{3!} * f'''(a) + \frac{(x-a)^4}{4!} * f^{(4)}(a) + \dots$

$$f(a+x) = f(a) + x * f'(a) + \frac{x^2}{2!} * f''(a) + \frac{x^3}{3!} * f'''(a) + \frac{x^4}{4!} * f^{(4)}(a) + \ldots$$

Let $f(x) = ln(x)$ then $f'(x) = \frac{1}{x}$ and $f'(a) = \frac{1}{a}$
With $f(x) = ln(x)$ and $a = 1$ then
$ln(1+x) = ln(1) + x * ln'(1) + \frac{x^2}{2!} * ln''(1) + \frac{x^3}{3!} * ln'''(1) + \ldots$

Recall

if $r(x) = \frac{1}{x^n} = x^{-n}$ then $r'(x) = -n * x^{-n-1} = \frac{-n}{x^{n+1}}$

With $f(x) = ln(x)$ and $a = 1$

$f'(x) = \frac{1}{x}$ $\qquad \therefore f'(1) = \frac{1}{1} = 1$

$f''(x) = \frac{-1}{x^2}$ $\qquad \therefore f''(1) = \frac{-1}{1^2} = -1$

$f'''(x) = \frac{2}{x^3}$ $\qquad \therefore f'''(1) = \frac{2}{1^3} = 2!$

$f^{(4)}(x) = \frac{-3*2}{x^4}$ $\qquad \therefore f^{(4)}(1) = \frac{-3*2}{1^4} = -3!$

Taylor Series for $ln(1+x)$ (Cont'd)

For small x, we can approximate, $f(a+x)$

$$f(a+x) = f(a)+x*f'(a)+\frac{x^2}{2!}*f''(a)+\frac{x^3}{3!}*f'''(a)+\frac{x^4}{4!}*f^{(4)}(a)+\dots$$

$\therefore$ with $f(x) = ln(x)$ and $a = 1$ then $f(1+x) = ln(1+x)$:

$$\boxed{f(1)=0 \quad f'(1)=1 \quad f''(1)=-1 \quad f'''(1)=2! \quad f^{(4)}(1)=3!}$$

$$\begin{aligned} ln(1+x) &= ln(1)+x*1+\frac{x^2}{2!}*(-1)+\frac{x^3}{3!}*(2!)+\frac{x^4}{4!}*(-3!)+\dots \\ &= x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots \end{aligned}$$

This is the same series as we had previously for $ln(1+x)$.

Taylor series $(1+x)^n$

Let $f(x) = x^n$ then $f(1+x) = (1+x)^n$

$$f(1+x) = f(1) + x * f'(1) + \frac{x^2}{2!} * f''(1) + \frac{x^3}{3!} * f'''(1) + \frac{x^4}{4!} * f^{(4)}(1) + \ldots$$

$$f'(x) = n * x^{n-1}$$
$$\therefore f'(1) = n * 1^{n-1} = n$$

$$f''(x) = n * (n-1) * x^{n-2}$$
$$\therefore f''(1) = n * (n-1)$$

$$f'''(x) = n * (n-1) * (n-2) * x^{n-3}$$
$$\therefore f'''(1) = n * (n-1) * (n-2)$$

Taylor series for $(1+x)^n$ (Cont'd)

$$f(1+x) = f(1) + x * f'(1) + \tfrac{x^2}{2!} * f''(1) + \tfrac{x^3}{3!} * f'''(1) + \tfrac{x^4}{4!} * f^{(4)}(1) + \ldots$$

With $f(x) = x^n$ and

$f(1) = 1$	$f'(1) = n$	$f''(1) = n(n-1)$	$f'''(1) = n(n-1)(n-2)$

$$\begin{aligned}(1+x)^n &= 1 + x * n + \frac{x^2}{2!} * n(n-1) + \frac{x^3}{3!} * n(n-1)(n-2) \ldots \\ &= 1 + n * x + \frac{n(n-1)}{2!} * x^2 + \frac{n(n-1)(n-2)}{3!} * x^3 + \ldots\end{aligned}$$

Note:
The exponent, n, in $(1+x)^n$ may be a negative Integer or a Rational number (a fraction).
e.g. $(1+x)^{-1}$ or $(1+x)^{\frac{1}{2}}$

Binonomial Series

$$(1+x)^n = 1 + n*x + \frac{n(n-1)}{2!}*x^2 + \frac{n(n-1)(n-2)}{3!}*x^3 + \dots$$

Series $tan^{-1}x$

$$f(x) = tan^{-1}x \therefore f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$$
$$f'(x) = (1+x^2)^{-1} = 1 - x^2 + \frac{(-1)(-2)}{2!}*x^4 + \frac{(-1)(-2)(-3)}{3!}*x^6 + \dots$$
$$\therefore f'(x) = 1 - x^2 + x^4 - x^6 + x^8 \dots \therefore f'(0) = 1$$

Differentiate term by term

$f''(x) = -2x + 4x^3 - 6x^5 + \dots$	$\therefore f''(0) = 0$
$f'''(x) = -2 + 4*3*x^2 + 6*5*x^4 + \dots$	$\therefore f'''(0) = -2$
$f^{(4)}(x) = 4*3*2*x - 6*5*4*x^3 + \dots$	$\therefore f^{(4)}(0) = 0$
$f^{(5)}(x) = 4! - 6*5*4*3*x^2 + \dots$	$\therefore f^{(5)}(0) = 4!$
$f^{(6)}(x) = -6!*x + \dots$	$\therefore f^{(6)}(0) = 0$

Maclaurin Series for $tan^{-1}x$ (Cont'd)

Maclaurin Series

$$f(x) = f(0) + x * f'(0) + \frac{x^2}{2!} * f''(0) + \frac{x^3}{3!} * f'''(0) + \frac{x^4}{4!} * f^{(4)}(0) \ldots$$

With $f(x) = tan^{-1}x$

$f(0) = 0$	$f'(0) = 1$	$f''(0) = 0$	$f'''(0) = -2$	$f^{(4)}(0) = 0$
$f^{(5)}(0) = 4!$	$f^{(6)}(0) = 0$			

$$tan^{-1}x = x * 1 + \frac{x^3}{3!} * (-2!) + \frac{x^5}{5!} * (4!) \ldots \therefore$$

$$tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$$

This is the same series as we had previously for $tan^{-1}x$.