Solving Linear Equations & Matrix Inverse using Gaussian (Gauss/Jordan) Elimination

Linear Equations via Gaussian Elimination

Using Gaussian Elimination, solve the following system of Linear Equations

$$\begin{aligned} x + y + 2 * z &= 8 \\ -x - 2 * y + 3 * z &= 1 \\ 3 * x - 7 * y + 4 * z &= 10 \end{aligned}$$

Reformulate as Augmented Matrix:

Augmented Matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right]$$

$$\{R2 := R2 + R1\}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 3 & -7 & 4 & 10 \end{array}\right]$$

$$\{R3 := R3 - 3 * R1\}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right]$$

$$\{R2 := R2 * (-1)\}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right]$$

Augmented Matrix (Cont'd)

$$\{R3 := R3 + 10 * R2\}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right]$$

$$\{R3 := R3 * \tfrac{-1}{52}\}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\{R2 := R2 + 5 * R3\}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\{R1 := R1 - 2 * R3\}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Augmented Matrix (Cont'd)

Finally,

$$\{R1 := R1 - R2\}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solution:
$x = 3,\ y = 1,\ z = 2$

Matrix Inverse is Unique

If a Matrix, M, has an inverse, it is unique.

Proof:

Assume a Matrix M has inverses X and Y i.e.

1. $X * M = Id$, also
2. $M * Y = Id \therefore$

$(X * M) * Y = Id * Y = Y$, from 1.
$X * (M * Y) = X * Id = X$, from 2.
$\therefore$ since $*$ is associative, we have $(X * M) * Y = X * (M * Y)$,
$\therefore$ we have: $Y = X$, i.e.
if M has more that one inverse then they are equal.

Finding Matrix Inverse by Gaussian Approach

Let M be a Matrix.

1. Create Augmented Matrix: $\left[\begin{array}{c|c} M & Id \end{array}\right]$
2. Reduce Augmented Matrix by Gaussian row operations to $\left[\begin{array}{c|c} Id & X \end{array}\right]$
3. Then X is inverse of M, i.e. $X = M^{-1}$.

Inverse Matrix (Cont'd)

Example:

Find the Inverse (by Gaussian Method) of: $M = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$

Augmented Matrix:

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right]$$

Inverse Matrix (Cont'd)

$$R1 := \frac{1}{2} * R1$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & \frac{1}{2} & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right]$$

$$R2 := R2 - 2 * R1$$
$$R3 := R3 - 2 * R1$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$R3 := R3 - R2$$
$$R1 := R1 - 3 * R2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & \frac{7}{2} & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$R1 := R1 - 3 * R3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{2} & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

Inverse Matrix (Cont'd)

With $M = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$ then

$$M^{-1} = \begin{bmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$