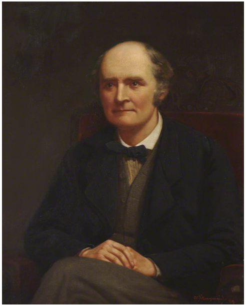
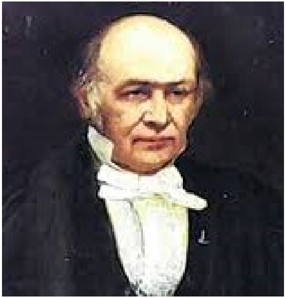
Cayley Hamilton Theorem

Cayley



1821 – 1895

Hamilton



1805 -1865

Cayley Hamilton Theorem

Let $A = [a_{ij}]_{n \times n}$ and let

$$p(t) = \begin{vmatrix} a_{11} - t & \dots & a_{in} \\ \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} - t \end{vmatrix}$$

be the **Characteristic Polynomial** of A then $p(A) = [0]_{n \times n}$
Each matrix, A, is a root of its own characteristic polynomial.
Note:
For an eigen value λ and corresponding eigen vector x,
$A * x = \lambda * x$.
For eigen value, λ, $p(\lambda) = 0$. Also, let $0 = [0]_{n \times n}$ then $p(A) = 0$

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ then characteristic polynomial is:

$$p(t) = \begin{vmatrix} 1-t & 2 \\ 3 & 2-t \end{vmatrix} = t^2 - 3*t - 4$$

$p(A) = A^2 - 3*A - 4*Id$ i.e.

$$\begin{aligned} & p(A) \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}^2 - 3*\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - 4*\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 6 \\ 9 & 10 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 9 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Example (Con'd)

Note:

$t^2 - 3 * t - 4 = (t - 4) * (t + 1) \therefore$

roots of $t^2 - 3 * t - 4$ are $t = 4$ and $t = -1$, i.e.

$p(4) = 0$ and $p(-1) = 0$.

General 2x2 matrices

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the characteristic polynomial is:

$$p(t) = \begin{vmatrix} a-t & b \\ c & d-t \end{vmatrix} = t^2 - (a+d)*t + (a*d - c*b)$$

Note: for 2×2 matrices

$$p(t) = t^2 - Tr(A)*t + det(A)$$

where $Tr(A) =$ sum along main diagonal of A i.e. the Trace of A.

Check Cayley Hamilton Property

$$p(A)$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 - (a+d) * \begin{bmatrix} a & b \\ c & d \end{bmatrix} + (a*d - c*b) * \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + b*c & a*b + b*d \\ c*a + d*c & c*b + d^2 \end{bmatrix}$$

$$- \begin{bmatrix} a^2 + a*d & a*b + b*d \\ c*a + d*c & a*d + d^2 \end{bmatrix}$$

$$+ \begin{bmatrix} a*d - c*b & 0 \\ 0 & a*d - c*b \end{bmatrix}$$

Check Cayley Hamilton Property (Cont;d)

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where

$$a_{11} = a^2 + b*c - a^2 - a*d + a*d - c*b = 0$$
$$a_{12} = a*b + b*d - a*b - b*d = 0$$
$$a_{21} = c*a + d*c - c*a - d*c = 0$$
$$a_{22} = c*b + d^2 - a*d - d^2 + a*d - c*b = 0$$

i.e.

$$p(A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Find Inverse via Cayley Hamilton

Example:

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, Find A^{-1}.

The characteristic polynomial for $A = t^2 - 3*t - 4 \therefore$

by Cayley Hamilton

$A^2 - 3*A - 4*Id = 0 \therefore$

$A^2 = 3*A + 4*Id \therefore$

$\{$Multiply across by A^{-1} $\}$

$A = 3*Id + 4*A^{-1}$

$A^{-1} = \frac{1}{4}*(A - 3*Id)$

Finding A^{-1} using Cayley Hamilton(Cont'd)

For $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$,

A^{-1}

$= \frac{1}{4} * (A - 3 * Id)$

$= \frac{1}{4} * \left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - 3 * \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$

$= \frac{1}{4} * \begin{bmatrix} -2 & 2 \\ 3 & -1 \end{bmatrix}$

Check: $A^{-1} * A$

$$\frac{1}{4} * \begin{bmatrix} -2 & 2 \\ 3 & -1 \end{bmatrix} * \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

$$= \frac{1}{4} * \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Note:

With $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

$|A| = -4, \therefore A^{-1} = \frac{1}{-4} * \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}$

Example: 2×2 Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then A satisifies its own characteristic equation, i.e.

$A^2 - (a + d) * A + (a * d - c * b) * Id) = 0 \therefore$

$A^2 - (a + d) * A = -(a * d - c * b) * Id) \therefore$

$\{$multiply on right by $A^{-1}\}$

$(A^2 - (a + d) * A) * A^{-1} = -(a * d - c * b) * Id) * A^{-1}$

$A - (a + d) * Id = -(a * d - c * b) * A^{-1} \therefore$

$\frac{-(A-(a+d)*Id)}{a*d-c*b} = A^{-1}$

Note: $a * d - c * b = |A|$

Check Inverse via Cayley Hamilton

Since $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$
\begin{aligned}
&A^{-1} \\
&= \frac{-1}{|A|} * (A - (a+d) * Id) \\
&= \frac{-1}{|A|} * \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - (a+d) * \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\
&= \frac{-1}{|A|} * \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a+d & 0 \\ 0 & a+d \end{bmatrix} \right) \\
&= \frac{-1}{|A|} * \begin{bmatrix} -d & b \\ c & -a \end{bmatrix} \\
&= \frac{1}{|A|} * \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\end{aligned}
$$

Example:

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, Find A^3.

The characteristic polynomial for $A = t^2 - 3 * t - 4 \therefore$

by Cayley Hamilton

$A^2 - 3 * A - 4 * Id = 0 \therefore$

$A^2 = 3 * A + 4 * Id \therefore$

A^3

$= 3 * A^2 + 4 * A$

$= 3 * (3 * A + 4 * Id) + 4 * A$

$= 9 * A + 12 * Id + 4 * A$

$= 13 * A + 12 * Id$

Finding A^k using Cayley Hamilton (Cont'd)

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, Find A^4.

Since

$A^2 = 3 * A + 4 * Id \therefore$

$A^3 = 13 * A + 12 * Id$

$\{$Multiply both sides by A $\}$

$$\begin{aligned} A^4 &= 13 * A^2 + 12 * A \\ &= 13 * (3 * A + 4 * Id) + 12 * A \\ &= 39 * A + 52 * Id + 12 * A \\ &= 51 * A + 52 * Id \end{aligned}$$

Finding A^k using Cayley Hamilton (Cont'd)

$$A^4$$

$$= 51 * A + 52 * Id$$

$$= 51 * \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 52 & 0 \\ 0 & 52 \end{bmatrix}$$

$$= \begin{bmatrix} 51 & 102 \\ 153 & 102 \end{bmatrix} + \begin{bmatrix} 52 & 0 \\ 0 & 52 \end{bmatrix}$$

$$= \begin{bmatrix} 103 & 102 \\ 153 & 154 \end{bmatrix}$$

Inverse 3×3 Example

Let $A = \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix}$

The characteristic polynomial

$$\begin{vmatrix} 5-\lambda & 6 & 2 \\ 0 & -1-\lambda & -8 \\ 1 & 0 & -2-\lambda \end{vmatrix}$$

$$= -\lambda^3 + 2*\lambda^2 + 15*\lambda - 36$$

Cont'd

The characteristic equation can be written as:
$-\lambda^3 + 2 * \lambda^2 + 15 * \lambda - 36 = 0.$

From the Cayley-Hamilton Theorem, for the matrix, A ,
$-A^3 + 2 * A^2 + 15 * A - 36 = 0 \therefore$
$36 = -A^3 + 2 * A^2 + 15 * A$

Multiply across by A^{-1} (A^{-1} exists as no eigen value is zero)
$36 * A^{-1} = -A^2 + 2 * A + 15 \therefore$
$A^{-1} = \frac{1}{36} * (-A^2 + 2 * A + 15)$

Calculating $-A^2 + 2 * A + 15$

$$A^2 = \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix} * \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 27 & 24 & -42 \\ -8 & 1 & 24 \\ 3 & 6 & 6 \end{bmatrix} \therefore$$

$-A^2 + 2 * A + 15$

$$= \begin{bmatrix} -27 & -24 & 42 \\ 8 & -1 & -24 \\ -3 & -6 & -6 \end{bmatrix} + 2 * \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 15 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -12 & 46 \\ 8 & 12 & -40 \\ -1 & -6 & 5 \end{bmatrix}$$

Cont'd

$$
\begin{aligned}
A^{-1} &= \frac{1}{36} * (-A^2 + 2 * A + 15) \\
&= \frac{1}{36} * \begin{bmatrix} -2 & -12 & 46 \\ 8 & 12 & -40 \\ -1 & -6 & 5 \end{bmatrix}
\end{aligned}
$$

Check that:

$$
\frac{1}{36} * \begin{bmatrix} -2 & -12 & 46 \\ 8 & 12 & -40 \\ -1 & -6 & 5 \end{bmatrix} * \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

Exercise: Cayley Hamilton

Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

1. Demonstrate that
   $A^3 = 3 * A^2 - 3 * A + Id$
   where Id is the 3×3 Identity matrix.
2. Express A^4 in terms of A^2, A and Id and hence calculate A^4 explicitly.
3. Use (1.) to find A^{-1} explicitly.