Probability Basics

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Probabilistic Inference

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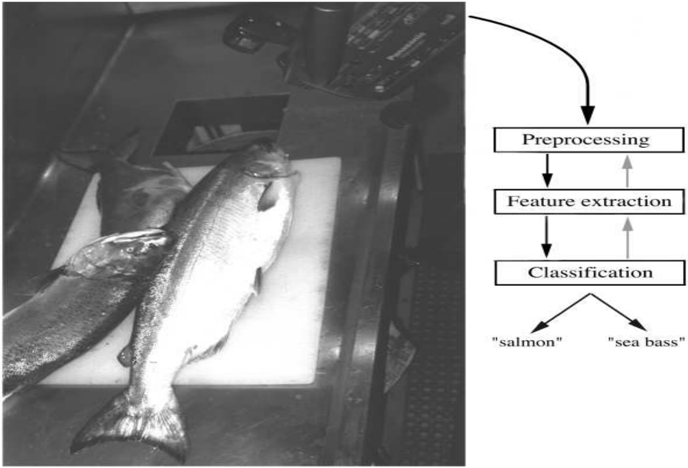
- Suppose there's a variable X whose value you would like to know, but don't
- Suppose there's another variable Y whose value you do know
- Suppose you know probabilities about how values of X and Y go together
- There's a standard way to use the probabilities to make a best guess about X
- In Speech Recognition you want to guess the words which were said, in Machine Translation you want to guess the best translation. To introduce the basic probabilistic framework we will first look though at entirely different kinds of example.

Duda and Hart's fish example

Suppose there are 2 types of fish. You might want to design a fish-sorter which seeks to distinguish between the 2 types of fish (eg. salmon vs. sea bass) by the value of some observable attribute, possibly an attribute a camera can easily measure (eg. lightness of skin)



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If you observe a fish with a particular value for x, what is the best way to use the observation to predict its category?

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$$\textit{choose} \arg\max_{\omega} P(\omega, x)$$

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So if you know both $P(x|\omega)$ and $P(\omega)$ for the two classes ω_1 and ω_2 can now pick the one which maximises $P(x|\omega)P(\omega)$

though widely given the name 'Bayesian Classifier' this really doing nothing more than saying pick the ω which makes the combination you are looking at as likely as possible.

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An equally sensible and in fact equivalent intuition for how to pick ω is to maximise conditional probability of ω given x ie.

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$$= \arg\max_{\omega} P(x|\omega)P(\omega)/P(x) \quad (3)$$

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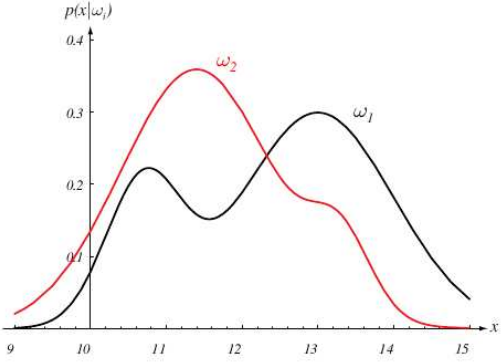
$$\begin{aligned} \arg\max_{\omega} P(\omega|x) &= \arg\max_{\omega} P(\omega, x)/P(x) && (2) \\ &= \arg\max_{\omega} P(x|\omega)P(\omega)/P(x) && (3) \\ &= \arg\max_{\omega} P(x|\omega)P(\omega) && (4) \end{aligned}$$

(2) is by definition of conditional probablility, (3) is by Product Rule, and (4) because denominator $P(x)$ does not mention ω, it does not vary with ω and can be left out

The following shows hypothetical plots of $P(x|\omega_1)$ and $P(x|\omega_2)$



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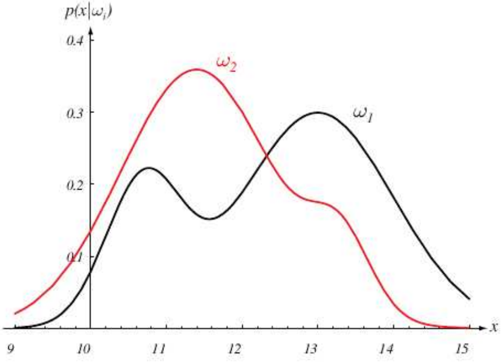

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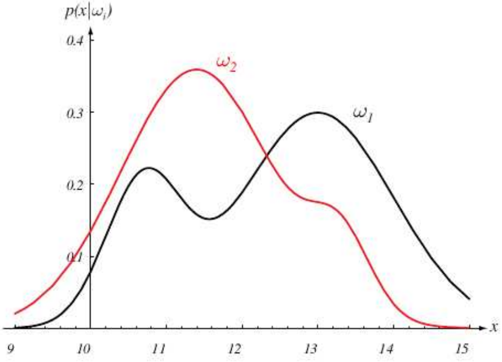

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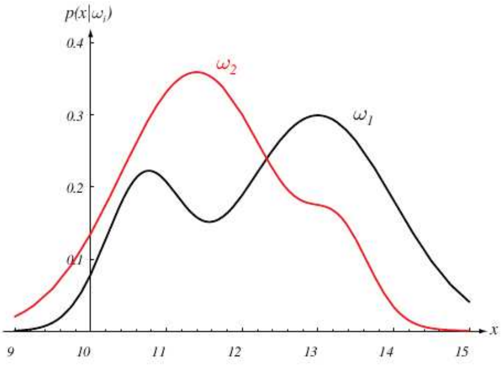

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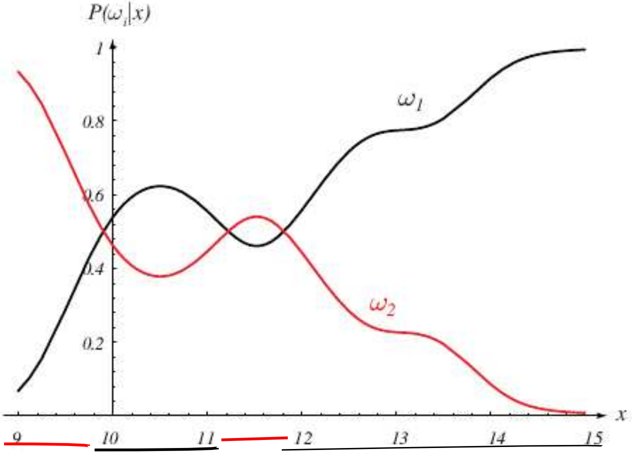
- but this does not mean ω_2 should be chosen for $x < 12.5$, and ω_1 otherwise.
- the plot shows only *half* of the $P(x|\omega)P(\omega)$ referred to in the decision function (1): the other factor is the a priori likelihood $P(\omega)$

Assuming a priori probs $P(\omega_1) = 2/3$, $P(\omega_2) = 1/3$, the plots below show $P(\omega_1, x)$ and $P(\omega_2, x)$, normalised at each x by $P(x)$ (ie. it shows $P(\omega|x)$)



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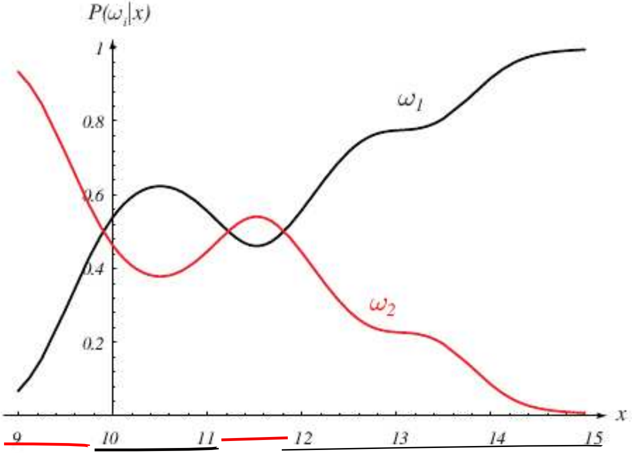

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- So roughly for $x < 10$ or $11 < x < 12$, ω_2 is the best-guess
- So roughly for $10 < x < 11$ or $12 < x$, ω_1 is the best-guess

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- if $p(\omega_1) = p(\omega_2)$, then the decision rests entirely on the class-conditionals: $p(x|\omega_1)$ vs. $p(x|\omega_2)$

'prior' and 'posterior'

- have seen that

$$\arg\max_{\omega} P(\omega|x) = \arg\max_{\omega}(p(x|\omega)p(\omega))$$

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- Without dividing through by $P(x)$ you get basically the much smaller joint probabilities. The maximum occurs for the same ω as the conditional probability and the ratios amongst them are the same as amongst the conditional probs.

Jedward example

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Then suppose these individual probabilities are known

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hence choose $\neg jed$

we have

$$p(omg|jed)p(jed) = 0.01, p(omg|\neg jed)p(\neg jed) = 0.099$$

both values are quite small, and they do not sum to 1

this is because they are alternate expressions for the joint probabilities $p(omg, jed)$ and $p(omg, \neg jed)$, and summing these gives the total *omg* probability, which is not that large.

if want real probability $p(jed|omg)$ summing these alternatives and normalising by this (effectively dividing by $p(omg)$) gives

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The posterior probability of $p(jed|omg)$ is quite small, even though $p(omg|jed)$ is large. This is due to the quite low prior $p(jed)$, and non negligible $p(omg|\neg jed) = 0.1$

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Raising the prior prob for jed to $p(jed) = 0.1$, changes the outcome to

$$p(jed|omg) = 0.526, p(\neg jed|omg) = 0.474$$

Or alternatively decreasing the prob of hearing *OMG* from anyone else to 0.001, changes the outcome to

$$p(jed|omg) = 0.917, p(\neg jed|omg) = 0.083$$

Recap

Joint Probability $P(X, Y)$

Marginal Probability $P(X) = \sum_Y P(X, Y)$

Conditional Probability $P(Y|X) = \dfrac{P(X, Y)}{P(X)}$ ... really $\dfrac{count(X, Y)}{count(X)}$

Product Rule $P(X, Y) = P(Y|X) \times P(X)$

Chain Rule $P(X, Y, Z) = p(Z|(X, Y)) \times P(X, Y) =$
$p(Z|(X, Y)) \times P(Y|X) \times p(X)$

Conditional Independence $P(X|Y, Z) = P(X|Z)$ ie. X ignores Y given Z

Bayesian Inversion $P(X|Y) = \dfrac{P(Y|X)P(X)}{P(Y)}$

Inference to infer X from Y choose $X = \arg\max_X P(Y|X)P(X)$

Further reading

see the course pages under 'Course Outline' for details on particular parts of of particular books which can serve as further sources of information concerning the topics introduced by the preceding slides