

CSU44004/CSU55004: FORMAL VERIFICATION

Lecture 2: Propositional Logic

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LOGICAL STATEMENTS

Propositions are **declarative statements**.

- “Alice is an engineer”
- “The sum of 3 and 5 is 8”
- “The train is late”

Declarative statements can be *declared* to be either **true** or **false** (but not both). These are the **truth-values** of propositions.

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Not all statements are declarative. The following cannot be declared true/false.

- “Let’s go to the cinema” (proposal)
- “Where is Soli?” (question)
- “Fantastic!” (exclamation)
- “It will probably rain tomorrow” (likelihood)

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Propositional logic involves **only** declarative statements.

Complex propositions can be constructed by simple ones using operators.

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We can examine whether such propositions are true or false when we know the values of the basic propositions.

- The train is not late (“the train is late” is **false**),
- Bob is not late to work (“Bob late to work” is **false**),
- Is p true? (without knowing whether there were any taxis in the station)

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We can examine the **necessary** values of the basic propositions that would make complex compositions be true/false.

- p is true,
- Bob is not late to work,
- The train is late
- Are there any taxis in the station?

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English (or any human language) is imprecise and subtle (verb tenses, etc.) and error prone.

A more mathematical language for logic would make the above arguments clear (“calculemus!” – stay tuned).

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- **Propositional logic** First developed by **Chrysippus of Soli** (3rd c. BC) and the Stoic philosophers
 - **conjunction, disjunction, negation**, different forms of implication



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 - **Leibniz** (17th-18th c.): he wanted to turn logic into something as precise as calculus: "Calculus Ratiocinator".
 - Believed all human ideas are made of small number of basic ideas (alphabet of human thought); complex ideas derived from basic ones with combinations similar to arithmetic multiplication.
 - Arguments would be solved by calculating "Calculemus!".
 - Symbolic rules for **conjunction, disjunction, negation**, ...



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 - He never publiced this! (200 years setback)
- **Boole and De Morgan** rediscovered and improved Leibniz calculus of logic (1847) – modern form of Propositional Logic.
- Developed more in the 20th c. (e.g., **Gerhard Gentzen**) influenced by other logicians/philosophers (e.g., Russell, Wittgenstein...)



Chrysippus of Soli – 3rd c. BC



Pierre Abelard – 12th c.



Gottfried Wilhelm Leibniz
– 17th/18th c.



George Boole, Augustus
De Morgan – 19th c.

Images from wikipedia

- Declarative statements have **no intrinsic truth-value**.
- They are simply a **string of symbols**, representing the declarative statements.
- A priori, all declarative statements (**propositions**) could be either **True** or **False** (but not both).
 - “Logic is interesting”
 - “This program terminates”
 - “Bob is male”
 - “Bob is female”
 - “Gates graduated Harvard”

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- We form complex propositions using the following **operators**
 - **Negation**: symbols: \neg \sim examples: $\neg p$ $\sim q$
 - **disjunction**: symbols: \vee **or** examples: $p \vee q$ $p' \text{ or } q'$
 - **conjunction**: symbols: \wedge **and** examples: $p \wedge q$ $p' \text{ and } q'$
 - **implication** or **conditional**: symbols: \rightarrow **implies**
examples: $p \rightarrow q$ $p' \text{ implies } q'$
 - **parantheses**: symbols: $($ $[$ $]$ $)$
- We will call atomic and complex propositions **formulas**

EXAMPLES OF SYMBOLIC PROPOSITIONS

p : "The train is late"

q : "There are taxis in the station"

r : "Bob is late to work"

$$(p \wedge \neg q) \rightarrow r$$

p' : "It is warm"

q' : "It is humid"

r' : "It is raining"

$$q' \rightarrow p'$$

$$(p' \wedge q') \rightarrow (r' \vee p')$$

$$(p' \text{ and } q') \text{ implies } (r' \text{ or } p')$$

To not use so many parentheses we use **binding priorities**:

\neg binds more tightly than

\wedge and \vee which bind more tightly than

\rightarrow which is right-associative

eg:

$$p_1 \wedge \neg q_1 \rightarrow r_1 \vee (p_2 \wedge \neg q_2) \rightarrow r_2$$

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Remember: everything so far is just syntax and syntactical conventions

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We have to be more rigorous:

We use a **formal grammar** that unambiguously accepts exactly the well-formed terms of our logic

To do this we need to define grammar (or meta-) **variables** that stand for **any term derivable from the grammar**. Variables:

$$A, C, B, \dots$$

(The book uses the greek letters ϕ, ψ, χ)

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The **logical formulas** of Propositional Logic are exactly those accepted by the following grammar in Backus Naur Form (BNF):

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We also bring back **binding priorities**:

- binds more tightly than
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- which is right-associative

EXAMPLE

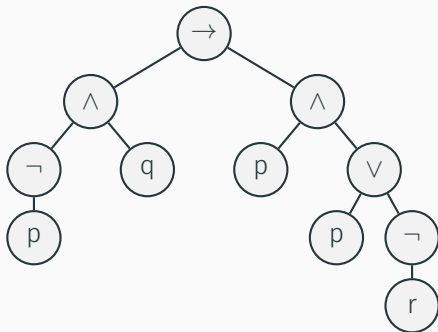
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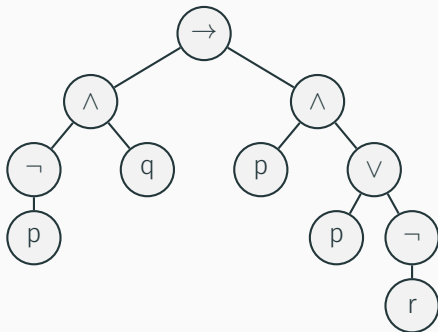
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With implicit binding priorities:

$$\neg p \wedge q \rightarrow (p \wedge (q \vee \neg r))$$

Write in Symbolic Propositional Logic

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2. "If the sun shines today then it won't shine tomorrow"

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Can you parse the following formulas assuming the implicit binding?

$$3 \quad p \vee q \rightarrow \neg p \wedge q$$

$$3 \quad p \wedge q \wedge q$$

SEMANTICS: THE MEANING OF FORMULAS

Now we are going to look at the truth values of propositional logic formulas A . We will write this as $\text{sem}(A)$

Definition

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- The meaning of each **logical operator** is a **predefined function** which maps the truth values of its parameters to the truth value of the formula obtained by applying the operator its parameters.
- The **valuation** or **model** of a formula A is an **assignment** of each propositional atom in A to a truth value.

The **meaning** of p : “It is warm” can be any of the following:

$$\rightarrow \text{sem}(p) = \text{T}$$

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A **model** of the formula $A = p$ can be any of the following

$$\rightarrow p \mapsto \text{T}$$

$$\rightarrow p \mapsto \text{F}$$

SEMANTICS OF COMPLEX FORMULAS

The meaning of operator \neg is a function f_{\neg} which takes one argument. This function is uniquely identified by the **truth table**:

A	$\neg A$
T	F
F	T

Here the A column lists all possible truth values of A. The $\neg A$ column the result of applying the f_{\neg} to these values; these are the truth values of the formula $\neg A$.

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The meaning of operators \wedge and \vee are two functions with two argumens:

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

A	B	$A \vee B$
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A **model** of the formula $A = \neg p \wedge (q \vee p)$ can be any of the following

$\rightarrow p \mapsto \text{T}, q \mapsto \text{T}$

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We can write truth tables of **composed operators** using multiple columns

e.g.: $\neg A \vee \neg B \rightarrow B$

A	B	$\neg A$	$\neg B$	$\neg A \vee \neg B$	$\neg A \vee \neg B \rightarrow B$
T	T	F	F	F	T
T	F	F	T	T	F
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Write the truth table of $\neg A \vee B \rightarrow B$

EXERCISE

Write the truth table of $\neg A \vee B \rightarrow B$

A	B	$\neg A$	$\neg A \vee B$	$\neg A \vee B \rightarrow B$
T	T			
T	F			
F	T			
F	F			

Complete the following truth table

A	B	$\neg A$	$\neg A \vee B$	$\neg A \vee B \rightarrow A$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	F
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Create the truth table of the formula:

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Give a model that makes the formula false.

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A is **satisfiable** when it has **a model** which makes it true.

A is **falsifiable** when it has **a model** which makes it false.

A is **valid** or **a tautology** when it has **no model** which makes it false.

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Q: show that if $A_1 \wedge (A_2 \wedge A_3)$ is satisfiable then $(A_1 \wedge A_2) \wedge A_3$ is satisfiable.

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Q: show that if $A_1 \wedge (A_2 \wedge A_3)$ is satisfiable then $(A_1 \wedge A_2) \wedge A_3$ is satisfiable.

Q: Let $(A_1 \wedge A_2) \rightarrow A_3$ be valid. Is it necessary that A_3 is satisfiable?