On explaining $P(A|B)$ as fraction $\frac{P(A,B)}{P(B)}$

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I mentioned in the lectures that $P(A|B)$ is *defined* as a ratio of two probabilities, $\frac{P(A,B)}{P(B)}$, though when you calculate you invariably work with a ratio of two *counts* $\frac{count(A,B)}{count(B)}$. Here's an attempt to recapitulate the intuition I gave in the lecture for why this ratio of probabilities idea makes sense.

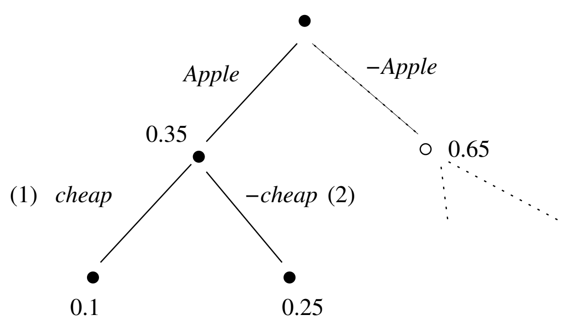
First suppose the following table represents a literal 'table' in a market on which second-hand mobile phones are placed. Suppose there are two brands *Apple* and *HTC* and that they fall into two prices *Expensive* and *Cheap*.

	Exp	Cheap
Apple	50 (.25)	20 (.1)
HTC	30 (.15)	100 (.5)

The cells of the table show primarily the *counts* of the different phones available. There's 200 in all and in brackets then a probability is shown, which divides the count by 200. You could think of this as the probability of grabbing a phone of that type (ie. brand and price-band) if you ran past this market stall and tried to steal one.

You should be able to readily work out things like $P(cheap|Apple)$, by just looking at the *Apple* row, which has 70 phones in all, noting that 20 of those are cheap so $P(cheap|Apple) = 20/70 = 2/7$. This is following the straightforward intuition that for this conditional probability you zero in on the *Apple* outcomes, counting how many of these there are, and then within these, you see how many are also *Cheap* outcomes, and take the ratio.

The picture below illustrates how things can be thought of in a hierarchical way



Moving from the top down to the left represesents the possibility the phone is an *Apple* phone, and the probability of that is 0.35. The two nodes beneath that then represent the price aspect, with the node to the left below saying the *Apple* is also *cheap* and the node to the right that the *Apple* is $\neg$*cheap*. These *joint* probabilities can be read off the table and are 0.1 and 0.25.

Now clearly there must be numbers (1) and (2) which *scale* the Apple prob of 0.35, down into these two alternatives:

$$0.35 \times (1) = 0.1 \quad \Rightarrow \quad (1) = \frac{0.1}{0.35} = \frac{2}{7}$$

$$0.35 \times (2) = 0.25 \quad \Rightarrow \quad (2) = \frac{0.25}{0.35} = \frac{5}{7}$$

Also, not at all accidentally, these two numbers (1) and (2) sum to 1, because $0.35 = 0.1 + 0.25$.

Thus for *cheap* you have a number (1) which turns $P(Apple)$ into $P(Apple, cheap)$, and for $\neg cheap$ you have a number (2) which turns $P(Apple)$ into $P(Apple, \neg cheap)$, and these sum to 1.

This motivates regarding (1) and (2) as a special kind of probability, for which people hit on the name *conditional probability*, and also hit on the notations $P(cheap|Apple)$ and $P(\neg cheap|Apple)$ for (1) and (2).

If you bear in mind this kind of motivation for conditional probability it also drives home the most conspicuous 'purpose' of $P(A|B)$, which is to be multiplied by $P(B)$ and give $P(A, B)$, that is, to generate a *joint* probability