UNIVERSITY OF DUBLIN

XMA2BA11

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

SF Computer Science (B.A.)
SF CSLL
SF ICT

Trinity Term 2007

MATHEMATICS 2BA1

Tuesday, May 22 RDS 09:30 — 12:30

Dr. D. R. Wilkins

Credit will be given for the best 5 questions answered.

Log tables are available from the invigilators, if required.

Students may avail of the HANDBOOK OF MATHEMATICS of Computer Science

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

1. (a) (10 marks) Use the Method of Mathematical Induction to prove that

$$\sum_{k=1}^{n} k(k+1) = \tfrac{1}{3}n(n+1)(n+2)$$

for all positive integers n.

(b) (10 marks) Obtain the general solution of the following ordinary differential equation:

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = \cos 2x.$$

2. (a) (6 marks) Suppose that m and n are non-negative integers. Prove that

$$\int_{-\pi}^{\pi} \cos mx \, \cos nx \, dx = 0 \quad \text{when } m \neq n,$$

and calculate the value of this integral when $m = n = 0$, and when m and n are strictly positive integers with $m = n$.

(b) (14 marks) Find the Fourier series of the periodic function $f\colon \mathbb{R} \to \mathbb{R}$ with period 2π which satisfies $f(x) = \pi - 2|x|$ for all real numbers x satisfying $-\pi \leq x \leq \pi$.

3. (a) (6 marks) What is meant by saying that a relation R on a set A is *reflexive*, *symmetric*, *anti-symmetric*, *transitive*, an *equivalence relation*, or a *partial order*?

   (b) (6 marks) Let R denote the relation on the set $\mathbb{Z}$ of integers, where integers x and y satisfy xRy if and only if $x^4 - y^4$ is an integer. Determine whether or not the relation R is reflexive, symmetric, anti-symmetric or transitive, whether or not it is an equivalence relation, and whether or not it is a partial order. [Justify your answers.]

   (c) (3 marks) Let $f\colon A \to B$ be a function from a set A to a set B. What is meant by saying that such a function is *injective*, that such a function is *surjective*, or that such a function is *bijective*?

   (d) (5 marks) Given any real numbers a and b with $a < b$, we define

   $$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}.$$

   Consider the functions $f\colon [0, 1] \to [0, 1]$ and $g\colon [0, 2] \to [0, 1]$, where $f(x) = 2x - x^2$ for all $x \in [0, 1]$, and $g(x) = 2x - x^2$ for all $x \in [0, 2]$. Which of the functions f and g are injective, which are surjective, and which are invertible? [Justify your answers.]

4. (a) (6 marks) What is a *monoid*? What is the *identity element* (or *neutral element*) of a monoid? What is meant by saying that an element y of a monoid is the *inverse* of an element x of that monoid? What is meant by saying that a monoid is a *group*?

   (b) (14 marks) Let $\otimes$ be the binary operation on the set $\mathbb{R}^2$ of ordered pairs of real numbers defined such that

   $$(a, b) \otimes (c, d) = (ac, d + bc)$$

   for all ordered pairs (a, b) and (c, d) of real numbers. Prove that $(\mathbb{R}^2, \otimes)$ is a monoid. What is the identity element of this monoid? Which are the invertible elements of this monoid? Is this monoid a group?

5. (a) (2 marks) Let m be a positive integer, and let x and y be integers. What is meant by saying that x and y are *congruent* modulo m (or, in symbols, that $x \equiv y \pmod{m}$).

   (b) (5 marks) Let m be a positive integer, and let x, x', y and y' be integers. Suppose that x is congruent to x' modulo m, and y is congruent to y' modulo m. Prove that xy is congruent to $x'y'$ modulo m. [Thus you are to prove that if $x \equiv x' \pmod{m}$ and $y \equiv y' \pmod{m}$, then $xy \equiv x'y' \pmod{m}$.]

   (c) (2 marks) What is meant by saying that two integers x and y are *coprime*?

   (d) (3 marks) List all integers x satisfying $0 \leq x < 18$ that are coprime to 18.

   (e) (4 marks) For each integer x satisfying $0 \leq x < 18$ that is coprime to 18, find an integer y satisfying $0 \leq y < 18$ with the property that $xy \equiv 1 \pmod{18}$.

   (f) (4 marks) Is it that case that, for each integer y, there exists an integer x satisfying $7x \equiv y \pmod{18}$? Is it that case that, for each integer z, there exists an integer x satisfying $4x \equiv z \pmod{18}$? [Justify your answers.]

6. (a) (5 marks) What is an *undirected graph*? What is meant by saying that an undirected graph is *connected*? What is meant by saying that an undirected graph is a *tree*? What is meant by saying that two graphs are *isomorphic*?

   (b) (3 marks) Consider the undirected graph whose vertices are a, b, c, d and e, and whose edges are ab, bc, bd, ce, and de. Is this graph connected? Is this graph a tree? [Justify your answers.]

   (c) (3 marks) If the edge bc is removed from from the graph specified in (b), is the resultant graph connected, and is it a tree? [Justify your answers.]

   (d) (3 marks) If the edges bc and bd are removed from from the graph specified in (b), is the resultant graph connected, and is it a tree? [Justify your answers.]

   (e) (6 marks) Is the graph specified in (b) isomorphic to the undirected graph whose vertices are p, q, r, s and t, and whose edges are pq, pr, qs, rs and st? [Justify your answer.]

7. (a) (6 marks) Describe the formal language over the alphabet $\{0, 1\}$ generated by the context-free grammar whose only non-terminal is $\langle S\rangle$, whose start symbol is $\langle S\rangle$ and whose productions are the following:

$$\begin{aligned} \langle S\rangle &\rightarrow 0 \\ \langle S\rangle &\rightarrow 0\langle S\rangle \\ \langle S\rangle &\rightarrow \langle S\rangle 11 \end{aligned}$$

Is this context-free grammar a regular grammar?

(b) (7 marks) Devise a regular grammar to generate the language L over the alphabet $\{a, b, c\}$ consisting of all finite strings, such as ab, $aabbb$, $aaaaab$, abc, $aabbbc$, that consist of one or more occurrences of the character a, followed by one or more occurrences of the character b, optionally followed by a single occurrence of the character c.

(c) (7 marks) Give the description of a finite state acceptor for the language L of (b), specifying the starting state, the finishing state or states, and the transition table for this finite state acceptor.