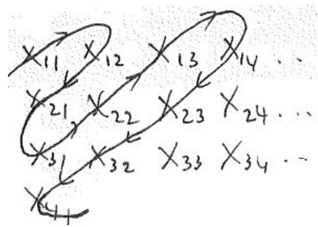
Notation: A sequence $\{x_1, x_2, \ldots\}$ can also be denoted by $\{x_i\}_{i=1,2,\ldots}$

Theorem:
Let $\{A_n\}_{n=1,2,\ldots}$ be a sequence of countably infinite sets. Let $S = \{A_1 \cup A_2 \cup \ldots \cup A_\infty\}$.
Then S is countably infinite.

Proof:
Each A_n is countably infinite $\Leftrightarrow A_n \sim J\ \forall n \geq 1 \Leftrightarrow A_n = \{x_{nk}\}_{k=1,2,\ldots} = \{x_{n1}, x_{n2}, \ldots\}$.
We use the indices like for the entries of a matrix. The first index tells us which A_n set the element belongs to, while the second index tells us where that element is in the enumeration (the counting) of A_n.



$$\{x_{11}, x_{12}, x_{21}, x_{31}, x_{22}, x_{13}, x_{14}, x_{23}, x_{32}, x_{41}, \ldots\}$$

$= \{A_1 \cup A_2 \cup \ldots \cup A_\infty\} = S$ is countably infinite because even if some x_{ij}'s are the same.
$A_n \subseteq S\ \forall n \geq 1$ and $A_n \sim J$.

Corollary 1: Suppose an indexing set **I** is countable, and $\forall i \in \mathbf{I}$, A_i is countable, then $T = \bigcup_{i \in I} A_i$ is countable.

Proof: The biggest set we can obtain here is when **I** is countably infinite and each A_i is countably infinite. By the previous theorem, T is countably infinite in that case. Therefore, T is at most countably infinite (may be finite if **I** is finite and each A_i is finite), so T is countable. (*q.e.d*)

Corollary 2: Let A be a countably infinite set, and let $A^n = A \times \ldots \times A$ (n times) = $\{(a_1, a_2, \ldots a_n) \mid a_1, a_2, \ldots, a_n \in A\}$. Then A^n is countably infinite.

Proof: We use induction.

Base Case: n=1 $A^1 = A \sim J \Rightarrow A^1$ is countably infinite.

Inductive Step: Assume A^{n-1} is countably infinite.
$A^n = A^{n-1} \times A = \{(b, a) \mid b \in A^{n-1}, a \in A\}$
$\forall b \in A^{n-1}\ S_b = \{(b, a) \in A^n \mid a \in A\} \sim J \sim A \Rightarrow S^b$ is countably infinite.
$A^n = \bigcup_{b \in A^{\wedge}(n-1)} S_b \sim J$ by Corollary 1, so A^n is indeed countably infinite as claimed. (*q.e.d*)

Corollary 3: N^n is countably infinite $\forall n \geq 1$.

Proof: N ~ J, so the result follows from Corollary 2. (*q.e.d*)

Corollary 4: Z^n is countably infinite $\forall n \geq 1$.

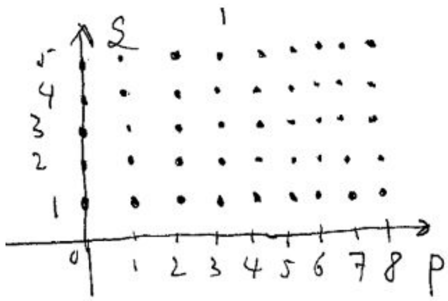
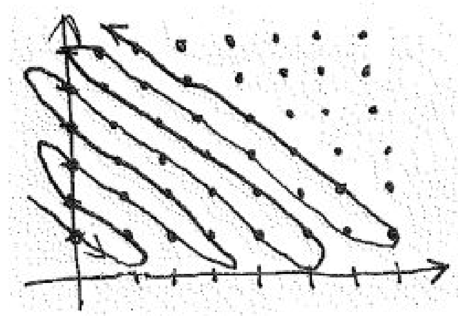
Proof: We showed Z ~ J, so the result follows from Corollary 2. (*q.e.d*)

Corollary 5: Q is countably infinite.

Proof: $Q = \{p/q \mid q \neq 0, p,q \in Z, (p, q)=1$ i.e. no common factors$\}$ but we can represent Q as $\{(p, q) \mid q \neq 0, p,q \in Z\} / \sim \subseteq Z^2$ where $(p_1, q_1) \sim (p_2, q_2) \Leftrightarrow p_1/q_1 = p_2/q_2$ by cross multiplication.

We also know $Z \subseteq Q$ (let q = 1). Therefore Q is sandwiched between $Z = Z^1$ and Z^2, both of which are countably infinite $\Rightarrow$ Q is countably infinite. (*q.e.d*)

Remark: We can give a visual representation of the previous argument as follows:

The dots are pairs (p, q) with $q \neq 0$, $p,q \in Z$ which form a lattice. We can use the snake trick from the theorem to show that the positive rationals $Q^+ = \{p/q \in Q \mid p/q > 0\}$ are countably infinite.

Similarly, we can show $Q^- = \{p/q \in Q \mid p/q < 0\}$ is countably infinite.

Then $Q = Q^- \cup \{0\} \cup \{Q^+\}$ is countably infinite by Corollary 1.

Next, show the set of sequences of 0s and 1s is uncountably infinite. We will use this result to show that other sets are uncountably infinite.

Theorem: Let A be the set of all sequences $s = \{x_1, x_2, ...\} = \{x_n\}_{n=1,2,...}$ such that $x_n \in \{0, 1\}$ $\forall n \geq 1$. Then A is uncountably infinite.

Remark: This result is proven via a clever construction, which is due to Georg Cantor (1845-1918), a very famous German mathematician who invented set theory. Cantor also came up with the diagonal argument (snake trick) which we used to prove that a countably infinite union of countably infinite sets is countably infinite, the idea that sizes of sets should be understood via bijections (A ~ B for A, B sets) as well as the notions of countably infinite and uncountably infinite.

Proof: Assume A is countable $\Leftrightarrow A = \{s_1, s_2, \ldots\}$ where $s_j = \{x^j_n\}_{n=1,2,...}$ for $x_n^j = 0$ or $x_n^j = 1$. We will now construct a sequence s_0 of 0s and 1s that cannot be in the enumeration $\{s_1, s_2, ...\}$. Let s_0 be such that $x_j^o = \{1 \text{ if } x_j^j = 0 \mid 0 \text{ if } x_j^j = 1\}$. In other words, s_0 differs from each s_j in the j^{th} element $\Rightarrow s_0 \notin \{s_1, s_2, ...\}$, but s_0 is a sequence of 0s and 1s $\Rightarrow s_0 \in A \Rightarrow\Leftarrow$ (*q.e.d*)

Corollary: The power set P(N) of N is uncountably infinite.

Remark:
Recall our proof that if B is a set with n elements, #(B) = n, then its power set P(B) has 2^n elements based on the "on/off" idea. For each element of B, we have the choice to include it in our subset ("on") or not to include it ("off"). Therefore we have 2 choices for each element and #(B) = n, so #P(B) = 2^n.

Proof: N ~ J, so we can write N = $\{x_1, x_2, \ldots\}$.
When we form a subset of N, for each i, we can include x_i or leave it out. Say we represent including x_i by 1 and leaving x_i out by 0. Then each subset of N can be represented uniquely as a sequence of 0s and 1s. In fact, there is a one-to-one correspondence between the subsets of N and the sequences of 0s and 1s. Therefore P(N) ~ A, where A is the set of all sequences of 0s and 1s, but we showed in the previous theorem that A is uncountably infinite ⇒ P(N) is uncountably infinite. (*q.e.d*)