XMA2C011, Annual Examination 2012: Worked Solutions

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1. (a) *Let A, B and C be sets. Prove that*

$$A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C).$$

[6 marks]

We show that every element of $A \cap (B \setminus C)$ is an element of $(A \cap B) \setminus (A \cap C)$, and vice versa.

Let $x \in A \cap (B \setminus C)$. Then $x \in A$ and $x \in B \setminus C$. Then $x \in B$ and $x \notin C$. But $x \in A$. It follows that $x \in A \cap B$ and $x \notin A \cap C$, and therefore $x \in (A \cap B) \setminus (A \cap C)$.

Now let $x \in (A \cap B) \setminus (A \cap C)$. Then $x \in A \cap B$ and $x \notin A \cap C$. Then $x \in A$ and $x \in B$, because $x \in A \cap B$. Moreover $x \in A$ and $x \notin A \cap C$, and therefore $x \notin C$. It follows that $x \in B \setminus C$. But also $x \in A$. It follows that $x \in A \cap (B \setminus C)$.

We have now shown that every element of one of the sets $A \cap (B \setminus C)$ and $(A \cap B) \setminus (A \cap C)$ is an element of the other. It follows that

$$A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C).$$

(b) *Let Q and R denote the relations on the set $\mathbb{R}$ of real numbers defined as follows:*

- *real numbers x and y satisfy xQy if and only if there exists some integer k such that $x = 2^k y$;*
- *real numbers x and y satisfy xRy if and only if there exists some non-negative integer k such that $x = 2^k y$.*

For each of the relations Q and R, determine whether or not that relation is *(i)* reflexive*,*

(ii) symmetric*,*

(iii) transitive*,*

(iv) anti-symmetric*,*

(v) an equivalence relation*,*

(vi) a partial order*,*

[Give appropriate short proofs and/or counterexamples to justify your answers.]

[14 marks]

[Recall that a relation R on a set X is an equivalence relation if and only if it is reflexive, symmetric and transitive. It is a partial order if and only if it is reflexive, anti-symmetric and transitive. A relation R on a set X is reflexive if and only if xRx for all $x \in X$; the relation is symmetric if and only if yRx for all $x, y \in X$ satisfying xRy; the relation is transitive if and only if xRz for all $x, y, z \in X$ satisfying xRy and yRz; the relation is anti-symmetric if and only if $x = y$ for all $x, y \in R$ satisfying xRy and yRx.]

Let x be a real number. Then $x = 2^k x$ for $k = 0$. It follows that xQx and xRx for all real numbers x. Thus the relations Q and R are *reflexive.*

Let x and y be real numbers satisfying xQy. Then there exists some integer k such that $x = 2^k y$. But then $y = 2^{-k}x$ and $-k$ is an integer, and therefore yQx. Thus the relation Q is *symmetric.*

Now $2R1$. But $1 \not R 2$, because there is no *non-negative* integer k satisfying $1 = 2^k \times 2$. Therefore the relation R is *not symmetric.*

Now $1Q2$ and $2Q1$ but $1 \neq 2$. It follows that the relation Q is *not anti-symmetric.*

Let x and y be real numbers. Suppose that xRy and yRx. Then there exist non-negative real numbers k and l such that $x = 2^k y$ and $y = 2^l x$. But then $x/y = 2^k$ and $x/y = 2^{-l}$, where $2^k \geq 1$ and $2^{-l} \leq 1$. It follows that $2^k = 2^{-l} = 1$, and therefore $x = y$. Thus the relation R is *anti-symmetric.*

Let x, y and z be real numbers. Suppose that xQy and yQz. Then there exist integers k and l such that $x = 2^k y$ and $y = 2^l z$. But then $x = 2^k 2^l z = 2^{k+l} z$, and $k + l$ is an integer, and therefore xQz. Thus the relation Q is *transitive.* Morever if k and l are both non-negative, then so is $k+l$. It follows that if xRy and yRz then xRz. Thus the relation R is *transitive.*

The relation Q on $\mathbb{R}$ is reflexive, symmetric and transitive. It is thus an *equivalence relation*. It is *not a partial order* because it is not anti-symmetric.

The relation R on $\mathbb{R}$ is reflexive, anti-symmetric and transitive. It is thus a *partial order*. It is *not an equivalence relation* because it is not symmetric.

2. (a) *Let $f\colon A \to B$ be a function from a set A to a set B. What is meant by saying that such a function is* injective*, and that such a function is* surjective*?*

[4 marks]

The function $f\colon A \to B$ is *injective* if and only if, given elements u and v of A for which $u \neq v$, the images $f(u)$ and $f(v)$ of those elements satisfy $f(u) \neq f(v)$. (An injective function $f\colon A \to B$ thus maps distinct elements of the set A to distinct elements of the set B.)

The function $f\colon A \to B$ is *surjective* if and only if, given any element b of B, there exists an element a of A satisfying $f(a) = b$.

(b) *Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function from the set $\mathbb{R}$ of real numbers to itself defined such that $f(x) = 3 - x^2$ for all real numbers x. Determine whether or not this function is injective, and whether or not it is surjective, giving brief reasons for your answers.*

[4 marks]

The function f is not injective. Indeed $f(1) = f(-1) = 2$, but $1 \neq -1$.

The function f is not surjective. Indeed 4 belongs to the codomain of this function, but is not in the range of the function. (Indeed $f(x) \leq 3$ for all real numbers x.)

(c) *What is a* monoid*? What is the* identity element *of a monoid? What is meant by saying that an element of a monoid is* invertible*?*

[6 marks]

A *monoid* $(A, *)$ consists of a set A on which is defined an associative binary operation $*$, for which there exists an *identity element* e, characterized by the property that $e * x = x * e = x$ for all $x \in A$.

(d) *Let $*$ denote the binary operation defined on the set $\mathbb{R}^2$ of ordered pairs of real numbers, where*

$$(x_1, y_1) * (x_2, y_2) = (x_1 x_2 - 3y_1 y_2, x_1 y_2 + y_1 x_2)$$

for all real numbers x_1, x_2, y_1 and y_2. Prove that $\mathbb{R}^2$, with the binary operation $*$, is a monoid. What is the identity element of this monoid?

[6 marks]

Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$. Then

$$\begin{aligned}
&((x_1, y_1) * (x_2, y_2)) * (x_3, y_3) \\
&\quad = (x_1 x_2 - 3y_1 y_2, x_1 y_2 + y_1 x_2) * (x_3, y_3) \\
&\quad = (x_1 x_2 x_3 - 3y_1 y_2 x_3 - 3x_1 y_2 y_3 - 3y_1 y_2 y_3, \\
&\qquad\quad x_1 x_2 y_3 - 3y_1 y_2 y_3 + x_1 y_2 x_3 + y_1 x_2 x_3) \\
&\quad = (x_1, y_1) * (x_2 x_3 - 3y_2 y_3, x_2 y_3 + y_2 x_3) \\
&\quad = (x_1 x_2 x_3 - 3x_1 y_2 y_3 - 3y_1 x_2 y_3 - 3y_1 y_2 x_3, \\
&\qquad\quad x_1 x_2 y_3 + x_1 y_2 x_3 + y_1 x_2 x_3 - 3y_1 y_2 y_3) \\
&\quad = ((x_1, y_1) * (x_2, y_2)) * (x_3, y_3).
\end{aligned}$$

Thus the binary operation $*$ on $\mathbb{R}^2$ is associative.
An element (e, f) of $\mathbb{R}^2$ is an identity element for the binary operation $*$ if and only if

$$(e, f) * (x, y) = (x, y) * (e, f) = (x, y)$$

for all $(x, y) \in \mathbb{R}^2$. This is the case if and only if

$$ex - 3fy = x \quad \text{and} \quad ey + fx = y$$

for all real numbers x and y. By inspection, we see that (e, f) is an identity element if and only if $e = 1$ and $f = 0$. Thus $(\mathbb{R}^2, *)$ is a monoid with identity element $(1, 0)$.

3. (a) *Describe the formal language over the alphabet $\{0, 1, 2\}$ generated by the context-free grammar whose non-terminals are $\langle S\rangle$ and $\langle A\rangle$, whose start symbol is $\langle S\rangle$ and whose productions are the following:*

$$\begin{aligned}
\langle S\rangle &\to 0\langle A\rangle \\
\langle A\rangle &\to 1\langle B\rangle \\
\langle B\rangle &\to 2\langle S\rangle \\
\langle B\rangle &\to 2
\end{aligned}$$

Is this context-free grammar a regular grammar?

[6 marks]

The formal language consists of all strings

$$012, \quad 012012, \quad 012012012, \ldots$$

that are the concatenation of one or more copies of the string 012.

This context-free grammar is a regular grammar since its productions involve only replacement of nonterminals by terminal followed by non-terminal, or a terminal alone.

(b) *Give the specification of a finite state acceptor that accepts the language over the alphabet* $\{0, 1\}$ *consisting of all non-empty finite sequences of binary digits such as*

$$0100, \quad 010001000, \quad 0, \quad 01001000$$

that do not contain two successive 1's.

In particular you should specify the set of states, the starting state, the finishing states, and the transition table that determines the transition function of the finite state acceptor.

[8 marks]

States: S, A, B, E

Starting state: S

Finishing states: A, B

Transition table:

	0	1
S	A	B
A	A	B
B	A	E
E	E	E

(The machine enters state A if the string so far entered is non-empty, terminates with 0, and no error has yet occurred. The machine enters state B if the string so far entered is non-empty, terminates with 1, and no error has yet occurred. If the machine is in state B and 1 is entered, then an error has occurred, and the machine transitions to the error state E. The error state E is not a finishing state, since one cannot finish with an acceptable string once the error state has been entered. The start state is not a finishing state, because the empty string is not acceptable. But if no error has occurred then one can finish at any time that a non-empty string has been entered, and therefore both A and B are finishing states.)

(c) *Devise a regular context-free grammar to generate the language over the alphabet $\{0, 1\}$ described above in (b).*

[6 marks]

Start symbol: $\langle S\rangle$.

Productions:

$$\begin{array}{rcl}
\langle S\rangle & \to & 0\langle A\rangle \\
\langle S\rangle & \to & 1\langle B\rangle \\
\langle A\rangle & \to & 0\langle A\rangle \\
\langle A\rangle & \to & 1\langle B\rangle \\
\langle B\rangle & \to & 0\langle A\rangle \\
\langle A\rangle & \to & \varepsilon \\
\langle B\rangle & \to & \varepsilon
\end{array}$$

4. In this question, all graphs are undirected graphs.

(a)

(i) What is meant by saying that a graph is complete*?*

(ii) What is meant by saying that a graph is regular*?*

(iii) What is meant by saying that a graph is connected*?*

(iv) What is meant by saying that a graph is a tree*?*

(v) Give the definition of an isomorphism *between two undirected graphs.*

[7 marks]

A graph is *complete* if, given any two distinct vertices of the graph, those vertices are endpoints of an edge of the graph.

A graph is *regular* if every vertex has the same degree.

A graph is *connected* if, given any two vertices in the graph, there is a path [or walk] in the graph from one vertex to the other.

A graph is a *tree* if it is connected and it has not circuits.

An *isomorphism* between two graphs is an invertible function from the vertices of one graph to the vertices of the other (establishing a one-to-one correspondence between the vertices of the two graphs) such that two vertices of one graph are endpoints of an edge of that graph if and only if the corresponding two vertices of the other graph are also endpoints of an edge.

(b) *Let G be the undirected graph whose vertices are a, b, c, d, e, f and g and whose edges are the following:*

$$a\,b, \quad a\,c, \quad b\,d, \quad c\,d, \quad d\,e, \quad e\,f. \quad e\,g.$$

(i) Is this graph complete?

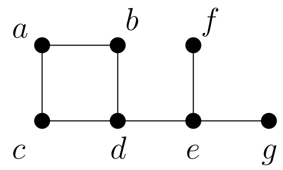
(ii) Is this graph regular?

(iii) Is this graph connected?

(iv) Is this graph a tree?

[Give brief reasons for each of your answers.]

[8 marks]



The graph is not complete. For example there is no edge $a\,d$.

The graph is not regular. The vertex d has degree 3, whereas the vertex g has degree 1.

The graph is connected. By inspection every vertex can be joined to the vertex d by a path of length at most two, and therefore any to vertices can by joined by a walk of length at most four.

The graph is not a tree because it has a circuit $a\,b\,d\,c\,a$.

(c) *Let V denote the set of vertices of the graph G defined in (b). Determine all possible isomorphisms $\varphi\colon V \to V$ from the graph G to itself that satisfy $\varphi(b) = c$.*

[5 marks]

There are two such isomorphisms. The vertex a is the only vertex of the graph of degree 2 adjacent to the vertex b. It therefore has to map to a vertex of degree 2 adjacent to the vertex c, and therefore has to map to itself. Similarly vertex d is the only vertex of degree 3 adjacent to b, and therefore has to map to itself. The other vertex e of degree 3 then has to map to itself. The isomorphism may then either leave the pendent vertices f and g unchanged, or else may switch these two vertices. Thus the two isomorphisms are $\varphi_1\colon V \to V$ and $\varphi_2\colon V \to V$, where

$$\varphi_1(a) = a,\ \ \varphi_1(b) = c,\ \ \varphi_1(c) = b,\ \ \varphi_1(d) = d,\ \ \varphi_1(e) = e,$$
$$\varphi_1(f) = f,\ \ \varphi_1(g) = g$$

and

$$\varphi_2(a) = a,\ \ \varphi_2(b) = c,\ \ \varphi_2(c) = b,\ \ \varphi_2(d) = d,\ \ \varphi_2(e) = e,$$
$$\varphi_2(f) = g,\ \ \varphi_2(g) = f.$$

5. (a) *Any function y of a real variable x that solves the differential equation*

$$\frac{d^3y}{dx^3} + 125y = 0$$

may be represented by a power series of the form

$$y = \sum_{n=0}^{+\infty} \frac{y_n}{n!} x^n,$$

where the coefficients $y_0, y_1, y_2, y_3, \ldots$ of this power series are real numbers.

Find values of these coefficients y_n for $n = 0, 1, 2, 3, 4, \ldots$ that yield a solution to the above differential equation with $y_0 = 1$, $y_1 = -5$ and $y_2 = 25$.

[8 marks]

Now

$$y' = \sum_{n=0}^{+\infty} \frac{y_{n+1}}{n!} x^n, \quad y'' = \sum_{n=0}^{+\infty} \frac{y_{n+2}}{n!} x^n \quad \text{and} \quad y''' = \sum_{n=0}^{+\infty} \frac{y_{n+3}}{n!} x^n.$$

A solution to the differential equation thus satisfies

$$\sum_{n=0}^{+\infty} \frac{y_{n+3} + 125 y_n}{n!} x^n$$

for all real numbers x, and therefore satisfies

$$y_{n+3} + 125 y_n = 0$$

for all natural numbers n. Now $y_0 = 1$, $y_1 = -5$, $y_2 = (-5)^2$ and $y_{n+3} = (-5)^3 y_n$ for all non-negative integers n. It follows that $y_n = (-5)^n$ for all non-negative integers n.
(It follows that

$$y = \sum_{n=0}^{+\infty} \frac{1}{n!}(-5x)^n = e^{-5x}.)$$

(b) *Find the general solution of the differential equation*

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = xe^{5x}.$$

[12 marks]

The general solution takes the form $y_P + y_C$, where y_C is the *complementary function* satisfying the differential equation $y_C'' - 4y_C' + 4y_C = 0$ and y_P is a *particular integral* of the differential equation.
The auxiliary polynomial $s^2 - 4s + 4$ has a repeated root equal to 2. Therefore

$$y_C = (Ax + B)e^{2x},$$

where A and B are arbitrary constants.
We look for a particular integral y_P of the form $y_P = (px + q)e^{5x}$. Then

$$\begin{aligned} y_P &= (px + q)e^{5x}, \\ y_P' &= (5px + p + 5q)e^{5x}, \\ y_P'' &= (25px + 10p + 25q)e^{5x}, \end{aligned}$$

and therefore

$$y_P'' - 4y_P' + 4y_P = (9px + 6p + 9q)e^{5x}.$$

It follows that $9p = 1$ and $6p + 9q = 0$, and therefore $p = \frac{1}{9}$ and $q = -\frac{2}{27}$. Thus $y_P = (\frac{1}{9}x - \frac{2}{27})e^{5x}$, and therefore

$$y = (\tfrac{1}{9}x - \tfrac{2}{27})e^{5x} + (Ax + B)e^{2x}.$$

6. (a) *Let n be an integer satisfying $n > 1$, and let ω be a complex number. Suppose that $\omega^n = 1$ and $\omega \neq 1$. Prove that $\sum_{j=0}^{n-1} \omega^j = 0$, where*

$$\sum_{j=0}^{n-1} \omega^j = 1 + \omega + \omega^2 + \cdots + \omega^{n-1}.$$

[6 marks]

$$\begin{aligned} (1-\omega)\sum_{j=0}^{n-1} \omega^j &= 1 + \sum_{j=1}^{n-1} \omega^j - \sum_{j=0}^{n-2} \omega^{j+1} - \omega^n \\ &= 1 + \sum_{j=1}^{n-1} \omega^j - \sum_{k=1}^{n-1} \omega^k - \omega^n \\ &= 1 - \omega^n = 0. \end{aligned}$$

But $1 - \omega \neq 0$. Therefore $\sum_{j=0}^{n-1} \omega^j = 0$.

(b) *Let $(z_n : n \in \mathbb{Z})$ be the doubly-infinite 3-periodic sequence with $z_0 = 2$, $z_1 = 2$ and $z_2 = 1$. Find values of a_0, a_1 and a_2 such that*

$$z_n = a_0 + a_1\omega^n + a_2\omega^{2n}$$

for all integers n, where $\omega = e^{2\pi i/3}$. (Note that $\omega = \frac{1}{2}(-1 + \sqrt{3}\, i)$, $\omega^2 = e^{-2\pi i/3} = \frac{1}{2}(-1 - \sqrt{3}\, i)$ and thus $\omega^3 = 1$ and $\omega + \omega^2 = -1$.)

[14 marks]

Using the standard formula for the coefficients of the Discrete Fourier Transform, we find that

$$a_j = \frac{1}{3}(z_0 + z_1\omega^{-n} + z_2\omega^{-2n})$$

for $j = 0, 1, 2$. Moreover $\omega^{-1} = \omega^2$ and $\omega^{-2} = \omega$, because $\omega^3 = 1$.

Therefore

$$\begin{aligned}
a_0 &= \frac{1}{3}(2+2+1) = \frac{5}{3} \\
a_1 &= \frac{1}{3}(2+2\omega^2+\omega) \\
&= \frac{1}{3}\left(2+2\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)+\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right) \\
&= \frac{1}{2}-\frac{\sqrt{3}}{2}i \\
a_2 &= \frac{1}{3}(2+2\omega+\omega^2) \\
&= \frac{1}{3}\left(2+2\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)+\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right) \\
&= \frac{1}{2}+\frac{\sqrt{3}}{2}i
\end{aligned}$$

7. (a) *Find the lengths of the vectors* $(1,2,3)$ *and* $(1,4,3)$ *and also the cosine of the angle between them.*

[6 marks]

Let $\vec{u} = (1,2,3)$ and $\vec{v} = (1,4,3)$, and let θ denotes the angle between these two vectors. Then $\vec{u}.\vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta$, where $|\vec{u}|$ and $|\vec{v}|$ denote the lengths of the vectors $\vec{u}$ and $\vec{v}$ respectively.
Now

$$\begin{aligned}
|\vec{u}|^2 &= 1^2+2^2+3^2 = 1+4+9 = 14, \\
|\vec{v}|^2 &= 1^2+4^2+3^2 = 1+16+9 = 26,
\end{aligned}$$

and thus $|\vec{u}| = \sqrt{14}$ and $|\vec{v}| = \sqrt{26}$. Now

$$\vec{u}.\vec{v} = 1\times 1 + 2\times 4 + 3\times 3 = 1+8+9 = 18.$$

Therefore

$$\cos\theta = \frac{18}{\sqrt{14\times 26}} = \frac{9}{\sqrt{7\times 13}} = \frac{9}{\sqrt{91}}.$$

(b) *Find the components of a non-zero vector that is orthogonal to the two vectors* $(1,2,3)$ *and* $(1,4,3)$.

[6 marks]

A non-zero vector orthogonal to $\vec{u}$ and $\vec{v}$ is the vector product $\vec{u} \times \vec{v}$, where $\vec{u} = (1, 2, 3)$ and $\vec{v} = (1, 4, 3)$. Now

$$\vec{u} \times \vec{v} = (2 \times 3 - 3 \times 4,\ \ 3 \times 1 - 3 \times 1,\ \ 1 \times 4 - 2 \times 1) = (-6, 0, 2).$$

To confirm this solution, one can easily check that the scalar product of $(-6, 0, 2)$ with each of $\vec{u}$ and $\vec{v}$ is zero.

(c) *Let the quaternions q and r be defined as follows:*

$$q = 1 - 4k, \quad r = i - j + 3k.$$

Calculate the quaternion products q^2, qr and rq. [Hamilton's basic formulae for quaternion multiplication state that

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.]$$

[8 marks]

$$\begin{aligned}
q^2 &= (1 - 4k)(1 - 4k) = 1 - 4k - 4k + 16k^2 \\
&= 1 - 8k - 16 \\
&= -15 - 8k, \\
qr &= (1 - 4k)(i - j + 3k) = i - j + 3k - 4ki + 4kj - 12k^2 \\
&= i - j + 3k - 4j - 4i + 12 \\
&= 12 - 3i - 5j + 3k \\
rq &= (i - j + 3k)(1 - 4k) = i - j + 3k - 4ik + 4jk - 12k^2 \\
&= i - j + 3k + 4j + 4i + 12 \\
&= 12 + 5i + 3j + 3k
\end{aligned}$$

8. (a) *Find an integer x such that $x \equiv 2$ (mod. 5), $x \equiv 1$ (mod. 13) and $x \equiv 5$ (mod. 17).*

[12 marks]

We find integers u_1, u_2 and u_3 such that

$$u_1 \equiv 1 \text{ (mod. 5)}, \quad u_1 \equiv 0 \text{ (mod. 13)}, \quad u_1 \equiv 0 \text{ (mod. 17)},$$

$$u_2 \equiv 0 \text{ (mod. 5)}, \quad u_2 \equiv 1 \text{ (mod. 13)}, \quad u_2 \equiv 0 \text{ (mod. 17)},$$

$$u_3 \equiv 0 \text{ (mod. 5)}, \quad u_3 \equiv 0 \text{ (mod. 13)}, \quad u_3 \equiv 1 \text{ (mod. 17)},$$

The integer u_1 must be divisible by both of the primes 13 and 17, and therefore must be divisible by 221 since $221 = 13 \times 17$. Now $221 \equiv 1$ (mod. 5). We may therefore take $u_1 = 221$.

Similarly the integer u_2 must be divisible by 85, since $85 = 5 \times 17$. Now $85 \equiv 7$ (mod. 13), $2 \times 17 = 14$, and $14 \equiv 1$ (mod. 13). Now $2 \times 85 = 170$. We conclude that $170 \equiv 1$ (mod. 13). We may therefore take $u_2 = 170$.

The integer u_3 must be divisible by 65, since $65 = 5 \times 13$. Now $65 \equiv 14$ (mod. 17), $11 \times 14 = 154$ and $154 \equiv 1$ (mod. 17). Now $11 \times 65 = 715$. We conclude that $715 \equiv 1$ (mod. 17). We may therefore take $u_3 = 715$.

We may then take

$$x = 2u_1 + u_2 + 5u_3 = 442 + 170 + 3575 = 4187.$$

(Let y be an integer. Then y satisfies the required congruences if and only if $y - 4187$ is divisble by 5, 13 and 17. Now $5 \times 13 \times 17 = 1105$. It follows that y satisfies the congruences if and only if $y - 4187$ is a multiply of 1105. Thus 872, 1977 and 3082 also satisfy the required congruences.)

(b) *Find the value of the unique integer x satisfying $0 \leq x < 13$ for which $3^{3002} \equiv x$ (mod. 13).*

[8 marks]

Now $3^3 = 27$ and $27 = 2 \times 13 + 1$. It follows that $3^3 \equiv 1$ (mod. 13), and therefore $3^{3k} \equiv 1$ (mod. 13). In particular $3^{3000} \equiv 1$ (mod. 13), and therefore $3^{3002} \equiv 3^2$ (mod. 13). It follows that $3^{3002} \equiv 9$ (mod. 13).