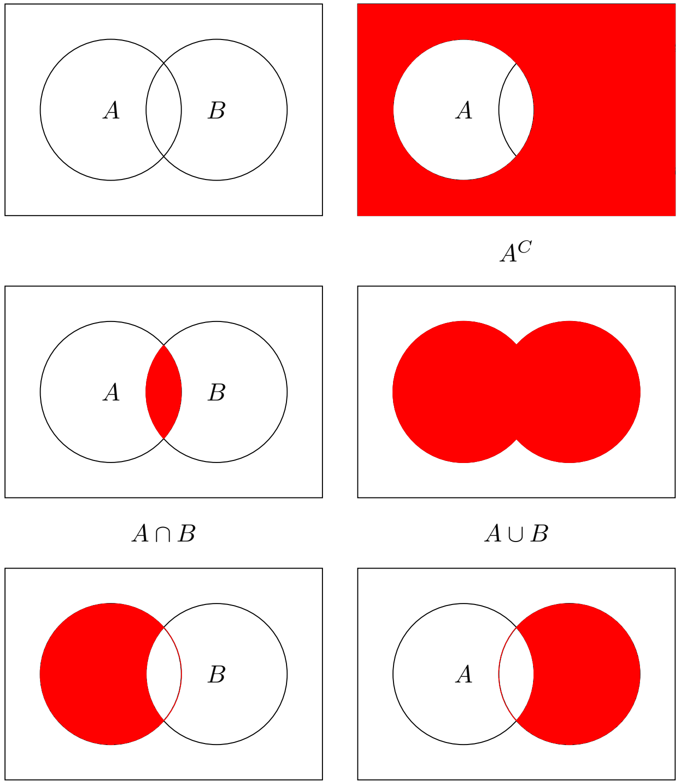
3.2.1 Venn Diagrams

Schematic representation of set operations.



A^C

$A \cap B$ $A \cup B$

$A \setminus B$ $B \setminus A$

Pros of Venn diagrams:

Very easy to visualize

Cons of Venn diagrams:

1. Misleading if for example $A \subset B$ or sets are in some other non standard configuration;

2. Not helpful if a lot of sets are involved;

3. Not helpful if sets are infinite or have some peculiar structure.

Moral of the story: Venn diagrams will **NOT** be accepted as proof of any statement in set theory. Instead, we will introduce rigorous ways of proving assertions in set theory.

3.2.2 Properties of Set Operations

Correspondence between Logic and Set Theory

Logical Connective	Set operation
$\wedge$	intersection $\cap$
$\vee$	union $\cup$
$\neg$	complement $(\ )^C$

As a result, various properties of set operations become obvious:

- Commutativity
  - $A \cap B = B \cap A$
  - $A \cup B = B \cup A$
- Associativity
  - $(A \cup B) \cup C = A \cup (B \cup C)$
  - $(A \cap B) \cap C = A \cap (B \cap C)$
- Distributivity
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup B)$
- De Morgan Laws in Set Theory
  - $(A \cap B)^C = A^C \cup B^C$
  - $(A \cup B)^C = A^C \cap B^C$
- Involutivity of the Complement
  - $(A^C)^C = A$

NB: An involution is a map such that applying it twice gives the identity. Familiar examples: reflecting across the x-axis, the y-axis, or the origin in the plane.

- Transitivity of Inclusion
  - $A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$
- Criterion for proving equality of sets, which comes from the tautology $(P \leftrightarrow Q) \leftrightarrow [(P \rightarrow Q) \wedge (Q \rightarrow P)]$
  - $A = B \leftrightarrow A \subseteq B \wedge B \subseteq A$
- Criterion for proving non-equality of sets
  - $A \neq B \leftrightarrow (A \backslash B) \cup (B \backslash A) \neq 0$

3.3 Example Proof in Set Theory

Proposition: $\forall A, B$ sets. $(A \cap B) \cup (A \backslash B) = A$

Proof: Use the criterion for proving equality of sets from above, **i.e.** inclusion in both directions.

<u>Show $(A \cap B) \cup (A \backslash B) \subseteq A$</u>: $\forall x \in (A \cap B) \cup (A \backslash B), x \in (A \cap B)$ or $x \in A \backslash B$. If $x \in (A \cap B)$, then clearly $x \in A$ as $A \cap B \subseteq A$ by definition. If $x \in A \backslash B$, then by definition $x \in A$ and $x \notin B$, so definitely $x \in A$. In both cases, $x \in A$ as needed.

<u>Show $A \subseteq (A \cap B) \cup (A \backslash B)$</u>: $\forall x \in A$, we have two possibilities, namely $x \in B$ or $x \notin B$. If $x \in B$, then $x \in A$ and $x \in B$, so $x \in A \cap B$. If $x \notin B$, then $x \in A$ and $x \notin B$, so $x \in A \backslash B$. In both cases, $x \in (A \cap B)$ or $x \in (A \backslash B)$, so $x \in (A \cap B) \cup (A \backslash B)$ as needed.

qed

3.4 The Power Set

Task: Understand what the power set of a set A is.

Definition: Let A be a set. The power set of A denoted $P(A)$ is the collection of all subsets of A.

Recall: $\emptyset \subseteq A$. It is also clear from the definition of a subset that $A \subseteq A$.

Examples:

1. $A = \{0, 1\}$
   $P(A) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$
2. $A = \{a, b, c\}$
   $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$
3. $A = \emptyset$
   $P(A) = \{\emptyset\}$
   $P(P(A)) = \{\emptyset, \{\emptyset\}\}$

NB: $\emptyset$ and $\{\emptyset\}$ are different objects. $\emptyset$ has no elements, whereas $\{\emptyset\}$ has one element.

Remark: $P(A)$ and A are viewed as living in separate worlds to avoid phenomena like Russell' paradox.

Q: If A has n elements, how many elements does $P(A)$ have?

A: 2^n

Theorem: Let A be a set with n elements, then $P(A)$ contains 2^n elements.

Proof: Based on the on/off switch idea.

$\forall x \in A$, we have two choices: either we include x in the subset or we don't (on vs off switch). A has n elements $\Rightarrow$ we have 2^n subsets of A.

qed

Alternate Proof: Using mathematical induction.

NB: It is an axiom of set theory (in the ZFC standard system) that every set has a power set, which implies no set consisting of all possible sets could exist, else what would its power set be?

3.5 Cartesian Products

Task: Understand sets like $\mathbb{R}^1$ in a more theoretical way.

Recall from Calculus:

$$\mathbb{R} = \mathbb{R}^1 \ni x$$

$$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 \ni (x_1, x_1)$$

$$\vdots$$

$$\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{\text{n times}} = \mathbb{R}^n \ni (x_1, x_2, ..., x_n)$$

These are examples of Cartesian products.

Definition: Let A, B be sets. The Cartesian product denoted by $A \times B$ consists of all ordered pairs (x, y) $s.t.$ $x \in A \wedge y \in B$, **i.e.** $A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$

Further Examples:

1. $A = \{1, 3, 7\}$
   $B = \{1, 5\}$
   $A \times B = \{(1, 1), (1, 5), (3, 1), (3, 5), (7, 1), (7, 5)\}$
   NB: The order in which elements in a pair matters: $(7, 1)$ is different from $(1, 7)$. This is why we call (x, y) an ordered pair.

2. $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \leftarrow$ circle of radius 1
$B = \{z \in \mathbb{R} \mid -2 \leq z \leq 2\} = \{-2, 2\} \leftarrow$ closed interval
$A \times B \leftarrow$ cylinder of radius 1 and height 4