9.8 Components of a graph

Task: Divide a graph into subgraphs that are isolated from each other.

Let (V, E) be an undirected graph. We define a relation $\sim$ on the set of vertices V, where $a, b \in V$ satisfy $a \sim b$ iff $\exists$ walk in the graph from a to b.

Lemma: Let (V, E) be an undirected graph. The relation $a \sim b$ or $a, b \in V$, which holds iff $\exists$ walk in the graph between a and b is an equivalence relation.

Proof: We must show $\sim$ is reflexive, symmetric, and transitive.

Reflexive: $\forall v \in V, v \sim v$ since the trivial walk is a walk from v to itself.

Symmetric: If $a \sim b$ for $a, b \in V$, then $\exists$ walk $v_0 v_1 ... v_n$ where $v_0 = a$ and $v_n = b$. This walk can be reversed to $v_n v_{n-1} ... v_1 v_0$, which now goes from $v_n = b$ to $v_0 = a$. Therefore, $b \sim a$ as needed.

Transitive: If $a \sim b$ and $b \sim c$, for $a, b, c \in V$, there $\exists$ walk $a v_1 v_2 ... v_{n-1} b$ from a to b and $\exists$ walk $b w_1 w_2 ... w_{m-1} c$ from b to c. We put these two walks together (concatenate them) to yield the walk $a v_1 v_2 ... v_{n-1} b w_1 w_2 ... w_{m-1} c$ from a to c. Therefore $a \sim c$.

qed

The equivalence relation $\sim$ on V partitions it into disjoint subsets $v_1, v_2, ... v_p$, where

1. $v_1 \cup v_2 \cup ... \cup v_p = V$
2. $v_i \cap v_j = \emptyset$ if $i \neq j$
3. Two vertices $a, b \in v_i \Leftrightarrow a \sim b$, **i.e.** $\exists$ walk in (V, E) from a to b

Note that an edge is a walk of length 1, so if $a, b \in V$ satisfy that $\exists\, ab \in E$, then a and b belong to the same v_i. As a result, we can partition the set of edges as follows:

$$E_i = \{ab \in E \mid a, b \in v_i\}$$

Clearly, $E_1 \cup E_2 \cup ... \cup E_p = E$ and $E_i \cap E_j = \emptyset$ if $i \neq j$. Furthermore, $(V_1, E_1), (V_2, E_2), ..., (V_p, E_p)$ are subgraphs of (V, E), and these subgraphs are disjoint since $V_i \cap V_j = \emptyset$ and $E_i \cap E_j = \emptyset$ if $i \neq j$. The subgraphs (V_i, E_i) are called the <u>components</u> (or <u>connected components</u>) of the graph (V, E).

Lemma: The vertices and edges of any walk in an undirected graph are all contained in a single component of that graph.

Proof: Let $v_0 v_1 ... v_n$ be a walk in a graph (V, E), then $v_0 v_1 ... v_r$ is a walk in (V, E) $\forall r\ 1 \leq r \leq n \Rightarrow v_0 \sim v_r$ $\forall r\ 1 \leq r \leq n \Rightarrow v_r$ belongs to the same component of the graph as v_0. The same is true for all the edges $v_{i-1} v_i$ for $1 \leq i \leq n$.

qed

Lemma: Each component of an undirected graph is connected.

Proof: Let (V, E) be a graph and let (V_1, E_i) be any component of (V, E). $\forall u, v \in V_i$, by definition $\exists$ walk in (V, E) between u and v. By previous lemma, however, all vertices and edges of this walk are in $(V_i, E_i) \Rightarrow$ the walk between u and v is a walk in (V_i, E_i), but this assertion is true $\forall u, v \in V_i \Rightarrow (V_i, E_i)$ is connected.

qed

Moral of the story Any undirected graph can be represented as a disjoint union of connected subgraphs, namely its components $\Rightarrow$ the study of undirected graphs reduces to the study of connected graphs, as components don't share either vertices or edges.

9.9 Circuits

Task: Use closed walks to understand the structure of graphs better.

Definition: Let (V, E) be a graph. A walk $v_0 v_1 ... v_n$ in (V, E) is called closed if $v_0 = v_n$, **i.e.** if it starts and ends at the same vertex.

Definition: Let (V, E) be a graph. A circuit is a nontrivial closed trail in (V, E), **i.e.** a closed walk with no repeated edges passing through at least two vertices.

Definition: A circuit is called simple if the vertices $v_0, v_1, v_2, ... v_{n-1}$ are distinct.

NB: This is the strangest condition regarding vertices that we can impose since $v_0 = v_n$.

Alternative terminology: Some authors use cycle to denote a simple circuit, which for others cycle denotes a circuit regardless of whether it is simple or not.

Q: When does a graph have simple circuits?

A: We can give 2 criteria for the existence of simple circuits:

1. Every vertex has degree ≥ 2.
2. $\forall u, v \in V$ s.t. $\exists$ 2 distinct paths from u to v.

Theorem: If (V, E) has no isolated or pendant vertices, **i.e.** $\forall v \in V$ deg $v \geq 2$, then (V, E) contains at least one simple circuit.

Proof: Consider all paths (V, E). The maximum length of a path is $\#(V) - 1$ since a path of length p passed through $p+1$ vertices. Take a path $v_0 v_1 ... v_m$ is (V, E) of maximum length, **i.e.** any other path in (V, E) has length $\leq m =$ length of $v_0 v_1 ... v_m$. Now consider the vertex v_m. deg $v_m \geq 2$ by assumption. We know v_{m-1} is adjacent to v_m since the edge $v_{m-1} v_m$ is part of the path $v_0 v_1 ... v_m$, but deg $v_m \geq 2$ means $\exists w \in V$ s.t. $w w_m \in E$. If $w \neq v_i$ for $0 \leq i \leq m-2$, then $v_0 v_1 ... v_m w$ is a path in (V, E) longer than $v_0 v_1 ... v_m \Rightarrow\Leftarrow$ to the fact that $v_0 v_1 ... v_m$ was chosen of maximal length. Therefore, $w = v_i$ for some $0 \leq i \leq m-2$, but then $v_i v_{i+1} ... v_m v_i$ is a simple circuit in the graph.

qed