Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Engineering, Mathematics and Science

School of Computer Science and Statistics

SF Integrated Computer Science **Trinity Term 2016**
SF CSL

MA2C03: Discrete Mathematics

Tuesday, May 3 **RDS** **14.00 — 17.00**

Prof. Andreea Nicoara and Prof. David Wilkins

Instructions to Candidates:

Credit will be given for the best 6 questions answered.

Materials Permitted for this Examination:

Formulae and Tables are available from the invigilators, if required.

Students may avail of the *Handbook of Mathematics* of Computer Science

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

1. (a) Let $f : [-1, 1] \to [0, 1]$ be the function defined by

$$f(x) = \frac{1}{2x^4 + 1} \text{ for all } x \in [-1, 1].$$

Determine whether or not this function is injective and whether or not it is surjective. Justify your answers.

(8 points)

(b) Let Q denote the relation on the set $\mathbb{Z}$ of integers, where integers x and y satisfy xQy if and only if

$$x - y = x^2 - y^2.$$

Determine the following:—

(i) whether or not the relation Q on $\mathbb{Z}$ is *reflexive*;

(ii) whether or not the relation Q on $\mathbb{Z}$ is *symmetric*;

(iii) whether or not the relation Q on $\mathbb{Z}$ is *anti-symmetric*;

(iv) whether or not the relation Q on $\mathbb{Z}$ is *transitive*;

(v) whether or not the relation Q on $\mathbb{Z}$ is an *equivalence relation*;

(vi) whether or not the relation Q on $\mathbb{Z}$ is a *partial order*.

Give appropriate short proofs and/or counterexamples to justify your answer.

(12 points)

(End of Question)

2. (a) Let A be a semigroup. Show that if A has an identity element then that identity element is unique.

(8 points)

(b) Let $A = \{1, 4, 7\}$. Consider the power set $\mathcal{P}(A)$ of A consisting of all subsets of A. Let $\mathcal{P}(A)$ be endowed with the set intersection operation $\cap$.

(i) Is $(\mathcal{P}(A), \cap)$ a semigroup? Justify your answer by appropriate short proofs and/or counterexamples.

(ii) Is $(\mathcal{P}(A), \cap)$ a monoid? Justify your answer by appropriate short proofs and/or counterexamples.

(iii) Is $(\mathcal{P}(A), \cap)$ a group? Justify your answer by appropriate short proofs and/or counterexamples.

(12 points)

(End of Question)

3. (a) Describe the formal language over the alphabet $\{a, l, p\}$ generated by the context-free grammar whose only non-terminal is $\langle S \rangle$, whose start symbol is $\langle S \rangle$, and whose production rules are the following:

   (1) $\langle S \rangle \to l$

   (2) $\langle S \rangle \to a\langle S \rangle$

   (3) $\langle S \rangle \to a\langle S \rangle p$

   In other words, describe the structure of the strings generated by this grammar.

   (6 points)

   (b) Let L be the language over the alphabet $\{0, 1\}$ consisting of all words, where the string 11 occurs at least once as a substring. In other words, the empty string ϵ, 1, and 0101 are not valid strings, but 11, 010110, and 1110110101 are valid strings. Draw a finite state acceptor that accepts the language L. Carefully label all the states including the starting state and the finishing states as well as all the transitions.

   (8 points)

   (c) Devise a regular grammar in normal form that generates the language L from part (b). Be sure to specify the start symbol, the non-terminals, and all the production rules.

   (6 points)

   (End of Question)

4. In this question, all graphs are undirected graphs.

   (a) (i) What is meant by saying that a graph is *bipartite*?

   (ii) What is meant by saying that a graph is *connected*?

   (iii) What is an *Eulerian trail* in a graph?

   (iv) What does it mean to say that (V', E') is a *subgraph* of a graph (V, E)?

(7 points)

(End of Question)

   (b) Let (V, E) be the graph with vertices a, b, c, d, and e and edges ab, bc, cd, de, and be.

   (i) Draw this graph. Write down its incidence table and its adjacency table.

   (ii) Is this graph complete? Justify your answer.

   (iii) Is this graph regular? Justify your answer.

   (iv) Does this graph have an Eulerian circuit? Justify your answer.

   (v) Does this graph have a Hamiltonian circuit? Justify your answer.

   (vi) Is this graph a tree? Justify your answer.

(8 points)

   (c) Let (V, E) be the graph defined in part (b). Give an example of an isomorphism $\varphi: V \to V$ from the graph (V, E) to itself that satisfies $\varphi(e) = c$.

(5 points)

(End of Question)

5. In this question, all graphs are undirected graphs.

   (a) Explain why, in any graph that is a tree, the number of edges is always one less than the number of vertices.

   (4 points)

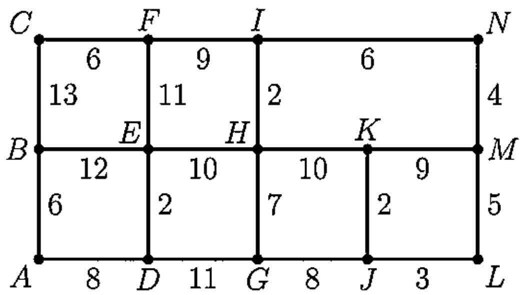
   (b) Explain why any connected graph in which the number of edges is one less than the number of vertices must be a tree.

   (4 points)

   (c) Consider the connected graph with vertices A, B, C, D, E, F, G, H, I, J, K, L, M and N and with edges, listed with associated costs, in the following table:

DE	HI	JK	JL	MN	LM	AB	IN	CF	GH
2	2	2	3	4	5	6	6	6	7
AD	GJ	KM	FI	EH	HK	DG	EF	BE	BC
8	8	9	9	10	10	11	11	12	13

   This graph is depicted below.



   Determine the minimum spanning tree generated by Prim's Algorithm, starting from the vertex A, where that algorithm is applied with the edges ordered as specified above. In particular, list the edges in the order in which they are added to the Prim subtree generated by the application algorithm, so that edges AB and AD as the first and second edges added to the Prim subtree.

   (12 points)

   (End of Question)

6. Let A, B, C, D, E, F G and H be eight points in three-dimensional Euclidean space whose Cartesian coordinates are as as follows:—

$$A = (2, 2, 3), \quad B = (3, 3, 5), \quad C = (4, 3, -1), \quad D = (5, 4, 1),$$

$$E = (5, 5, 7), \quad F = (6, 6, 9), \quad G = (7, 6, 3), \quad H = (8, 7, 5).$$

Note that

$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{GH} = \mathbf{u},$$

$$\overrightarrow{AC} = \overrightarrow{BD} = \overrightarrow{EG} = \overrightarrow{FH} = \mathbf{v},$$

$$\overrightarrow{AE} = \overrightarrow{BF} = \overrightarrow{CG} = \overrightarrow{DH} = \mathbf{w},$$

where

$$\mathbf{u} = (1, 1, 2), \quad \mathbf{v} = (2, 1, -4), \quad \mathbf{w} = (3, 3, 4).$$

It follows that A, B, C, D, E, F, G and H are the vertices of a parallelepiped in three-dimensional Euclidean space.

(a) Calculate the length of the line segments CD, CF, and the cosine of the angle between these two line segments at the point B.

(6 points)

(b) Calculate the equation of the plane passing through the points A, B and C, expressing the equation of the plane in the form $ax + by + cz = k$ for appropriate real constants a, b, c and k.

(8 points)

(c) Find the volume of the parallelepiped with vertices at A, B, C, D, E, F, G and H.

(6 points)

(End of Question)

7. In this question we consider differential equations of the form

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = (g + hx)e^{mx}$$

where the real numbers b, c, g, h and m are constants and $m^2 + bm + c \neq 0$. We suppose also that the auxiliary polynomial $z^2 + bz + c$ of this differential equation has two non-real roots $p + \sqrt{-1}\,q$ and $p - \sqrt{-1}\,q$, where $q \neq 0$.

(a) Prove that if $y_C = e^{px}(A\cos qx + B\sin qx)$, where A and B are real constants, then y_C satisfies the associated homogeneous linear differential equation

$$\frac{d^2y_C}{dx^2} + b\frac{dy_C}{dx} + cy_C = 0.$$

(6 points)

(b) Let y_P be a function of the form

$$y_P = (u + vx)e^{mx},$$

where u and v are constants to be determined. Calculate algebraic expressions that represent u and v in terms of b, c, g, h and m so as to ensure that y_P is a solution of the differential equation specified at the beginning of the question. [Your work should be fully justified.]

(10 points)

(c) Determine the form of the general solution of differential equations that have the form specified at the beginning of the question.

(4 points)

(End of Question)

8. (a) Find an integer x such that $x \equiv 2$ (mod. 3), $x \equiv 3$ (mod. 5) and $x \equiv 8$ (mod. 11).

(12 points)

(b) Find the value of the unique integer x satisfying $0 \leq x < 17$ for which $7^{80000003} \equiv x$ (mod. 17).

(8 points)

(End of Question)