UNIVERSITY OF DUBLIN

XMA2C011

TRINITY COLLEGE

FACULTY OF ENGINEERING, MATHEMATICS
AND SCIENCE

SCHOOL OF COMPUTER SCIENCE AND STATISTICS
DEPARTMENT OF COMPUTER SCIENCE

SF Integrated Computer Science Programme
SF CSLL

Trinity Term 2012

DISCRETE MATHEMATICS — MODULES MA2C01 AND MA2C02

Wednesday, May 2 SPORTS CTR 14:00 — 17:00

Dr. D. R. Wilkins

Credit will be given for the best **SIX** questions answered.

Each question is worth 20 marks.

Log tables are available from the invigilators, if required.

Students may avail of the HANDBOOK OF MATHEMATICS of Computer Science

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

1. (a) Let A, B and C be sets. Prove that

$$A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C).$$

[6 marks]

(b) Let Q and R denote the relations on the set $\mathbb{R}$ of real numbers defined as follows:

- real numbers x and y satisfy xQy if and only if there exists some integer k such that $x = 2^k y$;
- real numbers x and y satisfy xRy if and only if there exists some non-negative integer k such that $x = 2^k y$.

For each of the relations Q and R, determine whether or not that relation is

(i) *reflexive*,

(ii) *symmetric*,

(iii) *transitive*,

(iv) *anti-symmetric*,

(v) an *equivalence relation*,

(vi) a *partial order*,

[Give appropriate short proofs and/or counterexamples to justify your answers.]

[14 marks]

[Recall that a relation R on a set X is an equivalence relation if and only if it is reflexive, symmetric and transitive. It is a partial order if and only if it is reflexive, anti-symmetric and transitive. A relation R on a set X is reflexive if and only if xRx for all $x \in X$; the relation is symmetric if and only if yRx for all $x, y \in X$ satisfying xRy; the relation is transitive if and only if xRz for all $x, y, z \in X$ satisfying xRy and yRz; the relation is anti-symmetric if and only if $x = y$ for all $x, y \in R$ satisfying xRy and yRx.]

2. (a) Let $f\colon A \to B$ be a function from a set A to a set B. What is meant by saying that such a function is *injective*, and that such a function is *surjective*?

[4 marks]

(b) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function from the set $\mathbb{R}$ of real numbers to itself defined such that $f(x) = 3 - x^2$ for all real numbers x. Determine whether or not this function is injective, and whether or not it is surjective, giving brief reasons for your answers.

[4 marks]

(c) What is a *monoid*? What is the *identity element* of a monoid? What is meant by saying that an element of a monoid is *invertible*?

[6 marks]

(d) Let $*$ denote the binary operation defined on the set $\mathbb{R}^2$ of ordered pairs of real numbers, where

$$(x_1, y_1) * (x_2, y_2) = (x_1 x_2 - 3 y_1 y_2, x_1 y_2 + y_1 x_2)$$

for all real numbers x_1, x_2, y_1 and y_2. Prove that $\mathbb{R}^2$, with the binary operation $*$, is a monoid. What is the identity element of this monoid?

[6 marks]

3. (a) Describe the formal language over the alphabet $\{0, 1\}$ generated by the context-free grammar whose non-terminals are $\langle S\rangle$ and $\langle A\rangle$, whose start symbol is $\langle S\rangle$ and whose productions are the following:

$$\begin{aligned}\langle S\rangle &\to 0\langle A\rangle\\ \langle A\rangle &\to 1\langle B\rangle\\ \langle B\rangle &\to 2\langle S\rangle\\ \langle B\rangle &\to 2\end{aligned}$$

Is this context-free grammar a regular grammar?

[6 marks]

(b) Give the specification of a finite state acceptor that accepts the language over the alphabet $\{0, 1\}$ consisting of all non-empty finite sequences of binary digits such as

$$0100, \quad 010001000, \quad 0, \quad 01001000$$

that do not contain two successive 1's.

In particular you should specify the set of states, the starting state, the finishing states, and the transition table that determines the transition function of the finite state acceptor.

[8 marks]

(c) Devise a regular context-free grammar to generate the language over the alphabet $\{0, 1\}$ described above in (b).

[6 marks]

4. In this question, all graphs are undirected graphs.

(a)

(i) What is meant by saying that a graph is *complete*?

(ii) What is meant by saying that a graph is *regular*?

(iii) What is meant by saying that a graph is *connected*?

(iv) What is meant by saying that a graph is a *tree*?

(v) Give the definition of an *isomorphism* between two undirected graphs.

[7 marks]

(b) Let G be the undirected graph whose vertices are a, b, c, d, e, f and g and whose edges are the following:

$$ab, \quad ac, \quad bd, \quad cd, \quad de, \quad ef. \quad eg.$$

(i) Is this graph complete?

(ii) Is this graph regular?

(iii) Is this graph connected?

(iv) Is this graph a tree?

[Give brief reasons for each of your answers.]

[8 marks]

(c) Let V denote the set of vertices of the graph G defined in (b). Determine all possible isomorphisms $\varphi\colon V \to V$ from the graph G to itself that satisfy $\varphi(b) = c$.

[5 marks]

5. (a) Any function y of a real variable x that solves the differential equation

$$\frac{d^3y}{dx^3} + 125y = 0$$

may be represented by a power series of the form

$$y = \sum_{n=0}^{+\infty} \frac{y_n}{n!} x^n,$$

where the coefficients $y_0, y_1, y_2, y_3, \ldots$ of this power series are real numbers. Find values of these coefficients y_n for $n = 0, 1, 2, 3, 4, \ldots$ that yield a solution to the above differential equation with $y_0 = 1$, $y_1 = -5$ and $y_2 = 25$.

[8 marks]

(b) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = xe^{5x}.$$

[12 marks]

6. (a) Let n be an integer satisfying $n > 1$, and let ω be a complex number. Suppose that $\omega^n = 1$ and $\omega \neq 1$. Prove that $\sum_{j=0}^{n-1} \omega^j = 0$, where

$$\sum_{j=0}^{n-1} \omega^j = 1 + \omega + \omega^2 + \cdots + \omega^{n-1}.$$

[6 marks]

(b) Let $(z_n : n \in \mathbb{Z})$ be the doubly-infinite 3-periodic sequence with $z_0 = 2$, $z_1 = 2$ and $z_2 = 1$. Find values of a_0, a_1 and a_2 such that

$$z_n = a_0 + a_1\omega^n + a_2\omega^{2n}$$

for all integers n, where $\omega = e^{2\pi i/3}$. (Note that $\omega = \frac{1}{2}(-1+\sqrt{3}\,i)$, $\omega^2 = e^{-2\pi i/3} = \frac{1}{2}(-1-\sqrt{3}\,i)$ and thus $\omega^3 = 1$ and $\omega + \omega^2 = -1$.)

[14 marks]

7. (a) Find the lengths of the vectors $(1, 2, 3)$ and $(1, 4, 3)$ and also the cosine of the angle between them.

[6 marks]

(b) Find the components of a non-zero vector that is orthogonal to the two vectors $(1, 2, 3)$ and $(1, 4, 3)$.

[6 marks]

(c) Let the quaternions q and r be defined as follows:

$$q = 1 - 4k, \quad r = i - j + 3k.$$

Calculate the quaternion products q^2, qr and rq. [Hamilton's basic formulae for quaternion multiplication state that

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.]$$

[8 marks]

8. (a) Find an integer x such that $x \equiv 2$ (mod. 5), $x \equiv 1$ (mod. 13) and $x \equiv 5$ (mod. 17).

[12 marks]

(b) Find the value of the unique integer x satisfying $0 \leq x < 13$ for which $3^{3002} \equiv x$ (mod. 13).

[8 marks]