Theorem: Let (V, E) be an undirected graph and let $u, v \in V$ be vertices s.t. $u \neq v$ and $\exists$ at least two distinct paths in (V, E) from u to v. Then the graph contains at least one simple circuit.

Proof: Let $a_0a_1a_2...a_m$ and $b_0b_1...b_n$ be the two distinct paths in the graph between u and v, **i.e.** $a_0 = b_0 = u$ and $a_m = b_m = v$. WLOG let $m \leq n$. Since the paths are distinct $\exists\, i$ with $0 \leq i \leq m$ s.t. $a_i \neq b_i$. Choose the smallest i for which $a_i \neq b_i$, **i.e.** $a_0 = b_0, a_1 = b_1, ..., a_{i-1} = b_{i-1}$, but $a_i \neq b_i$. We have thus eliminated the redundancies at the beginning of the paths. We now need to eliminate redundancies at the other end of the paths. We know $a_m = b_n$ so $a_j \in \{b_k \mid i-1 < k \leq n\}$ is certainly satisfied for $j = m$, but we want to choose the <u>smallest</u> index for which this condition is satisfied. Let this index be $p \Rightarrow a_p \in \{b_k \mid i-1 < k \leq n\}$, **i.e.** $a_p = b_s$ for some s s.t. $i-1 < s \leq n$. Since p is the smallest index satisfying $a_p \in \{b_k \mid i-1 < k \leq n\}$, $a_i, a_{i+1}, ..., a_{p-1} \notin \{b_k \mid i-1 < k \leq n\} \Rightarrow \underbrace{a_{i-1}a_i...a_p}_{\text{indices running in increasing order}} \underbrace{b_{s-1}...b_i}_{\text{indices running in decreasing order}} a_{i-1}$ is a simple circuit in (V, E) (recall $a_p = b_s$ and $a_{i-1} = b_{i-1}$) $\Rightarrow (V, E)$ has at least one simple circuit.

qed

9.10 Eulerian trails and circuits

Task: Look at trails and circuits that traverse every edge of a graph. Derive criteria when such trails and circuits exist.

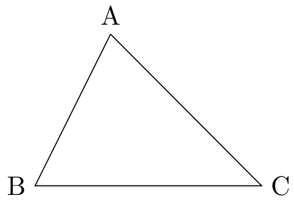
Definition: An <u>Eulerian trail</u> in a graph is a trail that traverses every edge of that graph. In other words, an Eulerian trail is a walk that traverses every edge of the graph exactly once.
Trail $\Rightarrow$ an edge is traversed at most once.
Eulerian $\Rightarrow$ every edge is traversed.

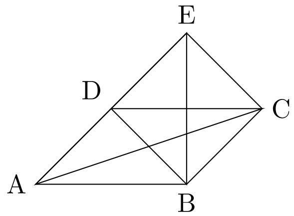
Definition: An <u>Eulerian circuit</u> is a graph is a circuit that traverses every edge of the graph.

Origin of the terminology: Eulerian comes from the Swiss mathematician Leonhard Euler (1707-1783) who solved the problem of the seven bridges of Königsberg/ Kaliningrad (then Prussia, now Russia) over the river Pregel in 1736. His negative solution is considered the beginning of graph theory as a subfield of mathematics. We will rederive Euler's results shortly. Google to see the configuration of the bridges on the river Pregel.

Examples:

1. $ABCA$ is an Eulerian circuit. The triangle is K_3.
2. Consider K_5, the complete graph with 5 vertices.
   $EABECDBCADE$ is an Eulerian circuit.





In both cases, the degree of the vertices is even for all vertices. We'll see this property is important and derive other necessary and sufficient conditions for the existence of Eulerian trails and circuits.

Theorem: Let (V, E) be a graph, and let $v_0 v_1 ... v_m$ be a trail in (V, E). Let $v \in V$ be a vertex, then the number of edges of the trail incident to v is even except when the trail is not closed and the trail starts or finishes at v, in which case the number of edges of the trail incident to the vertex v is odd.

Proof: Note that 0 is an even integer as $0 = 2 \times 0$.