9.11 Hamiltonian Paths and Circuits

Task: Look at paths and circuits that pass through every vertex of a graph.

Definition: A Hamiltonian path in a graph is a path that passed exactly once through every vertex of a graph.
Path $\Rightarrow$ we pass through a vertex at most once (no repeated vertices)
Hamiltonian $\Rightarrow$ we pass through every vertex.

Definition: A Hamiltonian circuit in a graph is a simple circuit that passes through every vertex of the graph.

Origin of the Terminology: Named after William Roman Hamilton (1805-1865) who showed in 1856 that such a circuit exists in the graph consisting of the vertices and edges of a dodecahedron (see page 88 in David Wilkins' notes for the picture of a Hamiltonian circuit on a dodecahedron). Hamilton developed a game called Hamilton's puzzle or the icosian game in 1857 whose object was to find Hamiltonian circuits in the dodecahedron (many solutions exist). This game was marketed in Europe as a pegboard with holes for each vertex of the dodecaherdron.
NB: The dodecahedron is a Platonic solid, and it turns out every Platonic solid has a Hamiltonian circuit. Recall that the Platonic solids are the tetrahedron (4 faces), the cube (6 faces), the octahedron (8 faces), the dodecahedron (12 faces), and the icosahedron (20 faces). Each of these is a regular graph.

Theorem: Every complete graph K_n for $n \geq 3$ has a Hamiltonian circuit.

Proof: Let $V = \{v_1, v_2, v_3, ...v_n\}$ be the set of vertices of K_n, then $v_1v_2v_3...v_nv_1$ is a Hamiltonian circuit. All edges in this circuit are part of K_n because K_n is complete.

qed

9.12 Forests and Trees

Task: Use the notion of a circuit to define trees.

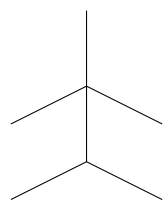
Definition: A graph is called acyclic if it contains no circuits (also known as cycles).

Definition: A <u>forest</u> is an acyclic graph.

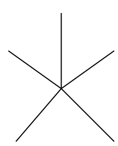
Definition: A <u>tree</u> is a connected forest.

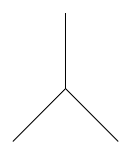
Examples:

1. Is a tree and a forest.



2. Is a forest with 2 connected components (**i.e.** it consists of 2 trees.)





Theorem: Every forest contains at least one isolated or pendant vertex.

Proof: Recall that when we studied circuits we proved a theorem that if (V, E) is a graph s.t. $\forall v \in V$ deg $v \geq 2$ (**i.e.** (V, E) has no isolated or pendant vertices), then (V, E) contains at least one simple circuit. The graph (V, E) is a forest, **i.e.** it contains no circuits $\Rightarrow \exists v \in V$ s.t. deg $v = 0$ or deg $v = 1$

qed

Theorem: A non-trivial tree contains at least one pendant vertex.

Proof: A non-trivial tree (V, E) must contain at least 2 vertices. Assume $\exists v \in V$ s.t. deg $v = 0$, **i.e.** v is isolated, then v forms a connected component by itself, but then (V, E) has at least 2 connected components as $\#(V) \geq 2 \Rightarrow\Leftarrow$ to the fact that a tree is by definition connected. Therefore, $\forall v \in V$, deg $v \geq 1$, but by the previous theorem $\exists v \in V$ s.t. $0 \leq$ deg $v \leq 1 \Rightarrow \exists v \in V$ s.t. deg $v = 1$ (since a tree is a forest).

qed

Theorem: Let (V, E) be a tree, then $\#(E) = \#(V) - 1$, where $\#(E)$ is the number of edges of the tree and $\#(V)$ is the number of vertices.

Proof: Use induction on $\#(V)$.

Base Case: $\#(V) = 1$. The graph is trivial $\Rightarrow \#(E) = 0$, so $0 = 1 - 1$ as needed.

Inductive Step: Suppose that every tree with m vertices ($\#(V) = m$) has $m - 1 = \#(v) - 1 = \#(E)$ edges. We seek to prove that if (V, E) is a tree with $m + 1$ vertices, then it has m edges.
By the previous theorem, (V, E) has one pendent vertex. Let that vertex be v. Since deg $v = 1$, then there is only one edge incident to v. Let vw be that edge. w is then the only vertex of (V, E) adjacent to v. We wish to reduce to the inductive hypothesis, the most natural way is to delete v from V and vw from E. Let $V' = V \backslash \{v\}$ and $E' = E \backslash \{vw\}$. (V', E') is a subgraph of (V, E) such that $\#(V') = \#(V) - 1$ and $\#(E') = \#(E) - 1$. To use the inductive hypothesis, we must show (V', E') is a tree, **i.e.** (V', E') is connected and $(V'E')$ contains no circuits. $\forall v_1, v_2 \in V'$, since (V, E) is a tree hence connected, $\exists$ path from v_1 to v_2 in (V, E). This path cannot pass through v because deg $v = 1 \Rightarrow$ it would have to pass through w twice contradicting the fact that it is a path (all vertices are distinct) $\Rightarrow$ this path is in $(V', E') \Rightarrow (V'E')$ is connected.
(V', E') is a subgraph of (V, E), which is a tree, hence does not contain any circuits, so (V', E') contains no circuits.
(V', E') is thus a tree, $\Rightarrow$ by the inductive hypothesis, $\#(V') = \#(V) - 1 = \#(E') - 1 = \#(E) - 1 - 1 = \#(E) - 2 \Rightarrow \#(V) - 1 = \#(E) - 2 \Leftrightarrow \#(V) = \#(E) - 1$ as needed.

qed

qed

Theorem: Let (V, E) be a tree, then $\#(E) = \#(V) - 1$, where $\#(E)$ is the number of edges of the tree and $\#(V)$ is the number of vertices.

Proof: Use strong induction of $\#(V)$.

Base Case: $\#(V) = 1$. The graph is trivial $\Rightarrow \#(E) = 0$, so $0 = 1 - 1$ as needed.

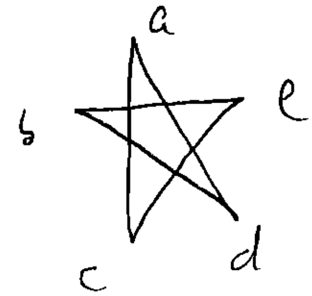
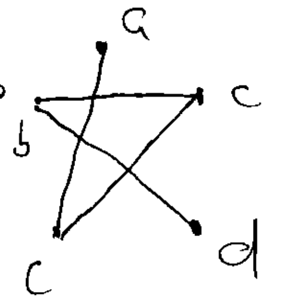
Inductive Step: Suppose that every tree with m vertices $(\#(V) = m)$ has $m - 1 = \#(v) - 1 = \#(E)$ edges. We seek to prove that if (V, E) is a tree with $m + 1$ vertices, then it has m edges.
By the previous theorem, (V, E) has one pendent vertex. Let that vertex be v. Since $\deg v = 1$, then there is only one edge incident to v. Let vw be that edge. w is then the only vertex of (V, E) adjacent to v. We wish to reduce to the inductive hypothesis, the most natural way is to delete v from V and vw from E. Let $V' = V \backslash \{v\}$ and $E' = E \backslash \{vw\}$. (V', E') is a subgraph of (V, E) such that $\#(V') = \#(V) - 1$ and $\#(E') = \#(E) - 1$. To use the inductive hypothesis, we must show (V', E') is a tree, **i.e.** (V', E') is connected and $(V'E')$ contains no circuits. $\forall v_1, v_2 \in V'$, since (V, E) is a tree hence connected, $\exists$ path from v_1 to v_2 in (V, E). This path cannot pass through v because $\deg v = 1 \Rightarrow$ it would have to pass through w twice contradicting the fact that it is a path (all vertices are distinct) $\Rightarrow$ this path is in $(V', E') \Rightarrow (V'E')$ is connected.
(V', E') is a subgraph of (V, E), which is a tree, hence does not contain any circuits, so (V', E') contains no circuits.
(V', E') is thus a tree, $\Rightarrow$ by the inductive hypothesis, $\#(V') = \#(V) - 1 = \#(E') - 1 = \#(E) - 1 - 1 = \#(E) - 2 \Rightarrow \#(V) - 1 = \#(E) - 2 \Leftrightarrow \#(V) = \#(E) - 1$ as needed.

qed

Spanning Trees

Task For any graph, construct a subgraph containing all the vertices of the original graph such that this subgraph is a tree.

Def A spanning tree in a graph (V, E) is a subgraph of the graph (V, E), which is a tree and includes every vertex in V.

Example  the pentagram has 

as a spanning tree (we delete the edge ad from the pentagram so that there is no circuit).

Remark A graph (V, E) may have more than one spanning tree, i.e. spanning trees are not unique.

Theorem Every connected graph contains a spanning tree.

Proof Let (V, E) be a connected graph. Let $\mathcal{C}$ be the collection of all connected subgraphs (V', E') of the graph (V, E) with $V' = V$ (i.e. containing all vertices of the original graph). The original graph $(V, E) \in \mathcal{C}$, so $\mathcal{C}$ is not empty. Choose (V, E') in $\mathcal{C}$ such that the number of edges $\#(E')$ is minimal, i.e. (V, E') is such that $\forall (V, E'') \in \mathcal{C}$, $\#(E') \leq \#(E'')$.

Claim (V, E') is the required spanning tree.

Proof of claim: (V, E') is connected and has the same vertices as (V, E) since it belongs to $\mathcal{C}$. We just need to show

that (V, E') is a tree, i.e. that it contains no circuits. (42)
We prove so indirectly, i.e. by contradiction. Assume (V, E') contains a circuit. Let vw be one of the edges traversed by a circuit in (V, E'). Let $E'' = E' - \{vw\}$ (we take out that edge). There still exists a walk from vertex v to vertex w via the remaining edges of the circuit. Note that since (V, E') is connected there exists a walk from every vertex in V to v via edges in E' and therefore to either v or w via edges in E''. Since there exists a walk from v to w via edges in E'', every vertex in V is connected to v via a walk whose edges belong to E'' $\Rightarrow$ (V, E'') is connected $\Rightarrow$ $(V, E'') \in \mathcal{C}$, but $\#(E'') = \#(E') - 1$ $\Rightarrow\Leftarrow$ as (V, E') was selected to be the graph in $\mathcal{C}$ w/ the least number of edges
$\Rightarrow$ (V, E') cannot contain a circuit $\Rightarrow$ (V, E') is the required spanning tree.

q.e.d.)