Corollary: A complete bipartite graph $k_{p,q}$ is regular $\Leftrightarrow p = q$

Proof: Recall that $V = V_1 \cup V_2$ $V_1 \cap V_2 = \emptyset$ for a bipartite graph, where $\#(V_1) = p$ and $\#(V_2) = q$.
"$\Leftarrow$" If $p = q, \forall v \in V_1$ satisfies that deg $v = p = q$ and $\forall v \in V_2$ satisfies that deg $v = p = q$ since the graph is complete $\Rightarrow k_{p,q}$ is p-regular.
"$\Rightarrow$" $k_{p,q}$ is regular $\Rightarrow \forall v \in V_1$ and $\forall v' \in V_2$, deg $v =$ deg v', but $k_{p,q}$ is complete $\Rightarrow v$ is adjacent to all vertices in V_2, **i.e.** deg $v = \#(V_2)$ and v' is adjacent to all vertices in V_1, **i.e.** deg $v' = \#(V_1)$. Therefore, $\#(V_1) = \#(V_2)$.

qed

9.6 Walks, trails and paths

Task: Make rigorous the notion of traversing parts of a graph in order to understand its structure better.

Definition: Let (V, E) be a graph. A walk $v_0v_1v_2...v_n$ of length n in the graph from vertex a to vertex b is determined by a finite sequence $v_0, v_1, v_1, ..., v_n$ of vertices of the graph s.t. $v_0 = a, v_n = b$ and $v_{i-1}v_i$ is an edge of the graph for $i = 1, 2, ..., n$.

Definition: A walk $v_0v_1v_2...v_n$ in a graph is said to traverse the edges $v_{i-1}v_i$ and to pass through the vertices $v_0, v_1, ..., v_n$. Length of walk = # of edges traversed $\Rightarrow$ the smallest possible number is zero edges. As a result, we have the following definition:

Definition: A walk that consists of a single vertex $v \in V$ and has length zero is called trivial.

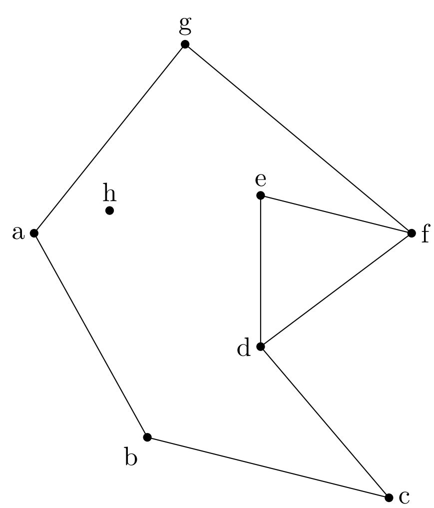
Definition: Let (V, E) be a graph. A trail $v_0v_1v_2...v_n$ of length n in the graph from some vertex a to some vertex b is a walk of length n from a to b with the property that edges $v_{i-1}v_i$ are distinct for $i = 1, 2, ..., n$. In other words, a trail is a walk in the graph, which traverses edges of the graph at most once.

Definition: Let (V, E) be a graph. A path $v_0v_1v_2...v_n$ of length n in the graph from some vertex a to some vertex b is a walk of length n from a to b with the property that vertices v_0, v_1, ..., v_n are distinct. In other words, a path is a walk in the graph, which passes through the vertices of the graph at most once.

Definition: A walk, trail or path is a graph is called trivial if it is a walk of length zero consisting of a single vertex $v \in V$; otherwise, the walk, trail, or path is called non-trivial.

Example:

1. h is a trivial walk/trail/path
2. $defd$ is a trail, but not a path because we pass through the vertex d twice.
3. def is a path
4. $gfdefdc$ is a walk but not a trail or a path

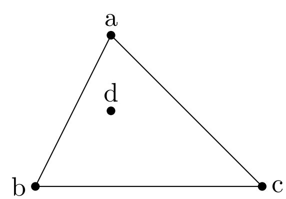


9.7 Connected Graphs

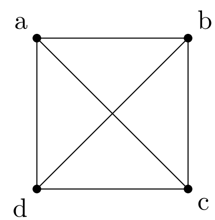
Task: Use the ideas above related to traversing parts of a graph in order to define a particularly important category of graphs.

Definition: An undirected graph (V, E) is called connected if $\forall u, v \in V$ vertices, $\exists$ path in the graph from u to v.

Examples: 1. Is not connected as d is not connected to any other vertex.



2. Is connected. $\exists$ path between any two of the vertices.



Theorem: Let (V, E) be a undirected graph, and let $u, v \in V$. $\exists$ path between u and v in the graph $\Leftrightarrow$ $\exists$ walk in the graph between u and v.

Proof: "$\Rightarrow$" trivial: A path is a walk.

"$\Leftarrow$" $\exists$ walk between u and v. Choose the walk of least length between u and v, (**i.e.** $\nexists$ a walk of lower length than this one) and prove it is a path. Let this walk be $a_0 a_1 ... a_n$ with $a_0 = u$ and $a_n = v$. Assume $\exists j, k$ with $a \leq j, k \leq n$ s.t. $j < k$ and $a_j = a_k$, but then $a_0 a_1 ... a_j a_{k+1} ... a_n$ would be a walk from u to v of <u>strictly smaller</u> length than $a_0 a_1 ... a_n$. $\Rightarrow\Leftarrow$ as we chose $a_0 a_1 ... a_n$ to be of minimal length $\Rightarrow a_j \neq a_k \forall j, k$ s.t. $0 \leq j, k \leq n \Rightarrow a_0 a_1 ... a_n$ is a path between u and v.

qed

Corollary: An undirected graph (V, E) is connected $\Leftrightarrow \forall u, v \in V$ $\exists$ walk in the graph between u and v.