UNIVERSITY OF DUBLIN

X-MA2C01-1

TRINITY COLLEGE

FACULTY OF ENGINEERING, MATHEMATICS
AND SCIENCE

SCHOOL OF COMPUTER SCIENCE AND STATISTICS

SF Integrated Computer Science Programme
SF CSLL

Trinity Term 2013

DISCRETE MATHEMATICS I & II

Wednesday, May 1 RDS 9.30–12.30

Dr. D. R. Wilkins

Instructions to Candidates:

Credit for module MA2C01 (Michaelmas Term) will be given for the best 3 questions answered from Section A.

Credit for module MA2C02 (Hilary Term) will be given for the best 3 questions answered from Section B.

Each question is worth 20 marks.

You may not start this examination until you are instructed to do so by the Invigilator.

Materials permitted for this examination:

Log tables are available from the invigilators, if required.

Students may avail of the HANDBOOK OF MATHEMATICS of Computer Science

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

SECTION A (Module MA2C01)

Credit will be given for the best 3 questions answered in this section.

1. (a) Let A, B and C be sets. Prove that

$$(A \setminus (B \cup C)) \cup (B \setminus C) = (A \cup B) \setminus C.$$

[Venn Diagrams, by themselves without an accompanying logical argument, do not provide sufficient justification to constitute a proof of the result.]

[8 marks]

(b) Let Q denote the relation on the set $\mathbb{Z}$ of integers, where integers x and y satisfy xQy if and only if both $x \leq y+3$ and $y \leq x+3$. Determine whether the relation Q is

(i) *reflexive*,

(ii) *symmetric*,

(iii) *transitive*,

(iv) *anti-symmetric*,

(v) an *equivalence relation*,

(vi) a *partial order*,

[Give appropriate short proofs and/or counterexamples to justify your answers.]

[12 marks]

[Recall that a relation R on a set X is an equivalence relation if and only if it is reflexive, symmetric and transitive. It is a partial order if and only if it is reflexive, anti-symmetric and transitive. A relation R on a set X is reflexive if and only if xRx for all $x \in X$; the relation is symmetric if and only if yRx for all $x, y \in X$ satisfying xRy; the relation is transitive if and only if xRz for all $x, y, z \in X$ satisfying xRy and yRz; the relation is anti-symmetric if and only if $x = y$ for all $x, y \in R$ satisfying xRy and yRx.]

2. (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function from the set $\mathbb{R}$ of real numbers to itself defined such that

$$f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 1; \\ x & \text{if } x \leq 1. \end{cases}$$

Determine whether or not this function is injective, whether or not it is surjective, and whether or not it is invertible, giving brief reasons for your answers.

[8 marks]

[Recall that a function $f\colon X \to Y$ between sets X and Y is *injective* if it maps distinct elements of X to distinct elements of Y, and is *surjective* if every element of Y belongs to the range of the function. A function $g\colon Y \to X$ is the inverse of the function f if and only if $g \circ f$ and $f \circ g$ are the identity functions of the sets X and Y respectively. The function f is *invertible* if such an inverse function exists.]

(b) Let X denote the set of odd integers, and let $\otimes$ be the binary operation on X defined such that $x \otimes y = \frac{1}{2}(xy + 3x + 3y + 3)$ for all odd integers x and y. Prove that $(X, \otimes)$ is a monoid. What is the identity element of this monoid? Determine the inverse of each invertible element of $(X, \otimes)$. Is $(X, \otimes)$ a group?

[12 marks]

[Recall that a monoid $(X, *)$ is a set on which is defined an associative binary operation $*$. An element e of X is said to be an *identity element* if $e * x = x * e = x$. An element y of X is said to be the inverse of some element x of X if $x * y = y * x = e$ for all $x \in X$. An element of a monoid is said to be *invertible* if it has an inverse. A group is a monoid in which every element is invertible.]

3. (a) Describe the formal language over the alphabet $\{0, 1\}$ generated by the context-free grammar whose non-terminals are $\langle S \rangle$ and $\langle A \rangle$, whose start symbol is $\langle S \rangle$ and whose productions are the following:

$$\begin{aligned} \langle S \rangle &\rightarrow 0\langle A \rangle \\ \langle A \rangle &\rightarrow 101\langle S \rangle \\ \langle S \rangle &\rightarrow \varepsilon \end{aligned}$$

(where ε denotes the empty word). Is this context-free grammar a regular grammar?

[6 marks]

(b) Give the specification of a finite state acceptor that accepts the language over the alphabet $\{a, c, g, t\}$ consisting of all strings made up of letters from this alphabet (including the empty string) that do not contain the substring 'cat'.

In particular you should specify the set of states, the starting state, the finishing states, and the transition table that determines the transition function of the finite state acceptor.

[8 marks]

(c) Devise a regular context-free grammar to generate the language over the alphabet $\{a, c, g, t\}$ described above in (b).

[6 marks]

4. In this question, all graphs are undirected graphs.

(a)

(i) What is meant by saying that a graph is *regular*?

(ii) What is an *Eulerian circuit* in a graph?

(ii) What is a *Hamiltonian circuit* in a graph?

(iv) What is meant by saying that a graph is a *tree*?

(v) Give the definition of an *isomorphism* between two undirected graphs.

[7 marks]

(b) Let G be the undirected graph whose vertices are a, b, c, d and e and whose edges are the following:

$$a\,b, \quad a\,e, \quad b\,c, \quad b\,e, \quad c\,d, \quad c\,e, \quad d\,e.$$

(i) Is this graph regular?

(ii) Does this graph have an Eulerian circuit?

(iii) Does this graph have a Hamiltonian circuit?

(iv) Is this graph a tree?

[Give brief reasons for each of your answers.]

[8 marks]

(c) Let V denote the set of vertices of the graph G defined in (b). Determine all possible isomorphisms $\varphi\colon V \to V$ from the graph G to itself that satisfy $\varphi(b) = c$.

[5 marks]

SECTION B (Module MA2C02)

Credit will be given for the best 3 questions answered in this section.

5. (a) Any function y of a real variable x that solves the differential equation

$$\frac{d^4y}{dx^4} - 16y = 0$$

may be represented by a power series of the form

$$y = \sum_{n=0}^{+\infty} \frac{y_n}{n!} x^n,$$

where the coefficients $y_0, y_1, y_2, y_3, \ldots$ of this power series are real numbers.

Find values of these coefficients y_n for $n = 0, 1, 2, 3, 4, \ldots$ that yield a solution to the above differential equation with $y_0 = 1$, $y_1 = 0$, $y_2 = -4$ and $y_3 = 0$.

[8 marks]

(b) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = \cos 5x.$$

[12 marks]

6. (a) Let $(z_n : n \in \mathbb{Z})$ be a doubly-infinite 3-periodic sequence of complex numbers (which thus has the property that $z_{n+3} = z_n$ for all integers n). Also let $\omega = \exp(2\pi i/3) = \frac{1}{2}(-1 + \sqrt{3}\, i)$, where $i = \sqrt{-1}$. Prove that

$$z_j = c_0 + c_1\omega^j + c_2\omega^{2j}$$

for all integers j, where

$$c_0 = \frac{1}{3}(z_0 + z_1 + z_2), \quad c_1 = \frac{1}{3}(z_0 + z_1\omega^2 + z_2\omega), \quad c_2 = \frac{1}{3}(z_0 + z_1\omega + z_2\omega^2).$$

[Note that $1+\omega+\omega^2 = 0$ and $\omega^3 = 1$. You should give a direct proof that utilizes these properties of the complex number ω, and does not merely claim the result of any more general theorem stated in the lecture notes.]

[12 marks]

(b) Let $(z_n : n \in \mathbb{Z})$ be the doubly-infinite 3-periodic sequence with $z_0 = 1$, $z_1 = 2$ and $z_2 = -2$. Find values of c_0, c_1 and c_2 such that

$$z_n = c_0 + c_1\omega^n + c_2\omega^{2n}$$

for all integers n, where $\omega = e^{2\pi i/3}$.

[8 marks]

7. (a) Find the lengths of the vectors $(4, 1, 2)$ and $(2, 2, 2)$ and also the cosine of the angle between them.

[6 marks]

(b) Find the components of a non-zero vector that is orthogonal to the two vectors $(3, 2, 5)$ and $(2, 1, 7)$, and determine the equation of the plane in $\mathbb{R}^3$ that contains the points $(3, 4, 6)$, $(6, 6, 11)$ and $(5, 5, 13)$.

[8 marks]

(c) Let the quaternions q and r be defined as follows:

$$q = i - 2k, \quad r = 1 - j + 5k.$$

Calculate the quaternion products qr and rq. [Hamilton's basic formulae for quaternion multiplication state that

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.]$$

[6 marks]

8. (a) Find an integer x such that $x \equiv 5$ (mod. 7), $x \equiv 2$ (mod. 17) and $x \equiv 3$ (mod. 19).

[12 marks]

(b) Find the value of the unique integer x satisfying $0 \leq x < 17$ for which

$$4^{1024000000002} \equiv x \pmod{17}.$$

[8 marks]