UNIVERSITY OF DUBLIN

XMA2C011

TRINITY COLLEGE

FACULTY OF ENGINEERING, MATHEMATICS
AND SCIENCE

SCHOOL OF COMPUTER SCIENCE AND STATISTICS
DEPARTMENT OF COMPUTER SCIENCE

SF Computer Science (B.A.)
SF CSLL

Trinity Term 2010

MODULES MA2C01 AND MA2C02

Wednesday, April 28 RDS—Main 14.00–17.00

Dr. D. R. Wilkins

Credit will be given for the best **SIX** questions answered.

Log tables are available from the invigilators, if required.

Students may avail of the HANDBOOK OF MATHEMATICS of Computer Science

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

1. (a) Let A, B and C be sets. Prove that

$$A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C).$$

[8 marks]

(b) Let $\natural$ be the relation on the set $\mathbb{Z}$ of integers, where integers x and y satisfy $x \natural y$ if and only if $y = x + k^2$ for some integer k. Determine whether or not the relation $\natural$ is an equivalence relation, and whether or not this relation is a partial order. [Give appropriate short proofs and/or counterexamples to justify your answers.]

[12 marks]

[Recall that a relation R on a set X is an equivalence relation if and only if it is reflexive, symmetric and transitive. It is a partial order if and only if it is reflexive, anti-symmetric and transitive. A relation R on a set X is reflexive if and only if xRx for all $x \in X$; the relation is symmetric if and only if yRx for all $x, y \in X$ satisfying xRy; the relation is transitive if and only if xRz for all $x, y, z \in X$ satisfying xRy and yRz; the relation is anti-symmetric if and only if $x = y$ for all $x, y \in R$ satisfying xRy and yRx.]

2. (a) Let $f \colon A \to B$ be a function from a set A to a set B. What is meant by saying that such a function is *injective*, and that such a function is *surjective*.

[4 marks]

(b) For each integer k, let $f_k \colon \mathbb{Z} \to \mathbb{Z}$ be the function from the set $\mathbb{Z}$ of integers to itself defined such that $f_k(x) = k(x - k)$ for all $x \in \mathbb{Z}$. In each of the three cases where $k = 0$, $k = -1$ and $k = 2$, determine whether or not the corresponding function $f_k \colon \mathbb{Z} \to \mathbb{Z}$ is injective, and whether or not this function is surjective, giving brief reasons for your answers.

[6 marks]

[Question 2 continues overleaf.]

(c) What is a *monoid*? What is the *identity element* of a monoid?

[4 marks]

(d) Let $\otimes$ denote the binary operation on the set $\mathbb{Z}$ of integers defined by $x \otimes y = 5xy - 9x - 9y + 18$. Prove that $\mathbb{Z}$, with the binary operation $\otimes$, is a monoid.

[6 marks]

3. (a) Describe the formal language over the alphabet $\{0, 1\}$ generated by the context-free grammar whose only non-terminal is $\langle S \rangle$, whose start symbol is $\langle S \rangle$ and whose productions are the following:

$$\begin{aligned} \langle S \rangle &\rightarrow \langle S \rangle 0 \\ \langle S \rangle &\rightarrow 1 \end{aligned}$$

Is this context-free grammar a regular grammar?

[6 marks]

(b) Give the specification of a finite state acceptor that accepts the language over the alphabet $\{0, 1\}$ whose words consist of m zeros followed by n 1's, where m and n are any positive integers. (In particular you should specify the set of states, the starting state, the finishing states, and the transition table that determines the transition function of the finite state acceptor.)

[8 marks]

(c) Devise a regular context-free grammar to generate the language over the alphabet $\{0, 1\}$ described above in (b).

[6 marks]

4. In this question, all graphs are undirected graphs.

(a)

(i) What is meant by saying that a graph is *complete*?

(ii) What is meant by saying that a graph is *regular*?

(iii) What is meant by saying that a graph is *connected*?

(iv) What is an *Eulerian circuit* in a graph?

(v) What is a *Hamiltonian circuit* in a graph?

[7 marks]

(b) Consider the graph G with vertices a, b, c, d and e, and edges ab, ad, ae, bc, be, cd, ce and de.

(i) Is this graph complete?

(ii) Is this graph regular?

(iii) Is this graph connected?

(iv) Does this graph have an Eulerian circuit?

(v) Does this graph have a Hamiltonian circuit?

[Give brief reasons for each of your answers.]

[13 marks]

5. (a) Any function y of a real variable x that solves the differential equation

$$\frac{d^3y}{dx^3} + 8y = 0$$

may be represented by a power series of the form

$$y = \sum_{n=0}^{+\infty} \frac{y_n}{n!} x^n,$$

where the coefficients $y_0, y_1, y_2, y_3, \ldots$ of this power series are real numbers.

Find values of these coefficients y_n for $n = 1, 2, 3, 4, \ldots$ that yield a solution to the above differential equation with $y_0 = 1$ and $y_1 = y_2 = 0$.

[8 marks]

(b) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \cos 4x.$$

[12 marks]

6. (a) Let n be an integer satisfying $n > 1$, and let ω be a complex number. Suppose that $\omega^n = 1$ and $\omega \neq 1$. Prove that $\sum_{j=0}^{n-1} \omega^j = 0$, where

$$\sum_{j=0}^{n-1} \omega^j = 1 + \omega + \omega^2 + \cdots + \omega^{n-1}.$$

[6 marks]

(b) Let $(z_n : n \in \mathbb{Z})$ be the doubly-infinite 3-periodic sequence with $z_0 = 2$, $z_1 = 1$ and $z_2 = 5$. Find values of a_0, a_1 and a_2 such that

$$z_n = a_0 + a_1\omega^n + a_2\omega^{2n}$$

for all integers n, where $\omega = e^{2\pi i/3}$. (Note that $\omega = \frac{1}{2}(-1+\sqrt{3}\,i)$, $\omega^2 = e^{-2\pi i/3} = \frac{1}{2}(-1-\sqrt{3}\,i)$ and thus $\omega^3 = 1$ and $\omega + \omega^2 = -1$.)

[14 marks]

7. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the periodic function with period 2π whose values on the interval $[-\pi, \pi]$ are given by the following formulae:

$$f(x) = \begin{cases} 0 & \text{if } x = 0,\quad \frac{1}{2}\pi < x \leq \pi \text{ or } -\pi \leq x < -\frac{1}{2}\pi; \\ 1 & \text{if } 0 < x < \frac{1}{2}\pi; \\ -1 & \text{if } -\frac{1}{2}\pi < x < \pi. \\ \dfrac{1}{2} & \text{if } x = \frac{1}{2}\pi; \\ -\dfrac{1}{2} & \text{if } x = -\frac{1}{2}\pi. \end{cases}$$

This function may be represented as a Fourier series of the form

$$f(x) = \sum_{n=1}^{+\infty} b_n \sin n\pi x.$$

Find the values of the coefficients b_n for $n = 1, 2, 3, 4, \ldots$.

[20 marks]

8. (a) Calculate the components of a non-zero vector (a, b, c) in $\mathbb{R}^3$ that is orthogonal to the vectors $(1, 0, 2)$ and $(3, 4, 1)$.

[7 marks]

(b) Let the quaternions q and r be defined as follows:

$$q = 1 - 2j, \quad r = i - 3k.$$

Calculate the quaternion products qr and rq. [Hamilton's basic formulae for quaternion multiplication state that

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.]$$

[6 marks]

(c) Find the greatest common divisor d of the integers 858 and 561, and find integers u and v such that $d = 858u + 561v$.

[7 marks]