UNIVERSITY OF DUBLIN

XMA2BA11

TRINITY COLLEGE

FACULTY OF ENGINEERING, MATHEMATICS
AND SCIENCE

SCHOOL OF COMPUTER SCIENCE AND STATISTICS

SF Computer Science (B.A.)
SF CSLL

Trinity Term 2008

MATHEMATICS 2BA1

Tuesday, May 20 RDS 9.30 – 12.30

Dr. D. R. Wilkins and Dr. V. Dotsenko

Credit will be given for the best 5 questions answered.

Log tables are available from the invigilators, if required.

Students may avail of the HANDBOOK OF MATHEMATICS of Computer Science

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

1. (a) (6 marks) What is meant by saying that a relation R on a set A is *reflexive*, *symmetric*, *anti-symmetric*, *transitive*, an *equivalence relation*, or a *partial order*?

   (b) (6 marks) Let Q denote the relation on the set $\mathbb{R}$ of real numbers, where real numbers x and y satisfy xQy if and only if there exists some non-negative integer k such that $y = \frac{1}{2}(3^k(2x - 1) + 1)$. Determine whether or not the relation Q is reflexive, symmetric, anti-symmetric or transitive, whether or not it is an equivalence relation, and whether or not it is a partial order. [Justify your answers.]

   (c) (3 marks) Let $f\colon A \to B$ be a function from a set A to a set B. What is meant by saying that such a function is *injective*, that such a function is *surjective*, or that such a function is *bijective*?

   (d) (5 marks) Given any integer k, let $k\mathbb{Z}$ denote the set of integer multiples of k (i.e., the set of all integers that are divisible by k). Consider the functions $f\colon 3\mathbb{Z} \to 12\mathbb{Z}$ and $g\colon 2\mathbb{Z} \to 8\mathbb{Z}$, where $f(x) = 8x - 12$ for all $x \in 3\mathbb{Z}$, and $g(x) = 4x + 24$ for all $x \in 2\mathbb{Z}$. Determine which (if any) of the functions f and g are injective, which are surjective, and which are invertible? [Briefly justify your answers.]

2. In this question, all graphs are undirected graphs.

   (a) (6 marks) What is a *walk* in a graph? What is a *trail* in a graph? What is a *path* in a graph? What is a *circuit* in a graph? What is an *Eulerian circuit* in a graph?

   (b) (10 marks) Consider the graph whose vertices are a, b, c and d, and whose edges are ab, ac, ad, bc, bd and cd. There are 15 trails in this graph from the vertex a to the vertex b. List all of these trails from a to b, and, for each of these trails, state whether or not it is a path.

   (c) (4 marks) If the edge ab is removed from the graph specified in (b), does the resulting graph have an Eulerian circuit? If the edges ab and cd are removed from the graph specified in (b), does the resulting graph have an Eulerian circuit? [Briefly justify your answers.]

3. (a) (5 marks) What is a *monoid*? What is the *identity element* (or *neutral element*) of a monoid? What is meant by saying that a function $f\colon A \to B$ from a monoid $(A, *_A)$ to a monoid $(B, *_B)$ is a *homomorphism*?

(b) (6 marks) Let $\otimes$ be the binary operation on the set $\mathbb{R}^2$ of ordered pairs of real numbers defined such that

$$(a_1, b_1) \otimes (a_2, b_2) = (a_1 a_2 + 2 b_1 b_2, a_1 b_2 + a_2 b_1 + b_1 b_2)$$

for all ordered pairs (a_1, b_1) and (a_2, b_2) of real numbers. Prove that $(\mathbb{R}^2, \otimes)$ is a monoid. What is the identity element of this monoid?

(c) (3 marks) Let $(\mathbb{R}, \times)$ be the monoid consisting of the real numbers with the usual multiplication operation, and let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined such that $f(a, b) = a + 2b$ for all $(a, b) \in \mathbb{R}^2$. Prove that the function f is a homomorphism from the monoid $(\mathbb{R}^2, \otimes)$ to the monoid $(\mathbb{R}, \times)$.

(d) (6 marks) Let a and b be real numbers, and let $(a_n, b_n) \in \mathbb{R}^2$ be defined recursively for all positive integers n such that $(a_1, b_1) = (a, b)$, and $(a_n, b_n) = (a, b) \otimes (a_{n-1}, b_{n-1})$ for all integers n satisfying $n > 1$. Prove (by induction on n, or otherwise) that

$$(a_n, b_n) = \left(\tfrac{2}{3}(a - b) + \tfrac{1}{3}(a + 2b), -\tfrac{1}{3}(a - b) + \tfrac{1}{3}(a + 2b)\right)$$

for all positive integers n.

4. (a) (6 marks) Describe the formal language over the alphabet $\{0, 1\}$ generated by the context-free grammar whose non-terminals are $\langle S\rangle$ and $\langle A\rangle$, whose start symbol is $\langle S\rangle$ and whose productions are the following:

$$\begin{aligned} \langle S\rangle &\to 0\langle S\rangle \\ \langle S\rangle &\to 0\langle A\rangle \\ \langle A\rangle &\to 1\langle A\rangle \\ \langle A\rangle &\to 0 \end{aligned}$$

Is this context-free grammar a regular grammar?

(b) (7 marks) Devise a regular grammar to generate the language L over the alphabet $\{a, b, c\}$ consisting of all finite strings involving the letters a, b and c in which abc does not occur as a substring.

(c) (7 marks) Give the description of a finite state acceptor for the language L of (b), specifying the starting state, the finishing state or states, and the transition table for this finite state acceptor.

5. (a) (10 marks) For the ordinary differential equation

$$f''(x) + f'(x) = 1,$$

find its general solution, and list all solutions that satisfy $f(0) = 1$, $f(1) = 2$.

(b) (10 marks) Obtain the general solution of the ordinary differential equation

$$f''(x) - 4f'(x) + 4f(x) = e^{2x} - e^{-x}.$$

6. (a) (8 marks) Show that the Fourier series of any trigonometric polynomial

$$a_0 + (a_1 \cos x + b_1 \sin x) + \ldots + (a_n \cos(nx) + b_n \sin(nx))$$

coincides with this polynomial.

(b) (12 marks) Consider the following 4-periodic doubly-infinite sequence a of complex numbers

$$\ldots, 1, 0, 3-i, 4, 1, 0, 3-i, \ldots$$

(where $a_0 = 1$, $a_1 = 0$ etc.). Compute its discrete Fourier transform, its convolution with itself $a \star a$ and the discrete Fourier transform of $a \star a$.

7. (a) (6 marks) Compute $\mathbf{uv}$ and $\mathbf{vu}$, where $\mathbf{u}$ and $\mathbf{v}$ are quaternions $2-3k$ and $1+i+j$ respectively.

(b) (6 marks) Use the Euclidean algorithm to compute the greatest common divisor of 273 and 313.

(c) (8 marks) Find all integers n congruent to 5 modulo 273 and to 15 modulo 313. In other words, solve the system of congruences

$$\begin{cases} n \equiv 5 \pmod{273}, \\ n \equiv 15 \pmod{313}. \end{cases}$$