5.3 Functions Defined on Finite Sets

Task: Derive conclusions about a function given the number of elements of the domain and codomain, if finite; understand the pigeonhole principle.

Proposition: Let A, B be sets and let $f : A \to B$ be a function. Assume A is finite. Then f is injective $\Leftrightarrow$ $f(A)$ has the same number of elements as A.

Proof:

A is finite so we can write it as $A = \{a_1, a_2, ..., a_p\}$ for some p. Then $f(A) = \{f(a_1), f(a_2), ..., f(a_p)\} \subseteq B$. A priori, some $f(a_i)$ might be the same as some $f(a_j)$. However, f injective $\Leftrightarrow f(a_i) \neq f(a_j)$ whenever $i \neq j \Leftrightarrow f(A)$ has exactly p elements just like A.

qed

Corollary 1 Let A, B be finite sets such that $\#(A) = \#(B)$. Let $f : A \to B$ be a function. f is injective $\Leftrightarrow$ f is bijective.

Proof:

"$\Rightarrow$" Suppose $f : A \to B$ is injective. Since A is finite, by the previous proposition, $f(A)$ has the same number of elements as A, but $f(A) \subseteq B$ and B has the same number of elements as $A \Rightarrow \#(A) = \#(f(A)) = \#(B)$, which means $f(A) = B$, **i.e.** f is also surjective $\Rightarrow$ f is bijective.

"$\Leftarrow$" f is bijective $\Rightarrow$ f is injective.

qed

Corollary 2 (The Pigeonhole Principle) Let A, B be finite sets, and let $f : A \to B$ be a function. If $\#(B) < \#(A)$, $\exists\, a, a' \in A$ with $a \neq a'$ such that $f(a) = f(a')$.

Remark: The name pigeonhole principle is due to Paul Erdös and Richard Rado. Before it was known as the principle of the drawers of Dirichlet. It has a simple statement, but it's a very powerful result in both mathematics and computer science.

Proof: Since $f(A) \subseteq B$ and $\#(B) < \#(A)$, $f(A)$ cannot hve as many elements as A, so by the proposition, f cannot be injective, namely $\exists\, a, a' \in A$ with $a \neq a'$ (**i.e.** distinct elements) s.t. $f(a) = f(a')$.

qed

Examples:

1. You have 8 friends. At least two of them were born the same day of the week. #(days of the week) $= 7 < 8$.
2. A family of five gives each other presents for Christmas. There are 12 presents under the tree. We conclude at least one person got three presents or more.
3. In a list of 30 words in English, at least two will begin with the same letter. #(Letter in the English alphabet) $= 26 < 30$.

5.4 Behaviour of Functions on Infinite Sets

Let A be a set, and $f : A \to A$ be a function. If A is finite, then corollary 1 tells us f injective $\Leftrightarrow$ f bijective. What if A is not finite?

5.4.1 Hilbert's Hotel problem (jazzier name: Hilbert's paradox of the Grand Hotel)

A fully occupied hotel with infinitely many rooms can always accommodate an additional guest as follows: The person in Room 1 moves to Room 2. The person in Room 2 moves to Room 3 and so on, **i.e.** if the rooms are $x_1, x_2, x_3...$ define the function $f(x_1) = x_2, f(x_2) = x_3, ..., f(x_m) = x_{m+1}$.

Claim: As defined f is injective but not surjective (hence not bijective!). Let $H = \{x_1, x_2, ...\}$ be the hotel consisting of infinitely many rooms. $f : H \to H$ is given by $f(x_n) = f(x_{n+1})$. $f(H) = H \backslash \{x_1\}$. We can use this idea to prove:

Proposition: A set A is finite $\Leftrightarrow$ $\forall f : A \to A$ an injective function is also bijective.

Proof: "$\Rightarrow$" If the set A is finite, then it follows immediately from Corollary 1 that every injective function $f : A \to A$ is bijective.
"$\Leftarrow$" We prove the contrapositive. Suppose that the set A is infinite. We shall construct an injective function that is not bijective. Since A is infinite, there exists some infinite sequence $x_1, x_2, x_3, ...$ consisting of distinct elements of A, i.e. an element of A occurs at most once in this sequence. Then there exists a function $f : A \to A$ such that $f(x_n) = x_{n+1}$ for all integers $n \geq 1$ and $f(x) = x$ if x is an element of A that is not in the sequence $x_1, x_2, x_3, ...$ If x is not a member of the infinite sequence $x_1, x_2, x_3, ...$, then the only element of A that gets mapped to x is the element x itself; if $x = x_n$, where $n > 1$, then the only element of A that gets mapped to x is x_{n-1}. It follows that the function f is injective. It is not surjective, however, since no element of A gets mapped to x_1. This function f is thus an example of a function from the set A to itself, which is injective but not bijective.