UNIVERSITY OF DUBLIN

XMA2BA11

TRINITY COLLEGE

FACULTY OF ENGINEERING, MATHEMATICS
AND SCIENCE

SCHOOL OF COMPUTER SCIENCE AND STATISTICS
DEPARTMENT OF COMPUTER SCIENCE

SF Computer Science (B.A.)
SF CSLL

Trinity Term 2009

MODULE MA2BA1

Tuesday, May 19 RDS 9:30-12:30

Dr. D. R. Wilkins

Credit will be given for the best 5 questions answered.

Log tables are available from the invigilators, if required.

Students may avail of the HANDBOOK OF MATHEMATICS of Computer Science

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

1. (a) (6 marks) What is meant by saying that a relation R on a set A is *reflexive*, *symmetric*, *anti-symmetric*, *transitive*, an *equivalence relation*, or a *partial order*?

(b) (6 marks) Let Q denote the relation on the set $\mathbb{Z}$ of integers, where integers x and y satisfy xQy if and only if $x^2 + 5y^2$ is divisible by 6. Determine whether or not the relation Q is reflexive, symmetric, anti-symmetric or transitive, whether or not it is an equivalence relation, and whether or not it is a partial order. [Justify your answers.]

(c) (3 marks) Let $f\colon A \to B$ be a function from a set A to a set B. What is meant by saying that such a function is *injective*, that such a function is *surjective*, or that such a function is *bijective*?

(d) (5 marks) Let $X = [1,2] \cup [3,4]$, and let $f\colon X \to [0,1]$ be the function defined such that

$$f(x) = \begin{cases} \frac{2}{3}x - \frac{1}{3} & \text{when } 1 \leq x \leq 2; \\ \frac{8}{3} - \frac{2}{3}x & \text{when } 3 \leq x \leq 4. \end{cases}$$

(Here $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ for all real numbers a and b satisfying $a \leq b$.) Is the function f injective? Is the function f surjective? Does the function f have a well-defined inverse?

2. In this question, all graphs are undirected graphs.

   (a) (6 marks)

   (i) What is meant by saying that a graph is *complete*?

   (ii) What is meant by saying that a graph is *regular*?

   (iii) What is meant by saying that a graph is *connected*?

   (iv) What is meant by saying that a graph is a *tree*?

   (v) Give the definition of an *isomorphism* between graphs.

   (b) (8 marks) Consider the graph G with vertices a, b, c, d, e and f, and edges $a\,b$, $a\,c$, $a\,d$, $b\,c$, $b\,f$, $c\,d$, $c\,e$, $c\,f$, $d\,e$ and $e\,f$.

   (i) Is this graph complete?

   (ii) Is this graph regular?

   (iii) Is this graph connected?

   (iv) Is this graph a tree?

   (c) (6 marks) Give an example of an isomorphism from the graph G defined in (b) to itself that sends the vertex a to the vertex e. You should specify the image of each vertex under the isomorphism.

3. (a) (5 marks) What is a *monoid*? What is the *identity element* of a monoid? What is meant by saying that an element x of a monoid $(A, *)$ is invertible? What is a *group*?

   (b) (10 marks) Let A be the set $\mathbb{Z} \times \mathbb{Z}$ consisting of all ordered pairs (a, b), where a and b are integers. Let $*$ denote the binary operation on A defined by

$$(a, b) * (c, d) = (ac, ad - 2a + b)$$

   for all integers a, b, c and d. Prove that $(A, *)$ is a monoid. What is its identity element?

   (c) (5 marks) Let $(A, *)$ be the monoid defined in (b). Prove that an element (a, b) of A is invertible if and only if either $a = 1$ or $a = -1$. Also find the inverse of each invertible element (a, b) of A.

4. (a) (6 marks) Describe the formal language over the alphabet $\{0, 1, 2\}$ generated by the context-free grammar whose non-terminals are $\langle S\rangle$, $\langle A\rangle$ and $\langle B\rangle$, whose start symbol is $\langle S\rangle$ and whose productions are the following:

$$\begin{array}{lcl} \langle S\rangle & \rightarrow & 0\langle A\rangle \\ \langle A\rangle & \rightarrow & 1\langle B\rangle \\ \langle B\rangle & \rightarrow & 2\langle S\rangle \\ \langle B\rangle & \rightarrow & 2 \end{array}$$

Is this context-free grammar a regular grammar?

(b) (7 marks) Give the specification of a finite state acceptor for the language L over the alphabet $\{p, q, r, s\}$ consisting of all non-empty finite strings involving the letters p, q and r and s in which the letter p is always followed by the letter q and the letter r is always followed by the letter s. In particular, you should specify the starting state, the finishing state or states, and the transition table for this finite state acceptor.

(c) (7 marks) Devise a regular grammar to generate the language L of (b).

5. (a) (8 marks) Prove by induction on n that

$$\sum_{i=1}^{n} 5^i(2i+1)^2 = \frac{5}{8}\left((8n^2+4n+3)5^n - 3\right)$$

for all positive integers n.

(b) (6 marks) Let the quaternions q and r be given by $q = i - k$, $r = 2i + j$. Calculate the products qr and rq. [Hamilton's basic formulae for quaternion multiplication state that

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.]$$

(c) (6 marks) Find the greatest common divisor d of the integers 1173 and 306, and find integers u and v such that $d = 1173u + 306v$.

6. Obtain the general solutions of the following ordinary differential equations:—

   (a) (6 marks) $\dfrac{d^2y}{dx^2} - 10\dfrac{dy}{dx} + 61y = 0$;

   (b) (7 marks) $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 8y = \sin 3x$;

   (c) (7 marks) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = xe^{3x}$.

   In the first case (a), find also the solution that satisfies the conditions $y(0) = 1$ and $y'(0) = 0$.

7. (a) (6 marks) Prove that

$$\int_{-\pi}^{\pi} \sin mx \, \cos nx \, dx = 0$$

   for all non-negative integers m and n.

   (b) (14 marks) Find the Fourier series of the periodic function $f\colon \mathbb{R} \to \mathbb{R}$ with period 2π which satisfies $f(x) = x + \pi$ when $-\pi \leq x \leq \frac{1}{2}\pi$ and $f(x) = 3(\pi - x)$ when $\frac{1}{2}\pi \leq x \leq \pi$.