

Discrete Maths Assignment 4
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1)

(a) Is $\{(x, \sin(\pi x)) \mid x \in \mathbb{R}\} \cap \{\mathbb{Q} \times \mathbb{Z}\}$ finite, countably infinite, or uncountably infinite? Justify your answer.

This set is countably infinite as both \mathbb{Q} and \mathbb{Z} are countably infinite sets, as a result the Cartesian product of these two sets also produces a set that is countably infinite.

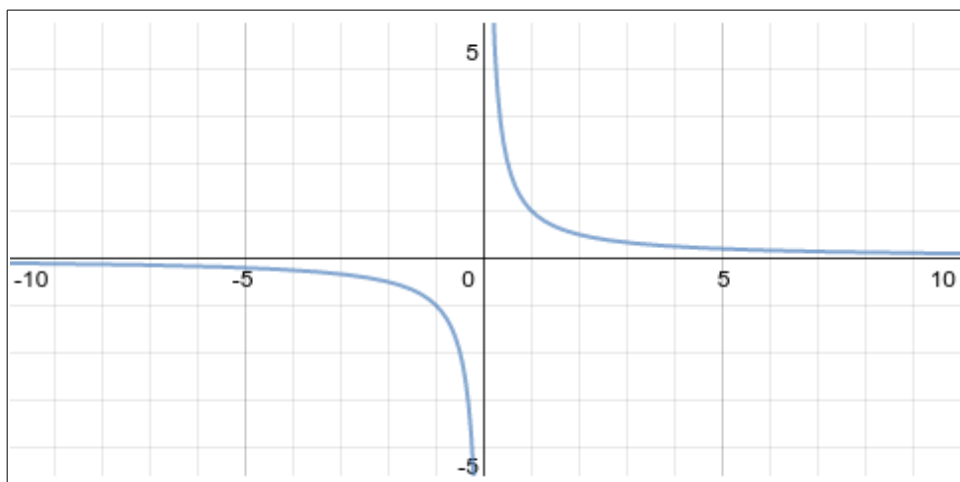
Although the set $\{(x, \sin(\pi x)) \mid x \in \mathbb{R}\}$ in itself is uncountably infinite, it's intersection with the set $\{\mathbb{Q} \times \mathbb{Z}\}$ produces a set that is within the bounds of $\{\mathbb{Q} \times \mathbb{Z}\}$ which as discussed above is countably infinite.

(b) Is $\{(x, y) \in \mathbb{R} \mid xy = 1\}$ finite, countably infinite, or uncountably infinite? Justify your answer.

This set \mathbb{R} itself is an uncountably infinite set as a result of the set of all irrational numbers being a subset of \mathbb{R} .

A set is countably infinite iff $\exists f: A \rightarrow \mathbb{N}$ a bijection, namely $A \sim \mathbb{N}$ s.t \mathbb{N} is the set of Natural Numbers.

As the bounds of this set are confined to $xy=1$ this implies that all elements of the set follow the form $(x, \frac{1}{x})$ since $xy=1$ iff $x = \frac{1}{y}$. This yields in itself an uncountably infinite set which can be illustrated under the following graph which is a representation of the function $xy=1$.

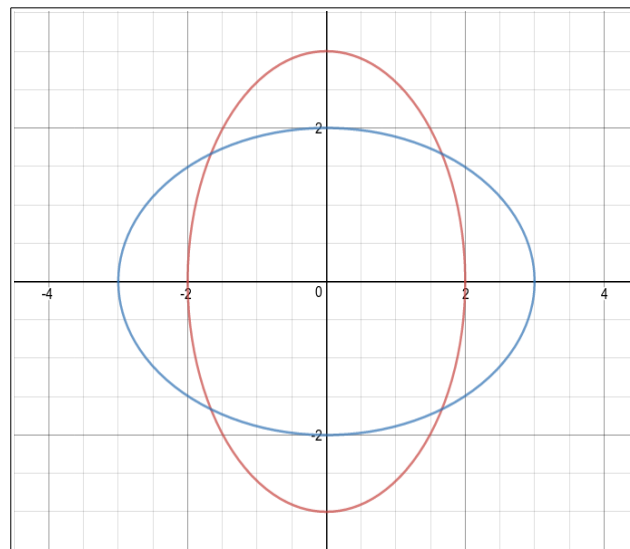


(c) Is $\{(x, y) \in \mathbb{R}^2 \mid x^2/4 + y^2/9 = 1\} \cap \{(x, y) \in \mathbb{R}^2 \mid x^2/9 + y^2/4 = 1\}$ finite, countably infinite, or uncountably infinite? Justify your answer.

The set \mathbb{R}^2 itself is uncountably infinite. Since, as discussed above, the set \mathbb{R} is uncountably infinite this yields that the Cartesian product \mathbb{R}^2 is also uncountably infinite.

However, the set $\{(x, y) \in \mathbb{R}^2 \mid x^2/4 + y^2/9 = 1\}$ and the set $\{(x, y) \in \mathbb{R}^2 \mid x^2/9 + y^2/4 = 1\}$ are the sets containing the co-ordinates of two ellipses, the first of which being with a vertical major axis and the second of which being with a horizontal major axis.

The intersection of these two sets is namely the set containing the co-ordinates of the intersections of the two ellipses which is a finite set containing four elements. This can be illustrated in the following graph.



(c) Let $A = \{0, 1\}$. Is $(0^* \circ 1^*) \cap \{A^* \circ 11 \circ A^*\}$ finite, countably infinite, or uncountably infinite? Justify your answer.

$(0^* \circ 1^*)$ describes a word containing an infinite amount of 0's concatenated with an infinite amount of 1's e.g. 0001111. This set itself is countably infinite as it is a language given under the regular expression $(0^* \circ 1^*)$ and the set of all regular languages over a finite alphabet is countably infinite..

$\{A^*\}$ is the set of all words over A . Since A is a finite alphabet, the set $\{A^*\}$ is a countably infinite set.

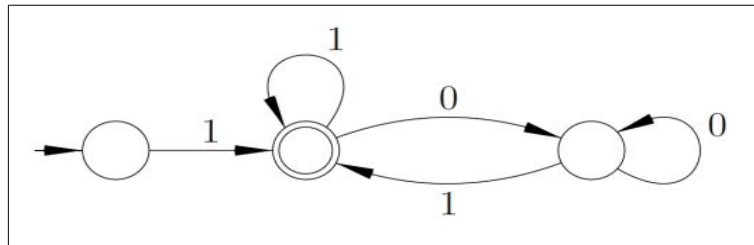
$\{A^* \circ 11 \circ A^*\}$ is also a countably infinite set, similar to what was discussed above it is a language defined under a regular expression and a regular language over a finite alphabet is countably infinite.

As a result the intersection of $(0^* \circ 1^*)$ and $\{A^* \circ 11 \circ A^*\}$ yields a set that is also countably infinite since the intersection of two countably infinite sets is itself a countably infinite set.

2)

(a) Let L be the language consisting of the binary representations of all odd natural numbers. Write down the algorithm of a Turing machine that recognizes L . Process the following strings according to your algorithm: empty string, 0, 11, and 100.

Binary representations of odd natural numbers can be described under the following expression $1\cup 1(0\cup 1)^*1$. An example of a finite state acceptor that accepts binary representations of odd natural numbers can be found below¹.



This can be described in the form of a Turing Machine M which performs the following algorithm:

1. If anything other than 1 is in the first cell REJECT.
2. Move right to rightmost 1, erase (write $_$) and move right. If none, REJECT.
3. If $_$ is in the current cell ACCEPT. Else REJECT.

Processing 'empty word':

1. Something other than 1 in the first cell – **REJECT**

Processing '0':

1. Something other than 1 in the first cell – **REJECT**

Processing '11':

1. 1 found in the first cell, erase and move right.
2. Move 1 place to rightmost 1, erase and move right.
3. $_$ found in current cell – **ACCEPT**

Processing '100':

1. 1 found in the first cell, erase and move right.
2. Couldn't find rightmost 1 – **REJECT**

¹ https://cs.stackexchange.com/questions/10983/regular-expression-for-odd-binary-numbers-without-leading-zeros?utm_medium=organic&utm_source=google_rich_qa&utm_campaign=google_rich_qa

(b) Write down the transition diagram of the Turing machine from part (a) carefully labelling the initial state, the accept state, the reject state, and all the transitions specified in your algorithm.

The Turing Machine above can be described under the 7-tuple $(S, A, \tilde{A}, t, i, S_{\text{accept}}, S_{\text{reject}})$:

$$S = \{S_1, S_2\}$$

$$A = \{0, 1\}$$

$$\tilde{A} = \{ _ \}$$

$$t = t : S \times \tilde{A} \rightarrow S \times \tilde{A} \times \{L, R\}$$

$$I = S_i$$

$$S_{\text{accept}} = S_{\text{accept}} <$$

$$S_{\text{reject}} = S_{\text{reject}}$$

