Similarly, we can define the adjacency table and the adjacency matrix of a graph:

Definition: Let (V, E) be an undirected graph with m vertices, and let these vertices be ordered as $v_1, v_2, ..., v_m$. The adjacency matrix for this graph is given by $\begin{pmatrix} b_{11} & b_{12} & ... & b_{1m} \\ b_{21} & b_{22} & ... & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & ... & b_{mm} \end{pmatrix}$ where $b_{ij} = 1$ if v_i and v_j are adjacent to each other and $b_{ij} = 0$ if v_i and v_j are not adjacent to each other.

Remark: "Being adjacent to" is a symmetric relation on the set of vertices V, so the adjacency matrix is symmetric, **i.e.** $b_{ij} = b_{ji} \quad \forall i, j \quad 1 \leq i, j \leq m$. It is not reflexive so all the entries on the diagonal are zero.

9.1 Complete graphs

Definition: A graph (V, E) is called complete if $\forall v, v' \in V$ s.t. $v \neq v'$, the edge $vv' \in E$. In other words, a complete graph has the highest number of edges possible given its number of vertices.

Examples:

1. The triangle is a complete graph.
2. The pentagram is not a complete graph.

Notation: A complete graph with n vertices is denoted by K_n.

Q: How does the adjacency matrix of a complete graph look like?

A: All entries are 1 except on the diagonal, where they are all zero.

9.2 Bipartite graphs

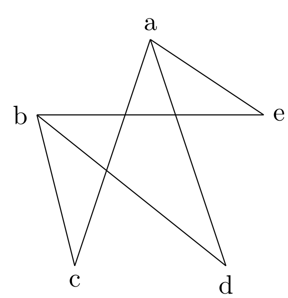
Definition: A graph (V, E) is called bipartite is $\exists$ subsets V_1 and V_2 s.t.

1. $V_1 \cup V_2 = V$
2. $V_1 \cap V_2 = \emptyset$
3. Every edge in E is of the form vw with $v \in V_1$ and $w \in V_2$.

A bipartite graph is called a complete bipartite graph if $\forall v \in V_1 \quad \forall w \in V_2 \quad \exists vw \in E$.

Notation: A complete bipartite graph where the set V_1 has p elements and the set V_2 has q elements is denoted by $K_{p,q}$.

Example:
$V_1 = \{a, b\}$
$V_2 = \{c, d, e\}$
$V = \{a, b, c, d, e\}$
$E = \{ac, ad, ae, bc, bd, be\}$
is a complete bipartite graph.



Next, relate graphs to each other via functions with special properties.

9.3 Isomorphisms of Graphs

Definition: An isomorphism between two graphs (V, E) and (V', E') is a bijective function $\varphi : V \to V'$ satisfying that $\forall a, b \in V$ with $a \neq b$ the edge $ab \in E \Leftrightarrow$ the edge $\varphi(a)\varphi(b) \in E'$.

Recall: A function $\varphi : V \to V'$ is bijective $\Leftrightarrow$ it has an inverse $\varphi^{-1} : V' \to V$. The bijection $\varphi : V \to V'$ that gives the isomorphism between (V, E) and $(V'E')$ thus sets up the following:

1. A 1-1 correspondence of the vertices V of (V, E) with the vertices V' of $(V', E') \rightsquigarrow$ comes from $\varphi : V \to V'$ being bijective.
2. A 1-1 correspondence of the edges E of (V, E) with the edges E' of $(V', E') \rightsquigarrow$ comes from the additional property in the definition of an isomorphism that $\forall a, b \in V$ with $a \neq b, ab \in E \Leftrightarrow \varphi(a)\varphi(b) \in E'$.

Remark: Just like an isomorphism of groups discussed earlier in the course, an isomorphism of graphs means (V, E) and (V', E') have the same "iso" form "morph$\bar{\text{e}}$". "Being isomorphic" is an equivalence relation, so we get classes of graphs that have the same form as our equivalence classes.

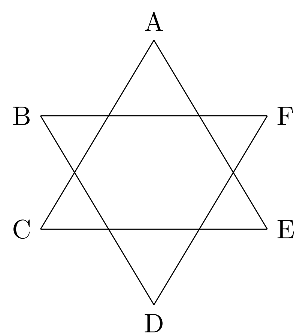
Definition: If there exists an isomorphism $\varphi : V \to V'$ between two graphs (V, E) and (V', E'), then (V, E) and (V', E') are called isomorphic.

9.4 Subgraphs

Task: Understand sub-objects of a graph.

Definition: Let (V, E) and (V', E') be graphs. The graph (V', E') is called a subgraph of (V, E) if $V' \subseteq V$ and $E' \subseteq E$, **i.e.** if (V', E') consists of a subset V' of the vertices of (V, E) and a subset E' of edges (V, E) between vertices in V'.

Example: Star of David on the flag of Israel
$V = \{a, b, c, d, e, f\}$
$E = \{ac, ce, ae, bf, fd, bd\}$



2 triangle subgraphs of the star of David:
$V' = \{a, c, e\}$ $E' = \{ac, ce, ae\}$
$V" = \{b, f, d\}$ $E" = \{bf, fd, bd\}$

9.5 Vertex Degrees

Task: Use numbers to understand incidence relationships.

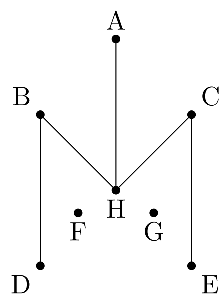
Definition: Let (V, E) be a graph. The <u>degree</u> $\deg v$ of a vertex $v \in V$ is defined as the number of edges of the graph that are incident to v, **i.e.** the number of edge with v as one of their endpoints.

Example:

def $f = \deg g = 0$
$\deg d = \deg e = \deg a = 1$
$\deg b = \deg c = 2$
$\deg h = 3$

Definition: A vertex of degree 0 is called an <u>isolated</u> vertex.

Definition: A vertex of degree 1 is called a <u>pendant</u> vertex.



Theorem: Let (V, E) be a graph. Then $\sum_{v \in V} \deg v = 2\#(E)$, where $\sum_{v \in V} \deg v$ is the sum of the degrees of all the vertices of the graph, and $\#(E)$ is the number of edges of the graph.

Proof: $\sum_{v \in V} \deg v$ is the sum of all the entries in the adjacency matrix. Every edge $vv' \in E$ contributes 2 to the sum $\sum_{v \in V} \deg v$, 1 for the vertex v and 1 for the vertex $v' \Rightarrow$ each edge must be counted twice, so $\sum_{v \in V} \deg v = 2\#(E)$.

qed

Corollary: $\sum_{v \in V} \deg v$ is an even integer.

Proof: Since $\sum_{v \in V} \deg v = 2\#(E)$, and $\#(E) \in \mathbb{N}$, the result follows.

qed

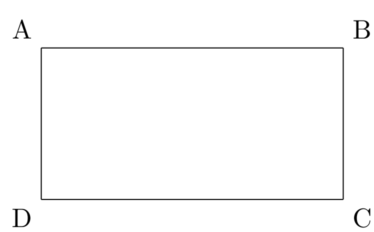
Corollary: In any graph, the number of vertices of odd degrees must be even.

Proof: Assume not, then $\sum_{v \in V} \deg v$ is an odd integer as $odd + even = odd \Rightarrow\Leftarrow$ to the previous corollary.

qed

Definition: A graph is called k-regular for some non-negative integer k if every vertex of the graph has degree equal to k.

Example: A rectangle is 2-regular.
$\deg a = \deg b = \deg c = \deg d = 2$.



Definition: A graph (V, E) is called regular is $\exists k \in \mathbb{N}$ s.t. (V, E) is k-regular.

Corollary: Let (V, E) be a k-regular graph. Then $k\#(V) = 2\#(E)$ where $\#(V)$ is the number of vertices and $\#(E)$ is the number of edges.

Proof: By the theorem, $\sum_{v \in V} \deg v = 2\#(E)$, but (V, E) is k-regular $\Rightarrow \deg v = k \; \forall v \in V$. Therefore $\sum_{v \in V} \deg v = \#(V) \times k = 2\#(E)$.

qed

Example: Consider a complete graph (V, E) with n vertices. (V, E) is $(n-1)$-regular because every vertex is adjacent to all the remaining $(n-1)$ vertices.

Corollary: A complete bipartite graph $k_{p,q}$ is regular $\Leftrightarrow p = q$

Proof: Recall that $V = V_1 \cup V_2 \; V_1 \cap V_2 = \emptyset$ for a bipartite graph, where $\#(V_1) = p$ and $\#(V_2) = q$.
"$\Leftarrow$" If $p = q, \forall v \in V_1$ satisfies that $\deg v = p = q$ and $\forall v \in V_2$ satisfies that $\deg v = p = q$ since the graph is complete $\Rightarrow (V, E)$ is p-regular.