Lemma D: Let (V, E) be a connected graph, and let some trail in this graph be given. Suppose that no vertex of the graph has the property that not all the edges of the graph incident to that vertex are traversed by the trail. Then the given trail is an Eulerian trail.

Proof: Let V_1 be the set of vertices through which the trail passes, and let V_2 be the set of vertices through which the trail does not pass. $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. The conclusion of Lemma D amounts to showing $V_2 = \emptyset$. $\forall u \in V_1$, u is incident to at least one edge traversed by the trail. $\Rightarrow$ All edges incident to the vertices in V_1 are traversed by the trail, but then every vertex in V adjacent to a vertex in V_1 must belong to $V_1 \Rightarrow$ no edge can join a vertex in V_1 to a vertex in V_2. If $V_2 \neq \emptyset$, then $\exists w \in V_2$, but then w cannot be joined by a path to any vertex in $V_1 \Rightarrow V_1$ and V_2 are in different connected components of the graph $\Rightarrow\Leftarrow$ since the graph is connected $\Rightarrow$ it has only one connected component. Therefore, $V_2 = \emptyset$.

qed

Finally, we can prove Euler's theorem:

Theorem A nontrivial connected graph contains an Eulerian circuit if the degree of every vertex of the graph is even.

Proof: Let (V, E) be a non-trivial connected graph s.t. $\forall v \in V$, deg v is even. By Lemma A, (V, E) contains at least one circuit. It therefore contains a circuit of maximal length (**i.e.** at least as long as any other circuit in the graph). We seek to prove that this circuit of maximal length is indeed Eulerian.
If the graph contains some vertex v s.t. some but not all of the edges of the graph incident to v are traversed by the circuit of maximal length, and v is a vertex on the circuit of maximal length, then by Lemma B $\exists$ a second circuit in (V, E) passing through v, which would not traverse any edge traversed by the circuit of maximal length. By Lemma C, however, we can concatenate the two circuits, obtaining a circuit of length strictly greater than the length of the circuit of maximal length $\Rightarrow\Leftarrow$ we concluse no vertex that belongs to the circuit of maximal length has the property that not all edges incident to it are traversed by the circuit of maximal length. Since (V, E) is connected, by Lemma D, the circuit of maximal length must be Eulerian.

qed

Corollary 2 along with this theorem together gives us:

Theorem: A non-trivial connected graph has an Eulerian circuit $\Leftrightarrow$ the degree of each of its vertices is even.

Corollary: Suppose a connected graph has exactly two vertices whose degree is odd. $\exists$ an Eulerian trail in the graph joining the two vertices with odd degrees.

Proof: We reduce this case to the previous one by embedding the graph (V, E) with vertices v, w that have odd degree into a graph (V', E') s.t. $v' = v \cup \{u\}$ for $u \notin V$ and $E' = E \cup \{uv, uw\}$. (V, E) is a subgraph of $(V'E')$ and (V', E') is connected and each one of its vertices has even degree by construction. By the theorem we just proved, (V', E') has an Eulerian circuit. We reorder the vertices so that the final two edges are the two added edges wu and uv. We now delete the edges wu and uv to obtain an Eulerian trail in the original graph (V, E) from v to w.

qed

9.11 Hamiltonian Paths and Circuits

Task: Look at paths and circuits that pass through every vertex of a graph.

Definition: A <u>Hamiltonian path</u> in a graph is a path that passed exactly once through every vertex of a graph.
Path $\Rightarrow$ we pass through a vertex at most once (no repeated vertices)
Hamiltonian $\Rightarrow$ we pass through every vertex.

Definition: A <u>Hamiltonian circuit</u> in a graph is a simple circuit that passes through every vertex of the graph.