UNIVERSITY OF DUBLIN

XMA2C011

TRINITY COLLEGE

FACULTY OF ENGINEERING, MATHEMATICS
AND SCIENCE

SCHOOL OF COMPUTER SCIENCE AND STATISTICS
DEPARTMENT OF COMPUTER SCIENCE

SF Computer Science (B.A.)
SF CSLL

Trinity Term 2011

DISCRETE MATHEMATICS — MODULES MA2C01 AND MA2C02

Thursday, May 5 RDS-MAIN 14:00 — 17:00

Dr. D. R. Wilkins

Credit will be given for the best **SIX** questions answered.

Log tables are available from the invigilators, if required.

Students may avail of the HANDBOOK OF MATHEMATICS of Computer Science

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

1. (a) Let A, B and C be sets. Prove that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

[8 marks]

(b) Let R denote the relation on the set $\mathbb{Z}^2$ of ordered pairs of integers, where elements (x, y) and (u, v) of $\mathbb{Z}^2$ satisfy $(x, y)\ R\ (u, v)$ if and only if either $x < u$ or else both $x = u$ and $y \leq v$. Thus, in the notation of formal logic,

$$(x, y)\ R\ (u, v) \iff (x < u) \vee ((x = u) \wedge (y \leq v)).$$

Determine whether or not this relation R is

(i) *reflexive*,

(ii) *symmetric*,

(iii) *transitive*,

(iv) *anti-symmetric*,

(v) an *equivalence relation*,

(vi) a *partial order*,

[Give appropriate short proofs and/or counterexamples to justify your answers.]

[12 marks]

[Recall that a relation R on a set X is an equivalence relation if and only if it is reflexive, symmetric and transitive. It is a partial order if and only if it is reflexive, anti-symmetric and transitive. A relation R on a set X is reflexive if and only if xRx for all $x \in X$; the relation is symmetric if and only if yRx for all $x, y \in X$ satisfying xRy; the relation is transitive if and only if xRz for all $x, y, z \in X$ satisfying xRy and yRz; the relation is anti-symmetric if and only if $x = y$ for all $x, y \in R$ satisfying xRy and yRx.]

2. (a) Let $f\colon A \to B$ be a function from a set A to a set B. What is meant by saying that such a function is *injective*, and that such a function is *surjective*?

[4 marks]

(b) Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be the function from the set $\mathbb{Z}$ of integers to itself defined such that $f(x) = x^3 - 1$ for all integers x. Determine whether or not this function is injective, and whether or not it is surjective, giving brief reasons for your answers.

[4 marks]

(c) What is a *monoid*? What is the *identity element* of a monoid? What is meant by saying that an element of a monoid is *invertible*?

[6 marks]

(d) Let $*$ denote the binary operation defined on the set $\mathbb{R}^2$ of ordered pairs of real numbers, where

$$(a, b) * (c, d) = (ac, ad + b)$$

for all real numbers a, b, c and d. Prove that $\mathbb{R}^2$, with the binary operation $*$, is a monoid. Also prove that an element (a, b) of $\mathbb{R}^2$ is an invertible element of this monoid if and only if $a \neq 0$.

[6 marks]

3. (a) Describe the formal language over the alphabet $\{0, 1\}$ generated by the context-free grammar whose non-terminals are $\langle S\rangle$ and $\langle A\rangle$, whose start symbol is $\langle S\rangle$ and whose productions are the following:

$$\begin{aligned}\langle S\rangle &\rightarrow 00\langle S\rangle 11\\ \langle S\rangle &\rightarrow \langle A\rangle\\ \langle A\rangle &\rightarrow 11\langle A\rangle\\ \langle A\rangle &\rightarrow 1\end{aligned}$$

Is this context-free grammar a regular grammar?

[6 marks]

(b) Give the specification of a finite state acceptor that accepts the language over the alphabet $\{a, 0, 1, (,)\}$ consisting of all words such as

$$aa0(10), \quad a101a10(10110), \quad a(1), \quad a1(10)$$

whose structure matches the following specification:

the character a, followed by
zero or more of the characters a, 0, 1, followed by
an opening parenthesis (, followed by
the binary digit 1
zero or more binary digits 0, 1, followed by
a closing parenthesis).

In particular you should specify the set of states, the starting state, the finishing states, and the transition table that determines the transition function of the finite state acceptor.

[8 marks]

(c) Devise a regular context-free grammar to generate the language over the alphabet $\{a, 0, 1, (,)\}$ described above in (b).

[6 marks]

4. In this question, all graphs are undirected graphs.

(a)

(i) What is meant by saying that a graph is *complete*?

(ii) What is meant by saying that a graph is *regular*?

(iii) What is meant by saying that a graph is *connected*?

(iv) What is meant by saying that a graph is a *tree*?

(v) Give the definition of an *isomorphism* between two undirected graphs.

[7 marks]

(b) Let G_1 be the undirected graph whose vertices are a, b, c, d, e and f, and whose edges are the following:

$$a\,b, \quad a\,c, \quad a\,d, \quad b\,c, \quad d\,e, \quad d\,f.$$

(i) Is this graph complete?

(ii) Is this graph regular?

(iii) Is this graph connected?

(iv) Is this graph a tree?

[Give brief reasons for each of your answers.]

[8 marks]

(c) Let G_2 be the undirected graph whose vertices are p, q, r, s, t and u and whose edges are the following:

$$p\,r, \quad p\,s, \quad p\,t, \quad q\,s, \quad q\,u, \quad s\,u.$$

Give an example of an isomorphism between the graph G_1 specified in (b) and the graph G_2.

[5 marks]

5. (a) Any function y of a real variable x that solves the differential equation

$$\frac{d^4y}{dx^4} - 16y = 0$$

may be represented by a power series of the form

$$y = \sum_{n=0}^{+\infty} \frac{y_n}{n!} x^n,$$

where the coefficients $y_0, y_1, y_2, y_3, \ldots$ of this power series are real numbers.

Find values of these coefficients y_n for $n = 0, 1, 2, 3, 4, \ldots$ that yield a solution to the above differential equation with $y_0 = 1$ $y_1 = 2$, $y_2 = 4$ and $y_3 = 8$.

[8 marks]

(b) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = x^2 e^{2x}.$$

[12 marks]

6. (a) Let n be an integer satisfying $n > 1$, and let ω be a complex number. Suppose that $\omega^n = 1$ and $\omega \neq 1$. Prove that $\sum_{j=0}^{n-1} \omega^j = 0$, where

$$\sum_{j=0}^{n-1} \omega^j = 1 + \omega + \omega^2 + \cdots + \omega^{n-1}.$$

[6 marks]

(b) Let $(z_n : n \in \mathbb{Z})$ be the doubly-infinite 4-periodic sequence with $z_0 = 2$, $z_1 = 1$ and $z_2 = 5$ and $z_3 = -2$. Find values of a_0, a_1, a_2 and a_3 such that

$$z_n = a_0 + a_1 i^n + a_2(-1)^n + a_3(-i)^n$$

for all integers n, where $i = \sqrt{-1}$.

[14 marks]

7. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the periodic function with period 2π whose values on the interval $[-\pi, \pi]$ are given by the following formulae:

$$f(x) = \begin{cases} 1 - \dfrac{3|x|}{\pi} & \text{if } -\frac{1}{3}\pi \leq x \leq \frac{1}{3}\pi; \\ 0 & \text{if } \frac{1}{3}\pi \leq x \leq \pi \text{ or } -\pi \leq x \leq -\frac{1}{3}\pi. \end{cases}$$

(Here $|x|$ denotes the absolute value of the real number x, equal to x when $x \geq 0$, and equal to $-x$ when $x \leq 0$.) This periodic function is an even function, and may therefore be represented as a Fourier series of the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{+\infty} a_n \cos nx,$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x) \cos nx \, dx.$$

Find the values of the coefficients a_n for $n = 0, 1, 2, 3, \ldots$.

(Note that $\cos \frac{1}{3}\pi = -\cos \frac{2}{3}\pi = -\cos \frac{4}{3}\pi = \cos \frac{5}{3}\pi = \frac{1}{2}$.)

[20 marks]

8. (a) Find the lengths of the vectors $(2, 4, 4)$ and $(6, 2, 9)$ and also the cosine of the angle between them.

[6 marks]

(b) Let the quaternions q and r be defined as follows:

$$q = 1 - 5k, \quad r = i + j - 2k.$$

Calculate the quaternion products qr and rq. [Hamilton's basic formulae for quaternion multiplication state that

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.]$$

[6 marks]

(c) Find the greatest common divisor d of the integers 2184 and 975, and find integers u and v such that $d = 2184u + 975v$.

[8 marks]