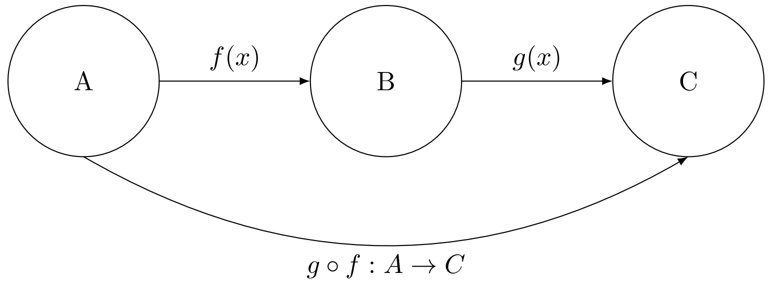
5.1 Composition of Functions

Task: Understand the natural operation that allows us to combine functions.



Example:

$f : \mathbb{R} \to \mathbb{R} \qquad f(x) = 2x$
$g : \mathbb{R} \to \mathbb{R} \qquad g(x) = \cos x$
$g \circ f(x) = g(f(x)) = g(2x) = \cos(2x)$
$f \circ g(x) = f(g(x)) = f(\cos x) = 2\,(\cos x) = 2\,\cos x$

5.2 Inverting Functions

Task: Figure out which properties a function has to satisfy so that its action can be undone, **i.e. when** we can define an inverse to the original function.

Given $f : A \to B$, want $f^{-1} : B \to A$ s.t. $f^{-1} \circ f : A \to A$ is the identity
$f^{-1} \circ f(x) = f^{-1}(f(x)) = x$
$A \xrightarrow{f} B \xrightarrow{f^{-1}} A$
It turns out f has to satisfy two properties for f^{-1} to exist:

1. Injective
2. Surjective

Definition: A function $f : A \to B$ is called <u>injective</u> or an injection (sometimes called one-to-one) if $f(x) = f(y) \Rightarrow x = y$

Examples:
$\sin x : [0, \frac{\pi}{2}] \to \mathbb{R}$ is injective
$\sin x : \mathbb{R} \to \mathbb{R}$ is not injective because $\sin 0 = \sin \pi = 0$

Definition: A function $f : A \to B$ is called <u>surjective</u> or a surjection (sometimes called onto) if $\forall z \in B\ \exists x \in A$ s.t. $\overline{f(x) = z}$.

Remark: f assigns a value to each element of A by its definition as a function, but it is not required to cover all of B. f is surjective if its range is all of B.

Examples:
$\sin x : \mathbb{R} \to [-1, 1]$ is surjective
$\sin x : \mathbb{R} \to \mathbb{R}$ is not surjective since $\nexists x \in \mathbb{R}$ s.t. $\sin x = 2$. We know $|\sin x| \leq 1\ \forall x \in \mathbb{R}$

Definition: A function $f : A \to B$ is called <u>bijective</u> or a bijection if f is <u>both</u> injective and surjective.

Example: $f : \mathbb{R} \to \mathbb{R}$ $\qquad f(x) = 2x + 1$ is bijective.

- Check injectivity: $f(x_1) = f(x_2) \Rightarrow 2x_1 + 1 = 2x_2 + 1 \Leftrightarrow 2x_1 = 2x_2 \Leftrightarrow x_1 = x_2$ as needed.
- Check surjectivity: $\forall z \in \mathbb{R}\ f(x) = z$ means $2x + 1 = z$.
  Solve for x: $2x = z - 1 \Rightarrow x = \frac{z-1}{2} \in \mathbb{R} \Rightarrow f$ is surjective.

Remark: All bijective functions have inverses because we can define the inverse of a bijection and it will be a function:

- Surjectivity ensures f^{-1} assigns an element to every element of B (its domain).
- Injectivity ensures f^{-1} assigns to each element of B <u>one and only one</u> element of A.

Conclusion: $f : A \to B$ bijective $\Rightarrow f^{-1}$ exists, **i.e.** f^{-1} is a function. It turns out (reverse the arguments above) that f^{-1} exists $\Rightarrow f : A \to B$ is bijective.
Altogether we get the following theorem:

Theorem: Let $f : A \to B$ be a function. f^{-1} exists $\Leftrightarrow f : A \to B$ is bijective.

Q: How do we find the inverse function f^{-1} given $f : A \to B$?

A: If $f(x) = y$, solve for x as a function of y since $f^{-1}(f(x)) = f^{-1}(y) = x$ as $f^{-1} \circ f$ is the identity.

Example: $f(x) = 2x + 1 = y$. Solve for x in terms of y.
$f : \mathbb{R} \to \mathbb{R}$
$2x = y - 1 \qquad x = \frac{y-1}{2}$

5.3 Functions Defined on Finite Sets

Task: Derive conclusions about a function given the number of elements of the domain and codomain, if finite; understand the pigeonhole principle.

Proposition: Let A, B be sets and let $f : A \to B$ be a function. Assume A is finite. Then f is injective $\Leftrightarrow f(A)$ has the same number of elements as A.

Proof:

A is finite so we can write it as $A = \{a_1, a_2, ..., a_p\}$ for some p. Then $f(A) = \{f(a_1), f(a_2), ..., f(a_p)\} \subseteq B$. A priori, some $f(a_i)$ might be the same as some $f(a_j)$. However, f injective $\Leftrightarrow f(a_i) \neq f(a_j)$ whenever $i \neq j \Leftrightarrow f(A)$ has exactly p elements just like A.

qed