4.3 Partial Orders

Task: Understand another type of relation with special properties.

Definition: Let A be a set. A relation R on A is called anti-symmetric if $\forall x, y \in A$ s.t. $xRy \wedge yRx$, then $x = y$.

Definition: A partial order is a relation on a set A that is reflexive, anti-symmetric, and transitive.

Examples:

1. $A = \mathbb{R}$ $\quad\quad\leq$ "less than or equal to" is a partial order
   (a) $\forall x \in \mathbb{R}, x \leq x \rightarrow$ reflexive
   (b) $\forall x, y \in \mathbb{R}$ s.t. $x \leq y \wedge y \leq x \implies x = y \rightarrow$ anti-symmetric
   (c) $\forall x, y, z \in \mathbb{R}$ s.t. $x \leq y \wedge y \leq z \implies x \leq z \rightarrow$ transitive
   Same conclusion if $A = \mathbb{Z}$ or $A = \mathbb{N}$
2. A is a set. Consider $P(A)$, the power set of A. The relation $\subseteq$ "being a subset of" is a partial order.
   (a) $\forall B \in P(A), B \subseteq B \rightarrow$ reflexive.
   (b) $\forall B, C \in P(A), B \subseteq C \wedge C \subseteq B \implies B = C$ (recall the criterion for proving equality of sets) $\rightarrow$ anti-symmetric
   (c) $\forall B, C, D \in P(A)$ s.t. $B \subseteq C \wedge C \subseteq D \implies B \subseteq D \rightarrow$ transitive

The most important example of a partial order is example (2) "being a subset of".

Q: Why is "being a subset of" a partial order as opposed to a total order?

A: There might exist subsets B, C of A s.t. neither $B \subseteq C$ nor $C \subseteq B$ holds, **i.e.** where B and C are not related via inclusion.

5 Functions

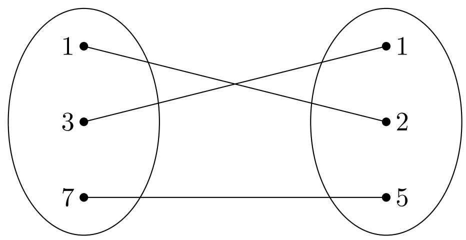
Task: Define a function rigorously and make sense of terminology associated to functions.

Definition: Let A, B be sets. A function $f : A \to B$ is a rule that assigns to every element of A one and only one element of B, **i.e.** $\forall x \in A \ \exists! y \in B$ s.t. $f(x) = y$. A is called the domain of f and B is called the codomain.

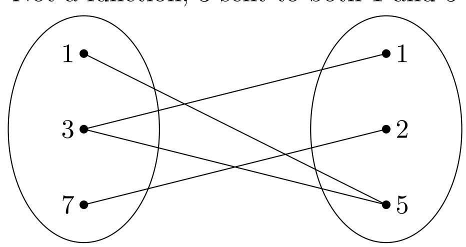
Examples:

1. $A = \{1, 3, 7\}$
   $B = \{1, 2, 5\}$

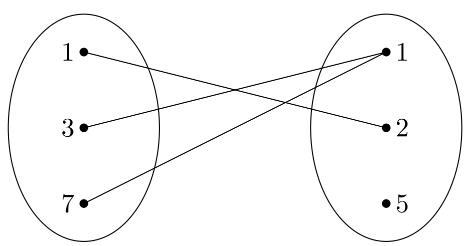
Is a function.



Not a function; 3 sent to both 1 and 5



Is a function.



2. $A = B = \mathbb{R}$ $\qquad F : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$ is called the identity function.

Definition: Let A, B be sets, and let $f : A \to B$ be a function. The range of f denoted by $f(A)$ is the subset of B defined by $f(A) = \{y \in B \mid \exists x \in A$ s.t. $f(x) = y\}$.

Definition: Let A be a set. A Boolean function on A is a function $F : A \to \{T, F\}$, which has A as its domain and the set of truth values $\{T, F\}$ as is codomain. $f : A \to \{T, F\}$ thus assigns truth values to the elements of A.

Function are often represented by graphs. If $f : A \to B$ is a function, the graph of f denoted $\Gamma(f)$ is the subset of the Cartesian product of the domain with the codomain $A \times B$ given by $\{(x, f(x)) \mid x \in A\}$.

Q: Is it possible to obtain every subset of $A \times B$ as the graph of some function?

A: No! For $f : A \to B$ to be a function $\forall x \in A \quad \exists! y \in B$ s.t. $f(x) = y$, so for $\Gamma \subseteq A \times B$ to be the graph of some function, Γ must satisfy that $\forall x \in A \quad \exists! y \in B$ s.t. $(x, y) \in \Gamma$. Then we can define f by letting $y = f(x)$.

NB For the usual set-up of a function $f : \mathbb{R} \to \mathbb{R}$, this observation amounts to the "vertical line test," which you have seen before coming to university.

5.1 Composition of Functions

Task: Understand the natural operation that allows us to combine functions.

$$\begin{tikzcd}
\text{A} \arrow[r, "f(x)"] \arrow[rr, bend right, "g \circ f : A \to C"'] & \text{B} \arrow[r, "g(x)"] & \text{C}
\end{tikzcd}$$