1 Review of Propositional Logic

Task: Recall enough propositional logic to see how it matches up with set theory.

Definition: A proposition is any declarative sentence that is either true or false.

1.1 Connectives

	Connectives		Notation in Maths
and	$\wedge$		
or	$\vee$	"Inclusive or"	
not	$\neg$	Sometimes denoted $\sim$	
implies	$\rightarrow$	if/then; called implication	$\Rightarrow$
if and only if	$\leftrightarrow$	Called equivalence	$\Leftrightarrow$

1.1.1 Truth Table of the Connectives

Let P, Q be propositions:

P	Q	P$\wedge$Q
F	F	F
F	T	F
T	F	F
T	T	T

P	Q	P$\vee$Q
F	F	F
F	T	T
T	F	T
T	T	T

P	$\neg$P
F	T
T	F

P	Q	P$\rightarrow$Q
F	F	T
F	T	T
T	F	F
T	T	T

P	Q	P$\leftrightarrow$Q
F	F	T
F	T	F
T	F	F
T	T	T

Priority of the Connectives

Highest to Lowest: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

1.2 Important Tautologies

$$\begin{array}{rcl} (P \rightarrow Q) & \leftrightarrow & (\neg P \vee Q) \\ (P \leftrightarrow Q) & \leftrightarrow & [(P \rightarrow Q) \wedge (Q \rightarrow P)] \\ \neg(P \wedge Q) & \leftrightarrow & (\neg P \vee \neg Q) \\ \neg(P \vee Q) & \leftrightarrow & (\neg P \wedge \neg Q) \end{array} \left.\vphantom{\begin{array}{c} a \\ b \end{array}}\right\}$$

De Morgan Laws (also appear in set theory)

As a result, $\neg$ and $\vee$ together can be used to represent all of $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.

Less obvious: One connective called the Sheffer stroke P|Q (which stands for "not both P and Q" or "P nand Q") can be used to represent all of $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ since $\neg$P $\leftrightarrow$ P|P and P $\vee$ Q $\leftrightarrow$ (P|P) | (Q|Q).

Recall that if P$\rightarrow$Q is a given implication, then Q$\rightarrow$P is called the converse of P$\rightarrow$Q, while $\neg$Q $\rightarrow$ $\neg$P is called the contrapositive of P $\rightarrow$ Q.

1.3 Indirect Arguments/Proofs by Contradiction/Reductio ad absurdum

Based on the tautology (P→Q) ↔ (¬Q → ¬P)

Example: Famous argument that $\sqrt{2}$ is irrational.

Proof:

Suppose $\sqrt{2}$ is rational, then it can be expressed in fraction form as $\frac{a}{b}$ with a and b integers, $b \neq 0$. Let us **assume** that our fraction is reduced, **i.e.** the only common divisor of the numerator a and denominator b is 1.

Then,

$$\sqrt{2} = \frac{a}{b}$$

Squaring both sides, we have

$$2 = \frac{a^2}{b^2}$$

Multiplying both sides by b^2 yields

$$2b^2 = a^2$$

Therefore, 2 divides a^2, i.e. a^2 is even. If a^2 is even, then a is also even, namely $a = 2k$ for some integer k.

Substituting the value of $2k$ for a, we have $2b^2 = (2k)^2$ which means that $2b^2 = 4k^2$. Dividing both sides by 2, we have $b^2 = 2k^2$. That means 2 divides b^2, so b is even.

This implies that both a and b are even, which means that both the numerator and the denominator of our fraction are divisible by 2. This contradicts our **assumption** that the numerator a and the denominator b have no common divisor except 1. Since we found a contradiction, our assumption that $\sqrt{2}$ is rational must be false. Hence the theorem is true.

qed

2 Predicate logic and Quantifiers

Task: Understand enough predicate logic to make sense of quantified statements.

In predicate logic, propositions depend on variables x, y, z, so their truth value may change depending on which values these variables assume: $P(x), Q(x, y), R(x, y, z)$

2.1 Introduce quantifiers

2.1.1 $\exists$ existential quantifier

Syntax: $\exists x P(x)$

Definition: $\exists x P(x)$ is true if $P(x)$ is true for some value of x. It is false otherwise.

2.1.2 $\forall$ universal quantifier

Syntax: $\forall x P(x)$

Definition: $\forall x P(x)$ is true if $P(x)$ is true for all allowable values of x. It is false otherwise.

2.1.3 $\exists!$ for one and only one (additional quantifier standard in maths)

Syntax: $\exists! x P(x)$

Definition: $\exists! x P(x)$ is true if $P(x)$ is true for exactly one value of x and false for all other values of x; otherwise, $\exists! x P(x)$ is false.

2.2 Alternation of Quantifiers

$\forall x \exists y \forall z \quad P(x, y, z)$

NB: The order <u>cannot</u> be exchanged as it might modify the truth value of the statement (think of examples with two quantifiers).

2.3 Negation of Quantifiers

$$\begin{array}{lcl} \neg(\exists x P(x)) & \leftrightarrow & \forall x \neg P(x) \\ \neg(\forall x P(x)) & \leftrightarrow & \exists x \neg P(x) \end{array}$$

3 Set Theory

Task: Understand enough set theory to make sense of other mathematical objects in abstract algebra, graph theory, etc.

Set theory started around 1870's $\rightarrow$ late development in mathematics but now taught early in one's maths education due to the Bourbaki school.

Definition: A set is a collection of objects. $x \in A$ means the element x is in the set A (**i.e.** belongs to A).

Examples:

1. All students in a class.
2. $\mathbb{N}$ the set of natural numbers starting at 0.

$\mathbb{N}$ is defined via the following two axioms:

(a) $0 \in \mathbb{N}$

(b) if $x \in \mathbb{N}$, then $x+1 \in \mathbb{N}$ ($x \in \mathbb{N} \to x+1 \in \mathbb{N}$)

3. $\mathbb{R}$ set of real numbers also introduced axiomatically. The hardest axiom is the last one: completeness. $\mathbb{R}$ is constructed from $\mathbb{Q}$ in one of two ways: via Dedekind cuts or Cauchy sequences.

   $\mathbb{R}$ is the set of real numbers. The axioms governing $\mathbb{R}$ are:

   (a) Additive closure: $\forall x, y\, \exists z(x+y=z)$

   (b) Multiplicative closure: $\forall x, y, \exists z(x \times y = z)$

   (c) Additive associativity: $\forall x, y, z \quad x+(y+z) = (x+y)+z$

   (d) Multiplicative associativity: $\forall x, y, z \quad x \times (y \times z) = (x \times y) \times z$

   (e) Additive commutativity: $\forall x, y \quad x+y = y+x$

   (f) Multiplicative commutativity: $\forall x, y \quad x \times y = y \times x$

   (g) Distributivity: $\forall x, y, z \quad x \times (y+z) = (x \times y) + (x \times z)$ and $(y+z) \times x = (y \times x) + (z \times x)$

   (h) Additive identity: There is a number, denoted 0, such that for all $x, x+0 = x$

   (i) Multiplicative identity: There is a number, denoted 1, such that for all $x, x \times 1 = 1 \times x = x$

   (j) Additive inverses: For every x there is a number, denoted $-x$, such that $x + (-x) = 0$

   (k) Multiplicative inverses: For every nonzero x there is a number, denoted x^{-1}, such that $x \times x^{-1} = x^{-1} \times x = 1$

   (l) $0 \neq 1$

   (m) Irreflexivity of $<$: $\sim (x < x)$

   (n) Transitivity of $<$: If $x < y$ and $y < z$, then $x < z$

   (o) Trichotomy: Either $x < y, y < x$, or $x = y$

   (p) If $x < y$, then $x + y < y + z$

   (q) If $x < y$ and $0 < z$, then $x \times z < y \times z$ and $z \times x < z \times y$

   (r) Completeness: If a nonempty set of real numbers has an upper bound, then it has a *least* upper bound.

4. $\emptyset$ is the empty set (The set with no elements).

Definition: Let A, B be sets. A=B if and only if all elements of A are elements of B and all elements of B are elements of A,
i.e. $A = B \leftrightarrow [\forall x(x \in A \to x \in B)] \wedge [\forall y(y \in B \to y \in A)]$

3.1 Two Ways to Describe Sets

1. The enumeration/roster method: list all elements of the set.
   NB: order is irrelevant.
   $A = \{0,1,2,3,4,5\} = \{5,0,2,3,1,4\}$

2. The formulaic/set builder method: give a formula that generates all elements of the set.
$A = \{x \in \mathbb{N} \mid 0 \leq x \wedge x \leq 5\} = \{0, 1, 2, 3, 4, 5\} = \{x \in \mathbb{N} : 0 \leq x \wedge x \leq 5\}$