To characterise $\mathbb{N}$ we need to recall the notion of a sequence:

Definition: A sequence is a set of elements $\{x_1, x_2, ...\}$ indexed by J, i.e. $\exists f : J \longrightarrow \{x_1, x_2, ...\}$ such that $f(n) = x_n \; \forall n \in J$.

Recall that sequences and their limits were used to define various notions in calculus (differentiation, interpretation, etc.) Also, calculators use sequences in order to compute with various rational and irrational numbers.

Examples

1. $\boldsymbol{\pi} \simeq 3.1415...$ i.e. instead of π we can work with the following sequence of rational numbers : $x_1 = 3$, $x_2 = 3.1$, $x_3 = 3.14$, $x_4 = 3.141$, $x_5 = 3.1415, ...$ $\lim_{n\to\infty} x_n = \pi$. π is irrational, i.e., $\pi \in \mathbb{R}\backslash\mathbb{Q}$.

2. $\frac{1}{3} \simeq 0.333...$ means we can set up the sequence of rational numbers $x_1 = 0$, $x_2 = 0.3$, $x_3 = 0.33$, $x_4 = 0.333$, $x_5 = 0.3333$ etc. such that $\lim_{n\to\infty} x_n = \frac{1}{3}$. Note that $\frac{1}{3} \in \mathbb{Q}$.

Restatement of the definition of countably infinite: A set A is countably infinite if its elements can be arranged in a sequence $\{x_1, x_2, ...\}$. This is another of saying A is in bijective correspondence with J, i.e $\exists f : A \longrightarrow J$ a bijection, namely $A \sim J$.

Application of the restatement: $\mathbb{Z} \sim \mathbb{N}$

Indeed, we can write $\mathbb{Z}$ as a sequence since $\mathbb{Z} = \{0, 1, -1, 2, -2, ...\}$ so $\mathbb{Z} \in [\mathbb{N}]$, $\mathbb{Z}$ is countably infinite like $\mathbb{N}$.

Big difference between finite and infinite sets: Let A, B be finite sets such that $A \subsetneqq B$, i.e. $A \subset B$ but $A \neq B$. Then $A \nsim B$ since $\#(A) < \#(B)$ and $J_n \nsim J_m$ if $n \neq m$. Let A, B be infinite sets such that $A \subsetneqq B$, $A \subset B$, but $A \neq B$. It is possible that $A \sim B$. We saw this behaviour in Hilbert's hotel problem (Hilbert's Paradox of the Grand Hotel): $\mathbb{N}^* \subsetneqq \mathbb{N}$, but $\mathbb{N} \sim \mathbb{N}^*$ via the bijection $f : \mathbb{N} \longrightarrow \mathbb{N}^*$ given by $f(n) = n + 1$, so $\{0, 1, 2, ...\} \sim \{1, 2, 3, ...\}$.

In the same vein, we get the following result:

Theorem: Every infinite subset of a countably infinite set is itself countably infinite.

Proof: Let $E \subseteq A$ be the subset in question, where E is infinite and A is countably infinite. A is countably infinite $\iff A \sim J \iff A = \{x_1, x_2, ...\}$.

To show E is countably infinite, we want to show we can represent $E = \{x_{n_1}, x_{n_2}, ...\}$. We construct this sequence of n_j's from the indices of the elements of A in the enumeration $\{x_1, x_2, ...\}$ as follows:

Let n_1 be the smallest integer in J such that $x_{n_1} \in E \subseteq A$. We construct the rest of the sequence of n_j's by induction. Say we have constructed $n_1, n_2, ..., n_{k-1} \in \mathbb{N}^*$ Let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$. By construction, $n_1 < n_2 < ...$ and $E = \{x_{n_1}, x_{n_2}, ...\}$

qed

Remark: $\{x_{n_1}, x_{n_2}, ...\}$ is called a subsequence of $\{x_1, x_2, ...\}$

Algorithmic restatement of previous proof:

Let $A = \{x_1, x_2, ...\}$ be an enumeration of A (i.e. writing the countably infinite set A as a sequence). We process $\{x_1, x_2, ...\}$ as a queue. First look at x_1. If $x_1 \in E$, keep x_1 and let $n_1 = 1$; otherwise, discard x_1. Process each x_i in turn, keeping only those that are in E. Their indices form the subsequence $\{n_j\}_{j=1,2,...}$ where $E = \{x_{n_1}, x_{n_2}, x_{n_3}, ...\}$.