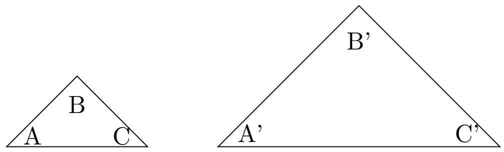


4. Let Γ be the set of all triangles in the plane. $ABC \sim A'B'C'$ if ABC and $A'B'C'$ are similar triangles, **i.e.** have equal angles.
   (a) $\forall ABC \in \Gamma, ABC \sim ABC$ so $\sim$ is reflexive
   (b) $ABC \sim A'B'C' \Rightarrow A'B'C' \sim ABC$ so $\sim$ is symmetric
   (c) $ABC \sim A'B'C'$ and $A'B'C' \sim A"B"C" \Rightarrow ABC \sim A"B"C"$, so $\sim$ is transitive

Clearly (a), (b), (c) use the fact that equality of angles is an equivalence relation.

Exercise: For various predicates you've encountered, check whether reflexive, symmetric or transitive. Examples of predicates include $\neq$, $<$, $>$, $\leq$, $\geq$, $\subseteq$, $\rightarrow$, $\leftrightarrow$

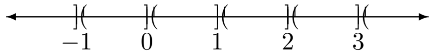
4.2 Equivalence Relations and Partitions

Task: Understand how equivalence relations divide sets.

Definition: Let A be a set. A partition of A is a collection of non-empty sets, any two of which are disjoint such that their union is A, **i.e.** $\lambda = \{A_\alpha \mid \alpha \in I\}$ s.t. $\forall \alpha, \alpha' \in I$ satisfying $\alpha \neq \alpha', A_\alpha \cap A_{\alpha'} = \emptyset$ and $\bigcup_{\alpha \in I} A_\alpha = A$

Here I is an indexing act (may be infinite). $\bigcup_{\alpha \in I} A_\alpha$ is the union of all the A_α's (possibly an infinite union)

Example $\{(n, n+1) \mid n \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$



$\bigcup_{n \in \mathbb{Z}} (n, n+1] = \mathbb{R}$

$(n, n+1] \cap (m, m+1] = \emptyset$ if $n \neq m$

Definition: If R is an equivalence relations on a set A and $x \in A$, the equivalence class of x denoted $[x]_R$ is the set $\{y \mid xRy\}$. The collection of all equivalence classes is called A modulo R and denoted A/R.

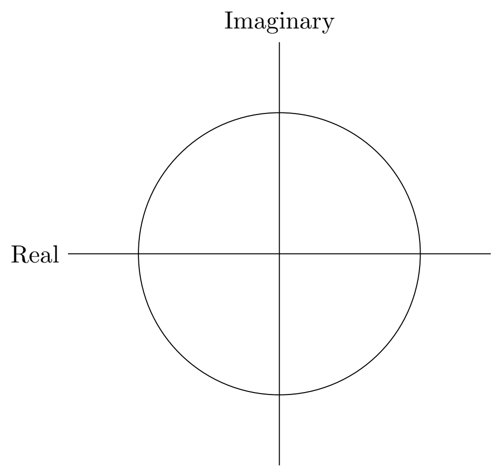
Examples:

1. $A = \mathbb{N} \qquad x \equiv y \bmod 3$
   We have the equivalence classes $[0]_R, [1]_R$ and $[2]_R$ given by the three possible remainders under division by 3.
   $[0]_R = \{0, 3, 6, 9, ...\}$
   $[1]_R = \{1, 4, 7, 10, ...\}$
   $[2]_R = \{2, 5, 8, 11, ...\}$
   Clearly $[0]_R \cup [1]_R \cup [2]_R = \mathbb{N}$ and they are mutually disjoint $\Rightarrow R$ gives a partition of $\mathbb{N}$.

2. $ABC \sim A'B'C'$
   $[ABC] = \{$The set of all triangles with angles of magnitude $\measuredangle ABC, \measuredangle BAC, \measuredangle ACB\}$
   The union over the set of all $[ABC]$ is the set of all triangles and $[ABC] \cap [A'B'C'] = \emptyset$ if $ABC \nsim A'B'C'$ since it means these triangles have at least one angle that is different.

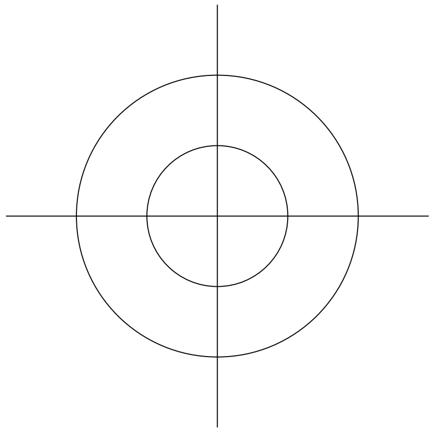
3. $A = \mathbb{C} \qquad x \sim y$ if $|x| = |y| \qquad$ equivalence relation
   $[x] = \{y \in \mathbb{C} \mid |x| = |y|\} = [r]$ for $r \in [0, +\infty)$ (meaning $r \geq 0$)

circle of radius $|x|$



$\bigcup\limits_{r \in [0, +\infty)} [r] = \mathbb{C}$
$[r_1] \cap [r_2] \neq \emptyset$ if $r_1 \neq r_2$ since two distinct circles in $\mathbb{C} \simeq \mathbb{R}^2$ with empty intersection.

circles $r_1 \wedge r_2$



Theorem: For any equivalence relation R on a set A, its equivalence classes form a partition of A, **i.e.**

1. $\forall x \in A, \exists y \in A$ s.t. $x \in [y]$ (every element of A sits somewhere)
2. $xRy \Leftrightarrow [x] = [y]$ (all elements related by R belong to the same equivalence class)
3. $\neg(xRy) \Leftrightarrow [x] \cap [y] = \emptyset$ (if two elements are not related by R, the they belong to disjoint equivalence classes)

Proof:

1. Trivial. Let $y = x$. $x \in [x]$ because R is an equivalence relation, hence reflexive, so xRx holds.
2. We will prove $xRy \Leftrightarrow [x] \subseteq [y]$ and $[y] \subseteq [x]$
   "$\Rightarrow$" Fix $x \in A, [x] = \{z \in A \mid xRz\} \Rightarrow \forall y \in A$ s.t. $xRy, y \in [x]$. Furthermore, $[y] = \{w \in A \mid yRw\}$
   $\Rightarrow \forall w \in [y], yRw$ but $xRy \Rightarrow xRw$ by transitivity. Therefore, $w \in [x]$. We have shown $[y] \subseteq [x]$.
   Since R is an equivalence relation, it is also symmetric. **i.e.** $xRy \Leftrightarrow yRx$. So by the same argument with x and y swapped $yRx \Rightarrow [x] \subseteq [y]$. Thus $xRy \Rightarrow [x] = [y]$.
   "$\Leftarrow$" $[x] = [y] \Rightarrow y \in [x]$ but $[x] = \{y \in A \mid xRy\}$
3. "$\Rightarrow$" We will prove the contrapositive. Assume $[x] \cap [y] \neq \emptyset \Rightarrow \exists z \in [x] \cap [y]$. $z \in [x]$ means xRz, whereas $z \in [y]$ means $yRz \Leftrightarrow zRy$ because R is symmetric. We thus have xRz and $zRy \Rightarrow xRy$ by the transitivity of R. xRy contradicts $\neg(xRy)$ so indeed $\neg(xRy) \Rightarrow [x] \cap [y] = \emptyset$
   "$\Leftarrow$" Once again we use the contrapositive:

Assume $\neg(\neg(xRy)) \Leftrightarrow xRy$. By part (b), $xRy \Rightarrow [x] = [y] \Rightarrow [x] \cap [y] \neq \emptyset$ since $x \in [x]$ and $y \in [y]$, **i.e.** these equivalence classes are non-empty. We have obtained the needed contradiction.

qed

Q: What partition does "=" impose on $\mathbb{R}$?

A: $[x] = \{x\}$ since $E = \{(x, x) \mid x \in \mathbb{R}\}$ the diagonal.
The one-element equivalence class is the smallest equivalence class possible (by definition, an equivalence class cannot be empty as it contains x itself). We call such a partition the <u>finest</u> possible partition.

Remark: The theorem above shows how every equivalence relation partitions a set. It turns out every partition of a set can be used to define an equivalence relation: xRy if x and y belong to the same subset of the partition (check this is indeed an equivalence relation!). Therefore, there is a 1-1 correspondence between partitions and equivalence relations: to each equivalence relation there corresponds a partition and vice versa.