**Theorem:** Let $(V, E)$ be a graph, and let $v_0v_1...v_m$ be a trail in $(V, E)$. Let $v \in V$ be a vertex, then the number of edges of the trail incident to $v$ is even except when the trail is not closed and the trail starts or finishes at $v$, in which case the number of edges of the trail incident to the vertex $v$ is odd.

**Proof:** Note that 0 is an even integer as $0 = 2 \times 0$.

**Case 1:** $v \neq v_0$ and $v \neq v_m$. If the trail does not pass through $v$, the number of edges incident to $v$ belonging to the trail is 0, which is even.
If the trail passes through $v$, then edges of the trail incident to $v$ are of the form $v_{i-1}v_i$ and $v_iv_{i+1}$ with $v = v_i$ and $0 < i < m$. Therefore, the number of edges of the trail incident to $v$ equals twice the number of integers $i$ among $1, 2, ..., m-1 (0 < i < m)$ s.t. $v = v_i \Rightarrow$ the number is even.

**Case 2:** $v = v_0$ and the trail is not closed, **i.e.** $v_m \neq v_0$. The edges incident to $v$ are $v_0v_1$ along with $v_{i-1}v_i$ and $v_iv_{i+1}$ whenever $v = v_i$, hence $1 + 2 \times \#(\text{instances when } v = v_i)$, which is odd.

**Case 3:** $v = v_m$ and the trail is not closed, **i.e.** $v_m \neq v_0$. Repeat the argument in case 2 with $v_{m-1}v_m$ replacing $v_0v_1$ to get that the number of edges incident to $v$ is odd.

**Case 4:** The trail is closed and $v = v_0 = v_m$. The edges incident to $v$ are $v_0v_1, v_{m-1}v_m$ as well as $v_{i-1}v_i$ and $v_iv_{i+1}$ for each $i$ s.t. $v = v_i \Rightarrow$ once again, the number of edges incident to $v$ is even.

**qed**

**Corollary 1:** Let $v$ be a vertex of the graph. Given any circuit in the graph, the number of edges incident to $v$ traversed by that circuit is even.

**Proof:** Apply the theorem to $v_0 v_1 ... v_m$ s.t. $v_0 = v_m$. We deduce that the number of edges incident to $v$ is even.

**Corollary 2:** If a graph admits an Eulerian circuit, then the degree of every vertex of that graph must be even.

**Proof:** Let $(V, E)$ be the graph. $\forall v \in V$, the number of edges of any Eulerian circuit incident to $v$ is even by the previous corollary. Since an Eulerian circuit by definition traverses every edge of the graph, every edge incident to $v$ is an edge of the Eulerian circuit $\Rightarrow$ deg $v$ is even $\forall v \in V$ (**NB:** deg $v$ could be zero if $v$ is an isolated vertex).

**Example:** By the previous corollary, $K_4$, the complete graph on four vertices, cannot have an Eulerian circuit since $\forall v$ in $K_4$, deg $v = 3$ ($K_4$ is 3-regular as we observed in a previous lecture).

**Corollary 3:** If a graph admits an Eulerian trail that is not a circuit, then the degrees of exactly two vertices of the graph must be odd, and the degrees of the remaining vertices must be even. The vertices with odd degrees are exactly the initial and final vertices of the Eulerian trail.

**Proof:** By the theorem, the initial and final vertices of the Eulerian trail have odd degree, whereas all vertices in between have even degrees.

**qed**

Next, prove the converse of corollary 2: A non-trivial connected graph has an Eulerian circuit if the degree of each of its vertices is even. The proof is carried out in a series of lemmas:

**Lemma A:** If the degree of each vertex is even, then $\exists$ circuit.

**Lemma B:** If the degree of each vertex is even, if $\exists$ circuit, and if $\exists$ edges not in the circuit incident to a vertex in the circuit, we can construct another circuit.

**Lemma C:** If we have two circuits with at least one vertex in common, we can combine them.

**Lemma D:** A criterion for when a trail is Eulerian in a connected graph.

**Lemma A:** Let $vw$ be an edge of a graph in which the degree of every vertex is even, then $\exists$ circuit of the graph that traverses the edge $vw$.

**Proof:** We construct the circuit starting with the edge $vw$. Let $v_0 = v$ and $v_1 = w$. Let $v_0v_1...v_k$ be any trail of length $k \geq 1$ traversing the edge $vw$. Suppose $v_k \neq v = v_0$. As we proved in the previous theorem, since $v_k$ is an endpoint of a non-closed trail, then the number of edges of the trail incident to $v_k$ is odd, but deg $v_k$ is even $\Rightarrow \exists$ edge of the graph incident to $v_k$ that is not traversed by the trail $v_0v_1...v_k$. Let $v_kv_{k+1}$ be this edge, then $v_0v_1...v_kv_{k+1}$ is a trail of length $k+1$ that starts at $v$ and traverses $vw$. Since every edge of the graph is traversed at most once by a trail, the length of any trail in the graph cannot be greater than the number of edges of the graph $\#(E)$. We have shown above that if our trail is not closed, then it can be extended. By successive extensions, we will eventually have constructed a trail that cannot be extended (in at most $\#(E) - 1$ steps). Therefore, that trail must be closed. As the edge $vw$ is traversed, this trail is nontrivial $\Rightarrow$ it is a circuit.

**qed**

**Lemma B:** Let $(V, E)$ be a connected graph s.t. $\forall v \in V$, deg $v$ is even, and let some circuit $v_0v_1...v_{m-1}v_0$ be given. Suppose that for some $i$ with $0 \leq i \leq m-1$, some but not all the edges of the graph incident to $v_i$ are traversed by $v_0v_1...v_{m-1}v_0$, then $\exists$ another circuit in $(V, E)$ passing through $v_i$ that does not traverse any edge traversed by $v_0v_1...v_{m-1}v_0$.

**Proof:** Let $E'$ be the set of edges not traversed by $v_0v_1...v_{m-1}v_0$. $(V, E')$ is a subgraph of $(V, E)$. $\forall v \in V$, # of edges of $v_0v_1...v_{m-1}v_0$ incident to $v = d(v) - d'(v)$, where $d(v) = \deg(v) = \#$ of edges in $(V, E)$ incident to $v$ and $d'(v) = \#$ of edges in $(V, E')$ incident to $v$. By Corollary 1, $d(v) - d'(v)$ is even, but by assumption $d(v) = \deg\ v$ is even $\Rightarrow d'(v)$ is even $\Rightarrow$ the degree of every vertex in the subgraph $(V, E')$ is even. Now consider the vertex $v_i$ in the statement of Lemma B. Some but not all edges incident to $v_i$ are traversed by $v_0v_1...v_{m-1}v_0 \Rightarrow d'(v_i) > 0$, **i.e.** at least one edge incident to $v_i$ is in the subgraph $(V, E')$. We are now exactly in the scenario described by Lemma A $\Rightarrow$ by Lemma A, $\exists$ circuit in $(V, E')$ passing through $v_i$. This circuit is also a circuit in $(V, E)$ as $(V, E')$ is a subgraph of $(V, E)$, and since all of its edges are in $E'$, this other circuit does not traverse any edge traversed by $v_0v_1...v_{m-1}v_0$

**qed**

**Lemma C:** Suppose that a graph contains a circuit of length $m$ and a circuit of length $n$. Suppose also that no edges of the graph is traversed by both circuits, and that at least one vertex of the graph is common to both circuits, then the graph contains a circuit of length $m + n$.

**Proof:** Let $v$ be a vertex of the graph that is common to both circuits. WLOG (without loss of generality) we assume both circuits start and finish at the vertex $v$. Let the first circuit be $vv_1...v_{m-1}v$, and let the second circuit be $vw_1w_2...w_{n-1}v$. We concatenate the two circuits obtaining a circuit $vv_1...v_{m-1}vw_1w_2...w_{n-1}v$ of length $m+n$.

**qed**

**Lemma D:** Let $(V,E)$ be a connected graph, and let some trail in this graph be given. Suppose that no vertex of the graph has the property that not all the edges of the graph incident to that vertex are traversed by the trail. Then the given trail is an Eulerian trail.

**Proof:** Let $V_1$ be the set of vertices through which the trail passes, and let $V_2$ be the set of vertices through which the trail does not pass. $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. The conclusion of Lemma D amounts to showing $V_2 = \emptyset$.