We shall also use the one-to-one correspondence with the set of sequences of 0s and 1s in order to prove R is uncountably infinite. The argument proceeds in two steps:

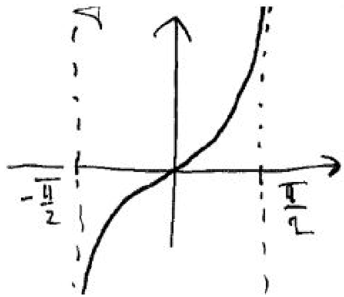
1. We show R ~ (0, 1) via a cleverly chosen bijection.
2. We set up a correspondence between (0, 1) and the set A of all sequences of 0s and 1s via a binary expansion.

Step 1 is the following proposition:

Proposition: R is in bijective correspondence with the interval (0, 1).

Remark:
$(0, 1) \subseteq R$, $(0, 1) \neq R$, but we saw infinite sets can be in one-to-one correspondence with one of their proper subsets.

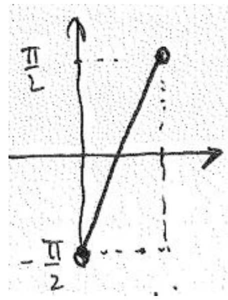
Proof: Recall from trigonometry that $\tan: (-\pi/2, \pi/2) \to R$ is a bijection. Here is the graph:



$x = -\pi/2$, $x = \pi/2$ are asymptotes of the graph.

$\tan x = \sin x / \cos x$
$\cos(-\pi/2) = \cos(\pi/2) = 0$

We now use a linear function, a bijection, to show $(0, 1) \sim (-\pi/2, \pi/2)$
$g(x) = \pi x - \pi/2$. Here is the graph:



The composition of two bijections is itself a bijection $\Rightarrow \tan(g(x)) = \tan(\pi x - \pi/2)$ is a bijection from $(0, 1)$ to R. The map we want $f: R \to (0, 1)$ is its inverse $f(x) = (\tan(\pi x - \pi/2))^{-1}$ as the inverse of a bijection is itself a bijection.

Step 2 is a bit more complicated. To each $x \in (0, 1)$, we want to associate $0.x_1x_2\ldots$ where after the decimal $\{x_1, x_2, \ldots\}$ is a sequence of 0s and 1s. In other words we are giving a binary expansion of every $x \in (0, 1)$ as $0.x_1x_2\ldots = 0 + x_1/2 + x_2/4 + x_3/8 + \ldots =$
$0 + x_1/2 + x_2/2^2 + x_3/2^3 + \ldots =$

$$0 + \sum_{n=1}^{\infty} 1/2^n \,.\, x_n = \sum_{n=1}^{\infty} 1/2^n\, x_n$$

Recall that ½ + ¼ + ⅛ + … = 1. This means that

$$1/2^k \sum_{n=1}^{\infty} 1/2^n \,.\, x_n = 1/2^{k+1} + 1/2^{k+2} + 1/2^{k+3} + \ldots = 1/2^k \quad \forall k \geq 1.$$

Thus 0.100000… and 0.0111111… both represent ½.
Similarly, and $x \in (0, 1)$ that is a sum of the form $1/2^{p1} + 1/2^{p2} + \ldots + 1/2^{pk}$ for $p_1, \ldots, p_k \in N^*$, $p_1 < p_2 < \ldots < p_k$ has two binary representations.

Question: Can we represent $x = 1/2^{p1} + 1/2^{p2} + \ldots + 1/2^{pk}$ in an easier to understand form?

Answer: Yes, we bring the fractions to the same denominator:
$x = 1/2^{p1} + 1/2^{p2} + \ldots + 1/2^{pk}$
$= 2^{pk-p1}/(2^{pk-p1}.2^{p1}) + 2^{pk-p2}/(2^{pk-p2}.2^{p2}) + \ldots + 2^{pk-pk-1}/(2^{pk-pk-1}.2^{pk-1}) + 1/2^{pk}$
$= (2^{pk - p1} + 2^{pk - p2} + \ldots + 2^{pk - pk-1} + 1)/2^{pk}$
= odd natural number / power of 2
$= m/2^n$ for $m \in N$ odd and $n \in N^*$ as $p_1 < p_2 < \ldots < p_k$ so the differences p_k-p_1, p_k-p_2, …, p_k-p_{k-1} are all positive integers. So the sequence in (0, 1) that has two decimal binary expansions is {½, ¼, ¾, ⅛, ⅜, ⅝, ⅞, ...} = B.
Note that B is countably infinite as each set $B_n = \{0 < odd/2^n < 1\}$ is finite, $B = \bigcup_{n=1}^{\infty} B_n$ is countable by our corollary, and the countably infinite sequence {½, ¼, ⅛, …} $\subseteq$ B, which means the countable set B must be countably infinite.