UNIVERSITY OF DUBLIN

XMA2BA11

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

SF Computer Science (B.A.)
SF CSLL
SF ICT

Trinity Term 2006

MATHEMATICS 2BA1

Tuesday, May 23 RDS 9.30 — 12.30

Dr. D. R. Wilkins

Credit will be given for the best 5 questions answered.

Log tables are available from the invigilators, if required.

Students may avail of the HANDBOOK OF MATHEMATICS of Computer Science

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

1. (a) Use the Method of Mathematical Induction to prove that

$$\sum_{k=1}^{n} 7^k k = \frac{7}{36}((6n-1)7^n + 1)$$

for all positive integers n.

(b) Let A, B and C be sets. Prove that

$$A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C).$$

2. (a) What is meant by saying that a relation R on a set A is *reflexive*, *symmetric*, *anti-symmetric*, *transitive*, an *equivalence relation*, or a *partial order*?

(b) Let R denote the relation on the set $\mathbb{Q}^*$ of non-zero rational numbers, where two non-zero rational numbers x and y satisfy the relation xRy if and only if there exists some integer k (whose value depends in general on the choice of x and y) such that $y = 2^k x$, Determine whether or not the relation R is reflexive, symmetric, anti-symmetric or transitive, whether or not it is an equivalence relation, and whether or not it is a partial order. [Justify your answers.]

(c) Let $f\colon A \to B$ be a function from a set A to a set B. What is meant by saying that such a function is *injective*, that such a function is *surjective*, or that such a function is *bijective*?

(d) Let $f\colon [-1,1] \to [3,-3]$ be the function defined by

$$f(x) = x^3 - 4x.$$

Is the function f injective? Is it surjective? Is it invertible?

3. (a) What is an *undirected graph*? What is meant by saying that an undirected graph is *connected*? What is meant by saying that an undirected graph is a *tree*?

(b) Which of the following graphs are connected? and which are trees?

(i) The graph (V, E_1), where $V = \{a, b, c, d\}$ and $E_1 = \{a\,d, b\,d, c\,d,\}$;

(ii) The graph (V, E_2), where $V = \{a, b, c, d\}$ and $E_2 = \{a\,d, a\,b, b\,d, c\,d\}$;

(iii) The graph (V, E_3), where $V = \{a, b, c, d\}$ and $E_3 = \{a\,b, c\,d\}$.

[Justify your answers.]

(c) Prove that if a graph is a tree, then the number $|V|$ of vertices of the graph and the number $|E|$ of edges satisfy the equation $|V| = |E| + 1$. [You may use, without proof, the result that any tree contains at least one pendant vertex.]

4. (a) What is a *semigroup*? What is a *monoid*? What is the *identity element* (or *neutral element*) of a monoid? What is meant by saying that an element of a monoid is *invertible*? What is a *group*?

(b) Let $\otimes$ denote the binary operation on the set $\mathbb{Z}$ of integers defined by $x \otimes y = 3xy + x + y$. Prove that $\mathbb{Z}$, with the binary operation $\otimes$, is a monoid. What is the identity element of this monoid? Is this monoid a group?

(c) Let the quaternions q and r be given by $q = i - 2j$, $r = i + k$. Calculate the products qr and rq. [Hamilton's basic formulae for quaternion multiplication state that

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.]$$

5. (a) Describe the language over the alphabet $\{0, 1\}$ generated by the context-free grammar whose only non-terminal is the start symbol $\langle S \rangle$ and whose productions are

$$\langle S \rangle \to \langle S \rangle \langle S \rangle, \quad \langle S \rangle \to 01.$$

(b) Let L be the language over the alphabet $\{0, 1\}$ consisting of those finite strings of binary digits that begin with the digit 1 and do not contain the substring 000. Construct a regular context-free grammar that generates the language L.

(c) Give the description of a finite state acceptor for the language L of (b), specifying the starting state, the finishing state or states, and the transition table for this finite state acceptor.

[Fully justify your answers.]

6. (a) Let y be a function that satisfies the differential equation

$$\frac{dy}{dx} + ay = 0.$$

Suppose that y is represented as the sum of an infinite series of the form

$$y = \sum_{n=0}^{\infty} \frac{y_n}{n!} x^n.$$

where $y_0, y_1, y_2, \ldots$ are real numbers. Suppose also that $y_0 = 1$. Determine the values of y_n for all non-negative integers n. [You may use the fact that this infinite series may be differentiated term-by-term, so the derivative of the function y is the sum of the derivatives of the individual terms.]

(b) Obtain the general solution of the following ordinary differential equation:—

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^{3x}.$$

7. (a) Let m be a positive integer. What is meant by saying that integers x and y are *congruent modulo* m?

(b) Let m be a fixed positive integer. Prove that the relation of congruence modulo m is an equivalence relation on the set $\mathbb{Z}$ of integers.

(c) What is meant by saying that two integers x and y are *coprime*?

(d) Let m be a positive integer, and let x and y be integers. Suppose that m divides the product xy and that m is coprime to x. Prove that m divides y. [You may use, without proof, the result that if two integers a and b are coprime, then there exist integers u and v such that $ua + vb = 1$.]

(e) Let m be a positive integer, and let x be an integer. Suppose that m and x are coprime. Prove that there exists some integer z such that the product xz is congruent to 1 modulo m (i.e., $xz \equiv 1 \pmod{m}$).

(f) List all integers x satisfying $0 < x < 12$ that are coprime to 12.

8. (a) Let $(z_n : n \in \mathbb{Z})$ be a doubly-infinite sequence

$$\ldots, z_{-3}, z_{-2}, z_{-1}, z_0, z_1, z_2, z_3, \ldots$$

of complex numbers which is m-periodic (so that $z_{k+m} = z_k$ for all integers k). Prove that

$$z_n = \sum_{k=0}^{m-1} c_k \omega_m^{nk},$$

for all integers n, where $\omega_m = e^{2\pi i/m}$ and

$$c_k = \frac{1}{m} \sum_{j=0}^{m-1} z_j \omega_m^{-jk}.$$

You may use, without proof, the result that, for any integer n,

$$\sum_{k=0}^{m-1} \omega_m^{nk} = \begin{cases} m & \text{if } n \text{ is divisible by } m; \\ 0 & \text{if } n \text{ is not divisible by } m. \end{cases}$$

(b) Let $(z_n : n \in \mathbb{Z})$ be the doubly-infinite 3-periodic sequence with $z_0 = 1$, $z_1 = 2$ and $z_2 = 6$. Find values of c_0, c_1 and c_2 such that

$$z_n = c_0 + c_1\omega^n + c_2\omega^{2n}$$

for all integers n, where $\omega = \omega_3 = e^{2\pi i/3}$. (Note that $\omega = \frac{1}{2}(-1 + \sqrt{3}\,i)$, $\omega^2 = e^{-2\pi ik/3} = \frac{1}{2}(-1 - \sqrt{3}\,i)$ and thus $\omega^3 = 1$ and $\omega + \omega^2 = -1$.)