TRINITY COLLEGE DUBLIN

THE UNIVERSITY OF DUBLIN

School of Computer Science and Statistics

SF Integrated Computer Science Programme **Trinity Term 2015**
SF CSLL

MA2C03 — Discrete Mathematics

Tuesday, April 28 **Sports Centre** **09.30 — 12.30**

Dr. D. R. Wilkins

Instructions to Candidates:

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Credit will be given for the best **SIX** questions answered.

Each question is worth 20 marks.

Materials Permitted for this Examination:

Formulae and Tables tables are available from the invigilators, if required.

Students may avail of the *Handbook of Mathematics* of Computer Science

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

1. (a) Let A and B and C be sets. Prove that

$$(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus (A \cup C)).$$

[Venn Diagrams, by themselves without an accompanying logical argument, do not provide sufficient justification to constitute a proof of the result.]

[8 marks]

(b) Let P denote the relation on the set $\mathbb{R}$ of real numbers, where real numbers x and y satisfy xPy if and only if

$$x^3 - y \geq y^3 - x.$$

Determine whether or not the relation P on $\mathbb{R}$ is

(i) *reflexive*,

(ii) *symmetric*,

(iii) *transitive*,

(iv) *anti-symmetric*,

(v) an *equivalence relation*,

(vi) a *partial order*,

[Give appropriate short proofs and/or counterexamples to justify your answers.]

[12 marks]

[A relation R on a set X is an equivalence relation if and only if it is reflexive, symmetric and transitive. It is a partial order if and only if it is reflexive, anti-symmetric and transitive. A relation R on a set X is reflexive if and only if xRx for all $x \in X$; the relation is symmetric if and only if yRx for all $x, y \in X$ satisfying xRy; the relation is transitive if and only if xRz for all $x, y, z \in X$ satisfying xRy and yRz; the relation is anti-symmetric if and only if $x = y$ for all $x, y \in R$ satisfying xRy and yRx.]

2. (a) Let $f\colon A \to B$ be a function from a set A to a set B. What is meant by saying that such a function is *injective*, and that such a function is *surjective*?

[4 marks]

(b) Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be the function from the set $\mathbb{Z}$ of integers to itself defined such that

$$f(x) = \begin{cases} x+1 & \text{if } x \text{ is even;} \\ x-1 & \text{if } x \text{ is odd.} \end{cases}$$

Determine whether or not this function is injective, and whether or not it is surjective, giving brief reasons for your answers.

[6 marks]

(c) Let $*$ denote the binary operation defined on the set $\mathbb{Z}$ of integers, where

$$x * y = 3xy - 5x - 5y + 10$$

for all integers x and y. Prove that $\mathbb{Z}$, with the binary operation $*$, is a monoid. What is the identity element of this monoid? Find all invertible elements of the monoid. Is the monoid a group?

[10 marks]

[A monoid is a set X on which is defined an associative binary operation $*$, where X contains an identity element e satisfying $e * a = a * e = a$ for all $a \in X$. An element a of X is *invertible* if and only if there exists some element b of X for which the equations $a * b = b * a = e$ are satisfied. A *group* is a monoid in which every element is invertible.]

3. (a) Describe the formal language over the alphabet $\{0,1\}$ generated by the context-free grammar whose non-terminals are $\langle S\rangle$ and $\langle A\rangle$, whose start symbol is $\langle S\rangle$ and whose productions are the following:

$$\begin{aligned}\langle S\rangle &\rightarrow 1\langle A\rangle 1\\ \langle A\rangle &\rightarrow \langle S\rangle\\ \langle A\rangle &\rightarrow 0\end{aligned}$$

Is this context-free grammar a regular grammar?

[6 marks]

(b) Give the specification of a finite state acceptor that accepts the language over the alphabet $\{0,1\}$ consisting of all non-empty finite strings of binary digits that begin and end with the digit 1 and which include at least two occurrences of the digit 0. In particular you should specify the set of states, the starting state, the finishing states, and the transition table that determines the transition function of the finite state acceptor.

[8 marks]

(c) Devise a regular context-free grammar to generate the language over the alphabet $\{0,1\}$ described above in (b).

[6 marks]

4. In this question, all graphs are undirected graphs.

(a) Answer the following:

(i) what is meant by saying that a graph is *complete*?

(ii) what is meant by saying that a graph is *regular*?

(iii) what is meant by saying that a graph is *connected*?

(iv) what is an *Eulerian circuit* in a graph?

(v) what is a *Hamiltonian circuit* in a graph?

(vi) what is meant by saying that a graph is a *tree*?

[8 marks]

(b) Let G be the undirected graph whose vertices are a, b, c, d and whose edges are the following:

$$a\,b, \quad a\,c, \quad a\,d, \quad b\,c, \quad b\,d, \quad c\,d.$$

(i) Is this graph complete?

(ii) Is this graph regular?

(iii) Is this graph connected?

(iv) Does this graph have an Eulerian circuit?

(v) Does this graph have a Hamiltonian circuit?

(vi) Is this graph a tree?

[Give brief reasons for each of your answers.]

[12 marks]

5. (a) Any function y of a real variable x that solves the differential equation

$$\frac{d^3y}{dx^3} + 64y = 0$$

may be represented by a power series of the form

$$y = \sum_{n=0}^{+\infty} \frac{y_n}{n!} x^n,$$

where the coefficients $y_0, y_1, y_2, y_3, \ldots$ of this power series are real numbers.

Find values of these coefficients y_n for $n = 0, 1, 2, 3, 4, \ldots$ that yield a solution to the above differential equation with $y_0 = 1$, $y_2 = 16$ and $y_4 = 256$.

[8 marks]

(b) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = xe^{3x}.$$

[12 marks]

6. (a) Let θ be a real number. Express $\cos 3\theta$ by a formula representing this function as a sum of terms, where each term is a product of the form $a \cos^p \theta \sin^q \theta$ for some real coefficient a and some non-negative integers p and q satisfying $p + q = 3$. [N.B., the powers of $\cos\theta$ and $\sin\theta$ of order zero are defined such that $\cos^0\theta = \sin^0\theta = 1$ for all real numbers θ.]

[4 marks]

(b) Let $\omega = e^{2\pi i/3} = \frac{1}{2}(-1 + \sqrt{3}\,i)$, where $i = \sqrt{-1}$. Show that $\omega^2 + \omega + 1 = 0$ and $\omega^3 = 1$.

[4 marks]

(c) Let $(z_n : n \in \mathbb{Z})$ be the doubly-infinite 3-periodic sequence with $z_0 = 1$, $z_1 = 6$ and $z_2 = -1$. Find values of a_0, a_1 and a_2 such that

$$z_n = a_0 + a_1\omega^n + a_2\omega^{2n}$$

for all integers n, where $\omega = e^{2\pi i/3}$.

[12 marks]

7. (a) Find the cosine of the angle between the vectors $(1,5,3)$ and $(1,-1,4)$.

[6 marks]

(b) Find the equation of the plane that contains the points $(1,2,4)$, $(1,1,3)$ and $(1,-1,2)$.

[8 marks]

(c) Let the quaternions q and r be defined as follows:

$$q = 2 - 3k, \quad r = i + 2j - k.$$

Calculate the quaternion products qr and rq. [Hamilton's basic formulae for quaternion multiplication state that

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.]$$

[6 marks]

8. (a) Find an integer x such that $x \equiv 2$ (mod. 3), $x \equiv 1$ (mod. 5) and $x \equiv 3$ (mod. 7).

[12 marks]

(b) Find the value of the unique integer x satisfying $0 \leq x < 13$ for which $5^{80000007} \equiv x$ (mod. 13).

[8 marks]