CS3081 – Computational Mathematics
2018 Exam Solutions

Question 1

> Determining the square root of a number p, $\sqrt{p}$, is the same as finding a solution to the equation $f(x) = x^2 - p = 0$.
> Write a MATLAB user-defined function that determines the square root of a positive number by solving the equation using Newton's method. Name the function $X_S = SquareRoot(p)$. The output argument X_S is the answer and the input argument p is the number whose square root is to be determined. Ensure that you comment your program extensively.
>
> The program should include the following features:
>
> - It should check if the number is positive. If not, the program should stop and display an error message.
> - The starting value of x for the iterations should be p.
> - The iterations should stop when the estimated relative error is smaller than 0.00001.
> - The number of iterations should be limited to 20. If a solution is not obtained in 20 iterations, the program should stop and display an error message.
>
> [50 Marks]

Newtons method of solving the roots to an equation is as follows:

$$f(x) = x^2 - p$$

$$f'(x) = 2x$$

Take an initial estimate for x_1 for the iterations (p) and solve:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

So, in our case our solution to the first iteration would be:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = p - \frac{p^2 - p}{2p}$$

This is repeated for 20 iterations until a solution is found or not found.

The iterations in Newtons method are stopped when the estimated relative error is smaller than a specified value (0.00001).

$$\left|\frac{x_{i+1} - x_i}{x_i}\right| < \varepsilon$$

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res = SquareRoot(3);
disp(res);

function Xs = SquareRoot(p)

  % Check if p is positive
  if p <= 0
    disp('Error: P must be positive');
    return;
  end

  % Initialise Xn to be p
  Xn = p;
  error = 0.00001;

  % Do iterations here
  for i=1:20

    fx = (Xn)^2 - p;
    fxPrime = 2*(Xn);

    % Xn+1 = Xn - f(Xn)/f'(Xn)
    Xn1 = Xn - (fx)/(fxPrime);
    fprintf('Iteration %d: Xn1 = %d\n\n',i,Xn1);

    % Check error value here, if less than specified, end
    if abs((Xn1 - Xn)/Xn) < error
      % If within bounds, solution found, return
      Xs = Xn1;
      fprintf('Soultion found on iteration %d: Xn1 = %d\n\n',i,Xn1);
      return;
    end;

    % Store Xn1 as Xn for next iteration
    Xn = Xn1;
  end

  % If we make it here, no solution in 20 iterations
  fprintf('No solution found');
  Xs = 'No solution';
end
```

Question 2

The power generated by a windmill varies with the wind speed. In an experiment, the following five measurements were obtained:

Wind Speed (Kmph)	14	22	30	38	46
Electric Power (W)	320	490	540	500	480

a) Derive a general expression for the (n-1)th order Lagrange polynomial passing through n points.

[35 Marks]

b) Use the result from part a) to calculate the power at a wind speed of 26 Kmph.

[15 Marks]

a) The n-1th order Lagrange polynomial passing through n points can be defined as:

$$f(x) = \frac{(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)}y_1 + \frac{(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)}y_2 + \cdots$$
$$+ \frac{(x-x_1)(x-x_2)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x-x_1)(x-x_2)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}y_i$$
$$+ \frac{(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{(x-x_1)(x-x_2)\ldots(x-x_{n-1})}y_n$$

The section highlighted in red just highlights the fact that at each iteration of a Lagrange polynomial you exclude the ${x_i}^{th}$ factor.

b) First select three points to use within our Lagrange polynomial:

- $(x_1, y_1) = (14, 320)$
- $(x_2, y_2) = (22, 490)$
- $(x_3, y_3) = (30, 540)$

Then, derive the second order Lagrange polynomial passing through 3 points:

$$f(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}y_3$$

Then, sub in the values from above:

$$f(x) = \frac{(x-22)(x-30)}{(14-22)(14-30)}320 + \frac{(x-14)(x-30)}{(22-14)(22-30)}490 + \frac{(x-14)(x-22)}{(30-14)(30-22)}540$$

Then, using the above equation, solve for f(26):

$$f(26) = \frac{(26-22)(26-30)}{(14-22)(14-30)}320 + \frac{(26-14)(26-30)}{(22-14)(22-30)}490 + \frac{(26-14)(26-22)}{(30-14)(30-22)}540$$

$$f(26) = \frac{(4)(-4)}{(-8)(-6)}320 + \frac{(12)(-4)}{(8)(-8)}490 + \frac{(12)(4)}{(16)(8)}540$$

$$f(26) = \frac{-320}{3} + \frac{735}{2} + \frac{405}{2}$$

$$\boldsymbol{f(26) \approx 463.33}$$

Therefore, using the second order Lagrange polynomial we have estimated that the power produced by the windmill when the wind speed is 26 km/h is approximately 463.33 W. However, from looking at the data points we can see that this is not likely as this is less power than when the wind is at 22 km/h. To improve accuracy we could have used a higher order polynomial.

Question 3

The following data show the number of female and male physicians in the U.S. for various years (American Medical Association):

Year	1980	1990	2000	2002	2003	2006	2008
Number of males	413,395	511,227	618,182	638,182	646,493	665,647	677,807
Number of females	54,284	104,194	195,537	215,005	225,042	256,257	276,417

a) Derive a three-point backward difference formula for the derivative with unequally spaced points and use it to calculate the rate of change of both male and female physicians in 2006.

[20 Marks]

b) Derive a three-point central difference formula for the derivative with unequally spaced points and use it with the result of part a) to calculate (predict) the number of male and female physicians in 2008.

[20 Marks]

c) Compare your answers with the given data and calculate the percentage error in both cases.

a) The three-point backward difference formula for the derivative with unequally spaced points is as follows:

$$f'(x_{i+2}) = \frac{x_{i+2} - x_{i+1}}{(x_i - x_{i+1})(x_i - x_{i+2})}\, y_i + \frac{x_{i+2} - x_i}{(x_{i+1} - x_i)(x_{i+1} - x_{i+2})}\, y_{i+1} + \frac{2x_{i+2} - x_i - x_{i+1}}{(x_{i+2} - x_i)(x_{i+2} - x_{i+1})}\, y_{i+2}$$

We can then use this to calculate the rate of change of **male physicians** in 2006 as follows:

- $(x_1, y_1) = (2002, 638{,}182)$
- $(x_2, y_2) = (2003, 646{,}493)$
- $(x_3, y_3) = (2006, 665{,}647)$

$$f'(x_3) = \frac{x_3 - x_2}{(x_1 - x_2)(x_1 - x_3)}\, y_1 + \frac{x_3 - x_1}{(x_2 - x_1)(x_2 - x_3)}\, y_2 + \frac{2x_3 - x_1 - x_2}{(x_3 - x_1)(x_3 - x_2)}\, y_3$$

$$f'(x_3) = \frac{2006 - 2003}{(2002 - 2003)(2002 - 2006)}\, y_1 + \frac{2006 - 2002}{(2003 - 2002)(2003 - 2006)}\, y_2 + \frac{2(2006) - 2002 - 2003}{(2006 - 2002)(2006 - 2003)}\, y_3$$

$$f'(x_3) = \frac{3}{4}\,(638182) - \frac{4}{3}\,(646493) + \frac{7}{12}\,(665647)$$

$$\boldsymbol{f'(x_3) \approx 4939.916}$$

Therefore, the rate of change of **male physicians** in 2006 is approximately 4939.916.

We can then also use this to calculate the rate of change of **female physicians** in 2006 as follows:

- $(x_1, y_1) = (2002, 215{,}005)$
- $(x_2, y_2) = (2003, 225, 042)$
- $(x_3, y_3) = (2006, 256, 257)$

$$f'(x_3) = \frac{3}{4}\,(215{,}005) - \frac{4}{3}\,(225, 042) + \frac{7}{12}\,(256, 257)$$

$$\boldsymbol{f'(x_3) = 10, 681}$$

Therefore, the rate of change of **female physicians** in 2006 is approximately 10,681.

b) The three-point central difference formula for the derivative with unequally spaced points is as follows:

- $(x_1, y_1) = (2003, 646{,}493)$
- $(x_2, y_2) = (2006, 665{,}647)$
- $(x_3, y_3) = (2008, ?)$

$$f'(x_{i+1}) = \frac{x_{i+1} - x_{i+2}}{(x_i - x_{i+1})(x_i - x_{i+2})}\, y_i + \frac{2x_{i+1} - x_i - x_{i+2}}{(x_{i+1} - x_i)(x_{i+1} - x_{i+2})}\, y_{i+1} + \frac{x_{i+1} - x_i}{(x_{i+2} - x_i)(x_{i+2} - x_{i+1})}\, y_{i+2}$$

Using this, we can predict the number of **male physicians** in 2008 as follows:

$$f'(x_2) = \frac{x_2 - x_3}{(x_1 - x_2)(x_1 - x_3)}\, y_1 + \frac{2x_2 - x_1 - x_3}{(x_2 - x_1)(x_2 - x_3)}\, y_2 + \frac{x_2 - x_1}{(x_3 - x_1)(x_3 - x_2)}\, y_3$$

$$f'(2006) = \frac{2006 - 2008}{(2003 - 2006)(2003 - 2008)}\, y_1 + \frac{2(2006) - 2003 - 2008}{(2006 - 2003)(2006 - 2008)}\, y_2 + \frac{2006 - 2003}{(2008 - x_1)(2008 - 2006)}\, y_3$$

$$f'(2006) = \frac{-2}{(-3)(-5)}\, y_1 + \frac{1}{(3)(-2)}\, y_2 + \frac{3}{(5)(2)}\, y_3$$

Given that we already know f'(2006) from above, we then sub this back in: and solve for y3:

$$4939.916 = \frac{3}{(5)(2)}\, y_3 - \frac{2}{15}\,(646493) - \frac{1}{6}(665{,}647)$$

$$\frac{3}{10}\, y_3 = 4939.916 + \frac{2}{15}\,(646493) + \frac{1}{6}(665{,}647)$$

$$\frac{3}{10}\, y_3 = 202{,}080.23$$

$$\boldsymbol{y_3 = 673{,}600.7}$$

Therefore, the estimated number of **male physicians** in 2008 is approximately 673,601.

Using the above formula, we can also predict the number of **female physicians** in 2008 as follows:

- $(x_1, y_1) = (2003, 225{,}042)$
- $(x_2, y_2) = (2006, 256{,}257)$
- $(x_3, y_3) = (2008, ?)$

$$f'(2006) = \frac{3y_3}{10} - \frac{2y_1}{15} - \frac{y_2}{6}$$

$$10{,}681 = \frac{3y_3}{10} - \frac{2(225{,}042)}{15} - \frac{(256{,}257)}{6}$$

$$\frac{3y_3}{10} = 83396.1$$

$$y_3 = 277{,}987$$

Therefore, the estimated number of **female physicians** in 2008 is approximately 227,987.

c) Using these predictions we can then calculate the error of the prediction using this method by comparing these to the actual numbers from the provided data:

$$Error_{male\_prediction} = \left|1 - \frac{malePrediction}{maleActual}\right|$$

$$Error_{male\_prediction} = \left|1 - \frac{673{,}600.7}{677807}\right|$$

$$Error_{male\_prediction} = 0.0062$$

$$\boldsymbol{Error_{male\_prediction}} = \mathbf{0.62}\%$$

$$Error_{female\_prediction} = \left|1 - \frac{femalePrediction}{femaleActual}\right|$$

$$Error_{female\_prediction} = \left|1 - \frac{277{,}987}{276417}\right|$$

$$Error_{female\_prediction} = 0.0057$$

$$\boldsymbol{Error_{female\_prediction}} = \mathbf{0.57}\%$$