CS3081 – Computational Mathematics
Recommended Questions Solutions

2.2 Apply the intermediate value theorem to show that the function $f(x) = \cos x - x^2$ has a root in the interval $[0, \pi/2]$.

The intermediate value theorem states that if $f(x)$ is continuous on the closed interval [a,b] and M is any number between $f(a)$ and $f(b)$ then there exists a c in [a,b] such that $f(c) = M$.

Consider the initial point of the interval:

$$f(0) = \cos(0) - (0)^2$$

$$= 1$$

Consider the final point of the interval:

$$f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\right)^2$$

$$= -\left(\frac{\pi}{2}\right)^2$$

The value of the function is positive in the initial point and negative at the final point. Thus, the function changes sign at some point during the interval. Therefore there exists a point c such that:

$$f(x) = 0$$

Thus, the function $f(x)$ has a root within the interval specified.

2.8 As a highway patrol officer, you are participating in a speed trap. A car passes your patrol car which you clock at 55 mph. One and a half minutes later, your partner in another patrol car situated two miles away from you, clocks the same car at 50 mph. Using the mean value theorem for derivatives (Eq. (2.4)), show that the car must have exceeded the speed limit of 55 mph at some point during the one and a half minutes it traveled between the two patrol cars.

The mean value theorem states that if a function $f(x)$ is continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b) then there exists a number $c \in [a, b]$ such that :

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The position of the car is denoted as:

$$s = f(x)$$

The distance between the two stations is:

$$f(b) - f(a) = 2\ miles$$

The time travelled is 1.5 minutes so:

$$b - a = \frac{1.5}{60} = 0.025\ hours$$

Plugging this into the above equation:

$$f'(c) = \frac{2}{0.025} = 80mph$$

If $f(x)$ denotes the position of the car, then $f'(x)$ denotes the speed of the car. As $c\ \in [a, b]$ and the speed of the car at point c is 80mph, the car must have exceeded 55mph within the 1.5 minute interval.

2.22 Write the Taylor's series expansion of the function $f(x) = \sin(ax)$ about $x = 0$, where $a \neq 0$ is a known constant.

Consider a function $f(x)$ that is differentiable $n + 1$ times in an interval containing the point $x = x_0$.

The Taylor theorem states that for each x in the interval, there exists a value $x = \xi$ between x and x_0 such that:

$$f(x) = \left[f(x_0) + (x - x_0)\frac{df}{dx}\bigg|_{x=x_0} + \frac{(x - x_0)^2}{2!}\frac{d^2 f}{dx^2}\bigg|_{x=x_0} + \frac{(x - x_0)^3}{3!}\frac{d^3 f}{dx^3}\bigg|_{x=x_0} + \cdots + \frac{(x - x_0)^n}{(n+1)!}\frac{d^n f}{dx^n}\bigg|_{x=x_0} + R_n(x) \right]$$

Where $R_n(x)$ called the remainder and is given by:

$$R_n(x) = \frac{(x - x_0)^{n+1}}{(n+1)!}\frac{d^{n+1} f}{dx^{n+1}}\bigg|_{x=\xi}$$

The Taylor series expansion of the function $f(x) = \sin\ (ax)$ about $x = 0$ (i.e $x_0 = 0$) is therefore as follows:

$$f(x) = \left[\sin\ (ax_0) + (x - x_0)\frac{d\sin\ (ax_0)}{dx}\bigg|_{x=x_0} + \frac{(x - x_0)^2}{2!}\frac{d^2\sin\ (ax_0)}{dx^2}\bigg|_{x=x_0} + \frac{(x - x_0)^3}{3!}\frac{d^3\sin\ (ax_0)}{dx^3}\bigg|_{x=x_0} + \cdots \right]$$

$$f(x) = \left[\sin(ax_0) + (x - x_0)\left[\mathrm{a}\cos(ax_0)|_{x=x_0}\right] + \frac{(x - x_0)^2}{2!}\left[-\mathrm{a}^2\sin(ax_0)|_{x=x_0}\right] + \frac{(x - x_0)^3}{3!}\left[-\mathrm{a}^3\cos(ax_0)|_{x=x_0}\right] + \cdots\right]$$

Substitute $x_0 = 0$ into this function:

$$f(x) = \left[\sin(0) + x[\mathrm{a}\cos(0)] + \frac{(x)^2}{2!}[-\mathrm{a}^2\sin(0)] + \frac{(x)^3}{3!}[-\mathrm{a}^3\cos(0)] + \cdots\right]$$

$$f(x) = 0 + x[\mathrm{a}] + \frac{(x)^3}{3!}[-\mathrm{a}^3] + \cdots$$

Therefore, the Taylor's series expansion of the function $f(x) = \sin(ax)$ is:

$$\boldsymbol{f(x) = \frac{(x)^1}{1!}[\mathbf{a}] - \frac{(x)^3}{3!}[\mathbf{a}^3] + \frac{(x)^5}{5!}[\mathbf{a}^5] + \cdots}$$

2.31 Write a user-defined MATLAB function that calculates the determinant of a square $(n \times n)$ matrix, where n can be 2, 3, or 4. For function name and arguments, use `D = Determinant(A)`. The input argument `A` is the matrix whose determinant is calculated. The function `Determinant` should first check if

```
function D = Determinant(A)

  % Get number of rows and columns
  [rows,columns] = size(A);

  % Verify square matrix & n inRange 2:4
  if rows == columns && (rows >= 2 && rows <= 4)
      if rows == 2
          D = det2x2(A);
      elseif rows == 3
          D = det3x3(A);
      else
          D = det4x4(A);
      end
  else
      D = 'Matrix must be square with 2 =< n <= 4';
  end

end

% 2x2 determinants
function D = det2x2(A)

  D = (A(1,1)*A(2,2)) - (A(1,2)*A(2,1))

end

% 3x3 determinants
function D = det3x3(A)

  aPart = A(1,1)*det2x2(A([2 3], 2:3));
  bPart = A(2,1)*det2x2(A([1 3], 2:3));
  cPart = A(3,1)*det2x2(A([1 2], 2:3));

  D = aPart - bPart + cPart;

end

% 4x4 determinants
function D = det4x4(A)

  aPart = A(1,1)*det3x3(A([2 3 4], 2:4));
  bPart = A(2,1)*det3x3(A([1 3 4], 2:4));
  cPart = A(3,1)*det3x3(A([1 2 4], 2:4));
  dPart = A(4,1)*det3x3(A([1 2 3], 2:4));

  D = aPart - bPart + cPart - dPart;

end
```

2.32 One important application involving the total differential of a function of several variables is estimation of uncertainty.

(*a*) The electrical power P dissipated by a resistance R is related to the voltage V and resistance by $P = V^2/R$. Write the total differential dP in terms of the differentials dV and dR, using Eq. (2.63).

(*b*) dP is interpreted as the uncertainty in the power, dV as the uncertainty in the voltage, and dR as the uncertainty in the resistance. Using the answer of part (*a*), determine the maximum percent uncertainty in the power P for $V = 400$ V with an uncertainty of 2%, and $R = 1000\ \Omega$ with an uncertainty of 3%.

Eq 2.63:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

a) Considering the electric power P dissipated by a resistance R is related to voltage V and resistance by:

$$P = \frac{V^2}{R}$$

We need to write the total differential dP in terms of the differentials dV and dR.

$$P = f(V,R) = \frac{V^2}{R}$$

$$dP = \frac{\partial f}{\partial V}dV + \frac{\partial f}{\partial R}dR$$

$$dP = \frac{\partial\left(\frac{V^2}{R}\right)}{\partial V}dV + \frac{\partial\left(\frac{V^2}{R}\right)}{\partial R}dR$$

$$dP = \frac{1}{R}(2V)dV + V^2\left(-\frac{1}{R^2}\right)dR$$

$$\boldsymbol{dP = \frac{2V}{R}dV - \frac{V^2}{R^2}dR}$$

b) From the question we can infer:

$$dV = 2\% = 0.02$$

$$V = 400V$$

$$dR = 3\% = 0.03$$

$$R = 1000\Omega$$

Using this, we can replace them in the equation above as:

$$dP = \frac{2(400)}{(1000)}(0.02) - \frac{(400)^2}{(1000)^2}0.03$$

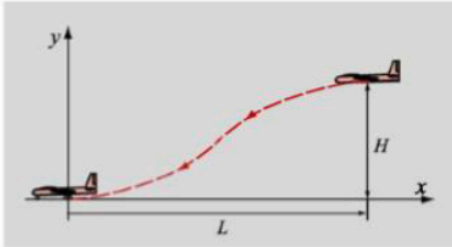
$$\boldsymbol{dP = 0.0112}$$

Therefore, the maximum percentage uncertainty in the power is 1.12%.

2.34 An aircraft begins its descent at a distance $x = L$ ($x = 0$ is the spot at which the plane touches down) and an altitude of H. Suppose a cubic polynomial of the following form is used to describe the landing:

$$y = ax^3 + bx^2 + cx + d$$

where y is the altitude and x is the horizontal distance to the aircraft. The aircraft begins its descent from a level position, and lands at a level position.



(*a*) Solve for the coefficients a, b, c, and d.

(*b*) If the aircraft maintains a constant forward speed ($\frac{dx}{dt} = u = constant$) and the magnitude of the vertical acceleration ($\frac{d^2y}{dt^2}$) is not to exceed a constant A, show that $\frac{6Hu^2}{L^2} \le A$.

(*c*) If $A = 0.3\ \text{ft/s}^2$, $H = 15000$ ft, and $u = 200$ mph, how far from the airport should the pilot begin the descent?

a) The height of the aircraft is described by the function:

$$f(x) = ax^3 + bx^2 + cx + d$$

At the distance $x = 0$, we know that the height of the aircraft is also 0. Therefore:

$$f(0) = 0$$

$$a(0)^3 + b(0)^2 + c(0) + d = 0$$

$$\boldsymbol{d = 0}$$

At the distance $x = 0$, we know that the slope of the aircraft is also 0. Therefore:

$$f'(0) = 0$$

$$\frac{d}{dx}[ax^3 + bx^2 + cx + d]_{x=0} = 0$$

$$[3ax^2 + 2bx + c]_{x=0} = 0$$

$$[3a(0)^2 + 2b(0) + c]_{x=0} = 0$$

$$\boldsymbol{c = 0}$$

At the distance $x = L$, we know that the slope of the aircraft is also 0. Therefore:

$$f'(L) = 0$$

$$\frac{d}{dx}[ax^3 + bx^2 + cx + d]_{x=L} = 0$$

$$[3ax^2 + 2bx + c]_{x=L} = 0$$

$$[3a(L)^2 + 2b(L)]_{x=0} = 0$$

$$3aL^2 + 2bL = 0$$

$$3aL + 2L = 0$$

$$\boldsymbol{a = -\frac{2}{3L}b}$$

At the $x = L$, we know that the height of the aircraft is H. Therefore:

$$f(L) = H$$

$$a(L)^3 + b(L)^2 + c(L) + d = H$$

$$aL^3 + bL^2 + 0 + 0 = H$$

$$\left(-\frac{2}{3L}b\right)L^3 + bL^2 = H$$

$$b\left[\left(-\frac{2}{3L}\right)L^3 + L^2\right] = H$$

$$b\left[\left(-\frac{2}{3}\right)L^2 + L^2\right] = H$$

$$bL^2\left[\left(-\frac{2}{3}\right) + 1\right] = H$$

$$b\frac{L^2}{3} = H$$

$$\boldsymbol{b = \frac{3H}{L^2}}$$

Now, plug this back into the equation for a to get it in terms of just H and L:

$$a = -\frac{2}{3L}b$$

$$a = -\frac{2}{3L}\left(\frac{3H}{L^2}\right)$$

$$\boldsymbol{a = -\frac{2H}{L^3}}$$

b) First, re-write our equation with our newfound values:

$$f(x) = ax^3 + bx^2 + cx + d$$

$$f(x) = \left(-\frac{2H}{L^3}\right)x^3 + \left(\frac{3H}{L^2}\right)x^2$$

Now, differentiate the above equation with respect to t:

$$\frac{dy}{dt} = \frac{d}{dt}\left[\left(-\frac{2H}{L^3}\right)x^3 + \left(\frac{3H}{L^2}\right)x^2\right]$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(-\frac{2H}{L^3}\right)x^3 + \frac{d}{dt}\left(\frac{3H}{L^2}\right)x^2$$

$$\boldsymbol{\frac{dy}{dt} = \left(-\frac{2H}{L^3}\right)3x^2\frac{dx}{dt} + \left(\frac{3H}{L^2}\right)2x\frac{dx}{dt}}$$

Again, differentiating with respect to t to find the acceleration:

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left[\left(-\frac{2H}{L^3}\right)3x^2\frac{dx}{dt} + \left(\frac{3H}{L^2}\right)2x\frac{dx}{dt}\right]$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left[\left(-\frac{2H}{L^3}\right)3x^2\boldsymbol{u} + \left(\frac{3H}{L^2}\right)2x\boldsymbol{u}\right]$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(-\frac{2H}{L^3}\right)3x^2\boldsymbol{u} + \frac{d}{dt}\left(\frac{3H}{L^2}\right)2x\boldsymbol{u}$$

$$\boldsymbol{\frac{d^2y}{dt^2} = \left(-\frac{2H}{L^3}\right)6xu^2 + \left(\frac{3H}{L^2}\right)2u^2}$$

The value of $\frac{d^2y}{dt^2}$ does not exceed a constant A, therefore:

$$\frac{d^2y}{dt^2} \leq A$$

$$\left(-\frac{2H}{L^3}\right)6xu^2 + \left(\frac{3H}{L^2}\right)2u^2 \leq A$$

$$\left[\left(-\frac{2H}{L^3}\right)6x + \left(\frac{3H}{L^2}\right)2\right]u^2 \leq A$$

$$\left[\left(-\frac{12H}{L^3}\right)x + \left(\frac{6H}{L^2}\right)\right]u^2 \leq A$$

$$\left[\left(-\frac{2}{L^3}\right)x + \left(\frac{1}{L^2}\right)\right]6Hu^2 \leq A$$

$$\left[\left(-\frac{2}{L}\right)x + 1\right]\frac{6Hu^2}{L^2} \leq A$$

$$\left[1 - \frac{2}{L}x\right]\frac{6Hu^2}{L^2} \leq A$$

Since $\left[1 - \frac{2}{L}x\right] \leq 1$ we can therefore infer that $\frac{6Hu^2}{L^2} \leq A$.

c) Given the following, calculate how far (x) the aircraft should begin its descent:

$$A = 0.3\ ft/s^2$$

$$H = 15{,}000\ ft$$

$$u = 200mph$$

Consider the above relation:

$$\frac{6Hu^2}{L^2} \leq A$$

Consider it in the form of:

$$\frac{6Hu^2}{L^2} = A$$

We can then re-arrange this in terms of L:

$$L = \sqrt{\frac{6Hu^2}{A}}$$

$$L = \sqrt{\frac{6(15000)(200)^2}{(0.3)}}$$

$$L = 1.6067 \times 10^5 \ ft$$

Therefore, the pilot should begin his descent from a distance of $1.6067 \times 10^5 \ ft$ from the airport.

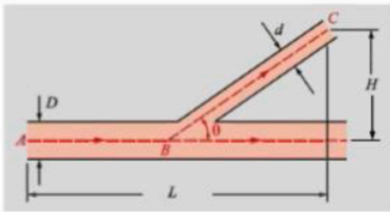
2.36 An artery that branches from another more major artery has a resistance for blood flow that is given by:

$$R_{flow} = K\left(\frac{L - H\cot\theta}{D^4} + \frac{H\csc\theta}{d^4}\right)$$

where R_{flow} is the resistance to blood flow from the major to the branching artery along path ABC (see diagram), d is the diameter of the smaller, branching artery, D is the diameter of the major artery, θ is the angle that the branching vessel makes with the horizontal, or axis, of the major artery, and L and H are the distances shown in the figure.

(*a*) Find the angle θ that minimizes the flow resistance in terms of d and D.

(*b*) If $\theta = 45°$, and $D = 5$ mm, what is the value of d that minimizes the resistance to blood flow?



a) The resistance to blood flow is defined under the equation:

$$R_{flow} = K\left(\frac{L - Hcot\theta}{D^4} + \frac{Hcsc\theta}{d^4}\right)$$

We need to find the angle θ that minimises the R_{flow} in terms of D and d. The derivative of R_{flow} with respect to the angle θ is:

$$\frac{d}{d\theta}R_{flow} = \frac{d}{d\theta}K\left(\frac{L - Hcot\theta}{D^4} + \frac{Hcsc\theta}{d^4}\right)$$

$$= \frac{d}{d\theta}K\left(\frac{L}{D^4} - \frac{Hcot\theta}{D^4} + \frac{Hcsc\theta}{d^4}\right)$$

$$= K\left(-\frac{H(-csc^2\theta)}{D^4} + \frac{H(-csc\theta(cot\theta)}{d^4}\right)$$

$$= K\left(\frac{H(csc^2\theta)}{D^4} - \frac{H(csc\theta(cot\theta)}{d^4}\right)$$

Now, equating this with 0 we have:

$$\frac{d}{d\theta}R_{flow} = 0$$

$$K\left(\frac{H(csc^2\theta)}{D^4} - \frac{H(csc\theta(cot\theta))}{d^4}\right) = 0$$

$$\frac{H(csc^2\theta)}{D^4} - \frac{H(csc\theta(cot\theta))}{d^4} = 0$$

$$\frac{H(csc^2\theta)}{D^4} = \frac{H(csc\theta(cot\theta))}{d^4}$$

$$\frac{(csc^2\theta)}{D^4} = \frac{csc\theta(cot\theta)}{d^4}$$

$$\frac{(csc^2\theta)}{csc\theta(cot\theta)} = \frac{D^4}{d^4}$$

$$\frac{csc\theta}{cot\theta} = \frac{D^4}{d^4}$$

$$\frac{\frac{1}{sin\theta}}{\frac{cos\theta}{sin\theta}} = \frac{D^4}{d^4}$$

$$\left(\frac{1}{sin\theta}\right)\left(\frac{sin\theta}{cos\theta}\right) = \frac{D^4}{d^4}$$

$$sec\theta = \frac{D^4}{d^4}$$

$$cos\theta = \frac{d^4}{D^4}$$

$$\boldsymbol{\theta = cos^{-1}\left(\frac{d^4}{D^4}\right)}$$

Now, find the second derivative at the point $\boldsymbol{\theta}$ as above:

$$\frac{d}{d\theta}\left(\frac{d}{d\theta}R_{flow}\right) = \frac{d}{d\theta}\left[K\left(\frac{H(csc^2\theta)}{D^4} - \frac{H(csc\theta(cot\theta))}{d^4}\right)\right]$$

$$= \frac{KH}{D^4}\frac{d}{d\theta}(csc^2\theta) - \frac{KH}{d^4}\frac{d}{d\theta}\left(csc\theta(cot\theta)\right)$$

$$= \frac{KH}{D^4}\left[-2\csc(\theta)^2\cot(\theta)\right]\frac{KH}{d^4}\left[-\csc(\theta)\,cot^2(\theta) + \csc(\theta)\,(-1 - cot^2(\theta))\right]$$

$$> 0$$

Therefore when

$$\theta = cos^{-1}\left(\frac{d^4}{D^4}\right)$$

the flow of resistance is at a minimum.

b) We need to find the value of d that minimises the resistance when:

$$\theta = 45°$$

$$D = 5mm$$

Re-write the equation for θ in terms of d:

$$\theta = cos^{-1}\left(\frac{d^4}{D^4}\right)$$

$$cos\theta = \left(\frac{d^4}{D^4}\right)$$

$$\cos\left(\frac{\pi}{4}\right) = \left(\frac{d^4}{(5)^4}\right)$$

$$0.7071 = \frac{d^4}{(5)^4}$$

$$d^4 = \frac{0.7071}{(5)^4}$$

$$d = \sqrt[4]{\frac{0.7071}{(5)^4}}$$

$$d = 4.580mm$$

Therefore, for $d = 4.580mm$ the resistance to blood flow is at a minimum.

3.2 Determine the root of $f(x) = x - 2e^{-x}$ by:

(a) Using the bisection method. Start with $a = 0$ and $b = 1$, and carry out the first three iterations.

(b) Using the secant method. Start with the two points, $x_1 = 0$ and $x_2 = 1$, and carry out the first three iterations.

(c) Using Newton's method. Start at $x_1 = 1$ and carry out the first three iterations.

a) Starting with the two points a and b calculate f(a) and f(b)

$$f(x) = x - 2e^{-x}$$

$$f(a) = a - 2e^{-a}$$

$$f(a) = 0 - 2e^{-0}$$

$$\boldsymbol{f(a) = -2}$$

$$f(b) = b - 2e^{-b}$$

$$f(b) = 1 - 2e^{-1}$$

$$\boldsymbol{f(b) = 0.2642}$$

Since the values of f(a) and f(b) are opposite signs, a solution lies within the range [0,1]. Calculate the average value of 0 and 1:

$$c = \frac{0+1}{2} = 0.5$$

Therefore, the average value of 0 and 1 is 0.5. Now, calculate f(c):

$$f(c) = 0.5 - 2e^{-0.5}$$

$$\boldsymbol{f(c) = -0.7131}$$

Since the values of f(b) and f(c) are of opposite signs, the root lies between b and c. Calculate the average of b and c:

$$d = \frac{1+0.5}{2} = 0.75$$

Therefore, the average value of 1 and 0.5 is 0.75. Now, calculate f(d):

$$f(d) = 0.75 - 2e^{-0.75}$$

$$\boldsymbol{f(d) = -0.1948}$$

Since the value of f(b) and f(d) are of opposite signs, a solution lies within the range of b and d. Calculate the average of these two values:

$$e = \frac{1 + 0.75}{2} = 0.875$$

Therefore, the average value of 1 and 0.75 is 0.875. Now, calculate f(e):

$$f(e) = 0.875 - 2e^{-0.875}$$

$$\boldsymbol{f(e) = 0.0412}$$

b) In the secant method the following formula is used:

$$f(x_{i+1}) = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

So for our given example we start off with $x_1 = 0$ and $x_2 = 1$:

$$f(x_1) = 0 - 2e^{-0}$$

$$\boldsymbol{f(x_1) = -2}$$

$$f(x_2) = 1 - 2e^{-1}$$

$$\boldsymbol{f(x_2) = 0.2642}$$

We can now solve for $\boldsymbol{x_3}$ as follows:

$$f(x_3) = x_2 - \frac{f(x_2)(x_1 - x_2)}{f(x_1) - f(x_2)}$$

$$f(x_3) = 1 - \frac{0.2642(0 - 1)}{-2 - 0.2642}$$

$$\boldsymbol{x_3 = 0.8833}$$

We can calculate the absolute error as:

$$\epsilon = \left|\frac{x_3 - x_2}{x_3}\right| * 100\%$$

$$\epsilon = \left|\frac{0.8833 - 1}{0.8833}\right| * 100\%$$

$$\epsilon = 13.21\%$$

We can then calculate x_4 using the above equation and continue the iterations.

3.14 Solve the following system of nonlinear equations:

$$x^2 + 2x + 2y^2 - 26 = 0$$
$$2x^3 - y^2 + 4y - 19 = 0$$

(*a*) Use Newton's method. Start at $x = 1,\ y = 1$, and carry out the first five iterations.
(*b*) Use the fixed-point iteration method. Start at $x = 1,\ y = 1$, and carry out the first five iterations.

a) First, represent the equations as a function of x and y:

$$f_1(x, y) = x^2 + 2x + 2y^2 - 26$$

$$f_2(x, y) = 2x^3 - y^2 + 4y - 19$$

Partially differentiate f_1 and f_2 with respect to x and y:

$$\frac{\partial f_1}{\partial x} = 2x + 2$$

$$\frac{\partial f_1}{\partial y} = 4y$$

$$\frac{\partial f_2}{\partial x} = 6x^2$$

$$\frac{\partial f_2}{\partial y} = -2y + 4$$

Write the Jacobian matrix:

$$J(f_1, f_2) = det\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

Substitute in the results from above:

$$J(f_1, f_2) = det\begin{bmatrix} 2x + 2 & 4y \\ 6x^2 & -2y + 4 \end{bmatrix}$$

$$= (2x + 2)(-2y + 4) - (6x^2)(4y)$$

$$= -4xy - 4y + 8x + 8 - 24xy^2$$

$$= -24xy^2 - 4xy + 8x - 4y + 8$$

Write the deltas:

$$\Delta x = \frac{-f_1(x_i, y_i) * \left.\frac{\partial f_2}{\partial y}\right|_{x_i, y_i} + f_2(x_i, y_i) * \left.\frac{\partial f_1}{\partial y}\right|_{x_i, y_i}}{J(f_1, f_2)}$$

$$\Delta y = \frac{-f_2(x_i, y_i) * \left.\frac{\partial f_1}{\partial x}\right|_{x_i, y_i} + f_1(x_i, y_i) * \left.\frac{\partial f_2}{\partial x}\right|_{x_i, y_i}}{J(f_1, f_2)}$$

Where for each iteration:

$$x_{i+1} = x_i + \Delta x$$

$$y_{i+1} = y_i + \Delta y$$

For the first iteration w/ x=1, y=1:

$$\Delta x = \frac{-(-21) * 2 + (-14) * 4}{-16} = 0.875$$

$$\Delta y = \frac{-(-14) * 4 + (-21) * 6}{-16} = 4.375$$

$$\boldsymbol{x_2 = 1 + 0.875 = 1.875}$$

$$\boldsymbol{y_2 = 1 + 4.375 = 5.375}$$

For the second iteration w/ x=1.875, y=5.375:

… Repeat steps as above until error $\boldsymbol{(x_2 - x_1) < 0.001}$

b) First, we must re-arrange the equation in terms of x and y:

$$x^2 + 2x + 2y^2 - 26 = 0 \ldots (1)$$

$$2x^3 - y^2 + 4y - 19 = 0 \ldots (2)$$

Re-arranging equation (1):

$$2y^2 = 26 - x^2 - 2x$$

$$y^2 = \frac{26 - x^2 - 2x}{2}$$

$$\boldsymbol{y = \sqrt{\frac{26 - x^2 - 2x}{2}}}$$

Re-arranging equation (2):

$$2x^3 - y^2 + 4y - 19 = 0$$

$$2x^3 = y^2 - 4y + 19$$

$$x^3 = \frac{y^2 - 4y + 19}{2}$$

$$\boldsymbol{x = \sqrt[3]{\frac{y^2 - 4y + 19}{2}}}$$

For the first iteration w/ $\boldsymbol{x = 1 , y = 1}$:

$$\boldsymbol{y = \sqrt{\frac{26 - (1)^2 - 2(1)}{2}} = 3.3911}$$

$$\boldsymbol{x = \sqrt[3]{\frac{(1)^2 - 4(1) + 19}{2}} = 2}$$

For the second iteration w/ $\boldsymbol{x = 2 , y = 3.3911}$:

… Repeat steps as above until error $(\boldsymbol{x_2 - x_1}) < \boldsymbol{0.001}$

4.2 Given the system of equations $[a][x] = [b]$, where $a = \begin{bmatrix} 2 & -2 & 1 \\ 3 & 2 & -5 \\ -1 & 2 & 3 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $b = \begin{bmatrix} 10 \\ -16 \\ 8 \end{bmatrix}$, determine the solution using the Gauss elimination method.

First, substitute the given matrices in $[a] * [x] = [b]$ form:

$$\begin{bmatrix} 2 & -2 & 1 \\ 3 & 2 & -5 \\ -1 & 2 & 3 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -16 \\ 8 \end{bmatrix} \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 \end{array}$$

We then want to eliminate the all of the bottom row ($R_3's$) elements besides the 3 so that we can get a value for x3. Therefore perform the following row operations on matrices $[a]$ and $[b]$:

- $R_2 = 2(R_2)$ - 3(R_1)
- $R_3 = 2(R_3)$ + R_1

$$\begin{array}{r} \rightarrow R_1 \\ \rightarrow \mathbf{2(R_2) - 3(R_1)} \\ \rightarrow \mathbf{2(R_3) - R_1} \end{array} \begin{bmatrix} 2 & -2 & 1 \\ 0 & 10 & -13 \\ 0 & 2 & 7 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -62 \\ 26 \end{bmatrix}$$

- $R_3 = 5(R_3)$ - R_2

$$\begin{array}{r} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow \mathbf{5(R_3) - R_2} \end{array} \begin{bmatrix} 2 & -2 & 1 \\ 0 & 10 & -13 \\ 0 & 0 & 48 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -62 \\ 196 \end{bmatrix}$$

From this we can now solve for x_3:

$$48x_3 = 196$$

$$\boldsymbol{x_3 = 4}$$

We can then sub this into row 2 and solve for x_2:

$$10x_2 - 13x_3 = -63$$

$$10x_2 - 13(4) = -63$$

$$\boldsymbol{x_2 = -1}$$

And then solve for x_1 using row 1:

$$2x_1 - 2x_2 + 1x_3 = 10$$

$$2x_1 - 2(-1) + 1(4) = 10$$

$$\boldsymbol{x_1 = 2}$$

4.13 Find the inverse of the matrix $\begin{bmatrix} 10 & 12 & 0 \\ 0 & 2 & 8 \\ 2 & 4 & 8 \end{bmatrix}$ using the Gauss–Jordan method.

First write out the matrix and the augmented matrix:

$$\begin{bmatrix} 10 & 12 & 0 \\ 0 & 2 & 8 \\ 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The aim of the game here is to reduce the LHS matrix to the RHS matrix by performing row operations:

- $R_1 = R_1/10$

$$\begin{matrix} \to R_1/10 \\ \to R_2 \\ \to R_3 \end{matrix} \begin{bmatrix} 1 & 1.2 & 0 \\ 0 & 2 & 8 \\ 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $R_3 = R_3 - 2(R_1)$

$$\begin{matrix} \to R_1 \\ \to R_2 \\ \to R_3 - 2(R_1) \end{matrix} \begin{bmatrix} 1 & 1.2 & 0 \\ 0 & 2 & 8 \\ 0 & 1.6 & 8 \end{bmatrix} \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.2 & 0 & 1 \end{bmatrix}$$

- $R_2 = R_2/2$

$$\begin{matrix} \to R_1 \\ \to R_2/2 \\ \to R_3 \end{matrix} \begin{bmatrix} 1 & 1.2 & 0 \\ 0 & 1 & 4 \\ 0 & 1.6 & 8 \end{bmatrix} \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.5 & 0 \\ -0.2 & 0 & 1 \end{bmatrix}$$

- $R_1 = R_1 - 1.2(R_2)$

$$\begin{matrix} \to R_1 - 1.2(R_2) \\ \to R_2 \\ \to R_3 \end{matrix} \begin{bmatrix} 1 & 0 & -4.8 \\ 0 & 1 & 4 \\ 0 & 1.6 & 8 \end{bmatrix} \begin{bmatrix} 0.1 & -0.6 & 0 \\ 0 & 0.5 & 0 \\ -0.2 & 0 & 1 \end{bmatrix}$$

- $R_3 = R_3 - 1.6(R_2)$

$$\begin{matrix} \to R_1 \\ \to R_2 \\ \to R_3 - 1.6(R_2) \end{matrix} \begin{bmatrix} 1 & 0 & -4.8 \\ 0 & 1 & 4 \\ 0 & 0 & 1.6 \end{bmatrix} \begin{bmatrix} 0.1 & -0.6 & 0 \\ 0 & 0.5 & 0 \\ -0.2 & -0.8 & 1 \end{bmatrix}$$

- $R_3 = R_3/1.6$

$$\begin{array}{r} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3/1.6 \end{array} \begin{bmatrix} 1 & 0 & -4.8 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1 & -0.6 & 0 \\ 0 & 0.5 & 0 \\ -0.125 & -0.5 & 0.625 \end{bmatrix}$$

- $R_1 = R_1 + 4.8(R_3)$
- $R_2 = R_2 + 4(R_3)$

$$\begin{array}{r} \rightarrow R_1 + 4.8(R_3) \\ \rightarrow R_2 + 4(R_3) \\ \rightarrow R_3 \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -0.5 & -3 & 3 \\ 0.5 & 2.5 & -2.5 \\ -0.125 & -0.5 & 0.625 \end{bmatrix}$$

4.19 **Find the condition number of the matrix in Problem 4.13 using the 1-norm.**

$$a = \begin{bmatrix} 10 & 12 & 0 \\ 0 & 2 & 8 \\ 2 & 4 & 8 \end{bmatrix}$$

We know from 4.13 that the inverse of a is:

$$a^{-1} = \begin{bmatrix} -0.5 & -3 & 3 \\ 0.5 & 2.5 & -2.5 \\ -0.125 & -0.5 & 0.625 \end{bmatrix}$$

Consider the formula for the condition number of a matrix:

$$cond[a] = \|[a]\|\|[a]^{-1}\| \ldots$$

Where:

- $\|[a]\|$ denotes the norm of matrix a
- $\|[a]^{-1}\|$ denotes the norm of inverse of matrix a

Write the formula to find the 1-norm of a matrix:

$$\|[a]\|_1 = \max_{1 \le j \le n} \sum_{j=1}^{n} |a_{ij}|$$

Consider the matrix a as stated above and 3 for n:

$$\|[a]\|_1 = \max_{1 \le j \le 3} \sum_{j=1}^{3} |a_{ij}|$$

$$\|[a]\|_1 = \max_{1 \le j \le 3} [(|10| + |0| + |2|), (|12| + |2| + |4|), (|0| + |8| + |8|)]$$

$$\|[a]\|_1 = \max_{1 \le j \le 3} [12, 18, 16]$$

$$\boldsymbol{\|[a]\|_1 = 18}$$

Therefore, the 1-norm of matrix a is 18.

4.25 Write a user-defined MATLAB function that calculates the 1-norm of any matrix. For the function name and arguments use `N = OneNorm(A)`, where `A` is the matrix and `N` is the value of the norm. Use the function for calculating the 1-norm of:

(*a*) The matrix $A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1.5 \end{bmatrix}$. (*b*) The matrix $B = \begin{bmatrix} 4 & -1 & 0 & 1 & 0 \\ -1 & 4 & -1 & 0 & 1 \\ 0 & -1 & 4 & -1 & 0 \\ 1 & 0 & -1 & 4 & -1 \\ 0 & 1 & 0 & -1 & 4 \end{bmatrix}$.

```
function N = OneNorm(A)

  % Find the row and column dimensions
  [row, cols] = size(A);

  % Iterate over columns
  for j=1:cols
    % Calculate sum of columns and store in array
    column_sums(j) = sum(abs(A(:,j)));
  end;

  % Initialise max
  max = column_sums(1);

  % Find the largest value amongst the sums
  for j=1:cols-1
    if column_sums(j+1) > max
      max = column_sums(j+1);
    end
  end

  % Return max
  N = max;

end
```

5.3 Find the eigenvalues of the following matrix by solving for the roots of the characteristic equation.

$$\begin{bmatrix} 10 & 0 & 0 \\ 1 & -3 & -7 \\ 0 & 2 & 6 \end{bmatrix}$$

The expression for the characteristic equation to find the Eigen values is:

$$det[a - \lambda I] = 0$$

$$|a - \lambda I| = 0$$

As the matrix a is of order 3x3, the order of the identity matrix is also 3x3:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Calculate the matrix $a - \lambda I$:

$$a - \lambda I = \begin{bmatrix} 10 & 0 & 0 \\ 1 & -3 & -7 \\ 0 & 2 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$a - \lambda I = \begin{bmatrix} 10 - \lambda & 0 & 0 \\ 1 & -3 - \lambda & -7 \\ 0 & 2 & 6 - \lambda \end{bmatrix}$$

Using this calculate the characteristic equation $|a - \lambda I| = 0$:

$$|a - \lambda I| = 0$$

$$\begin{vmatrix} 10 - \lambda & 0 & 0 \\ 1 & -3 - \lambda & -7 \\ 0 & 2 & 6 - \lambda \end{vmatrix} = 0$$

$$10 - \lambda[(-3 - \lambda)(6 - \lambda) - (-7)(2)] - 0 + 0 = 0$$

$$10 - \lambda[3\lambda - 6\lambda + \lambda^2 - 18 + 14] = 0$$

$$(10 - \lambda)(\lambda^2 - 3\lambda - 4) = 0$$

$$(10 - \lambda)(\lambda + 1)(\lambda - 4) = 0$$

$$\boldsymbol{\lambda = 10, \lambda = 4, \lambda = -1}$$

Therefore, the Eigen values of the matrix are 10, 4 and -1

5.7 Apply the power method to find the largest eigenvalue of the matrix from Problem 5.2 starting with the vector $[1\ 1\ 1]^T$.

$$a = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

For $i = 1$, the Eigen vector $[x]_1 = [1 \quad 1 \quad 1]^T$ that is:

$$[x]_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Calculate the next Eigen vector $[x]_2$ for $i = 2$

$$[x]_2 = a[x]_1$$

$$[x]_2 = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$[x]_2 = \begin{bmatrix} 1+1+2 \\ 4+1+5 \\ 0+0+1 \end{bmatrix}$$

$$[x]_2 = \begin{bmatrix} 4 \\ 10 \\ 1 \end{bmatrix}$$

We must then extract the highest element from the vector and normalise the vector:

$$[x]_2 = \begin{bmatrix} 4 \\ \mathbf{10} \\ 1 \end{bmatrix}$$

$$[x]_2 = 10 \begin{bmatrix} 4/10 \\ \mathbf{10/10} \\ 1/10 \end{bmatrix}$$

$$[x]_2 = 10 \begin{bmatrix} 0.4 \\ 1 \\ 0.1 \end{bmatrix}$$

Therefore, the normalized unit vector of $[x]_2$ is $\begin{bmatrix} 0.4 \\ 1 \\ 0.1 \end{bmatrix}$

We then repeat these steps, for the next Eigen vector $[x]_3$ for $i = 3$

$$[x]_3 = a[x]_2$$

$$[x]_3 = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.4 \\ 1 \\ 0.1 \end{bmatrix}$$

$$[x]_3 = \begin{bmatrix} 0.4 + 1 + 0.2 \\ 1.6 + 1 + 0.5 \\ 0 + 0 + 0.1 \end{bmatrix}$$

$$[x]_3 = \begin{bmatrix} 1.6 \\ 3.1 \\ 0.1 \end{bmatrix}$$

We must then extract the highest element from the vector and normalise the vector:

$$[x]_3 = \begin{bmatrix} 1.6 \\ \mathbf{3.1} \\ 0.1 \end{bmatrix}$$

$$[x]_3 = 3.1 \begin{bmatrix} 1.6/\mathbf{3.1} \\ 3.1/\mathbf{3.1} \\ 0.1/\mathbf{3.1} \end{bmatrix}$$

$$[x]_3 = 3.1 \begin{bmatrix} 0.516 \\ 1 \\ 0.0323 \end{bmatrix}$$

Therefore, the normalized unit vector of $[x]_3$ is $\begin{bmatrix} 0.516 \\ 1 \\ 0.0323 \end{bmatrix}$

We then repeat these steps, for the next Eigen vector $[x]_4$ for $i = 4$

$$[x]_4 = a[x]_3$$

$$[x]_4 = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.516 \\ 1 \\ 0.0323 \end{bmatrix}$$

$$[x]_4 = \begin{bmatrix} 0.516 + 1 + 0.0646 \\ 2.064 + 1 + 0.1615 \\ 0 + 0 + 0.0323 \end{bmatrix}$$

$$[x]_4 = \begin{bmatrix} 1.58 \\ 3.225 \\ 0.0323 \end{bmatrix}$$

We must then extract the highest element from the vector and normalise the vector:

$$[x]_4 = \begin{bmatrix} 1.58 \\ 3.225 \\ 0.0323 \end{bmatrix}$$

$$[x]_4 = 3.225 \begin{bmatrix} 1.58/3.225 \\ 3.225/3.225 \\ 0.0323/3.225 \end{bmatrix}$$

$$[x]_4 = 3.225 \begin{bmatrix} 0.49 \\ 1 \\ 0.01 \end{bmatrix}$$

Therefore, the normalized unit vector of $[x]_4$ is $\begin{bmatrix} 0.49 \\ 1 \\ 0.01 \end{bmatrix}$

This is repeated and on our 6th iteration we get:

$$[x]_6 = \begin{bmatrix} 1.5066 \\ 3.0165 \\ 0.0033 \end{bmatrix}$$

$$[x]_6 = 3.0165 \begin{bmatrix} 1.5066/3.0165 \\ 3.0165/3.0165 \\ 0.0033/3.0165 \end{bmatrix}$$

$$[x]_6 = 3.0165 \begin{bmatrix} 0.5 \\ 1 \\ 0.001 \end{bmatrix}$$

Since the difference between the multiplicative factors (was 3.01 for $[x]_5$) is relatively small we can terminate our iterations here at $[x]_6$.

We can also say that the largest Eigenvalue for the given matrix a is:

3.0165

5.9 Apply the inverse power method to find the smallest eigenvalue of the matrix from Problem 5.3 starting with the vector $[1\ 1\ 1]^T$. The inverse of the matrix in Problem 5.3 is:

$$\begin{bmatrix} 0.1 & 0 & 0 \\ 0.15 & -1.5 & -1.75 \\ -0.05 & 0.5 & 0.75 \end{bmatrix}$$

$$a = \begin{bmatrix} 10 & 0 & 0 \\ 1 & -3 & -7 \\ 0 & 2 & 6 \end{bmatrix}$$

$$a^{-1} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.15 & -1.5 & -1.75 \\ -0.05 & 0.5 & 0.75 \end{bmatrix}$$

For $i = 1$, the Eigen vector $[x]_1 = [1 \quad 1 \quad 1]^T$ that is:

$$[x]_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Calculate the next Eigen vector $[x]_2$ for $i = 2$

$$[x]_2 = a^{-1}[x]_1$$

$$[x]_2 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.15 & -1.5 & -1.75 \\ -0.05 & 0.5 & 0.75 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$[x]_2 = \begin{bmatrix} 0.1 \\ -3.1 \\ 1.2 \end{bmatrix}$$

We must then extract the highest element from the vector and normalise the vector:

$$[x]_2 = \begin{bmatrix} 0.1 \\ -3.1 \\ 1.2 \end{bmatrix}$$

$$[x]_2 = -3.1 \begin{bmatrix} 0.1/-3.1 \\ -3.1 \\ 1.2/-3.1 \end{bmatrix}$$

$$[x]_2 = -3.1 \begin{bmatrix} -0.0323 \\ 1 \\ -0.3871 \end{bmatrix}$$

Therefore, the normalized unit vector of $[x]_2$ is $\begin{bmatrix} -0.0323 \\ 1 \\ -0.3871 \end{bmatrix}$

We then repeat these steps, for the next Eigen vector $[x]_3$ for $i = 3$

$$[x]_3 = a^{-1}[x]_2$$

$$[x]_3 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.15 & -1.5 & -1.75 \\ -0.05 & 0.5 & 0.75 \end{bmatrix} \begin{bmatrix} -0.0323 \\ 1 \\ -0.3871 \end{bmatrix}$$

$$[x]_3 = \begin{bmatrix} -0.00323 \\ -0.8274 \\ 0.2113 \end{bmatrix}$$

We must then extract the highest element from the vector and normalise the vector:

$$[x]_3 = \begin{bmatrix} -0.00323 \\ -0.8274 \\ 0.2113 \end{bmatrix}$$

$$[x]_2 = -0.8274 \begin{bmatrix} -0.00323/-0.8274 \\ -0.8274/-0.8274 \\ 0.2113/-0.8274 \end{bmatrix}$$

$$[x]_2 = -0.8274 \begin{bmatrix} 0.0039 \\ 1 \\ -0.2554 \end{bmatrix}$$

Therefore, the normalized unit vector of $[x]_2$ is $\begin{bmatrix} 0.0039 \\ 1 \\ -0.2554 \end{bmatrix}$

These steps are repeated until the difference between the multiplicative factors becomes significantly small. In our case this happens to be when $[x]_9$'s multiplicative factor is -1 and $[x]_8$'s multiplicative factor is -1.0002. We can terminate the iterations here and confidently say this is approximately the largest Eigenvalue and Eigenvector.

$$[x]_9 = -1 \begin{bmatrix} 0 \\ 1 \\ -0.2857 \end{bmatrix}$$

We can also say that the largest Eigenvalue for the given matrix a is:

-1

5.10 Write a user-defined MATLAB function that determines the largest eigenvalue of an $(n \times n)$ matrix by using the power method. For the function name and argument use `e = MaxEig(A)`, where `A` is the matrix and `e` is the value of the largest eigenvalue. Use the function `MaxEig` for calculating the largest eigenvalue of the matrix of Problem 5.8. Check the answer by using MATLAB's built-in function for finding the eigenvalues of a matrix.

```
function e = MaxEig(A)

  % Get rows and columns
  [row, col] = size(A);

  % Initial vector [1, 1, 1]
  x = [1; 1; 1];

  % Perform 8 iterations
  for j=1:8
    % Multiply matrix by x
    p = A * x;

    % Get the maximum element
    [maxVal, index] = max(abs(p(:)))
    maxVal = maxVal * sign(p(index))

    % Divide each element in vector by max multiplicative factor
    for i=1:cols
      x(i) = p(i)/maxVal;
    end
  end

  e = a;
end
```

5.18 Suppose there are N web sites that are linked to each other. One (overly simplified) method to assess the importance of a particular web site i is as follows. If web site j links (or points) to web site i, a quantity $[a_{ij}]$ can be set to 1 whereas if j does not link to website k, than $[a_{jk}]$ is set to 0. Thus, if $[x_2]$ stands for the importance of web site 2, and web sites 1 and 4 point to web site 2, then $x_2 = x_1 + x_4$, and so on. Consider four web sites, larry.com, bill.com, steve.com, and mark.com linked as shown in the directed graph below.

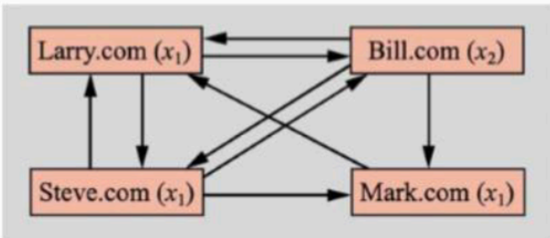
Let x_1 be the importance of larry, x_2 be the importance of bill, x_3 be the importance of steve, and x_4 be the importance of mark. The above directed graph[2] when converted to a set of equations using the scheme described before results in the following equations:

$x_1 = x_2 + x_3 + x_4$, $x_2 = x_1 + x_3$, $x_3 = x_1 + x_2$, and $x_4 = x_2 + x_3$, which can be written as:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = [A][x]$$

(*a*) Find the eigenvalues and the corresponding eigenvectors of $[A]$, using MATLAB's built-in function `eig`.

(*b*) Find the eigenvector from part (*a*) whose entries are all real and of the same sign (it does not matter if they are all negative or all positive), and rank the web sites in descending order of importance based on the indices of the web sites corresponding to the largest to the smallest entries in that eigenvector.



a) The MATLAB code is as follows:

$$\gg A = [0\ 1\ 1\ 1; 1\ 0\ 1\ 0; 1\ 1\ 0\ 0; 0\ 1\ 1\ 0];$$

$$\gg [V, D] = eig(A);$$

This will return the following Eigenvectors in V:

$$\begin{bmatrix} -0.6060 & -0.1914+0.5371i & -0.1914-0.5371i & 0 \\ -0.4774 & 0.2171-0.2367i & 0.2171+0.2367i & -0.7071 \\ -0.4774 & 0.2171-0.2367i & 0.2171+0.2367i & -0.7071 \\ -0.4027 & -0.6844 & -0.6844 & 0 \end{bmatrix}$$

And this will return the following Eigenvalues in D:

$$\boxed{2.2695, -0.6348+0.6916i, -0.6348-0.6916i, \ -1}$$

b) From the matrix of Eigenvectors the only entry which is real and all of the same sign is:

$$\begin{bmatrix} -0.6060 \\ -0.4744 \\ -0.4744 \\ -0.4027 \end{bmatrix}$$

Assuming that the Eigenvectors indicate the importance of the website, they can be ranked in **descending order** as follows:

Rank	Team
1	x_4
2	x_2 or x_3
3	x_2 or x_3
4	x_1

6.3 The following data give the approximate population of China for selected years from 1900 until 2010:

Year	1900	1950	1970	1980	1990	2000	2010
Population (millions)	400	557	825	981	1135	1266	1370

Assume that the population growth can be modeled with an exponential function $p = be^{mx}$, where x is the year and p is the population in millions. Write the equation in a linear form (Section 6.3), and use linear least-squares regression to determine the constants b and m for which the function best fits the data. Use the equation to estimate the population in the year 1985.

Consider the function:

$$p = be^{mx}$$

Apply the natural logarithm on both sides to bring down the exponent:

$$\ln(p) = \ln(be^{mx})$$

$$\ln(p) = \ln(b) + \ln(e^{mx})$$

$$\ln(p) = \ln(b) + mx\ln(e)$$

$$\ln(p) = \ln(b) + mx$$

Now, define the terms as follows:

$$\ln(p) = Y$$

$$\ln(b) = B$$

We can now write the equation in linear form as:

$$Y = mx + B$$

Re-write the data in the table taking the natural logarithm of each entry:

Year (x)	1900	1950	1970	1980	1990	2000	2010
Population ln(p) = Y	5.9915	6.3226	6.7154	6.8886	7.0344	7.1436	7.2226

Now, apply the least square regression method. First, calculate the value of S_x

$$S_x = \sum_{i=1}^{7} x_i$$

$$S_x = 1900 + 1950 + 1970 + 1980 + 1990 + 2000 + 2010$$

$$\boldsymbol{S_x = 13,800}$$

Then, calculate the value of S_y

$$S_y = \sum_{i=1}^{7} y_i$$

$$S_y = 5.9915 + 6.3226 + 6.7154 + 6.8886 + 7.0344 + 7.1436 + 7.2226$$

$$\boldsymbol{S_y = 47.3816}$$

Then, calculate the value of S_{xx}

$$S_{xx} = \sum_{i=1}^{7} {x_i}^2$$

$$S_{xx} = (1900)^2 + \cdots + (2010)^2$$

$$\boldsymbol{S_{xx} = 27214000}$$

Then, calculate the value of S_{xy}

$$S_{xy} = \sum_{i=1}^{7} x_i y_i$$

$$S_{xy} = (1900)(5.9915) + \cdots + (2010)(7.2226)$$

$$\boldsymbol{S_{xy} = 9.3384 \times 10^4}$$

Then, calculate the value of m

$$m = \frac{nS_{xy} - S_x S_y}{nS_{xx} - (S_x)^2}$$

$$m = \frac{7(9.3384 \times 10^4) - (13800)(47.3186)}{7(27214000) - (13800)^2}$$

$$\boldsymbol{m = 0.0119}$$

Then, calculate the value of B

$$B = \frac{S_{xx} S_y - S_{xy} S_x}{nS_{xx} - (S_x)^2}$$

$$B = \frac{(27214000)(47.3186) - (9.3384 \times 10^4)(13800)}{7(27214000) - (13800)^2}$$

$$\boldsymbol{B = -16.7383}$$

Now, using our original equations solve the value of b:

$$B = \ln(b)$$

$$b = e^B$$

$$b = e^{-16.7383}$$

$$\boldsymbol{b = 5.3785 \times 10^{-8}}$$

Therefore, using our newfound values for b and m, the new relation is:

$$p = 5.3785 \times 10^{-8} e^{0.0119x}$$

We can then, using this determine an estimation of the population in the year 1985 by letting x equal to 1985 in the above relation:

$$p = 5.3785 \times 10^{-8} e^{0.0119x}$$

$$p = 5.3785 \times 10^{-8} e^{0.0119(1985)}$$

$$p = 975.7718$$

Therefore, we can estimate that the population in 1985 will be approximately 975.7718 million

6.8 Water solubility in jet fuel, W_S, as a function of temperature, T, can be modeled by an exponential function of the form $W_S = be^{mT}$. The following are values of water solubility measured at different temperatures. Using linear regression, determine the constants m and b that best fit the data. Use the equation to estimate the water solubility at a temperature of 10° C. Make a plot that shows the function and the data points.

T (°C)	–40	–20	0	20	40
W_S (% wt.)	0.0012	0.002	0.0032	0.006	0.0118

Consider the function:

$$W_S = be^{mT}$$

Apply the natural logarithm to both sides:

$$\ln(W_S) = \ln(be^{mT})$$

$$\ln(W_S) = \ln(b) + \ln(e^{mT})$$

$$\ln(W_S) = \ln(b) + mT\ln(e)$$

$$\ln(W_S) = \ln(b) + mT$$

Let:

$$\ln(W_S) = Y$$

$$\ln(b) = B$$

We can now express our function as a linear expression (let T =X):

$$Y = mX + B$$

Re-write the data in the table taking the natural logarithm of each entry:

Temp (X)	-40	-20	0	20	40
ln(Ws) (Y)	6.7254	6.2146	5.7446	5.1160	4.4397

Now, apply the least square regression method. First, calculate the value of S_x

$$S_x = \sum_{i=1}^{7} x_i$$

$$S_x = -40 - 20 + 20 + 40$$

$$\boldsymbol{S_x = 0}$$

Then, calculate the value of S_y

$$S_y = \sum_{i=1}^{7} y_i$$

$$S_y = -6.725 - 6.2146 - 5.7446 - 5.1160 - 4.4397$$

$$\boldsymbol{S_y = -28.2403}$$

Then, calculate the value of S_{xx}

$$S_{xx} = \sum_{i=1}^{7} {x_i}^2$$

$$S_{xx} = (-40)^2 + \cdots + (40)^2$$

$$\boldsymbol{S_{xx} = 4000}$$

Then, calculate the value of S_{xy}

$$S_{xy} = \sum_{i=1}^{7} x_i y_i$$

$$S_{xy} = (-40)(-6.7254) + \cdots + (2010)(7.2226)$$

$$\boldsymbol{S_{xy} = 113.4034}$$

Then, calculate the value of m

$$m = \frac{nS_{xy} - S_x S_y}{nS_{xx} - S_x S_x}$$

$$m = \frac{5(113.4034) - (0)(28.2403)}{5(4000) - (0)^2}$$

$$\boldsymbol{m = 0.0284}$$

Then, calculate the value of B

$$B = \frac{S_{xx} S_y - S_{xy} S_x}{nS_{xx} - S_x S_x}$$

$$B = \frac{(4000)(47.3186) - (113.4034)(0)}{5(4000) - (0)^2}$$

$$\boldsymbol{B = -5.6481}$$

Now, using our original equations solve the value of b:

$$B = \ln(b)$$

$$b = e^{B}$$

$$b = e^{-5.6481}$$

$$\boldsymbol{b} = \mathbf{0.0035}$$

Therefore, using our newfound values for b and m, the new relation is:

$$W_S = 0.0035e^{0.0284T}$$

We can then, using this determine an estimation of the water solubility in jet fuel at a temperature of 10° C:

$$W_S = 0.0035e^{0.0284T}$$

$$W_S = 0.0035e^{0.0284(10)}$$

$$W_S = 0.0046$$

6.13 The power generated by a windmill varies with the wind speed. In an experiment, the following five measurements were obtained:

Wind Speed (mph)	14	22	30	38	46
Electric Power (W)	320	490	540	500	480

Determine the fourth-order polynomial in the Lagrange form that passes through the points. Use the polynomial to calculate the power at a wind speed of 26 mph.

The n-1th order Lagrange polynomial passing through n points can be defined as:

$$f(x) = \left[\frac{(x-x_2)(x-x_3)\dots(x-x_n)}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_n)}y_1 + \frac{(x-x_1)(x-x_3)\dots(x-x_n)}{(x_2-x_1)(x_2-x_3)\dots(x_2-x_n)}y_2 + \dots + \frac{(x-x_1)(x-x_2)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x-x_1)(x-x_2)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}y_i + \frac{(x-x_1)(x-x_2)\dots(x-x_{n-1})}{(x-x_1)(x-x_2)\dots(x-x_{n-1})}y_n\right]$$

This can also be written as:

$$f(x) = \sum_{i=1}^{n} y_i L_i(x) = \sum_{i=1}^{n} y_i \prod_{j=1, j\neq i}^{n} \frac{(x-x_j)}{(x_i-x_j)}$$

Therefore, the 4th order Lagrange polynomial passing through the five points $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ and (x_5, y_5) is:

$$f(x) = \begin{bmatrix} \dfrac{(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}y_1 + \\ \dfrac{(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)}y_2 + \\ \dfrac{(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)}y_3 + \\ \dfrac{(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)}y_4 + \\ \dfrac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}y_5 \end{bmatrix}$$

Using the data from the supplied table we then sub in the values for the following:

- $(x_1, y_1) = (14, 320)$
- $(x_2, y_2) = (22, 490)$
- $(x_3, y_3) = (30, 540)$
- $(x_4, y_4) = (38, 500)$
- $(x_5, y_5) = (46, 480)$

When we multiply this out, we are then left with the fourth order Lagrange polynomial as follows:

$$f(x) = 8.98 \times 10^{-4}x^4 - 0.085x^3 + 2.016x^2 + 10.468x - 23.212$$

We can then use this polynomial to estimate the power at a wind speed of 26mph:

$$f(26) = 8.98 \times 10^{-4}(26)^4 - 0.085(26)^3 + 2.016(26)^2 + 10.468(26) - 23.212$$

$$f(26) = 528.18\,W$$

Thus, using the Lagrange fourth order polynomial we can estimate that the power at a wind speed of 26mph is approximately 528.18W.

6.41 The following measurements were recorded in a study on the growth of trees.

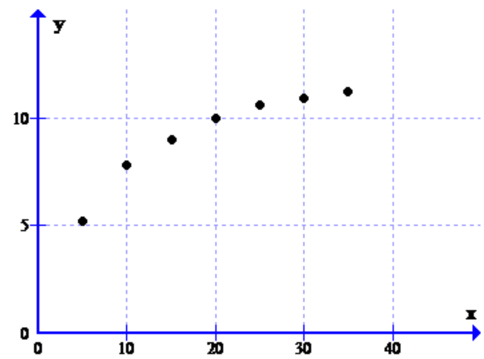
Age (year)	5	10	15	20	25	30	35
Height (m)	5.2	7.8	9	10	10.6	10.9	11.2

The data is used for deriving an equation $H = H(Age)$ that can predict the height of the trees as a function of their age. Determine which of the nonlinear equations that are listed in Table 6-2 can best fit the data and determine its coefficients. Make a plot that shows the data points (asterisk marker) and the equation (solid line).

Table 6-2: Transforming nonlinear equations to linear form.

Nonlinear equation	Linear form	Relationship to $Y = a_1X + a_0$	Values for linear least-squares regression	Plot where data points appear to fit a straight line
$y = bx^m$	$\ln(y) = m\ln(x) + \ln(b)$	$Y = \ln(y)$, $X = \ln(x)$ $a_1 = m$, $a_0 = \ln(b)$	$\ln(x_i)$ and $\ln(y_i)$	y vs. x plot on logarithmic y and x axes. $\ln(y)$ vs. $\ln(x)$ plot on linear x and y axes.
$y = be^{mx}$	$\ln(y) = mx + \ln(b)$	$Y = \ln(y)$, $X = x$ $a_1 = m$, $a_0 = \ln(b)$	x_i and $\ln(y_i)$	y vs. x plot on logarithmic y and linear x axes. $\ln(y)$ vs. x plot on linear x and y axes.
$y = b10^{mx}$	$\log(y) = mx + \log(b)$	$Y = \log(y)$, $X = x$ $a_1 = m$, $a_0 = \log(b)$	x_i and $\log(y_i)$	y vs. x plot on logarithmic y and linear x axes. $\log(y)$ vs. x plot on linear x and y axes.
$y = \frac{1}{mx+b}$	$\frac{1}{y} = mx + b$	$Y = \frac{1}{y}$, $X = x$ $a_1 = m$, $a_0 = b$	x_i and $1/y_i$	$1/y$ vs. x plot on linear x and y axes.
$y = \frac{mx}{b+x}$	$\frac{1}{y} = \frac{b}{m}\frac{1}{x} + \frac{1}{m}$	$Y = \frac{1}{y}$, $X = \frac{1}{x}$ $a_1 = \frac{b}{m}$, $a_0 = \frac{1}{m}$	$1/x_i$ and $1/y_i$	$1/y$ vs. $1/x$ plot on linear x and y axes.

In order to choose which function would be best suited to model the above data points, first plot x vs y:



The scatter plot shows that the graph is similar to the nonlinear equation from the table:

$$y = \frac{mx}{b + x}$$

The linear form of this equation is:

$$\frac{1}{y} = \frac{b}{m}\frac{1}{x} + \frac{1}{m}$$

We can then consider this:

- $Y = \frac{1}{y}$
- $X = \frac{1}{x}$
- $M = \frac{b}{m}$
- $C = \frac{1}{m}$

And from that, the equation can be considered linear as:

$$Y = MX + C$$

Now, apply the least square regression method. First, calculate the value of S_x

$$S_x = \sum_{i=1}^{7} x_i$$

Since X = 1/x, we then have:

$$S_x = \frac{1}{5} + \cdots + \frac{1}{35}$$

Then, calculate the value of S_y

$$S_y = \sum_{i=1}^{7} y_i$$

Since Y = 1/y, we then have:

$$S_y = \frac{1}{5.2} + \cdots + \frac{1}{11.2}$$

Then, calculate the value of S_{xx}

$$S_{xx} = \sum_{i=1}^{7} {x_i}^2$$

$$S_{xx} = \left(\frac{1}{5}\right)^2 + \cdots + \left(\frac{1}{35}\right)^2$$

Then, calculate the value of S_{xy}

$$S_{xy} = \sum_{i=1}^{7} x_i y_i$$

$$S_{xy} = \left(\frac{1}{5}\right)\left(\frac{1}{5.2}\right) + \cdots + \left(\frac{1}{35}\right)\left(\frac{1}{11.2}\right)$$

Then, calculate the value of M

$$M = \frac{nS_{xy} - S_x S_y}{nS_{xx} - S_x S_x}$$

$$\boldsymbol{M = 0.6025}$$

Then, calculate the value of C

$$C = \frac{S_{xx} S_y - S_{xy} S_x}{nS_{xx} - S_x S_x}$$

$$\boldsymbol{C = 0.0706}$$

Now, using our original equations solve the value of m:

$$C = \frac{1}{m}$$

$$0.0706 = \frac{1}{m}$$

$$m = \frac{1}{0.0706}$$

$$\boldsymbol{m = 14.164}$$

We can also solve the value of b:

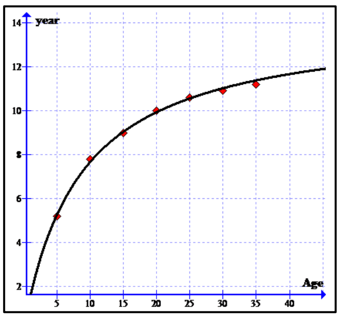
$$0.6025 = \frac{b}{14.164}$$

$$\boldsymbol{b = 8.534}$$

Thus, we can now write our equation as:

$$\boldsymbol{y = \frac{14.164x}{8.534 + x}}$$

By plotting the data points, and this given equation we can show that we have defined an accurate function to represent H=H(age):



6.21 Write a MATLAB user-defined function that determines the best fit of a power function of the form $y = bx^m$ to a given set of data points. Name the function `[b m]= PowerFit(x,y)`, where the input arguments `x` and `y` are vectors with the coordinates of the data points, and the output arguments `b` and `m` are the values of the coefficients. The function `PowerFit` should use the approach that is described in Section 6.3 for determining the value of the coefficients. Use the function to solve Problem 6.3.

The exponential function is as follows:

$$y = bx^m$$

This equation can be written in linear form as follows:

$$\ln(y) = \ln(bx^m)$$

$$\ln(y) = \ln(b) + \ln\,(x^m)$$

$$\ln(y) = \ln(b) + \mathrm{m}\ln(x)$$

$$\ln(y) = \mathrm{m}\ln(x) + \ln(b)$$

Now what we need to do is to determine the constants m and b. For a linear equation y = mx + b the coeffecients are:

$$m = \frac{n\sum_{i=1}^{n} x_i y_i - (\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} y_i)}{n\sum_{i=1}^{n} {x_i}^2 - (\sum_{i=1}^{n} x_i)^2}$$

$$b = \frac{(\sum_{i=1}^{n} {x_i}^2)(\sum_{i=1}^{n} y_i) - (\sum_{i=1}^{n} x_i y_i)(\sum_{i=1}^{n} x_i)}{n\sum_{i=1}^{n} {x_i}^2 - (\sum_{i=1}^{n} x_i)^2}$$

For the equation $\ln(y) = m\ln x + \ln(b)$, the constants are,

$$m = \frac{n\sum_{i=1}^{n}\ln(x_i)\ln(y_i) - \left(\sum_{i=1}^{n}\ln(x_i)\right)\left(\sum_{i=1}^{n}\ln(y_i)\right)}{n\sum_{i=1}^{n}(\ln x_i)^2 - \left(\sum_{i=1}^{n}\ln(x_i)\right)^2}$$

$$\ln(b) = \frac{\left(\sum_{i=1}^{n}(\ln x_i)^2\right)\left(\sum_{i=1}^{n}\ln(y_i)\right) - \left(\sum_{i=1}^{n}\ln(x_i)\ln(y_i)\right)\left(\sum_{i=1}^{n}\ln(x_i)\right)}{n\sum_{i=1}^{n}(\ln x_i)^2 - \left(\sum_{i=1}^{n}\ln(x_i)\right)^2}$$

The MATLAB function PowerFit can be defined as follows:

```
function [b,m] = PowerFit(x,y);

  a = length(x);
  b = length(y);

  if a~=b
    error('SIZE OF x AND y MUST BE THE SAME');
  end

  sumx = 0;
  sumy = 0;
  sumxy = 0;
  sumx_2 = 0;

  % Calculate sum of log(x)'s and log(ys)
  for i=1:a
    sumx = sumx + log(x(i));
    sumy = sumy + log(y(i));

    sumx_2 = sumx_2 + ((log(x(i)))^2);
    sumxy = sumxy + (log(x(i)) * log(y(i)));
  end

  m = ((b*sumxy)-(sumx*sumy)) / ((b*sumx_2)-((sumx)^2));
  b = ((sumx_2*sumy)-(sumxy*sumx)) / ((b*sumx_2)-((sumx)^2));
  b = exp(b);
end
```

```
>> x = [1900 1950 1970 1980 1990 2000 2010];
p = [400 557 825 981 1135 1266 1370];
[b m] = PowerFit(x,p)
```

The following is the MATLAB output:

```
b =

   7.2065e-75


m =

   23.3944
```

Therefore, the values of the coefficients are,

$$\boxed{\begin{aligned} m &= 23.3944 \\ b &= 7.2065\times10^{-75} \end{aligned}}$$

8.3 The following data show estimates of the population of Liberia in selected years between 1960 and 2010:

Year	1960	1970	1980	1990	2000	2010
Population (millions)	1.1	1.4	1.9	2.1	2.8	4

Calculate the rate of growth of the population in millions per year for 2010.
(*a*) Use two-point backward difference formula.
(*b*) Use three-point backward difference formula.
(*c*) Using the slope in 2010 from part (*b*), apply the two-point central difference formula to extrapolate and predict the population in the year 2020.

The formula for the two point backward difference formula for the first derivative is as follows:

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h} + O(h)$$

We can neglect O(h) here. Then, substitute 2010 for x_i and 2000 for x_{i-1} and 10 for h (distance between points):

$$f'(2010) = \frac{f(2010) - f(2000)}{10}$$

$$f'(2010) = \frac{4 - 2.8}{10}$$

$$\boldsymbol{f'(2010) = 0.12}$$

The formula for the three point backward difference formula for the first derivative is as follows:

$$f'(x_i) = \frac{f(x_{i-2}) - 4f(x_{i-1}) + 3f(x_i)}{2h} + O(h^2)$$

We can neglect O(h) here. The, substitute 2010 for x_i, 2000 for x_{i-1}, 1990 for x_{i-2} and 10 for h (distance between points):

$$f'(2010) = \frac{f(1990) - 4f(2000) + 3f(2010)}{2(10)}$$

$$f'(2010) = \frac{2.1 - 4(2.8) + 3(4)}{2(10)}$$

$$\boldsymbol{f'(2010) = 0.145}$$

The formula for the two point central difference formula for the first derivative is as follows:

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} + O(h^2)$$

We can neglect O(h) here. The, substitute 2010 for x_i, for x_{i-1}, 1990, 10 for h (distance between points) and we will solve for x_{i+1} :

$$f'(2010) = \frac{f(2020) - f(1990)}{2(10)}$$

$$0.145 = \frac{f(2020) - 2.1}{20}$$

$$f(2020) = 20(0.145) + 2.8$$

$$f(2020) = 5.7$$

Therefore, our prediction for the population of Liberia in 2020 is approximately 5.7 million.

8.5 Given three ***unequally*** spaced points (x_i, y_i), (x_{i+1}, y_{i+1}), and (x_{i+2}, y_{i+2}), use Taylor series expansion to develop a finite difference formula to evaluate the first derivative dy/dx at the point $x = x_i$. Verify that when the spacing between these points is equal, the three-point forward difference formula is obtained. The answer should involve y_i, y_{i+1}, and y_{i+2}.

Consider three unequally spaced points $(x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2})$.

Using Taylor series expansion, develop a finite difference formula, to evaluate the first derivative dy/dx at the point $x = x_i$.

Given that the points are unequally spaced this means that we will have different values ofr h as follows:

$$h_1 = x_{i+1} - x_i$$

$$h_2 = x_{i+2} - x_{i+1}$$

Using the Taylor series expansion we can find the value of the function and its derivatives at point x_i.

$$f(x_{i+1}) = f(x_i) + f'(x_i)h_1 + \frac{f''(x_i)}{2!}{h_1}^2 + \frac{f'''(x_i)}{3!}{h_1}^3 \ldots (1)$$

$$f(x_{i+2}) = f(x_i) + f'(x_i)(h_1 + h_2) + \frac{f''(x_i)}{2!}(h_1 + h_2)^2 + \frac{f'''(x_i)}{3!}(h_1 + h_2)^3 \ldots (2)$$

Multiply equation (1) by $(h_1 + h_2)^2$:

$$(h_1 + h_2)^2 f(x_{i+1}) = \left[f(x_i)(h_1 + h_2)^2 + f'(x_i)h_1(h_1 + h_2)^2 + (h_1 + h_2)^2\frac{f''(x_i)}{2!}{h_1}^2 + (h_1 + h_2)^2\frac{f'''(x_i)}{3!}{h_1}^3 \ldots\right]$$

Multiply equation (2) by $(h_1)^2$:

$$(h_1)^2 f(x_{i+2}) = \left[f(x_i)(h_1)^2 + f'(x_i)(h_1 + h_2)(h_1)^2 + (h_1)^2\frac{f''(x_i)}{2!}(h_1 + h_2)^2 + (h_1)^2\frac{f'''(x_i)}{3!}(h_1 + h_2)^3\right]$$

Subtract equation (2) from equation (1)

$$(h_1 + h_2)^2 f(x_{i+1}) - (h_1)^2 f(x_{i+2}) = f(x_i)(2h_1h_2 + {h_2}^2) + f'(x_i)({h_1}^2h_2 + {h_2}^2h_1) + O(h^2)$$

Re-writing this in terms of $f'(x_i)$ gives us:

$$f'(x_i)\left({h_1}^2 h_2 + {h_2}^2 h_1\right) = (h_1 + h_2)^2 f(x_{i+1}) - (h_1)^2 f(x_{i+2}) - f(x_i)(2h_1 h_2 + {h_2}^2) + O(h^2)$$

$$\boldsymbol{f'(x_i) = \frac{(h_1 + h_2)^2 f(x_{i+1}) - (h_1)^2 f(x_{i+2}) - (2h_1 h_2 + {h_2}^2) f(x_i)}{\left({h_1}^2 h_2 + {h_2}^2 h_1\right)} + O(h^2)}$$

8.19 Write a MATLAB user-defined function that determines the first and second derivatives of a function that is given by a set of discrete points with equal spacing. For the function name use `[yd,ydd] = FrstScndDeriv(x,y)`. The input arguments `x` and `y` are vectors with the coordinates of the points, and the output arguments `yd` and `ydd` are vectors with the values of the first and second derivatives, respectively, at each point. For calculating both derivatives, the function should use the finite difference formulas that have a truncation error of $O(h^2)$.

(*a*) Use the function `FrstScndDeriv` to calculate the derivatives of the function that is given by the data in Problem 8.18.

(*b*) Modify the function (rename it `FrstScndDerivPt`) such that it also creates three plots (one page in a column). The top plot should be of the function, the second plot of the first derivative, and the third of the second derivative. Apply the function `FrstScndDerivPt` to the data in Problem 8.18.

```
function [yd,ydd] = FirstSecondDeriv(x,y)

  n = length(x);

  % All points equally spaced
  h = (x(2)-x(1));

  % Use the three point forward difference formula for 1st and 2nd derivative
  yd(1) = (-3*y(1) + 4*y(2) - y(3)) / 2*h
  ydd(1) = (y(1) - 2*y(2) + y(3)) / 2*h

  % Loop over the rest of the vectors
  for i=2:n-1
    % Use two point central difference formula for 1st derivative
    yd(i) = (y(i+1) - y(i-1)) / 2*h;

    % Use the three point central difference formula for 2nd derivative
    ydd(i) = (y(i-1) - 2*y(i) + y(i+1)) / h*h;
  end

  % For last entry use three point backward difference formula for 1st deriv
  yd(n) = (y(n-2) - 4*y(n-1) + 3*y(n)) / 2*h;

  % For last entry use three point backward difference formula for 2nd deriv
  ydd(n) = (y(n-2) - 2*y(n-1) + y(n)) / h*h;
end
```

To then test this code to get the first and second derivatives do the following:

```
x=[-1 -0.5 0 0.5 1 1.5 2 2.5 3 3.5 4 4.5];
y=[-3.632 -0.3935 1 0.6487 -1.282 -4.518 -8.611 -12.82 -15.91 -15.88 -9.402 9.017];
[a,b]=FrstScndDeriv(x,y)
```

The MATLAB output for the derivatives of the function.

a =

2.0805 2.3160 0.5211 -1.1410 -2.5833 -3.6645 -4.1510 -3.6495 -1.5300 3.2540 12.4485 12.1947

b =

-7.3800 -7.3800 -6.9792 -6.3176 -5.2212 -3.4280 -0.4640 4.4760 12.4800 25.7920 47.7640
47.7640

Thus, the second derivatives of the function are

$$\boxed{\begin{array}{l}-7.3800,\ -7.3800,\ -6.9792,\ -6.3176,\ -5.2212,\ -3.4280,\\ -0.4640,\ 4.4760,\ 12.4800,\ 25.7920,\ 47.7640,\ 47.7640\end{array}}.$$

To then graph the results we would do the following:

```
x=[-1 -0.5 0 0.5 1 1.5 2 2.5 3 3.5 4 4.5];
y=[-3.632 -0.3935 1 0.6487 -1.282 -4.518 -8.611 -12.82 -15.91 -15.88 -9.402 9.017];
[a,b]=FrstScndDeriv(x,y)
```

The MATLAB output for the derivatives of the function.

```
a =
2.0805 2.3160 0.5211 -1.1410 -2.5833 -3.6645 -4.1510 -3.6495 -1.5300 3.2540 12.4485 12.1947
b =
-7.3800 -7.3800 -6.9792 -6.3176 -5.2212 -3.4280 -0.4640 4.4760 12.4800 25.7920 47.7640
47.7640
```

Thus, the second derivatives of the function are

$$\boxed{\begin{array}{l}-7.3800, -7.3800, -6.9792, -6.3176, -5.2212, -3.4280, \\ -0.4640, \; 4.4760, 12.4800, 25.7920, 47.7640, \; 47.7640\end{array}}.$$

This then produces the plots:

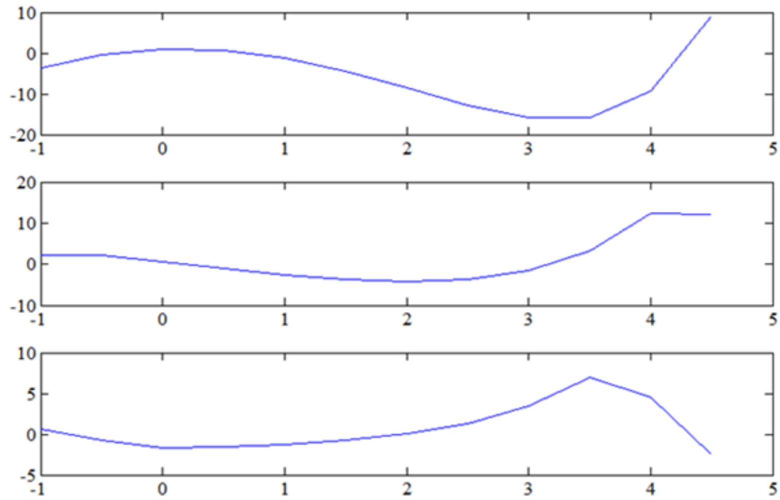


Figure 1

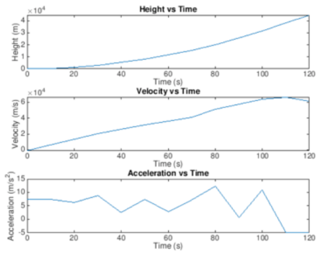
8.31 The altitude of the space shuttle during the first two minutes of its ascent is displayed in the following table (www.nasa.gov):

t (s)	0	10	20	30	40	50	60	70	80	90	100	110	120
h (m)	–8	241	1,244	2,872	5,377	8,130	11,617	15,380	19,872	25,608	31,412	38,309	44,726

Assuming the shuttle is moving straight up, determine its velocity and acceleration at each point. Display the results in three plots (h versus time, velocity versus time, and acceleration versus time).
(*a*) Solve by using the user-defined function `FrstScndDerivPt` that was written in Problem 8.19.
(*b*) Solve by using the MATLAB built-in function `diff`.

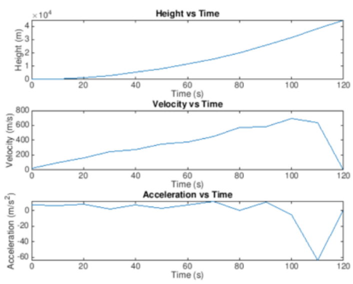
```
time = [0 10 20 30 40 50 60 70 80 90 100 110 120];
height = [-8 241 1244 2872 5377 8130 11617 15380 19872 25608 31412 38309 44726];

% Generate velocity (first derivative)
% Generate acceleration (second derivative)
[vel,acc] = FirstSecondDeriv(time, height);

disp(vel);
disp(acc);

% Plot the given data
subplot(3,1,1);
plot(time,height);
title('Height vs Time');
xlabel('Time (s)');
ylabel('Height (m)');

% Plot the velocity (first derivative) vs time
subplot(3,1,2);
plot(time,vel);
title('Velocity vs Time');
xlabel('Time (s)');
ylabel('Velocity (m/s)');

% Plot the acceleration (second derivative) vs time
subplot(3,1,3);
plot(time,acc);
title('Acceleration vs Time');
xlabel('Time (s)');
ylabel('Acceleration (m/s^2)');
```

```
   -1280    6260   13155   20665   26290   31200   36250   41275   51140   57700   63505   66570   61770
  7.5400  7.5400  6.2500  8.7700  2.4800  7.3400  2.7600  7.2900 12.4400  0.6800 10.9300 -4.8000 -4.8000
```




```
time = [0 10 20 30 40 50 60 70 80 90 100 110 120];
height = [-8 241 1244 2872 5377 8130 11617 15380 19872 25608 31412 38309 44726];

% Generate velocity (first derivative)
vel = diff(height)./diff(time);
vel(13) = 0;

% Generate acceleration (second derivative)
acc = diff(vel)./diff(time);
acc(13) = 0;

disp(vel);
disp(acc);

% Plot the given data
subplot(3,1,1);
plot(time,height);
title('Height vs Time');
xlabel('Time (s)');
ylabel('Height (m)');

% Plot the velocity (first derivative) vs time
subplot(3,1,2);
plot(time,vel);
title('Velocity vs Time');
xlabel('Time (s)');
ylabel('Velocity (m/s)');

% Plot the acceleration (second derivative) vs time
subplot(3,1,3);
plot(time,acc);
title('Acceleration vs Time');
xlabel('Time (s)');
ylabel('Acceleration (m/s^2)');
```

```
 24.9000 100.3000 162.8000 250.5000 275.3000 348.7000 376.3000 449.2000 573.6000 580.4000 689.7000 641.7000        0
  7.5400   6.2500   8.7700   2.4800   7.3400   2.7600   7.2900  12.4400   0.6800  10.9300  -4.8000 -64.1700        0
```



9.1 The function $f(x)$ is given in the following tabulated form. Compute $\int_0^{1.8} f(x)dx$ with $h = 0.3$ and with $h = 0.4$.
(*a*) Use the composite rectangle method.
(*b*) Use the composite trapezoidal method.
(*c*) Use the composite Simpson's 3/8 method.

x	0	0.3	0.6	0.9	1.2	1.5	1.8
$f(x)$	0.5	0.6	0.8	1.3	2	3.2	4.8

a) The formula to determine the integration using **composite rectangle method** is:

$$I(f) = \sum_{a}^{b} f(x)\,dx$$

$$\approx h \sum_{i=1}^{N} f(x_i)$$

We can then determine N for h=0.3 as follows:

$$h = \frac{b-a}{N}$$

$$0.3 = \frac{1.8-0}{N}$$

$$N = 6$$

Obtain the summation of the data for h=0.3 and N=6:

$$I(f) \approx h \sum_{i=1}^{N} f(x_i)$$

$$I(f) \approx 0.3(0.5 + 0.6 + 0.8 + 1.3 + 2 + 3.2)$$

$$\boldsymbol{I(f) \approx 2.52}$$

We can then determine N for h=0.4 as follows:

$$h = \frac{b-a}{N}$$

$$0.4 = \frac{1.8-0}{N}$$

$$N = 4.5 \ \approx 5$$

We must first calculate the values of $f(x)$ over 5 points with a width of 0.4 as follows:

x	0	0.4	0.8	1.2	1.6
$f(x)$	0.5857	0.5492	1.0587	2.1142	3.7157

Obtain the summation of the data for h=0.4 and N=5:

$$I(f) \approx h\sum_{i=1}^{N} f(x_i)$$

$$I(f) \approx 0.4(0.5857 + 0.5492 + 1.0587 + 2.1142 + 3.7157)$$

$$\boldsymbol{I(f) \approx 3.2094}$$

b) The formula to determine the integration using composite trapezoidal method is:

$$I(f) = \frac{h}{2}[f(a) + f(b)] + h\sum_{i=2}^{N} f(x_i)$$

For h=0.3 and N=6 we have:

$$I(f) = \frac{0.3}{2}[f(a) + f(b)] + 0.3\sum_{i=2}^{6} f(x_i)$$

$$I(f) \approx \frac{0.3}{2}[0.5857 + 3.7157] + 0.3(0.6 + 0.8 + 1.3 + 2 + 3.2)$$

$$\boldsymbol{I(f) \approx 3.165}$$

For h=0.4 and N=5 we have:

$$I(f) = \frac{0.4}{2}[0.5 + 4.8] + 0.4(0.5492 + 1.05857 + 2.1142 + 3.7157)$$

$$\boldsymbol{I(f) \approx 3.015}$$

c) The formula to determine the integration using Simpsons method is:

$$I(f) = \frac{3h}{8}\left[f(a) + f(b) + 3\sum_{i=2,5,8}^{N-1}[f(x_i) + f(x_{i+1})] + 2\sum_{j=4,7,10}^{N-2} f(x_j)\right]$$

For h=0.3 and N=6 we have:

$$I(f) \approx \frac{3(0.3)}{8}\left[0.5 + 4.8 + 3[(0.6 + 0.8) + (2 + 3.2)] + 2[1.3]\right]$$

$$\boldsymbol{I(f) \approx 3.116}$$

For h=0.4 and N=5 we have:

$$I(f) \approx \frac{3(0.4)}{8}\left[0.5857 + 3.7157 + 3[(0.5492 + 1.0587)] + 2[2.1142]\right]$$

$$\boldsymbol{I(f) \approx 2.003}$$