ST3009 - Statistical Methods for Computer Science
Week 1 Questions

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Question 1 - *A substitution cypher is derived from orderings of the first 10 letters of the alphabet. How many ways can the 10 letters be ordered if each letter appears exactly once and:*

(a) With no other restrictions and given that each letter can appear exactly once the first letter can be chosen from an option of 10 letters, the second from the remaining 9, third from remaining 8 etc… The product rule states that the total number of outcomes of multiple experiments is the product of the number of total possible outcomes. This leads us with 10 factorial as seen below.

10x9x8x7x6x5x4x3x2x1 = 10! = 3628800

(b) Since E and F must be beside each other we consider them as the one box as they must be stuck together. This leaves us with 9 total options for how all letters can be arranged as E and F are considered as the one letter. Since there is also two ways that E and F can be arranged this must also be accounted for. This produces 9 factorial for the arrangement of EF together and a further 9 factorial for the arrangement of FE together.

(9x8x7x6x5x4x3x2x1) x 2 = 9! x 2 = 362880 x 2 = 725760

(c) Since we are looking for an arrangement of 6 letters the result is in theory 6 factorial. However since there are 3 occurrences of A and 2 occurrences of N these can also be considered unique arrangements if we are to consider each A as $A_1A_2A_3$ they can be arranged in 3! different ways and similarly with the two N's as N_1N_2 as 2!. To account for this we divide the original 6 factorial by (3! x 2!).

$$\frac{\mathbf{6!}}{\mathbf{3! \times 2!}}$$

(d) With no repeated letters in the sequence ABCDE we do not need to account for repeated occurrences as above. There are n(n-1)(n-2)…(n-k+1) ways that a group of k items can be selected from n items when order matters. Each group of k items will be counted k! times so we need to divide by this number to get the number of unordered groups. This leads us with the equation:

$$\frac{\boldsymbol{n(n-1)(n-2)\ldots(n-k+1)}}{\boldsymbol{k!}} = \frac{\boldsymbol{n!}}{\boldsymbol{(n-k)k!}}$$

Since we are choosing 3 objects from a selection of 5 we set n = 5 and k = 3. This leaves us with:

$$\frac{\mathbf{5!}}{\mathbf{(5-3)3!}} = \binom{\mathbf{5}}{\mathbf{3}} = \mathbf{10}$$

Question 2 – *A six sided die is rolled four times.*

(a) The first roll of the dice can produce one of 6 possible results. The second roll of the die can also produce one of 6 possible results irrespective of the first roll. By the product rule this means that the total number of potential sequences / outcomes is:

6x6x6x6 = 6^4 = 1296

(b) Since we want two of the four rolls to produce exactly two occurrences of 3 we can consider these rolls as a combination since there are 4 possible slots for two threes to occur. This can be calculated by using the following equation with n = 4 and k = 2:

$$\frac{\boldsymbol{n}!}{(\boldsymbol{n}-\boldsymbol{k})\boldsymbol{k}!} = \frac{\mathbf{4!}}{\mathbf{(4-2)2!}} = \binom{\mathbf{4}}{\mathbf{2}} = \mathbf{6}$$

After we have chosen the 3's we have two more slots/rolls to execute. These rolls must produce a number in the set { 1, 2, 4, 5, 6 } and as a result have 5 total possibilities. This means that there are 5x5 different sequences of rolls for the remaining slot. Accounting for the previous 6 different combinations calculated above produces a total number of occurrences as follows:

$$\mathbf{6 \times 5^2 = 150}$$

(c) This can be calculated by determining the total number of outcomes with no 3's and the number of outcomes with exactly one 3 and subtracting this from the total number of possible outcomes. The total number of outcomes with no 3's is calculated using the product rule by considering that in each of the four roll the outcome produces a number in the range 1-5:

5x5x5x5 = 5^4 = 625

To calculate the number of possible outcomes with exactly one occurrence of 3 can be done similar to part (b) where we want exactly one of the four rolls to produce a 3 so we want the number of different combinations in 4 rolls for one output to occur:

$$\frac{\boldsymbol{n}!}{(\boldsymbol{n}-\boldsymbol{k})\boldsymbol{k}!} = \frac{\mathbf{4!}}{\mathbf{(4-1)1!}} = \binom{\mathbf{4}}{\mathbf{1}} = \mathbf{4}$$

After this we have three more rolls to account for in which we want the outcome to be a number in the set { 1, 2, 4, 5, 6 } i.e a total of 5 possible outcomes. This means that there are 5x5x5 possible outcomes for these remaining three rolls. This combined with the previous combination for one occurrence produces:

$$\mathbf{4 \times 5^3 = 500}$$

Adding both of these results together (625 + 500) gives us the total number of outcomes that **do not** contain at least two occurrences of 3. By subtracting this from the total number of possible outcomes we can get the number of possible outcomes that **do** contain at least two occurrences of the number 3.

$$\mathbf{1296 - (625 + 500) = 171}$$

Question 3 – *You are counting cards in a card game that uses two decks of cards. Each deck has 4 cards (the ace from each of 4 suits), so there are 8 cards total. Cards are only distinguishable based on their suit, not which deck they came from.*

(a) Since cards are not distinguishable by the deck they came from and only their suit this means that there are 4 possible options for each occurrence with each suit only being allowed to appear twice. With 8 possible slots to fill this means there are 8! ways of selecting the cards.

8x7x6x5x4x3x2x1 = 8! = 40320

However, since there are two occurrences of each suit i.e each ace A_1 and A_2 can be arranged as A_1 A_2 and also A_2A_1 .This means that there are 2! ways of ordering these individual ace's. So the total number of distinct arrangements of the 8 cards is as follows:

$$\mathbf{\frac{8!}{2! \times 2! \times 2! \times 2!} = 2520}$$

(b) There are n(n-1)(n-2)…(n-k+1) ways that a group of k items can be selected from n items when order matters. Each group of k items will be counted k! times so we need to divide by this number to get the number of unordered groups. This leads us with the equation:

$$\boldsymbol{\frac{n(n-1)(n-2)\ldots(n-k+1)}{k!} = \frac{n!}{(n-k)k!}}$$

Since the order of the cards does not matter and there are 4 possible cards with which you can be dealt we are choosing 2 objects from a selection of 4 we set n = 4 and k = 2 This leaves us with:

$$\frac{\mathbf{4!}}{\mathbf{(4-2)2!}} = \binom{\mathbf{4}}{\mathbf{2}} = \mathbf{6}$$

(c) Using the answer and method from part (b) there are a total of 6 distinct pairs of cards that you can be dealt from the two decks. These pairs include all possible suits. However, if we wanted to find the pairs that only contained “good” cards i.e Hearts and Diamonds we would have to half this number since 50% of our original “good” cards from (b) have been removed. This thus leaves us with:

$$\frac{\mathbf{4!}}{\mathbf{(4-2)2!}} = \binom{\mathbf{4}}{\mathbf{2}} = \mathbf{6}$$

$$\mathbf{6} \times \frac{\mathbf{1}}{\mathbf{2}} = \mathbf{3}$$