CS3081 – Computational Mathematics
Recommended Questions Solutions

2.22 Write the Taylor's series expansion of the function $f(x) = \sin(ax)$ about $x = 0$, where $a \neq 0$ is a known constant.

Consider a function $f(x)$ that is differentiable $n+1$ times in an interval containing the point $x = x_0$.

The Taylor theorem states that for each x in the interval, there exists a value $x = \xi$ between x and x_0 such that:

$$f(x) = \left[f(x_0) + (x - x_0)\frac{df}{dx}\bigg|_{x=x_0} + \frac{(x-x_0)^2}{2!}\frac{d^2f}{dx^2}\bigg|_{x=x_0} + \frac{(x-x_0)^3}{3!}\frac{d^3f}{dx^3}\bigg|_{x=x_0} + \cdots + \frac{(x-x_0)^n}{(n+1)!}\frac{d^nf}{dx^n}\bigg|_{x=x_0} + R_n(x) \right]$$

Where $R_n(x)$ called the remainder and is given by:

$$R_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!}\frac{d^{n+1}f}{dx^{n+1}}\bigg|_{x=\xi}$$

The Taylor series expansion of the function $f(x) = \sin(ax)$ about $x = 0$ (i.e $x_0 = 0$) is therefore as follows:

$$f(x) = \left[\sin(ax_0) + (x - x_0)\frac{d\sin(ax_0)}{dx}\bigg|_{x=x_0} + \frac{(x-x_0)^2}{2!}\frac{d^2\sin(ax_0)}{dx^2}\bigg|_{x=x_0} + \frac{(x-x_0)^3}{3!}\frac{d^3\sin(ax_0)}{dx^3}\bigg|_{x=x_0} + \cdots \right]$$

$$f(x) = \left[\sin(ax_0) + (x - x_0)\left[\mathrm{a}\cos(ax_0)|_{x=x_0}\right] + \frac{(x-x_0)^2}{2!}\left[-\mathrm{a}^2\sin(ax_0)|_{x=x_0}\right] + \frac{(x-x_0)^3}{3!}\left[-\mathrm{a}^3\cos(ax_0)|_{x=x_0}\right] + \cdots \right]$$

Substitute $x_0 = 0$ into this function:

$$f(x) = \left[\sin(0) + x[\mathrm{a}\cos(0)] + \frac{(x)^2}{2!}[-\mathrm{a}^2\sin(0)] + \frac{(x)^3}{3!}[-\mathrm{a}^3\cos(0)] + \cdots \right]$$

$$f(x) = 0 + x[\mathrm{a}] + \frac{(x)^3}{3!}[-\mathrm{a}^3] + \cdots$$

Therefore, the Taylor's series expansion of the function $f(x) = \sin(ax)$ is:

$$\boldsymbol{f(x) = \frac{(x)^1}{1!}[\mathrm{a}] - \frac{(x)^3}{3!}[\mathrm{a}^3] + \frac{(x)^5}{5!}[\mathrm{a}^5] + \cdots}$$

2.31 Write a user-defined MATLAB function that calculates the determinant of a square $(n \times n)$ matrix, where n can be 2, 3, or 4. For function name and arguments, use `D = Determinant(A)`. The input argument `A` is the matrix whose determinant is calculated. The function `Determinant` should first check if

```
function D = Determinant(A)

  % Get number of rows and columns
  [rows,columns] = size(A);

  % Verify square matrix & n inRange 2:4
  if rows == columns && (rows >= 2 && rows <= 4)
      if rows == 2
          D = det2x2(A);
      elseif rows == 3
          D = det3x3(A);
      else
          D = det4x4(A);
      end
  else
      D = 'Matrix must be square with 2 =< n <= 4';
  end

end

% 2x2 determinants
function D = det2x2(A)

  D = (A(1,1)*A(2,2)) - (A(1,2)*A(2,1))

end

% 3x3 determinants
function D = det3x3(A)

  aPart = A(1,1)*det2x2(A([2 3], 2:3));
  bPart = A(2,1)*det2x2(A([1 3], 2:3));
  cPart = A(3,1)*det2x2(A([1 2], 2:3));

  D = aPart - bPart + cPart;

end

% 4x4 determinants
function D = det4x4(A)

  aPart = A(1,1)*det3x3(A([2 3 4], 2:4));
  bPart = A(2,1)*det3x3(A([1 3 4], 2:4));
  cPart = A(3,1)*det3x3(A([1 2 4], 2:4));
  dPart = A(4,1)*det3x3(A([1 2 3], 2:4));

  D = aPart - bPart + cPart - dPart;

end
```

4.2 Given the system of equations $[a][x] = [b]$, where $a = \begin{bmatrix} 2 & -2 & 1 \\ 3 & 2 & -5 \\ -1 & 2 & 3 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $b = \begin{bmatrix} 10 \\ -16 \\ 8 \end{bmatrix}$, determine the solution using the Gauss elimination method.

First, substitute the given matrices in $[a] * [x] = [b]$ form:

$$\begin{bmatrix} 2 & -2 & 1 \\ 3 & 2 & -5 \\ -1 & 2 & 3 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -16 \\ 8 \end{bmatrix} \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 \end{array}$$

We then want to eliminate the all of the bottom row ($R_3's$) elements besides the 3 so that we can get a value for x3. Therefore perform the following row operations on matrices $[a]$ and $[b]$:

- $R_2 = 2(R_2)$ - 3(R_1)
- $R_3 = 2(R_3) + R_1$

$$\begin{array}{r} \rightarrow R_1 \\ \rightarrow \mathbf{2(R_2) - 3(R_1)} \\ \rightarrow \mathbf{2(R_3) - R_1} \end{array} \begin{bmatrix} 2 & -2 & 1 \\ 0 & 10 & -13 \\ 0 & 2 & 7 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -62 \\ 26 \end{bmatrix}$$

- $R_3 = 5(R_3)$ - R_2

$$\begin{array}{r} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow \mathbf{5(R_3) - R_2} \end{array} \begin{bmatrix} 2 & -2 & 1 \\ 0 & 10 & -13 \\ 0 & 0 & 48 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -62 \\ 196 \end{bmatrix}$$

From this we can now solve for x_3:

$$48x_3 = 196$$

$$\boldsymbol{x_3 = 4}$$

We can then sub this into row 2 and solve for x_2:

$$10x_2 - 13x_3 = -63$$

$$10x_2 - 13(4) = -63$$

$$\boldsymbol{x_2 = -1}$$

And then solve for x_1 using row 1:

$$2x_1 - 2x_2 + 1x_3 = 10$$

$$2x_1 - 2(-1) + 1(4) = 10$$

$$\boldsymbol{x_1 = 2}$$

4.13 Find the inverse of the matrix $\begin{bmatrix} 10 & 12 & 0 \\ 0 & 2 & 8 \\ 2 & 4 & 8 \end{bmatrix}$ using the Gauss–Jordan method.

First write out the matrix and the augmented matrix:

$$\begin{bmatrix} 10 & 12 & 0 \\ 0 & 2 & 8 \\ 2 & 4 & 8 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The aim of the game here is to reduce the LHS matrix to the RHS matrix by performing row operations:

- $R_1 = R_1/10$

$$\begin{array}{r} \to R_1/10 \\ \to R_2 \\ \to R_3 \end{array}\begin{bmatrix} 1 & 1.2 & 0 \\ 0 & 2 & 8 \\ 2 & 4 & 8 \end{bmatrix}\begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $R_3 = R_3 - 2(R_1)$

$$\begin{array}{r} \to R_1 \\ \to R_2 \\ \to R_3 - 2(R_1) \end{array}\begin{bmatrix} 1 & 1.2 & 0 \\ 0 & 2 & 8 \\ 0 & 1.6 & 8 \end{bmatrix}\begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.2 & 0 & 1 \end{bmatrix}$$

- $R_2 = R_2/2$

$$\begin{array}{r} \to R_1 \\ \to R_2/2 \\ \to R_3 \end{array}\begin{bmatrix} 1 & 1.2 & 0 \\ 0 & 1 & 4 \\ 0 & 1.6 & 8 \end{bmatrix}\begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.5 & 0 \\ -0.2 & 0 & 1 \end{bmatrix}$$

- $R_1 = R_1 - 1.2(R_2)$

$$\begin{array}{r} \to R_1 - 1.2(R_2) \\ \to R_2 \\ \to R_3 \end{array}\begin{bmatrix} 1 & 0 & -4.8 \\ 0 & 1 & 4 \\ 0 & 1.6 & 8 \end{bmatrix}\begin{bmatrix} 0.1 & -0.6 & 0 \\ 0 & 0.5 & 0 \\ -0.2 & 0 & 1 \end{bmatrix}$$

- $R_3 = R_3 - 1.6(R_2)$

$$\begin{array}{r} \to R_1 \\ \to R_2 \\ \to R_3 - 1.6(R_2) \end{array}\begin{bmatrix} 1 & 0 & -4.8 \\ 0 & 1 & 4 \\ 0 & 0 & 1.6 \end{bmatrix}\begin{bmatrix} 0.1 & -0.6 & 0 \\ 0 & 0.5 & 0 \\ -0.2 & -0.8 & 1 \end{bmatrix}$$

- $R_3 = R_3/1.6$

$$\begin{array}{r} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3/1.6 \end{array} \begin{bmatrix} 1 & 0 & -4.8 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1 & -0.6 & 0 \\ 0 & 0.5 & 0 \\ -0.125 & -0.5 & 0.625 \end{bmatrix}$$

- $R_1 = R_1 + 4.8(R_3)$
- $R_2 = R_2 + 4(R_3)$

$$\begin{array}{r} \rightarrow R_1 + 4.8(R_3) \\ \rightarrow R_2 + 4(R_3) \\ \rightarrow R_3 \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -0.5 & -3 & 3 \\ 0.5 & 2.5 & -2.5 \\ -0.125 & -0.5 & 0.625 \end{bmatrix}$$

4.19 **Find the condition number of the matrix in Problem 4.13 using the 1-norm.**

$$a = \begin{bmatrix} 10 & 12 & 0 \\ 0 & 2 & 8 \\ 2 & 4 & 8 \end{bmatrix}$$

We know from 4.13 that the inverse of a is:

$$a^{-1} = \begin{bmatrix} -0.5 & -3 & 3 \\ 0.5 & 2.5 & -2.5 \\ -0.125 & -0.5 & 0.625 \end{bmatrix}$$

Consider the formula for the condition number of a matrix:

$$cond[a] = \|[a]\|\|[a]^{-1}\| \ldots$$

Where:

- $\|[a]\|$ denotes the norm of matrix a
- $\|[a]^{-1}\|$ denotes the norm of inverse of matrix a

Write the formula to find the 1-norm of a matrix:

$$\|[a]\|_1 = \max_{1\le j\le n} \sum_{j=1}^{n} |a_{ij}|$$

Consider the matrix a as stated above and 3 for n:

$$\|[a]\|_1 = \max_{1\le j\le 3} \sum_{j=1}^{3} |a_{ij}|$$

$$\|[a]\|_1 = \max_{1\le j\le 3}[(|10| + |0| + |2|), (|12| + |2| + |4|), (|0| + |8| + |8|)]$$

$$\|[a]\|_1 = \max_{1\le j\le 3}[12, 18, 16]$$

$$\boldsymbol{\|[a]\|_1 = 18}$$

Therefore, the 1-norm of matrix a is 18.

4.25 Write a user-defined MATLAB function that calculates the 1-norm of any matrix. For the function name and arguments use `N = OneNorm(A)`, where `A` is the matrix and `N` is the value of the norm. Use the function for calculating the 1-norm of:

(*a*) The matrix $A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1.5 \end{bmatrix}$. (*b*) The matrix $B = \begin{bmatrix} 4 & -1 & 0 & 1 & 0 \\ -1 & 4 & -1 & 0 & 1 \\ 0 & -1 & 4 & -1 & 0 \\ 1 & 0 & -1 & 4 & -1 \\ 0 & 1 & 0 & -1 & 4 \end{bmatrix}$.

```
function N = OneNorm(A)

  % Find the row and column dimensions
  [row, cols] = size(A);

  % Iterate over columns
  for j=1:cols
    % Calculate sum of columns and store in array
    column_sums(j) = sum(abs(A(:,j)));
  end;

  % Initialise max
  max = column_sums(1);

  % Find the largest value amongst the sums
  for j=1:cols-1
    if column_sums(j+1) > max
      max = column_sums(j+1);
    end
  end

  % Return max
  N = max;

end
```

5.3 Find the eigenvalues of the following matrix by solving for the roots of the characteristic equation. $$\begin{bmatrix} 10 & 0 & 0 \\ 1 & -3 & -7 \\ 0 & 2 & 6 \end{bmatrix}$$

The expression for the characteristic equation to find the Eigen values is:

$$det[a - \lambda I] = 0$$

$$|a - \lambda I| = 0$$

As the matrix a is of order 3x3, the order of the identity matrix is also 3x3:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Calculate the matrix $a - \lambda I$:

$$a - \lambda I = \begin{bmatrix} 10 & 0 & 0 \\ 1 & -3 & -7 \\ 0 & 2 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$a - \lambda I = \begin{bmatrix} 10 - \lambda & 0 & 0 \\ 1 & -3 - \lambda & -7 \\ 0 & 2 & 6 - \lambda \end{bmatrix}$$

Using this calculate the characteristic equation $|a - \lambda I| = 0$:

$$|a - \lambda I| = 0$$

$$\begin{vmatrix} 10 - \lambda & 0 & 0 \\ 1 & -3 - \lambda & -7 \\ 0 & 2 & 6 - \lambda \end{vmatrix} = 0$$

$$10 - \lambda[(-3 - \lambda)(6 - \lambda) - (-7)(2)] - 0 + 0 = 0$$

$$10 - \lambda[3\lambda - 6\lambda + \lambda^2 - 18 + 14] = 0$$

$$(10 - \lambda)(\lambda^2 - 3\lambda - 4) = 0$$

$$(10 - \lambda)(\lambda + 1)(\lambda - 4) = 0$$

$$\boldsymbol{\lambda = 10, \lambda = 4, \lambda = -1}$$

Therefore, the Eigen values of the matrix are 10, 4 and -1

5.7 Apply the power method to find the largest eigenvalue of the matrix from Problem 5.2 starting with the vector $[1\ 1\ 1]^T$.

$$a = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

For $i = 1$, the Eigen vector $[x]_1 = [1 \quad 1 \quad 1]^T$ that is:

$$[x]_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Calculate the next Eigen vector $[x]_2$ for $i = 2$

$$[x]_2 = a[x]_1$$

$$[x]_2 = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$[x]_2 = \begin{bmatrix} 1+1+2 \\ 4+1+5 \\ 0+0+1 \end{bmatrix}$$

$$[x]_2 = \begin{bmatrix} 4 \\ 10 \\ 1 \end{bmatrix}$$

We must then extract the highest element from the vector and normalise the vector:

$$[x]_2 = \begin{bmatrix} 4 \\ \mathbf{10} \\ 1 \end{bmatrix}$$

$$[x]_2 = 10 \begin{bmatrix} 4/10 \\ \mathbf{10/10} \\ 1/10 \end{bmatrix}$$

$$[x]_2 = 10 \begin{bmatrix} 0.4 \\ 1 \\ 0.1 \end{bmatrix}$$

Therefore, the normalized unit vector of $[x]_2$ is $\begin{bmatrix} 0.4 \\ 1 \\ 0.1 \end{bmatrix}$

We then repeat these steps, for the next Eigen vector $[x]_3$ for $i = 3$

$$[x]_3 = a[x]_2$$

$$[x]_3 = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.4 \\ 1 \\ 0.1 \end{bmatrix}$$

$$[x]_3 = \begin{bmatrix} 0.4 + 1 + 0.2 \\ 1.6 + 1 + 0.5 \\ 0 + 0 + 0.1 \end{bmatrix}$$

$$[x]_3 = \begin{bmatrix} 1.6 \\ 3.1 \\ 0.1 \end{bmatrix}$$

We must then extract the highest element from the vector and normalise the vector:

$$[x]_3 = \begin{bmatrix} 1.6 \\ \mathbf{3.1} \\ 0.1 \end{bmatrix}$$

$$[x]_3 = 3.1 \begin{bmatrix} 1.6/\mathbf{3.1} \\ 3.1/\mathbf{3.1} \\ 0.1/\mathbf{3.1} \end{bmatrix}$$

$$[x]_3 = 3.1 \begin{bmatrix} 0.516 \\ 1 \\ 0.0323 \end{bmatrix}$$

Therefore, the normalized unit vector of $[x]_3$ is $\begin{bmatrix} 0.516 \\ 1 \\ 0.0323 \end{bmatrix}$

We then repeat these steps, for the next Eigen vector $[x]_4$ for $i = 4$

$$[x]_4 = a[x]_3$$

$$[x]_4 = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.516 \\ 1 \\ 0.0323 \end{bmatrix}$$

$$[x]_4 = \begin{bmatrix} 0.516 + 1 + 0.0646 \\ 2.064 + 1 + 0.1615 \\ 0 + 0 + 0.0323 \end{bmatrix}$$

$$[x]_4 = \begin{bmatrix} 1.58 \\ 3.225 \\ 0.0323 \end{bmatrix}$$

We must then extract the highest element from the vector and normalise the vector:

$$[x]_4 = \begin{bmatrix} 1.58 \\ 3.225 \\ 0.0323 \end{bmatrix}$$

$$[x]_4 = 3.225 \begin{bmatrix} 1.58/3.225 \\ 3.225/3.225 \\ 0.0323/3.225 \end{bmatrix}$$

$$[x]_4 = 3.225 \begin{bmatrix} 0.49 \\ 1 \\ 0.01 \end{bmatrix}$$

Therefore, the normalized unit vector of $[x]_4$ is $\begin{bmatrix} 0.49 \\ 1 \\ 0.01 \end{bmatrix}$

This is repeated and on our 6th iteration we get:

$$[x]_6 = \begin{bmatrix} 1.5066 \\ 3.0165 \\ 0.0033 \end{bmatrix}$$

$$[x]_6 = 3.0165 \begin{bmatrix} 1.5066/3.0165 \\ 3.0165/3.0165 \\ 0.0033/3.0165 \end{bmatrix}$$

$$[x]_6 = 3.0165 \begin{bmatrix} 0.5 \\ 1 \\ 0.001 \end{bmatrix}$$

Since the difference between the multiplicative factors (was 3.01 for $[x]_5$) is relatively small we can terminate our iterations here at $[x]_6$.

We can also say that the largest Eigenvalue for the given matrix a is:

3.0165

5.9 Apply the inverse power method to find the smallest eigenvalue of the matrix from Problem 5.3 starting with the vector $[1\ 1\ 1]^T$. The inverse of the matrix in Problem 5.3 is:

$$\begin{bmatrix} 0.1 & 0 & 0 \\ 0.15 & -1.5 & -1.75 \\ -0.05 & 0.5 & 0.75 \end{bmatrix}$$

$$a = \begin{bmatrix} 10 & 0 & 0 \\ 1 & -3 & -7 \\ 0 & 2 & 6 \end{bmatrix}$$

$$a^{-1} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.15 & -1.5 & -1.75 \\ -0.05 & 0.5 & 0.75 \end{bmatrix}$$

For $i = 1$, the Eigen vector $[x]_1 = [1 \quad 1 \quad 1]^T$ that is:

$$[x]_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Calculate the next Eigen vector $[x]_2$ for $i = 2$

$$[x]_2 = a^{-1}[x]_1$$

$$[x]_2 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.15 & -1.5 & -1.75 \\ -0.05 & 0.5 & 0.75 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$[x]_2 = \begin{bmatrix} 0.1 \\ -3.1 \\ 1.2 \end{bmatrix}$$

We must then extract the highest element from the vector and normalise the vector:

$$[x]_2 = \begin{bmatrix} 0.1 \\ -3.1 \\ 1.2 \end{bmatrix}$$

$$[x]_2 = -3.1 \begin{bmatrix} 0.1/-3.1 \\ -3.1 \\ 1.2/-3.1 \end{bmatrix}$$

$$[x]_2 = -3.1 \begin{bmatrix} -0.0323 \\ 1 \\ -0.3871 \end{bmatrix}$$

Therefore, the normalized unit vector of $[x]_2$ is $\begin{bmatrix} -0.0323 \\ 1 \\ -0.3871 \end{bmatrix}$

We then repeat these steps, for the next Eigen vector $[x]_3$ for $i = 3$

$$[x]_3 = a^{-1}[x]_2$$

$$[x]_3 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.15 & -1.5 & -1.75 \\ -0.05 & 0.5 & 0.75 \end{bmatrix} \begin{bmatrix} -0.0323 \\ 1 \\ -0.3871 \end{bmatrix}$$

$$[x]_3 = \begin{bmatrix} -0.00323 \\ -0.8274 \\ 0.2113 \end{bmatrix}$$

We must then extract the highest element from the vector and normalise the vector:

$$[x]_3 = \begin{bmatrix} -0.00323 \\ -0.8274 \\ 0.2113 \end{bmatrix}$$

$$[x]_2 = -0.8274 \begin{bmatrix} -0.00323/-0.8274 \\ -0.8274/-0.8274 \\ 0.2113/-0.8274 \end{bmatrix}$$

$$[x]_2 = -0.8274 \begin{bmatrix} 0.0039 \\ 1 \\ -0.2554 \end{bmatrix}$$

Therefore, the normalized unit vector of $[x]_2$ is $\begin{bmatrix} 0.0039 \\ 1 \\ -0.2554 \end{bmatrix}$

These steps are repeated until the difference between the multiplicative factors becomes significantly small. In our case this happens to be when $[x]_9$'s multiplicative factor is -1 and $[x]_8$'s multiplicative factor is -1.0002. We can terminate the iterations here and confidently say this is approximately the largest Eigenvalue and Eigenvector.

$$[x]_9 = -1 \begin{bmatrix} 0 \\ 1 \\ -0.2857 \end{bmatrix}$$

We can also say that the largest Eigenvalue for the given matrix a is:

-1

5.10 Write a user-defined MATLAB function that determines the largest eigenvalue of an $(n \times n)$ matrix by using the power method. For the function name and argument use `e = MaxEig(A)`, where `A` is the matrix and `e` is the value of the largest eigenvalue. Use the function `MaxEig` for calculating the largest eigenvalue of the matrix of Problem 5.8. Check the answer by using MATLAB's built-in function for finding the eigenvalues of a matrix.

```
function e = MaxEig(A)

  % Get rows and columns
  [row, col] = size(A);

  % Initial vector [1, 1, 1]
  x = [1; 1; 1];

  % Perform 8 iterations
  for j=1:8
    % Multiply matrix by x
    p = A * x;

    % Get the maximum element
    [maxVal, index] = max(abs(p(:)))
    maxVal = maxVal * sign(p(index))

    % Divide each element in vector by max multiplicative factor
    for i=1:cols
      x(i) = p(i)/maxVal;
    end
  end

  e = a;
end
```

6.3 The following data give the approximate population of China for selected years from 1900 until 2010:

Year	1900	1950	1970	1980	1990	2000	2010
Population (millions)	400	557	825	981	1135	1266	1370

Assume that the population growth can be modeled with an exponential function $p = be^{mx}$, where x is the year and p is the population in millions. Write the equation in a linear form (Section 6.3), and use linear least-squares regression to determine the constants b and m for which the function best fits the data. Use the equation to estimate the population in the year 1985.

Consider the function:

$$p = be^{mx}$$

Apply the natural logarithm on both sides to bring down the exponent:

$$\ln(p) = \ln(be^{mx})$$

$$\ln(p) = \ln(b) + \ln(e^{mx})$$

$$\ln(p) = \ln(b) + mx\ln(e)$$

$$\ln(p) = \ln(b) + mx$$

Now, define the terms as follows:

$$\ln(p) = Y$$

$$\ln(b) = B$$

We can now write the equation in linear form as:

$$Y = mx + B$$

Re-write the data in the table taking the natural logarithm of each entry:

Year (x)	1900	1950	1970	1980	1990	2000	2010
Population ln(p) = Y	5.9915	6.3226	6.7154	6.8886	7.0344	7.1436	7.2226

Now, apply the least square regression method. First, calculate the value of S_x

$$S_x = \sum_{i=1}^{7} x_i$$

$$S_x = 1900 + 1950 + 1970 + 1980 + 1990 + 2000 + 2010$$

$$\boldsymbol{S_x = 13,800}$$

Then, calculate the value of S_y

$$S_y = \sum_{i=1}^{7} y_i$$

$$S_y = 5.9915 + 6.3226 + 6.7154 + 6.8886 + 7.0344 + 7.1436 + 7.2226$$

$$\boldsymbol{S_y = 47.3816}$$

Then, calculate the value of S_{xx}

$$S_{xx} = \sum_{i=1}^{7} {x_i}^2$$

$$S_{xx} = (1900)^2 + \cdots + (2010)^2$$

$$\boldsymbol{S_{xx} = 27214000}$$

Then, calculate the value of S_{xy}

$$S_{xy} = \sum_{i=1}^{7} x_i y_i$$

$$S_{xy} = (1900)(5.9915) + \cdots + (2010)(7.2226)$$

$$\boldsymbol{S_{xy} = 9.3384 \times 10^4}$$

Then, calculate the value of m

$$m = \frac{nS_{xy} - S_x S_y}{nS_{xx} - (S_x)^2}$$

$$m = \frac{7(9.3384 \times 10^4) - (13800)(47.3186)}{7(27214000) - (13800)^2}$$

$$\boldsymbol{m = 0.0119}$$

Then, calculate the value of B

$$B = \frac{S_{xx} S_y - S_{xy} S_x}{nS_{xx} - (S_x)^2}$$

$$B = \frac{(27214000)(47.3186) - (9.3384 \times 10^4)(13800)}{7(27214000) - (13800)^2}$$

$$\boldsymbol{B = -16.7383}$$

Now, using our original equations solve the value of b:

$$B = \ln(b)$$

$$b = e^{B}$$

$$b = e^{-16.7383}$$

$$\boldsymbol{b = 5.3785 \times 10^{-8}}$$

Therefore, using our newfound values for b and m, the new relation is:

$$p = 5.3785 \times 10^{-8} e^{0.0119x}$$

We can then, using this determine an estimation of the population in the year 1985 by letting x equal to 1985 in the above relation:

$$p = 5.3785 \times 10^{-8} e^{0.0119x}$$

$$p = 5.3785 \times 10^{-8} e^{0.0119(1985)}$$

$$p = 975.7718$$

Therefore, we can estimate that the population in 1985 will be approximately 975.7718 million

6.8 Water solubility in jet fuel, W_S, as a function of temperature, T, can be modeled by an exponential function of the form $W_S = be^{mT}$. The following are values of water solubility measured at different temperatures. Using linear regression, determine the constants m and b that best fit the data. Use the equation to estimate the water solubility at a temperature of 10° C. Make a plot that shows the function and the data points.

T (°C)	–40	–20	0	20	40
W_S (% wt.)	0.0012	0.002	0.0032	0.006	0.0118

Consider the function:

$$W_S = be^{mT}$$

Apply the natural logarithm to both sides:

$$\ln(W_S) = \ln(be^{mT})$$

$$\ln(W_S) = \ln(b) + \ln\,(e^{mT})$$

$$\ln(W_S) = \ln(b) + mT\ln\,(e)$$

$$\ln(W_S) = \ln(b) + mT$$

Let:

$$\ln(W_S) = Y$$

$$\ln(b) = B$$

We can now express our function as a linear expression (let T =X):

$$Y = mX + B$$

Re-write the data in the table taking the natural logarithm of each entry:

Temp (X)	-40	-20	0	20	40
ln(Ws) (Y)	6.7254	6.2146	5.7446	5.1160	4.4397

Now, apply the least square regression method. First, calculate the value of S_x

$$S_x = \sum_{i=1}^{7} x_i$$

$$S_x = -40 - 20 + 20 + 40$$

$$\boldsymbol{S_x = 0}$$

Then, calculate the value of S_y

$$S_y = \sum_{i=1}^{7} y_i$$

$$S_y = -6.725 - 6.2146 - 5.7446 - 5.1160 - 4.4397$$

$$\boldsymbol{S_y = -28.2403}$$

Then, calculate the value of S_{xx}

$$S_{xx} = \sum_{i=1}^{7} {x_i}^2$$

$$S_{xx} = (-40)^2 + \cdots + (40)^2$$

$$\boldsymbol{S_{xx} = 4000}$$

Then, calculate the value of S_{xy}

$$S_{xy} = \sum_{i=1}^{7} x_i y_i$$

$$S_{xy} = (-40)(-6.7254) + \cdots + (2010)(7.2226)$$

$$\boldsymbol{S_{xy} = 113.4034}$$

Then, calculate the value of m

$$m = \frac{nS_{xy} - S_x S_y}{nS_{xx} - S_x S_x}$$

$$m = \frac{5(113.4034) - (0)(28.2403)}{5(4000) - (0)^2}$$

$$\boldsymbol{m = 0.0284}$$

Then, calculate the value of B

$$B = \frac{S_{xx} S_y - S_{xy} S_x}{nS_{xx} - S_x S_x}$$

$$B = \frac{(4000)(47.3186) - (113.4034)(0)}{5(4000) - (0)^2}$$

$$\boldsymbol{B = -5.6481}$$

Now, using our original equations solve the value of b:

$$B = \ln(b)$$

$$b = e^{B}$$

$$b = e^{-5.6481}$$

$$\boldsymbol{b = 0.0035}$$

Therefore, using our newfound values for b and m, the new relation is:

$$W_S = 0.0035e^{0.0284T}$$

We can then, using this determine an estimation of the water solubility in jet fuel at a temperature of 10° C:

$$W_S = 0.0035e^{0.0284T}$$

$$W_S = 0.0035e^{0.0284(10)}$$

$$W_S = 0.0046$$

6.13 The power generated by a windmill varies with the wind speed. In an experiment, the following five measurements were obtained:

Wind Speed (mph)	14	22	30	38	46
Electric Power (W)	320	490	540	500	480

Determine the fourth-order polynomial in the Lagrange form that passes through the points. Use the polynomial to calculate the power at a wind speed of 26 mph.

The n-1th order Lagrange polynomial passing through n points can be defined as:

$$f(x) = \left[\frac{(x-x_2)(x-x_3)\dots(x-x_n)}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_n)}y_1 + \frac{(x-x_1)(x-x_3)\dots(x-x_n)}{(x_2-x_1)(x_2-x_3)\dots(x_2-x_n)}y_2 + \dots + \frac{(x-x_1)(x-x_2)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x-x_1)(x-x_2)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}y_i + \frac{(x-x_1)(x-x_2)\dots(x-x_{n-1})}{(x-x_1)(x-x_2)\dots(x-x_{n-1})}y_n\right]$$

This can also be written as:

$$f(x) = \sum_{i=1}^{n} y_i L_i(x) = \sum_{i=1}^{n} y_i \prod_{j=1, j\neq i}^{n} \frac{(x-x_j)}{(x_i-x_j)}$$

Therefore, the 4th order Lagrange polynomial passing through the five points $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ and (x_5, y_5) is:

$$f(x) = \begin{bmatrix} \frac{(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}y_1 + \\ \frac{(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)}y_2 + \\ \frac{(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)}y_3 + \\ \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)}y_4 + \\ \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}y_5 \end{bmatrix}$$

Using the data from the supplied table we then sub in the values for the following:

- $(x_1, y_1) = (14, 320)$
- $(x_2, y_2) = (22, 490)$
- $(x_3, y_3) = (30, 540)$
- $(x_4, y_4) = (38, 500)$
- $(x_5, y_5) = (46, 480)$

When we multiply this out, we are then left with the fourth order Lagrange polynomial as follows:

$$f(x) = 8.98 \times 10^{-4}x^4 - 0.085x^3 + 2.016x^2 + 10.468x - 23.212$$

We can then use this polynomial to estimate the power at a wind speed of 26mph:

$$f(26) = 8.98 \times 10^{-4}(26)^4 - 0.085(26)^3 + 2.016(26)^2 + 10.468(26) - 23.212$$

$$f(26) = 528.18\,W$$

Thus, using the Lagrange fourth order polynomial we can estimate that the power at a wind speed of 26mph is approximately 528.18W.

6.41 The following measurements were recorded in a study on the growth of trees.

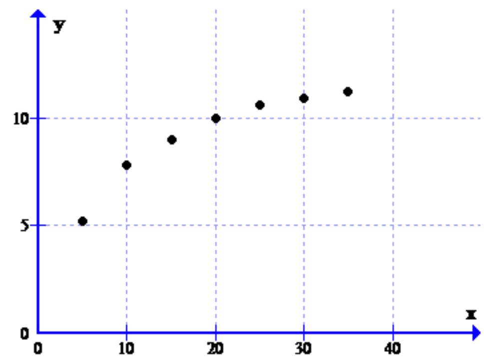
Age (year)	5	10	15	20	25	30	35
Height (m)	5.2	7.8	9	10	10.6	10.9	11.2

The data is used for deriving an equation $H = H(Age)$ that can predict the height of the trees as a function of their age. Determine which of the nonlinear equations that are listed in Table 6-2 can best fit the data and determine its coefficients. Make a plot that shows the data points (asterisk marker) and the equation (solid line).

Table 6-2: Transforming nonlinear equations to linear form.

Nonlinear equation	Linear form	Relationship to $Y = a_1X + a_0$	Values for linear least-squares regression	Plot where data points appear to fit a straight line
$y = bx^m$	$\ln(y) = m\ln(x) + \ln(b)$	$Y = \ln(y)$, $X = \ln(x)$ $a_1 = m$, $a_0 = \ln(b)$	$\ln(x_i)$ and $\ln(y_i)$	y vs. x plot on logarithmic y and x axes. $\ln(y)$ vs. $\ln(x)$ plot on linear x and y axes.
$y = be^{mx}$	$\ln(y) = mx + \ln(b)$	$Y = \ln(y)$, $X = x$ $a_1 = m$, $a_0 = \ln(b)$	x_i and $\ln(y_i)$	y vs. x plot on logarithmic y and linear x axes. $\ln(y)$ vs. x plot on linear x and y axes.
$y = b10^{mx}$	$\log(y) = mx + \log(b)$	$Y = \log(y)$, $X = x$ $a_1 = m$, $a_0 = \log(b)$	x_i and $\log(y_i)$	y vs. x plot on logarithmic y and linear x axes. $\log(y)$ vs. x plot on linear x and y axes.
$y = \frac{1}{mx+b}$	$\frac{1}{y} = mx + b$	$Y = \frac{1}{y}$, $X = x$ $a_1 = m$, $a_0 = b$	x_i and $1/y_i$	$1/y$ vs. x plot on linear x and y axes.
$y = \frac{mx}{b+x}$	$\frac{1}{y} = \frac{b}{m}\frac{1}{x} + \frac{1}{m}$	$Y = \frac{1}{y}$, $X = \frac{1}{x}$ $a_1 = \frac{b}{m}$, $a_0 = \frac{1}{m}$	$1/x_i$ and $1/y_i$	$1/y$ vs. $1/x$ plot on linear x and y axes.

In order to choose which function would be best suited to model the above data points, first plot x vs y:



The scatter plot shows that the graph is similar to the nonlinear equation from the table:

$$y = \frac{mx}{b + x}$$

The linear form of this equation is:

$$\frac{1}{y} = \frac{b}{m}\frac{1}{x} + \frac{1}{m}$$

We can then consider this:

- $Y = \frac{1}{y}$
- $X = \frac{1}{x}$
- $M = \frac{b}{m}$
- $C = \frac{1}{m}$

And from that, the equation can be considered linear as:

$$Y = MX + C$$

Now, apply the least square regression method. First, calculate the value of S_x

$$S_x = \sum_{i=1}^{7} x_i$$

Since X = 1/x, we then have:

$$S_x = \frac{1}{5} + \cdots + \frac{1}{35}$$

Then, calculate the value of S_y

$$S_y = \sum_{i=1}^{7} y_i$$

Since Y = 1/y, we then have:

$$S_y = \frac{1}{5.2} + \cdots + \frac{1}{11.2}$$

Then, calculate the value of S_{xx}

$$S_{xx} = \sum_{i=1}^{7} {x_i}^2$$

$$S_{xx} = \left(\frac{1}{5}\right)^2 + \cdots + \left(\frac{1}{35}\right)^2$$

Then, calculate the value of S_{xy}

$$S_{xy} = \sum_{i=1}^{7} x_i y_i$$

$$S_{xy} = \left(\frac{1}{5}\right)\left(\frac{1}{5.2}\right) + \cdots + \left(\frac{1}{35}\right)\left(\frac{1}{11.2}\right)$$

Then, calculate the value of M

$$M = \frac{nS_{xy} - S_x S_y}{nS_{xx} - S_x S_x}$$

$$\boldsymbol{M = 0.6025}$$

Then, calculate the value of C

$$C = \frac{S_{xx} S_y - S_{xy} S_x}{nS_{xx} - S_x S_x}$$

$$\boldsymbol{C = 0.0706}$$

Now, using our original equations solve the value of m:

$$C = \frac{1}{m}$$

$$0.0706 = \frac{1}{m}$$

$$m = \frac{1}{0.0706}$$

$$\boldsymbol{m = 14.164}$$

We can also solve the value of b:

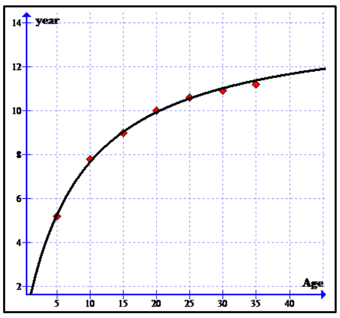
$$0.6025 = \frac{b}{14.164}$$

$$\boldsymbol{b = 8.534}$$

Thus, we can now write our equation as:

$$\boldsymbol{y = \frac{14.164x}{8.534 + x}}$$

By plotting the data points, and this given equation we can show that we have defined an accurate function to represent H=H(age):



6.21 Write a MATLAB user-defined function that determines the best fit of a power function of the form $y = bx^m$ to a given set of data points. Name the function `[b m]= PowerFit(x,y)`, where the input arguments `x` and `y` are vectors with the coordinates of the data points, and the output arguments `b` and `m` are the values of the coefficients. The function `PowerFit` should use the approach that is described in Section 6.3 for determining the value of the coefficients. Use the function to solve Problem 6.3.

The exponential function is as follows:

$$y = bx^m$$

This equation can be written in linear form as follows:

$$\ln(y) = \ln(bx^m)$$

$$\ln(y) = \ln(b) + \ln\,(x^m)$$

$$\ln(y) = \ln(b) + \mathrm{m}\ln(x)$$

$$\ln(y) = \mathrm{m}\ln(x) + \ln(b)$$

Now what we need to do is to determine the constants m and b. For a linear equation y = mx + b the coeffecients are:

$$m = \frac{n\sum_{i=1}^{n} x_i y_i - (\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} y_i)}{n\sum_{i=1}^{n} {x_i}^2 - (\sum_{i=1}^{n} x_i)^2}$$

$$b = \frac{(\sum_{i=1}^{n} {x_i}^2)(\sum_{i=1}^{n} y_i) - (\sum_{i=1}^{n} x_i y_i)(\sum_{i=1}^{n} x_i)}{n\sum_{i=1}^{n} {x_i}^2 - (\sum_{i=1}^{n} x_i)^2}$$

For the equation $\ln(y) = m\ln x + \ln(b)$, the constants are,

$$m = \frac{n\sum_{i=1}^{n}\ln(x_i)\ln(y_i) - \left(\sum_{i=1}^{n}\ln(x_i)\right)\left(\sum_{i=1}^{n}\ln(y_i)\right)}{n\sum_{i=1}^{n}(\ln x_i)^2 - \left(\sum_{i=1}^{n}\ln(x_i)\right)^2}$$

$$\ln(b) = \frac{\left(\sum_{i=1}^{n}(\ln x_i)^2\right)\left(\sum_{i=1}^{n}\ln(y_i)\right) - \left(\sum_{i=1}^{n}\ln(x_i)\ln(y_i)\right)\left(\sum_{i=1}^{n}\ln(x_i)\right)}{n\sum_{i=1}^{n}(\ln x_i)^2 - \left(\sum_{i=1}^{n}\ln(x_i)\right)^2}$$

The MATLAB function PowerFit can be defined as follows:

```
function [b,m] = PowerFit(x,y);

  a = length(x);
  b = length(y);

  if a~=b
    error('SIZE OF x AND y MUST BE THE SAME');
  end

  sumx = 0;
  sumy = 0;
  sumxy = 0;
  sumx_2 = 0;

  % Calculate sum of log(x)'s and log(ys)
  for i=1:a
    sumx = sumx + log(x(i));
    sumy = sumy + log(y(i));

    sumx_2 = sumx_2 + ((log(x(i)))^2);
    sumxy = sumxy + (log(x(i)) * log(y(i)));
  end

  m = ((b*sumxy)-(sumx*sumy)) / ((b*sumx_2)-((sumx)^2));
  b = ((sumx_2*sumy)-(sumxy*sumx)) / ((b*sumx_2)-((sumx)^2));
  b = exp(b);
end
```

```
>> x = [1900 1950 1970 1980 1990 2000 2010];
p = [400 557 825 981 1135 1266 1370];
[b m] = PowerFit(x,p)
```

The following is the MATLAB output:

```
b =

   7.2065e-75


m =

   23.3944
```

Therefore, the values of the coefficients are,

$$\boxed{\begin{array}{l} m = 23.3944 \\ b = 7.2065 \times 10^{-75} \end{array}}$$