Chapter 9

Numerical Integration

Core Topics

(i) Rectangle and Midpoint Methods (9.2)

(ii) The Trapezoidal Method (9.3)

(iii) Simpson’s Methods (9.4)

(iv) Gaussian Quadrature (9.5)

(v) Evaluation of Multiple Integrals (9.6)

(vi) Use of MATLAB built-in Functions for Integration (9.7)

(vii) Estimation of Error (9.8)

(viii) Richardson Extrapolation (9.9)

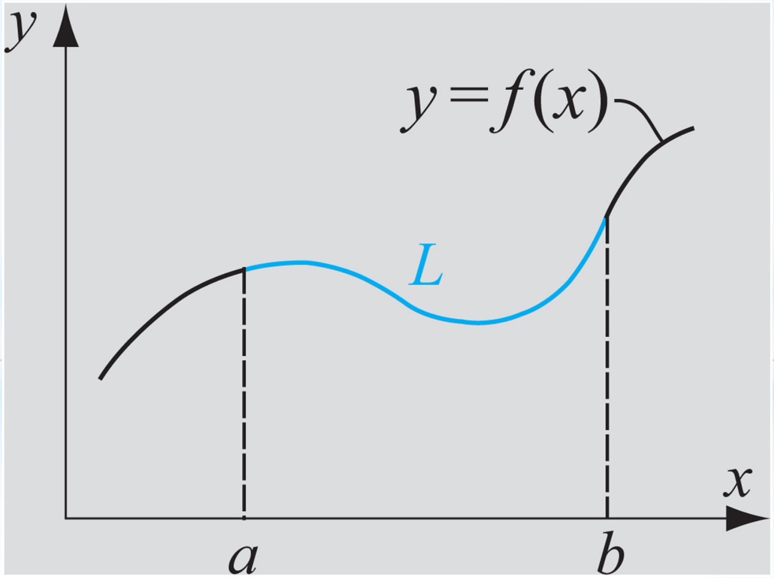
(ix) Romberg Integration (9.10)

(x) Improper Integrals (9.11)

Background:

Integration is frequently encountered in problems in engineering and science. For example the length of a curve between points a and b is given by:

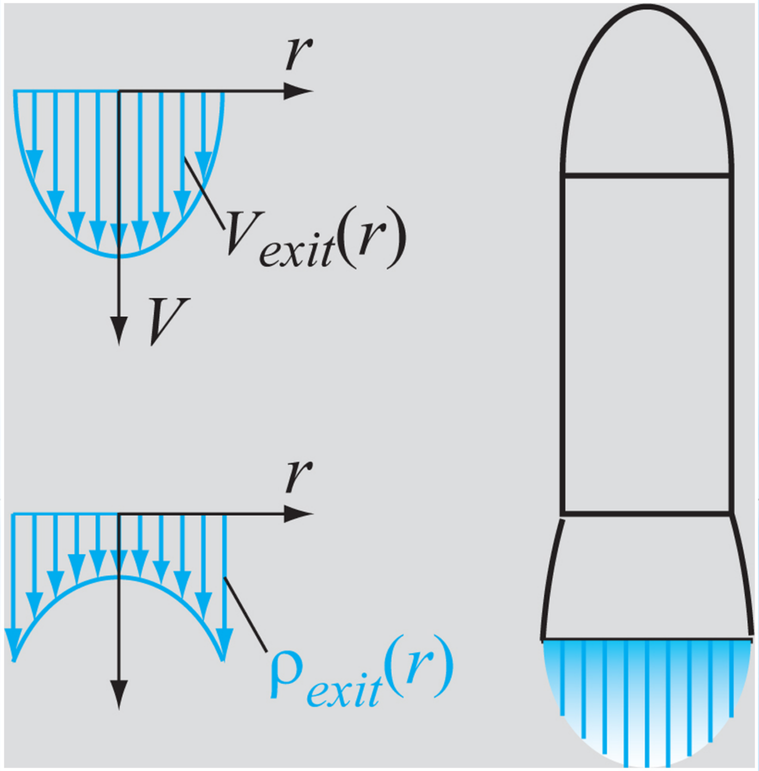
$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$



Background:

Another example is in calculating the thrust of a rocket which is expressed as a function of the exit velocity of the exhaust and the density of the exhaust over the exhaust aperture.

$$T = \int_0^R 2\pi\rho(r)V_{exit}^2(r)rdr$$

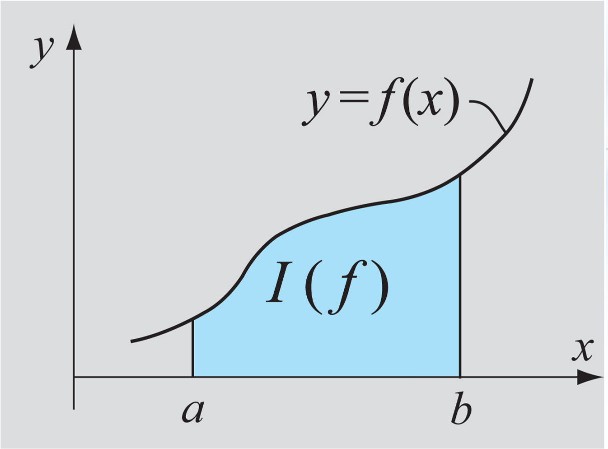


Background:

The general form of a definite integral (also called an antiderivative) is:

$$I(f)=\int_a^b f(x)dx \qquad (9.1)$$

where $f(x)$, called the integrand, is a function of the independent variable x. a and b are the limits of the integration. Graphically the value of the integral corresponds to the area under the curve between a and b when we integrate over the x-axis.

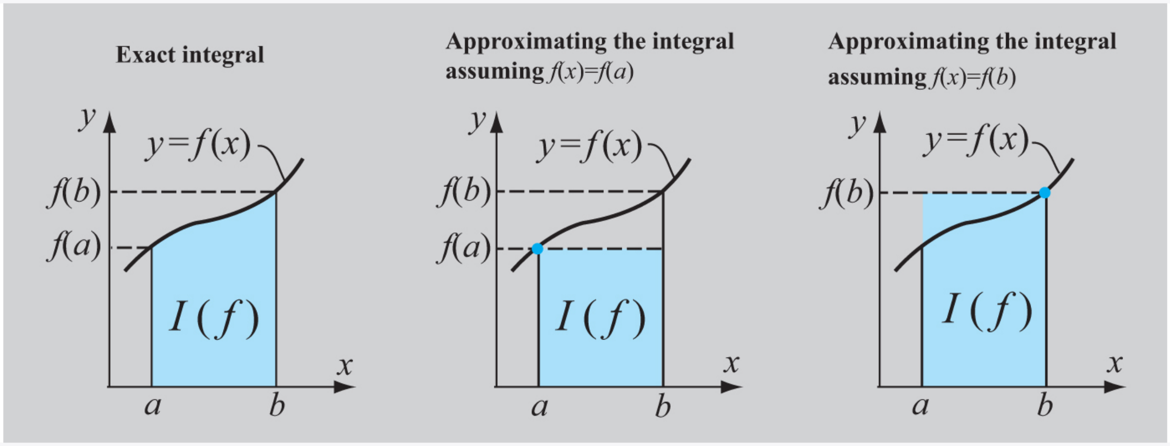


Background:

The integrand can be an analytical function or a set of discrete points (tabulated data). Numerical integration is needed when the integrand is discrete and/or where analytical integration is difficult, numerous or impossible.

Rectangle and Midpoint Methods:

The simplest approximation for $\int_a^b f(x)dx$ is to take $f(x)$ over the interval $x \in [a, b]$ as a constant equal to the value of $f(x)$ at either one of the endpoints. This is known as the Rectangle Method.

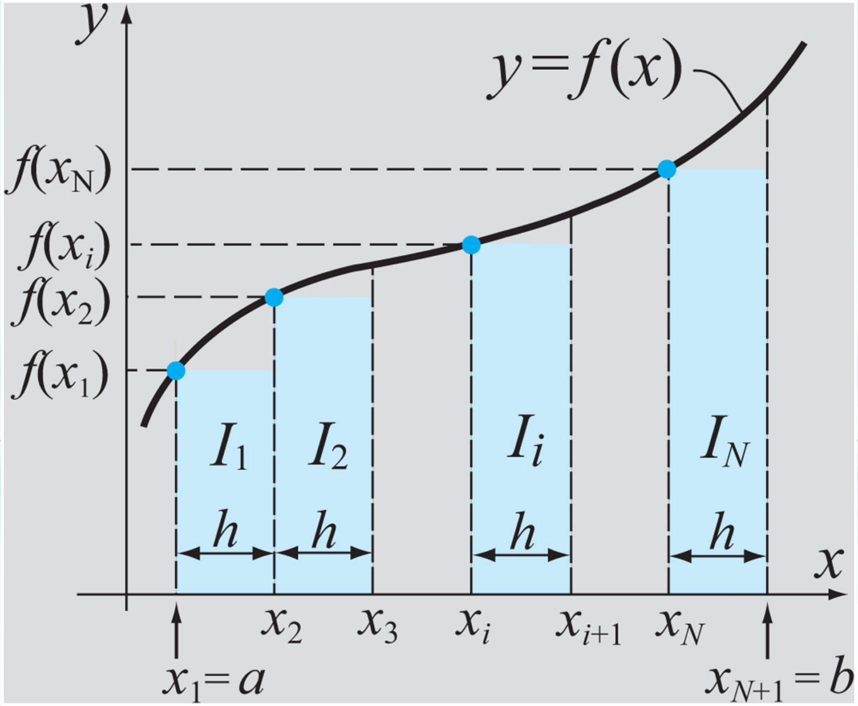


The integral is then:

$$I(f) = \int_a^b f(a)dx = f(a)(b-a) \quad \text{or} \quad I(f) = \int_a^b f(b)dx = f(b)(b-a) \tag{9.2}$$

Rectangle and Midpoint Methods:

A more accurate method is to use the Composite Rectangle Method in which the domain $[a, b]$ is divided into N subintervals and evaluating the integrals over each of the subintervals using the Rectangle Method and summing the results to get the integral over the complete domain.



Rectangle and Midpoint Methods:

Then:

$$(f) = \int_a^b f(x)dx \approx \overbrace{f(x_1)(x_2 - x_1)}^{I_1} + \overbrace{f(x_2)(x_3 - x_2)}^{I_2} + \ldots + \overbrace{f(x_i)(x_{i+1} - x_i)}^{I_i}$$

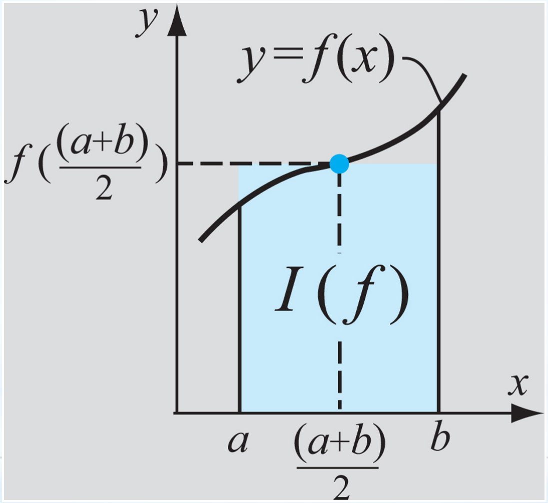
$$+ \ldots + \underbrace{f(x_N)(x_{N+1} - x_N)}_{I_\mathrm{N}} = \sum_{i=1}^{N} [f(x_i)(x_{i+1} - x_i)] \qquad (9.3)$$

or (when the subintervals have the same width h:

$$I(f) = \int_a^b f(x)dx \approx h\sum_{i=1}^{N} f(x_i) \qquad (9.4)$$

Rectangle and Midpoint Methods:

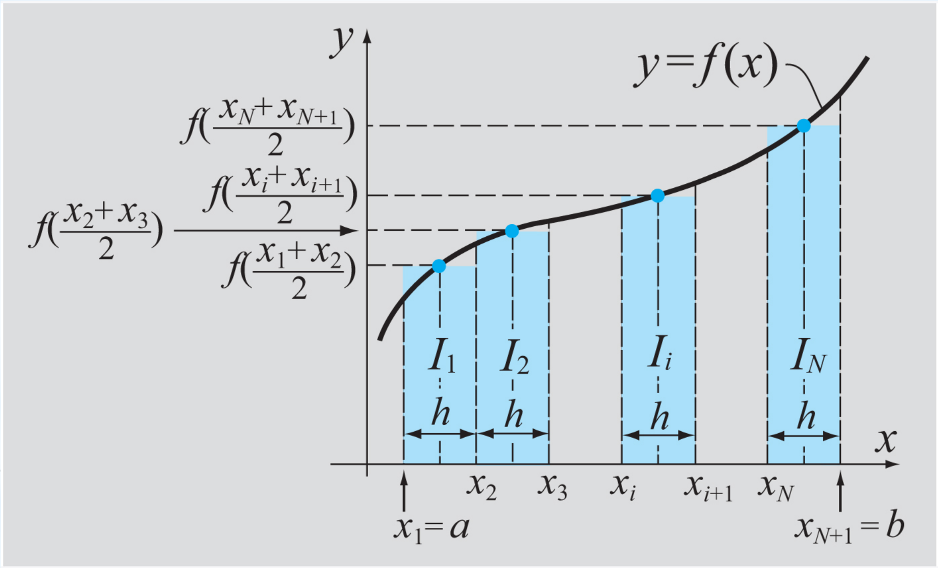
An improvement over the Rectangle Method is to use the Midpoint Method. Instead of approximating the integrand by the values of the function at $x = a$ or $x = b$, the value of the integrand at the middle of the interval, that is, $f\left(\frac{a+b}{2}\right)$ is used:



$$I(f) = \int_a^b f(x)dx \approx \int_a^b f\left(\frac{a+b}{2}\right)dx = f\left(\frac{a+b}{2}\right)(b-a) \qquad (9.5)$$

Rectangle and Midpoint Methods:

Again this can be improved upon by dividing the domain into N subintervals and performing a composite integration:



Rectangle and Midpoint Methods:

Mathematically this is:

$$I(f) = \int_a^b f(x)dx \approx \overbrace{f\left(\frac{x_1+x_2}{2}\right)(x_2-x_1)}^{I_1} + \overbrace{f\left(\frac{x_2+x_3}{2}\right)(x_3-x_2)}^{I_2} + \ldots$$

$$+ \overbrace{f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)}^{I_i} + \ldots + \overbrace{f\left(\frac{x_N+x_{N+1}}{2}\right)(x_{N+1}-x_N)}^{I_\mathrm{N}}$$

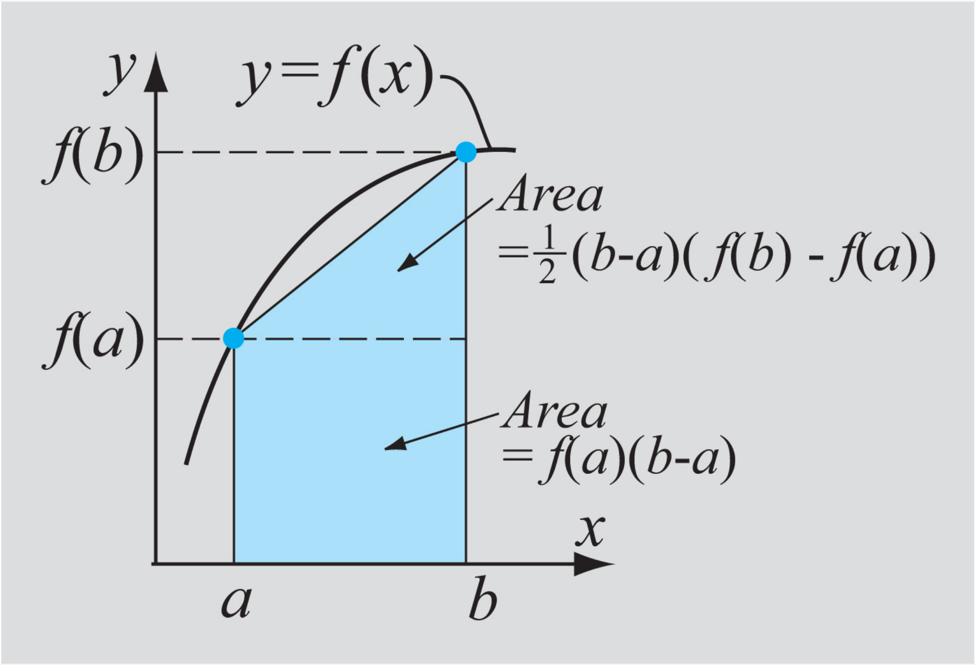
$$= \sum_{i=1}^{N}\left[f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)\right] \qquad (9.6)$$

If each of the subintervals have the same length h we can write:

$$I(f) = \int_a^b f(x)dx \approx h\sum_{i=1}^{N} f\left(\frac{x_i+x_{i+1}}{2}\right) \qquad (9.7)$$

The Trapezoidal Method:

A refinement over the simple Rectangle and Midpoint Methods is to use a linear function to approximate the integrand over the interval of integration.



The Trapezoidal Method:

The equation of a line connecting two points with x-ordinates a and b is:

$$f(x) \approx f(a) + (x-a) f[a, b] = f(a) + (x-a)\frac{[f(b) - f(a)]}{b-a} \qquad (9.8)$$

Substituting Equation (9.8) into (9.1) and integrating analytically gives:

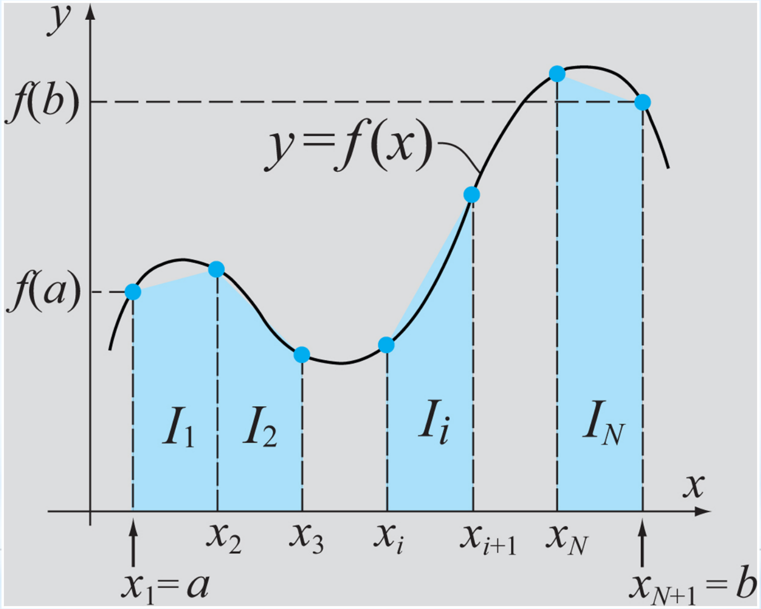
$$I(f) \approx \int_a^b \left(f(a) + (x-a)\frac{[f(b) - f(a)]}{b-a} \right) dx$$
$$= f(a)(b-a) + \frac{1}{2}[f(b) - f(a)](b-a) \qquad (9.9)$$

Simplifying the result gives the Trapezoidal rule:

$$\boxed{I(f) \approx \frac{[f(a) + f(b)]}{2}(b-a)} \qquad (9.10)$$

The Trapezoidal Method:

Again a more accurate result can be performed by composite integration yielding the Composite Trapezoidal Method:



The Trapezoidal Method:

Mathematically:

$$I(f) = \int_a^b f(x)dx = \underbrace{\int_{x_1=a}^{x_2} f(x)dx}_{I_1} + \underbrace{\int_{x_2}^{x_3} f(x)dx}_{I_2} + \dots + \underbrace{\int_{x_i}^{x_{i+1}} f(x)dx}_{I_i}$$

$$+ \dots + \underbrace{\int_{x_N}^{x_{N+1}} f(x)dx}_{I_N} = \sum_{i=1}^{N} \int_{x_i}^{x_{i+1}} f(x)dx \quad (9.11)$$

Applying the Trapezoidal Method to each subinterval yields:

$$I(f) = \int_a^b f(x)dx \approx \frac{1}{2} \sum_{i=1}^{N} [f(x_i) + f(x_{i+1})](x_{i+1} - x_i) \quad (9.12)$$

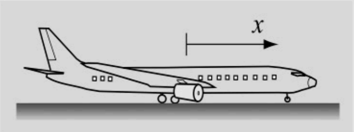
The Trapezoidal Method:

With equally sized subintervals this can be reduced to:

$$I(f) \approx \frac{h}{2}[f(a) + f(b)] + h\sum_{i=2}^{N} f(x_i) \qquad (9.13)$$

Example 9-1: Distance traveled by a decelerating airplane.

A Boeing 737-200 airplane of mass $m = 97000$ kg lands at a speed of 93 m/s (about 181 knots) and applies its thrust reversers at $t = 0$. The force F that is applied to the airplane, as it decelerates, is given by $F = -5v^2 - 570000$, where v is the airplane's velocity. Using Newton's second law of motion and flow dynamics, the relationship between the velocity and the position x of the airplane can be written as:



$$mv\frac{dv}{dx} = -5v^2 - 570000$$

where x is the distance measured from the location of the jet at $t = 0$.
Determine how far the airplane travels before its speed is reduced to 40 m/s (about 78 knots) by using the composite trapezoidal method to evaluate the integral resulting from the governing differential equation.

SOLUTION

Even though the governing equation is an ODE, it can be expressed as an integral in this case. This is done by separating the variables such that the speed v appears on one side of the equation and x appears on the other.

$$\frac{97000v\,dv}{(-5v^2 - 570000)} = dx$$

Next, both sides are integrated. For x the limits of integration are from 0 to an arbitrary location x, and for v the limits are from 93 m/s to 40 m/s.

$$\int_0^x dx = -\int_{93}^{40}\frac{97000v}{(5v^2+570000)}dv = \int_{40}^{93}\frac{97000v}{(5v^2+570000)}dv \tag{9.14}$$

The objective of this example is to show how the definite integral on the right-hand side of the equation can be determined numerically using the composite trapezoidal method. In this problem, however, the integration can also be carried out analytically. For comparison, the integration is done both ways.

Analytical Integration

The integration can be carried out analytically by using substitution. By substituting $z = 5u^2 + 570000$, the integration can be performed to obtain the value $x = 574.1494$ m.

Numerical Integration

To carry out the numerical integration, the following user-defined function, named `trapezoidal`, is created.

Program 9-1: Function file, integration trapezoidal method.

```
function I = trapezoidal(Fun,a,b,N)
% trapezoidal numerically integrate using the composite trapezoidal method.
% Input Variables:
% Fun  Name for the function to be integrated.
% (Fun is assumed to be written with element-by-element calculations.)
% a  Lower limit of integration.
% b  Upper limit of integration.
% N  Number of subintervals.
% Output Variable:
% I  Value of the integral.

h = (b-a)/N;                      Calculate the width h of the subintervals.
x = a:h:b;                        Create a vector x with the coordinates of the subintervals.
F = Fun(x);                       Create a vector F with the values of the integrand at each point x.

I=h*(F(1)+F(N+1))/2+h*sum(F(2:N));   Calculate the value of the integral according to Eq. (9.13).
```

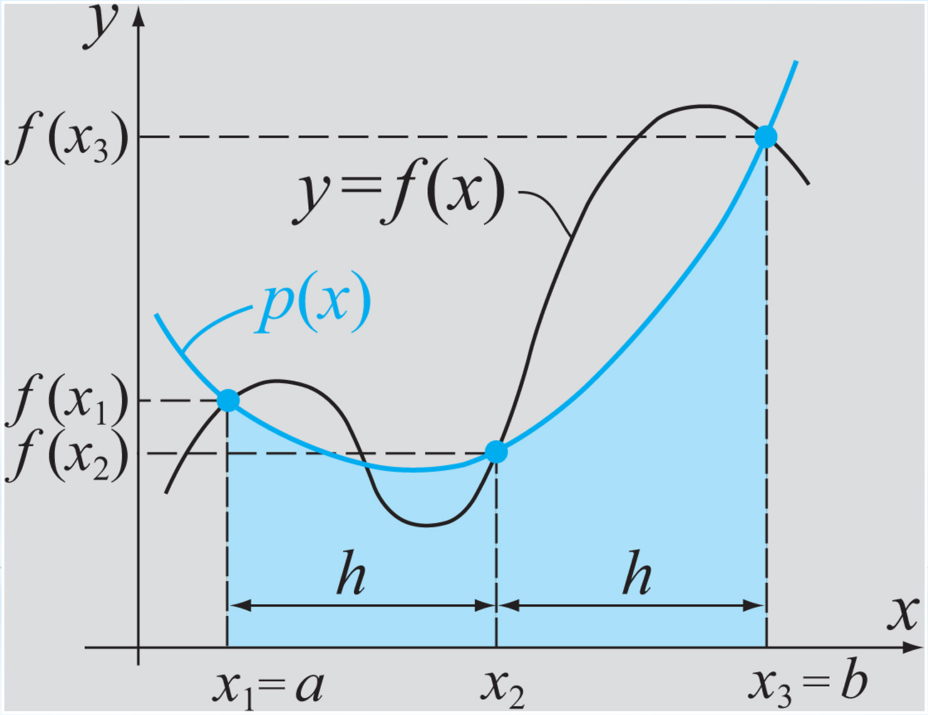
The function `trapezoidal` is used next in the Command Window to determine the value of the integral in Eq. (9.14). To examine the effect of the number of subintervals on the result, the function is used three times using $N = 10$, 100, and 1000. The display in the Command Window is:

```
>> format long g
>> Vel = @ (v) 97000*v./(5*v.^2+570000);     Define an anonymous function for the integrand.
                                             Note element-by-element calculations.
>> distance = trapezoidal(Vel,40,93,10)
distance =
        574.085485133712
>> distance = trapezoidal(Vel,40,93,100)
distance =
        574.148773931409
>> distance = trapezoidal(Vel,40,93,1000)
distance =
        574.149406775129
```

As expected, the results show that the integral is evaluated more accurately as the number of subintervals is increased. When $N = 1000$, the answer is the same as that calculated analytically to four decimal places.

Simpson's Methods:

With Simpson's Methods the integrand is approximated by a quadratic or cubic polynomial.



Simpson's Methods:

Simpson's 1/3 Method:

In this method a quadratic polynomial is used to approximate the integrand. Three points from the domain $[a, b]$ are used. These are the endpoints $x_1 = a$, $x_3 = b$ and the midpoint $x_2 = (a + b)/2$.

The polynomial can be expressed in Newton's form (it could just as well be expressed in standard or Lagrange form):

$$p(x) = \alpha + \beta(x - x_1) + \gamma(x - x_1)(x - x_2) \qquad (9.15)$$

Where α, β and γ are unknown constants evaluated by applying the conditions that the polynomial passes through the points $p(x_1) = f(x_1)$, $p(x_2) = f(x_2)$ and $p(x_3) = f(x_3)$.

Simpson's Methods:

These conditions yield:

$$\alpha = f(x_1), \qquad \beta = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

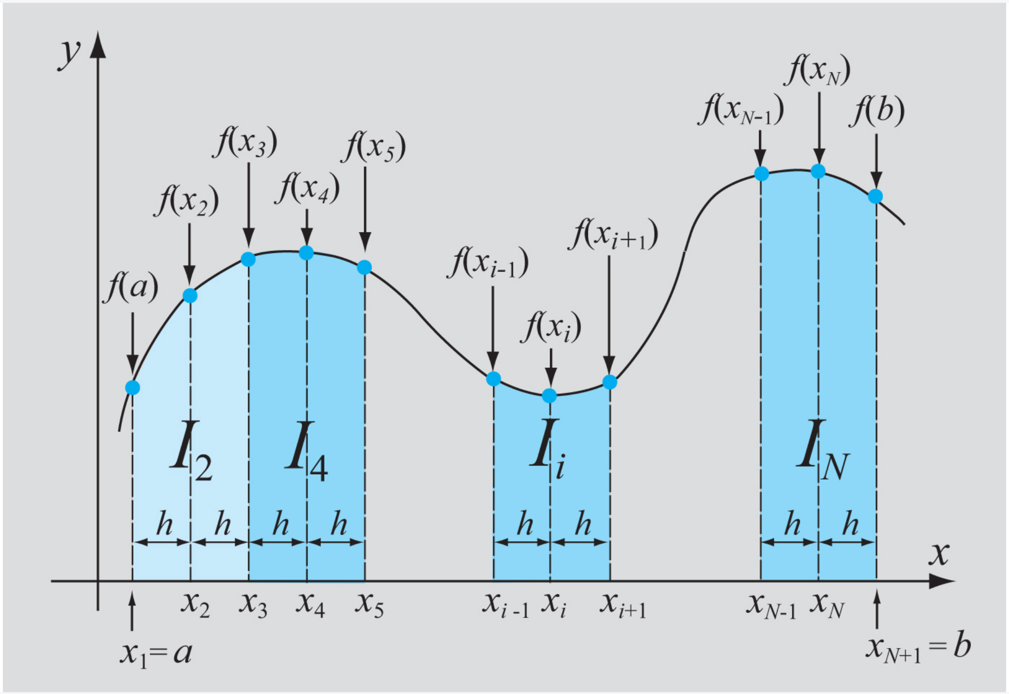
$$\gamma = \frac{f(x_3) - 2f(x_2) + f(x_1)}{2h^2}$$

where $h = \frac{b-a}{2}$. Substituting the values for the constants back into Equation (9.15) and integrating $p(x)$ over the interval $[a, b]$ gives:

$$\begin{aligned} I &= \int_{x_1}^{x_3} f(x)dx \approx \int_{x_1}^{x_3} p(x)dx = \frac{h}{3}[f(x_1) + 4f(x_2) + f(x_3)] \\ &= \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \end{aligned} \tag{9.16}$$

Simpson's Methods:

Simpson's Methods can be extended to a composite form by dividing the interval $[a, b]$ into N subintervals of length h.



Simpson's Methods:

Since Simpson's 1/3 Method is applied to two subintervals at a time it is necessary that the total number of subintervals is even. The integral over the whole interval is then written:

$$I(f) = \int_a^b f(x)dx = \overbrace{\int_{x_1=a}^{x_3} f(x)dx}^{I_2} + \overbrace{\int_{x_3}^{x_5} f(x)dx}^{I_4} + \ldots + \overbrace{\int_{x_{i-1}}^{x_{i+1}} f(x)dx}^{I_i} + \ldots$$

$$+ \underbrace{\int_{x_{N-1}}^{x_{N+1}=b} f(x)dx}_{I_N} = \sum_{i=2,4,6}^{N} \int_{x_{i-1}}^{x_{i+1}} f(x)dx \qquad (9.17)$$

Simpson's Methods:

Rewriting Equation (9.16) for two adjacent subintervals of equal length gives:

$$\text{by:}_i(f) = \int_{x_{i-1}}^{x_{i+1}} f(x)dx \approx \frac{h}{3}[f(x_{i-1}) + 4f(x_i) + f(x_{i+1} \qquad (9.18)$$

where $h = x_{i+1} - x_i = x_i - x_{i-1}$.

Substituting Equation (9.18) into (9.17) for each of the integrals and collecting similar terms gives the composite Simpson's 1/3 formula:

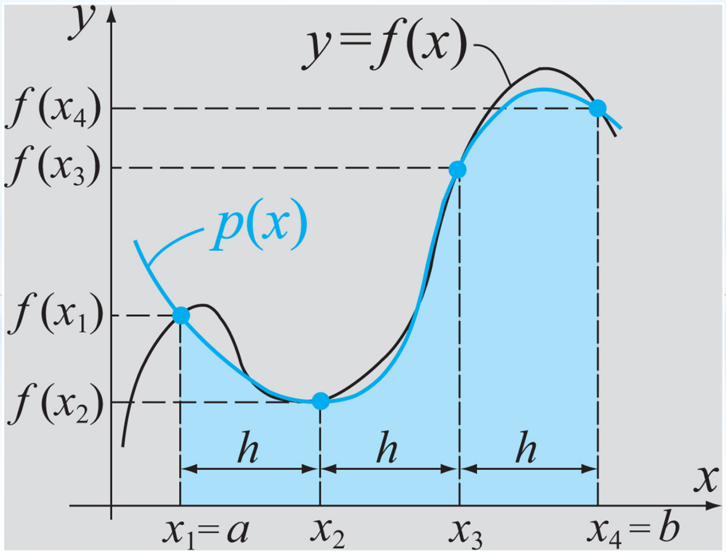
$$(f) \approx \frac{h}{3}\left[f(a) + 4\sum_{i=2,4,6}^{N} f(x_i) + 2\sum_{j=3,5,7}^{N-1} f(x_j) + f(b)\right] \qquad (9.19)$$

where $h = (b - a)/N$

Simpson's Methods:

Simpson's 3/8 Method:

In this method a cubic polynomial is used to approximate the integrand. A third order polynomial can be determined using four points. These are the endpoints $x_1 = a,\ x_4 = b$ and the two points x_2 and x_3 that divide the interval into three equal sections.



Simpson's Methods:

The polynomial can be written in the form:

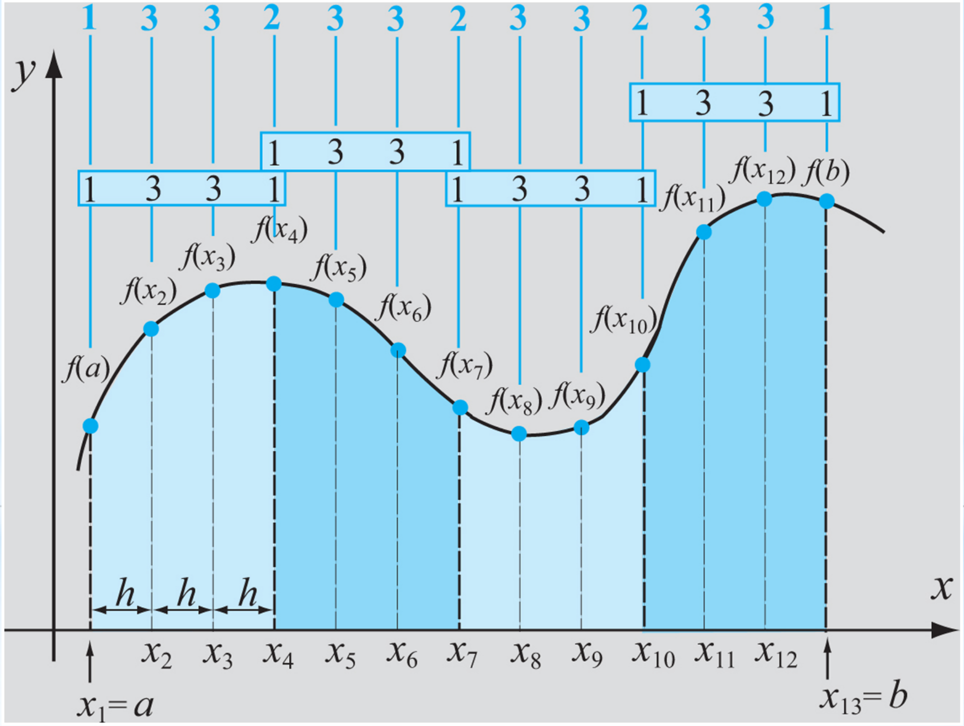
$$p(x) = c_3x^3 + c_2x^2 + c_1x + c_0$$

where c_0 to c_3 are determined from the conditions that the polynomial passes through the points, $p(x_1) = f(x_1), \dots, p(x_4) = f(x_4)$. Once these constants are determined the polynomial can be easily integrated to give:

$$= \int_a^b f(x)dx \approx \int_a^b p(x)dx = \frac{3}{8}h[f(a) + 3f(x_2) + 3f(x_3) + f(b \quad (9.20)$$

Simpson's Methods:

A composite form of Simpson's 3/8 Method can be achieved by dividing the interval $[a, b]$ into N subintervals. In general these can have arbitrary width but we concern ourselves with equal spacings here.



Simpson's Methods:

The integration in each group of three intervals is done by applying Equation (9.20). The integral over the whole domain is obtained by summing the integrals over the subintervals.

In general (where N is is divisible by 3):

$$(f) \approx \frac{3h}{8}\left[f(a) + 3\sum_{i=2,5,8}^{N-1}[f(x_i) + f(x_{i+1})] + 2\sum_{j=4,7,10}^{N-2} f(x_j) + f(b)\right] \quad (9.22)$$

Gauss Quadrature:

The general form of Gauss Quadrature is a weighted sum:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} C_i f(x_i) \tag{9.23}$$

The coefficients C_i are the weights and x_i are specific points in the interval known as Gauss points.

We obtain weights C_i and abscissae x_i that make Equation (9.23) exact for polynomials $x^0, x^1, x^2, \ldots$. The hope is that because these will yield exact results for polynomials they can then be used for other functions that are well approximated by polynomials.

Gauss Quadrature:

Beginning with the domain $[-1, 1]$ the form of Gauss quadrature is:

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} C_i f(x_i) \qquad (9.24)$$

We now force Equation (9.24) to be exact for the cases where $f(x) = 1, x, x^2, x^3, \ldots$. The number of cases that have to be considered depends on n. For example when $n = 2$ we need to determine four values:

$$\int_{-1}^{1} f(x)dx \approx C_1 f(x_1) + C_2 f(x_2) \qquad (9.25)$$

Gauss Quadrature:

The four constants C_1, C_2, x_1 and x_2 are determined by enforcing Equation (9.25) to be exact when applied to the following four cases:

Case 1: $f(x) = 1$ $\int_{-1}^{1}(1)dx = 2 = C_1 + C_2$

Case 2: $f(x) = x$ $\int_{-1}^{1} xdx = 0 = C_1x_1 + C_2x_2$

Case 3: $f(x) = x^2$ $\int_{-1}^{1} x^2dx = \frac{2}{3} = C_1x_1^2 + C_2x_2^2$

Case 4: $f(x) = x^3$ $\int_{-1}^{1} x^3dx = 0 = C_1x_1^3 + C_2x_2^3$

Gauss Quadrature:

This is a system of four equations with four unknowns so we can solve for C_1, C_2, x_1 and x_2. However since the equations are nonlinear multiple solutions can exist. A particular solution can be obtained by imposing the condition that x_1 and x_2 are symmetrically located about $x = 0$ i.e. $x_1 = -x_2$. This requirement implies that $C_1 = C_2$. Solving gives:

$$C_1 = C_2 = 1, x_1 = -\frac{1}{\sqrt{3}}, \quad x_2 = \frac{1}{\sqrt{3}}$$

Substituting back into Equation (9.25) gives (for $n = 2$):

$$\int_{-1}^{1} f(x)dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \qquad (9.26)$$

This equation is exact for $f(x) = 1, f(x) = x, f(x) = x^2$ and $f(x) = x^3$.

Gauss Quadrature:

Example:

Let $f(x) = \cos x$.

$$\int_{-1}^{1} \cos x \, dx = \sin x \Big|_{-1}^{1} = \sin(1) - \sin(-1) = 1.68294197$$

The approximate value using Gauss quadrature is:

$$\cos\left(\frac{-1}{\sqrt{3}}\right) + \cos\left(\frac{1}{\sqrt{3}}\right) = 1.67582366$$

This error is within 4.5% of the exact solution.

Gauss Quadrature:

The accuracy of Gauss quadrature is increased by using a higher value of n. For $n = 3$ the equation has the form:

$$\int_{-1}^{1} f(x)dx \approx C_1 f(x_1) + C_2 f(x_2) + C_3 f(x_3) \qquad (9.28)$$

In this case six values have to be determined: C_1, C_2, C_3 and x_1, x_2, x_3.

The values are determined as before by forcing Equation (9.28) to be exact when $f(x) = 1, f(x) = x, f(x) = x^2, \ldots, f(x) = x^5$. This results in a set of six equations with six unknowns. The values that are determined are given in the following table for this and a number of points up to $n = 6$.

Gauss Quadrature:

The general formula to use in conjunction with the table for Gauss quadrature is:

$$\int_{-1}^{1} f(x)dx \approx C_1 f(x_1) + C_2 f(x_2) + C_3 f(x_3) + \ldots + C_n f(x_n \qquad (9.30)$$

n (Number of points)	Coefficients C_i (weights)	Gauss points x_i
2	C_1 = 1 C_2 = 1	x_1 = –0.57735027 x_2 = 0.57735027
3	C_1 = 0.5555556 C_2 = 0.8888889 C_3 = 0.5555556	x_1 = –0.77459667 x_2 = 0 x_3 = 0.77459667
4	C_1 = 0.3478548 C_2 = 0.6521452 C_3 = 0.6521452 C_4 = 0.3478548	x_1 = –0.86113631 x_2 = –0.33998104 x_3 = 0.33998104 x_4 = 0.86113631
5	C_1 = 0.2369269 C_2 = 0.4786287 C_3 = 0.5688889 C_4 = 0.4786287 C_5 = 0.2369269	x_1 = –0.90617985 x_2 = –0.53846931 x_3 = 0 x_4 = 0.53846931 x_5 = 0.90617985
6	C_1 = 0.1713245 C_2 = 0.3607616 C_3 = 0.4679139 C_4 = 0.4679139 C_5 = 0.3607616 C_6 = 0.1713245	x_1 = –0.93246951 x_2 = –0.66120938 x_3 = –0.23861919 x_4 = 0.23861919 x_5 = 0.66120938 x_6 = 0.93246951

Gauss Quadrature:

Evaluating $\int_{-1}^{1} \cos x \; dx$ again using Gauss quadrature with three points we get:

$$\int_{-1}^{1} \cos x \; dx$$

$$\approx 0.5555556 cos(-0.77459667) + 0.8888889 cos(0) + 0.5555556 cos(0.77459667)$$

$$= 1.68285982$$

This is an almost exact result.

Gauss Quadrature:

When the domain of integration is not $[-1, 1]$ then a change of variables is necessary. In other words the integral $\int_a^b f(x)dx \ -> \int_{-1}^{1} g(t)dt$.

The change in variable is:

$$x = \frac{1}{2}[t(b-a) + a + b] \quad \text{and} \quad dx = \frac{1}{2}(b-a)dt \qquad (9.31)$$

The Gauss quadrature is then performed using the transformed integral.

Example:

Example 9-2: Evaluation of a single definite integral using fourth-order Gauss quadrature.

Evaluate $\int_0^3 e^{-x^2} dx$ using four-point Gauss quadrature.

SOLUTION

Step 1: Since the limits of integration are $[0, 3]$, the integral has to be transformed to the form $\int_{-1}^{1} f(t)dt$. In the present problem $a = 0$ and $b = 3$. Substituting these values in Eq. (9.31) gives:

$$x = \frac{1}{2}[t(b-a)+a+b] = \frac{1}{2}[t(3-0)+0+3] = \frac{3}{2}(t+1) \quad \text{and} \quad dx = \frac{1}{2}(b-a)dt = \frac{1}{2}(3-0)dt = \frac{3}{2}dt$$

Substituting these values in the integral gives:

$$I = \int_0^3 e^{-x^2} dx = \int_{-1}^{1} f(t)dt = \int_{-1}^{1} \frac{3}{2} e^{-\left[\frac{3}{2}(t+1)\right]^2} dt$$

Step 2: Use four-point Gauss quadrature to evaluate the integral. From Eq. (9.30), and using Table 9-1:

$$I = \int_{-1}^{1} f(t)dt \approx C_1 f(t_1) + C_2 f(t_2) + C_3 f(t_3) + C_4 f(t_4) = 0.3478548 \cdot f(-0.86113631)$$
$$+ 0.6521452 \cdot f(-0.33998104) + 0.6521452 \cdot f(0.33998104) + 0.3478548 \cdot f(0.86113631)$$

Evaluating $f(t) = \frac{3}{2} e^{-\left[\frac{3}{2}(t+1)\right]^2}$ gives:

$$I = 0.3478548 \frac{3}{2} e^{-\left[\frac{3}{2}((-0.86113631)+1)\right]^2} + 0.6521452 \frac{3}{2} e^{-\left[\frac{3}{2}((-0.33998104)+1)\right]^2}$$
$$+ 0.6521452 \frac{3}{2} e^{-\left[\frac{3}{2}(0.33998104+1)\right]^2} + 0.3478548 \frac{3}{2} e^{-\left[\frac{3}{2}(0.86113631+1)\right]^2} = 0.8841359$$

The exact value of the integral (when carried out analytically) is 0.8862073. The error is only about 1%.

Improper Integrals:

So far we have considered integrals of well behaved functions with finite limits. Situations arise however where functions are not well behaved (discontinuous or singular) and/or have infinite limits.

Integrals with Singularities:

If the integral $\int_a^b f(x)dx$ is singular at $x = c \in [a, b]$ then the integration can be performed over two intervals $[a, c]$ and $[c, b]$. Mathematically integrals that have a singularity at one of the endpoints may or may not have a finite value. For example the function $1/\sqrt{x}$ is singular at $x = 0$ but the integral of the function over $[0,2]$ has a value of 2.
That is: $\int_0^1 \frac{1}{\sqrt{x}} dx = 2$. An estimate of the integral can be obtained using any of the methods described here that do not use endpoints e.g. composite midpoint or Gauss quadrature. Otherwise we can choose points very close to the endpoints that do not include the singularity and integrate over this domain.
On the other hand $\int_0^1 \frac{1}{x} dx$ does not have a finite value (i.e. $= \infty$).

Improper Integrals:

Integrals with Unbounded Limits:

Again integrals with infinite limits can converge (have a finite value) or may diverge (have an infinite value). In the case of the former the integration can be performed over a finite domain where the function has values that are not close to zero. This is a typical situation where the integrand has a finite value over a relatively small domain and close to zero everywhere else. An example is the Gaussian pdf in statistics:

$$\int_{-\infty}^{b} \frac{1}{\sqrt{2\pi}} e^{\left(\frac{-x^2}{2}\right)} dx$$

The computation of the integral stops where successive iterations yield only a small (tolerable) change in the result.

Recommended Problems:

Problems to be solved by hand (do at least 2):
9.1, 9.5, 9.7, 9.10

Problems to be programmed in MATLAB:
9.23, 9.26

Problems in Science and Engineering (do at least 1):
9.30(b), 9.35

You should pick out problems that you find interesting/challenging and do these too.