

Injective types

I will not present
[1] and [2] in order.

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I'll intersperse the
results.

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- [1] M-H. Escardó. Injective types in univalent mathematics.
Mathematical structures in Computer Science
Vol 31, Issue 1, pp. 89–111, 2021

- [2] Tom de Jong & M-H.E. <http://cs.bham.ac.uk/~mhe/TypeTopology>
more recently, in Agda. We are formalizing it into a paper.

Extension problem

embedding:



$j^{-1}y$ is a proposition,

$$j^{-1}y \stackrel{\text{def}}{=} \sum_{x \in X} jx = y \quad (\text{fiber})$$

⋮

- D is injective if for every f and j there is some \bar{f} .

- D is algebraically injective if for any f and j there is a designated \bar{f} .

Σ

we denote it by
 f/j as $f/j \circ j = f$.

Injectivity and excluded middle

T.F.A.E.

- excluded middle holds (i.e. we are working in a boolean topos).
- The algebraically injective types are precisely the pointed types.

Also if excluded middle holds then the injective types are precisely the non-empty types.

Injectivity under propositional resizing

(Any proposition in my universe has an equivalent copy in my universe we please.)

1. Injectivity is equivalent to the propositional truncation of algebraic injectivity. (This is a particular case of choice that just holds.)
2. The algebraically injectives are precisely the retracts of types of the form $X \rightarrow U$ with U a type universe.
 - (a) The alg. injective sets are the retracts of powersets.
 - (b) The alg. inj. $(n+1)$ -types are the retracts of types of the form $X \rightarrow U$ with U a universe of n -types

ctd

3. The algebraically injective types are precisely the underlying types of the algebras of the partial-map classifier monad.

$$\mathcal{L}X \stackrel{\text{def}}{=} \sum_{P:\Omega} (P \rightarrow X)$$

type of partial elements
of X .

type of propositions

4. Any universe \mathcal{U} is a retract of the next universe \mathcal{U}^+ .

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\quad \iota \quad} & \mathcal{U}^+ \\ id \Downarrow & & \pi \end{array}$$

because \mathcal{U} is injective.

In the absence of propositional resizing

We are forced to keep track of universe levels.

$$\begin{array}{ccc} \mathcal{U} & & \mathcal{V} \\ X & \xrightarrow{j} & Y \\ f \searrow & & \swarrow f/j \\ D & & W \end{array}$$

D is (algebraically) \mathcal{U}, \mathcal{V} -injective if for every $x : \mathcal{U}, y : \mathcal{V}$,

$j : x \hookrightarrow Y, f : X \rightarrow D$, there is a (designated) f/j

such that the above diagram commutes.

In the presence of propositional resizing

(Algebraic) infectivity is universe independent:

D is \mathcal{U}, \mathcal{V} -infective iff

D is $\mathcal{U}'-\mathcal{V}'$ -infective, for any universes $\mathcal{U}, \mathcal{V}, \mathcal{U}', \mathcal{V}'$.

From now on:

1. We will not assume propositional resizing.

2. We will work exclusively with algebraic infectivity,
perversely abbreviated as infectivity.

(Requires univalence)

Universes are injective

In many ways there are two extreme ways.

Left Kan extension

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{j} & \mathcal{V} \\ f \downarrow & & \downarrow f \setminus j(\mathcal{Y}) \stackrel{\text{def}}{=} \sum_{(x,-) : j^{-1}(\mathcal{Y})} f^x \end{array}$$

Recall:
 $j^{-1} \text{dis}$
 $j(\mathcal{Y}) = \sum_{x : X} f^x = \mathcal{Y}$

Right Kan extension

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{j} & \mathcal{V} \\ f \downarrow & & \downarrow f \setminus j(\mathcal{Y}) \stackrel{\text{def}}{=} \prod_{(x,-) : j^{-1}(\mathcal{Y})} f^x \end{array}$$

j is embedding

proposition because

Particular case \mathcal{U} is injective

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{j} & \mathcal{U} \\ X & \searrow f & \swarrow i' \\ & \mathcal{U} & \end{array}$$

This doesn't work (in the absence of propositional resizing)

if we promote X or Y to live in universes larger than \mathcal{U} .

$\Omega_{\mathcal{U}}$ -resizing

The type $\Omega_{\mathcal{U}} \stackrel{\text{def}}{=} \sum_{x:\mathcal{U}} \text{is-prop } x \times (\neg\neg x \rightarrow x)$,
whose native universe is \mathcal{U}^+ , has an equivalent copy in \mathcal{U} .

Not provable or disprovable.

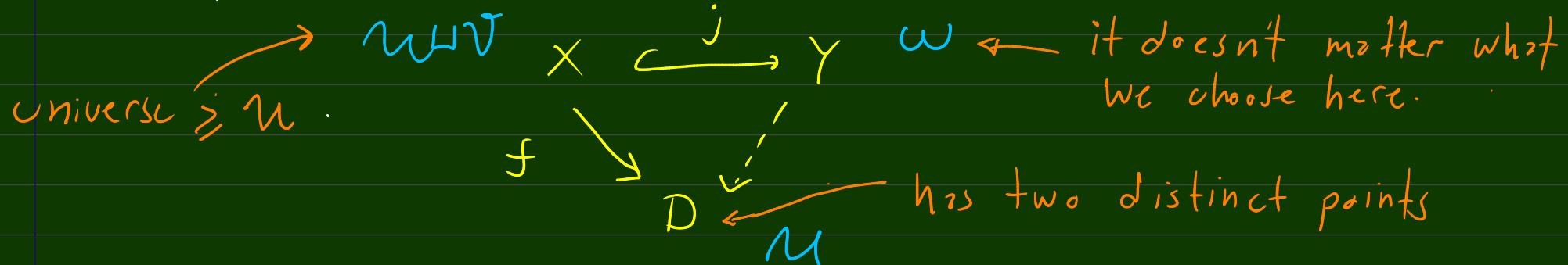
[1] M. Rathjens. Proof theory of constructive systems: inductive types & univalence (2017)

[2] A. Swan. Double negation stable h-propositions in cubical sets (2022) arxiv

No-go theorem

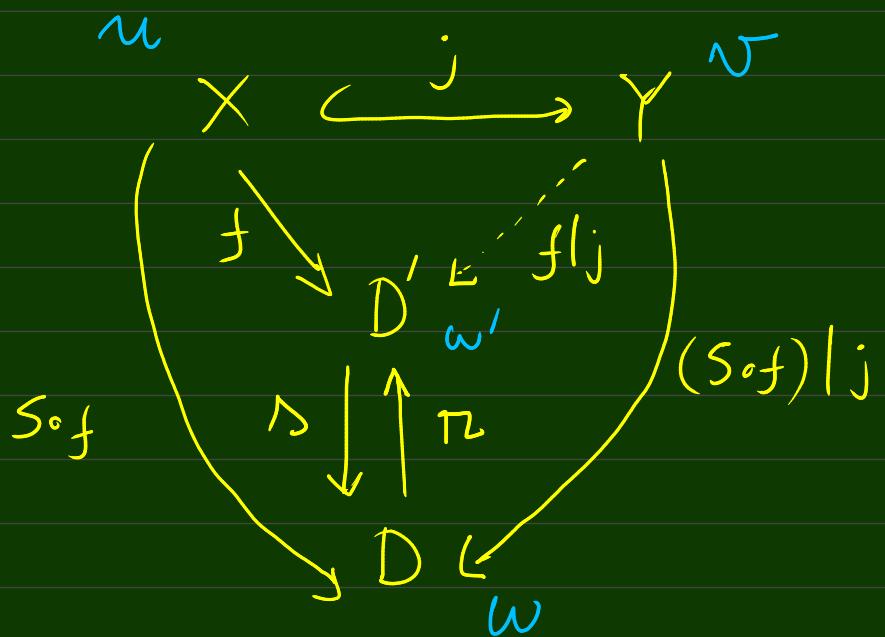
There are no small injectives in general.

Suppose we have the following injectivity situation.



Then $\Omega_{\mathcal{M}}$ -resizing holds.

Retracts of injectives are injective



If D is injective, then so is my retract D' of D .

Products of injectives are injective

$$\begin{array}{ccc} m & & n \\ X \xrightarrow{j} Y & & X \xrightarrow{j} Y \\ f_i \searrow & \swarrow f_i|j & f \searrow \\ D_i & & \prod_i D_i \\ w & & \text{wrt} \end{array}$$

Index type in universe \mathcal{T} .

Every injective type is a retract of any type it is embedded into

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\delta} & \mathcal{V} \\ D & \hookrightarrow & Y \\ id & \searrow & \downarrow \pi^{\text{def}} = \delta \circ id \\ & D & \mathcal{U} \end{array}$$

Or: every embedding of an injective type into any type
is a section.

The identity type former is an embedding

Found independently by Egbert Rijke

$$X \hookrightarrow (X \rightarrow U)$$

$$x \mapsto (y \mapsto x = y)$$

Corollary. If $D:U$ is U, U^+ -injective
then D is a retract of $D \rightarrow U$.

Because

$$D \hookrightarrow (D \rightarrow U)$$

$$\begin{array}{ccc} id & \searrow & \check{\pi} \\ & D & \end{array}$$

N.B. Because
of the no-go
theorem, the
hypothesis can't
be fulfilled without
some form of
propositional
resizing.

More examples

Non-exhaustive list among
the examples we know.

The following types, which live in the universe \mathcal{U}^+ ,
are \mathcal{U} , \mathcal{U} -injective.

0. The type of propositions in \mathcal{U} .

1. The type of ordinals in \mathcal{U} .

2. The type of iterative (multi)sets in \mathcal{U} .

3. The type \mathcal{U}_\bullet of pointed types in \mathcal{U} .

4. The types of ∞ -magmas, pointed ∞ -magmas,
monoids, groups in \mathcal{U} .

5. The type $\mathcal{Z}X \stackrel{\text{def}}{=} \sum P : \Omega_{\mathcal{U}}, (P \rightarrow X)$ of
partial elements of any type $X : \mathcal{U}$.

6. The carriers of suplattices in \mathcal{U}^+ .
/pointed dcpos

Some counter-examples

(WEM) Weak excluded middle $\stackrel{\text{def}}{=}$ for every $P : \Omega_n$, $\neg P$ or $\neg\neg P$.

Equivalent to De Morgan Law.

If any of the following types is injective, then WEM holds.

1. $2 \stackrel{\text{def}}{=} 1+1$.
2. The simple types, obtained from IN by iterating " \rightarrow ".
3. The type of Dedekind reals.
4. The type $\text{IN}_\infty \stackrel{\text{def}}{=} \sum \alpha : \text{IN} \rightarrow 2, (\prod i : \text{IN}, \alpha_i \geq \alpha_{i+1})$.
5. Any type with an apartness relation and two points apart.
6. Any type of the form $X + Y$ with X and Y pointed.

Another counter-example

All propositions are projective $\stackrel{\text{def}}{=}$

$$\prod P : \mathcal{D}_n, \prod Y : P \rightarrow n,$$

$$(\prod P : P, \|Y_P\|)$$

$$\rightarrow \|\prod P : P, Y_P\|.$$

When $\|Y_P\| \cong \mathcal{D}\|$, this is known as the world's simplest axiom

which is not valid in some toposes (Fournier & Scedrov 1982)
of choice,

If the type $\sum x : u . \|X\|$ of inhabited types
is injective, then all propositions are projective.

An example related to the previous counter-example

The type $\sum_{x:\mathbb{N}} \neg\neg X$ of non-empty types
is injective.

This gives an illustration of the difference between
propositional truncation and double negation.

(They are equivalent iff excluded middle holds.)

Indecomposability of injective types

In computation, it is important to identify decidable properties of types.

Injective types have no non-trivial decidable properties.

"Rice's Theorem"

for injective types.

$$\text{decomposition } X \stackrel{\text{def}}{=} \sum f: X \rightarrow 2, f^{-1}0 \times f^{-1}1$$

$$\simeq \sum_{X_0, X_1: u} (X_0 + X_1 \simeq X) \times X_0 \times X_1$$

If an injective type has a decomposition,

then weak excluded middle holds.

When is Σ subtype of an infective type itself infective?

Not always. We have seen that \mathcal{U} is infective, but its subtype $\sum x : \mathcal{U} . ||x||$ is not in general infective.

A subtype $\sum d : D . P_d$ of an infective type $D : \mathcal{U}^+$,
with $P : D \rightarrow \Omega_{\mathcal{U}}$, is \mathcal{U}, \mathcal{U} -infective iff

there is a designated function $f : D \rightarrow D$ s.t. for all $d : D$,

1. $P(f d)$ holds, and

2. P_d implies $f d = d$.

E.g. $\bullet f x = (\lambda x \rightarrow x) \quad \checkmark$
 $\bullet f x = (||x|| \rightarrow x) \quad \times$
Needs propositions projective.

(Algebraically) flabby types

A type D is (algebraically) \mathcal{U} -flabby if the extension problem

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{P \in} & 1 \\ & \searrow & \downarrow \\ & D & \mathcal{U} \end{array}$$

can be solved for every proposition P .

Every partial element of D can be extended to a (total) element of D .

Algebraic flabbiness is witnessed by a function $\sqcup : (P \rightarrow D) \rightarrow D$
 s.t. for every $f : P \rightarrow D$ and every $p : P$, we have $\sqcup f = f p$.

F is bby \Leftrightarrow injective

1. The direction \Leftarrow is by definition.

But let's record the universe levels.

If $D : \mathcal{U}$ is \mathcal{M}, \mathcal{V} -injective then it is \mathcal{U} -flabby.

(so notice that the universe \mathcal{V} is forgotten in the conclusion.)

2. For the direction \Rightarrow , we perform the following construction.

$$f: X \xrightarrow{j} Y$$

$$f|_D: f(j(y)) \stackrel{\text{def}}{=} \bigsqcup_{(x,-) : (j^{-1}y)} f_x$$

\in proposition

For $D = \mathcal{U}$ we
 took $\bigsqcup = \Sigma$
 and $\bigsqcup = \prod$
 as possible choices.

The second direction with explicit universe levels

2. If a type $D : W$ is $\mathcal{M} \sqcup \mathcal{T}$ -flabby then it is also \mathcal{M}, \mathcal{T} -injective.

We get the following [resizing theorem] by going back and forth with the above constructions.

$1 \Rightarrow 2 \Rightarrow 1$. If a type $D : W$ is $\mathcal{M} \sqcup \mathcal{T}$, \mathcal{T} -injective, then it is also \mathcal{M}, \mathcal{T} -injective.

Notice that, again, the universe \mathcal{T} is forgotten.

Flabbiness of \sum types

We give a sufficient condition, which we know not to be

necessary (by one of the examples already given). ① flabby

$$\begin{array}{ccc}
 P & \xrightarrow{\quad} & 1 \\
 & \searrow & \downarrow \\
 < f, g > & & (\times, a) \\
 & \swarrow & \downarrow \\
 & \sum_{x:X} A x &
 \end{array}$$

$f: P \rightarrow X$
 $g: \prod_{p:P} A(f p)$

② compatibility dots

The canonical map $\rho: A(\sqcup_f) \rightarrow \prod_{p:P} A(f p)$

has a given section.

* The type \mathcal{U}_0 is flabby and $\simeq \sum_{(x,-)} (\sum_{x:\mathcal{U}} , ||x||), X$

not flabby

$\left(\sum_{x:\mathcal{U}} , ||x|| \right), X$

Flabbiness of \sum -types of the form $\sum x:U. A x$

E.g take $A x \stackrel{\text{def}}{=} (x \times x \rightarrow x)$ for the type of ∞ -magma.

Define $T: \prod_{X,Y} X \simeq Y \rightarrow A x \rightarrow A y$ using univalence as expected.

Then $T_{x,y} f$ is an equivalence $A x \simeq A y$.

Lemm. For any T of the same type as γ ,

if $T \text{id} \sim \text{id}$, then $T f \sim \gamma f$ for any $f: x \simeq y$.

ctd.

$P: \Omega$

For any $P_0: P$ we have a canonical equivalence $(\overline{\Pi}_{P: P} F_P)^\varphi \cong F_{P_0}$.
 and $F: P \rightarrow \mathcal{U}$

Lemma: The compatibility condition holds if the following map has a section for all $P: \Omega_{\mathcal{U}}$ and $F: P \rightarrow \mathcal{U}$:

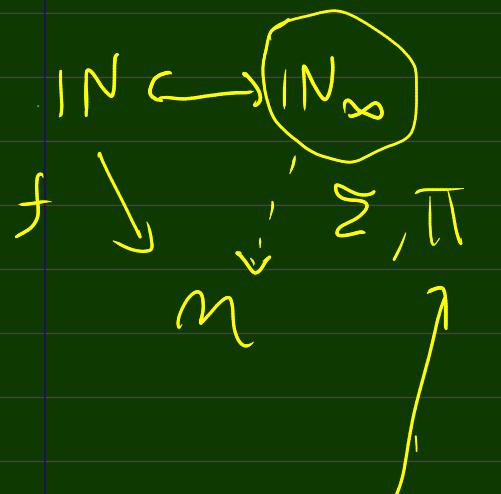
$$P: A(\Pi_{P: P} F_P) \rightarrow \overline{\Pi}_{P: P} A F_P$$

$$a \longmapsto (P_0 \mapsto \overline{T}^\varphi a)$$

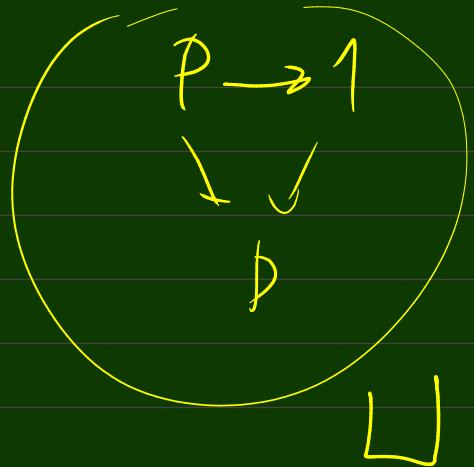
This condition is very easy to check for magmas, monoids, groups etc.
 The only "art" is to choose a suitable \overline{T} with $\overline{T} \text{id} \sim \text{id}$.

Discussion

1. My original interest in injectivity comes from my work on searchable types in which I use injectivity to construct plenty of them, using ordinals to measure their complexity.
These types are totally separated (the functions into 2 separate the points) even though injective types are indecomposable.
2. There are equations for algebraic fibriness that make the notion even more algebraic, in fact coinciding with algebras of the partial-map classifier monad \mathcal{L} .
3. While a paper is not available, we invite you to read TypeTopology.ij



$$\begin{array}{ccc} X & \xhookrightarrow{j} & Y & \xhookrightarrow{k} & Z \\ f \searrow & & \downarrow f|j & & \\ D & & & & (f|j)|k = f|(k \circ j) \end{array}$$



want this

$$\begin{array}{ccc} X & \xhookrightarrow{j} & Y \\ f \downarrow & \swarrow & f|j \stackrel{?}{=} y \mapsto \square \\ D & & (x, -) : j^{-1} y \end{array}$$

(written during question time)