Methodol Comput Appl Probab (2010) 12:681–694
DOI 10.1007/s11009-009-9134-6

Multivariate Hierarchical Copulas with Shocks

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Received: 13 December 2008 / Revised: 7 May 2009 /
Accepted: 7 May 2009 / Published online: 16 May 2009

Abstract A transformation to obtain new multivariate hierarchical copulas, starting with an arbitrary copula, is introduced. In addition to the hierarchical structure, the presented construction principle explicitly supports singular components. These may be interpreted as the effect of local or global shocks to the underlying random variables. A large spectrum of dependence patterns can be achieved by the presented transformation, which seems promising for practical applications. Moreover, copulas arising from this construction are similarly admissible with respect to analytical tractability and sampling routines as the original copula. Finally, several well-known families of copulas may be interpreted as special cases.

Keywords Copula · Hierarchical structure · Marshall-Olkin distribution · Shock model · Singular component

Mathematics Subject Classifications (2000) 60E05 · 62H99 · 65C10 · 65C99 · 91B70

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1 Introduction

Copulas are distribution functions of random vectors on $[0, 1]^d$, $d \geq 2$, with standard uniform univariate margins. They are mainly used for two reasons: the first being to study the dependence structure of a random vector separately from the marginal distributions; the second to construct new families of multivariate distributions. Both are well-known applications inferred from Sklar's Theorem, see Sklar (1959). The construction of new copulas, motivated by the quest for theoretical insight or for practical applications, has been the objective of several investigations over the last years. However, most constructions have focused on the bivariate case and just a few parametric families of copulas are known in dimension $d \geq 3$.

Recently, several authors have considered transformations to construct new copulas from given ones, in order to enlarge the number of parameters or to allow for asymmetries, see e.g. Morillas (2005), Liebscher (2008), Aas et al. (2009), and Rodríguez-Lallena and Úbeda-Flores (2009). The presented transformation has a similar intention. Additionally, it allows for an interpretation in terms of local or global shocks, accompanying the analytical form, and for a hierarchical structure. The demand for hierarchical structures in applications was recently addressed by Embrechts (2009), and for nested Archimedean copulas particularly by Joe (1997, pp. 87), Savu and Trede (2006), McNeil (2008), Hofert (2008), and Hofert and Scherer (2008). Asymmetric copulas are particularly of interest for the modeling of classified components where the categories are exposed to diverse effects. Examples for such hierarchical structures naturally arise e.g. when data from different geographic regions or industry sectors is collected, see Hofert and Scherer (2008) for an application in credit-risk modeling.

The presented construction principle, inspired by Durante et al. (2007), further incorporates singular components. For other copulas with this property, see e.g. Cuadras and Augé (1981), Marshall and Olkin (1967), McNeil and Nešlehová (2009), and Mai and Scherer (2009). In the current framework, singular components allow for the intuitive interpretation of being the result of a global or local shock.

The probabilistic representation of the presented transformation is useful for two reasons. Firstly, it allows to understand and interpret the induced dependence structure. Secondly, it suggests a natural sampling strategy for the resulting copula. For practical applications it is convenient that sampling routines for the original copula can easily be adapted to sample the transformation with negligible additional computational costs. Other authors considering the relevant, yet difficult, problem of sampling copulas in large dimensions include e.g. Marshall and Olkin (1988), McNeil et al. (2005, pp. 193), McNeil (2008), McNeil and Nešlehová (2009), and Mai and Scherer (2009). For instance, efficient sampling routines for multivariate distributions are popular in mathematical finance, where the treatment of large portfolios (for risk management applications or the pricing of basket options) often requires Monte Carlo simulations. This article also addresses how the transformation affects measures of association such as Kendall's tau, Spearman's rho, and the tail dependence coefficients.

The remaining paper is organized as follows: Section 2 formally introduces the transformation as a main result and presents various examples as corollaries. Measures of association of the transformation are investigated in Section 3. A sampling routine, illustrated by two examples, is given in Section 4. Finally, Section 5 concludes.

2 The Construction Principle

We consider a d-dimensional random vector $\boldsymbol{X}$ on the unit d-cube of type

$$\boldsymbol{X} = \left(X_{11}, \ldots, X_{1d_1}, \ldots, X_{J1}, \ldots, X_{Jd_J}\right) \in [0,1]^d, \; d = \sum_{j=1}^{J} d_j, \tag{1}$$

where $d_j \geq 1$ for $j = 1, \ldots, J$. Suppose that the vector $\boldsymbol{X}$ is distributed according to

$$F(\boldsymbol{u}) = C_0\left(F_1\left(u_{11}\right), \ldots, F_1\left(u_{1d_1}\right), \ldots, F_J\left(u_{J1}\right), \ldots, F_J\left(u_{Jd_J}\right)\right), \; \boldsymbol{u} \in [0,1]^d, \tag{2}$$

for some copula C_0 and some distribution functions F_j on $[0, 1]$, $j \in \{1, \ldots, J\}$, such that $X_{ji} \sim F_j$ for $i \in \{1, \ldots, d_j\}$. Since all involved random variables have support on $[0, 1]$ we only consider $[0, 1]^d$ for notational simplicity; the extension of the respective distribution functions to $\mathbb{R}^d$ is straightforward. Intuitively speaking, we may consider $\boldsymbol{X}$ as a portfolio of fractional losses partitioned in J homogeneous sectors, i.e. components belonging to the same sector are identically distributed and dependent according to the j-th margin $C_0^{(j1,\ldots,jd_j)}$ of C_0. For d assets, insurance claims, etc., with nominals $N_{11}, \ldots, N_{Jd_J}$ one might interpret $X_{ji} \in [0, 1]$, i.e. X_{ji} takes values in $[0, 1]$, as the fractional loss of N_{ji}.

We now expose the vector $\boldsymbol{X}$ to different kinds of independent shocks. Local shocks affecting sector $j \in \{1, \ldots, J\}$ are modeled by a random variable $Z_j \in [0, 1]$ with distribution function $G(t)/F_j(t)$, $t \in [0, 1]$. Global shocks affecting all components of $\boldsymbol{X}$ simultaneously are modeled by a random variable $Z \in [0, 1]$ with distribution function $t/G(t)$, $t \in [0, 1]$, under some weak assumptions on F_j and G as stated in Theorem 1, where $\boldsymbol{X}$, Z_j, $j \in \{1, \ldots, J\}$, and Z are assumed to be mutually independent. Recalling our interpretation for $\boldsymbol{X}$, some adverse local or global event may cause additional losses in an insurance portfolio. More formally, in our framework shocks are incorporated by means of the shock transformation

$$Y_{ji} = \max\{X_{ji}, Z_j, Z\} \in [0,1], \; i \in \{1, \ldots, d_j\}, \; j \in \{1, \ldots, J\}. \tag{3}$$

This shock transformation obviously alters the dependence structure of the components of $\boldsymbol{X}$. The main result, i.e. Theorem 1, addresses the distribution function of the vector $\boldsymbol{Y} = (Y_{11}, \ldots, Y_{Jd_J}) \in [0, 1]^d$.

Theorem 1 (Shock transformation) *Let $\boldsymbol{X} \in [0, 1]^d$ be given as in Eqs. 1 and 2. Assume that $F_j : [0, 1] \to [0, 1]$, $j \in \{1, \ldots, J\}$, are distribution functions and $G : [0, 1] \to [0, 1]$ is a function such that $t/G(t)$ and $G(t)/F_j(t)$, $j \in \{1, \ldots, J\}$, are distribution functions on $[0, 1]$. Then, the distribution function of the vector $\boldsymbol{Y} \in [0, 1]^d$ of Eq. 3 is given by*

$$C(\boldsymbol{u}) = C_0\left(\left(F_j\left(u_{j1}\right), \ldots, F_j\left(u_{jd_j}\right)\right)_{j=1}^{J}\right) \cdot \frac{\min_{j,i} u_{ji}}{G\left(\min_{j,i} u_{ji}\right)} \cdot \prod_{j=1}^{J} \frac{G\left(\min_i u_{ji}\right)}{F_j\left(\min_i u_{ji}\right)} \tag{4}$$

for $\boldsymbol{u} \in [0,1]^d$. Moreover, the vector $\boldsymbol{Y}$ has univariate uniform marginals, i.e. C is a copula.

Proof First note that for $\boldsymbol{u} = (u_{11}, \ldots, u_{Jd_J}) \in [0,1]^d$

$$\mathbb{P}\left(Y_{ji} \le u_{ji}\ \forall i, j\right) = \mathbb{P}\left(X_{ji} \le u_{ji}\ \forall i, j\right) \cdot \mathbb{P}\left(Z \le \min_{j,i} u_{ji}\right) \cdot \prod_{j=1}^{J} \mathbb{P}\left(Z_j \le \min_i u_{ji}\right)$$

$$= C_0\left(\left(F_j(u_{j1}), \ldots, F_j(u_{jd_j})\right)_{j=1}^{J}\right) \cdot \frac{\min_{j,i} u_{ji}}{G\left(\min_{j,i} u_{ji}\right)}$$

$$\cdot \prod_{j=1}^{J} \frac{G\left(\min_i u_{ji}\right)}{F_j\left(\min_i u_{ji}\right)}.$$

Moreover, put $\tilde{\boldsymbol{u}}_{ji} = (1, \ldots, 1, u_{ji}, 1, \ldots, 1) \in [0,1]^d$ and observe that

$$C\left(\tilde{\boldsymbol{u}}_{ji}\right) = C_0\left(1, \ldots, 1, F_j(u_{ji}), 1, \ldots, 1\right) \cdot \frac{u_{ji}}{G\left(u_{ji}\right)} \cdot \frac{G\left(u_{ji}\right)}{F_j\left(u_{ji}\right)} = u_{ji},$$

so the univariate margins of C are uniform on $[0,1]$, and, hence, C is a copula. □

Remark 1

(i) Note that an alternative construction as in Eq. 3, using $\boldsymbol{X} \sim F$, $Z_j \sim F_{Z_j}$, and $Z \sim F_Z$ for some distribution functions F_{Z_j}, $j \in \{1, \ldots, J\}$, and F_Z, does not generally lead to an explicit form of the copula as in construction (4). This is due to the fact that the (k,l)-th margin of the resulting distribution function is given by $F_k(u_{kl}) \cdot F_Z(u_{kl}) \cdot F_{Z_k}(u_{kl})$. By means of our construction, $F_{Z_k}(u_{kl}) = G(u_{kl})/F_k(u_{kl})$ and $F_Z(u_{kl}) = u_{kl}/G(u_{kl})$, so this product simplifies and a copula emerges.

(ii) The assumption of $t/G(t)$ being a distribution function of a random variable on $[0,1]$ implies that $t/G(t) \le 1$, i.e. $t \le G(t)$ for all $t \in [0,1]$. Similarly, $G(t)/F_j(t) \le 1$ implies $G(t) \le F_j(t)$ for all $t \in [0,1]$. Hence,

$$t \le G(t) \le F_j(t) \le 1,\ \forall t \in [0,1],\ \ j \in \{1, \ldots, J\}. \tag{5}$$

Observe that the condition of $t/G(t)$ being increasing is implied e.g. by G being concave, see Marshall and Olkin (1979, p. 453). Further note that the condition of G/F_j being increasing is equivalent (assuming differentiability of F_j and G) to the fact that $(G'F_j - GF_j')/F_j^2 \ge 0$, which in turn is equivalent to $(\log F_j(t))' \le (\log G(t))'$ for all $t \in [0,1]$. Further note that $t/F_j(t)$ is increasing for all $j \in \{1, \ldots, J\}$ as both $t/G(t)$ and $G(t)/F_j(t)$ are assumed to be so.

(iii) Another hierarchical model can be constructed using additive factors via

$$Y_{ji} = X_{ji} + Z_j + Z,\ i \in \{1, \ldots, d_j\},\ \ j \in \{1, \ldots, J\}.$$

However, it is generally difficult to determine the copula of the vector $\boldsymbol{Y}$ in this framework, since already the univariate marginals are convolutions. Moreover, it is not possible to introduce a singular component to the copula of $\boldsymbol{Y}$ by this construction, if the copula C_0 of $\boldsymbol{X}$ does not already have this property. In the following we therefore focus on transformation (3).

From Eqs. 4 and 5 we infer several special cases. These may be intuitively interpreted in terms of possible applications.

(i) If only the global shock Z (but no local shock Z_j) affects the random vector $\boldsymbol{X}$, this corresponds to the case where $Z_j \equiv 0$, $j \in \{1, \dots, J\}$, which in turn happens if and only if $F_j(t) = G(t)$ for all $t \in [0, 1]$, $j \in \{1, \dots, J\}$. The resulting copula is given by

$$C(\boldsymbol{u}) = C_0\left(\left(G\left(u_{ji}\right)\right)_{j=1,i=1}^{J,d_j}\right) \cdot \frac{\min_{j,i} u_{ji}}{G\left(\min_{j,i} u_{ji}\right)}, \ \boldsymbol{u} \in [0, 1]^d,$$

In particular, another specific form is obtained if $G(t) = 1$ for all $t \in [0, 1]$. By Eq. 5 this implies that $F_j(t)$ also equals one for all j. Hence, C agrees with the Upper Fréchet copula $M(\boldsymbol{u}) = \min_{j,i} u_{ji}$.

(ii) If $G(t) = t$ for all $t \in [0, 1]$, then

$$C(\boldsymbol{u}) = C_0\left(\left(F_j\left(u_{j1}\right), \dots, F_j\left(u_{jd_j}\right)\right)_{j=1}^{J}\right) \cdot \prod_{j=1}^{J} \frac{\min_i u_{ji}}{F_j\left(\min_i u_{ji}\right)}, \ \boldsymbol{u} \in [0, 1]^d.$$

In terms of the model, this means that $\boldsymbol{X}$ is affected by local shocks but no global shock is lurking. If $F_j(t) = t$ for all $t \in [0, 1]$, $j \in \{1, \dots, J\}$, Eq. 4 reduces to $C(\boldsymbol{u}) = C_0(\boldsymbol{u})$, corresponding to the case where the hierarchical model for $\boldsymbol{X}$ is neither affected by global nor local shocks. Further, if $F_j(t) = 1$ for all $t \in [0, 1]$, $j \in \{1, \dots, J\}$, then Eq. 4 reduces to

$$C(\boldsymbol{u}) = \prod_{j=1}^{J} \min_i u_{ji}, \ \boldsymbol{u} \in [0, 1]^d,$$

which corresponds to independent comonotone sectors.

(iii) If $C_0 \equiv \prod_{j=1}^{J} C_j$, i.e. the sectors are independent, then

$$C(\boldsymbol{u}) = \prod_{j=1}^{J} C_j\left(F_j\left(u_{j1}\right), \dots, F_j\left(u_{jd_j}\right)\right) \cdot \frac{\min_{j,i} u_{ji}}{G\left(\min_{j,i} u_{ji}\right)} \cdot \prod_{j=1}^{J} \frac{G\left(\min_i u_{ji}\right)}{F_j\left(\min_i u_{ji}\right)}$$

for all $\boldsymbol{u} \in [0, 1]^d$. If additionally $d_j = 1$, $j \in \{1, \dots, J\}$, i.e. the sectors only consist of a single element, then

$$\begin{aligned} C(\boldsymbol{u}) = C\left(u_1, \dots, u_J\right) &= \prod_{j=1}^{J} F_j\left(u_j\right) \cdot \frac{u_{(1)}}{G\left(u_{(1)}\right)} \cdot \prod_{j=1}^{J} \frac{G\left(u_j\right)}{F_j\left(u_j\right)} \\ &= \frac{u_{(1)}}{G\left(u_{(1)}\right)} \cdot \prod_{j=1}^{J} G\left(u_j\right) = u_{(1)} \cdot \prod_{j=2}^{J} G\left(u_{(j)}\right), \ \boldsymbol{u} \in [0, 1]^d, \end{aligned}$$

where the subscript (j) denotes the j-th order statistics. This equals the construction of Durante et al. (2007).

Example 1 (Shocks with polynomial distribution functions) Assume that $F_j(t) = t^{\alpha_j}$ and $G(t) = t^{\beta}$, $t \in [0, 1]$, where $0 \leq \alpha_j \leq \beta \leq 1$, $j \in \{1, \ldots, J\}$. Then, the copula C of $\boldsymbol{Y}$ is given by

$$C(\boldsymbol{u}) = C_0\left(\left(u_{j1}^{\alpha_j}, \ldots, u_{jd_j}^{\alpha_j}\right)_{j=1}^{J}\right) \cdot \left(\min_{j,i} u_{ji}\right)^{1-\beta} \cdot \prod_{j=1}^{J}\left(\min_{i} u_{ji}\right)^{\beta-\alpha_j}, \quad \boldsymbol{u} \in [0, 1]^d. \tag{6}$$

From Eq. 6 we may infer several special cases, which may be interpreted in terms of possible applications.

(i) If C_0 is an extreme-value copula, i.e. $C_0(\boldsymbol{u}^t) = C_0^t(\boldsymbol{u})$ with $\boldsymbol{u}^t$ to be meant componentwise, then C is also an extreme-value copula. Interesting applications for this setup can be found e.g. in hydrology, one given by Durante and Salvadori (2009).

(ii) If $C_0 \equiv \prod_{j=1}^{J} C_j$, i.e. the sectors are independent, and $C_j \equiv M$, $j \in \{1, \ldots, J\}$, then

$$C(\boldsymbol{u}) = \left(\min_{j,i} u_{ji}\right)^{1-\beta} \cdot \prod_{j=1}^{d}\left(\min_{i} u_{ji}\right)^{\beta}, \quad \boldsymbol{u} \in [0, 1]^d.$$

(iii) If $C_0(\boldsymbol{u}) = \Pi(\boldsymbol{u}) = \prod_{j,i=1}^{J,d_J} u_{ji}$, i.e. all components X_{ji} of $\boldsymbol{X}$ are independent, then

$$C(\boldsymbol{u}) = \left(\min_{j,i} u_{ji}\right)^{1-\beta} \cdot \prod_{j=1}^{J} u_{j(1)}^{\beta} u_{j(2)}^{\alpha_j} \cdots u_{j(d_j)}^{\alpha_j}, \quad \boldsymbol{u} \in [0, 1]^d.$$

This copula is of Marshall-Olkin type and might be interpreted as independent components affected by a single global shock, see Marshall and Olkin (1967).

3 Properties of C

In this section, properties of the copula transformation of Eq. 4 are studied. In several computations it turns out to be convenient to impose the following assumptions on the model.

(A1) $C_0(\boldsymbol{u}) = \prod_{j=1}^{J} C_j(u_{j1}, \ldots, u_{jd_j})$, i.e. initially independent sectors are assumed.
(A2) $C_0 \equiv \Pi$, i.e. all components are initially independent.
(B) $F_j(t) = t^{\alpha_j}$, $G(t) = t^{\beta}$, $t \in [0, 1]$, $0 \leq \alpha_j \leq \beta \leq 1$, $j \in \{1, \ldots, J\}$.

Note that (A2) obviously implies (A1) and (B) corresponds to Example 1 with shocks following polynomial distribution functions. Now consider the bivariate margin $C^{(j_1 i_1, j_2 i_2)}$ of the components i_1 in sector j_1 and i_2 in sector j_2. If $j_1 = j_2 = j$, then

$$C^{(ji_1, ji_2)}\left(u_{ji_1}, u_{ji_2}\right) = C_0^{(ji_1, ji_2)}\left(F_j\left(u_{ji_1}\right), F_j(u_{ji_2})\right) \cdot \frac{\min\{u_{ji_1}, u_{ji_2}\}}{F_j\left(\min\{u_{ji_1}, u_{ji_2}\}\right)} \tag{7}$$

$$\overset{(A2)}{=} F_j\left(\max\{u_{ji_1}, u_{ji_2}\}\right) \cdot \min\{u_{ji_1}, u_{ji_2}\} \tag{8}$$

$$\underset{(B)}{\overset{(A2)}{=}} \max\{u_{ji_1}, u_{ji_2}\}^{\alpha_j} \cdot \min\{u_{ji_1}, u_{ji_2}\}. \tag{9}$$

Equation 8 represents a bivariate copula from the class studied by Durante (2007), see also Durante et al. (2008). In particular, if additionally (B) is assumed it reduces to Eq. 9, which agrees with the bivariate Cuadras-Augé copula with parameter α_j, see Cuadras and Augé (1981). Equation 7 represents a new kind of bivariate copula having an interesting property: If $C_0^{(ji_1,ji_2)}$ is positive quadrant dependent (PQD), i.e. $C_0 \geq \Pi$ in the pointwise order, then the inequality

$$C^{(ji_1,ji_2)}\left(u_{ji_1},u_{ji_2}\right) \geq \min\{u_{ji_1},u_{ji_2}\} F_j\left(\max\{u_{ji_1},u_{ji_2}\}\right) \geq u_{ji_1}u_{ji_2}$$

holds, which implies that $C^{(ji_1,ji_2)}(u_{ji_1},u_{ji_2})$ is also PQD. This property is as expected, since a local shock typically increases the amount of positive dependence among the random variables in the same sector.

If $j_1 \neq j_2$, i.e. components belonging to different sectors are considered, then

$$\begin{aligned} C^{(j_1i_1,j_2i_2)}\left(u_{ji_1},u_{ji_2}\right) &= C_0^{(j_1i_1,j_2i_2)}\left(F_{j_1}(u_{j_1i_1}),F_{j_2}(u_{j_2i_2})\right)\cdot\min\{u_{j_1i_1},u_{j_2i_2}\} \\ &\quad \cdot \frac{\max\left\{G(u_{j_1i_1}),G(u_{j_2i_2})\right\}}{F_{j_1}(u_{j_1i_1})\cdot F_{j_2}(u_{j_2i_2})} \end{aligned} \tag{10}$$

$$\overset{(A1)}{=} \min\{u_{j_1i_1},u_{j_2i_2}\}\cdot\max\left\{G(u_{j_1i_1}),G(u_{j_2i_2})\right\} \tag{11}$$

$$\underset{(B)}{\overset{(A1)}{=}} \min\{u_{j_1i_1},u_{j_2i_2}\}\cdot\max\{u_{j_1i_1},u_{j_2i_2}\}^{\beta}. \tag{12}$$

Further, if $C_0^{(j_1i_1,j_2i_2)}$ is PQD, Eq. 5 implies

$$C^{(j_1i_1,j_2i_2)}\left(u_{ji_1},u_{ji_2}\right) \geq \min\{u_{j_1i_1},u_{j_2i_2}\}\cdot\max\left\{G(u_{j_1i_1}),G(u_{j_2i_2})\right\} \geq u_{j_1i_1}u_{j_2i_2},$$

so $C^{(j_1i_1,j_2i_2)}$ is PQD as well. Again, this property reflects the observation that global and local shocks increase the amount of positive dependence among the random variables.

Regarding the investigation of the dependence structure C as a result of transformation (3) applied to C_0, note that for the bivariate measures of monotone dependence Spearman's rho and Kendall's tau, see Joe (1997, p. 32), it turns out to be difficult to obtain a general closed form, even for components belonging to the same sector $j \in \{1,\ldots,J\}$. However, under assumptions (A2) and (B) one can show that the bivariate margins of Eq. 4 are Marshall-Olkin type copulas, see Marshall and Olkin (1967). For these copulas, Spearman's rho and Kendall's tau are e.g. computed in Embrechts et al. (2001) and given by $\rho_{C^{(ji_1,ji_2)}} = 12/(3+\alpha_j)-3$, $\rho_{C^{(j_1i_1,j_2i_2)}} = 12/(3+\beta)-3$, $\tau_{C^{(ji_1,ji_2)}} = (1-\alpha_j)/(1+\alpha_j)$, and $\rho_{C^{(j_1i_1,j_2i_2)}} = (1-\beta)/(1+\beta)$. From this we may again infer that even under the assumption of independent components of $\boldsymbol{X}$, the copula transformation (3) is able to introduce the whole possible range of positive dependence to the components of $\boldsymbol{Y}$, which is intuitively clear in terms of the shocks Z_j and Z.

Allowing for tail dependence is often a desired property of copulas; especially in applications with rare events, such as credit-risk modeling, where the tails of the copula influence common extremes of the underlying random variables. For a bivariate copula C, the parameters λ_L and λ_U of lower and upper tail dependence, respectively, can be computed via $\lambda_L = \lim_{u\searrow 0} C(u,u)/u$, $\lambda_U = \lim_{u\nearrow 1}(1-2u+C(u,u))/(1-u)$, see Joe (1997, p. 33). In the sequel, we investigate how lower and upper tail dependence parameters change under the transformation of Eq. 3.

As these parameters describe the extremal dependence between two components, we distinguish between the cases where the components belong to the same or to different sectors, respectively, i.e. we have to consider two sorts of bivariate margins. The results are presented in the following proposition.

Proposition 1 *Assume as given a copula C_0 with tail dependence parameters $\lambda_L^{C_0}$ and $\lambda_U^{C_0}$. The bivariate tail dependence parameters $\lambda_L^{C^{(\cdot,\cdot)}}$ and $\lambda_U^{C^{(\cdot,\cdot)}}$ of the copula transformation C are then given as follows, where $C^{(ji_1,ji_2)}$ and $C^{(j_1i_1,j_2i_2)}$ denote the corresponding bivariate margins of C as listed in Eqs. 7 to 12 and assuming that all involved derivatives and limits exist.*

(1) *If $\lim_{t\searrow 0} F_j(t) = 0$, then $\lambda_L^{C^{(ji_1,ji_2)}} = \lambda_L^{C_0^{(ji_1,ji_2)}}$, i.e. the degree of lower tail dependence between components belonging to the same sector is invariant under the shock transformation.*

(2) (2.1) $\lambda_L^{C^{(j_1i_1,j_2i_2)}} = \lim_{t\searrow 0} C_0^{(j_1i_1,j_2i_2)}(F_{j_1}(t), F_{j_2}(t))G(t)/(F_{j_1}(t)F_{j_2}(t))$.

(2.2) *If $F_{j_1}(t) \le F_{j_2}(t)$ in a neighborhood of zero and $\lim_{t\searrow 0} F_{j_2}(t) = 0$, then $\lambda_L^{C^{(j_1i_1,j_2i_2)}} \le \lambda_L^{C_0^{(j_1i_1,j_2i_2)}}$. The same result holds if $F_{j_2}(t) \le F_{j_1}(t)$ in a neighborhood of zero and $\lim_{t\searrow 0} F_{j_1}(t) = 0$.*

(2.3) *If (A1) holds, then $\lambda_L^{C^{(j_1i_1,j_2i_2)}} = \lim_{t\searrow 0} G(t)$.*

(2.4) *If $\lim_{t\searrow 0} G(t)/F_{j_k}(t) = 0$ for at least one $k \in \{1,2\}$, or if (B) holds with $\alpha_{j_1} + \alpha_{j_2} < \beta$, then $\lambda_L^{C^{(j_1i_1,j_2i_2)}} = 0$.*

(3) $\lambda_U^{C^{(ji_1,ji_2)}} = 1 - \lim_{t\nearrow 1} F_j'(t)(1 - \lambda_U^{C_0^{(ji_1,ji_2)}})$. *Concluding, even if C_0 does not allow for upper tail dependence, the transformed copula C may exhibit this property.*

(4) (4.1) *Let $C_{0,u_{j_ki_k}}^{(j_1i_1,j_2i_2)}$, $k \in \{1,2\}$, denote the k-th partial derivative of $C_0^{(j_1i_1,j_2i_2)}$, then*

$$\begin{aligned}\lambda_U^{C^{(j_1i_1,j_2i_2)}} &= 1 - \lim_{t\nearrow 1}\left(G'(t) - F_{j_1}'(t) - F_{j_2}'(t) + \frac{d}{dt}C_0^{(j_1i_1,j_2i_2)}\left(F_{j_1}(t), F_{j_2}(t)\right)\right)\\ &= 1 - \lim_{t\nearrow 1}\Big(G'(t) - F_{j_1}'(t)\big(1 - C_{0,u_{j_1i_1}}^{(j_1i_1,j_2i_2)}\big(F_{j_1}(t), F_{j_2}(t)\big)\big)\\ &\qquad - F_{j_2}'(t)\big(1 - C_{0,u_{j_2i_2}}^{(j_1i_1,j_2i_2)}\big(F_{j_1}(t), F_{j_2}(t)\big)\big)\Big).\end{aligned}$$

(4.2) *If (A1) holds, then $\lambda_U^{C^{(j_1i_1,j_2i_2)}} = 1 - \lim_{t\nearrow 1} G'(t)$.*

(4.3) *If (A1) and (B) hold, then $\lambda_U^{C^{(j_1i_1,j_2i_2)}} = 1 - \beta$.*

Proof For notational simplicity, we drop some indices and write C_0 instead of $C_0^{(ji_1,ji_2)}$ (for 1 and 3) and C_0 instead of $C_0^{(j_1i_1,j_2i_2)}$ (for 2 and 4); similarly for C.

(1) Assuming $\lim_{t\searrow 0} F_j(t) = 0$, it follows that $\lambda_L^C = \lim_{t\searrow 0} C_0(F_j(t), F_j(t))/F_j(t) = \lim_{t\searrow 0} C_0(t,t)/t = \lambda_L^{C_0}$.

(2) (2.1) By definition, $\lambda_L^C = \lim_{t\searrow 0} C_0(F_{j_1}(t), F_{j_2}(t))G(t)/(F_{j_1}(t)F_{j_2}(t))$.

(2.2) Without loss of generality, assume $F_{j_1}(t) \le F_{j_2}(t)$ in a neighborhood of zero. Using the fact that G/F_{j_1} is bounded above by one and $F_{j_1}(t) \le F_{j_2}(t)$, $\lambda_L^C \le \lim_{t\searrow 0} C_0(F_{j_2}(t), F_{j_2}(t))/F_{j_2}(t)$. As $\lim_{t\searrow 0} F_{j_2}(t) = 0$, the right-hand side equals $\lambda_L^{C_0}$ which establishes $\lambda_L^C \le \lambda_L^{C_0}$.

(2.3) Assume (A1) holds. Then $\lambda_L^C = \lim_{t\searrow 0}(F_{j_1}(t)F_{j_2}(t)G(t))/(F_{j_1}(t)F_{j_2}(t)) = \lim_{t\searrow 0} G(t)$.

(2.4) Assume without loss of generality that $\lim_{t\searrow 0} G(t)/F_{j_1}(t) = 0$. Then $0 \leq \lambda_L^C \leq \lim_{t\searrow 0}(F_{j_2}(t)G(t))/(F_{j_1}(t)F_{j_2}(t)) = 0$. If (B) holds and $\beta > \alpha_{j_1} + \alpha_{j_2}$, then $0 \leq \lambda_L^C = \lim_{t\searrow 0} C_0(t^{\alpha_1}, t^{\alpha_2})t^{\beta-(\alpha_1+\alpha_2)} \leq \lim_{t\searrow 0} t^{\beta-(\alpha_1+\alpha_2)} = 0$.

(3) By l'Hospital's Rule, $\lambda_U^C = 2 - \lim_{t\nearrow 1} \frac{d}{dt}(C_0(F_j(t), F_j(t))t/F_j(t))$. Taking the derivative and using the fact that $\lim_{t\nearrow 1} F_j(t) = 1$, by Eq. 5, leads to $\lambda_U^C = 2 - \left(\lim_{t\nearrow 1} \frac{d}{ds} C_0(s, s)|_{s=F_j(t)} \lim_{t\nearrow 1} F_j'(t) + 1 - \lim_{t\nearrow 1} F_j'(t)\right)$. As $\lim_{t\nearrow 1} F_j(t) = 1$, $\lim_{t\nearrow 1} \frac{d}{ds} C_0(s, s)|_{s=F_j(t)} = \lim_{t\nearrow 1} \frac{d}{dt} C_0(t, t) = 2 - \lambda_U^{C_0}$ and the claim follows.

(4) (4.1) By l'Hospital's Rule, $\lambda_U^C = 2 - \lim_{t\nearrow 1} \frac{d}{dt}\big(C_0(F_{j_1}(t), F_{j_2}(t))tG(t)/(F_{j_1}(t) \cdot F_{j_2}(t))\big)$. Again taking the derivative and using the fact that $\lim_{t\nearrow 1} F_j(t) = 1$ implies $\lambda_U^C = 1 - \lim_{t\nearrow 1}\big(G'(t) - F_{j_1}'(t) - F_{j_2}'(t) + \frac{d}{dt} C_0 (F_{j_1}(t), F_{j_2}(t))\big)$. The next statement of the claim follows by an application of the chain rule.

(4.2) Application of 4.1.

(4.3) Application of 4.1. □

4 Sampling *C*

The probabilistic interpretation of the transformation via the separation of components distributed according to C_0 and shocks allows to efficiently sample the copula transformation C of Eq. 4, as long as sampling routines for C_0 and the shocks are available. The following algorithm then generates vectors of random variates from C.

Algorithm 1 (Sampling C)

(1) Sample the global shock $Z \in [0, 1]$ with distribution function $t/G(t)$.

(2) Sample independent local shocks $Z_j \in [0, 1]$ with distribution functions $G(t)/F_j(t)$, $j \in \{1, \dots, J\}$, independent of Z.

(3) Sample $(U_{11}, \dots, U_{Jd_J}) \sim C_0$ and set $X_{ji} = F_j^{[-1]}(U_{ji})$ for all $i \in \{1, \dots, d_j\}$, $j \in \{1, \dots, J\}$.

(4) Set $Y_{ji} = \max\{X_{ji}, Z_j, Z\}$, $i \in \{1, \dots, d_j\}$, $j \in \{1, \dots, J\}$, and return the vector $(Y_{11}, \dots, Y_{Jd_J})$.

Example 2 (Shock-transforming an exchangeable Archimedean copula) In this example we transform a three-dimensional exchangeable Archimedean copula of the form

$$C_0(\boldsymbol{u}) = \psi\left(\psi^{-1}(u_1) + \psi^{-1}(u_2) + \psi^{-1}(u_3)\right),$$

where the generator $\psi : [0, \infty) \to [0, 1]$ belongs to Clayton's family with parameter $\vartheta = 4/3$, corresponding to a Kendall's tau of 0.4. Figures 1 and 2 show cloud plots of 1,000 vectors of random variates from this distribution affected by different shock scenarios. The plot on the left-hand side of Fig. 1 shows the vectors of random variates of the exchangeable Clayton copula without shocks. For this setup, choose $J = 1$ and, trivially, $F_J(t) = G(t) = t, t \in [0, 1]$, in Algorithm 1. All other plots illustrate the effect of different shocks on the copula. As global shock, we apply $Z \sim t/G(t)$ with $G(t) = t^\beta$ and $\beta = 0.6$. Note that sampling such a polynomial distribution function boils down to applying the inversion principle, namely generating

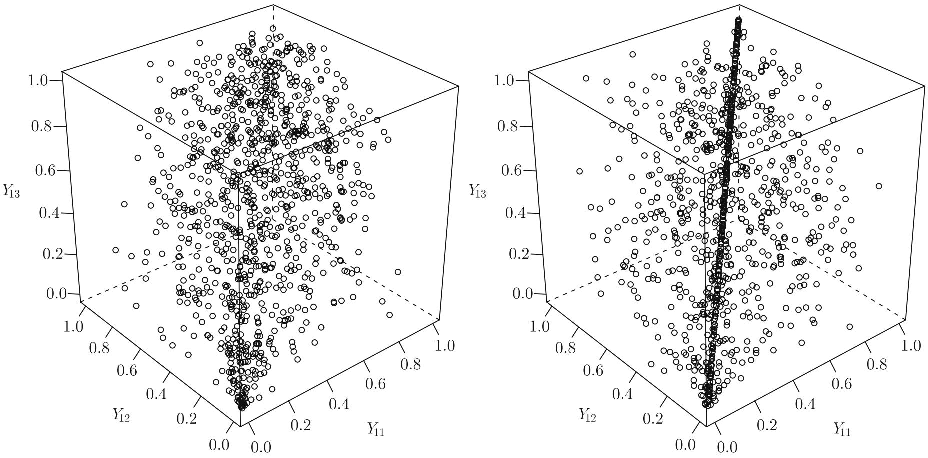

Fig. 1 1,000 vectors of random variates from an exchangeable Clayton copula without shocks (*left*) and with a global shock (*right*), respectively

$U \sim U[0, 1]$ and returning $Z = U^{1/\beta}$. The plot on the right-hand side of Fig. 1 shows the corresponding effect on the exchangeable Clayton copula. In this case, put $J = 1$ and $F_J \equiv G$. In Fig. 2, left-hand side, a local shock of the form $Z_2 \sim G(t)/F_2(t)$ affects the last two components, where we choose $F_2(t) = t^{\alpha_2}$ with $\alpha_2 = 0.2$. For this case, use $J = 2$ and $F_1(t) = G(t) = t$, $t \in [0, 1]$. On the right-hand side, the exchangeable Clayton copula is shown, affected by both the global and the local shock.

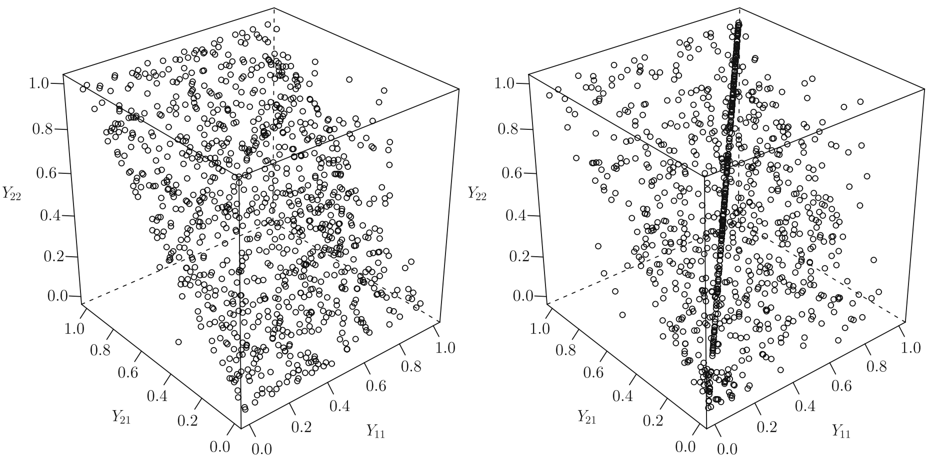

Fig. 2 1,000 vectors of random variates from an exchangeable Clayton copula with a local shock affecting the last two components (*left*) and both a global and a local shock affecting the last two components (*right*), respectively

Table 1 Empirical versions of measures of association under different shock scenarios for the exchangeable Clayton copula based on 100,000 vectors of random variates

$d = 3$	Shock scenarios			
Measures	None	Global	Local	Global and local
$\hat{\tau}_{11,21}$ $\hat{\rho}_{11,21}$	0.3990 0.5627	0.4799 0.6268	0.0809 0.1211	0.3316 0.4422
$\hat{\tau}_{11,22}$ $\hat{\rho}_{11,22}$	0.4011 0.5651	0.4803 0.6276	0.0808 0.1210	0.3318 0.4425
$\hat{\tau}_{21,22}$ $\hat{\rho}_{21,22}$	0.4024 0.5667	0.4839 0.6312	0.8080 0.7243	0.7245 0.8088
$\hat{\mu}_{21,22}$	0.0000	0.3112	0.7021	0.7029
$\hat{\mu}_{11,21,22}$	0.0000	0.2645	0.0000	0.2546

Table 1 presents numerical results for Kendall's tau, Spearman's rho, and the singular component. Concerning the latter, $\hat{\mu}_{j_1 i_1, j_2 i_2}$ denotes the fraction of simulated random vectors for which $Y_{j_1 i_1} = Y_{j_2 i_2}$ holds, similarly with three components using $\hat{\mu}_{j_1 i_1, j_2 i_2, j_3 i_3}$. All reported statistics are based on 100,000 generated vectors of random variates. For our set of parameters, Table 1 shows that the presence of the global shock increases the amount of pairwise dependence, whereas the local shock increases or decreases the amount of dependence, depending on whether components belonging to the same or different sectors are considered, respectively. The interaction of both effects can be studied within the setup with both global and local shocks. Moreover, as soon as a pair of variables is affected by the same shock, a singular component is implied.

Example 3 (Shock-transforming a nested Archimedean copula) In this example we illustrate Algorithm 1 using as original copula C_0 a four-dimensional partially nested Archimedean copula, see e.g. McNeil (2008) or Hofert (2008), with two sectors of dimension two. The parametric form of C_0 is given by

$$C_0(\boldsymbol{u}) = \psi_0\left(\psi_0^{-1}\left(\psi_1\left(\psi_1^{-1}(u_{11}) + \psi_1^{-1}(u_{12})\right)\right) + \psi_0^{-1}\left(\psi_2\left(\psi_2^{-1}(u_{21}) + \psi_2^{-1}(u_{22})\right)\right)\right)$$

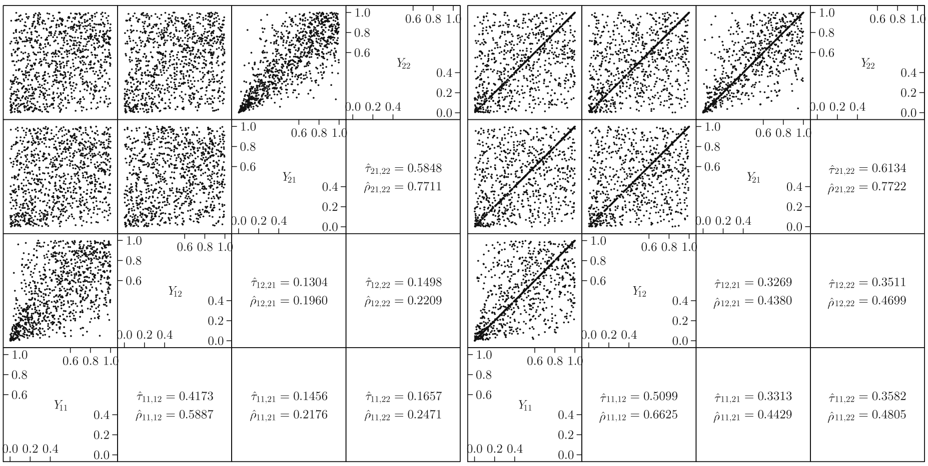

Fig. 3 1,000 vectors of random variates from a partially nested Clayton copula without shocks (*left*) and with a global shock (*right*), respectively

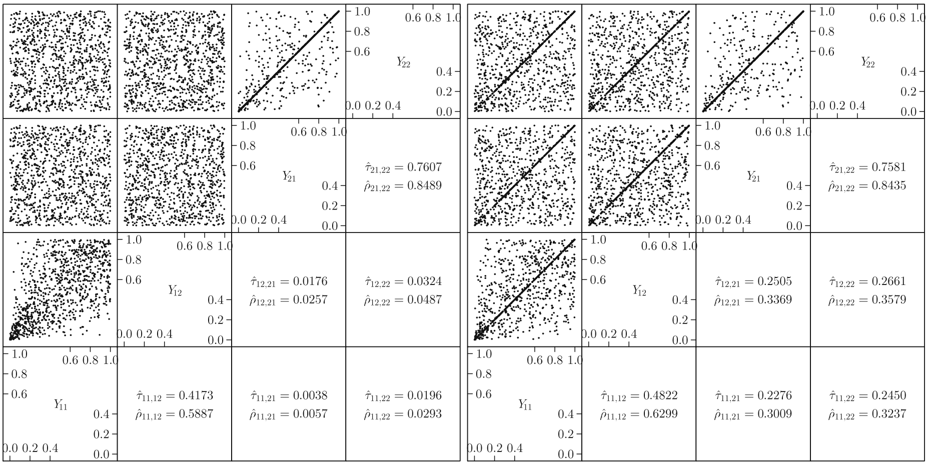

Fig. 4 1,000 vectors of random variates from a partially nested Clayton copula with only a local shock affecting the last two components (*left*) and both a global and a local shock affecting the last two components (*right*), respectively

for $\boldsymbol{u} \in [0,1]^4$, where we assume that all Archimedean generators ψ_i with corresponding parameters ϑ_i, $i \in \{0,1,2\}$, belong to Clayton's family with $\vartheta_0 = 2/9$, $\vartheta_1 = 4/3$, and $\vartheta_2 = 3$. For an intuition on the degree of dependence of the partially nested Archimedean copula C_0, note that the first two components have Kendall's tau equal to 0.4, the last two of 0.6, and components belonging to different sectors have Kendall's tau equal to 0.1. Figures 3 and 4 show scatterplot matrices of 1,000 vectors of random variates of the original copula and three shock transformations with corresponding pairwise sample versions of Kendall's tau and Spearman's rho. The plot on the left-hand side of Fig. 3 shows vectors of random variates of the partially nested Clayton copula without shocks. The remaining plots illustrate how the copula is affected by different kinds of shocks. As a global shock we apply $Z \sim t/G(t)$ with $G(t) = t^\beta$ and $\beta = 0.6$. The plot on the right-hand side of Fig. 3 shows the corresponding effect on the partially nested Clayton copula. In Fig. 4, left-hand side, a local shock of the form $Z_2 \sim G(t)/F_2(t)$ is applied to the last two

Table 2 Empirical versions of measures of association under different shock scenarios for the nested Clayton copula based on 100,000 vectors of random variates

$d = 4$	Shock scenarios			
Measures	None	Global	Local	Global and local
$\hat{\tau}_{11,12}$ $\hat{\rho}_{11,12}$	0.4012 0.5653	0.4844 0.6317	0.4012 0.5653	0.4820 0.6290
$\hat{\tau}_{11,21}$ $\hat{\rho}_{11,21}$	0.0997 0.1490	0.3002 0.4001	0.0174 0.0261	0.2651 0.3539
$\hat{\tau}_{11,22}$ $\hat{\rho}_{11,22}$	0.1011 0.1509	0.3026 0.4035	0.0193 0.0290	0.2664 0.3556
$\hat{\tau}_{12,21}$ $\hat{\rho}_{12,21}$	0.1019 0.1521	0.3028 0.4039	0.0199 0.0298	0.2684 0.3583
$\hat{\tau}_{12,22}$ $\hat{\rho}_{12,22}$	0.1016 0.1517	0.3028 0.4039	0.0200 0.0300	0.2684 0.3584
$\hat{\tau}_{21,22}$ $\hat{\rho}_{21,22}$	0.5998 0.7865	0.6245 0.7849	0.7670 0.8505	0.7669 0.8503
$\hat{\mu}_{11,12}$	0.0000	0.3117	0.0000	0.3097
$\hat{\mu}_{21,22}$	0.0000	0.3417	0.7236	0.7235
$\hat{\mu}_{11,12,21,22}$	0.0000	0.1966	0.0000	0.1971

components, where we choose $F_2(t) = t^{\alpha_2}$ and $\alpha_2 = 0.2$. On the right-hand side, the partially nested Clayton copula is shown, affected by both the global and the local shock.

Table 2 summarizes the effect of incorporating shocks on Kendall's tau, Spearman's rho, and the singular component. As before, the reported statistics are based on 100,000 vectors of random variates.

5 Conclusion

We introduced a new transformation on the set of d-dimensional copulas. Given some initial copula, this transformation allows to define new copulas with hierarchical structure and singular components; both properties are interesting for practical applications. Besides the analytical form, we also present a probabilistic interpretation in terms of global and local shocks. It is shown how several well-known copulas arise as special cases of this construction principle. Moreover, the effect of the shock-transformation on different measures of association such as Kendall's tau, Spearman's rho, and the tail dependence coefficients is studied and illustrated in some examples. The underlying construction principle has the appealing property of being easily sampled, as long as a sampling strategy for the initial copula is known. An explicit sampling algorithm was derived. As an application, we considered two examples involving Archimedean copulas.

References

Aas K, Czado C, Frigessi A, Bakken H (2009) Pair-copula constructions of multiple dependence. Insur Math Econ 44(2):182–198

Cuadras CM, Augé J (1981) A continuous general multivariate distribution and its properties. Commun Stat Theory Methods 10(4):339–353

Durante F (2007) A new family of symmetric bivariate copulas. C R Math 344(3):195–198

Durante F, Salvadori G (2009) On the construction of multivariate extreme value models via copulas. Environmetrics (in press)

Durante F, Quesada-Molina JJ, Úbeda-Flores M (2007) On a family of multivariate copulas for aggregation processes. Inf Sci 177:5715–5724

Durante F, Kolesárová A, Mesiar R, Sempi C (2008) Semilinear copulas. Fuzzy Sets Syst 159(1): 63–76

Embrechts P (2009) Copulas: a personal view. J Risk Insur (in press)

Embrechts P, Lindskog F, McNeil AJ (2001) Modelling dependence with copulas and applications to risk management. http://www.risklab.ch/ftp/papers/DependenceWithCopulas.pdf

Hofert M (2008) Sampling Archimedean copulas. Comput Stat Data Anal 52(12):5163–5174

Hofert M, Scherer M (2008) CDO pricing with nested Archimedean copulas. http://www.mathematik.uni-ulm.de/numerik/preprints/2008/CDOpricingAC.pdf

Joe H (1997) Multivariate models and dependence concepts. Chapman & Hall/CRC, London

Liebscher E (2008) Construction of asymmetric multivariate copulas. J Multivar Anal 99(10):2234–2250

Mai JF, Scherer M (2009) Lévy-frailty copulas. J Multivar Anal 100:1567–1585

Marshall AW, Olkin I (1967) A multivariate exponential distribution. J Am Stat Assoc 62:30–44

Marshall AW, Olkin I (1979) Inequalities: theory of majorization and its applications. In: Mathematics in science and engineering, vol 143. Academic [Harcourt Brace Jovanovich], New York

Marshall AW, Olkin I (1988) Families of multivariate distributions. J Am Stat Assoc 83:834–841

McNeil AJ (2008) Sampling nested Archimedean copulas. J Stat Comput Simul 78:567–581

McNeil AJ, Nešlehová J (2009) Multivariate Archimedean copulas, d-monotone functions and l_1-norm symmetric distributions. Ann Stat (in press)

McNeil AJ, Frey R, Embrechts P (2005) Quantitative risk management: concepts, techniques, and tools. Princeton University Press, Princeton

Morillas PM (2005) A method to obtain new copulas from a given one. Metrika 61(2):169–184

Rodríguez-Lallena JA, Úbeda-Flores M (2009) Multivariate copulas with quadratic section in one variable. Metrika (in press)

Savu C, Trede M (2006) Hierarchical Archimedean copulas. http://www.uni-konstanz.de/micfinma/conference/Files/papers/Savu_Trede.pdf

Sklar A (1959) Fonctions de répartition à n dimensions et leurs marges. Publ Inst Stat Univ Paris 8:229–231