Daniel Opdahl
Financial Planning Project - Math 351

Email to clients

Dear Mr. and Mrs. Nobody,

Congratulations on the birth of Junior! Now down to business.
If you invest $100 per month starting now, I predict that you will be able to accrue enough return on investment so that you have at least $47,000 in your account by the time Junior turns 18 and goes to college. Of course, nothing in the market is assured, so I can't absolutely guarantee anything. However, there is a 75% chance that you will have at least $47,130.77 in your account at the end of these next 18 years based on what returns the market has yielded in the past. You can expect about that amount in your account in 18 years time, probably even more.
In order for you to have at least $60,000 in your account by the time Junior turns 18 and goes to college, I recommend saving $160 each month. This figure lends itself to an 80% likelihood based on 7,000 computer simulations of the market.
I look forward to managing your account!
Happy birthday, Junior!

- Daniel Opdahl

Deterministic Case

The case where investments are made in a financial account on a monthly basis and where investments grow at a certain rate per year can be modeled by the following linear differential equation, where *A(t)* is the amount in the account at time *t*, *M* is the monthly payment, *r* is the growth rate per year, and *t* is time in years.

$$A(t)' = 12M + rA(t) \qquad (1)$$

Note that the equation above assumes that both the growth rate and the monthly investments are constant processes, rather than discrete events.

For the Nobody's particular case, they are investing \$100 each month, and start with no money in their account when $t = 0$. Thus, we can model their situation with the following Initial Value Problem.

$$A(t)' = 1200 + rA(t) \ , \ \ A(0) = 0 \qquad (2)$$

The solution to this IVP is shown below.

$$A(t) = \frac{1200}{r} e^{-rt} - \frac{1200}{r} \qquad (3)$$

This equation represents the amount *A* in their account at some time *t* based on the monthly payments of \$100 (\$1200 annually), and the investment growth rate per year *r* for an 18 year period.

In order to determine a historic value of *r* for an 18 year period, I looked at the annual returns on investment of the S&P 500 since 1926. Using Sage, I calculated the geometric mean of $1 + r$ for every 18 year period in the data set.

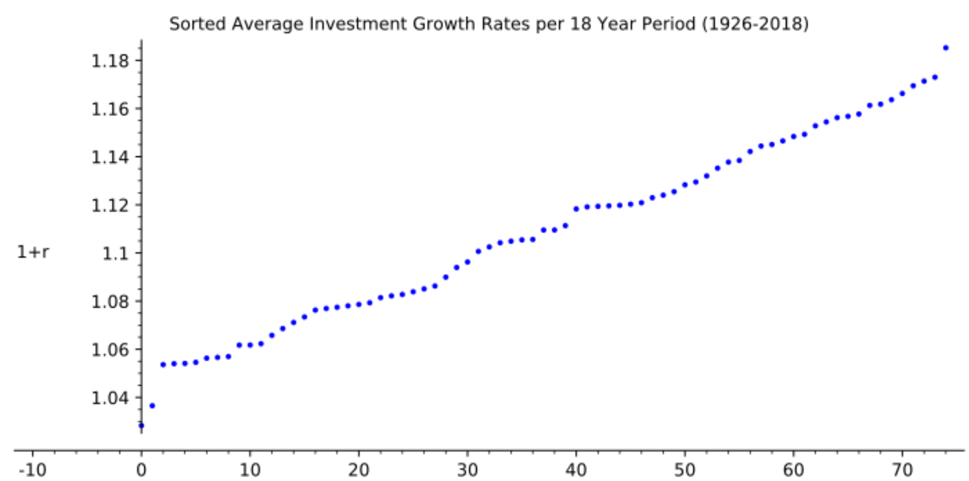

Figure 1: Geometric means of 1+r values for all 18 year periods, sorted from least to greatest (not chronologically)

Using these resulting r values, I can calculate various values for the amount the Nobody's will have saved at the end of 18 years at various confidence levels. For instance, the median value of $1 + r$ in the above data set is 1.1095. Because 50% of $1 + r$ values are above this value, I can be 50% certain that the Nobody's will have at least this value for the 18 year window for which their investments will be in the market. Using this value of r in the equation for *A*, I find that $A(18) = 67738.4586$, so I can be 50% sure that the Nobody's investments will return \$67738.46 at the end of 18 years. The table below illustrates more values of $1 + r$, their corresponding confidence values, and their expected return on investment after 18 years.

$1+r$	$A(18)$ (USD)	Confidence (%)	Percentile
1.18525750332828	175,333.06	1.35	99th
1.14031900582799	98,348.77	25	75th
1.10954510118909	67,738.46	50	50th
1.07776507665079	47,130.77	75	25th
1.02833854264591	28,178.47	98.5	1st

Table1: Confidence values and expected return on investment after 18 years of selected values of 1+*r* for the deterministic model

Introducing Stochasticity

Due to the unpredictable nature of the market, our model must be able to account for random changes in the market. In order to do this, I used Euler's method with the following discretized version of Equation 1 plus a stochastic term that introduces randomness.

$$\Delta A = (rA + 12M)\Delta t + \sigma A \Delta B \qquad (4)$$

Where ΔB is a normally distributed random variable with mean 0 and standard deviation $\sqrt{\Delta t}$, and where σ represents the volatility of the investment. Using Euler's method with Equation 4, a random sample from ΔB will be selected at every step, resulting in a different solution path every time the method is used to approximate a possible solution.

I used Euler's method and Equation 4 to generate 10,000 possible solution paths and recorded the value of $A(18)$ for each generated path. I assumed volatility of $\sigma = 0.15$, average growth rate of $r = 0.1$, monthly savings of $M = 100$, step size of $\Delta t = \frac{1}{24}$, and initial value of $A(0) = 0$. The sorted results of these 10,000 generated solution paths are shown in the figure below, and some important values from the data set are shown in the table below.

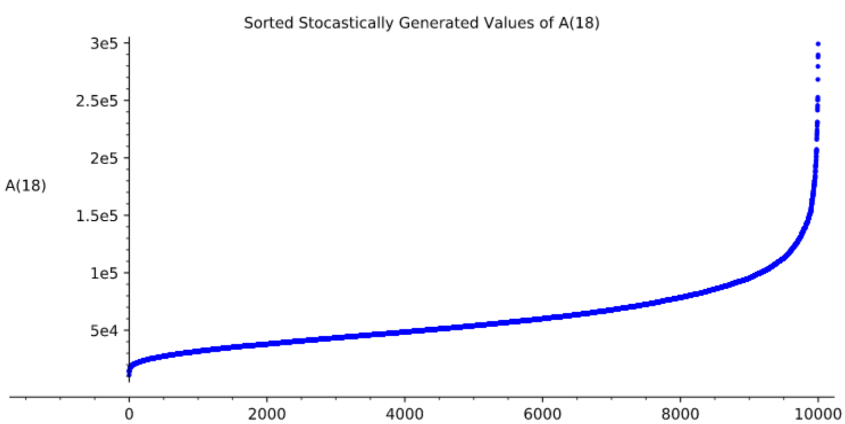

Figure 2: Expected return on investment after 18 years of 10,000 generated solutions (as a reference for the reader, 100000

Mean:	60161.8638984251
Median:	53928.7344522508
75th Percentile:	72541.2688877077
25th Percentile:	40738.6983826554

Table 2: Values taken from the data set shown in Figure 2

From the shape of the curve in Figure 2 we can tell that there are relatively few outliers, but those outliers are large compared to the rest of the data. This will raise the value of the mean, making it a less effective predictor of what outcomes are most likely. A far more representative measure of what the most likely outcomes are is the median. For $r = 0.1$, our deterministic case yields $A(18) = 60595.77$, whereas our median value for our stochastically generated data is $A(18) = 53928.73$. To be 75% certain of a prediction based on our stochastically generated data, we look at the 25th percentile, or the data point that has 75% of all data points above it, meaning that since every data point is equally likely, there is a 75% chance that $A(18)$ will be *at least* the value at the 25th percentile, which is $A(18) = 40738.70$.

In order for the Nobody's to be "reasonably confident" (which we will define here as having 80% confidence) that they will have \$50,000 after 18 years ($A(18) \geq 50000$), we need to do an analysis similar to the one we just did and calculate the 20th percentile values; however this time, we need to explore different values of M. Again I employed Euler's method, but this time I explored six different values of M: \$100, \$110, \$120, \$130, \$140, \$150. For each value of M, I generated 1,000 possible solution paths and recorded the value of $A(18)$ for each generated path. I assumed volatility of $\sigma = 0.15$, average growth rate of $r = 0.1$, step size of $\Delta t = \frac{1}{12}$, and initial value of $A(0) = 0$. The values of M and their corresponding 20th percentile values are shown in the table below.

M	100	110	120	130	140	150	160
20th Percentile	37344.43	41859.03	45572.74	50154.12	53286.42	57798.13	60180.25

Table 3: 20th percentile values for various values of M

Based on these numbers, for the Nobody's to be 80% certain of having \$50,000 in their account by the end of 18 years, I would recommend that they save \$130 per month. Because 80% of values generated by the model for $M = 130$ were above 50,000, we can be 80% certain that if the Nobody's invest \$130 per month, they will have, according to the model, at least \$50,154.12.

<u>Role of Volatility</u>

To discover how long-term investments are affected by volatility, we can again use a stochastic Euler's method to model many different solution paths for a given value of sigma. I used this stochastic Euler's method to generate 2,000 possible solution paths over the course of 35 years for each value of sigma $\sigma = 0,\ 0.25,\ 0.5,\ 1.0$ and recorded the final balance in the account, $A(35)$, for each solution path. I assumed average growth rate of $r = 0.1$, monthly savings of $M = 100$, step size of $\Delta t = \frac{1}{12}$, and initial value of $A(0) = 0$. The results are shown in Figures 3-6.

Sigma: 0
Average: 379663.805179954
Median Value: 379663.805179954

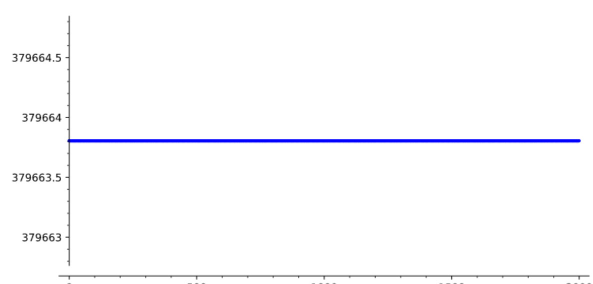

Figure 3: Final values of 2,000 solution paths after 35 years with sigma $\sigma = 0$ (Deterministic case)

Sigma: 0.25
Average: 353361.502062663
Median Value: 149528.820781543

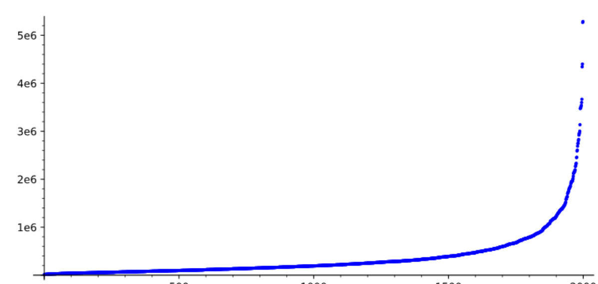

Figure 4: Final values of 2,000 solution paths after 35 years with sigma $\sigma = 0.25$

Sigma: 0.5
Average: 308177.350431715
Median Value: 174468.955053986

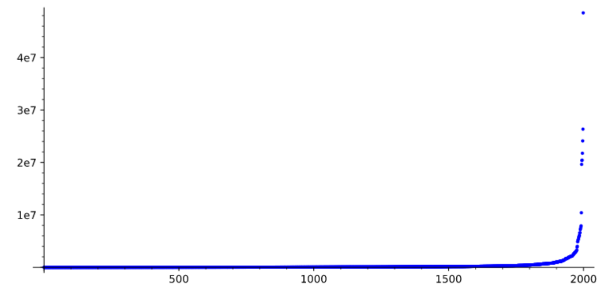

Figure 5: Final values of 2,000 solution paths after 35 years with sigma $\sigma = 0.5$

Sigma: 1.0
Average: 49876.4888673780
Median Value: 12538.0023285515

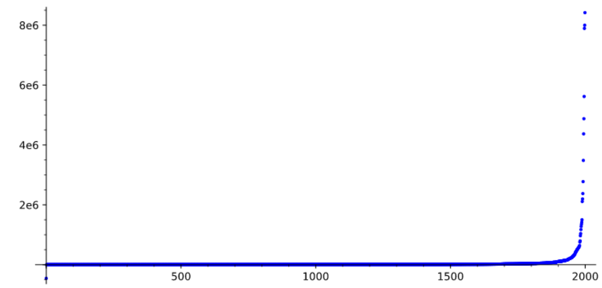

Figure 6: Final values of 2,000 solution paths after 35 years with sigma $\sigma = 1.0$

A pattern is clear in our data. As volatility (sigma) increases, both average and median value drop. Although there are occasional outliers (with especially big winners in more volatile markets), the overwhelming likelihood is that a more volatile market will result in less return on investment, and in the case where sigma $\sigma = 1.0$, almost no average return on investment.

<u>What Type of Investment</u>

To investigate which fund would be best for the Nobody's we can again use a stochastic Euler's method to model many different solution paths for various values of r and σ that characterize the 3 different funds.

Fund 1:
$\sigma = 0.35$, $r = 0.15$

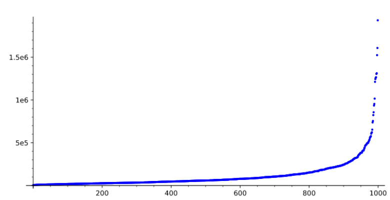

Figure 7: Final values of 1,000 solution paths with $\sigma = 0.35$, $r = 0.15$

Average: 113626.722078453
Median: 152875.451588826

Fund 2:
$\sigma = 0.15$, $r = 0.1$

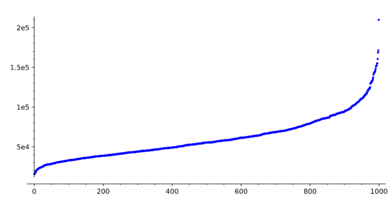

Figure 8: Final values of 1,000 solution paths with $\sigma = 0.15$, $r = 0.1$

Average: 60300.4206604741
Median: 73891.1188675798

Fund 3:
$\sigma = 0.03$, $r = 0.05$

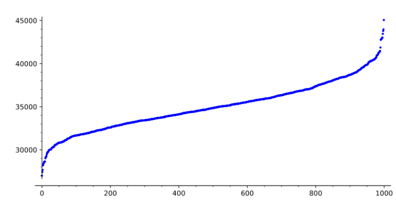

Figure 9: Final values of 1,000 solution paths with $\sigma = 0.03$, $r = 0.05$

Average: 35006.7726535612
Median: 39321.2069821313

The figures above shows that over the course of 18 years, Fund 1 has the highest median and average returns on investment. I believe this is because 18 years is a long enough timeframe for the volatility to "even out" and for the investment to really benefit from the higher r value. There may be large swings in the middle, but over the course of 18 years, there is enough time for the low swings to balance out with the high swings. If my client was worried about having a higher risk investment, I would recommend that he invest something like 80% in Fund 1 and 20% in Fund 2, but I would always recommend that he put as many funds as he feels comfortable with in Fund 1.

Nonconstant Contributions

If we wish to account for an increasing monthly contribution (i.e., $M(t)$ is a function of t), we can modify our original deterministic case using $r = 0.07776507665079$ as well as using p to represent the percentage increase in the amount saved per year ($M(t) = 1200tp$).

$$A(t) = \frac{1200 + (1200tp)}{r} e^{-rt} - \frac{1200 + (1200tp)}{r} \qquad (5)$$

The values of $A(18)$ for the above equation for various p values appear in the table below.

p	0.0	0.01	0.03	0.05	0.075	0.1
$A(18)$	\$47,130.77	\$55,614.30	\$72,581.38	\$89,548.45	\$110,757.30	\$131,966.14

Table 4: Funds in account after 18 years for various percentage increases in amount saved per year

Based on the data in Table 4, if the Nobody's wish to have at least \$60,000 in their account after 18 years and want to begin by saving \$100 per month, I would advise them to increase the amount they invest in the account by about 2% aggregate every year, or 0.167% every month. Of course, these recommendations do assume that $r = 0.07776507665079$. But, based on the discussion in the Deterministic Case section, there is a little bit more than a 75% chance of $r \geq 0.07776507665079$, so we can feel comfortable with these predictions.