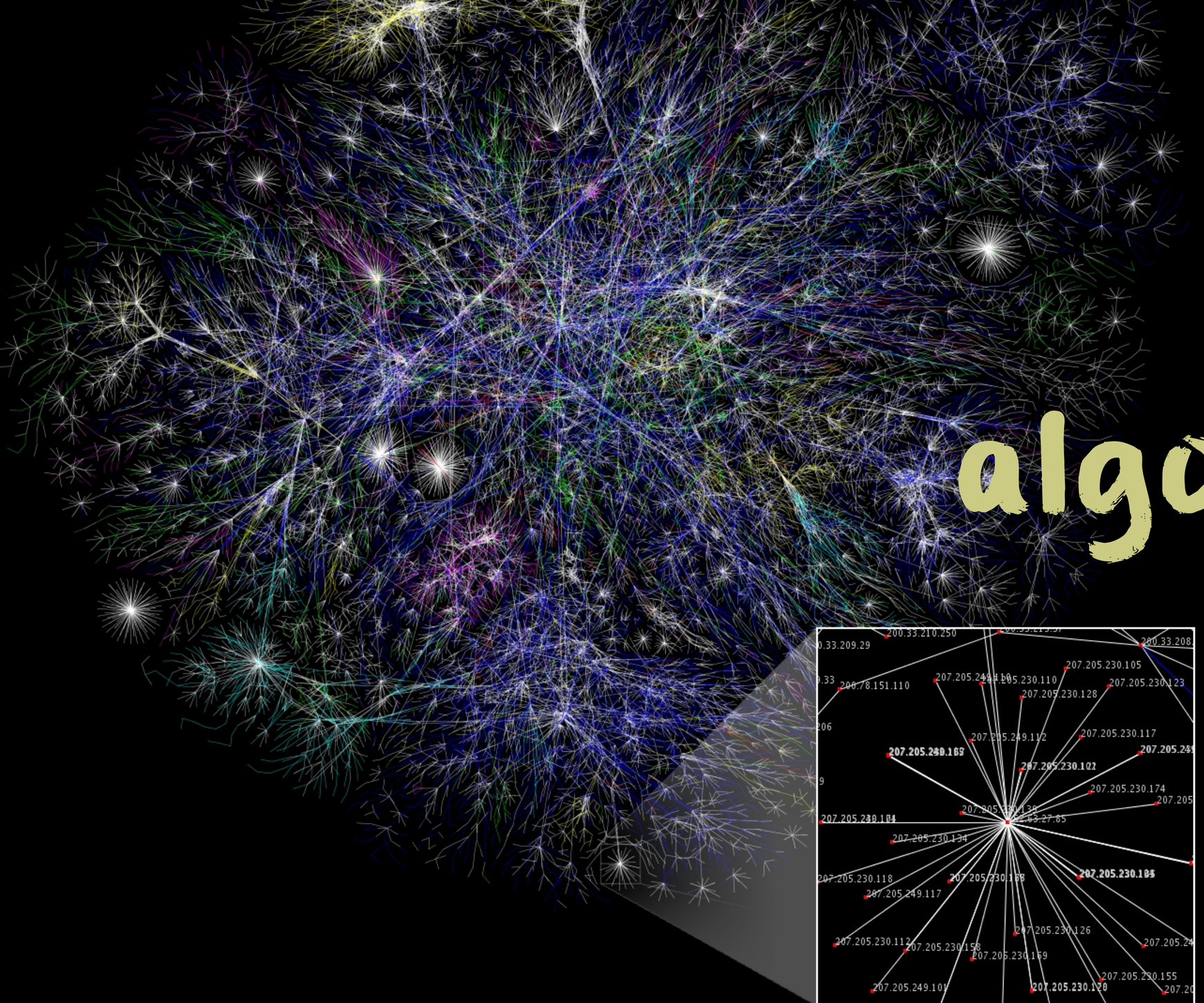
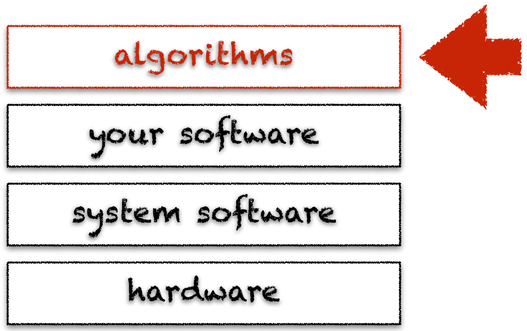
graph algorithm

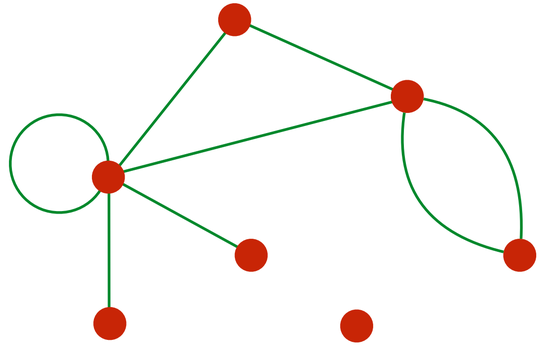


learning objectives



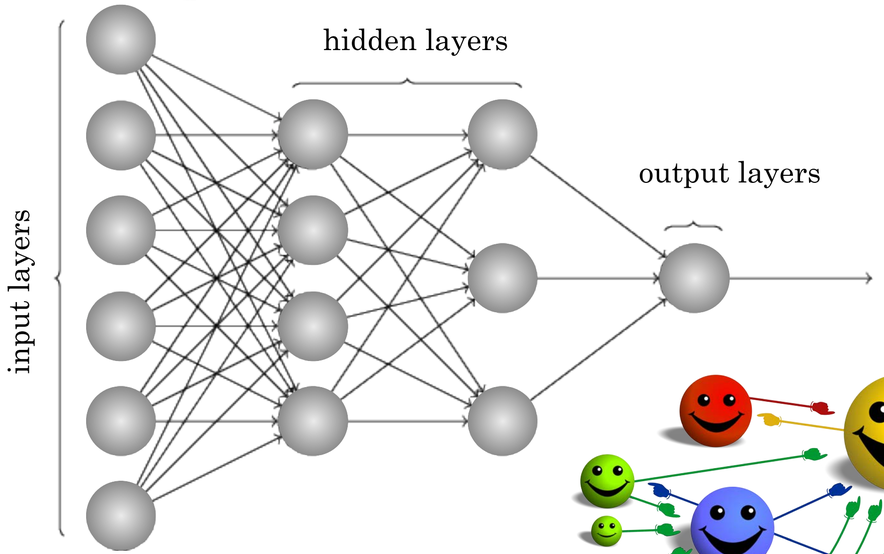
- learn what graphs are in mathematical terms
- learn how to represent graphs in computers
- learn about typical graph algorithms

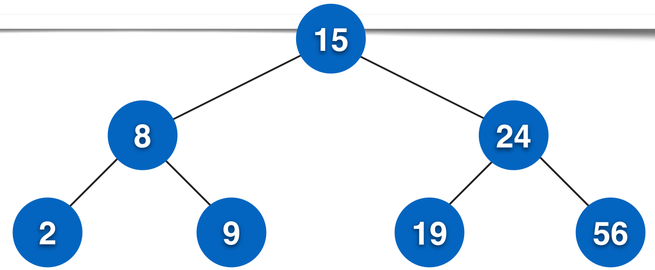
why graphs?

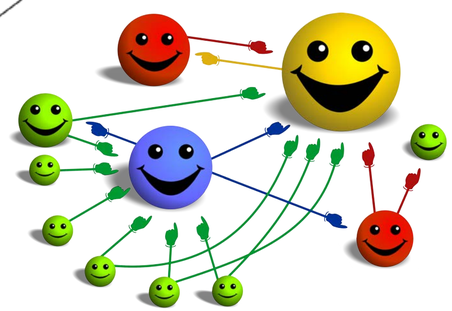


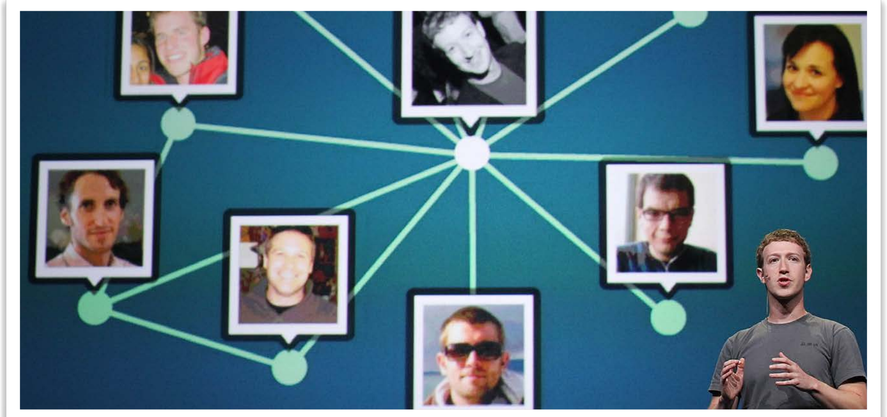
intuitively, a graph is formed by
vertices and edges between vertices

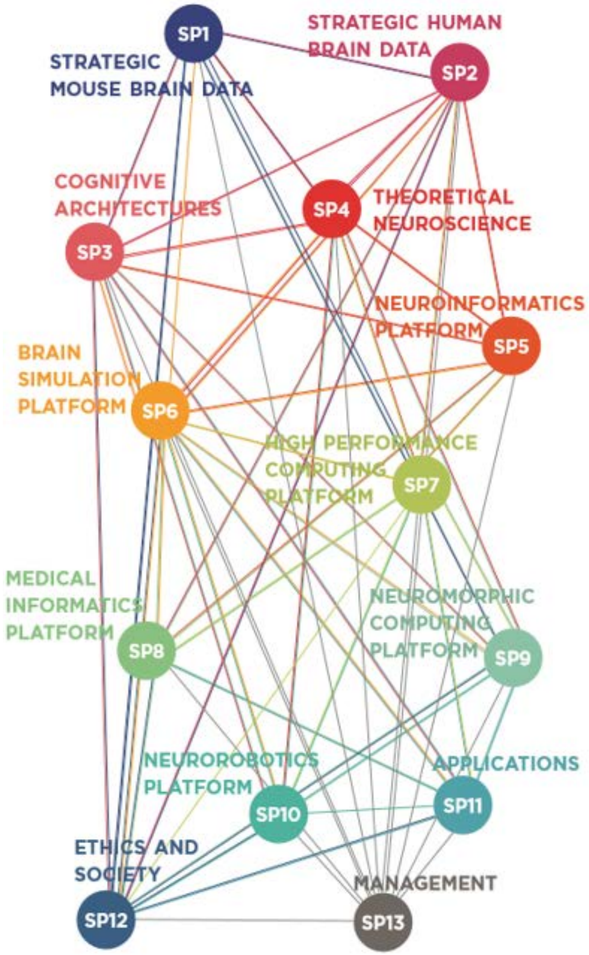
graphs are used in numerous fields to model
relationships (edges) between elements (vertices)







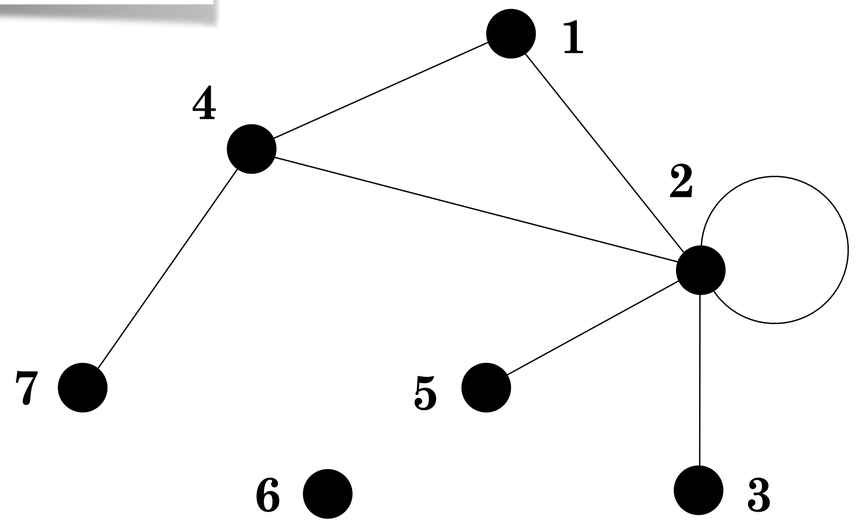




what's a graph?

formally, a graph is a tuple $G = (V,E)$ of sets, where V is a set of vertices (or nodes or points) and E is a set of edges such that:

$$E \subseteq V \times V$$



example:

$$V = \{1, 2, 3, 4, 5, 6, 7\}$$

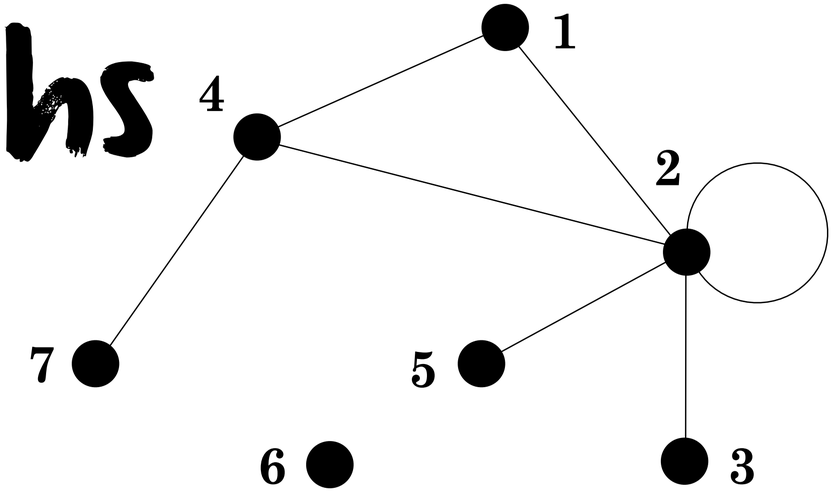
$$E = \{\{1, 2\}, \{1, 4\}, \{2\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{4, 7\}\}$$

types of graphs

undirected:

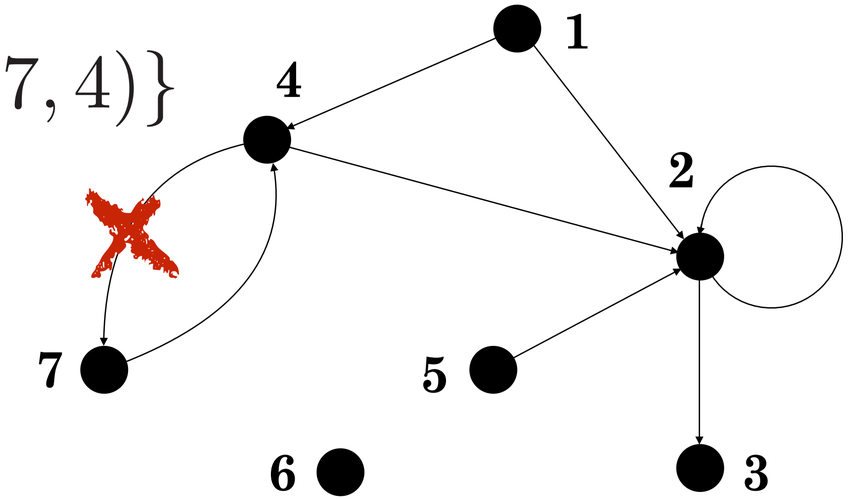
$V = \{1, 2, 3, 4, 5, 6, 7\}$

$E = \{\{1, 2\}, \{1, 4\}, \{2\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{4, 7\}\}$



directed:

$E = \{(1, 2), (1, 4), (2, 2), (2, 3), (4, 2), (4, 7), (5, 2), (7, 4)\}$



oriented:

$E = \{(1, 2), (1, 4), (2, 2), (2, 3), (4, 2), (4, 7), (5, 2), (7, 4)\}$

notations & metrics

let G be graph, $G.V$ denotes its set of vertices and $G.E$ its set of edges

the edge between vertices x and y is noted $\{x,y\}$, (x,y) or simply xy

the order of G, written $|G|$, is the number of its vertices, whereas $\|G\|$ denotes its number of edges

graph G is sparse if $\|G\| \ll |G|^2$ and it is dense if $\|G\| \approx |G|^2$

two vertices x and y are adjacent or neighbors if $xy \in G$

if all the vertices of G are pairwise adjacent, then G is complete

notations & metrics

a path from vertex x to vertex y is a sequence $\langle v_0, v_1, \ldots, v_k \rangle$ of vertices $v_i \in V$ where $x = v_0$ and $y = v_k$, such that $\forall\, i \in \{1, \ldots, k\} : (v_{i-1}, v_i) \in E$

a graph is connected if every pair of vertices is connected via a path

a path $\langle v_0, v_1, \ldots, v_k \rangle$ is a cycle if vertices $v_0 = v_k$

we can store attributes in vertices and edges using the dotted notation, e.g., $v.color$ stores a $color$ attribute in vertex v, while $e.weight$ and $(x,y).weight$ store a $weight$ attribute in edge e and edge (x,y) respectively

notations & metrics

let $G = (V,E)$ and $G' = (V',E')$ be two graphs, if $V' \subseteq V$ and $E' \subseteq E$, then G' is a subgraph of G, which we write $G' \subseteq G$

let $G = (V,E)$ and $G' = (V',E')$ be two graphs and $G' \subseteq G$, if $V' = V$, G' is a spanning subgraph of G

the degree (or valency) of a vertex v is the number of neighbors of v and is noted $d(v)$

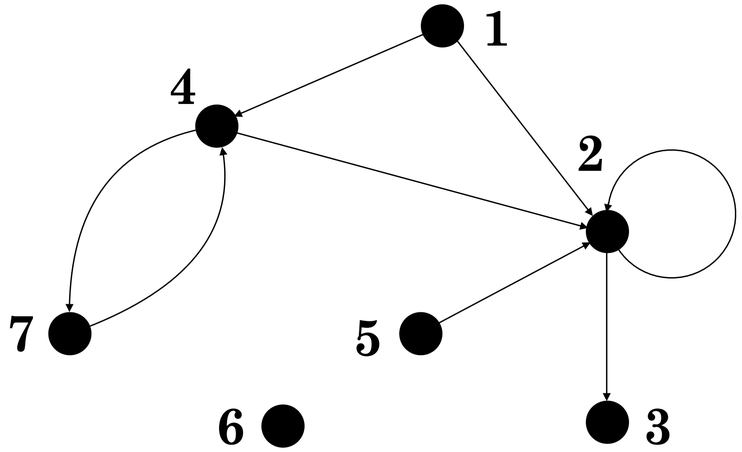
we defined $\delta(G) = \min \{ d(v) \mid v \in V \}$ as the minimum degree of G

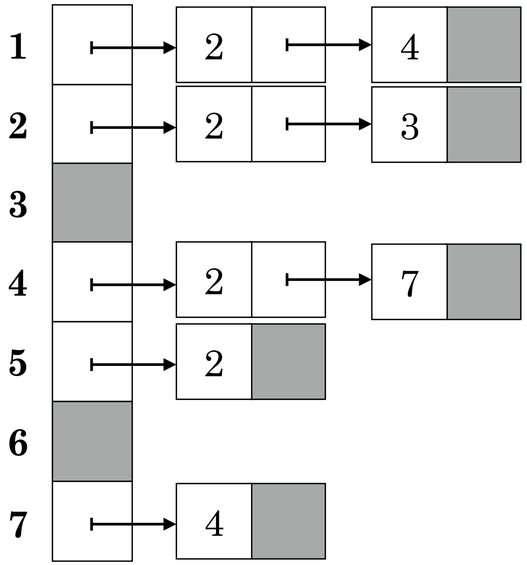
we defined $\Delta(G) = \max \{ d(v) \mid v \in V \}$ as the maximum degree of G

we defined $d(G) = \frac{1}{|V|} \sum_{v \in V} d(v)$ as the average degree of G

representing graphs

directed





adjacency list

	1	2	3	4	5	6	7
1	0	1	0	1	0	0	0
2	0	1	1	0	0	0	0
3	0	0	0	0	0	0	0
4	0	1	0	0	0	0	1
5	0	1	0	0	0	0	0
6	0	0	0	0	0	0	0
7	0	0	0	1	0	0	0

adjacency matrix

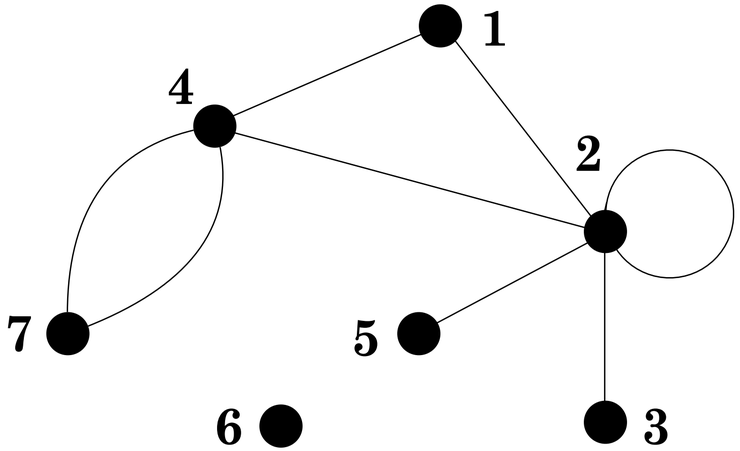
an adjacency list is best suited for representing a sparse graph

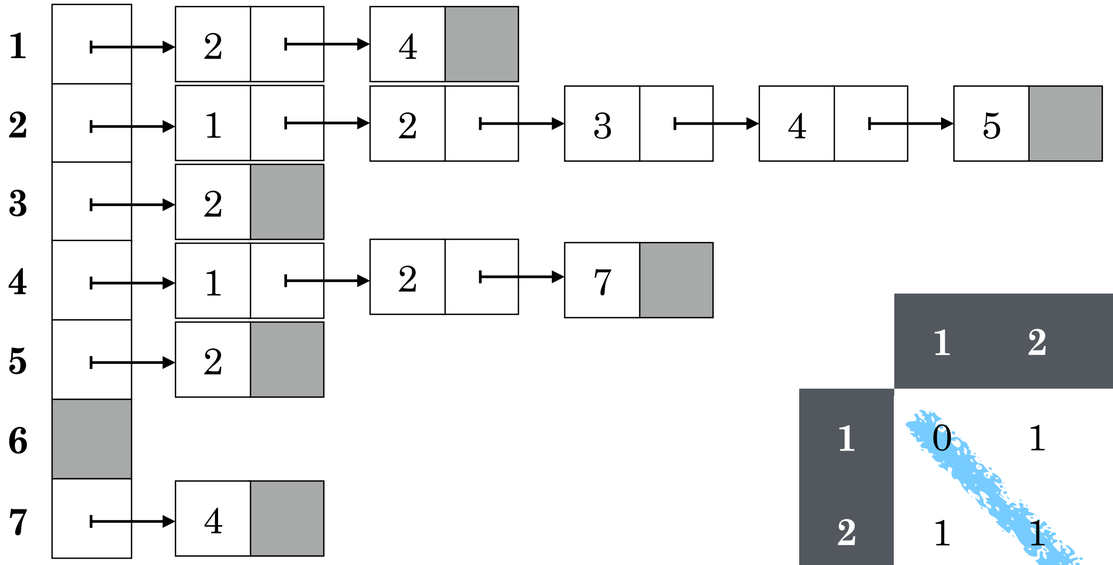
most graph algorithms rely on adjacency lists

an adjacency matrix is best suited for representing a dense graph or when the algorithm needs to know quickly if there exists an edge connecting two vertices

representing graphs

undirected





adjacency list

adjacency matrix

	1	2	3	4	5	6	7
1	0	1	0	1	0	0	0
2	1	1	1	1	1	0	0
3	0	1	0	0	0	0	0
4	1	1	0	0	0	0	1
5	0	1	0	0	0	0	0
6	0	0	0	0	0	0	0
7	0	0	0	1	0	0	0

typical problems

breadth-first search

minimum spanning tree

single-source shortest paths

breadth-first search

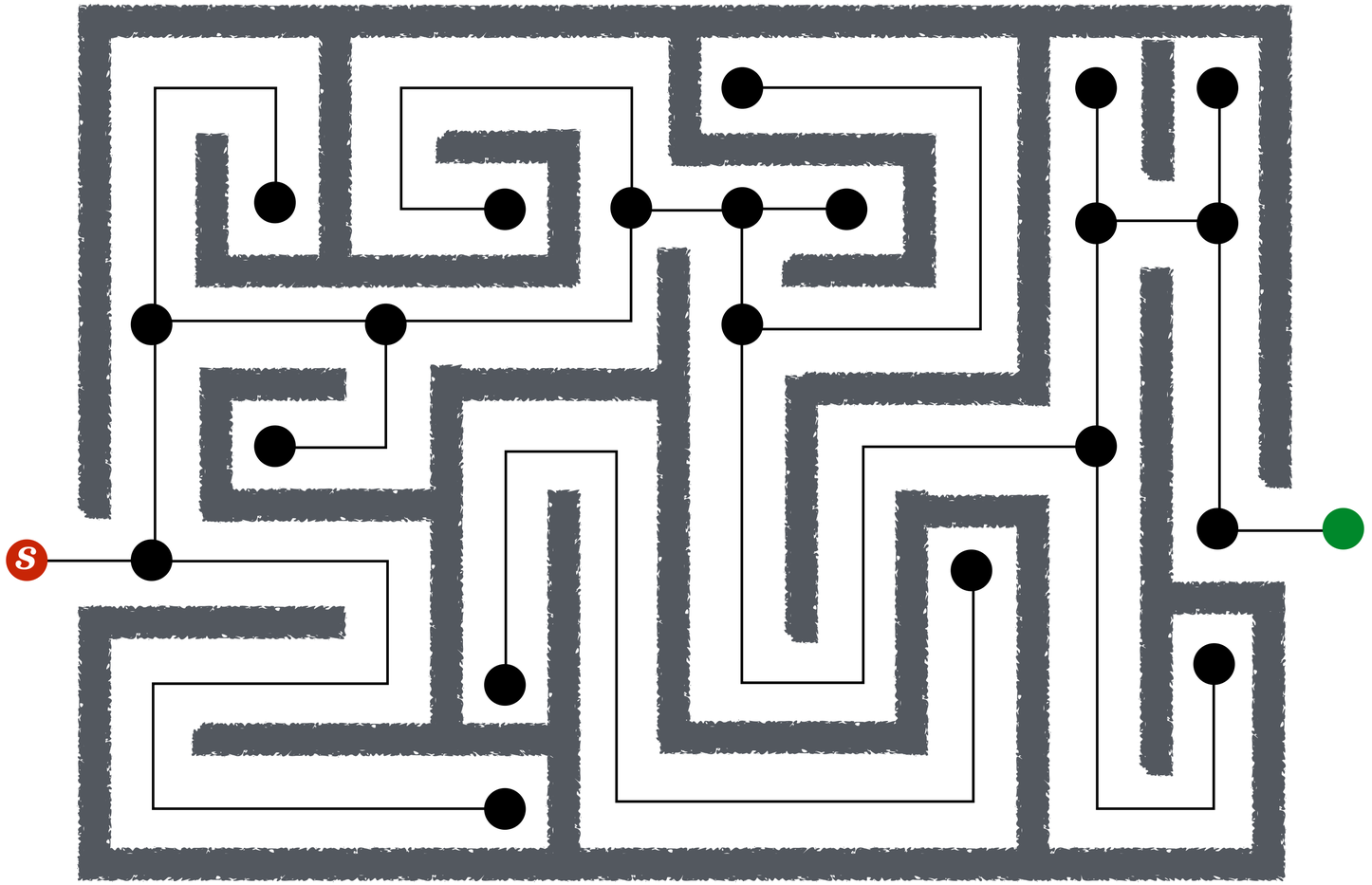
given graph G and a source vertex $s \in G$,
it discover every vertex reachable from s

it computes the distance from s to every vertex $v \in G$

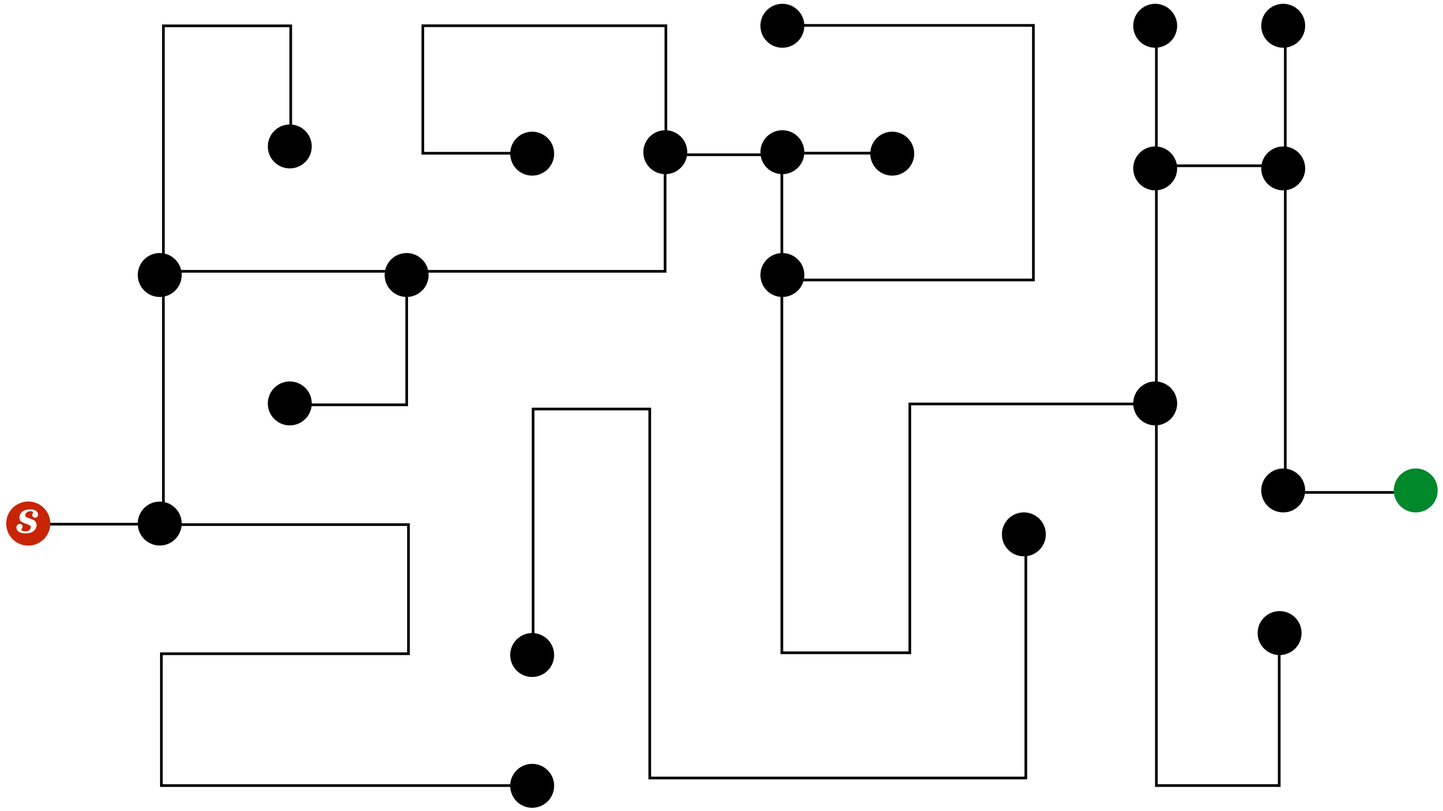
it produces a breadth-first tree rooted at s
that contains all reachable vertices from s

the search is said to be breadth-first because it
discovers all vertices at distance k from s before
discovering any vertices at distance $k+1$

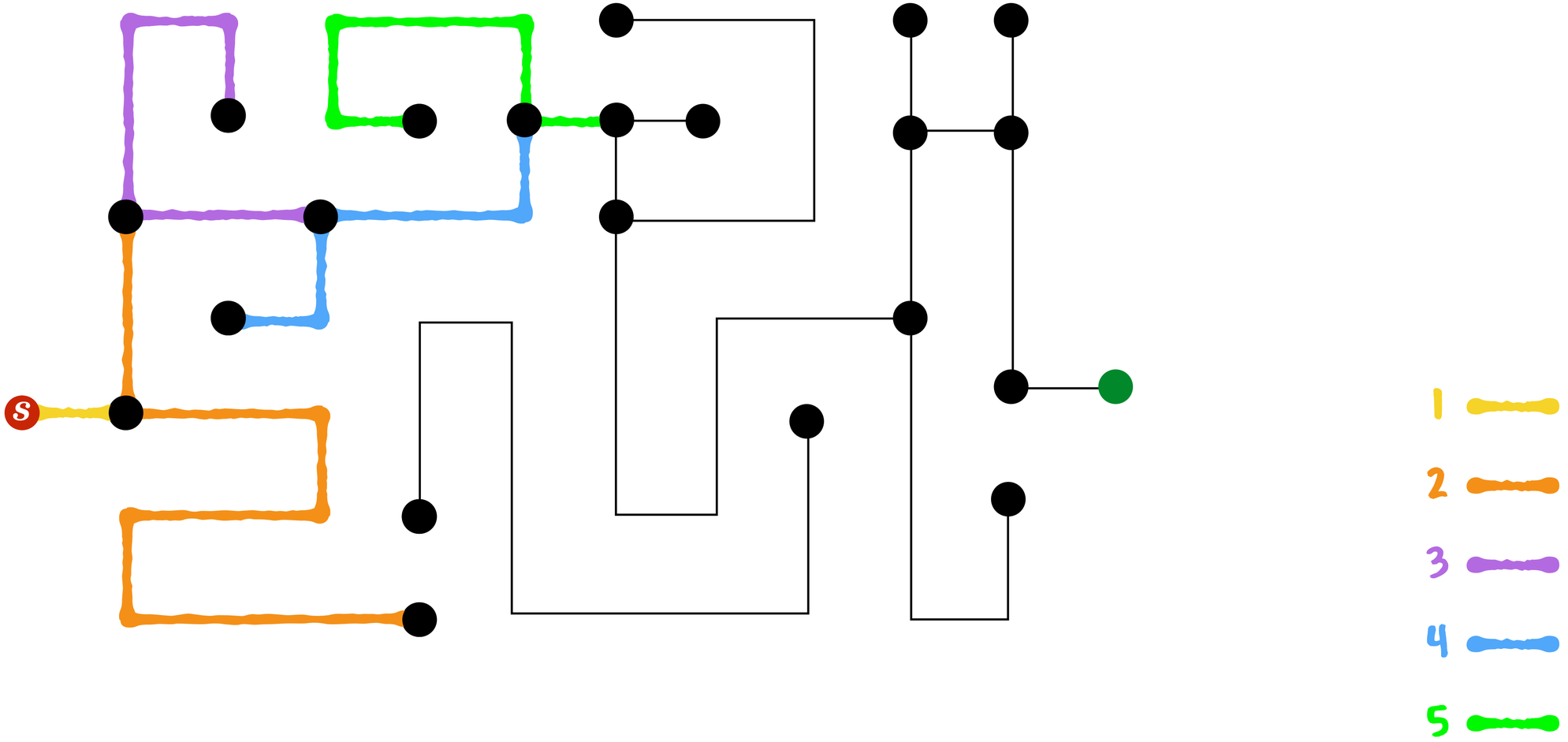
breadth-first search



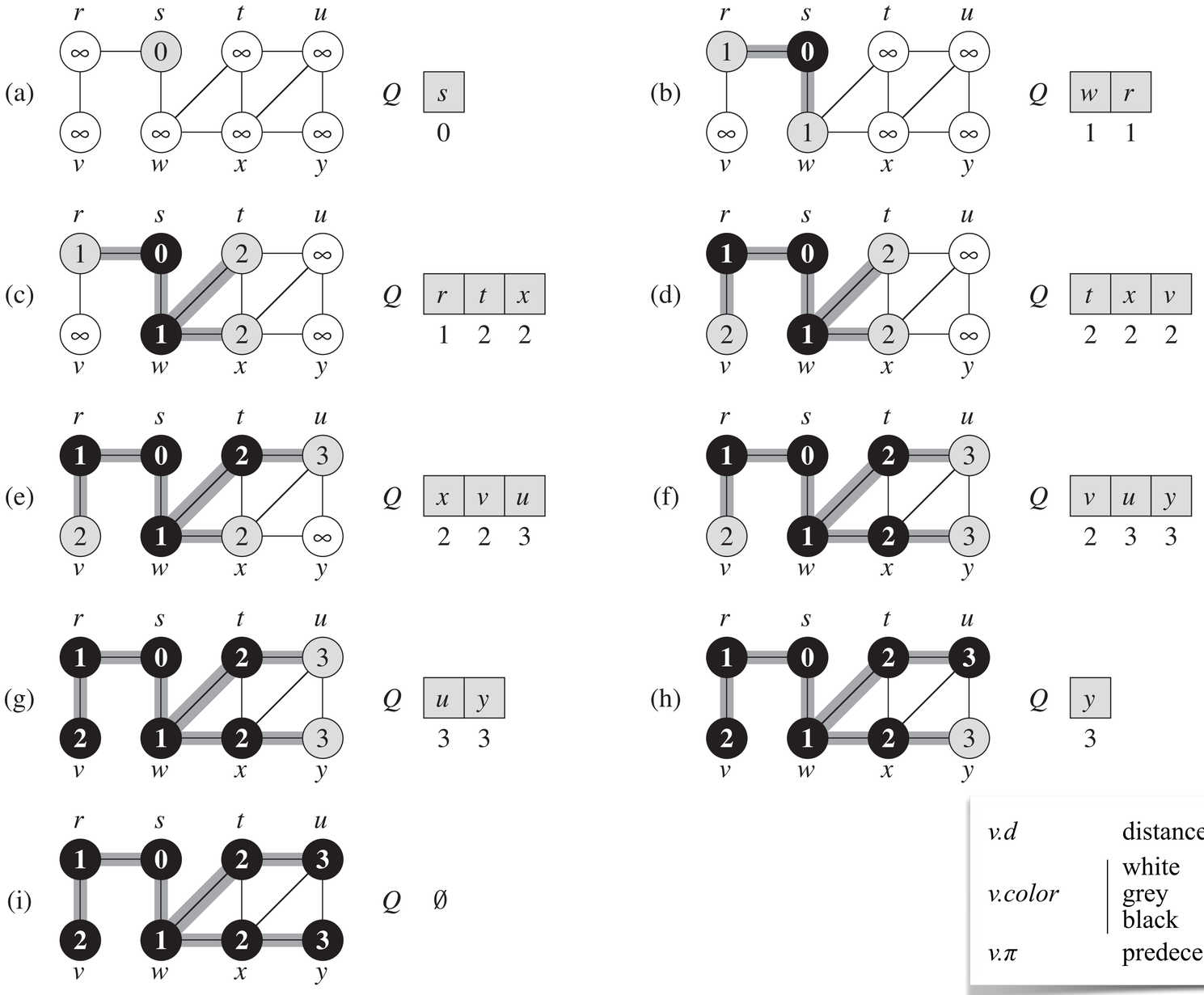
breadth-first search



breadth-first search



breadth-first search




```
BFS(G, s)
 1  for each vertex u ∈ G.V − {s}
 2      u.color = WHITE
 3      u.d = ∞
 4      u.π = NIL
 5  s.color = GRAY
 6  s.d = 0
 7  s.π = NIL
 8  Q = ∅
 9  ENQUEUE(Q, s)
10  while Q ≠ ∅
11      u = DEQUEUE(Q)
12      for each v ∈ G.Adj[u]
13          if v.color == WHITE
14              v.color = GRAY
15              v.d = u.d + 1
16              v.π = u
17              ENQUEUE(Q, v)
18      u.color = BLACK
```

$v.d$	distance from source s
$v.color$	white : undiscovered grey : discovered with some neighbors discovered black : discovered with all neighbors discovered
$v.\pi$	predecessor in bread-first tree

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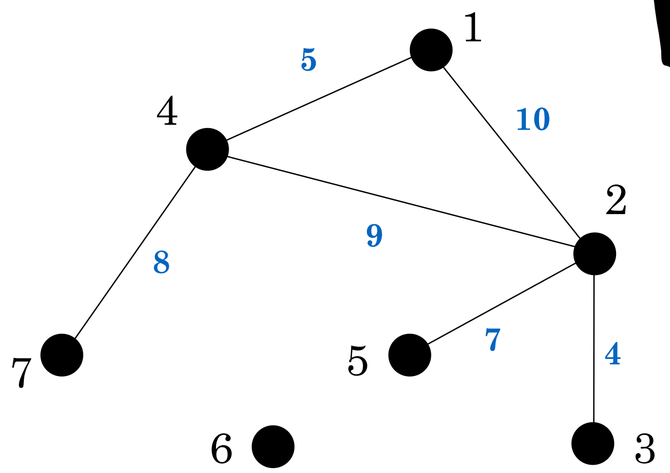
typical problems

breadth-first search

minimum spanning tree

single-source shortest paths

minimum spanning tree



a **weighted graph** $G_w = (G, w)$ is a tuple composed of a graph $G = (V, E)$ and of a **function** $w : E \to \mathbb{R}$ associating **a weight** w_e to each edge $e \in E$

---

a **minimum (weight) spanning tree** of graph $G_w = (G, w)$ is a connected subgraph (V', E') such that:

1. $V' = V$
2. (V', E') does not contain any cycles
3. $\sum_{e \in E'} w_e$ is **minimal across all subgraphs** fulfilling 1 and 2

minimum spanning tree

a disjoint-set data structure maintains a collection $S = \{S_1, S_2, \ldots, S_k\}$ of disjoint dynamic sets where each set is identified by a member of the set known as its representative

a disjoint-set data structure supports the following operations:

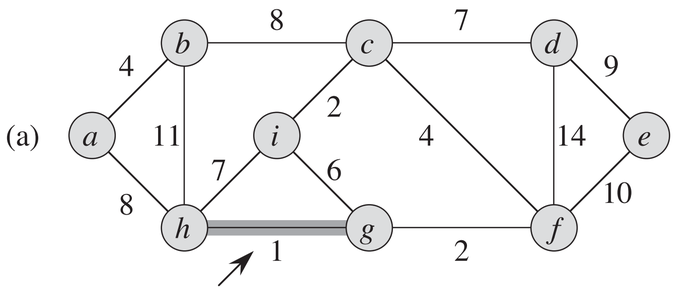
MAKE-SET(x) creates a new set whose only member and its representative is x

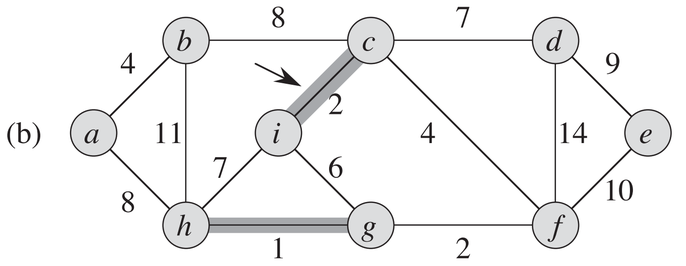
UNION(x,y) merges the dynamic sets that contain x and y, say S_x and S_y, into a new set that is the union of these two sets

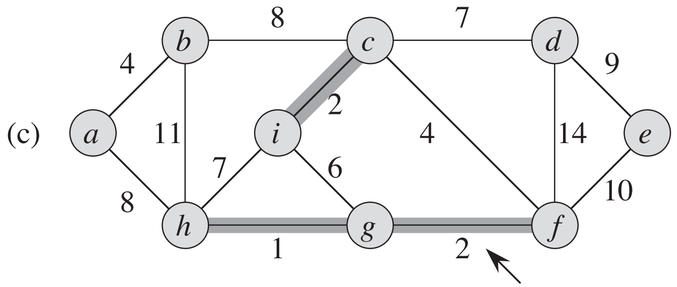
FIND-SET(x) returns the representative of the set containing x

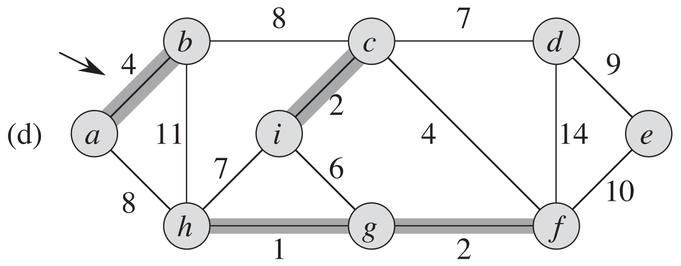
minimum spanning tree - Kruskal's algorithm

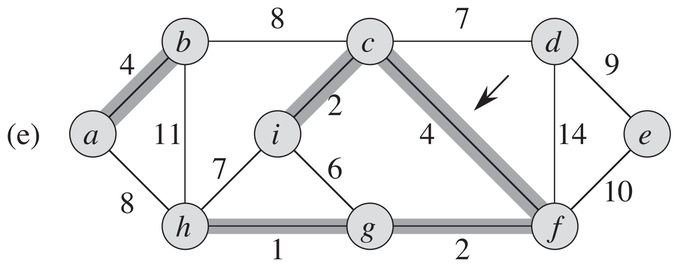
```
MST-KRUSKAL(G, w)
1  A = ∅
2  for each vertex v ∈ G.V
3       MAKE-SET(v)
4  sort the edges of G.E into nondecreasing order by weight w
5  for each edge (u, v) ∈ G.E, taken in nondecreasing order by weight
6       if FIND-SET(u) ≠ FIND-SET(v)
7            A = A ∪ {(u, v)}
8            UNION(u, v)
9  return A
```

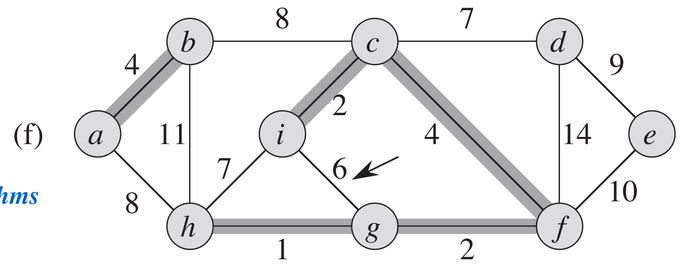


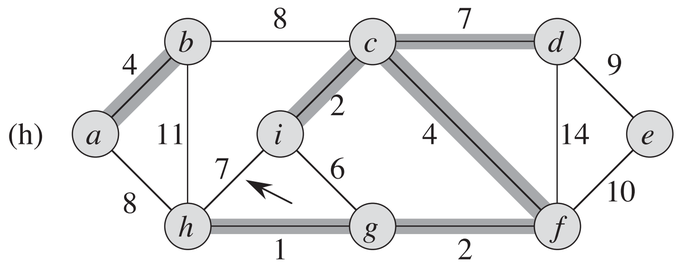


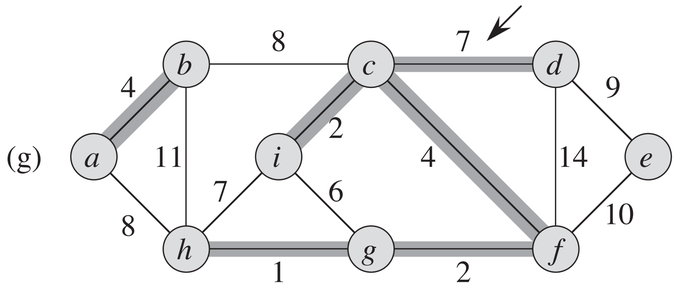












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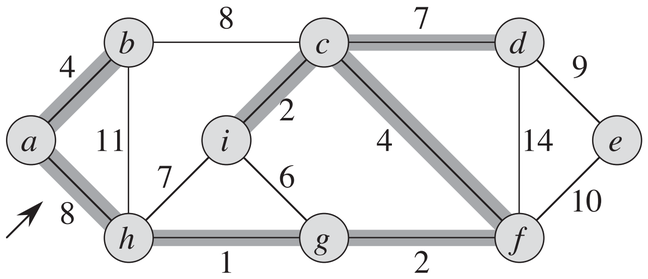
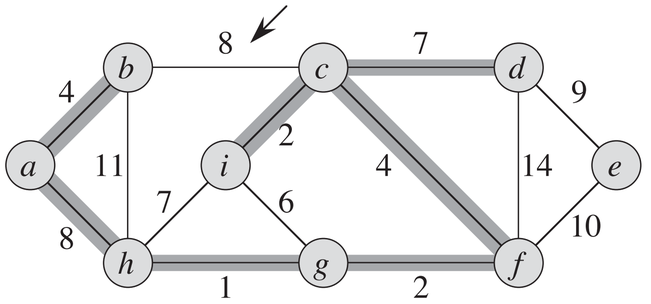
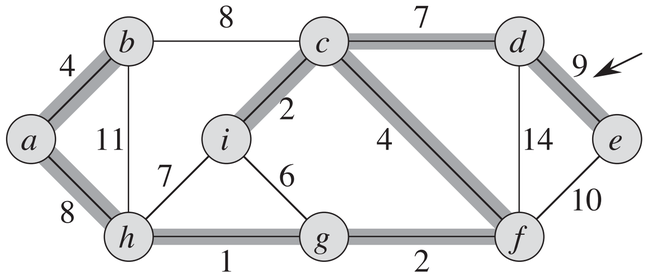
(i)



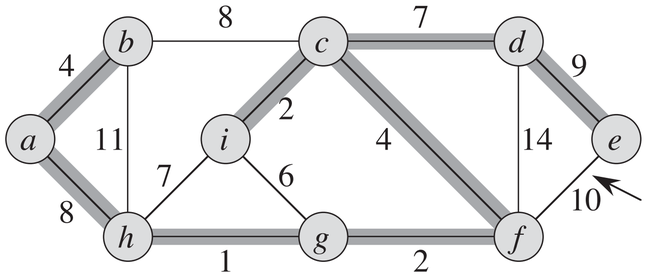
(j)



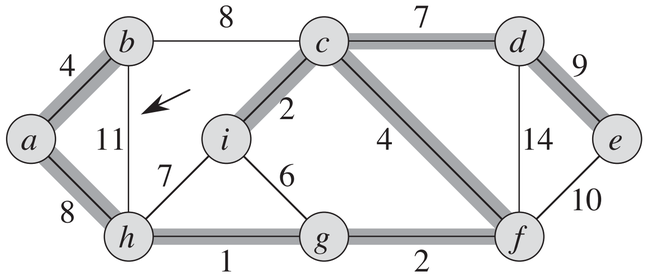
(k)



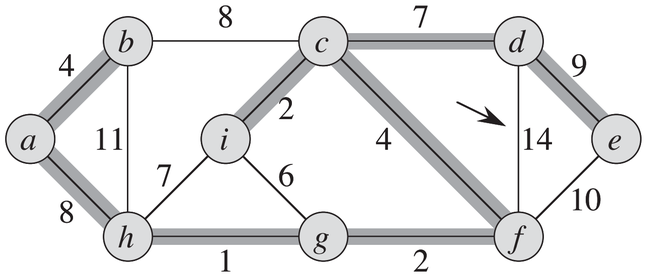
(l)



(m)



(n)



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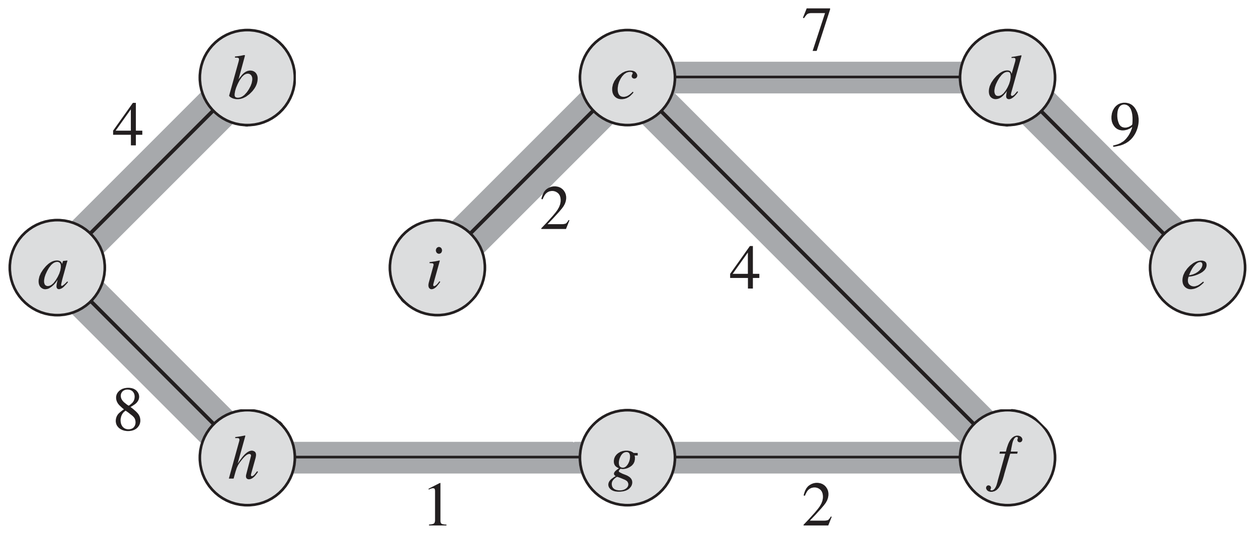

the Kruskal's algorithm is greedy, i.e., it makes locally optimal choice at each step

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