Modularity Maximization

$$\Delta Q = \frac{1}{2} s^T \hat{B}[g] s = \frac{1}{2} \sum_{i=1}^{n_g} s_i \left(\sum_{j=1}^{n_g} \hat{B}[g]_{i,j} s_j \right)$$

$$s'_i = \begin{cases} s_i & i \neq k \\ -s_i & i = k \end{cases}$$

$$s'_j = \begin{cases} s_j & j \neq k \\ -s_j & j = k \end{cases}$$

$$\frac{1}{2} s'^T \hat{B}[g] s' = \frac{1}{2} s'^T \left(\sum_{j=1}^{n_g} \hat{B}[g]_{i,j} s'_j \right)_{i=1}^{n_g} = \frac{1}{2} \sum_{i=1}^{n_g} s'_i \sum_{j=1}^{n_g} \hat{B}[g]_{i,j} s'_j \overbrace{=}$$

$$= \frac{1}{2} \sum_{i=1}^{n_g} s'_i \left(\left(\sum_{j \in \{1,\dots,n_g\} \setminus \{k\}} \hat{B}[g]_{i,j} s_j \right) - \hat{B}[g]_{i,k} s_k \right)$$

$$= \frac{1}{2} \left(\left(\sum_{i \in \{1,\dots,n_g\} \setminus \{k\}} s_i \left(\left(\sum_{j \in \{1,\dots,n_g\} \setminus \{k\}} \hat{B}[g]_{i,j} s_j \right) - \hat{B}[g]_{i,k} s_k \right) \right) - s_k \left(\left(\sum_{j \in \{1,\dots,n_g\} \setminus \{k\}} \hat{B}[g]_{k,j} s_j \right) - \hat{B}[g]_{k,k} s_k \right) \right) =$$

$$= \frac{1}{2} \left(\left(\sum_{i \in \{1,\dots,n_g\} \setminus \{k\}} s_i \left(\left(\sum_{j=1}^{n_g} \hat{B}[g]_{i,j} s_j \right) - 2\hat{B}[g]_{i,k} s_k \right) \right) - s_k \left(\left(\sum_{j=1}^{n_g} \hat{B}[g]_{k,j} s_j \right) - 2\hat{B}[g]_{k,k} s_k \right) \right) =$$

$$= \frac{1}{2} \left(\left(\sum_{i=1}^{n_g} s_i \left(\left(\sum_{j=1}^{n_g} \hat{B}[g]_{i,j} s_j \right) - 2\hat{B}[g]_{i,k} s_k \right) \right) - 2 s_k \left(\left(\sum_{j=1}^{n_g} \hat{B}[g]_{k,j} s_j \right) - 2\hat{B}[g]_{k,k} s_k \right) \right) =$$

$$= \frac{1}{2} \left(\left(\sum_{i=1}^{n_g} s_i \left(\sum_{j=1}^{n_g} \hat{B}[g]_{i,j} s_j \right) - \sum_{i=1}^{n_g} s_i 2\hat{B}[g]_{i,k} s_k \right) - 2 s_k \left(\sum_{j=1}^{n_g} \hat{B}[g]_{k,j} s_j \right) + 4 s_k \hat{B}[g]_{k,k} s_k \right) =$$

$$= \frac{1}{2} \left(\left(s^T \hat{B}[g] s - \sum_{i=1}^{n_g} s_i 2\hat{B}[g]_{i,k} s_k \right) - 2 s_k \left(\sum_{j=1}^{n_g} \hat{B}[g]_{k,j} s_j \right) + 4 s_k \hat{B}[g]_{k,k} s_k \right) =$$

$$= \frac{1}{2} s^T \hat{B}[g] s - \sum_{i=1}^{n_g} s_i \hat{B}[g]_{i,k} s_k - s_k \left(\sum_{j=1}^{n_g} \hat{B}[g]_{k,j} s_j \right) + 2 s_k \hat{B}[g]_{k,k} s_k =$$

$$= \Delta Q - s_k \left(\sum_{i=1}^{n_g} s_i \hat{B}[g]_{i,k} - \sum_{j=1}^{n_g} \hat{B}[g]_{k,j} s_j + 2\hat{B}[g]_{k,k} s_k \right) =$$

$$= \Delta Q - s_k \left(\sum_{i=1}^{n_g} \left(s_i \hat{B}[g]_{i,k} - \hat{B}[g]_{k,i} s_i \right) + 2\hat{B}[g]_{k,k} s_k \right) = \Delta Q - s_k \left(\sum_{i=1}^{n_g} s_i \left(\hat{B}[g]_{i,k} - \hat{B}[g]_{k,i} \right) + 2\hat{B}[g]_{k,k} s_k \right)$$

$$\Longrightarrow \arg\max_{1 \leq k \leq n_g} \left(\Delta Q \mid_{s'_k = -s_k} - \Delta Q \right) = \arg\max_{1 \leq k \leq n_g} \left(\Delta Q - s_k \left(\sum_{i=1}^{n_g} s_i \left(\hat{B}[g]_{i,k} - \hat{B}[g]_{k,i} \right) + 2\hat{B}[g]_{k,k} s_k \right) - \Delta Q \right) =$$

$$= \arg\max_{1 \leq k \leq n_g} \left(-s_k \left(\sum_{i=1}^{n_g} s_i \left(\hat{B}[g]_{i,k} - \hat{B}[g]_{k,i} \right) + 2\hat{B}[g]_{k,k} s_k \right) \right) =$$

$$= \arg\max_{1 \leq k \leq n_g} \left(-s_k \sum_{i=1}^{n_g} s_i \left(\hat{B}[g]_{i,k} - \hat{B}[g]_{k,i} \right) - 2 s_k \hat{B}[g]_{k,k} s_k \right) = \arg\max_{1 \leq k \leq n_g} \left(s_k \sum_{i=1}^{n_g} s_i \left(\hat{B}[g]_{k,i} - \hat{B}[g]_{i,k} \right) - 2\hat{B}[g]_{k,k} \right)$$

pseodo-code:

input: $\left(s, \hat{B}[g]\right)$, output: s'

1. $bestS \leftarrow s$
2. $bestM \leftarrow s^T \hat{B}[g] s$
3. $s' \leftarrow s$
4. do
5. $\quad s \leftarrow bestS$
6. $\quad M \leftarrow bestM$
7. $\quad$ hasMoved$\leftarrow (0)_{i=1}^{n_g}$
8. $\quad$ moved$\leftarrow 0$
9. $\quad$ while moved$< n_g$
11. $\quad\quad$ init $\delta \leftarrow \left(-2\hat{B}[g]_{k,k}\right)_{k=1}^{n_g}$
12. $\quad\quad$ for $1 \leq i, j \leq n_g$

$\quad\quad\quad$ if $k = i$:

13. $\quad\quad\quad\quad \delta[i] \leftarrow \delta[i] + s'_i s'_j \hat{B}[g]_{i,j}$

$\quad\quad\quad$ else if $k = j$:

$\quad\quad\quad\quad \delta[j] \leftarrow \delta[j] - s'_i s'_j \hat{B}[g]_{i,j}$

14. $\quad\quad K \leftarrow \underset{1 \leq k \leq n_g \wedge \text{hasMoved}[k]=0}{\arg\max} \delta[k]$
15. $\quad\quad M \leftarrow M + K$
16. $\quad\quad s'_K \leftarrow -s'_K$
17. $\quad\quad$ hasMoved$[K] \leftarrow 1$
18. $\quad\quad$ moved $\leftarrow$ moved $+ 1$
19. $\quad\quad$ if $M > bestM$:
20. $\quad\quad\quad bestS \leftarrow s'$
21. $\quad\quad\quad bestM \leftarrow M$
22. while $bestS$! = s
23. return $bestS$