Modularity Maximization

$$\Delta Q = \frac{1}{2} s^T \hat{B}[g] s = \frac{1}{2} \sum_{i=1}^{n_g} s_i \left(\sum_{j=1}^{n_g} \hat{B}[g]_{i,j} s_j \right)$$

$$s'_i = \begin{cases} s_i & i \neq k \\ -s_i & i = k \end{cases}$$

$$\frac{1}{2} s'^T \hat{B}[g] s' = \frac{1}{2} s'^T \left(\sum_{j=1}^{n_g} \hat{B}[g]_{i,j} s'_j \right)_{i=1}^{n_g} = \frac{1}{2} \sum_{i=1}^{n_g} s'_i \sum_{j=1}^{n_g} \hat{B}[g]_{i,j} s'_j \overbrace{=}^{s'_j = \begin{cases} s_j & j \neq k \\ -s_j & j = k \end{cases}}$$

$$= \frac{1}{2} \sum_{i=1}^{n_g} s'_i \left(\left(\sum_{j \in \{1,\ldots,n_g\} \setminus \{k\}} \hat{B}[g]_{i,j} s_j \right) - \hat{B}[g]_{i,k} s_k \right)$$

$$= \frac{1}{2} \left(\left(\sum_{i \in \{1,\ldots,n_g\} \setminus \{k\}} s_i \left(\left(\sum_{j \in \{1,\ldots,n_g\} \setminus \{k\}} \hat{B}[g]_{i,j} s_j \right) - \hat{B}[g]_{i,k} s_k \right) \right) - s_k \left(\left(\sum_{j \in \{1,\ldots,n_g\} \setminus \{k\}} \hat{B}[g]_{k,j} s_j \right) - \hat{B}[g]_{k,k} s_k \right) \right) =$$

$$= \frac{1}{2} \left(\left(\sum_{i \in \{1,\ldots,n_g\} \setminus \{k\}} s_i \left(\left(\sum_{j=1}^{n_g} \hat{B}[g]_{i,j} s_j \right) - 2\hat{B}[g]_{i,k} s_k \right) \right) - s_k \left(\left(\sum_{j=1}^{n_g} \hat{B}[g]_{k,j} s_j \right) - 2\hat{B}[g]_{k,k} s_k \right) \right) =$$

$$= \frac{1}{2} \left(\left(\sum_{i=1}^{n_g} s_i \left(\left(\sum_{j=1}^{n_g} \hat{B}[g]_{i,j} s_j \right) - 2\hat{B}[g]_{i,k} s_k \right) \right) - 2 s_k \left(\left(\sum_{j=1}^{n_g} \hat{B}[g]_{k,j} s_j \right) - 2\hat{B}[g]_{k,k} s_k \right) \right) =$$

$$= \frac{1}{2} \left(\sum_{i=1}^{n_g} s_i \left(\sum_{j=1}^{n_g} \hat{B}[g]_{i,j} s_j \right) - \sum_{i=1}^{n_g} s_i 2\hat{B}[g]_{i,k} s_k - 2 s_k \left(\sum_{j=1}^{n_g} \hat{B}[g]_{k,j} s_j \right) + 4 s_k \hat{B}[g]_{k,k} s_k \right) =$$

$$= \frac{1}{2} \left(s^T \hat{B}[g] s - \sum_{i=1}^{n_g} s_i 2\hat{B}[g]_{i,k} s_k - 2 s_k \left(\sum_{j=1}^{n_g} \hat{B}[g]_{k,j} s_j \right) + 4 s_k \hat{B}[g]_{k,k} s_k \right) =$$

$$= \frac{1}{2} s^T \hat{B}[g] s - \sum_{i=1}^{n_g} s_i \hat{B}[g]_{i,k} s_k - s_k \left(\sum_{j=1}^{n_g} \hat{B}[g]_{k,j} s_j \right) + 2 s_k \hat{B}[g]_{k,k} s_k =$$

$$= \Delta Q - s_k \left(\sum_{i=1}^{n_g} s_i \hat{B}[g]_{i,k} + \sum_{j=1}^{n_g} \hat{B}[g]_{k,j} s_j - 2\hat{B}[g]_{k,k} s_k \right) =$$

$$= \Delta Q - s_k \left(\sum_{i=1}^{n_g} \left(s_i \hat{B}[g]_{i,k} + \hat{B}[g]_{k,i} s_i \right) - 2\hat{B}[g]_{k,k} s_k \right) = \Delta Q - s_k \left(\sum_{i=1}^{n_g} s_i \left(\hat{B}[g]_{i,k} + \hat{B}[g]_{k,i} \right) - 2\hat{B}[g]_{k,k} s_k \right) =$$

$$\overbrace{=}^{\hat{B}[g] \text{ is cymetrical}} \Delta Q - s_k \left(\sum_{i=1}^{n_g} s_i 2\hat{B}[g]_{i,k} - 2\hat{B}[g]_{k,k} s_k \right) = \Delta Q - 2 s_k \sum_{i=1}^{n_g} s_i \hat{B}[g]_{k,i} + 2 s_k \hat{B}[g]_{k,k} s_k =$$

$$= \Delta Q - 2 s_k \sum_{i=1}^{n_g} s_i \hat{B}[g]_{k,i} + 2 s_k^2 \hat{B}[g]_{k,k} \overbrace{=}^{s_k \in \{-1,1\}} \Delta Q - 2 s_k \sum_{i=1}^{n_g} s_i \hat{B}[g]_{k,i} + 2\hat{B}[g]_{k,k}$$

$$\Longrightarrow \underset{1 \le k \le n_g}{\arg\max} \left(\Delta Q \mid_{s'_k = -s_k} - \Delta Q \right) = \underset{1 \le k \le n_g}{\arg\max} \left(\Delta Q - 2 s_k \sum_{i=1}^{n_g} s_i \hat{B}[g]_{k,i} + 2\hat{B}[g]_{k,k} - \Delta Q \right) =$$

$$= \underset{1 \le k \le n_g}{\arg\max} \left(2\hat{B}[g]_{k,k} - 2 s_k \sum_{i=1}^{n_g} s_i \hat{B}[g]_{k,i} \right)$$

pseodo-code:

input: $\left(s, \hat{B}[g]\right)$, output: s'

$bestS \leftarrow s$

$bestM \leftarrow s^T \hat{B}[g] s$

$s' \leftarrow s$

do

  $s \leftarrow bestS$

  $M \leftarrow bestM$

  hasMoved$\leftarrow (0)_{i=1}^{n_g}$

  moved$\leftarrow 0$

  while moved$< n_g$

    init $\delta \leftarrow \left(2\hat{B}[g]_{k,k}\right)_{k=1}^{n_g}$

    for $1 \leq i, j \leq n_g$

      $\delta[i] \leftarrow \delta[i] - 2s'_i s'_j \hat{B}[g]_{i,j}$

    $K \leftarrow \underset{1 \leq k \leq n_g \wedge \text{hasMoved}[k]=0}{\arg\max} \delta[k]$

    $M \leftarrow M + \delta[K]$

    $s'_K \leftarrow -s'_K$

    hasMoved$[K] \leftarrow 1$

    moved $\leftarrow$ moved $+ 1$

    if $M > bestM$:

      $bestS \leftarrow s'$

      $bestM \leftarrow M$

while $bestS$ $!=$ s

return $bestS$