The essential details of syntax, operational semantics, and typing for indexed natural numbers.

Type syntax

$$
\begin{aligned}
N &::= \mathbb{N} \mid N + N' \mid N * N' \mid \alpha && \text{(type-level naturals)} \\
\tau &::= \mathsf{Nat}\ N \mid \alpha \mid \ldots && \text{(types)}
\end{aligned}
$$

Term syntax

$$e ::= zero \mid succ(e) \mid ind(e, zero \mapsto e', (succ(x), y) \mapsto e'')$$

The first two are constructors. The last construct is the induction principle for eliminating natural numbers.

Operational semantics

$ind(zero, zero \mapsto e_0, (succ(x), y) \mapsto e_1) \rightsquigarrow e_0$

$ind(succ(n), zero \mapsto e_0, (succ(x), y) \mapsto e_1) \rightsquigarrow e_1[n/x][ind(n, zero \mapsto e_0, (succ(x), y) \mapsto e_1))/y]$

$$\frac{e \rightsquigarrow e'}{ind(e, zero \mapsto e_0, (succ(x), y) \mapsto e_1) \rightsquigarrow ind(e', zero \mapsto e_0, (succ(x), y) \mapsto e_1)}$$

Typing rules

$$\frac{}{\emptyset \vdash zero : \mathsf{Nat}\ 0}$$

$$\frac{\Gamma \vdash e : \mathsf{Nat}\ n}{\emptyset \vdash succ(e) : \mathsf{Nat}\ (n+1)}$$

$$\frac{\begin{array}{l}\Gamma \vdash e : \mathsf{Nat}\ n \\ \Gamma \vdash e_0 : \tau[0/\alpha] \\ \Gamma, x : \mathsf{Nat}\ m, y : \tau[m/\alpha] \vdash e_1 : \tau[(m+1)/\alpha]\end{array}}{\Gamma \vdash ind(e, zero \mapsto e_0, (succ(x), y) \mapsto e_1) : \tau[n/\alpha]}$$