Indexed natural numbers

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The standard essential details of syntax, operational semantics, and typing for indexed natural numbers.

Type syntax

$$N ::= \mathbb{N} \mid N + N' \mid N * N' \mid \alpha \qquad \text{(type-level naturals)}$$
$$\tau ::= \mathsf{Nat}\ N \mid \alpha \mid \ldots \qquad \text{(types)}$$

where α are type variables that can be substituted for.

Term syntax

$$e ::= \texttt{zero} \mid \texttt{succ}(e) \mid \texttt{ind}(e, \texttt{zero} \mapsto e', (\texttt{succ}(x), y) \mapsto e'')$$

The first two are constructors. The last construct is the induction principle for eliminating natural numbers where the first expression is the natural number to eliminate, the second part is for handling the zero case, and the last is for handling the successor case where y gets bound to the recursive result.

Operational semantics (call-by-value)

Values for natural numbers:

$$n ::= \texttt{zero} \mid \texttt{succ}(n)$$

Reductions:

$$\texttt{ind}(\texttt{zero}, \texttt{zero} \mapsto e_0, (\texttt{succ}(x), y) \mapsto e_1) \rightsquigarrow e_0 \qquad (\beta_{\texttt{zero}})$$

$$\texttt{ind}(\texttt{succ}(n), \texttt{zero} \mapsto e_0, (\texttt{succ}(x), y) \mapsto e_1) \rightsquigarrow e_1[n/x][\texttt{ind}(n, \texttt{zero} \mapsto e_0, (\texttt{succ}(x), y) \mapsto e_1)/y] \qquad (\beta_{\texttt{succ}})$$

$$\frac{e \rightsquigarrow e'}{\texttt{ind}(e, \texttt{zero} \mapsto e_0, (\texttt{succ}(x), y) \mapsto e_1) \rightsquigarrow \texttt{ind}(e', \texttt{zero} \mapsto e_0, (\texttt{succ}(x), y) \mapsto e_1)} \qquad \text{(cong)}$$

Typing rules

$$\frac{}{\emptyset \vdash \texttt{zero} : \mathsf{Nat}\ 0}$$

$$\frac{\Gamma \vdash e : \mathsf{Nat}\ n}{\emptyset \vdash \texttt{succ}(e) : \mathsf{Nat}\ (n+1)}$$

$$\frac{\begin{array}{l}\Gamma \vdash e : \mathsf{Nat}\ n \\ \Gamma \vdash e_0 : \tau[0/\alpha] \\ \Gamma, x : \mathsf{Nat}\ m, y : \tau[m/\alpha] \vdash e_1 : \tau[(m+1)/\alpha]\end{array}}{\Gamma \vdash \texttt{ind}(e, \texttt{zero} \mapsto e_0, (\texttt{succ}(x), y) \mapsto e_1) : \tau[n/\alpha]}$$