Indexed natural numbers

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The standard essential details of syntax, operational semantics, and typing for indexed natural numbers.

Type syntax

$$N ::= \mathbb{N} \mid N + N' \mid N * N' \mid \alpha \qquad \text{(type-level naturals)}$$
$$\tau ::= \mathsf{Nat}\ N \mid \alpha \mid \ldots \qquad \text{(types)}$$

Term syntax

$$e ::= zero \mid succ(e) \mid ind(e, zero \mapsto e', (succ(x), y) \mapsto e'')$$

The first two are constructors. The last construct is the induction principle for eliminating natural numbers.

Operational semantics

$$ind(zero, zero \mapsto e_0, (succ(x), y) \mapsto e_1) \rightsquigarrow e_0$$

$$ind(succ(n), zero \mapsto e_0, (succ(x), y) \mapsto e_1) \rightsquigarrow e_1[n/x][ind(n, zero \mapsto e_0, (succ(x), y) \mapsto e_1))/y]$$

$$\frac{e \rightsquigarrow e'}{ind(e, zero \mapsto e_0, (succ(x), y) \mapsto e_1) \rightsquigarrow ind(e', zero \mapsto e_0, (succ(x), y) \mapsto e_1)}$$

Typing rules

$$\frac{}{\emptyset \vdash zero : \mathsf{Nat}\ 0}$$

$$\frac{\Gamma \vdash e : \mathsf{Nat}\ n}{\emptyset \vdash succ(e) : \mathsf{Nat}\ (n + 1)}$$

$$\frac{\Gamma \vdash e : \mathsf{Nat}\ n \quad \Gamma \vdash e_0 : \tau[0/\alpha] \quad \Gamma, x : \mathsf{Nat}\ m, y : \tau[m/\alpha] \vdash e_1 : \tau[(m + 1)/\alpha]}{\Gamma \vdash ind(e, zero \mapsto e_0, (succ(x), y) \mapsto e_1) : \tau[n/\alpha]}$$