Classical Particles, Fields, and Matter I

Based on lectures by Dr. Barry Ritchie
Notes taken by Daniel Moore

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Topic

etc..

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1.1 The Fundamental Goal of Mechanics

1.2 Resistance to Motion Through a Fluid ("drag")

When a real object moves through a fluid (including air), it experiences a resistive force, often called *drag* with a few basic properties:

(i) It is velocity-dependent.

(ii) It usually acts in the direction opposite to the velocity.

The general notation for drag, taking these two into account, is then $\mathbf{F_{drag}} = -f(v)\hat{\mathbf{v}}$, where $f(v)$ is the magnitude of the velocity-dependent force, and $-\hat{\mathbf{v}}$ is the unit vector in the direction opposite to the velocity.

We can approximate this drag fore with a Taylor series, by writing

$$f(v) = a + bv + v^2 + \dots$$

Sine $f(v = 0) = 0$, we know $a = 0$, and because we're working at relatively low speeds, we can ignore anything above v^2. This leaves us with a general expression for the magnitude of the drag force:

$$f(v) = bv + cv^2 = f_{lin} + f_{quad}$$

In linear drag, the fluid breaks up into "sheets," which slide past each other. Stokes' law for laminar flow of a sphere is written as

$$f(v) = bv = 3\pi\eta Dv$$

where η is the viscosity of the fluid, and D is the diameter of the sphere.

In quadratig drag, the fluid breaks up into vortices, or "eddies," staying in motion after the object passes. Newton's general equation for quadratic drag is:

$$f(v) = cv^2 = k\rho Av^2$$

where k is a geometric constant, ρ is the density of the fluid, and A is the cross-sectional area of the object.

To determine when each of these should be used, we can take the ratio of the quadratic drag to the linear drag—or, at least, something proportional to it. If we do this for a sphere, we find

$$\begin{aligned} \frac{f_{quad}}{f_{lin}} &= \frac{k\rho Av^2}{3\pi\eta Dv} \\ &\propto \frac{\rho D^2 v^2}{\eta Dv} = \frac{\rho Dv}{\eta} \end{aligned}$$

This final fraction, more generally written, is alled Reynold's Number:

$$R_e = \frac{\rho Lv}{\eta}$$

where L is the "characteristic dimension."

- $R_e \gg 1 \implies \mathbf{F_{drag}} \approx -cv^2\hat{\mathbf{v}}$
- $R_e \ll 1 \implies \mathbf{F_{drag}} \approx -bv\hat{\mathbf{v}}$
- $R_e \approx 1 \implies \mathbf{F_{drag}} \approx -(bv + cv^2)\hat{\mathbf{v}}$

1.2.1 Motion with Linear Drag

First, we will look at the first term, linear drag, also called "Stokes drag," "low Reynolds number drag," or "laminar flow." We can write the general equation(s) for this (in the air) as

$$\begin{aligned}
\mathbf{F} &= m\ddot{\mathbf{r}} = m\mathbf{g} - b\mathbf{v} \\
&= m\dot{\mathbf{v}} = m\mathbf{g} - b\mathbf{v} \\
&= \begin{cases} m\dot{v_x} = -bv_x \\ m\dot{v_y} = -mg - bv_y \end{cases}
\end{aligned}$$

Because these are separable, first-order differential equations, we can easily split them into their components and look at those separately.

Horizontal Motion with Linear Drag

Separating first the horizontal equation from above, we can consider a simple example, like a cart moving across a fritionless track, experiencing air resistance. Eventually, the cart will eventually slow down, and we can find the rate by which it slows. There are two ways to do this, and I'll show both of them. First, by integration:

$$\begin{aligned}
F = m\frac{\mathrm{d}v}{\mathrm{d}t} &= -bv \\
\frac{\mathrm{d}v}{v} &= -\frac{b}{m}\mathrm{d}t \\
\int_{v_0}^{v(t)} \frac{1}{v'}\mathrm{d}v' &= -\frac{b}{m}\int_0^t \mathrm{d}t' \\
\ln(\frac{v(t)}{v_0}) &= -\frac{b}{m}t \\
e^{\ln(\frac{v(t)}{v_0})} &= e^{-\frac{b}{m}t} \\
v(t) &= v_0 e^{-t/\tau}
\end{aligned}$$

where $\tau = \frac{m}{b}$ is the *characteristic time*. Alternatively,

$$\begin{aligned}
m\dot{v} &= -bv \\
\dot{v} &= -\frac{b}{m}v \\
\implies v(t) &= Ae^{-\frac{b}{m}t} \\
v(0) = v_0 \implies v(t) &= v_0 e^{-t/\tau}
\end{aligned}$$

From here, we can easily find the position function by integrating:

$$\begin{aligned}
v(t) = \frac{\mathrm{d}x}{\mathrm{d}t} &= v_0 e^{-t/\tau} \\
\mathrm{d}x &= v_0 e^{-t/\tau} dt \\
\int_0^{x(t)} \mathrm{d}x' &= v_0 \int_0^t e^{-t'/\tau} \mathrm{d}t' \\
x(t) &= -v_0 \tau (e^{-t/\tau} - 1) \\
x(t) &= x_\infty (1 - e^{-t/\tau})
\end{aligned}$$

where $x_\infty = v_o \tau$.

Vertical Motion with Linear Drag

Similarly, we can onsider a simple example with only vertical drag, like a ball thrown straight up into the air, which falls straight back down. Likewise, there are multiple methods we can use to solve this, but I will only show one. We will assume for now that the positive y-axis is pointed upwards.

$$\begin{aligned}
F = m\frac{\mathrm{d}v}{\mathrm{d}t} &= -mg - bv \\
\frac{1}{mg/b + v}\mathrm{d}v &= -\frac{b}{m}\mathrm{d}t \\
\int_{v_0}^{v(t)} \frac{1}{mg/b + v'}\mathrm{d}v' &= -\frac{b}{m}\int_0^t \mathrm{d}t' \\
\ln\left(\frac{\frac{mg}{b} + v(t)}{\frac{mg}{b} + v_0}\right) &= -\frac{t}{\tau} \\
v(t) &= -\frac{mg}{b} + (\frac{mg}{b} + v_0)e^{-t\tau}
\end{aligned}$$

I'm going to stop here to talk about *terminal velocity*—that is, the maximum velocity an object can have. To find the terminal velocity, we want to take $\lim_{t\to\infty} v(t) = -\frac{mg}{b}$. We thus define terminal velocity as $v_{ter} = \frac{mg}{b}$, which lets us re-write our $v(t)$ equation:

$$\begin{aligned}
v(t) &= -v_{ter} + (v_{ter} + v_0)e^{-t/\tau} \\
v(t) &= v_0 e^{-t/\tau} + v_{ter}(e^{-t/\tau} - 1)
\end{aligned}$$

Once again, from here we can easily find the position function by integrating again:

$$\begin{aligned}
v(t) = \frac{\mathrm{d}y}{\mathrm{d}t} &= v_0 e^{-t/\tau} + v_{ter}(e^{-t/\tau} - 1) \\
\int_0^{y(t)} \mathrm{d}y' &= \int_0^t v_0 e^{-t/\tau} + v_{ter}(e^{-t/\tau} - 1)\mathrm{d}t \\
y(t) &= v_0\tau(1 - e^{-t/\tau}) + v_{ter}\tau(1 - e^{-t/\tau}) - v_{ter}t \\
&= (v_0 + v_{ter})\tau(1 - e^{-t/\tau}) - v_{ter}t
\end{aligned}$$

Projectile Motion with Linear Drag

Now, we can consider motion in both the x and y directions with linear drag. We already know our equations of motion:

$$\begin{aligned}\mathbf{F}(t) = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= m\mathbf{g} - b\mathbf{v} \\ &= m\left[\frac{\mathrm{d}v_x}{\mathrm{d}t}\hat{\mathbf{x}} + \frac{\mathrm{d}v_y}{\mathrm{d}t}\hat{\mathbf{y}}\right] = -mg\hat{\mathbf{y}} - b(v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}})\end{aligned}$$

If we separate these into their x and y components:

$$\begin{cases} m\frac{\mathrm{d}v_x}{\mathrm{d}t} = -bv_x \\ m\frac{\mathrm{d}v_y}{\mathrm{d}t} = -mg - bv_y \end{cases}$$

And we just solved for those! As a reminder, we currently know

$$\begin{aligned} &v_x(t) = v_{x0}e^{-t/\tau} && x(t) = x_\infty(1 = e^{-t/\tau}) \\ &v_y(t) = v_{y0}e^{-t/\tau} + v_{ter}(e^{-t/\tau} - 1) && y(t) = (v_{y0} + v_{ter})\tau(1 - e^{-t/\tau}) - v_{ter}t \end{aligned}$$

where $\tau = m/b$, $v_{ter} = mg/b$, $x_\infty = v_{x0}\tau$, $v_{x0} = v_0\cos\theta$, and $v_{y0} = v_0\sin\theta$.

We can use these equations to find the trajectory—that is, a function $y(x)$–by using the function $x(t)$ to find the funtion $t(x)$. If we do this, we find

$$y(x) = \frac{v_{y0} + v_{ter}}{v_{x0}}x + v_{ter}\tau\ln\left(1 - \frac{x}{v_{x0}\tau}\right)$$

This is transcendental equation, so we can only find information about it graphically, numerically, or approximately.

1.2.2 Motion with Qudratic Drag

Next, let's look at quadratic drag, also called "Newtonian drag," or "high Reynolds number drag." We can write the general equation(s) for this, assuming a positive y direction as downwards, as

$$\begin{aligned}\mathbf{F} &= m\ddot{\mathbf{r}} \\ &= m\dot{\mathbf{v}} = m\mathbf{g} - cv^2\hat{\mathbf{v}}\end{aligned}$$

Because these are no longer linear differential equations, we may have to go about solving them a little differently.

Horizontal Motion with Quadratic Drag

We can consider an example problem similar to the last: a cart is moving across a frictionless track, experiencing air resistance. We can write and solve this equation of motion as

$$\begin{aligned}
m\frac{\mathrm{d}v}{\mathrm{d}t} &= -cv^2 \\
m\frac{1}{v^2}\mathrm{d}v &= -c\mathrm{d}t \\
m\int_{v_0}^{v(t)} \frac{1}{v'^2}\mathrm{d}v' &= -c\int_0^t \mathrm{d}t' \\
m\left(\frac{1}{v_0} - \frac{1}{v}\right) &= -ct \\
v(t) &= \frac{v_0}{1+cv_0 t/m} = \frac{v_0}{1_t/\tau}
\end{aligned}$$

where $\tau - \frac{m}{cv_0}$ is given by a different set of constants, but represents roughly the same thing.

To find $x(t)$, we can integrate again:

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{v_0}{1_t/\tau} \\
\int_{x_0}^{x(t)} \mathrm{d}x' &= \int_0^t \frac{v_0}{1+t'/\tau}\mathrm{d}t' \\
x(t) - x_0 &= v_0\int_0^t \frac{1}{1+t'/\tau}\mathrm{d}t' \\
x(t) = x_0 + v_0\tau\ln\left(1+\frac{t}{\tau}\right)
\end{aligned}$$

If we take $x_0 = 0$, we can exclude the x_0 term, which will will do for the future.

Vertical Motion with Quadratic Drag

Assuming a downwards y is positive, we can write the equation of motion as

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - cv^2$$

Since we know terminal velocity will show up in this, as it did in linear drag, let's find that first:

$$\begin{aligned}
mg - cv^2 &= 0 \\
mg &= cv^2 \\
v^2 &= \frac{mg}{c} \\
v_{ter} &= \sqrt{\frac{mg}{c}} \implies c & = \frac{mg}{v_{ter}^2}
\end{aligned}$$

This, conveniently, lets us simplify out equation of motion:

$$\begin{aligned}
m\frac{\mathrm{d}v}{\mathrm{d}t} &= mg - m\frac{gv^2}{v_{ter}^2} \\
\frac{\mathrm{d}v}{\mathrm{d}t} &= g\left(1 - \frac{v^2}{v_{ter}^2}\right) \\
\frac{1}{1 - v^2/v_{ter}^2}\mathrm{d}v &= g\mathrm{d}t
\end{aligned}$$

Assuming the ball starts from rest, and skipping the full derivation (left as an exercise to the reader?), we find that

$$\begin{aligned}
v(t) &= v_{ter}\tanh\left(\frac{gt}{v_{ter}}\right) \\
y(t) &= \frac{v_{ter}^2}{g}\ln\left(\cosh\left(\frac{gt}{v_{ter}}\right)\right)
\end{aligned}$$

Projectile Motion with Quadratic Drag

Recall:

$$m\dot{\mathbf{v}} = m\mathbf{g} - cv\mathbf{v}$$

But now, $v = \sqrt{v_x^2 + v_y^2}$, so when we try to separate our equations, we get

$$\begin{cases}
m\dot{v_x} = -cv_x\sqrt{v_x^2 + v_y^2} \\
m\dot{v_y} = -mg - c\sqrt{v_x^2 + v_y^2}
\end{cases}$$

These equations are coupled, and are not easily decoupled, so they can only be solved numerically.

1.3 Motion of a Charged Particle in a Uniform Magnetic Field

1.3.1 Solving with Differential Equations

For some particle of charge q moving in a magnetic field $\mathbf{B}$ pointing in the positive z direction, the net force is given by the Lorentz force:

$$\mathbf{F} = m\dot{\mathbf{v}} = q\mathbf{v} \times \mathbf{B}$$

We can separate these equations to:

$$\begin{cases}
m\dot{v_x} = qBv_y \\
m\dot{v_y} = -qBv_x \\
m\dot{v_z} = 0 \qquad\qquad \implies v_z \text{ is constant}
\end{cases}$$

$$\begin{cases}
\ddot{x} = \frac{qB}{m}\dot{y} = \omega\dot{y} \\
\ddot{y} = -\frac{qB}{m}\dot{x} = -\omega\dot{x}
\end{cases}$$

Where $\omega = \frac{qB}{m}$ is the cyclotron frequency. These equations are coupled but separble:

$$\ddot{x} = \omega\dot{y} \implies \dot{x} = \omega y + c$$
$$\ddot{y} = -\omega\dot{x} \implies \dot{y} = -\omega y + d$$

We can combine all of these equations to say:

$$\begin{aligned} &\ddot{x} = \omega(-\omega x + d) = -\omega^2 x + \omega d \\ \implies &x(t) = R\cos(\omega t - \phi) + a \\ &\ddot{y} = -\omega(\omega y + c) = -\omega^2 y - \omega c \\ \implies &y(t) = -R\sin(\omega t - \phi) + b \end{aligned}$$

This describes clockwise circular motion with radius R centered on (a, b) with frequency ω. We can find R as

$$R = \frac{x^2 + y^2}{\omega} = \frac{v}{\omega} = \frac{mv}{qB}$$

1.3.2 Solving using Complex Variables

We begin the same way: from the magnetic component of the Lorentz force, $\mathbf{F} = m\dot{\mathbf{v}} = q(\mathbf{v} \times \mathbf{B})$, we can derive the equations of motion,

$$\begin{cases} m\dot{v_x} = qBv_y \\ m\dot{v_y} = -qBv_x \end{cases}$$
$$\begin{cases} \ddot{x} = \frac{qB}{m}\dot{y} = \omega\dot{y} \\ \ddot{y} = -\frac{qB}{m}\dot{x} = -\omega\dot{x} \end{cases}$$

To decouple this, rather than the method we used before, we can use complex variables, though. Let's define a complex variable $s = \dot{x} + i\dot{y}$. This means that

$$\begin{aligned} &\ddot{s} = \ddot{x} + i\ddot{y} = \omega\dot{y} + i\omega\dot{x} = -i\omega(\dot{x} + i\dot{y}) \\ &\ddot{s} = -i\omega s s = Ae^{-i\omega t} \end{aligned}$$

Where A is a complex number. Now, let's define another complex variable, $u = x + iy$. Similarly, this means that

$$\begin{aligned} &\dot{u} = \dot{x} + i\dot{y} = s \\ &u = \int s\mathrm{d}t = \int Ae^{-i\omega t}\mathrm{d}t = \frac{iA}{\omega}e^{-i\omega t} + C \end{aligned}$$

We can write C, a complex constant, as $C = a + ib$. We can also write $\frac{iA}{\omega}$, another complex constant, as $Re^{-i\varphi}$, where R is purely real. This means that we can re-write u as

$$\begin{aligned} u(t) &= \frac{iA}{\omega}e^{-i\omega t} + C \\ &= Re^{-i\varphi}e^{-i\omega t} + a + ib \\ &= Re^{-i(\omega t + \varphi)} + a + ib \\ &= R\cos(\omega t + \varphi) - iR\sin(\omega t + \varphi) \end{aligned}$$

Since $\mathrm{Re}(u) = x$ and $\mathrm{Im}(u) = y$, this means that

$$\begin{aligned} x(t) &= R\cos(\omega t + \varphi) + a \\ y(t) &= -R\sin(\omega t + \varphi) + b \\ z(t) &= 0 \end{aligned}$$

Which is also what we got the other way!

2 Momentum and Angular Momentum

2.1 Momentum for a System of Particles

2.1.1 Conservation of Momentum

Newtonian Mechanics focuses on momentum, $\mathbf{p} = m\mathbf{v}$. The three laws can be written for a system of particles in terms of momentum:

(i) $\sum F = 0 \implies \mathbf{p} = m\mathbf{v}$ is constant

(ii) $\sum F_{ext} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$

(iii) For any two particles in a system, $\mathbf{F_{ij}} = -\mathbf{F_{ji}}$. That is, if they're interacting, they should exert equal and opposite forces on each other.

The second law is often also reffered to the *Law (or Principle) of Conservation of Momentum.* In words, more explicitly, the Principle of Conservation of Momentum says that if the net external force on a particle or a system of particles is equal to zero, then the system's momentum will remain constant.
The third law is, in reality, not always followed as such, but we will look primarily at situations where the third law is applicable.

2.1.2 Center of Mass

The *center of mass*, $\mathbf{R}$, is the mass-averaged position of a system of particles. It is not a real object, rather a mathematical point, but the center of mass is the location in space that, can be used to represent a system of particles as a single particle. We can find the center of mass as:

$$\mathbf{R} = \frac{\sum_{i=1}^{N} m_i \mathbf{r}_i}{\sum_{i=1}^{N} m_i} = \frac{1}{M}\sum_{i=1}^{N} m_i \mathbf{r}_i$$

Where M is the total mass of the system, or $M = \sum_{i=0}^{N} m_i$.
$\mathbf{R}$ is then a vector with the components

$$\begin{cases} R_x = \frac{1}{M}\sum_{i=1}^{N} m_i x_i \\ R_y = \frac{1}{M}\sum_{i=1}^{N} m_i y_i \\ R_z = \frac{1}{M}\sum_{i=1}^{N} m_i z_i \end{cases}$$

For a continuous body, we would take those sums as the distance between the sums is infinitesimally small, which is the definition of an integral, so we can write

$$\mathbf{R} = \frac{1}{M}\int_M dM = \frac{1}{M}\int_M \mu \mathrm{d}V$$

Where μ is the density of the continuous body.

The derivative of the center of mass is called the *center-of-mass velocity*, and is given by

$$\dot{\mathbf{R}} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{M}\sum_{i=1}^{N} m_i \mathbf{r}_i\right] = \frac{1}{M}\sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i$$

2.1.3 Momentum for a System of Particles

The linear momentum for a system of particles is defined as the sum of the linear momentum of each of the particles inside it:

$$\mathbf{P} = \sum_{i=1}^{N} \mathbf{p}_i = \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i = M\dot{\mathbf{R}}$$

On any arbitrary particle in the system, there will be two types of forces acting on it: the external forces ($\mathbf{F}_i$) and the internal forces ($\mathbf{F_{ij}}$). From the second law, the sum of the forces can be written as

$$\mathbf{F}_i + \sum_{i\neq j}^{N} \mathbf{F}_{ij} = \dot{\mathbf{p}}_i$$

If we sum this over the full system to find the total force on the ith particle, we get

$$\sum \left[\mathbf{F}_i + \sum_{i\neq j}^{N} \mathbf{F}_{ij} \right] = \sum \dot{\mathbf{p}}_i$$

The second term in the sum, $\sum_{i\neq j}^{N} \mathbf{F}_{ij}$, will have equal and opposite pairs as long as the system obeys the third law, so if we assume for now that all systems obey the third law, we can take that term as being equal to 0, meaning we can write the total force on the system as

$$\sum_{i=1}^{N} \dot{\mathbf{P}} = \sum_{i=1}^{N} \dot{\mathbf{F}}_i = M\ddot{\mathbf{R}} = \mathbf{F_{ext}}$$

2.1.4 The Rocket Problem

The rocket problem is a representation of a system with conserved momentum and changing mass. A rocket propels itself using no external forces, so the total momentum is constant, but it has changing mass and somehow still propels itself forward. For our basic analysis, we'll look at an example with only horizontal motion and no wind resistance (like it's in space or something).

The total mass of the system, $m(t)$, is changing by an amount $\mathrm{d}m$ (implicitly negative) in the time period $\mathrm{d}t$. $\mathrm{d}m$ is being ejected at an exhaust velocity, v_{ex}, relative to the rocket. At the initial time t, the mass is m, and the velocity is v. The momentum is given by

$$P(t) = mv$$

At time $t + \mathrm{d}t$, the mass become $m + \mathrm{d}m$, and its velocity becomes $v - v_{ex}$. The momentum now becomes

$$\begin{aligned} P(t+dt) &= (m + \mathrm{d}m)(v + \mathrm{d}v) - \mathrm{d}m(v - v_{ex}) \\ &= mv + m\mathrm{d}v + \mathrm{d}mv - \mathrm{d}mv + \mathrm{d}mv_{ex} + \cancelto{0}{\mathrm{d}m\mathrm{d}v} \\ &= mv + m\mathrm{d}v + \mathrm{d}mv_{ex} \end{aligned}$$

If no external forces are acting, as we said, total momentum must be conserved, or $P(t) \stackrel{!}{=} P(t + \mathrm{d}t)$. Let's see what happens if we try to force this to be true:

$$\begin{aligned}
mv &\stackrel{!}{=} mv + m\mathrm{d}v + \mathrm{d}mv_{ex} \\
0 &= m\mathrm{d}v + \mathrm{d}mv_{ex} \\
m\mathrm{d}v &= -\mathrm{d}mv_{ex} \implies \text{“The Rocket Equation”} \\
m\frac{\mathrm{d}v}{\mathrm{d}t} &= -\frac{\mathrm{d}m}{\mathrm{d}t}v_{ex} \\
m\dot{v} &= -\dot{m}v_{ex}
\end{aligned}$$

Even though there are no external forces here, we still have a term that looks like a “force” on the left-hand side. We definethis equation as a sort of force called “thrust.” This equation tells us that in order for momentum to remain constant, the remaining mass must increase in velocity to mtch the momentum being ejected. We can continue to find the required change in velocity:

$$\begin{aligned}
m\mathrm{d}v &= -\mathrm{d}mv_{ex} \\
\mathrm{d}v &= -\frac{\mathrm{d}m}{m}v_{ex} \\
\int_{v_0}^{v(t)} \mathrm{d}v' &= -v_{ex}\int_{m_0}^{m} \frac{1}{m'}\mathrm{d}m' \\
v - v_0 = \Delta v &= v_{ex} \ln\left(\frac{m_0}{m}\right) \implies \text{“The Rocket Equation”}
\end{aligned}$$

2.2 Angular Momentum for a System of Particles

2.2.1 Angular Momentum for a Single Particle

Conservation of angular momentum is remarkably similar to conservation of linear momentum, but let's make sure we talk about it first. Angular momentum for a single particle is defined as

$$\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v}$$

Notice that since we're dependent on the position $\mathbf{p}$ here, angular momentum, unlike linear momentum, will depend on our origin/coordinate system.

We can easily find the time derivative of angular momentum:

$$\begin{aligned}
\dot{\boldsymbol{\ell}} = \frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{r} \times \mathbf{p}] &= (\dot{\mathbf{r}} \times \mathbf{p}) + (\mathbf{r} \times \dot{\mathbf{p}}) \\
&= \cancelto{0}{(\dot{\mathbf{r}} \times m\dot{\mathbf{r}})} + (\mathbf{r} \times \mathbf{F}) \equiv \Gamma
\end{aligned}$$

The first term goes to zero because we are left with the cros product of parallel vectors, and in the second term, we use our definition $\mathbf{F} = \dot{\mathbf{p}}$. Γ is defined as the net torque on a particle about O.

2.2.2 Angular Momentum for a System of Particles

For a system of particles, we define angular momentum similarly to how we total linear momentum for a system:

$$\mathbf{L} = \sum_{i=1}^{N} \boldsymbol{\ell}_\mathbf{i} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i = \sum_{i=1}^{N} \mathbf{r}_i \times m\mathbf{v}_i$$

Differentiating with respect to t gives us

$$\begin{aligned}\dot{\mathbf{L}} = \sum_{i=1}^{N} \dot{\boldsymbol{\ell}}_\mathbf{i} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{i=1}^{N} \mathbf{r}_i \times m_i\mathbf{v}_i\right] \\ &= \sum_{i=1}^{N} \cancelto{0}{\dot{\mathbf{r}}_i \times m_i\mathbf{v}_i} + \sum_{i=1}^{N} \mathbf{r}_i \times m_i\dot{\mathbf{v}}_i \\ &= \sum_{i=1}^{N} \mathbf{r}_i \times \dot{\mathbf{p}}_i\end{aligned}$$

From our discussion of linear momentum, we know that the derivative of $\mathbf{p}$ is equal to the sum of the forces, or for a system, the sum of the external and internal forces, written

$$\sum_{i=1}^{N} \dot{\mathbf{p}}_\mathbf{i} = \sum \left[\mathbf{F}_i + \sum_{i\neq j}^{N} \mathbf{F}_{ij}\right]$$

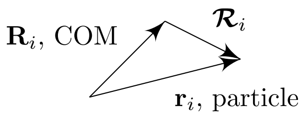
so we can simplidy our equation for total angular momentum to:

$$\begin{aligned}\dot{\mathbf{L}} &= \sum_{i=1}^{N} \mathbf{r}_i \times \dot{\mathbf{p}}_i = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}^{net} \\ &= \sum_{i=1}^{N} \mathbf{r}_i \times \left[\mathbf{F}_i + \sum_{i\neq j}^{N} \mathbf{F}_{ij}\right] \\ &= \underbrace{\sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i}_{\boldsymbol{\Gamma}^{ext}} + \sum_{i=1}^{N}\sum_{i\neq j}^{N} \mathbf{r}_i \times \mathbf{F}_{ij}\end{aligned}$$

Newton's third law, once again, doesn't discuss this second term! But, like with linear angular momentum, in a system wher the third law holds, we will end up with opposite pairs of internal forces, with one outlier ($\mathbf{r}_{ij} \times \mathbf{F}_{ji}$). However, if we simplify our system such that all forces are *central forces*, then those two vectors will be parallel, so their cross-product will also go to zero. We can say, then, that in a system where Newton's third law holds, and all internal forces are central forces,

$$\dot{\mathbf{L}} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i = \boldsymbol{\Gamma}^{ext}$$

The principle of conservation of angular momentum here is implied, but if we write it out, we say that if the net external torque on an N-particle system is zero, the system's total angular momentum is constant.



Angular Momentum about the Center of Mass

We will find it convenient to define a vector $\boldsymbol{\mathcal{R}}_i = \mathbf{r}_i - \mathbf{R}$, or $\mathbf{r}_i = \mathbf{R} + \boldsymbol{\mathcal{R}}_i$. Taking the time derivatives, we can easily see that it follows that $\mathbf{v}_i = \mathbf{v}_{cm} + \dot{\boldsymbol{\mathcal{R}}}_i$. Using these definitions, we can re-write our definition of a system's total angular momentum:

$$\begin{aligned}\mathbf{L} &= \sum_{i=1}^{N} \mathbf{r}_i \times m_i \mathbf{v}_i = \sum_{i=1}^{N} (\mathbf{R} + \boldsymbol{\mathcal{R}}_i) \times m_i(\mathbf{v}_{cm} + \dot{\boldsymbol{\mathcal{R}}}_i) \\ &= \sum_{i=1}^{N} \left[\mathbf{R} \times \mathbf{v}_{cm} + \mathbf{R} \times m_i \dot{\boldsymbol{\mathcal{R}}}_i + \boldsymbol{\mathcal{R}}_i \times m_i \mathbf{v}_{cm} + \boldsymbol{\mathcal{R}}_i \times m_i \dot{\boldsymbol{\mathcal{R}}}_i \right]\end{aligned}$$

Although this is dauntingly large, we can simplify it significantly. Let's take it in parts:

(i) $\mathbf{R} \times \mathbf{v}_{cm}$
This is written solely in terms of center-of-mass quantities, and can be slightly simplified to $\mathbf{R} \times \mathbf{P}$

(ii) $\mathbf{R} \times m_i \dot{\boldsymbol{\mathcal{R}}}_i$
From the diagram, we know that $\dot{\boldsymbol{\mathcal{R}}}_i = \mathbf{v}_i - \mathbf{v}_{cm}$, so we can re-write the second term as

$$\sum_{i=1}^{N} m_i \dot{\boldsymbol{\mathcal{R}}}_i = \sum_{i=1}^{N} m_i \mathbf{v}_i - \sum_{i=1}^{N} m_i \mathbf{v}_{cm} = M\mathbf{v}_{cm} - M\mathbf{v}_{cm} = 0$$

(iii) $\mathbf{R} \times m_i \dot{\boldsymbol{\mathcal{R}}}$
From the diagram, we know that $\boldsymbol{\mathcal{R}}_i = \mathbf{r}_i - \mathbf{R}$, so we can re-write the second term as

$$\sum_{i=1}^{N} m_i \boldsymbol{\mathcal{R}}_i = \sum_{i=1}^{N} m_i \mathbf{r}_i - \sum_{i=1}^{N} m_i \mathbf{R} = M\mathbf{R} - M\mathbf{R} = 0$$

(iv) $\sum_{i=1}^{N} \boldsymbol{\mathcal{R}}_i \times m_i \dot{\boldsymbol{\mathcal{R}}}_i$
This is written solely in terms of quantities relative to the center of mass and the particle, so we will choose not to simplify this any further.

Combining all of those simplifications, we find that

$$\mathbf{L} = \underbrace{\mathbf{R} \times \mathbf{P}}_{\substack{\text{"Orbital Angular Momentum":} \\ \text{System angular} \\ \text{momentum about} \\ \text{the origin}}} + \underbrace{\sum_{i=1}^{N} \boldsymbol{\mathcal{R}}_i \times m_i \dot{\boldsymbol{\mathcal{R}}}_i}_{\substack{\text{"Spin Angular Momentum":} \\ \text{System angular} \\ \text{momentum about} \\ \text{the center of mass}}}$$

3 Energy

3.1 Kinetic Energy and Work

According to Newton's second law, the net force on a given particle is equal to its change in momentum. Similarly, we define the net change in force of a particle as its *kinetic energy* (T). We can derive this to get a general equation for kinetic energy:

$$\begin{aligned} F &= m\frac{\mathrm{d}v}{\mathrm{d}t} = m\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= m\frac{\mathrm{d}v}{\mathrm{d}x}v = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}mv^2\right] = \frac{\mathrm{d}T}{\mathrm{d}x} \\ \implies T &\equiv \frac{1}{2}mv^2 \end{aligned}$$

We also often want to discuss the changes in kinetic energy, which we define as the *work*. We can fairly easily derive this as well:

$$\begin{aligned} F = \frac{\mathrm{d}T}{\mathrm{d}x} &\implies F\mathrm{d}x = \mathrm{d}T \\ \int_{x_1}^{x_2} F\mathrm{d}x &= \int_{T(x_1)}^{T(x_2)} \mathrm{d}T \\ &= T(x_2) - T(x_2) = \Delta T \\ \implies \int_{x_1}^{x_2} F\mathrm{d}x &= \Delta T = W \end{aligned}$$

This final equation is called the Work-Energy theorem (or sometimes the Work-KE theorem). Although we did this in one dimension, it is easily extendable into multiple dimensions. Skipping the derivations, we get similar results:

$$\begin{aligned} T &= \frac{1}{2}v^2 = \frac{1}{2}\mathbf{v}\cdot\mathbf{v} \\ W = \Delta T &= \int_{\mathbf{r}_1\ C}^{\mathbf{r}_2} \mathbf{F}\cdot\mathbf{r} \end{aligned}$$

Note here that for the work, we must now integrate over a particular path.

If we want to talk about how the kinetic energy changes over time, we define the *power* as

$$P(t) = \frac{\mathrm{d}T}{\mathrm{d}t} = m\dot{\mathbf{v}}\cdot\mathbf{v}$$

3.2 Potential Energy

3.2.1 Time-Independent Potential Energy

The first condition for a force to be *conservative* (we'll define this more explicitly later) is that it depends only on the position, $\mathbf{r}$, and not explicitly on time t. The second condition is that the work done by the force on an object going from $\mathbf{r}_1 \to \mathbf{r_2}$ should be the same regardless of the path taken—that is, it should be *path-independent*.

Proposition. $\mathbf{F} = -\boldsymbol{\nabla} U$, where U is a function of its coordinates and not an explicit function of time.

For this to be true, then in Cartesian coordinates, we would write

$$\mathbf{F} = -\left(\frac{\partial U}{\partial x}\hat{\mathbf{x}} + \frac{\partial U}{\partial y}\hat{\mathbf{y}} + \frac{\partial U}{\partial z}\hat{\mathbf{z}}\right)$$

This means that using our definition of work from the previous section, we can write

$$\begin{aligned} W &= \int_{\mathbf{r}_1\ C}^{\mathbf{r}_2} -\left(\frac{\partial U}{\partial x}\hat{\mathbf{x}} + \frac{\partial U}{\partial y}\hat{\mathbf{y}} + \frac{\partial U}{\partial z}\hat{\mathbf{z}}\right) \\ &= -\int_{\mathbf{r}_1\ C}^{\mathbf{r}_2} dU(x, y, z) \\ &= -\Delta U = -U(\mathbf{r}_2 - \mathbf{r}_1) \end{aligned}$$

Since $\mathbf{F} - \boldsymbol{\nabla} U$ can be written as the gradient of a scalar function, we know that $\boldsymbol{\nabla} \times \mathbf{F} = \boldsymbol{\nabla} \times (-\boldsymbol{\nabla} U) = -\boldsymbol{\nabla} \times (\boldsymbol{\nabla} U) = 0$. One way to prove that a force is path-independent is by taking the line integral around a closed path, which should come out to zero. I we do that using Stokes's theorem, then we can say that

$$\begin{aligned} \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \hat{\mathbf{n}} \\ &= \iint_S (\mathbf{0}) \cdot \hat{\mathbf{n}} \\ &= 0 \end{aligned}$$

So, if we can write F such that $F = -\boldsymbol{\nabla} U(\mathbf{r})$, then F satisfies both conditions to be a conservative force.

Combining our equations for work, $W = \Delta T$ and $W = -\Delta U$, we can derive the *Law of Conservation of Energy*:

$$\begin{aligned} T(\mathbf{r}_2) - T(\mathbf{r}_1) &= U(\mathbf{r}_1) - U(\mathbf{r}_2) \\ T(\mathbf{r}_1) + U(\mathbf{r}_1) &= T(\mathbf{r}_2) + U(\mathbf{r}_2) \\ \implies E &\equiv T + U \text{ is constant} \end{aligned}$$

This is why such a force is called conservative: if it satisfies the two conditions, total mechanical energy is conserved. We can write the conditions a few different ways:

(i) $\mathbf{F}$ is a function of position, and not an explicit function of time or velocity: $\mathbf{F} = \mathbf{F}(\mathbf{r}), \mathbf{F} \neq \mathbf{F}(t), \mathbf{F} \neq \mathbf{F}(\mathbf{v})$

(ii) $\mathbf{F}$ is path-independent:

$$\begin{cases} \mathbf{F}(\mathbf{r}) = -\boldsymbol{\nabla} U \\ W(\mathbf{r}_2 \to \mathbf{r}_1) = -\Delta U = \Delta T \\ \boldsymbol{\nabla} \times \mathbf{F}(\mathbf{r}) = 0 \\ \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \hat{\mathbf{n}} = 0 \end{cases}$$

3.2.2 Time-Dependent Potential Energy

Suppose we do have a potential energy function $U(\mathbf{r}, t)$ that is an explicit function of time. How does total mechanical energy change? In terms of differentials, since we know $E = T + U$, we can write $\mathrm{d}E = \mathrm{d}T + \mathrm{d}U$. We can use this to explore the time rate of change by saying:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}T}{\mathrm{d}t} + \frac{\mathrm{d}U}{\mathrm{d}t}$$

We'll find each of these separately, starting with $\frac{\mathrm{d}T}{\mathrm{d}t}$:

$$\begin{aligned}
T &= \frac{1}{2}mv^2 \\
\frac{\mathrm{d}T}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}mv^2\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}m\mathbf{v}\cdot\mathbf{v}\right] \\
&= \frac{1}{2}\left(\dot{\mathbf{v}}\cdot\mathbf{v} + \mathbf{v}\cdot\dot{\mathbf{v}}\right) \\
&= m\dot{\mathbf{v}}\cdot\mathbf{v} \\
&= \mathbf{F}\cdot\mathbf{v}
\end{aligned}$$

And following with $\frac{\mathrm{d}U}{\mathrm{d}t}$:

$$\begin{aligned}
\mathrm{d}U &= \left(\frac{\partial U}{\partial x}\mathrm{d}x + \frac{\partial U}{\partial y}\mathrm{d}y + \frac{\partial U}{\partial z}\mathrm{d}z + \frac{\partial U}{\partial t}\mathrm{d}t\right) \\
&= \boldsymbol{\nabla}U\cdot\mathrm{d}\mathbf{r} + \frac{\partial U}{\partial t}\mathrm{d}t \\
&= -\mathbf{F}\cdot\mathrm{d}\mathbf{r} + \frac{\partial U}{\partial t}\mathrm{d}t
\end{aligned}$$

$$\frac{\mathrm{d}U}{\mathrm{d}t} = -\mathbf{F}\cdot\mathbf{v} + \frac{\partial U}{\partial t}$$

Combining these to get $\frac{\mathrm{d}E}{\mathrm{d}t}$, we find

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \mathbf{F}\cdot\mathbf{v} - \mathbf{F}\cdot\mathbf{v} + \frac{\partial U}{\partial t} \\
&= \frac{\partial U}{\partial t}
\end{aligned}$$

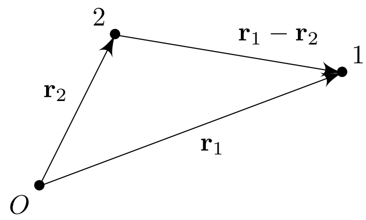
3.3 Central Forces

We will only touch on the idea of central forces in this chapter. A *central force* is a force which

(i) Only depends on $\mathbf{r}$ ($\mathbf{F} = \mathbf{F}(\mathbf{r})$)

(ii) Is everywhere directed either towards or away from the center ($\mathbf{F}(\mathbf{r}) = f(\mathbf{r})\hat{\mathbf{r}}$)

Proposition. Any central force is a conservative force. This will be proven later.



3.4 Energy in Multi-Particle Systems

3.4.1 Energy in Two-Particle Systems

If we have 6 of the irght variables, we can specify the locations of the two particles and find some relavent terms:

$$\begin{aligned}
\mathbf{r}_1 &= x_1\hat{\mathbf{x}} + y_1\hat{\mathbf{y}} + z_1\hat{\mathbf{z}} \\
\mathbf{r}_2 &= x_2\hat{\mathbf{x}} + y_2\hat{\mathbf{y}} + z_2\hat{\mathbf{z}} \\
\boldsymbol{\nabla}_1 &= \left[\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1} + \frac{\partial}{\partial z_1}\right] \\
\boldsymbol{\nabla}_2 &= \left[\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2} + \frac{\partial}{\partial z_2}\right]
\end{aligned}$$

For a real example, we'll pretend that the two particles are interacting only through gravty. Gravitational force is an example of a central force, which we know is a force which is a function only of the

Important to note: this makes a central force *translationally invariant*, meaning the force will be the same regardless of the reference frame, since we only care about the difference between the points, not where the points are located withrespect to any origin or anything.

We can look first at the force of 2 on 1, if we put $\mathbf{r}_2$ at the origin, meaning $\mathbf{F}_{12}(\mathbf{r}_1 - \mathbf{r}_2) = \mathbf{F}_{12}(\mathbf{r}_1)$

We've already seen what happens to single particles in a conservative field, and since a central force is also conservative, those properties should hold:

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{F} &= 0 \\
\implies \boldsymbol{\nabla}_1 \times \mathbf{F}_{12}(\mathbf{r}_1) &= 0
\end{aligned}$$

$$\begin{aligned}
\mathbf{F}(\mathbf{r}) &= -\boldsymbol{\nabla} U(\mathbf{r}) \\
\mathbf{F}_{12}(\mathbf{r}_1) &= -\boldsymbol{\nabla}_1 U(\mathbf{r}_1)
\end{aligned}$$

We can also generalize this if we don't set particle 2 as the origin:

$$\begin{aligned}
\boldsymbol{\nabla}_1 \times \mathbf{F}_{12}(\mathbf{r}_1 - \mathbf{r}_2) = 0 \\
\mathbf{F}_{12}(\mathbf{r}_1 - \mathbf{r}_2) = \boldsymbol{\nabla}_1 U(\mathbf{r}_1 - \mathbf{r}_2)
\end{aligned}$$

We can repeat this process again for the force on particle 2, and we should get basically the same results, but flipped:

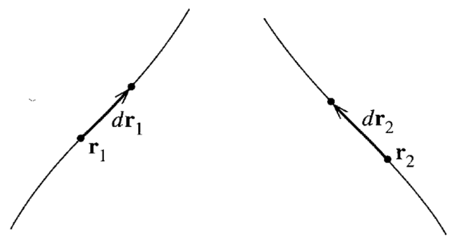
$$\begin{aligned}
\boldsymbol{\nabla}_2 \times \mathbf{F}_{21}(\mathbf{r}_2 - \mathbf{r}_1) = 0 \\
\mathbf{F}_{21}(\mathbf{r}_2 - \mathbf{r}_1) = \boldsymbol{\nabla}_2 U(\mathbf{r}_2 - \mathbf{r}_1)
\end{aligned}$$

This tells us something really useful: there is only one potential energy function, and the forces are just derived from taking partial derivatives with respect to each of the six variables. The only real difference between the two forces is the input. We can use a property,

$$\boldsymbol{\nabla}_1 f(\mathbf{r}_1 - \mathbf{r}_2) = -\boldsymbol{\nabla}_2 f(\mathbf{r}_1 - \mathbf{r}_2)$$

to find that $\mathbf{F}_{12} = -\mathbf{F}_{21}$, which is exactly Newton's third law.

We can also do some energy analysis on these particles. If we draw the path of the two particles, it will look something like this:



We know that the toal work done will be equal to the change in kinetic energy for each of the particles,and we can do some math to simplify that:

$$\begin{aligned}
W_{tot} &= \mathrm{d}T = \mathrm{d}T_1 + \mathrm{d}T_2 \\
&= \mathbf{F}_{12} \cdot \mathrm{d}\mathbf{r}_1 + \mathbf{F}_{21} \cdot \mathrm{d}\mathbf{r}_2 \\
&= \mathbf{F}_{12} \cdot (\mathrm{d}\mathbf{r}_1 - \mathrm{d}\mathbf{r}_2) \\
&= \mathbf{F}_{12} \cdot \mathrm{d}(\mathbf{r}_1 - \mathbf{r}_2) \\
&= -\boldsymbol{\nabla}_1 U(\mathbf{r}_1 - \mathbf{r}_2) \cdot \mathrm{d}(\mathbf{r}_1 - \mathbf{r}_2)
\end{aligned}$$

This result is *exactly* what we got for a single particle, just with $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Since we just showed the relationship between the changes in kinetic energy and the change in potential energy, we can use that to show that the total change in energy will always be 0 in this case, so energy will be conserved.

$$E \equiv T_1 + T_2 + U_{12}$$

3.4.2 Energy in N-Particle Systems

Kinetic energy for an N-particle system is easily scalable:

$$T = \sum_{\alpha=1}^{N} T_\alpha$$

If there's no external forces, then the total potential energy, U^{int}, must be taken for each set of particles, so it is

$$U^{int} = \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} U_{\alpha\beta}(\mathbf{r}_\alpha - \mathbf{r}_\beta)$$

If there is a net external force, we would also want to add

$$U^{ext} = \sum_{\alpha=1}^{N} U_\alpha^{ext}$$

That means that

$$
\begin{aligned}
U &= U^{int} + U^{ext} \\
&= \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N} U_{\alpha\beta}(\mathbf{r}_\alpha - \mathbf{r}_\beta) + \sum_{\alpha=1}^{N} U_\alpha^{ext} \\
F_\alpha &= -\boldsymbol{\nabla}_x \left[U_\alpha + \sum_{\alpha\neq\beta}^{N} U_{\alpha\beta}(\mathbf{r}_\alpha - \mathbf{r}_\beta) \right]
\end{aligned}
$$

So if we have all conservative forces,

$$
\begin{aligned}
E = T = U &= T + U^{ext} + U^{int} \\
&= \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha v_\alpha^2 + \sum_{\alpha=1}^{N} U_\alpha^{ext} + \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N} U_{\alpha\beta}(\mathbf{r}_\alpha - \mathbf{r}_\beta)
\end{aligned}
$$

In rigid bodies, the internal potential energy will remain constant, so we can analyze rigid bodies as a single object or particle.

4 Oscillations

4.1 Simple Harmonic Motion (SHM)

4.1.1 Simple Harmonic Oscillation and Oscillators

Definition (Simple Harmonic Motion)**.** *Simple harmonic motion* is periodic, symmetric motion about an equilibrium point with a period independent of the magnitude of the motion.

In one dimension, we write the equations of motion using Hooke's law:

$$F = m\ddot{x} = -k(x - x_0)$$
$$U = \frac{1}{2}k(x - x_0)^2$$

Proposition. For any potential function with one or more minimum, small displacements about any minimum will exhibit SHM.

Proof. If a function $U(x)$ has a minimum at $x = x_0$, then for small displacements, we can approximate $U(x)$ with a Taylor series:

$$U(x) \approx U(x)\bigg|_{x=x_0} + \frac{\mathrm{d}U(x)}{\mathrm{d}x}\bigg|_{x=x_0}(x - x_0) + \frac{\mathrm{d}^2U(x)}{\mathrm{d}x^2}\bigg|_{x=x_0}(x - x_0)^2$$

We know that $U(x_0) = U_0$ is constant.
We also know that $\frac{\mathrm{d}U(x_0)}{\mathrm{d}x} = 0$ for equilibrium terms, so the term vanishes.
We finally know that $\frac{\mathrm{d}^2U(x_0)}{\mathrm{d}x^2} > 0$ for a stable equilibrium.
If we define the final term as k, then we're left with

$$U(x) \approx U_0 + \frac{1}{2}k(x - x_0)^2$$

which is exactly our equation for simple harmonic motion. □

Our equation of motion in one dimension,

$$F = m\ddot{x} = -kx$$
$$\implies m\ddot{x} + kx = 0$$

is a homogeneous, ordinary, second-order linear differential equation, meaning it can always be solved for two linearly independent solutions with two arbitrary constants. Given the form of the equation, we can assume a solution like $x \sim e^{qt}$. By substituion, we can get the characteristic equation,

$$mq^2 = -k$$
$$q = \pm i\omega_0 : \omega_0 = \sqrt{\frac{k}{m}}$$
$$\implies \ddot{x} = -\omega_0^2 x$$

ω_0 is usually called the "natural frequency."

The solutions to this differential equation can be written in a number of different, but equivalent, ways:

$$\begin{aligned}
x &= C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t} \\
x &= B_1 \cos(\omega_0 t) + B_2 \cos(\omega_0 t) \\
x &= A\cos(\omega_0 t - \delta) \\
x &= \mathrm{Re}[Ae^{i(\omega_0 t - \delta)}]
\end{aligned}$$

There are a number of constants in there we want to define real quick:

$$\begin{aligned}
A &\rightarrow \text{Amplitude (where the turning points are} \pm A) \\
\omega_0 = \sqrt{\frac{k}{m}} &\rightarrow \text{Natural (Angular) Frequency} \\
\delta &\rightarrow \text{Phase Shift} \\
\tau = \frac{2\pi}{\omega_0} &\rightarrow \text{Period}
\end{aligned}$$

For now, we'll look at the simplest one, $x = A\cos(\omega_0 t - \delta)$. To fix the constants, we need the initial conditions. As an example, let's assume that $x(0) = x_0$ and that $\dot{x}(0) = v(0) = v_0$. THen,

$$\begin{aligned}
x_0 &= A\cos(\delta) \\
v(t) &= -\omega_0 A \sin(\omega_0 t - \delta) \\
\implies v_0 &= \omega_0 A \sin(\delta)
\end{aligned}$$

We can solve this system of equations to find the amplitude. I'll skip a little math here, but if we multiply the first by ω_0, square the two equations, then add them, then that will give us an expression for A:

$$A^2 = x_0^2 + \left(\frac{v_0}{\omega_0}\right)^2$$

4.1.2 Energy in Simple Harmonic Motion

We already know that if we have something like a spring-mass system (a simple example of something approximating simple harmonic motion), the total energy will be given by $E = T + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$. Is energy conserved in SHM, though? Let's look:

$$\begin{aligned}
m\ddot{x} + kx &= 0 \\
\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right] \\
&= m\ddot{x}\dot{x} + kx\dot{x} \\
&= \dot{x}(m\ddot{x} + kx) \\
&= 0
\end{aligned}$$

So, we can see that yes, in idealized simple harmonic motion, energy is conserved. If a system is motionless at the moment $t = a$, then

$$E = U(a) + \frac{1}{2}kx_{max}^2 = \frac{1}{2}kA^2$$

At any given time, then, we can relate the speed and position:

$$\begin{aligned}\frac{1}{2}kA^2 &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\\ A^2 &= \frac{m}{k}\dot{x}^2 + x^2\\ &= \left(\frac{\dot{x}}{\omega_0}\right)^2 + x^2\end{aligned}$$

4.2 Damped Harmonic Motion (DHM)

4.2.1 Damped Harmonic Oscillation and Oscillators

SHM is fairly ubiquitous, but it's an idealized case. In reality, oscillators would lose energy to their surroundings, which we call "damping." Although oscillators can experience both linear and quadratic drag, quadratic drag is difficult to calculate, so we'll mostly look at linear (Stokes's) drag.

First, let's look at the equations of motion for DHM in one dimension:

$$\begin{aligned}F =& m\ddot{x} = -kx - b\dot{x}\\ & m\ddot{x} + b\dot{x} + kx = 0\end{aligned}$$

This is a homogeneous, ordinary, second-order linear differential equation with constant coefficients, meaning it will have two linearly independent solutions with two arbitrary constants, and, more importantly, it can always be solved.

So let's solve it! With our given equation of motion, we can guess a solution like $x \sim e^{qt}$. If we plug that in, and take out the common factors, we end up with

$$\begin{aligned}mq^2 + bq + k &= 0\\ q^2 + \frac{4b}{2m}q + \omega_0^2 &= 0\\ q^2 + 2\beta q + \omega_0^2 &= 0\\ \implies \ddot{x} + 2\beta\dot{x} + \omega_0^2 x &= 0\end{aligned}$$

Where $\beta = \frac{b}{2m}$ and $\omega_0^2 = \frac{k}{m}$, both of which have units of 1/sec, and both of which indicate resistance to movement (ie. drag). This means that we can find $q = \beta \pm \sqrt{\beta^2 - \omega_0^2}$, and the general solution will be

$$x(t) = e^{-\beta t}\left(C_1 e^{t\sqrt{\beta^2-\omega_0^2}} + C_2 e^{-t\sqrt{\beta^2-\omega_0^2}}\right)$$

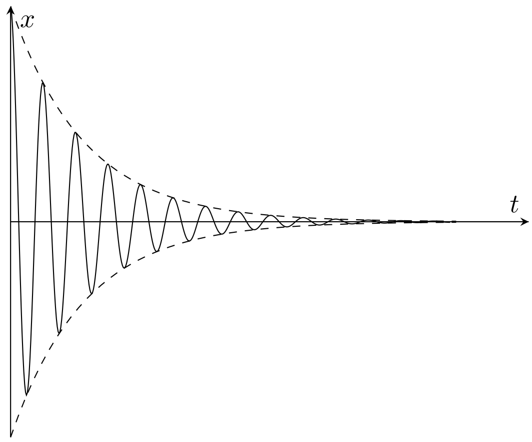
For DHM, there are three potential situations, which we'll explore separately:
Case 1: Underdamped

$$\begin{aligned}&\beta^2 > \omega_0^2\\ \implies\ & q = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \text{ is imaginary}\end{aligned}$$

Often, we will write $q = -\beta \pm i\omega_1$, where ω_1 is the real part of the imaginary square root (that is, $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$). This means that we can re-write the general solution as

$$\begin{aligned} x(t) &= e^{-\beta t}\left(C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t}\right) \\ &= Ae^{-\beta t}\cos(\omega_1 t - \delta) \end{aligned}$$

This is almost exactly the same as normal simple harmonic motion, except the amplitude is decaying with time. This type of motion will osccilate at an angular frequency $\omega_1 < \omega_0$. If $x_0 \neq 0$ and $v_0 = 0$, then this will look like the following graph:



The dashed line on the outside is called the *envelope*, and is the decaying amplitude function $Ae^{-\beta t}$.

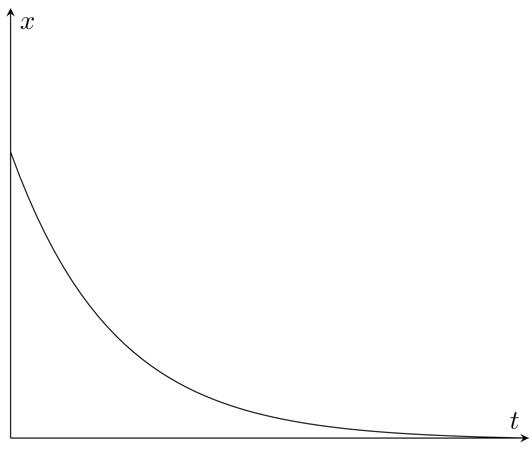
Case 2: Overdamped

$$\begin{gathered} \beta^2 > \omega_0^2 \\ \implies q = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \text{ is real} \end{gathered}$$

In this case, the exponential term is real, so we will end up with a non-oscillatory exponential function. We can write our equation as

$$x(t) = e^{\beta t}\left(C_1 e^{t\sqrt{\beta^2-\omega_0^2}} + C_2 e^{-t\sqrt{\beta^2-\omega_0^2}}\right)$$

Given the same initial conditions, $x_0 \neq 0$ and $v_0 = 0$, the graph will look like



Case 3: Critically Damped

$$\beta^2 = \omega_0^2$$
$$\implies q = -\beta \pm \sqrt{\beta^2 - \omega_0^2} = -\beta$$

The exponential term is once again real and non-oscillatory. It will look very similar to the previous graph, but it wil reach equilibrium ($x = 0$) quicker than its overdamped counterpart.

4.2.2 Energy in Damped Harmonic Motion

We know that

$$\begin{aligned} E &= T + U \\ &= \frac{1}{2}m\dot{v}^2 + \frac{1}{2}kx^2 \\ \implies \frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right] \\ &= m\ddot{x}\dot{x} + kx\dot{x} \\ &= \dot{x}(m\ddot{x} + kx) \\ &= -b\dot{x}^2 \end{aligned}$$

This means that for any damped harmonic motion, the total change in energy is negative, so the system is losing energy.

4.3 Driven Damped Harmonic Motion - Sinusoidal Driving Force

4.3.1 Damped Harmonic Motion with a Sinusoidal Driving Force

We'll be looking at damped harmonic motion with an external sinusoidal driving force. The equation of motion for this will look like

$$\begin{aligned} m\ddot{x} + b\dot{x} + kx &= F_0 e^{i\omega t} \\ &= F_0 \cos(\omega t) + iF_0 \sin(\omega t) \end{aligned}$$

This is a weird way of writing it, but the idea is that if we have a cosine driving force, we'll be look at the real part of the RHS, and subsequently the real part of the solution. Similarly, if we have a sine driving force, we'll be looking at the imaginary part of the RHS, and subsequently the imaginary part of the solution.

The equation of motion is a non-homogeneous, ordinary second-order linear differential equation with constant coefficients. This means that the solution will be the sum of a homogeneous solution and a particular solution. The homogeneous solution (that is, the solution when the RHS is equal to zero) will be the general damped harmonic oscillator solution from the last section:

$$x_h(t) = A_h e^{\beta t}\left(C_1 e^{t\sqrt{\beta^2 - \omega_0^2}} + C_2 e^{-t\sqrt{\beta^2 - \omega_0^2}}\right)$$

For the particular solution, we can use an ansatz (that is, an educated guess) and say that

$$x_p(t) = A_1 e^{i(\omega t - \delta)} = A_1(\cos(\omega t - \delta) + i\sin(\omega t - \delta))$$

Substitution into the equation of motion will give us an equation that can be split into real and imaginary parts:

$$
\begin{aligned}
\text{Re}: \quad & A_1(k - m\omega^2) = F_0 \cos(\delta) \\
\text{Im}: \quad & A_1(b\omega) = F_0 \sin(\delta)
\end{aligned}
$$

We can combine these to find that

$$
\begin{aligned}
\tan(\delta) &= \frac{b\omega}{k - m\omega^2} \\
&= \frac{2\beta\omega}{\omega_0^2 - \omega^2} \\
\implies \delta &= \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)
\end{aligned}
$$

We can also use these equations to find the generalized form of A_1:

$$
\begin{aligned}
A_1^2(k - m\omega)^2 + A_1^2(b\omega)^2 &= F_0^2(\cos^2(\delta) + \sin^2(\delta)) \\
A_1^2\left((k - m\omega)^2 + (b\omega)^2\right) &= F_0^2 \\
A_1(\omega) &= \frac{F_0}{\sqrt{(k - m\omega)^2 + (b\omega)^2}} \\
&= \frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}
\end{aligned}
$$

This will often be written more simply as

$$
A_1(\omega) = \frac{F_0/m}{D(\omega)^{\frac{1}{2}}}
$$

where $D(\omega) = (\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2$.

This is everything we need to be able to find the particular solution. If the driving force is a sine function, then we can say that

$$
x_p(t) = A_1 \sin(\omega t - \delta)
$$

and if the driving force is a cosine function,

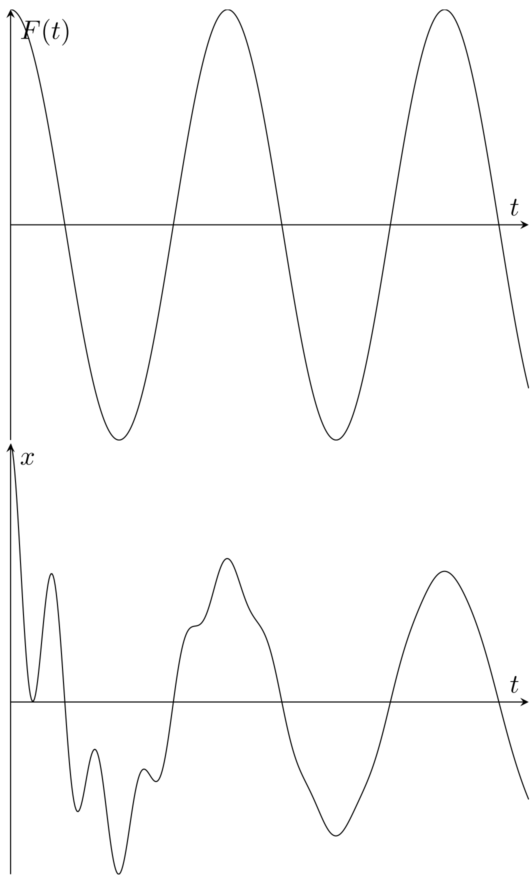
$$
x_p(t) = A_1 \cos(\omega t - \delta)
$$

with A_1 and δ being the constants we just found.

The homogeneous term, or the decaying damped harmonic oscillator term, will go to zero as $t \to \infty$, so we call it the *transient term*.

The remaining particular term, after long enough, will be the only piece determining the motion, so we call it the *steady-state solution*.

We can graph an example function showing the driving force and the position function:



A_1 is maximized where $\frac{\mathrm{d}A_1}{\mathrm{d}\omega} = 0$:

$$\left.\frac{\mathrm{d}A_1}{\mathrm{d}\omega}\right|_{\omega=\omega_R} = \frac{\mathrm{d}}{\mathrm{d}\omega}\left[\frac{F_0/m}{(D(\omega))^{1/2}}\right]_{\omega=\omega_R} = -\frac{F_0}{2m}[D(\omega)]^{-3/2}\left.\frac{\mathrm{d}D(w)}{\mathrm{d}\omega}\right|_{\omega=\omega_R} = 0$$

We know that the denominator function, $D(\omega)$ can never be 0, since that would give us an undefined function, so for this to be true, that means $\frac{\mathrm{d}D(\omega)}{\mathrm{d}\omega}$ must instead be equal to zero:

$$\begin{aligned}\left.\frac{\mathrm{d}D(\omega)}{\mathrm{d}\omega}\right|_{\omega=\omega_R} &= [2(\omega_0 - \omega^2)(-2\omega) + 8\beta^2\omega]_{\omega=\omega_R} \\ &= \omega_0^2 - \omega_R^2 + 8\beta^2 = 0 \\ \omega_R &= \sqrt{\omega_0^2 - 2\beta^2} = \sqrt{\omega_1^2 - \beta^2}\end{aligned}$$

This ω_R, where A_1 is maximized is called the "resonant frequency." When the system is driven at this frequency, it is operating at its maximum, and

$$A_1(\omega_R) = \frac{F_0}{2\beta m\omega_1}$$

In the case of weak damping (where $\beta \ll \omega$, like Barton's pendulum), $\omega_R \approx \omega_1 \approx \omega_0$, so $A_1(\omega_R) \approx \frac{F_0}{2\beta m\omega_0}$. If $\omega_0^2 < 2\beta^2$, when no resonance is even possible.

4.3.2 The Phase Shift

$$\delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

(i) If $\omega \ll \omega_0$, $\delta \approx 0$, so the motion is *in phase* with the driving force

(ii) If $\omega \approx \omega_0$, $\delta \approx \frac{\pi}{2}$, so the motion is 90° *out of phase* with the driving force

(iii) If $\omega \gg \omega_0$, $\delta \approx \pi$, so the motion is 180° *out of phase* with the driving force

4.3.3 Energy in the Steady-State Driven Damped Harmonic Oscillator

If we have a driven damped harmonic oscillator in a steady state with $F(t) = F_0 \cos(\omega t)$, then we know that

$$\begin{aligned} x_{ss}(t) &= A_1 \cos(\omega t - \delta) \\ \dot{x}_{ss}(t) &= -A_1 \omega \sin(\omega t - \delta) \end{aligned}$$

The work per unit time done by the driving force can be found with

$$\begin{aligned} \frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}T}{\mathrm{d}t} &= Fv = F(t)\dot{x} \\ &= F_0 \cos(\omega t)[-A_1\omega \sin(\omega t - \delta)] \end{aligned}$$

The change in energy per unit time can be found similarly:

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}T}{\mathrm{d}t} + \frac{\mathrm{d}U}{\mathrm{d}t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right] \\ &= (m\ddot{x} + kx)\dot{x} = -b\dot{x}^2 + F(t)\dot{x} \end{aligned}$$

In a steady state, the LHS averaged over a complete cycle (or period) should b 0. That is, the energy lost by damping should be perfectly matched by the external forced averaged over one period, or

$$\left\langle \frac{\mathrm{d}E}{\mathrm{d}t} \right\rangle = \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right] \right\rangle = \langle -b\dot{x}^2 + F(t)\dot{x} \rangle = 0$$

Proof.

$$\begin{aligned} \dot{x} &= -A_1\omega \sin(\omega t - \delta) \\ -b\dot{x}^2 + F(t)\dot{x} &= -b[-A_1\omega\sin(\omega t - \delta)]^2 + F_0\cos(\omega t)[-A\omega\sin(\omega t - \delta)] \\ &= -bA_1^2\omega^2\sin^2(\omega t - \delta) + A_1F_0\omega\cos(\omega t)\sin(\omega t - \\ &= -bA_1^2\omega^2\sin^2(\omega t - \delta) + A_1F_0\omega\left[\cos^2(\omega t)\sin(\delta) - \frac{1}{2}\cos(2\omega t)\cos(\delta)\right] \end{aligned}$$

We know that $\langle \sin^2(\omega t)\rangle = \langle \cos^2(\omega t)\rangle = 0$, and we can find similarly that $\frac{\sin(2\omega t)}{2}\cos(\delta)\Big\rangle = 0$, so the average of this becomes

$$\begin{aligned} \left\langle \frac{\mathrm{d}E}{\mathrm{d}t} \right\rangle &= -\frac{1}{2}bA_1^2\omega^2 + \frac{1}{2}A_1F_0\omega\sin(\delta) \\ &= \frac{A_1\omega}{2}(-bA_1\omega + F_0\sin(\delta)) = 0 \end{aligned}$$

Note that the final equality to zero works because the imaginary part to our earlier solution was $A_1(b\omega) = F_0 \sin\delta$. □

4.4 Driven Damped Harmonic Motion - Non-Sinusoidal Periodic Driving Force

4.5 Driven Damped Harmonic Motion - Impulse Forces

5 Lagrange's Equations

5.1 Derivation of Lagrange's Equations

5.1.1 Generalized Coordinates

If we're considering N particles in k dimensions, we will always need, at most, $n \cdot k$ coordinates to find a complete solution. For example, if we use 3-dimensional Cartesian coordinates for a system of N particles, we will need three dimensions for each of the N particles, (x_N, y_N, z_N), or $3N$ coordinates. That's often more than we actually need, though. If we have equations that relate two or more coordinates (or *constraints*), then we can subtract the number of constraints, giving us the minimum number of required coordinates, or *degrees of freedom.*

Definition (Degrees of Freedom)**.** The *degrees of freedom* for a system are the minimum number of coordinates needed to totally define the system, given by

$$f = n \cdot k - c$$

where n is the number of particles, k is the number of coordinates, and c is the number of constraints.

Once we have the degrees of freedom, applying the constraints, what we're left with will be the *generalized coordinates.* There can be multiple sets of generalized coordinates (eg. we can represent a system in either Cartesian or polar coordinates), but they can all be expressed in f coordinates.

Transformations between Coordinates

Specifically, we'll be looking at transformations between Cartesian coordinates and other potential generalized coordinates. To start, we will define

$$\{x_j\} = \{x_1, x_2, x_3, x_4, x_5, x_6, \ldots, x_{3N}\} = \{x_1, y_1, z_1, x_2, y_2, z_2, \ldots, z_N\}$$

Now, if we want to go from $\{x_j\}$ to $\{q_k\}$, then we write

$$\begin{aligned} \mathrm{d}x_j &= \sum_{k=1}^{f} \frac{\partial x_j}{\partial q_k} \mathrm{d}q_k \\ \dot{x}_j &= \sum_{k=1}^{f} \left[\frac{\partial x_j}{\partial q_k} \frac{\partial q_k}{\partial t} \right] + \frac{\partial x_j}{\partial t} \end{aligned}$$

Noting that the only time $\frac{\partial x_j}{\partial t}$ appears is in a moving coordinate system. The infinitesimals are then written as

$$\delta x_j = \sum_{k=1}^{f} \frac{\partial x_j}{\partial q_k} \delta q_k$$

5.1.2 Generalized Kinetic Energy

In Cartesian coordinates, we know that

$$T = \frac{1}{2} \sum_{i=1}^{n} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$$

Using our just-developed expression to convert between Cartesian and generalized coordinates, we can write this as

$$T = \sum_{k=1}^{3n}\sum_{\ell=1}^{3n}\left[\frac{1}{2}A_{k\ell}\dot{q}_k\dot{q}_\ell\right] + \sum_{k=1}^{3n}[B_k + \dot{q}_k] + T_0$$

Where A_k, B_k, and T_0 are defined as

$$\begin{aligned}
A_{k\ell} &= \sum_{i=1}^{3n} m_i\left[\frac{\partial x_i}{\partial q_k}\frac{\partial x_i}{\partial q_\ell} + \frac{\partial y_i}{\partial q_k}\frac{\partial y_i}{\partial q_\ell} + \frac{\partial z_i}{\partial q_k}\frac{\partial z_i}{\partial q_\ell}\right] \\
B_k &= \sum_{i=1}^{n} m_i\left[\frac{\partial x_i}{\partial t}\frac{\partial x_i}{\partial q_k} + \frac{\partial y_i}{\partial t}\frac{\partial y_i}{\partial q_k} + \frac{\partial z_i}{\partial t}\frac{\partial z_i}{\partial q_k}\right] \\
T_0 &= \sum_{i=1}^{n}\frac{1}{2}m_i\left[\left(\frac{\partial x_i}{\partial t}\right)^2 + \left(\frac{\partial y_i}{\partial t}\right)^2 + \left(\frac{\partial z_i}{\partial t}\right)^2\right]
\end{aligned}$$

If $A_{k\ell}$ is nonzero only when $k = 1$, then $\{q_k\}$ are orthogonal. If the generalized coordinates don't contain explicit functions of time, we call them "scleronomic" and the terms B_k and T_0 go away. Otherwise, they're called "rheonomic."

5.1.3 Generalized Momentum

The momentum of a single particle in the x direction is

$$p_x = \frac{\partial T}{\partial \dot{x}} = m\dot{x}$$

Similarly, for a system,

$$p_x = \sum_{i=1}^{n}\frac{\partial T}{\partial \dot{x}_i}$$

And for a system in generalized coordinates,

$$p_k = \sum_{k=1}\frac{\partial T}{\partial \dot{q}_k}$$

5.1.4 Generalized Forces

To talk about generalized forces, we first want to talk about work. Work is generally defined as

$$\begin{aligned}
W &= \int \mathbf{F}\cdot \mathrm{d}\mathbf{r} \\
\delta W &= \sum_{i=1}^{n}(F_{ix}\delta x_i + F_{iy}\delta y_i + F_{iz}\delta z_i)
\end{aligned}$$

We know from earlier that we can re-write the differential of x_i in terms of generalized coordinates with

$$\delta x_i = \sum_{k=1}^{f}\frac{\partial x_i}{\partial q_k}\delta q_k$$

This means that we can write the differential of the work function as

$$\begin{aligned}
\delta W &= \sum_{i=1}^{n}\left[F_{ix}\sum_{k=1}^{f}\frac{\partial x_i}{\partial q_k}\delta q_k + F_{iy}\sum_{k=1}^{f}\frac{\partial y_i}{\partial q_k}\delta q_k + F_{iz}\sum_{k=1}^{f}\frac{\partial z_i}{\partial q_k}\delta q_k\right] \\
\delta W &= \sum_{k=1}^{f} Q_k q_k
\end{aligned}$$

This, of course, implies that

$$Q_k = \sum_{k=1}^{n}\left(F_{ix}\frac{\partial x_i}{\partial q_k} + F_{iy}\frac{\partial y_i}{\partial q_k} + F_{ix}\frac{\partial x_i}{\partial q_k}\right)$$

What else do we know about work, though? Well, if we're working with only conservative forces, we know that

$$\begin{aligned}
\delta W &= -\delta U \\
&= -\sum_{i=1}^{n}\left(\frac{\partial U}{\partial x_i}\delta x_i + \frac{\partial U}{\partial y_i}\delta y_i + \frac{\partial U}{\partial z_i}\delta z_i\right) \\
&= -\sum_{i=1}^{n}\left(\frac{\partial U}{\partial x_i}\sum_{k=1}^{f}\frac{\partial x_i}{\partial q_k}\delta q_k + \frac{\partial U}{\partial y_i}\sum_{k=1}^{f}\frac{\partial x_i}{\partial q_k}\delta q_k + \frac{\partial U}{\partial z_i}\sum_{k=1}^{f}\frac{\partial x_i}{\partial q_k}\delta q_k\right) \\
&= -\sum_{i=1}^{n}\sum_{k=1}^{f}\left(\frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_k}\delta q_k + \cdots\right) \\
\delta U_k &= \frac{\partial U}{\partial q_k}\delta q_k
\end{aligned}$$

We can combine these things we know about the differential for work and say that for conservative forces,

$$\begin{aligned}
\delta W_k = Q_k\delta q_k = \delta U_k &= -\frac{\partial U}{\partial q_k}\delta q_k \\
\implies Q_k &= -\frac{\partial U}{\partial q_k}
\end{aligned}$$

If we have a mixture of conservative and non-conservative forces, this doesn't work exactly, but it's pretty close:

$$Q_k = \underbrace{Q'_k}_{\text{Non-Conservative}} - \underbrace{\frac{\partial U}{\partial q_k}}_{\text{Conservative}}$$

5.1.5 Lagrange's Equations

We know from Newton's second law that

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$$

We also know that we defined the generalized momentum as

$$p_k = \frac{\partial T}{\partial \dot{q}_k}$$

Combining the two of those, we can find some interesting and useful expressions, and by extension, start to really develop a useful set of equations for Lagrangian dynamics. This is pretty easy in Cartesian coordinates:

$$\begin{aligned} T &= \frac{1}{2}\sum_{i=1}^{N} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) \\ \frac{\partial T}{\partial \dot{q}_k} &= \sum_{i=1}^{N} m_i \left(\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_k} + \dot{y}_i \frac{\partial \dot{y}_i}{\partial \dot{q}_k} + \dot{z}_i \frac{\partial \dot{z}_i}{\partial \dot{q}_k} \right) \end{aligned}$$

We have a definition for $\dot{x}_i$,

$$\dot{x}_i = \sum_{k=1}^{f} \frac{\partial x_i}{\partial q_k}\dot{q}_k + \frac{\partial x_i}{\partial t}$$

If we take the partial of that with respect to $\dot{q}_k$, we can find that

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_k} = \frac{\partial x_i}{\partial q_k}$$

If we plug this in for the partial of T with respect to $\dot{q}_k$, and take the time-derivative to find the generalized momentum:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial T}{\partial \dot{q}_k}\right] = &\sum_{i=1}^{N} m_i \left(\ddot{x}_i \frac{\partial x_i}{\partial q_k} + \ddot{y}_i \frac{\partial y_i}{\partial q_k} + \ddot{z}_i \frac{\partial z_i}{\partial q_k} \right) \\ &+ \sum_{i=1}^{N} m_i \left(\dot{x}_i \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial x_i}{\partial q_k}\right] + \dot{y}_i \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial y_i}{\partial q_k}\right] + \dot{z}_i \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial z_i}{\partial q_k}\right] \right) \end{aligned}$$

That's a lot. But we do know that the first term of that equation is

$$\sum_{i=1}^{N} m_i \left(\ddot{x}_i \frac{\partial x_i}{\partial q_k} + \ddot{y}_i \frac{\partial y_i}{\partial q_k} + \ddot{z}_i \frac{\partial z_i}{\partial q_k} \right) = \sum_{i=1}^{N} F_{ix}\frac{\partial x_i}{\partial q_k} + F_{iy}\frac{\partial y_i}{\partial q_k} + F_{iz}\frac{\partial z_i}{\partial q_k} = Q_k$$

To solve the second part, we can remember that the order of differentiation isn't actually important, so we can write

$$\dot{x}_i \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial x_i}{\partial q_k}\right] = \dot{x}_i \frac{\mathrm{d}}{\mathrm{d}q_k}\left[\frac{\partial x_i}{\partial t}\right] = \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_k}$$

So, putting this in the bigger part of this equation,

$$\begin{aligned} &\sum_{i=1}^{N} m_i \left(\dot{x}_i \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial x_i}{\partial q_k}\right] + \dot{y}_i \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial y_i}{\partial q_k}\right] + \dot{z}_i \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial z_i}{\partial q_k}\right] \right) \\ =&\sum_{i=1}^{N} m_i \left(\dot{x}_i \frac{\partial \dot{x}_i}{\partial q_k} \dot{y}_i \frac{\partial \dot{y}_i}{\partial q_k} \dot{z}_i \frac{\partial \dot{z}_i}{\partial q_k} \right) = \frac{\partial T}{\partial q_k} \end{aligned}$$

Putting those two together, we get *Lagrange's Equation*:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial T}{\partial \dot{q}_k}\right] &= Q_k + \frac{\partial T}{\partial q_k} \\ \implies \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial T}{\partial \dot{q}_k}\right] - \frac{\partial T}{\partial q_k} &= Q_k \end{aligned}$$

If we're working with all conservative forces, so that we can write $Q_k = -\frac{\partial U}{\partial q_k}$, then we can instead write

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial T}{\partial \dot{q}_k}\right] - \frac{\partial T}{\partial q_k} = -\frac{\partial U}{\partial q_k}$$

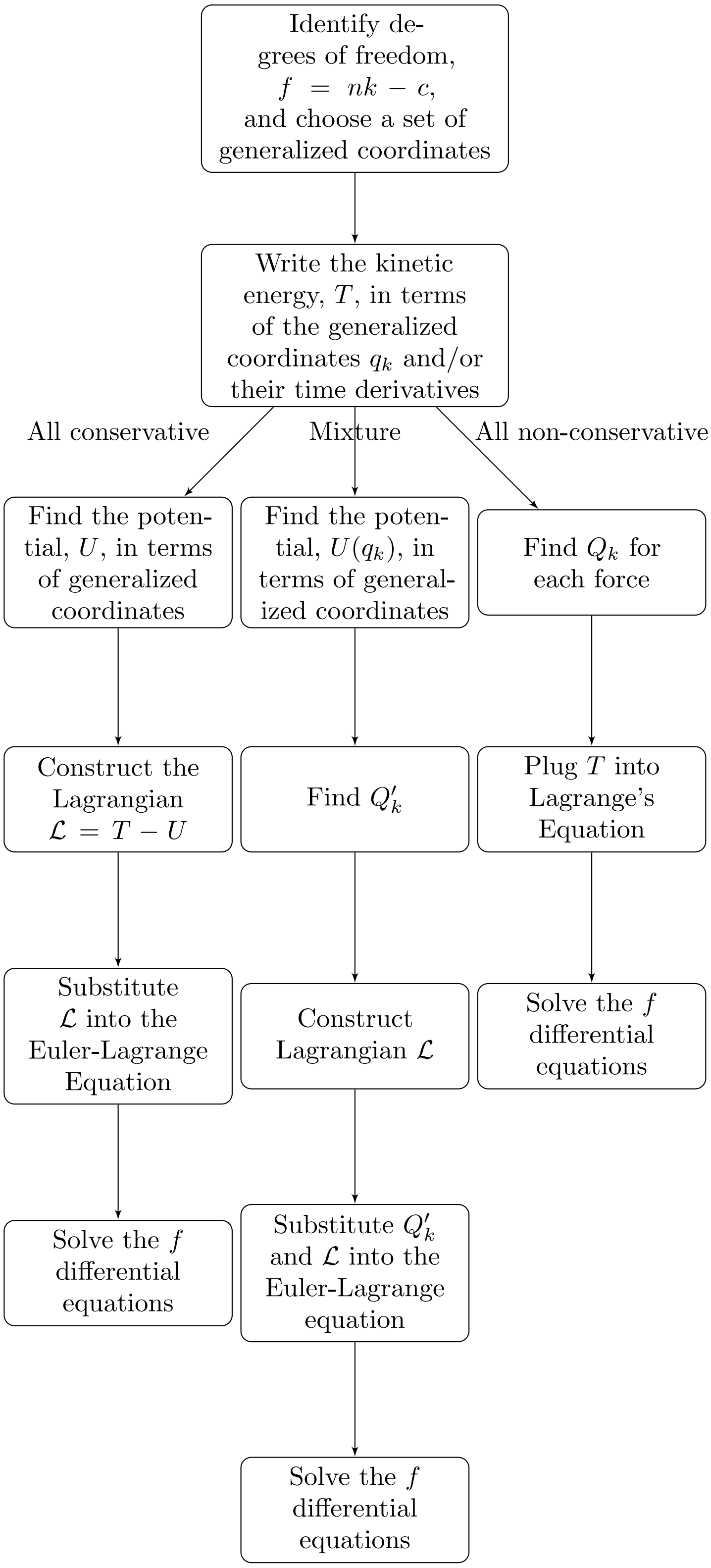
But we can make this even better. If the potential isn't defined in terms of velocity, then we define what we call the *Lagrangian*, $\mathcal{L}$:

$$\mathcal{L} = T - U$$

If we use this definition in Lagrange's equation, we get the Euler-Lagrange Equation for, respectively, non-conservative and conservative forces:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right] - \frac{\partial \mathcal{L}}{\partial q_k} = Q'_k$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right] - \frac{\partial \mathcal{L}}{\partial q_k} = 0$$

These sets of equations are applicable to any set of generalized coordinates. Note that Lagrange's Equation requires no reatructions on U, while the Euler-Lagrange Equation does. In general, we can define a problem-solving strategy using this definition of Lagrangian dynamics.


Identify degrees of freedom, $f = nk - c$, and choose a set of generalized coordinates
Write the kinetic energy, T, in terms of the generalized coordinates q_k and/or their time derivatives
All conservative
Mixture
All non-conservative
Find the potential, U, in terms of generalized coordinates
Find the potential, $U(q_k)$, in terms of generalized coordinates
Find Q_k for each force
Construct the Lagrangian $\mathcal{L} = T - U$
Find Q'_k
Plug T into Lagrange's Equation
Substitute $\mathcal{L}$ into the Euler-Lagrange Equation
Construct Lagrangian $\mathcal{L}$
Solve the f differential equations
Solve the f differential equations
Substitute Q'_k and $\mathcal{L}$ into the Euler-Lagrange equation
Solve the f differential equations

5.2 Applications of Lagrange's Equations

5.2.1 Elementary Examples

For these first few elementary examples, we will deal exclusively with conservative forces.

1. Find the equations of motion for a one-dimensional mass-spring system with mass m and spring constant k.

   i. Identify the degrees of freedom
   There is one "particle" (the mass), moving in one dimension, and there are no constraints on that movement, so $n = 1$, $k = 1$, and $c = 0$, meaning $f = nk - c = 1$. We can choose this coordinate to be anything, but it's probably easiest to choose it as x.

   ii. Write the kinetic energy in terms of generalized coordinates
   $$T = \frac{1}{2}m\dot{x}^2$$

   iii. All forces are conservative, so find the potential in terms of generalized coordinates
   $$U = \frac{1}{2}kx^2$$

   iv. Construct the Lagrangian
   $$\mathcal{L} = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

   v. Substitute into the Euler-Lagrange equation
   $$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\right] - \frac{\partial\mathcal{L}}{\partial q_k} = 0$$

   $$\begin{aligned}
   \frac{\partial\mathcal{L}}{\partial\dot{x}} &= \frac{\partial}{\partial\dot{x}}\left[\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right] = m\dot{x}\\
   \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\mathcal{L}}{\partial\dot{x}}\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left[m\dot{x}\right] = m\ddot{x}\\
   \frac{\partial\mathcal{L}}{\partial x} &= \frac{\partial}{\partial x}\left[\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right] = -kx
   \end{aligned}$$

   $$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\right] - \frac{\partial\mathcal{L}}{\partial q_k} = m\ddot{x} + kx = 0$$

   vi. Solve the differential equation
   As we saw in chapter 5, we can solve this by
   $$x(t) = A\cos(\omega t - \delta)$$
   where $\omega = \sqrt{k/m}$ and A and δ are fixed by the initial conditions.

2. Find the equation(s) of motion for a simple pendulum of length ℓ and mass m.

   i. Identify degrees of freedom
   There is, one particle, the mass, moving now in two dimensions (x and y), but now there is also an extra constraint: the distance from the rotation point to the mass must always be ℓ. Therefore, the degrees of freedom are equal to $f = nk - c = 1$. We will use θ, the angle the mass and string make with the vertical, for our generalized coordinate.

   ii. Write the kinetic energy in terms of the generalized coordinate
   $$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\ell\dot{\theta})^2 = \frac{1}{2}m\ell^2\dot{\theta}^2$$

   iii. All forces are conservative, so find the potential in terms of general coordinates
   $$U = mgl(1 - \cos\theta)$$
   (Gravitational potential energy)

   iv. Construct the Lagrangian
   $$\mathcal{L} = T - U = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell(1 - \cos\theta)$$

   v. Substitute into the Eurler-Lagrange equation
   $$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\right] - \frac{\partial\mathcal{L}}{\partial q_k} = 0$$
   $$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= m\ell^2\dot{\theta}\\ \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right] &= m\ell^2\ddot{\theta}\\ \frac{\partial\mathcal{L}}{\partial\theta} &= -mg\ell\sin\theta\\ \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\right] - \frac{\partial\mathcal{L}}{\partial q_k} &= m\ell^2\ddot{\theta} + mg\ell\sin\theta = 0\end{aligned}$$

   vi. Solve the differential equation
   $$\begin{aligned}m\ell^2\ddot{\theta} &= -mg\ell\sin\theta\\ \ell\ddot{\theta} &= -g\sin\theta\end{aligned}$$
   For small θ values,
   $$\begin{aligned}\ell\ddot{\theta} &= -g\theta\\ \theta(t) &= A\cos(\omega t - \delta)\end{aligned}$$
   where $\omega = \sqrt{g/\ell}$

5.2.2 Ignorable Coordinates

When we're looking at situations involving central forces, we often want to use spherical polar coordinates. We have an expression for kinetic energy in generalized coordinates that we can apply to SP coordinates specifically:

$$T = \sum_{k=1}^{3n} \sum_{\ell=1}^{3n} \frac{1}{2} A_{k\ell} \dot{q}_k \dot{q}_\ell + \sum_{k=1}^{3n} B_k \dot{q}_k + T_0$$

The last two terms we said would only appear if we had a coordinate system that was changing in time, which we generally don't with SP coordinates, so they can be ignored. Using that general form, with expressions from the metric tensor for SP coordinates:

$$T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\varphi}^2)$$

The potential energy we can't know generally, so we'll just call it $U(r, \theta, \varphi)$ for now, with the note that if we're working with central forces, it'll be $U(r)$. That means we have everything we need to write a general version of the Lagrangian in SP coordinates:

$$\mathcal{L} = T - U = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\varphi}^2) - U(r, \theta, \varphi)$$

As an aside, we have defined the generalized momentum as

$$p_k = \frac{\partial T}{\partial \dot{q}_k}$$

Alternatively, if we follow some textbooks and assume that the potential energy function has no explicit velocity or time dependence, then we can say equivalently that

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k}$$

If we decide to use this definition of momentum, then we can re-write the Euler-Lagrange equation:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right] - \frac{\partial \mathcal{L}}{\partial q_k} &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right] &= \frac{\partial \mathcal{L}}{\partial q_k} \\
\frac{\mathrm{d}p_k}{\mathrm{d}t} &= \frac{\partial \mathcal{L}}{\partial q_k}
\end{aligned}$$

Why is this important? It makes this next section a lot more clear. If the time derivative of momentum is equal to zero (and thus its implied values are equal to zero), then... there is no changeing momentum, and therefore no changing velocity, so there's no net force or acceleration in that direction! We call any coordinate for which this is true "ignorable." That name is a little bit misleading, because we don't actually want to totally ignore it, but it's not really that important when we're developing our equations of motion, because there is always constant velocity. Why did we spend so much time developing the

Lagrangian for SP coordinates just to talk about ignorable coordinates? Let's see!

Using the Lagrangian we constructed for SP coordinates earlier, if we try to find the EOMs for each of the three coordinates, we get some interesting results. For r:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{r}}\right] = \frac{\mathrm{d}p_r}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[m\dot{r}\right] = m\ddot{r}$$
$$m\ddot{r} = \frac{\partial \mathcal{L}}{\partial r} = mr\dot{\theta}^2 + mr\sin^2\theta\dot{\varphi}^2 - \frac{\partial U}{\partial r}$$

For θ:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right] = \frac{\mathrm{d}}{\mathrm{d}t}[p_\theta] = \frac{d}{\mathrm{d}t}[mr^2\dot{\theta}]$$
$$\frac{d}{\mathrm{d}t}[mr^2\dot{\theta}] = m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) = \frac{\partial \mathcal{L}}{\partial \theta} = mr^2\sin\theta\cos\theta\dot{\varphi}^2 - \frac{\partial U}{\partial \theta}$$

For φ:

$$\frac{\mathrm{d}}{\mathrm{d}t}[mr^2\sin^2\theta\dot{\varphi}] = \frac{\partial \mathcal{L}}{\partial t} = -\frac{\partial U}{\partial \dot{\varphi}}$$

This doesn't tell us that much on its surface. But, if we have exclusively central forces, it does tell us something important. Recall that central forces imply that $U = U(r)$, so in that case, $\frac{\partial U(r)}{\partial \dot{\varphi}}$ would vanish, meaning

$$\frac{\partial \mathcal{L}}{\partial t} = \frac{\partial p_\varphi}{\partial t} = 0$$

This makes φ an ignorable coordinate!

5.2.3 Electromagnetic Forces and Velocity-Dependent Potentials

6 Two-Body, Central-Force Motion

7 Mechanics in Non-Intertial Frames

7.1 Basic Equations for Non-Inertial Reference Frames

7.1.1 Definitions and Standards

Definition (Reference Frame)**.** A *reference frame* is a choice of coordinate axes by which the position and motion of an object can be specified.

Definition (Inertial Reference Frame)**.** An *inertial reference frame* (IRF) is a reference frame in which Newton's first law (the law of inertia, "in the absence of external forces, a particle moves with constant velocity") holds. Generally, this is a non-rotational reference frame moving at a onstant velocity (including 0).

Definition (Non-Inertial Reference Frame)**.** A *non-inertial reference frame* (NIRF) is a reference frame in which Newton's first law does not hold. Generally, this is a rotating or accelerating reference frame.

For the purposes of this chapter (and the remainder of this class), we'll follow Taylor and go against many other textbooks by using primes to refer to IRFs, and normal coordinates to refer to NIRFs. So, for example, $\mathbf{v}'$ would describe the velocity of a particle in an IRF, whereas $\mathbf{v}$ would describe the velocity of a particle in a NIRF.

7.1.2 Translational Motion Between Reference Frames

Consider two reference frames: one stationary IRF, S', and one accelerating NIRF, S, whose origin is a distance $\mathbf{R}'$ from the origin of S', moving at a rate of $\mathbf{V}'$ with respect to the origin of S'.
A particle, P can be measured at $\mathbf{r}$ in the S frame, or at $\mathbf{r}'$ in the S' frame. Based on the figure, we can say that

$$\begin{aligned}
\mathbf{r}' &= \mathbf{R}' + \mathbf{r} & &\implies \mathbf{r}' - \mathbf{R}' = \mathbf{r} \\
\mathbf{v}' &= \mathbf{V}' + \mathbf{v} & &\implies \mathbf{v}' - \mathbf{V}' = \mathbf{v} \\
\mathbf{a}' &= \mathbf{A}' + \mathbf{a} & &\implies \mathbf{a}' - \mathbf{A}' = \mathbf{a}
\end{aligned}$$

If Newton's second law holds in the NIRF(and we assume that it always does), then

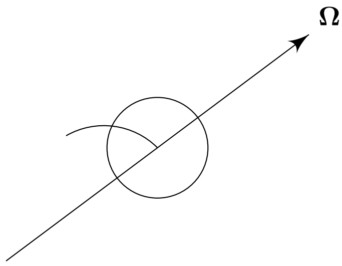
$$\begin{aligned}
\mathbf{F} = m\mathbf{a} &= m(\mathbf{a}' - \mathbf{A}') \\
&= m\mathbf{a}' - m\mathbf{A}' \\
&= m\mathbf{a}' + \mathbf{F}_{inertial}
\end{aligned}$$

Where $\mathbf{F}_{inertial} = -m\mathbf{A}'$ is called the "fictitious," "pseudo," or "apparent" force.

7.1.3 Rotational Motion Between Reference Frames

Consider two frames, S and S', with the same origin, but S is rotating about an axis $\hat{\mathbf{u}}$, with an angular velocity of $\mathbf{\Omega}'$ with respect to the S' frame. Since the origins coincide, and there is no translational motion, $\mathbf{R}' = 0$, $\mathbf{V}' = 0$, $\mathbf{A}' = 0$, and for some particle P, $\mathbf{r} = \mathbf{r}'$. From this, we can derive the expressions

$$\begin{aligned}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{r}'}{\mathrm{d}t} = \mathbf{v}' &= \frac{\mathrm{d}}{\mathrm{d}t}\left[x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}\right] \\ &= \frac{\mathrm{d}x}{\mathrm{d}t}\hat{\mathbf{x}} + \frac{\mathrm{d}y}{\mathrm{d}t}\hat{\mathbf{y}} + \frac{\mathrm{d}z}{\mathrm{d}t}\hat{\mathbf{z}} + x\frac{\mathrm{d}\hat{\mathbf{x}}}{\mathrm{d}t} + y\frac{\mathrm{d}\hat{\mathbf{y}}}{\mathrm{d}t} + z\frac{\mathrm{d}\hat{\mathbf{z}}}{\mathrm{d}t} \\ &= \mathbf{v} + x\frac{\mathrm{d}\hat{\mathbf{x}}}{\mathrm{d}t} + y\frac{\mathrm{d}\hat{\mathbf{y}}}{\mathrm{d}t} + z\frac{\mathrm{d}\hat{\mathbf{z}}}{\mathrm{d}t}\end{aligned}$$



In a time $\mathrm{d}t$, $\hat{\mathbf{x}}$ changes by $\mathrm{d}\hat{\mathbf{x}}$, such that

$$|\mathrm{d}\hat{\mathbf{x}}| \approx |\hat{\mathbf{x}}|\Omega \sin\alpha \mathrm{d}t$$

Where $\approx$ becomes $=$ if we take $\mathrm{d}t$ to be infinitesimally small. By extension,

$$\begin{aligned}\left|\frac{\mathrm{d}\hat{\mathbf{x}}}{\mathrm{d}t}\right| &= \Omega\sin\alpha \\ \left|\frac{\mathrm{d}\hat{\mathbf{x}}}{\mathrm{d}t}\right| &= \boldsymbol{\Omega}\times\hat{\mathbf{x}}\end{aligned}$$

We can expand this to more than juust $\hat{\mathbf{x}}$ and use this result to simplify the expression we found earlier for $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$:

$$\begin{aligned}\frac{\mathrm{d}\mathbf{r}'}{\mathrm{d}t} &= \mathbf{v} + \boldsymbol{\Omega}\times\left[x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}\right] \\ &= \mathbf{v} + \boldsymbol{\Omega}\times\mathbf{r}\end{aligned}$$

This is super important because *any* vector can be written this way!

Theorem (Coriolis's Theorem)**.** Any arbitrary vector $\mathbf{Q}'$ in the IRF can be written in the NIRF as

$$\frac{\mathrm{d}\mathbf{Q}'}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t} + \Omega\times\mathbf{Q}$$

To convert forces between the frames, we start with the fact that in the IRF, we know that Newton's second law must hold, so

$$\mathbf{F}' = m\mathbf{a}' = m\frac{\mathrm{d}\mathbf{v}'}{\mathrm{d}t}$$

Using the Coriolus theorem, we can find that

$$\begin{aligned}\mathbf{a}' = \frac{\mathrm{d}\mathbf{v}'}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{v} + \boldsymbol{\Omega}\times\mathbf{r}] + \boldsymbol{\Omega}\times(\mathbf{v} + \Omega\times\mathbf{r}) \\ \mathbf{a}' &= \mathbf{a} + \left(\frac{\mathrm{d}\Omega}{\mathrm{d}t}\times\mathbf{r}\right) + \left(\boldsymbol{\Omega}\times\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right) + (\boldsymbol{\Omega}\times\mathbf{v}) + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})\end{aligned}$$

Putting this into the definition of a force, this means that

$$\begin{aligned}\mathbf{F}' &= m\mathbf{a}' \\ &= \mathbf{F} - m(\dot{\boldsymbol{\Omega}} \times \mathbf{r}) - 2m(\boldsymbol{\Omega} \times \mathbf{v}) - m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})\end{aligned}$$

Where every term after the $\mathbf{F}$ is called a "fictitious force," and specifically:

(i) $-m(\dot{\boldsymbol{\Omega}} \times \mathbf{r})$ is called the "transverse force," and is zero unless the rotational velocity is changing.

(ii) $-2m(\boldsymbol{\Omega} \times \mathbf{v})$ is called the "Coriolis force," and is zero unless the particle moves in the NIRF

(iii) $-m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$ is called the "centrifugal force," and is the force due to the presence of the rotation of the NIRF.

7.1.4 General Motion Between Reference Frames

We've seen how to convert motion where the NIRF has only translational motion, and where the NIRF has only rotational motion, so if we want to know how to examine one with both, we can just add those together to get the *very* important result:

$$\mathbf{F} = \mathbf{F}' - m\mathbf{A}' - m(\dot{\boldsymbol{\Omega}} \times \mathbf{r}) - 2m(\boldsymbol{\Omega} \times \mathbf{v}) - m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$$

7.2 Motion in a Non-Intertial Reference Frame: The Earth

7.2.1 Considerations for Earth as a Non-Inertial Reference Frame

(i) The Earth rotates about its axis at an angular velocity of $\Omega_E = \frac{2\pi}{85164}\mathrm{s}^{-1} = 7.29 \times 10^{-5}\mathrm{s}^{-1}$

(ii) For the purposes of this problem, $\Omega_E^2 = 5.3 \times 10^{-9}$ is small enough to be ignorable

(iii) Similarly, the Earth's rotational angular momentum doesn't change enough for us to care about it right now, so we can consider it ignorable

(iv) Two products show up quite a bit so it's nice to have them recorded here and now:

$$\begin{aligned}\Omega_E R_E &= 464\mathrm{m} \cdot \mathrm{s}^{-1} \\ \Omega_E R_E^2 &= 3.4 \times 10^{-3}\mathrm{g}\end{aligned}$$

7.2.2 The Plumb Line (and the Centrifugal Force)

Definition (Plumb line)**.** A *plumb line* is a tool resembling a pendulum that is used to find the "local vertical."

Problem setup:

– Inertial reference frame: The origin at th center of the Earth, the $\hat{\mathbf{z}}'$ axis pointing out the north pole

– Non-inertial reference frame: The origin at the surface of the Earth at a colatitude (angle from $\hat{\mathbf{z}}'$) of θ, the $\hat{\mathbf{z}}$ axis pointing up the local vertical, and the $\hat{\mathbf{y}}$ axis pointing north

Since the bob of the plumb line is relatively stationary, $\mathbf{v} = 0$, and the term $m(\mathbf{\Omega} \times \mathbf{v}) = 0$.
Earth's angular velocity will have components along both the $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ axes, so in the NIRF,

$$\mathbf{\Omega}_E = \begin{pmatrix} 0 \\ \Omega \sin\theta \\ \Omega \cos\theta \end{pmatrix}$$

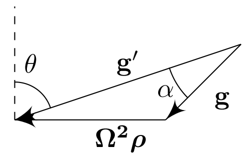
The NIRF force equation then becomes

$$\begin{aligned} m\ddot{\mathbf{r}} &= \mathbf{F}' - m\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) \\ m\mathbf{g} &= m\mathbf{g}' - m\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) \end{aligned}$$

Examining some of these individually, $\mathbf{g}' = 9.81\mathrm{m} \cdot \mathrm{s}^{-1}$ points towards th center of the Earth. $\mathbf{g}$, however, is the overall NIRF acceleration which includes both $\mathbf{g}'$ and the acceleration of the centrifugal force, meaning it won't point exactly towards the center of the Earth. Choosing spherical coordinates for our approximately-spherical Earth,

$$\begin{aligned} \mathbf{F}_{cf} &= -m\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = -m\Omega^2 \rho \hat{\boldsymbol{\rho}} \\ m\mathbf{g} &= m\mathbf{g}' - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) \\ \mathbf{g} &= \mathbf{g}' - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) \\ \mathbf{g} &= \mathbf{g}' + \Omega^2 \rho \hat{\boldsymbol{\rho}} \end{aligned}$$

In a picture,



Using this picture, we know from the law of sines that

$$\begin{aligned} \frac{\Omega^2 \rho}{\sin\alpha} &= \frac{\mathbf{g}}{\sin\left(\frac{\pi}{2} - \theta\right)} = \frac{\mathbf{g}}{\cos\theta} \\ \sin\alpha &= \frac{\Omega^2 (R_E \sin\theta)\cos\theta}{g} = \frac{\Omega^2 R_E \sin(2\theta)}{2g} \end{aligned}$$

If we assume that $\mathbf{g} = \mathbf{g}'$, which is true to a few parts per thousand, then the biggest α will get, at $\theta = 45°$, will be is $\alpha \approx 0.1°$. The $\mathbf{F}_{cf}$ is, therefore, existant but largely ignorable on the Earth, even at the equator (where $\theta = 45°$). This means that, to a pretty accurate precision, we can simply write $\mathbf{F} = \mathbf{F}'$ on the surface of the Earth.

7.2.3 General Motion at the Earth's Surface (and the Coriolis Force)

Without ignoring the Coriolis force, we know that

$$\begin{aligned} m\ddot{\mathbf{r}} &\approx m\mathbf{g} - 2m(\mathbf{\Omega} \times \mathbf{v}) \\ \ddot{\mathbf{r}} &\approx -g\hat{\mathbf{z}} - 2(\mathbf{\Omega} \times \mathbf{v}) \end{aligned}$$

The Coriolis acceleration here is

$$-2(\mathbf{\Omega} \times \mathbf{v}) = 2\Omega \begin{pmatrix} \dot{y}\cos\theta - \dot{z}\sin\theta \\ -\dot{x}\cos\theta \\ -\dot{x}\sin\theta \end{pmatrix}$$

Altogether, including the Coriolis aceleration, we can write

$$\ddot{\mathbf{r}} = \begin{pmatrix} 2\dot{y}\Omega\cos\theta - 2\dot{z}\Omega\sin\theta \\ -2\dot{x}\Omega\cos\theta \\ -g + 2\dot{x}\Omega\sin\theta \end{pmatrix}$$

Example. Let's say we drop a rock from the height $z = h$. Initially,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We can solve the coupled equations of motion easily, but lets try a slightly different approach. Let's say that $\dot{x} = \dot{y} = 0$, and $\dot{z} = -gt$. Then,

$$\begin{aligned} \ddot{\mathbf{r}} &= \begin{pmatrix} 2gt\Omega\sin\theta \\ 0 \\ -g \end{pmatrix} \\ \dot{\mathbf{r}} &= \begin{pmatrix} gt^2\Omega\sin\theta \\ 0 \\ -gt \end{pmatrix} \\ \mathbf{r} &= \begin{pmatrix} \frac{1}{3}gt^3\Omega\sin\theta \\ 0 \\ h - \frac{1}{2}gt^2 \end{pmatrix} \end{aligned}$$

So, in this non-inertial reference frame, when an object is dropped, it will move to the east by

$$x \approx \frac{1}{3}gt^3\Omega\sin\theta = \frac{\Omega g}{3}\left(\frac{2h}{g}\right)^{3/2}\sin\theta$$

7.2.4 The Foucault Pendulum

8 Rotational Motion of Rigid Bodies

8.1 Angular Momentum and Energy of a Rotating Body

8.1.1 General Notes on Rigid Bodies

Definition (Rigid body)**.** A *rigid body* is a system of N particles which retain their relative position with respect to each other.

In a rigid body, the location of the ith particle is

$$\mathbf{r}_i = x_i\hat{\mathbf{x}} + y_i\hat{\mathbf{y}} + z_i\hat{\mathbf{z}}$$

8.1.2 Rotation About a Fixed Axis

If a fixed body rotates about the z axis, which passes through it, with angular velocity $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$, then from what we've learned before, the velocity of the ith particle is

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i = \begin{pmatrix} -\omega y_i \\ \omega x_i \\ 0 \end{pmatrix}$$

The total kinetic energy of this system is then

$$T = \sum_{i=1}^{N} m_i v_i^2 = \sum_{i=1}^{N} \frac{1}{2} m_i (x_i^2 + y_i^2)\omega^2 = \frac{1}{2} I_{zz}\omega^2$$

Where $I_{zz} = \sum_{i=1}^{N} m_i(x_i^2 + y_i^1)$ is called the "moment of inertia about the z axis." Similarly, we can find the total angular momentum:

$$\begin{aligned}
\mathbf{L} &= \sum_{i=1}^{N} \mathbf{L}_i = \sum_{i=1}^{N} \mathbf{r}_i \times m_i \mathbf{v}_i \\
&= \sum_{i=1}^{N} m_i\{[-\omega z_i x_i]\hat{\mathbf{x}} + [\omega z_i y_i]\hat{\mathbf{y}} + [\omega(x_i^2 + y_i^2)]\hat{\mathbf{z}}\} \\
&= I_{zx}\omega\hat{\mathbf{x}} + I_{zy}\omega\hat{\mathbf{y}} + I_{zz}\omega\hat{\mathbf{z}}
\end{aligned}$$

Where I_{zx} and I_{zy} are called "products of inertia." Note that if either (or both) of the products of inertia is not equal to zero, $\mathbf{L}$ is not parallel to $\boldsymbol{\omega}$, and the direction of $\mathbf{L}$ will be continuously changing unless a sufficient torque is applied.

If we have a continuous object rather than a system of N objects, then we take the integral over the mass distribution $\varrho \mathrm{d}m$:

$$\sum_{i=1}^{N} m_i \to \int \mathrm{d}m$$

$$I_{zx} = -\sum_{i=1}^{N} m_i(z_i x_i) \to -\int \mathrm{d}m zx = -\int \varrho \mathrm{d}V zx$$

8.1.3 Rotation About an Arbitrary Axis

If we have the same rigid body, but we decide to instead rotate about some other arbitrary axis such that

$$\boldsymbol{\omega} = \omega_x \hat{\mathbf{x}} + \omega_y \hat{\mathbf{y}} + \omega_z \hat{\mathbf{z}}$$

To get the angular momentum of the ith particle, we can find

$$\begin{aligned}
\mathbf{L}_i = &\sum_{i=1}^{N} \mathbf{r}_i \times m_i \mathbf{v}_i = \mathbf{r}_i \times m_i(\boldsymbol{\omega} \times \mathbf{r}_i) \\
&= m_i[(y_i^2 + z_i^2)\omega_x - x_i y_i \omega_y - x_i z_i \omega_z]\hat{\mathbf{x}} \\
&\quad + m_i[-y_i x_i \omega_x + (z_i^2 + x_i^2)\omega_y - y_i z_i \omega_z]\hat{\mathbf{y}} \\
&\quad + m_i[-z_i x_i \omega_x - z_i y_i \omega_y + (x_i^2 + y_i^2)\omega_z]\hat{\mathbf{z}} \\
&= L_{xi}\hat{\mathbf{x}} + L_{yi}\hat{\mathbf{y}} + L_{zi}\hat{\mathbf{z}}
\end{aligned}$$

To find the total angular momentum, we would need to sum over all of these individual angular momenta:

$$\begin{aligned}
\mathbf{L} = &\sum_{i=1}^{N} \mathbf{L}_i \\
= &\left\{ \left[\sum_{i=1}^{N} m_i(y_i^2 + z_i^2)\right]\omega_x - \left[\sum_{i=1}^{N} m_i x_i y_y\right]\omega_y - \left[\sum_{i=1}^{N} m_i x_i z_i\right]\omega_z \right\}\hat{\mathbf{x}} \\
&+ \left\{ \left[\sum_{i=1}^{N} m_i y_i x_i\right]\omega_x - \left[\sum_{i=1}^{N} m_i(z_i^2 + x_i^2)\right]\omega_y - \left[\sum_{i=1}^{N} m_i y_i z_i\right]\omega_z \right\}\hat{\mathbf{y}} \\
&+ \left\{ \left[\sum_{i=1}^{N} m_i z_i x_i\right]\omega_x - \left[\sum_{i=1}^{N} m_i z_i y_i\right]\omega_y - \left[\sum_{i=1}^{N} m_i(x_i^2 + y_i^2)\right]\omega_z \right\}\hat{\mathbf{z}} \\
= &[I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z]\hat{\mathbf{x}} \\
&+ [I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z]\hat{\mathbf{y}} \\
&+ [I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z]\hat{\mathbf{z}}
\end{aligned}$$

That's a lot of work to write and it also looks a whole lot like a linear combination that can be written as the product of a matrix and a vector. That's good, because it is. We can write this as the product of the vector $\boldsymbol{\omega}$ and the moment of inertia tensor, $\mathbb{I}$, where

$$\mathbb{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}, \qquad \boldsymbol{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Note that $\mathbb{I}$ is diagonally symmetric and it depends on the chosen set of axes. I won't go through the work here, but we can use a similar process to show that

$$T = \frac{1}{2}\boldsymbol{\omega} \times \mathbf{L} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbb{I}\boldsymbol{\omega}$$

As before, in the case of continuous bodies, we would replace the $I_{x_i x_j}$ terms with integrals instead of sums.

Example. Find $\mathbb{I}$ for a solid cube rotating about 1 corner. Use axes parallel to the cube's edge. Sides have a length a, the origin is placed at one corner. The total mass of the object is M, so $\varrho = \frac{M}{a^3}$. Find the the total angular momentum, $\mathbf{L}$, if the object is rotated about the x axis, and about the main diagonal.

$$\begin{aligned} I_{xx} &= \int_0^a \mathrm{d}x \int_0^a \mathrm{d}y \int_0^a \mathrm{d}z \varrho(y^2 + z^2) \\ &= \frac{2\varrho a^5}{3} = \frac{2}{3} M a^3 \end{aligned}$$

The symmetry of the cube implies that $I_{xx} = I_{yy} = I_{zz}$, so we've successfully found the diagonal components of $\mathbb{I}$, or the moments of inertia.

$$\begin{aligned} I_{xy} = -\int \mathrm{d}m xy &= -\int_0^a \mathrm{d}x \int_0^a \mathrm{d}y \int_0^a \mathrm{d}z \varrho xy \\ &= -\varrho \left(\frac{a^2}{2}\right)\left(\frac{a^2}{2}\right) \\ &= -\frac{1}{4} M a^2 \end{aligned}$$

Similarly, symmetry implies that the other products of inertia will all be the same.

So, we can write $\mathbb{I}$ as:

$$\mathbb{I} = Ma^2 \begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix} = \frac{Ma^2}{12} \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}$$

If this cube is rotated about the x axis, such that

$$\boldsymbol{\omega} = \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix}$$

then

$$\mathbf{L} = \mathbb{I}\boldsymbol{\omega} = \frac{Ma^3\omega}{12} \begin{pmatrix} 8 \\ -3 \\ -3 \end{pmatrix}$$

Note that $\mathbf{L}$ is not parallel to $\boldsymbol{\omega}$, so a torque will be needed to keep it rotating about the x axis.

If this cube is rotated about its main diagonal, such that

$$\boldsymbol{\omega} = \frac{\omega}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

then

$$\mathbf{L} = \mathbb{I}\boldsymbol{\omega} = \frac{Ma^2\omega}{12\sqrt{3}}\begin{pmatrix}2\\2\\2\end{pmatrix} = \frac{Ma^2}{6}\boldsymbol{\omega}$$

Note that $\mathbf{L}$ *is* parallel to $\boldsymbol{\omega}$, so the cube will spin stably on that axis.

8.1.4 Principal Axes

Definition (Principal axis)**.** A *principal axis* is an axis of rotation for an object around which the angular momentum vector $\mathbf{L}$ is parallel.

Note that a principal axis only occurs if the products of inertia in $\mathbb{I}$ associated with that axis vanish. Since $\mathbb{I}$ is a square 3x3 symmetric matrix, and since any square matrix can be diagonalized, there must be 3 principal axes about a particular point whose principal moments of inertia are the non-zero diagonal elements.

There are three primary ways of finding the principle axes:

(i) Using symmetries
This works especially well if O passes through the center of mass of the object. If there is a rotational symmetry axis passing through O, that is always a principal axis, and its perpendicular plane will contain the other two. If a body has two perpendicular planes of symmetry that meet at point O, the 3 axes for those planes are the principal axes.

(ii) If you know one principal axis for the point O
Let's label the principle axes as $\{\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}}\}$, with moments of inertia λ_1, λ_2, and λ_3. If we already know that, say, $\hat{\mathbf{3}}$ is a principal axis which is aligned with the $\hat{\mathbf{z}}$ axis, then the other two axes must be in the xy plane passing through O. Let's say that the angle between $\hat{\mathbf{1}}$ and $\hat{\mathbf{x}}$ is θ.

Since $\hat{\mathbf{1}}$ is a principal axis, we know that if we rotate about $\hat{\mathbf{1}}$,

$$\mathbf{L} = \begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}\begin{pmatrix}\omega_1\\0\\0\end{pmatrix} = \lambda_1\boldsymbol{\omega}_1$$

In the xyz basis,

$$\boldsymbol{\omega}_1 = \begin{pmatrix}\omega_x\\ \omega_y\\ 0\end{pmatrix}$$

with $\tan\theta = \frac{\omega_y}{\omega_x}$. This means that in the xyz basis,

$$\mathbf{L} = \lambda_1\boldsymbol{\omega}_{\mathbf{1}} = \begin{pmatrix}I_{xx} & I_{xy} & 0\\ I_{yx} & I_{yy} & 0\\ 0 & 0 & I_{zz}\end{pmatrix}\begin{pmatrix}\omega_x\\ \omega_y\\ 0\end{pmatrix}$$

Supposing we are able to find I_{xx}, I_{yy}, and I_{xy}, we would find that

$$\begin{aligned}\lambda_1\omega_x &= I_{xx}\omega_x + I_{xy}\omega_y \\ \lambda_1\omega_y &= I_{xy}\omega_x + I_{yy}\omega_y\end{aligned}$$

$$\begin{aligned}\lambda_1 &= I_{xx} + I_{xy}\frac{\omega_y}{\omega_x} = I_{xx} + I_{xy}\tan\theta \\ \lambda_1 &= I_{yy} + I_{yy}\frac{\omega_x}{\omega_y} = I_{xy} + I_{yy}\cot\theta\end{aligned}$$

$$\begin{aligned}I_{xx} + I_{xy}\tan\theta &= I_{xy}\cot\theta + I_{yy} \\ I_{xx} - I_{yy} &= I_{xy}(\cot\theta - \tan\theta) \\ &= I_{xy}\frac{2}{\tan 2\theta} \\ \tan 2\theta &= \frac{2I_{xy}}{I_{xx} - I_{yy}}\end{aligned}$$

There will be two θ solutions to this, corresponding to the two remaining axes.

(iii) Brute force with the eigenvalue equation:

$$\begin{aligned}\mathbf{L} = \mathbb{I}\boldsymbol{\omega} &= \lambda\boldsymbol{\omega} \\ (\mathbb{I} - \lambda\hat{\mathbf{1}})\omega &= 0 \\ |\mathbb{I} - \lambda\hat{\mathbf{1}}| &= 0\end{aligned}$$

8.1.5 Terminology Notes

Definition (Spherical top)**.** A *spherical top* is an object for which $\lambda_1 = \lambda_2 = \lambda_3$. Examples include basketbals, baseballs, and dice.

Definition (Symmetric top)**.** A *symmetric top* is an object for which $\lambda_1 = \lambda_2 \neq \lambda_3$. Examples include footballs, frisbees, and normal tops.

Definition (Asymmetric top)**.** An *asymmetric top* is an object for which $\lambda_1 \neq \lambda_2 \neq \lambda_3$. Examples include a falling rigid cat, I guess?

Definition (Rotor)**.** A *rotor* is an object for which $\lambda_1 = 0, \lambda_2 = \lambda_3$. Examples include dumbells and diatomic molecules.

8.2 Analysis of Rotational Rigid Body Motion

8.2.1 Euler's Equations

If we apply a torque to a rotating rigid body, $\mathbf{L}$ changes according to

$$\boldsymbol{\Gamma} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}$$

If we fix the $\{\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}}\}$ coordinate system such that it rotates with the body (the non-inertial "body frame"), and fix the $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ coordinate system outside (the inertial "space frame"), then with respect to the space frame, Coriolis's theorem tells us that

$$\mathbf{\Gamma} = \mathbf{\Gamma}' = \frac{\mathrm{d}\mathbf{L}'}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} + \boldsymbol{\omega} \times \mathbf{L}$$

For a rigid body, $\mathbb{I}$ doesn't change, so $\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbb{I}\dot{\boldsymbol{\omega}}$. In body coordinates,

$$\mathbf{\Gamma} = \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 \dot{\omega_1} \\ \lambda_2 \dot{\omega_2} \\ \lambda_3 \dot{\omega_3} \end{pmatrix} + \begin{pmatrix} \omega_2\omega_3(\lambda_3 - \lambda_2) \\ \omega_3\omega_1(\lambda_1 - \lambda_3) \\ \omega_1\omega_2(\lambda_2 - \lambda_1) \end{pmatrix}$$

As a series of equations, we can write this as

$$\begin{aligned} \Gamma_1 &= \lambda_1 \dot{\omega_1} + \omega_2\omega_3(\lambda_3 - \lambda_2) \\ \Gamma_2 &= \lambda_2 \dot{\omega_2} + \omega_3\omega_1(\lambda_1 - \lambda_3) \\ \Gamma_3 &= \lambda_3 \dot{\omega_3} + \omega_1\omega_2(\lambda_2 - \lambda_1) \end{aligned}$$

These are called Euler's equations and they're *super* important!

Example. In the absence of any applied torquesm prove that the magnitue of $|\mathbf{L}|$ and T are constant.

(i) $|\mathbf{L}|$:
Since there are no torques,

$$\begin{aligned} 0 &= \lambda_1 \dot{\omega_1} + \omega_2\omega_3(\lambda_3 - \lambda_2) \\ 0 &= \lambda_2 \dot{\omega_2} + \omega_3\omega_1(\lambda_1 - \lambda_3) \\ 0 &= \lambda_3 \dot{\omega_3} + \omega_1\omega_2(\lambda_2 - \lambda_1) \end{aligned}$$

Multiplying by $\lambda_i \omega_i$,

$$\begin{aligned} 0 &= \lambda_1^2 \omega_1 \dot{\omega_1} + \lambda_1\omega_1\omega_2\omega_3(\lambda_3 - \lambda_2) \\ 0 &= \lambda_2^2 \omega_2 \dot{\omega_2} + \lambda_2\omega_2\omega_3\omega_1(\lambda_1 - \lambda_3) \\ 0 &= \lambda_3^2 \omega_3 \dot{\omega_3} + \lambda_3\omega_3\omega_1\omega_2(\lambda_2 - \lambda_1) \end{aligned}$$

If we add these together, the final term cancels, so

$$\begin{aligned} 0 &= \lambda_1^2 \omega_1 \dot{\omega_1} + \lambda_2^2 \omega_2 \dot{\omega_2} + \lambda_3^2 \omega_3 \dot{\omega_3} \\ 0 &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\lambda_1^2\omega_1^2 + \lambda_2^2\omega_2^2 + \lambda_3^2\omega_3^2] \\ 0 &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[L^2] \end{aligned}$$

Which means $|\mathbf{L}|$ is a constant

(ii) T:
Starting from the same initial Euler equations equal to zero again, if we instead multiply everything simply by ω_i,

$$\begin{aligned} 0 &= \lambda_1 \omega_1 \dot{\omega_1} + \omega_1\omega_2\omega_3(\lambda_3 - \lambda_2) \\ 0 &= \lambda_2 \omega_2 \dot{\omega_2} + \omega_2\omega_3\omega_1(\lambda_1 - \lambda_3) \\ 0 &= \lambda_3 \omega_3 \dot{\omega_3} + \omega_3\omega_1\omega_2(\lambda_2 - \lambda_1) \end{aligned}$$

If we add these together again, we find

$$\begin{aligned}
0 &= \lambda_1\omega_1\dot{\omega_1} + \lambda_2\omega_2\dot{\omega_2} + \lambda_3\omega_3\dot{\omega_3} \\
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2)\right] \\
0 &= \frac{\mathrm{d}T}{\mathrm{d}t}
\end{aligned}$$

Which means that T is a constant.

Example. A dumbell is composed of a massless rod of length $2b$ connecting two equal masses of mass m. The rod is the $\hat{\mathbf{3}}$ axis and makes an angle α with the $\hat{\mathbf{z}}$ axis. The dumbell spins at a constant ω about the $\hat{\mathbf{z}}$ axis. What torque is needed to maintain this motion?

8.2.2 Free Rotation

8.2.3 Euler Angles

8.2.4 Example: Spinning Symmetric Top

8.2.5 Example: Gyroscope