- Counting (See Page 16)
- Conditional Probability

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

- Independence

$$\Pr(A \cap B) = \Pr(A)\Pr(B)$$

- Baye's Rule

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(B|A)\Pr(A)}{\Pr(B)}$$

Counting

Important: When the sample space n contains **equally probable** outcomes. The probability function for set $A \subset \mathcal{S}$ is found by counting the number of elements in A by the number of elements in $\mathcal{S}$.

Theorem (Fundamental Theorem of Counting)

If a job consists of k separate tasks and the i^{th} task can be completed in n_i ways, $i = 1, \ldots, k$, then the entire job can be done in $n_1 \times n_2 \times \cdots \times n_k$ ways.

Example (Lottery Tickets)

Consider picking six numbers from $\{1, 2, \ldots, 44\}$
Ordered: The order of the numbers is significant,
$(1, 5, 10) \neq (1, 10, 5)$
Unordered: The order of the numbers is not significant,
$(1, 5, 10) = (1, 10, 5)$
With Replacement: The same number can be picked multiple times. $(1, 5, 5)$ is possible.
Without Replacement: The same number cannot be picked multiple times. $(1, 5, 5)$ is not possible.
In general we are picking r item's from a total n items.

- Ordered sampling without replacement:
  $44 \times 43 \times \ldots \times 39 = \frac{44!}{38!}$ or $\frac{n!}{(n-r)!}$
- Ordered sampling with replacement: 44^6 or n^r
- Unordered sampling without replacement: $\binom{44}{6}$ or $\binom{n}{r}$
  Start with ordered sampling without replacement $\frac{44!}{38!}$ and then count the number of ordered samples which would produce the same unordered samples, $\frac{6!}{0!} = 6!$. Divide to get the number of unique unordered samples

$$\binom{44}{6} = \frac{44!}{(44-6)!6!} \text{ or } \binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

- Unordered sampling with replacement: $\binom{49}{6}$ or $\binom{n+r-1}{r}$

Where does this come from? This is the hardest case to count. Imagine 44 cups or boxes end to end. The number of possible lottery tickets is equal to the number of ways that we can put the six markers into the 44 boxes.

For example, two such arrangements could be

1.

	MM	M					M	
1	2	3	4	5	. . .	42	43	44

|	M	M	|	M	|	|	|	. . .	|	|	M	|
1	2	3	4	5	6	7	8	. . .	46	47	48	49

2.

M			MMM					
1	2	3	4	5	. . .	42	43	44

M	|	|	|	M	M	M	|	|	. . .	|	|	|
1	2	3	4	5	6	7	8	9	. . .	47	48	49

All we really need to keep track of is the arrangements of the markers and the walls of the boxes. The two outermost walls are not important. So, we have to count all of the arrangements of 43 walls (44 bins give 45 walls, minus the outer two) and 6 markers. We need to the number of select 6 out of the 49 positions for the 6 markers. This is simple unordered sampling without replacement.

$$\binom{49}{6} \text{ or } \binom{n+r-1}{r}$$

There is a neat table on page 16 that shows the number of possible arrangements of size r from n objects.

	Replacement W/O	Replacement W
Ordered	$\frac{n!}{(n-r)!}$	n^r
Unordered	$\binom{n}{r}$	$\binom{n+r-1}{r}$

Example (Poker)

A standard deck of playing cards (52 cards–13 in each of 4 suits: spades, hearts, diamonds and clubs. The 13 denominations are ace,K,Q,J,10,9,8,7,6,5,4,3,2.) Player A deals 5 cards to Player B (5 card stud). Find the following probabilities.

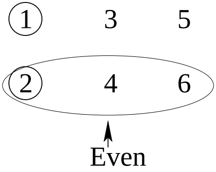
- $P(\text{royal flush}) = \frac{\text{number of ways to get a royal flush}}{\text{number of possible poker hands}} = \frac{4}{\binom{52}{5}} = 1.54 \times 10^{-6}$
- $P(\text{4 of a kind}) = \frac{\binom{13}{1}\binom{48}{1}}{\binom{52}{5}} = 2.4 \times 10^{-4}$
- $P(\text{3 of a kind}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}4^2}{\binom{52}{5}} = 0.02113$
- $P(\text{full house}) = P(\text{3 of a kind and 1 pair}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}} = 1.44 \times 10^{-3}$

- $P(\text{straight flush}) =$
  $P(\text{5 consecutive cards of the same suit}) = \frac{4 \cdot 9}{\binom{52}{5}} =$
  1.385×10^{-5}
- $P(\text{flush}) = P(\text{5 cards of the same suit}) = \frac{4\binom{13}{5} - 36 - 4}{\binom{52}{5}} =$
  1.97×10^{-3}
- $P(\text{straight}) = P(\text{5 consecutive cards}) = \frac{10(4^5) - 36 - 4}{\binom{52}{5}} =$
  3.92×10^{-3}
- $P(\text{2 pair}) = \frac{\binom{13}{2}\binom{4}{2}\binom{4}{2} \cdot 44}{\binom{52}{5}} = 0.0475$
- $P(\text{1 pair}) = \frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}4^3}{\binom{52}{5}} = 0.423$

Conditional Probability

- If we throw a fair die: $\Pr(1) = \Pr(2) = 1/6$
- However, given the outcome is even $\{2, 4, 6\}$, then

$$\Pr(1|\{2,4,6\}) = 0$$
$$\Pr(2|\{2,4,6\}) = 1/3$$



- Why is the $\Pr(1|\{2,4,6\}) = 0$?
  Well, given that the die came up even then the intersection between the two events is $\emptyset$.
- Why is the $\Pr(2|\{2,4,6\}) = 1/3$?
  Conditioning on $\{2,4,6\}$ changed the sample space from $\{1,2,3,4,5,6\}$ to $\{2,4,6\}$.

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Note $\Pr(A|B)$ is a probability function

Definition (Bayes Theorem)

$$\Pr(A|B) = \frac{\Pr(B \cap A)}{\Pr(B)} = \frac{\Pr(B|A)\Pr(A)}{\Pr(B)}$$

This can generalized. Let $A_1, A_2, \ldots$ be a partition A. then

$$\Pr(A_i|B) = \frac{\Pr(B|A_i)\Pr(A_i)}{\sum_{j=1}^{\infty}\Pr(B|A_j)\Pr(A_j)}.$$

Monty Hall Problem

- Three doors and behind one of the doors is a valuable prize and the other two doors are worthless prizes (goats)
- The contestant selects one of the doors (Door 1)
- The host will then reveal one of the goats. (Door 3)
- What is the probability that the contestant will win the valuable prize if they switch?

	Reveal			
Car	1	2	3	
1	0	$1/3 \times 1/2 = 1/6$	$1/3 \times 1/2 = 1/6$	$1/3$
2	0	$1/3 \times 0 \quad = 0$	$1/3 \times 1 \quad = 1/3$	$1/3$
3	0	$1/3 \times 1 \quad = 1/3$	$1/3 \times 0 \quad = 0$	$1/3$
	0	1/2	1/2	1

$$\Pr(\text{Switch Wins}|\text{Reveal 3}) = \frac{\Pr(C2 \cap R3)}{\Pr(R3)} = \frac{1/3}{1/2} = 2/3$$

- Suppose the car is behind a randomly selected door
- You select door number one
- The host will reveal the goat behind the highest numbered door.
  - If goats are behind both doors two and three the host will reveal door number three.
- The host reveals door number three, should you switch?

	Reveal		
Car	2	3	
1	0	$1 \times 1/3 = 1/3$	$1/3$
2	0	$1 \times 1/3 = 1/3$	$1/3$
3	$1 \times 1/3 = 1/3$	0	$1/3$
	$1/3$	$2/3$	1

$$\Pr(\text{Switch Wins}|\text{Reveal 3}) = \frac{\Pr(C2 \cap R3)}{\Pr(R3)} = \frac{1/3}{2/3} = 1/2$$

$$\Pr(\text{Switch Wins}|\text{Reveal 2}) = \frac{\Pr(C3 \cap R2)}{\Pr(R2)} = \frac{1/3}{1/3} = 1$$

Example (Car Rental)

Suppose there are three rental car agencies (A, B, and C) in a certain town. Suppose that agency A has probability 0.1, agency B has probability 0.08, and agency C has probability of 0.125 of renting a customer an "unsafe" car. An agency is chosen at random and a car rented from it is found to be unsafe. What is the conditional probability that the car came from agency A? From agency B? From agency C?

	Agency			
Cond.	A	B	C	
Safe	$.9\times 1/3$ $= 0.3$	$.92\times 1/3$ $\simeq 0.30667$	$.875\times 1/3$ $\simeq .29167$	0.89833
Unsafe	$0.1\times 1/3$ $\simeq 0.03333$	$.08\times 1/3$ $\simeq 0.02667$	$.125\times 1/3$ $\simeq 0.04167$	0.10167
	$1/3$	$1/3$	$1/3$	1.00

Given a rental car is unsafe, the probability that the car was rented from agency A is

$$P(A|U) = \frac{\Pr(A \cap U)}{\Pr(U)} = \frac{0.03333}{0.10167} = 0.328$$

The probability that the unsafe car was rented from agency B is

$$P(B|U) = \frac{P(B \cap U)}{P(U)} = \frac{0.02667}{0.10167} = 0.262$$

The probability that the unsafe car was rented from agency C is

$$P(C|U) = \frac{P(C \cap U)}{P(U)} = \frac{0.04167}{0.10167} = 0.410$$

Independence

- Two events, A and B, are independent if

$$\Pr(A \cap B) = \Pr(A)\Pr(B).$$

Definition (Independence)

A collection of events $A_1, \ldots, A_n$ are **mutually independent** if for any subcollection $A_{i1}, \ldots, A_{ik}$, we have

$$\Pr\left(\bigcap_{j=1}^{k} A_{i_j}\right) = \prod_{j=1}^{k} \Pr\left(A_{i_j}\right).$$

Example (Counterfeiter)

(a) There are 100 coins per box (1 false, 99 gold). One coin in each of 100 boxes is examined. The probability the counterfeiter is caught $= P$(King selects at least one false coin) = 1-P(no false coins are selected) $= 1 - \left(\frac{99}{100}\right)^{100} \approx 0.634$. (b) Replace 100 by n and let $n \to \infty$. This time, P(caught) $= 1 - \left(\frac{n-1}{n}\right)^{n}$. As $n \to \infty$, $1 - \left(\frac{n-1}{n}\right)^{n} \to 1 - e^{-1} \approx 0.632121.$
Helpful limit: $\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^{n} = e^{a}$

- Counting
  - Table on Page 16!
  - Replacement and Order
  - Probabilities assume events are equally likely
    Usually not satisfied with UW
- Conditional Probability

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

- Independence

$$\Pr(A \cap B) = \Pr(A)\Pr(B)$$

- Baye's Rule (Still just conditional probability!)

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(B|A)\Pr(A)}{\Pr(B)}$$