Introduction to Data Processing and Representation
(236201)
Winter 2021-2022

Homework 3

- **Published date: 30/12/2021**
- **Deadline date: 13/01/2022**

- Submission is in pairs only.
- Submit your entire solution (including the theoretical part, the Python part and the Python code) electronically via the course website. The file should be a zip file containing the your PDF submission and Python code.
- The submission should be in English and in a clear printed form (recommended) or a clear hand-writing.
- Rigorous mathematical proofs and reasoning are required for theoretical questions. Vague answers and unjustified claims will not be accepted.

I Theory

1. On Circulant Matrices

In this exercise, we use the normalised convention for the DFT matrix.

a. Consider the matrix $J = \begin{pmatrix} 0 & \dots & \dots & 0 & 1 \\ 1 & 0 & \ddots & \ddots & 0 \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{n \times n}$. For $k \in \mathbb{N}$, compute J^k.

   In particular, what is J^n?

b. Compute the eigenvalues of J.

c. Do the full eigendecomposition of J. Is J diagonalisable? If yes, can it be diagonalised in a unitary basis?

d. Consider the general circulant matrix $H = \begin{pmatrix} h_0 & h_{n-1} & h_{n-2} & \dots & h_1 \\ h_1 & h_0 & h_{n-1} & \dots & h_2 \\ h_2 & h_1 & h_0 & \dots & h_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{n-1} & h_{n-2} & h_{n-3} & \dots & h_0 \end{pmatrix}$. Show that H and J are linked by a polynomial expression, i.e. find a polynomial P such that $H = P(J)$.

e. Compute the full eigendecomposition of H. Is it diagonalisable and if so is it in a unitary basis?

f. Show that the diagonalisation basis matrix B can be chosen as the DFT^* matrix.

g. Prove that the eigenvalues, stacked in a column, equals to the product of B and the first row of H rewritten in column form up to a normalisation constant, i.e. if $\lambda_0, \dots, \lambda_{n-1}$ are the eigenvalues of H then:

$$\begin{pmatrix} \lambda_0 \\ \vdots \\ \lambda_{n-1} \end{pmatrix} = \sqrt{n} B \begin{pmatrix} h_0 \\ h_{n-1} \\ \dots \\ h_1 \end{pmatrix}$$

.

h. Consider two circulant matrices H_1 and H_2. Show that they commute. Compute $H_1 H_2$, is this matrix circulant?

i. Compute DFT^k for $k \in \mathbb{N}$. What are the resulting matrices?

j. Prove that a convolution of n-dimensional signals can be computed by point-wise multiplication of the signals in the Fourier domain, up to normalisation. This means prove that if $z = x \otimes y$ where $\otimes$ the convolution operator, then $(DFT)z = \sqrt{n}(DFT)x \odot (DFT)y$ where $\odot$ is the Hadamard product.

2. Fourier Transform

a. Given two functions $f(t)$ and $g(t)$ and denote the convolution of the two functions by $h(t)$, that is

$$f(t) * g(t) = h(t).$$

What is $f(t-1) * g(t+1)$ in terms of $h(t)$?

b. Given two functions $f(t)$ and $g(t)$, show that the following condition holds

$$\int_{-\infty}^{\infty} f(t)g(-t)dt = \int_{-\infty}^{\infty} \mathcal{F}(u)\mathcal{G}(u)du,$$

where $\mathcal{F}(u)$ and $\mathcal{G}(u)$ are the Fourier transform of $f(t)$ and $g(t)$ respectively.

3. Discrete Fourier Transform

Denote a 1D signal with $2N$ elements as $\boldsymbol{\phi} \in \mathbb{R}^{2N}$ given by

$$\boldsymbol{\phi} = [1, \frac{1}{2}, 0, \ldots, 0, \frac{1}{2}]^\top$$

We use zero-based indexing in this exercise.

a. What is the DFT of $\boldsymbol{\phi}$?

b. Consider another 1D signal $\boldsymbol{\psi}$ with N elements and denote its DFT-domain representation by $\boldsymbol{\psi}^F$. Consider a new signal $\boldsymbol{\gamma}$ by inserting zeros between the elements of $\boldsymbol{\psi}$, i.e.,

$$\boldsymbol{\gamma} = [\boldsymbol{\psi}_0, 0, \boldsymbol{\psi}_1, 0, \boldsymbol{\psi}_2, 0, \cdots, \boldsymbol{\psi}_{N-1}, 0]^\top \in \mathbb{R}^{2N}$$

Find the DFT of $\boldsymbol{\gamma}$ in terms of $\boldsymbol{\psi}^F$.

c. Show that the convolution of $\boldsymbol{\gamma}$ and $\boldsymbol{\phi}$ is the linear interpolation of $\boldsymbol{\psi}$, that is,

$$\boldsymbol{h} = \boldsymbol{\gamma} * \boldsymbol{\phi} = [\boldsymbol{\psi}_0, \frac{\boldsymbol{\psi}_0 + \boldsymbol{\psi}_1}{2}, \boldsymbol{\psi}_1, \frac{\boldsymbol{\psi}_1 + \boldsymbol{\psi}_2}{2}, \boldsymbol{\psi}_2, \cdots, \boldsymbol{\psi}_{N-1}, \frac{\boldsymbol{\psi}_{N-1} + \boldsymbol{\psi}_0}{2}]^\top$$

d. Find the DFT of $\boldsymbol{h}$ in terms of $\boldsymbol{\psi}^F$.

II Implementation

The purpose of this exercise is to get familiar with the concept of functional maps. We will transform images/signal in some ways and consider these transforms as the result of remapping the image after some changes in the parametrisation. In class, you saw that when the transform is known then the link between the coefficients of orthonormal basis in the first image/signal and in the second image/signal are linked by a linear transform, i.e. a matrix called the functional map. However, here we do not provide you with the corresponding transform but you can still recover approximately the functional map.

Instruction:

- Figures should be titled in a appropriate font size.
- Submit the code and a report describing the results and your understanding of the exercise.
- Note that the additional image files and audio files provided in the course website with this homework assignment.
- It is recommended to normalize the graylevels to the range [0,1] after loading the images.

1. Image example

Consider the mandril image and its distorted version. The distortion was found by applying a transform of the x parameter space, by squeezing and strecthing the x axis. Here, we consider one signal as a row of the image. Thus each row of the original image has been squeezed and stretched in the same way, giving the rows of the distorted image.

The basis of interest, both in the original image domain and in the distorted image domain is the DFT matrix basis. The DFT has as many columns (and rows) than the signals have entries, i.e. as many columns as the images.

a. Compute the DFT representation coefficients for each row signal in the original and distorted image. Denote $\alpha_{r,i}$ (resp. $\beta_{r,i}$) the i-th coefficient of the r-th row in the original (resp. distorted) image. As usual with the DFT, we start indexing at 0.

b. Denote, using the notations in the lecture, $\widetilde{c}_{i,j}$ the i,j-th coefficient of the functional map C. Recall that then $\beta_{r,j} = \sum_i \alpha_{r,k}\widetilde{c}_{k,j}$. We can rewrite this in a more compact matrix formulation for each row r as $\widetilde{b_r} = a_r C$, where a_r (resp. $\widetilde{b_r}$) is the representation row of coefficients of the r-th row of the original (resp. distorted) row in the

DFT basis. We can stack this matrix formulation into an even more compact formulation by writing $\widetilde{B} = AC$ where the r-th row of A (resp. $\widetilde{B}$) is a_r (resp. $\widetilde{b_r}$). When the transform mapping is known, C can be explicitly computed. Unfortunately, it is here considered as unknown and slight errors may exist leading to noise effects. Thus stacking each row allows to improve the stability of the recovered functional map. We wish to compute empirically C using a least squares approach. That is compute $\widehat{C}$ such that $\widehat{C} \in \underset{C}{\operatorname{argmin}} ||\widetilde{B} - AC||_2^2$. Least squares is a well-known problem and it can be solved, with some assumptions on the rank of A with pseudo inversion. Show that the matrix A is full rank, and so computing the pseudo inverse is legal. If it is not full rank remove redundant columns from the problem until you get a full rank matrix. Find then $\widehat{C}$. You are allowed to use build-in pseudo inverse operators to perform the inversion such as the Moore-Penrose pseudoinverse.

c. Distort the original image rows using your approximation of the functional map $\widehat{C}$. Note that you must go into the DFT domain to do your calculations and back to the normal domain to display an image understandable for a human being. Compare your result with the given distorted image.

d. Distort the 'butterfly' image by distorting its rows using $\widehat{C}$.

2. Audio example

Consider two audio signals (skycastle.wav and skycastle-distortion.wav) note that skycastle-distortion signal is the distortional signal of skycastle. Assume that the period of the distortion is known and it is 512 in entry unit.

Utilizing the same approach as with the image case, estimate an approximate functional map between the clean signal and the noisy signal. Use the DFT basis for the basis functions. (Hint: Note that the distortion is independent from the true signal and it is periodic, then the transformation can be done periodically as well. You might want to divide your signal into successive non overlapping small signals.) Use the matrices to denoise the other audio signal (totoro-distortion.wav). Compare the obtained result with the corresponding true audio signal (totoro.wav) by graph. What is the MSE error?

Use built-in functions to read and sound audio files.

For these audio signals, set the sample frequency to be $Fs = 48000$ when sounding them (since it is the sampling frequency).