Signal, Image, and Data Processing (236201)

Winter 2021-2022

Homework 2

- **Published date: 08/12/2021**
- **Deadline date: 23/12/2021**

Guidelines:

- Submission is in pairs only.
- Submit your entire solution (including the theoretical part, the Python part and the Python code) electronically via the course website. The file should be a zip file containing the your PDF submission and Python code.
- The submission should be in English and in a clear printed form (recommended) or a clear hand-writing.
- Rigorous mathematical proofs and reasoning are required for theoretical questions. Vague answers and unjustified claims will not be accepted.

I Theory

1. k-term best approximation in L^2

Consider the space of squared integrable functions $E = L^2(\mathbb{R}, \mathbb{C})$, to which we associate the natural Hermitian product. Let $f \in E$ and F be a subspace of E of finite dimension n.

a. Consider and fix a finite family of orthonormal functions $\beta_1, \ldots, \beta_n \in F$ such that $F = \mathrm{Vec}(\beta_1, \ldots, \beta_n)$. Let $k \in \{1, \ldots, n\}$.

   (a) Let $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ be a set of k increasing integers between 1 and n. What is the k-term approximation of f in F using $\mathrm{Vec}(\beta_{i_1}, \ldots, \beta_{i_k})$? What is the associated SE (squared-error)?

   (b) Which, of the $\binom{n}{k}$, k-approximation of f in F is best in the SE sense? Is it unique? What is the associated SE?

b. Consider and fix two different finite families of orthonormal functions $\beta_1, \ldots, \beta_n \in F$ and $\widetilde{\beta}_1, \ldots, \widetilde{\beta}_n \in F$.

   (a) Compare the n-approximations of f, in the SE sense, using the β family on one hand and the $\widetilde{\beta}$ family on the other.

   (b) What can you say about the k-term approximation on each family, where $k \in \{1, \ldots, n-1\}$?

2. Haar matrix and Walsh-Hadamard matrix

Given $t \in [0, 1]$, consider the signal as

$$\phi(t) = a + b\cos(2\pi t) + c\cos^2(\pi t) \tag{1}$$

where a, b, and $c \in \mathbb{R}$ are constants. The procedures considered in this question for the approximation of $\phi(t)$ should be optimal with respect to the minimization of the approximation MSE, calculated over the continuous domain $[0, 1]$.

a. The 4×4 Haar matrix is given by

$$\boldsymbol{H}_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix} \tag{2}$$

and its columns are used to form a set of 4 orthonormal functions, $\{\psi_i^H(t)\}_{i=1}^4$, defined for $t \in [0, 1]$, by using the change of basis from the standard basis with this matrix.

(i) Prove that $\boldsymbol{H}_4$ is unitary.

(ii) Show the set of orthonormal Haar functions $\{\psi_i^H(t)\}_{i=1}^4$. The functions should be presented using graphs with explicit notation of relevant values on the two axes.

(iii) What is the best approximation of ϕ using this Haar basis? What is the associated MSE?

(iv) Assume $a \geq b \geq 0$ and $c \geq 0$. What is the best 1-term approximation of ϕ? What is the best 2-term approximation of ϕ? What is the best 3-term approximation of ϕ? What is the best 4-term approximation of ϕ?

(v) Assume $a = \frac{1}{\pi}$, $b = 1$, and $c = \frac{3}{2}$. What is the best 1-term approximation of ϕ? What is the best 2-term approximation of ϕ? What is the best 3-term approximation of ϕ? What is the best 4-term approximation of ϕ?

b. The 4×4 Walsh-Hadamard matrix is given by

$$\boldsymbol{W}_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \tag{3}$$

and its columns are used to form a set of 4 orthonormal functions, $\{\chi_i^W(t)\}_{i=1}^4$, defined for $t \in [0, 1]$.

(i) Prove that $\boldsymbol{W}_4$ is unitary.

(ii) Show the set of orthonormal Walsh-Hadamard functions $\{\psi_i^W(t)\}_{i=1}^4$. The functions should be presented using graphs with explicit notation of relevant values on the two axes.

(iii) What is the best approximation of ϕ using this Walsh-Hadamard basis? What is the associated MSE?

(iv) Assume $a \geq b \geq 0$ and $c \geq 0$. What is the best 1-term approximation of ϕ? What is the best 2-term approximation of ϕ? What is the best 3-term approximation of ϕ? What is the best 4-term approximation of ϕ?

(v) Assume $a = \frac{1}{\pi}$, $b = 1$, and $c = \frac{3}{2}$. What is the best 1-term approximation of ϕ? What is the best 2-term approximation of ϕ? What is the best 3-term approximation of ϕ? What is the best 4-term approximation of ϕ?

3. On Hadamard matrices

Let $n \in \mathbb{N}^*$ a positive integer and $N = 2^n$. Consider the Hadamard matrix of dimension $H_{2^n} = H_N$.

a. Prove that H_N a symmetric, real, and unitary matrix. Prove also that it can be written as $H_N = \lambda_N A$ where $\lambda_N \in \mathbb{R}$ a constant (give its explicit value) and A a matrix with only ± 1 entries.

b. For a sequence, s, of digit numbers taking the value ± 1, we denote $S(s)$ the number of changes of sign in s.

   (i) Denote s_1, s_2 two sequences of numbers of same length. What is $S(s_1 s_2)$, where $s_1 s_2$ the concatenation of both sequences? Hint: you might want to consider several cases.

   (ii) Denote r_i the i-th row of H_N. Prove the ensemble equality:

$$\{S(r_1), \dots, S(r_N)\} = \{0, \dots, N-1\},$$

   i.e. that the number of changes of sign in the rows of H_N are the first N integers starting at 0.

4. On Haar matrices

Haar matrices are traditionally defined as $H_{2(N+1)} = \begin{pmatrix} H_{2N} \otimes (1,1) \\ I_{2N} \otimes (1,-1) \end{pmatrix}$, with $H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Recall the definition of the Kronecker product between A and B is

$$A \otimes B = \begin{pmatrix} a_{1,1}B & a_{1,2}B & \dots & a_{1,n}B \\ a_{2,1}B & a_{2,2}B & \dots & a_{2,n}B \\ \vdots & \ddots & \ddots & \vdots \\ a_{n,1}B & a_{n,2}B & \dots & a_{n,n}B \end{pmatrix}.$$

a. Consider $N \geq 1$. Is H_{2N} symmetric?

b. Consider $N \geq 1$. Is H_{2N} orthogonal?

c. Consider $N \geq 1$. Is H_{2N} unitary?

d. In cases when H_{2N} is not normalised, we scale each of its rows to have unit norm. This operation produces the matrix $\tilde{H}_{2N}$. Provide a simple recursive equation between $\tilde{H}$ matrices in matrix form using Kronecker products. HINT: It should resemble the equation between the H matrices.

e. Prove that for any two matrices A and B, we have $(A \otimes B)^\top = A^\top \otimes B^\top$.

f. Given the convention in the course, we like to change basis by applying transpose matrix multiplication. As such, for us, we prefer to use $\hat{H}_{2N} = \tilde{H}_{2N}^\top$. Provide a simple recursive equation between $\hat{H}$ matrices im matrix form using Kronecker products. HINT: It should resemble the equation between the H matrices.

II Implementation

Instruction:

- Figures should be titled in a appropriate font size.
- Submit the code and a report describing the results and your understanding of the exercise.

1. Numerical and Practical Bit Allocation for Two-Dimensional Signals

Consider a function

$$\phi(x, y) = A \cos(2\pi\omega_x x) \sin(2\pi\omega_y y) \quad \text{for} \quad (x, y) \in [0, 1] \times [0, 1] \tag{4}$$

where $A = 2500$, $\omega_x = 2$ and $\omega_y = 7$.

a. Mathematically develop formulas for derivatives and integrals to calculate the value-range, the horizontal-derivative energy and the vertical-derivative energy.

b. Approximate the continuous-domain signal $\phi(x, y)$ by a very high resolution digitalization. Present the signal as an image using the *cv2.imshow* function (use an appropriate gray-level scaling that suits the value of A).

c. Numerically calculate the value-range, the horizontal-derivative energy and the vertical-derivative energy. Compare these numerical results to the analytically calculated values from the question a.

d. Use the numerical approximations and numerically solve the bit-allocation optimization to determine N_x, N_y and b.

e. Consider two bit-allocation procedures with the bit-budgets $B_{low} = 5000$ and $B_{high} = 50000$. Write the obtained values of N_x, N_y and b.

f. Implement a searching procedure that finds the best bit-allocation parameters by practically evaluating the bit-allocation MSE for many combinations of parameters.

g. Apply the practical searching procedure for two bit-budgets $B_{low} = 5000$ and $B_{high} = 50000$. For each of the two bit-budgets, what are the optimal values of N_x, N_y and b? Are these similar to the corresponding values from the question e? Explain it in detail. Present the reconstructed images obtained in the experiments.

h. Consider the same function but with different parameters: $A = 2500$, $\omega_x = 7$ and $\omega_y = 2$. Repeat the analysis from question a to question g and compare the results. Explain the differences.

2. Hadamard, Hadamard-Walsh, and Haar matrices

a. Implement Hadamard matrices $\boldsymbol{H}_{2^n}$. This should be a function taking as input the level n. This function should return a $2^n \times 2^n$ matrix.

b. Take the two orthonormal families $\boldsymbol{H}_{2^n}$ and $\{\sqrt{2^n}\mathbf{1}_{\Delta_i}(t)\}_{i=1}^{2^n}$ into a new set of functions $\{h_i(t)\}_{i=1}^{2^n}$ by

$$\begin{pmatrix} h_1(t) \\ h_2(t) \\ \vdots \\ h_{2^n}(t) \end{pmatrix} = \boldsymbol{H}_{2^n}^\top \begin{pmatrix} \sqrt{2^n}\mathbf{1}_{\Delta_1}(t) \\ \sqrt{2^n}\mathbf{1}_{\Delta_2}(t) \\ \vdots \\ \sqrt{2^n}\mathbf{1}_{\Delta_{2^n}}(t) \end{pmatrix} \tag{5}$$

Plot the functions $\{h_i(t)\}_{i=1}^{2^n}$ for $n = 2, ..., 6$.

c. Implement Walsh-Hadamard matrices $\widetilde{\boldsymbol{H}}_{2^n}$. This should be a function taking as input Hadamard matrices $\boldsymbol{H}_{2^n}$. This function should return a $2^n \times 2^n$ matrix.

d. Take the two orthonormal families $\widetilde{\boldsymbol{H}}_{2^n}$ and $\{\sqrt{2^n}\mathbf{1}_{\Delta_i}(t)\}_{i=1}^{2^n}$ into a new set of functions $\{hw_i(t)\}_{i=1}^{2^n}$ by

$$\begin{pmatrix} hw_1(t) \\ hw_2(t) \\ \vdots \\ hw_{2^n}(t) \end{pmatrix} = \widetilde{\boldsymbol{H}}_{2^n}^\top \begin{pmatrix} \sqrt{2^n}\mathbf{1}_{\Delta_1}(t) \\ \sqrt{2^n}\mathbf{1}_{\Delta_2}(t) \\ \vdots \\ \sqrt{2^n}\mathbf{1}_{\Delta_{2^n}}(t) \end{pmatrix} \tag{6}$$

Plot the functions $\{hw_i(t)\}_{i=1}^{2^n}$ for $n = 2, ..., 6$.

e. Implement Haar matrices $\hat{\boldsymbol{H}}_{2^n}$ as defined in the theory part in Exercise 4. question f. This should be a function taking as input the level n. This function should return a $2^n \times 2^n$ matrix.

f. Take the two orthonormal families $\hat{\boldsymbol{H}}_{2^n}$ and $\{\sqrt{2^n}\mathbf{1}_{\Delta_i}(t)\}_{i=1}^{2^n}$ into a new set of functions $\{ha_i(t)\}_{i=1}^{2^n}$ by

$$\begin{pmatrix} ha_1(t) \\ ha_2(t) \\ \vdots \\ ha_{2^n}(t) \end{pmatrix} = \hat{\boldsymbol{H}}_{2^n}^\top \begin{pmatrix} \sqrt{2^n}\mathbf{1}_{\Delta_1}(t) \\ \sqrt{2^n}\mathbf{1}_{\Delta_2}(t) \\ \vdots \\ \sqrt{2^n}\mathbf{1}_{\Delta_{2^n}}(t) \end{pmatrix} \tag{7}$$

Plot the functions $\{ha_i(t)\}_{i=1}^{2^n}$ for $n = 2, ..., 6$.

g. Given $t \in [-4, 5]$, consider a function

$$\phi(t) = t \exp(t). \tag{8}$$

Consider $n = 2$, what are the best k-term approximation of $\phi(t)$ for $k = 1, ..., 2^n$ in each basis? Present the results on a graph. What are the corresponding MSE errors?