Parallax Determination for an Asteroid

James. R. Graham

2014/10/13

1 Orbital motion

This section describes the motion of two point masses orbiting under the influence of Newtonian gravity and governed by Newton's laws of motion. Consider the orbital motion of masses m_1 and m_2, located at $\mathbf{r}_1$ and $\mathbf{r}_2$ with respect to the center of mass of the system. If no external forces act, then by conservation of momentum the center of mass is fixed, i.e.,

$$\mathbf{r}_{CM} = \left(m_1\mathbf{r}_1 + m_2\mathbf{r}_2\right)/\left(m_1 + m_2\right) = 0 \ , \tag{1}$$

and $\mathbf{r}_1 = -m_2\mathbf{r}/\left(m_1 + m_2\right)$ and $\mathbf{r}_2 = m_1\mathbf{r}/\left(m_1 + m_2\right)$, where $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ is the vector jointing the two objects.

Each mass experiences the mutual gravitational force of the other, and the relative motion of the two is described by

$$\ddot{\mathbf{r}} = -k^2 \frac{\mathbf{r}}{r^3} \ , \tag{2}$$

where dots denote time derivatives. The constant $k^2 = G(m_1+m_2)$ and G is Newton's constant. In this application we consider the case where m_1 is the mass of the sun and $m_1 >> m_2$.

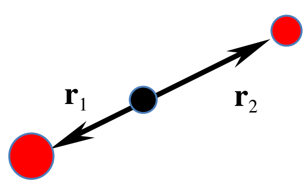

Figure 1: Two masses (red circles) orbit about the center of mass (black dot) at positions defined by $\mathbf{r}_1$ and $\mathbf{r}_2$.

We use units where mass, length, and time are measured in solar masses, AU, and days, respectively and $k = \sqrt{GM_\odot} = 0.017\,202\,098\,950$ AU$^{3/2}$ d^{-1} (the Gaussian gravitational constant.) In the Gaussian system of units the orbital period (in days) is given by Kepler's third law stated as $p = 2\pi\, a^{3/2}/k$, where a is the semimajor axis (in AU).

1.1 Heliocentric and geocentric coordinates

Let $\mathbf{R} = (X_{eq}, Y_{eq}, Z_{eq})$ be the position[1] vector of the Earth relative to the Sun (see Figure 2) and let the geocentric position vector of the asteroid be $\rho\mathbf{s}$, then

$$\mathbf{r} = \mathbf{R} + \rho\mathbf{s} \ , \tag{3}$$

[1] The JPL HORIZONS ephemeris lists the Cartesian components of the sun-earth vector $\mathbf{R}$ in either ecliptic or equatorial coordinates. Be sure to choose a consistent coordinate system or use Eqs. (16) or (18) transform to the correct frame.

where **s** is the unit vector from the earth towards the asteroid.

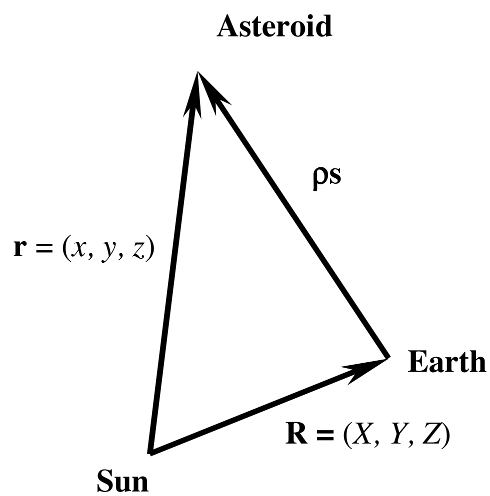

Figure 2: The relative position of the sun, earth (observer), and the target asteroid.

From the components of Eq. (3) in the equatorial coordinate system reference frame we have

$$\rho\mathbf{s} = \begin{pmatrix} x_{eq} - X_{eq} \\ y_{eq} - Y_{eq} \\ z_{eq} - Z_{eq} \end{pmatrix} = \rho \begin{pmatrix} \cos\alpha\cos\delta \\ \sin\alpha\cos\delta \\ \sin\delta \end{pmatrix} \tag{4}$$

(compare with Eq. (14)). Geocentric equatorial coordinates are found using inverse trig functions from

$$\tan\alpha = \frac{y_{eq} - Y_{eq}}{x_{eq} - X_{eq}} \tag{5}$$

and

$$\sin\delta = \frac{z_{eq} - Z_{eq}}{\rho} \tag{6}$$

where

$$\rho^2 = \left(x_{eq} - X_{eq}\right)^2 + \left(y_{eq} - Y_{eq}\right)^2 + \left(z_{eq} - Z_{eq}\right)^2 . \tag{7}$$

In applications like Eq. (5) be sure to use the two-argument function `numpy.arctan2(y,x)` to get the sign of the angles correct. The function `numpy.arctan2` returns an angle in the range –π to π, whereas, by convention, the right ascension is in the range 0 to 2π.

1.2 Measuring Parallax

The distance to a solar system body can be determined from measurements of the object's position on the celestial sphere, Newtonian dynamics, and knowledge of the earth's orbit about the sun. If **R** and **r** are the position vectors of the Earth and the target body relative to the sun (see Figure 2), then equations of motion are (compare with Eq. (2))

$$\ddot{\mathbf{r}} = -k^2 \frac{\mathbf{r}}{r^3} \quad \text{and} \quad \ddot{\mathbf{R}} = -k^2 \frac{\mathbf{R}}{R^3}.$$

Our equations neglect the mass of the smaller body in both cases. We assume that the earth-sun vector, **R**, is known; our astrometric data are the measurements of **s**, the unit vector from the Earth to the target body.

Differentiating Eq. (3) with respect to time gives the orbital velocity with respect to the sun

$$\dot{\mathbf{r}} = \dot{\mathbf{R}} + \rho\dot{\mathbf{s}} + \dot{\rho}\mathbf{s}\ , \tag{8}$$

and differentiating again gives the acceleration

$$\ddot{\mathbf{r}} = \ddot{\mathbf{R}} + \ddot{\rho}\mathbf{s} + 2\dot{\rho}\dot{\mathbf{s}} + \rho\ddot{\mathbf{s}}.$$

Substituting into the equations of motion and eliminating **r** leads to

$$\mathbf{s}\left(\ddot{\rho} + k^2 \frac{\rho}{r^3}\right) + 2\dot{\rho}\dot{\mathbf{s}} + \rho\ddot{\mathbf{s}} = k^2\mathbf{R}\left(\frac{1}{R^3} - \frac{1}{r^3}\right).$$

To eliminate the first term take the cross product with **s** and then the dot product[2] with $d\mathbf{s}/dt$

$$\rho = k^2\left(\frac{1}{R^3} - \frac{1}{r^3}\right)\frac{\dot{\mathbf{s}}\cdot(\mathbf{R}\times\mathbf{s})}{\dot{\mathbf{s}}\cdot(\ddot{\mathbf{s}}\times\mathbf{s})}. \tag{9}$$

Together with

$$r^2 = \rho^2 + R^2 + 2\rho\mathbf{R}\cdot\mathbf{s}\ , \tag{10}$$

which is derived by squaring Eq. (3) we can solve iteratively for ρ, given an initial guess for r.

1.3 Finding unit vectors and their time derivatives

To find the parallax we need to measure **s** and its first and second time derivatives. The Cartesian components of **s** are found from Eq. (4). To measure a position needs a measurement at one epoch, to measure a velocity requires two epochs, and to measure an acceleration requires three measurements, say at t_1, t_2, and t_3. The velocity and acceleration are expressed using Taylor series approximations for the observed unit vectors $\mathbf{s}_1$, $\mathbf{s}_2$, and $\mathbf{s}_3$ and solving the resultant simultaneous equations for the time derivatives. Thus, if $\tau_1 = t_2 - t_1$ and $\tau_3 = t_3 - t_2$ then

[2] Vector triple products can be computed in Python using `numpy.linalg.det()`.

$$\begin{aligned}\mathbf{s}_1 &= \mathbf{s}_2 - \tau_1\dot{\mathbf{s}}_2 + \frac{1}{2}\tau_1^2\ddot{\mathbf{s}}_2 \\ \mathbf{s}_3 &= \mathbf{s}_2 + \tau_3\dot{\mathbf{s}}_2 + \frac{1}{2}\tau_3^2\ddot{\mathbf{s}}_2\end{aligned},$$

yielding

$$\begin{aligned}\dot{\mathbf{s}}_2 &= \frac{\tau_3\left(\mathbf{s}_2 - \mathbf{s}_1\right)}{\tau_1\left(\tau_1 + \tau_3\right)} + \frac{\tau_1\left(\mathbf{s}_3 - \mathbf{s}_2\right)}{\tau_3\left(\tau_1 + \tau_3\right)} \\ \ddot{\mathbf{s}}_2 &= \frac{2\left(\mathbf{s}_3 - \mathbf{s}_2\right)}{\tau_3\left(\tau_1 + \tau_3\right)} - \frac{2\left(\mathbf{s}_2 - \mathbf{s}_1\right)}{\tau_1\left(\tau_1 + \tau_3\right)}.\end{aligned} \tag{11}$$

2 Appendix: Ecliptic & equatorial coordinates

The most convenient frame of reference for describing orbital motion is the ecliptic frame. The x-axis direction corresponds to the line defined by the intersection of the celestial equator and the orbital plane of the earth (the ecliptic plane). The perpendicular to the ecliptic plane defines the z-axis. The y-axis forms a right-handed set with x and z thus defined. The positive x direction is defined by the earth-sun direction when the sun appears to cross the celestial equator at the Vernal equinox.

From inspection of Figure 3, the conversion between polar ecliptic and Cartesian ecliptic coordinates is given by

$$\begin{aligned} x &= \cos\lambda\cos\beta \\ y &= \sin\lambda\cos\beta \;. \\ z &= \sin\beta \end{aligned} \tag{12}$$

Note that the components of the unit vector **s** in the ecliptic Cartesian system in §1.2 are

$$\mathbf{s} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{13}$$

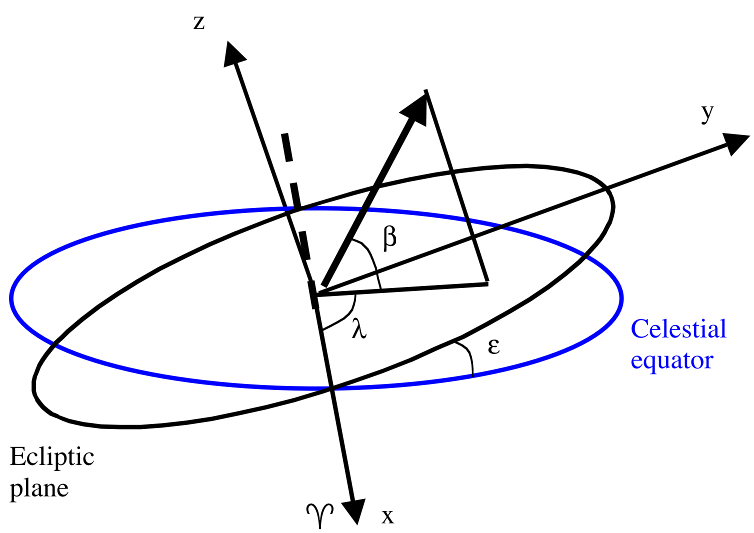

Figure 3: Ecliptic coordinates. The ecliptic represents the orbital plane of the earth about the sun, and the equator is the celestial equator. The angles λ and β are ecliptic longitude and latitude, respectively. The x-axis points towards ♈, the vernal equinox, and the *z*-axis is the ecliptic pole. The obliquity of the ecliptic or the angle between the celestial equator and the ecliptic plane is ε = +23°.43929111 for equinox J2000.

The coordinates of astronomical objects are typically measured by the angles known as right ascension, α, and declination, δ, in the equatorial system defined by reference stars measured in the International Celestial Reference System, taken at epoch 2000, so the Cartesian equatorial coordinates are

$$
\begin{aligned}
x_{eq} &= \cos\alpha\cos\delta \\
y_{eq} &= \sin\alpha\cos\delta \\
z_{eq} &= \sin\delta
\end{aligned} \quad . \tag{14}
$$

The ecliptic and equatorial systems are related by a rotation, ε, about the x-axis, so a rotation matrix gives the transformation from equatorial to ecliptic coordinates

$$\mathbf{x} = \mathbf{T}_x(\varepsilon)\mathbf{x}_{eq} \tag{15}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & \sin\varepsilon \\ 0 & -\sin\varepsilon & \cos\varepsilon \end{pmatrix} \begin{pmatrix} x_{eq} \\ y_{eq} \\ z_{eq} \end{pmatrix}, \tag{16}$$

where $\varepsilon = 23°.43929111$ for the equinox 2000.

Your measurements are in celestial coordinates—(α, δ). Do not convert from celestial coordinates to ecliptic coordinates, (λ, β) because all you need are the components of **s** in the ecliptic frame, i.e., you first compute (x_{eq}, y_{eq}, z_{eq}) from (α, δ) using Eq. (14), and then use Eq. (16) to find the components of **s** in the ecliptic Cartesian frame—(x, y, z).

Recall that the rotation matrices are orthogonal matrices, i.e., $\mathbf{A}^T\mathbf{A}=1$, and the coordinate transformation from ecliptic coordinates to equatorial coordinates is

$$\mathbf{T}_x^T(\varepsilon)\mathbf{x} = \mathbf{T}_x^T(\varepsilon)\mathbf{T}_x(\varepsilon)\mathbf{x}_{eq} = \mathbf{x}_{eq}, \tag{17}$$

so that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_{eq} \\ y_{eq} \\ z_{eq} \end{pmatrix} \tag{18}$$

and we can find (α, δ) from Eq. (14)

$$
\begin{aligned}
\tan\alpha &= y_{eq}/x_{eq} \\
\sin\delta &= z_{eq}
\end{aligned} \quad . \tag{19}
$$

Remember that α is defined in the range 0–2π, but inverse trig functions typically give the angle in the range $-\pi$ to π.

3 Appendix: The JPL Horizons Ephemeris

Ephemeris information can be generated either using the HORIZONS web interface, or you can send an email to horizons@ssd.jpl.nasa.gov with subject set to `JOB`. The example shown below requests the (X, Y, Z) position vector once a day in the heliocentric, ecliptic coordinate system. The cryptic keywords `COMMAND= '399'` defines the target as the earth, and `CENTER='500@10'` sets the sun as the origin. This request by default also generates the velocity components (V_X, V_Y, V_Z).

```
From: James Graham <jrg@berkeley.edu>
Date: November 1, 2012 10:12:52 PM EDT
To: Horizons System Ephemeris <horizons@ssd.jpl.nasa.gov>
Subject: JOB

!$$SOF
! Comments start with an exclamation point. Don't
! delete the magic start and end strings
EMAIL_ADDR = 'jrg@berkeley.edu'
! Add the edmail address you want the response sent to
COMMAND    = '399'
! Object 399 is the earth
OBJ_DATA   = 'NO'
! Don't print the summary data for the earth
TABLE_TYPE = 'VECTORS'
! Return (X,Y,Z) and (VX,VY,VZ)
REF_PLANE  = 'ECLIPTIC'
! Ecliptic coordinates
MAKE_EPHEM = 'YES'
! Return the computation
CENTER     = '500@10'
! Coordinate systems is centered on the sun
START_TIME = '2012-AUG-24 0:00'
STOP_TIME  = '2012-AUG-26 0:00'
STEP_SIZE  = '1 day'
! Start, stop, and interval
REF_SYSTEM = 'J2000'
! Equinox is 2000.0
VEC_LABELS = 'NO'
! Don't print X=, Y=, Z=, &c.
OUT_UNITS  = 'AU-D'
! Use AU and days instead of km and km/s
!$$EOF
```

4 References

The treatment is based on Chs. 6 & 7 of “Spherical Astronomy”, Robin M. Green, Cambridge, 1985. The main difference is that derivations using spherical trigonometry have been replaced using Cartesian coordinates and rotation matrices (yeah!).