Quaternions: Curbing the limitations of rotational algebra

To what extent is Quaternion geometry capable of replacing Euler angles in 3-D modeling?

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Abstract

Initially, I began this exploration to investigate vectors and how they are used in 3-D animation and movie-making, but as I delved deeper into the world of modern geometry, I came across the dilemma programmers faced of having to transition to more effective and efficient methods of 3-D programming. The dilemma introduced me to the concept of how simple vector geometry can be used to model rotations in 3-D space, which was a concept I had never explored in-class or ever considered in my life.

This past decade, leading changes in field of Mathematics and Applied Mathematics have largely and most significantly dealt with replacing crude methods of modeling 3-dimensional Euclidean Space for more versatile and useful methods. For example, most notably, the use of Euler Angles has become gradually outdated in computer programming and graphic design, and many are switching over to 4-D vectors called quaternions. With the sheer pace of advancement in STEM fields, mathematical models are in dire need to keep up, and quaternions may be the solution. This Extended Essay will explore the topic of replacing Euler Angles with Quaternions and thereby shedding past methods of geometry for more modern and sophisticated ones. The aim of this exploration is to determine:

To what extent is Quaternion geometry capable of replacing Euler angles in 3-D modeling?

This will be achieved by first explaining the derivation, and mathematical rules behind these two subjects, and then comparing how each method functions based on specific criterion such as avoiding mechanical issues (Gimbal Lock), or efficiency and effectiveness of use. Overall, a comparative and analytical method, in addition to knowledge of vector operations learned in the Math IB course will be employed to produce a conclusive answer to the research question.

Word count: 290

Introduction

Since the dawn of ancient geometry and the implementation of mathematical constructs by architectural civilizations to decipher the three-dimensional world, the field of three-dimensional geometry has been pondered, postulated, and most significantly, enhanced. When the ancient Greeks, instead of dipping their toes to test the water temperature, dived head first into an ocean of knowledge years ahead of their time and built the first pyramids of Egypt, the study of Geometry became more of a mining operation than an expedition. The inevitable discovery of the area of the triangle, the compilation of the Moscow and Rhind Papyrus containing the Greek's miracle methods on shapes and fractions, the works of Renaissance Men: Pythagoras, Plato, Zeno, Euclid, Archimedes, and modern geometers: Rene Descartes, Carl Gauss, Georg Reimann, all contributed to the molding of 3-D Geometry into its most modern and sophisticated form.

Today, the most modern form of 3-D Geometry is **non-Euclidean geometry** which largely encompasses the methods discovered by Carl Gauss and Georg Reimann during the transitional period from the use of classical geometry to modern geometry methods (Heinrich, 2016). During this shift from the purely concrete, compass-and-pen methods of modeling shapes in Euclidean plane to the modern methods of linear algebra, conceptual Cartesian co-ordinates and equations, the axiomatic system of Euclidean geometry became less popular.

Euclid's Parallel Postulate for plane geometry (The Verdict? Outdated)

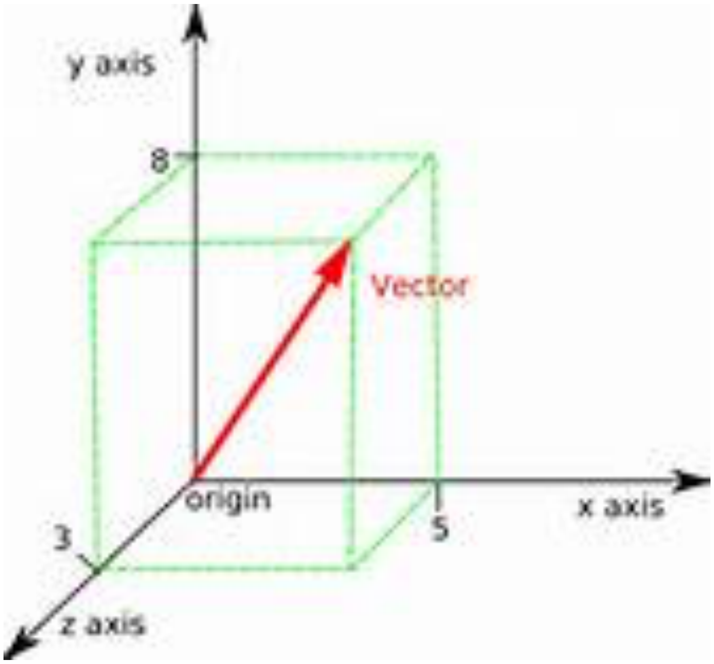
Parallel Postulate: That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Modern Rules for defining Three-Dimensional Space

1. Euclidean Space Is not all encompassing: there exists only one for each dimension, including dimensions above three. In one dimension, it is a real line. In two dimensions, the Cartesian plane, and finally in three dimensions, it is the co-ordinate space with real number co-ordinates.
2. A sequence of n numbers can be a representation of n-dimensional space. If n=3, then the three-dimensional Euclidean Space $\mathbb{R}^3$ is denoted.
3. In $\mathbb{R}^3$ Euclidean Space, three co-ordinate parameters are required to determine the location of a point: width, height, depth. All known matter in the physical universe exist in this three-parameter space.

Perhaps the most versatile tool and advantage over classical geometry that modern models give us is the ability to represent three-dimensional space in the form of **vectors**. This is only possible because in the remodified version of Euclidean Space we have today, we can define three-dimensional space as being a construct of three independent variables rather than an abstract concept. Basically, the length of a box does not depend on its width or depth, and conversely, the width or depth of a box do not depend on its length.

In linear algebra, 3-D space described by three independent vectors:
Figure 1. Vector Space Concept (http://intmstat.com/vectors/235-unit-vector.gif)



1. A vector can be imagined as an arrow that has a magnitude equal to its length, and a direction denoted by the direction of the arrow. In $\mathbb{R}^3$ it is denoted by an ordered triple, for example, Vector A= (A1, A2, A3) in the form (x,y,z).
2. The Cross Product of two Vectors (multiplication), is shown:

   A=(A1, A2, A3)   B=(B1, B2, B3)

   A*B= (A1*B1, A2*B2, A3*B3)

3. The magnitude or length of a vector is denoted by ||A||, where ||A|| is equal to the square root of the cross product of vector A with itself:

   $||A||=\sqrt{(A*A)} = \sqrt{(A1^2+A2^2+A3^2)}$

4. A unit vector is a vector of unit length: ||A||=1
5. Any vector can be made into a unit vector by dividing it by its own magnitude: $\frac{u}{|u||} = A$
6. The addition of vectors geometrically indicates extending the added vector from the head of the previous vector, and then forming a resultant vector that closes the triangle by touching both the tail of the original vector and the head of the added vector.
7. Any vector can be extended by multiplying it by a scalar quantity larger than one.
8. The cross product of two vectors forms a vector that is perpendicular to both original vectors in 3-D space. Generally, the cross product of a vector is calculated by:

   $a \times b = |a|\ |b| \sin(\theta)$, where θ is the angle between the two original vectors.

The reason vectors are so versatile and useful is because they can be used to model 3-D space in terms of equations. For example, many Canadian High Schools teach plane geometry in the form of vector equations, which really heightens the intuitive way that mathematics can be learned.

While throughout this introduction we have explored the usage of vectors in 3-Dimensional space, the main aim of this EE investigation takes a step out of the comfortable confines of curriculum-based vectors and even 3-D space. What this exploration seeks to confirm is the potential of adding an extra dimension of rotation beyond the limits of three-dimensions and how these 4-dimensional vectors called Quaternions have become applicable in the real world through inertial motion sensing, robotics, and computer programming/graphic design. This exploration will focus on the central theme prevalent in the introduction: *To what extent have the modern world applications of Quaternions created a shift in conventional usage of linear algebra and how effectively can it replace the now archaic Euler Angles steeped in classical geometry?*

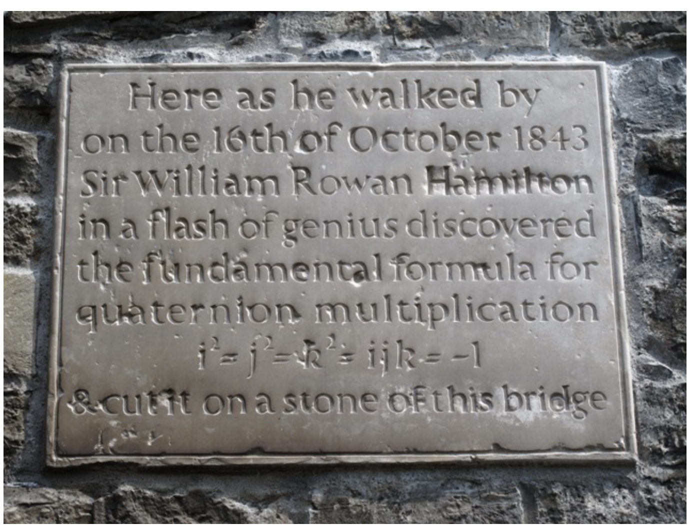
Curricular Extension: 3-D vectors to Quaternions

Curriculum-based Vector Geometry deals with translational vectors and how the 3-D world can be defined by Vector equations. Similarly, in rotational Geometry, an independent vector of rotation is required. Euler defined rotational vectors as:

The rotational relationship between any two coordinate frames can be described by a unit vector about which rotation takes place and a total rotation angle.

While a conventional vector may have 3 parameters, a 4-dimensional Vector, coined "Quaternion" by William Rowan Hamilton after he discovered it in 1843 while walking across a town bridge with his Lady, has 4 parameters (Buchmann, 2009). For example, the vector A= (A_1, A_2, A_3) in Euclidean Space has dimensions (x, y, z) denoting co-ordinates on each distinctive axes. Therefore, in order to add the rotational dimension, we require A4, giving A= (A_1, A_2, A_3, A_4).

Figure 2. Brougham Bridge plaque. Photo by *Tevian Dray*



Quaternion Properties:

1. The Unit Quaternion

   Hamilton discovered that by using a unit vector *e* to represent a unit of length 1 on each of the three x, y, and z axes, he could then associate a rotational dimension with each of those axes. This rotational dimension he discovered turned out to be $\mathrm{Sin}\frac{\theta}{2}$ for the vector components A1, A2 and A3, and $\mathrm{Cos}\frac{\theta}{2}$ for the separate scalar component A4, which is distinctive to the added rotational dimension.

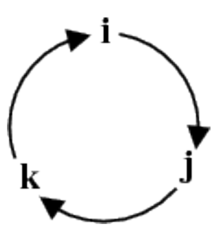
   $Q_A = (A_1, A_2, A_3, A_4)$
   $= [e*(\mathrm{Sin}\frac{\theta}{2}), \mathrm{Cos}\frac{\theta}{2}]$
   $= [e_1*\mathrm{Sin}\frac{\theta}{2}, e_2*\mathrm{Sin}\frac{\theta}{2}, e_3*\mathrm{Sin}\frac{\theta}{2}, \mathrm{Cos}\frac{\theta}{2}]$
   $= (A_1i+A_2j+A_3k+A_4)$

2. Parameters i, j, and k

   $Q_A= A_1i+A_2j+A_3k+A_4$, where $i^2 = j^2 = k^2 = ijk = -1$

   The parameters i, j, and k can be thought of as unit vectors, hence their multiplication, called the **Quaternion product**, is similar to the cross product of vectors.

   Figure 3. Clockwise indicates multiplication to yield a positive result (eg: i*j=k)
   Counter-clockwise indicates multiplication to yield a negative result (eg: j*i=-k).



3. Multiplication of Quaternions

   The multiplication of two or more quaternions indicates successive rotations. For example, $Q_n=Q_1Q_2Q_3…Q_n$ translates to *n* successive rotations. To multiply two quaternions, simply multiply each term in quaternion A by each term quaternion B.

   $Q_a*Q_b= (Q_1i+Q_2j+Q_3k+Q_4)_a*(Q_1i+Q_2j+Q_3k+Q_4)_b$
   $= (- Q_1^2+ Q_1 Q_2k-Q_1Q_3j+ Q_1 Q_4i- Q_1 Q_2k- Q_2^2+ Q_2 Q_3i+ Q_2 Q_4j+ Q_3 Q_1j- Q_3 Q_2i+$
   $- Q_3^2+ Q_3 Q_4k+ Q_1 Q_4i+ Q_2 Q_4j+ Q_3 Q_4k+Q_4^2)$
   $= (2 Q_1 Q_4i+2 Q_2 Q_4j+2 Q_3 Q_4k- Q_1^2- Q_2^2- Q_3^2 +Q_4^2)$
   $= (2 Q_1 Q_4, 2Q_2 Q_4, 2Q_3 Q_4, - Q_1^2- Q_2^2- Q_3^2 +Q_4^2)$

4. Commutativity and Associativity

Almost intuitively, we understand that vectors are both commutative and associative because vector operations lie within the fields of real and complex axioms. For example, the multiplication of vector A and Vector B yields the same result as the multiplication of Vector B and Vector A. If you reverse the order of multiplication, the product remains unchanged. However, for quaternions, because the imaginary numbers i, j, and k have such unique parameters, and follow the multiplication rules described in figure 2, the multiplication of quaternions is not commutative.

$Q_A*Q_B \neq Q_B*Q_A$

$(Q_A*Q_B)*Q_C= Q_A*(Q_B*Q_C)$

5. Quaternion Normalization

For a quaternion to represent a rotation, it must have a magnitude of 1.0 in other words, any quaternion that represents a rotation must be **normalized.** Dividing a quaternion by its magnitude, similar to dividing a vector by its magnitude, will yield a unit quaternion.

$Q_{normal}=(Q_1,Q_2,Q_3,Q_4)/(\sqrt{\ } Q_1^2+ Q_2^2+ Q_3^2+ Q_4^2)$

6. Quaternion multiplicative identity

As with vectors, all quaternions have a conjugate, and this is represented by:

$Q^{-1}=Q^*=(-Q_1i-Q_2j-Q_3k+Q_4)$

Next, the quaternion identity is a zero angle rotation, denoted by $Q_4=1$

$Q_{Identity}=(0,0,0,1)$

Hence,

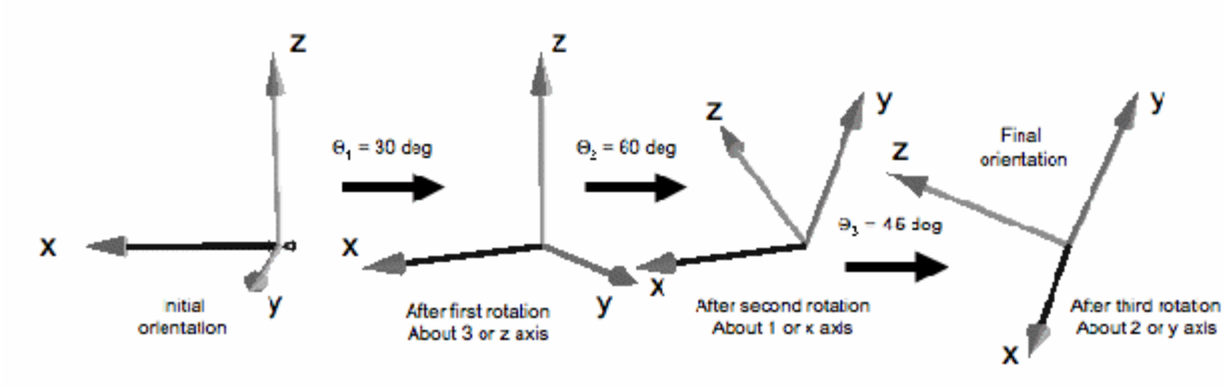
$QxQ^*= Q^*xQ= Q_{Identity}=(0,0,0,1)$

These are the only multiplicative operations that are commutative.

7. Co-ordinate frames (also known as inertial frames when referring to rotational mechanics of objects in 3-D space)

Quaternion operations aren't just limited to rotating vectors; they can also rotate the frame of reference that vector is in. By denoting a co-ordinate frame with a quaternion, and then multiplying it by a rotational quaternion, we can obtain a resultant and rotated co-ordinate frame. Generally, if a quaternion can be thought of as a rotational operation on a vector relative to a fixed frame of reference, then the inverse is also true. Relative to a fixed vector in space, a frame of reference can also rotate using quaternions.

Figure 4. Rotating a co-ordinate frame (http://www.euclideanspace.com/maths/geometry/rotations/conversions/quaternionToEuler/nhughes/quat_2_euler_64d9275a.gif)



Suppose there are two co-ordinate frames, Q_A and Q_B, where Q_B is the rotational result of Q_A. Hence, we know that quaternion A was multiplied by a rotational quaternion Q_{AB} and thus rotated to position B.

$Q_B = Q_A Q_{AB}$

Multiplying both sides by the conjugate of Q_A,

$Q_A^{-1} Q_B = Q_A^{-1} Q_A Q_{AB}$

$Q_{AB} = Q_A^{-1} Q_B$, where Q_{AB} is the rotational quaternion.

8. Vector Transformations using quaternions

Now that we have established how rotating a vector using a quaternion can also be thought of as rotating the co-ordinate frame of that vector, we can transform vectors directly.

Problem: Suppose a space-craft was facing directly skyward (90 degrees z-axis) according to Earth's inertial frame of reference, where the z- axis is the Earth's rotational axis, and axes x, y, and z are mutually perpendicular. How can the space-craft be rotated 90 degrees about the z-axis in any direction?

First, we start off with the unit vector denoting the space-craft (this also ensures that the resultant quaternion is normalized):

e= (0,0,1)

Then, applying the rotational quaternion,

$Q_{spacecraft}$= (A_1, A_2, A_3, A_4)
= [e*(Sin$\frac{\theta}{2}$), Cos$\frac{\theta}{2}$]
= [e_1*Sin$\frac{\theta}{2}$, e_2*Sin$\frac{\theta}{2}$, e_3*Sin$\frac{\theta}{2}$, Cos$\frac{\theta}{2}$]

Substituting $\theta = 90$ degrees,

$Q_{spacecraft}$= (0*sin (45), 0*sin(45), 1*sin(45), cos(45))
= (0, 0, 0.707107, 0.707107)

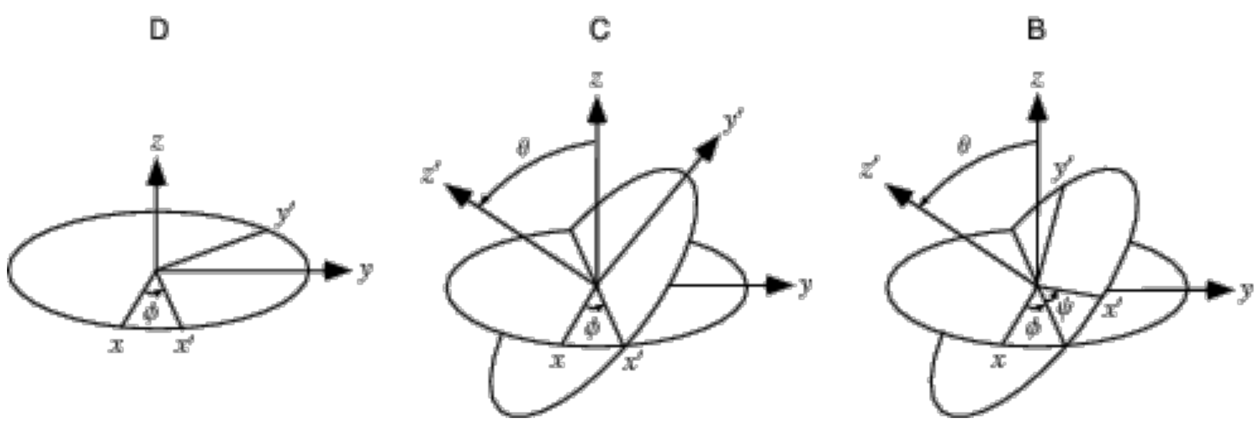
These co-ordinates describe precisely how the co-ordinate frame, in this case, the Earth-Centered Frame was rotated. We just rotated the Earth!

Conclusively, we have examined the various properties of quaternions and determined, most significantly, that quaternions are non-commutative, they must be normalized to denote a rotation, and they can represent a rotational transformation to a co-ordinate or inertial frame of reference. Unlike vectors, quaternions have a must wider range of mathematical possibilities because they are not restricted to a limited set of 3 parameters, and rather than being a descriptor of 3-D space, the actively reshape and transform it. Overall, the real world applications of quaternions, such as determining the orbital position of satellites, rotating inertial frames of spacecraft and modelling accurate astronomy, all contribute to the modern status of quaternions as a convenient tool in 3-D mechanics and mathematics.

Euler Angles in Three-Dimensional linear algebra

As opposed to using the vector-based and often less intuitive quaternion methods of rotation in 3-D space, we may take a step back into Euler's realm of classical mechanics and simplify the 3-D model to 3 parameters, pitch, roll and yaw. These three angles are defined by **Euler's Rotation Theorem** as rotational **matrices.** That means any rotation can be described by 3 angles, B, C, and D which are in matrix form.

Figure 5. Rotational matrices D, C, and B (Image from *Wolfram MathWorld*)



Matrix A: General rotation

$$\mathbf{A} \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Matrix D Yaw Rotation: Rotation about the z axis, produces new co-ordinate axis x', y'

$$\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: the number 1 in the last column of the last row indicates the z-axis being rotated. Generally, the first column represents the x-axis, and the second column represents the y-axis.

Matrix C Roll/Pitch Rotation: Rotation about the x-axis (in this case the x' axis), produces new z' axis

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

Matrix B Yaw: Rotation about the z' axis, also shifts the position of x'

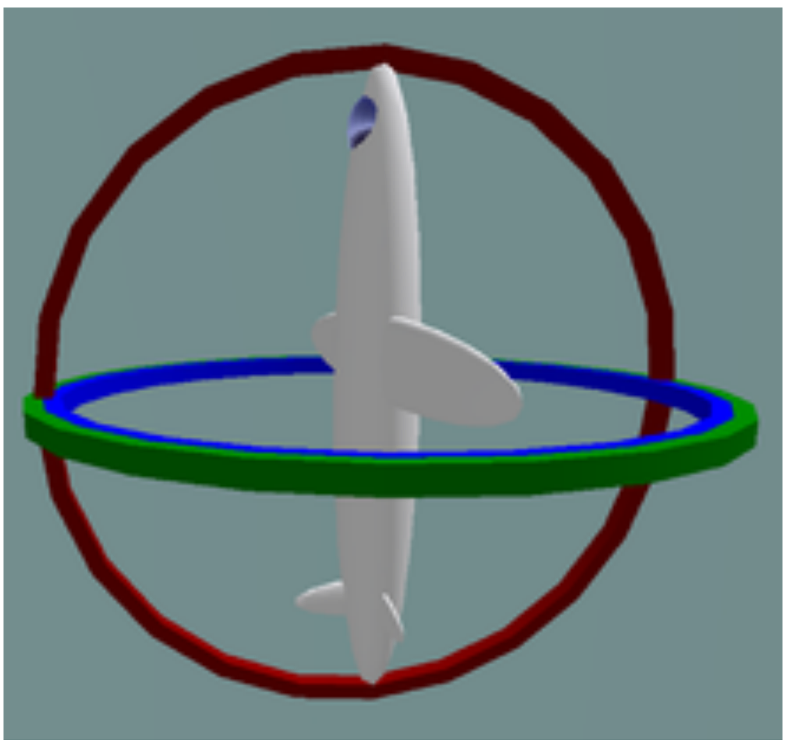
$$\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

Gimbal Lock

"Gimbal lock occurs when the orientation of the sensor cannot be uniquely represented using Euler Angles." (CH robotics, 2015)

Already, through our investigation on Euler Angles, we can see that this method has many limitations. For example, once one of the angles reaches 90 degrees, and 2 axes coincide, those two axes will share a parameter of rotation. In other words, when the pitch angle is 90 degrees, both yaw and roll will cause the frame of reference to rotate the exact same way. This becomes a problem in the field of robotics where rotational mechanics dictate the movement of mechanical objects because with Gimbal Lock, one of the three degrees of freedom is lost. The three-gimbal mechanism, shown in figure 5, is locked when two of the axes are driven into a parallel configuration, "locking" the system and essentially making it two-dimensional. The lack of a gimbal to accommodate a one-axis rotation becomes the major issue.

Figure 6. Gimbal apparatus
(https://en.wikipedia.org/wiki/Gimbal_lock#/media/File:Gimbal_lock.png)



Consider figure 5: before, the plane's pitch, roll and yaw (Euler angles) all had angles equal to zero, meaning the three gimbal axes dictating its rotation were mutually perpendicular. However once the aircraft pitches 90 degrees, yaw can no longer change.

Loss of a degree of freedom (represented using Euler angles)

Now, suppose we have a general rotation R=product of rotational matrices.

Based on Euler's rotational theorem,

$$R=\begin{bmatrix}1 & 0 & 0\\ 0 & cos\emptyset & -sin\emptyset\\ 0 & sin\emptyset & cos\emptyset\end{bmatrix}*\begin{bmatrix}cos\beta & 0 & sin\beta\\ 0 & 1 & 0\\ -sin\beta & 0 & cos\beta\end{bmatrix}*\begin{bmatrix}cos\gamma & -sin\gamma & 0\\ sin\gamma & cos\gamma & 0\\ 0 & 0 & 1\end{bmatrix}$$

When we set the pitch angle β to 90 degrees, knowing that cos (90) =0 and sin (90) =1,

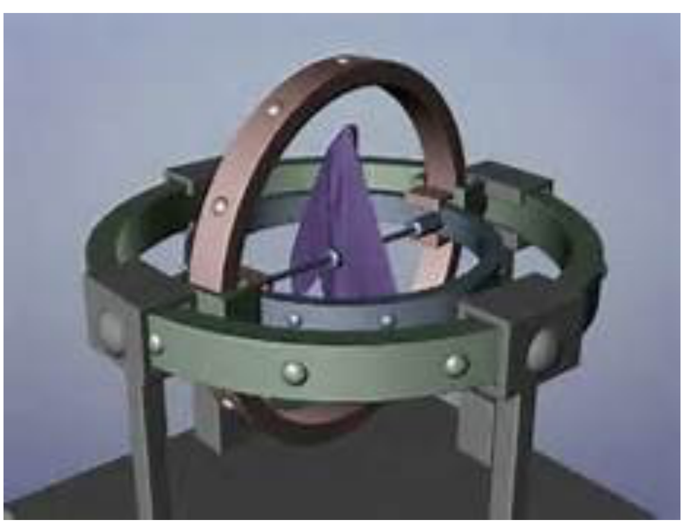
$$R==\begin{bmatrix}1 & 0 & 0\\ 0 & cos\emptyset & -sin\emptyset\\ 0 & sin\emptyset & cos\emptyset\end{bmatrix}*\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ -1 & 0 & 0\end{bmatrix}*\begin{bmatrix}cos\gamma & -sin\gamma & 0\\ sin\gamma & cos\gamma & 0\\ 0 & 0 & 1\end{bmatrix}$$

Hence with matrix multiplication (Dot product with 1 row x 1 column and so on)

$$R=\begin{bmatrix}0 & 0 & 1\\ sin\emptyset & cos\emptyset & 0\\ -cos\emptyset & sin\emptyset & 0\end{bmatrix}*\begin{bmatrix}cos\gamma & -sin\gamma & 0\\ sin\gamma & cos\gamma & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0 & 1\\ \sin(\emptyset+\gamma) & \cos(\emptyset+\gamma) & 0\\ -\cos(\emptyset+\gamma) & \sin(\emptyset+\gamma) & 0\end{bmatrix}$$

From the resultant matrix, we can see that while the angle of rotation changes $(\emptyset+\gamma)$, the rotational axis stays in the z-axis direction. The last column and the first row of the matrix will not be altered no matter what angles are placed in terms of $(\emptyset+\gamma)$, because the z-axis would no longer be affected by rotations.

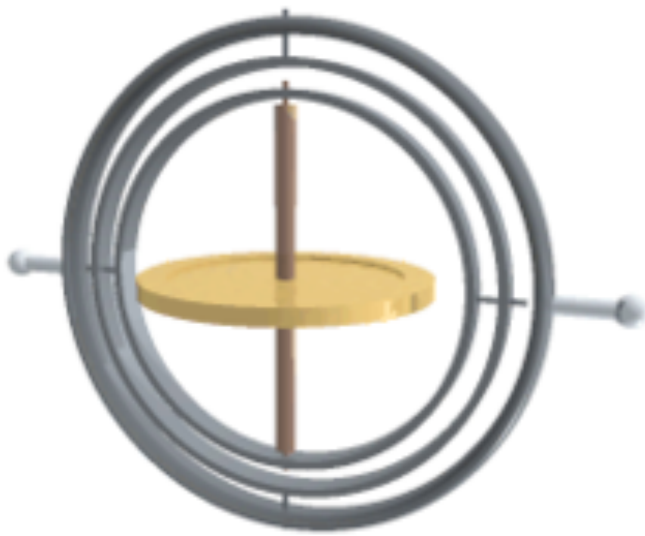
Figure 7. Another Gimbal Lock
(http://media-cache-ak0.pinimg.com/736x/ac/ff/3a/acff3a37a50ae4e90ea19c7f28229b0a.jpg)



Solution

There are many solutions to Gimbal lock, for example, one of them is overcoming the "locked" state by adding a fourth gimbal, which can compensate and replace any of the locked rotational axes. This is shown in Figure 8. Another solution is to manually move one of the gimbals to an arbitrary position every-time one reaches 90 degrees, but this quickly becomes impossible as we deal with real-world mechanics and functional machines instead of models.

Figure 8. 4-Gimbal apparatus (solves gimbal lock, but requires 4^{th} gimbal to be in perpetual motion) (https://en.wikipedia.org/wiki/Gimbal_lock#/media/File:Gimbal_lock_still_occurs_with_4_axis.png) (for above image)



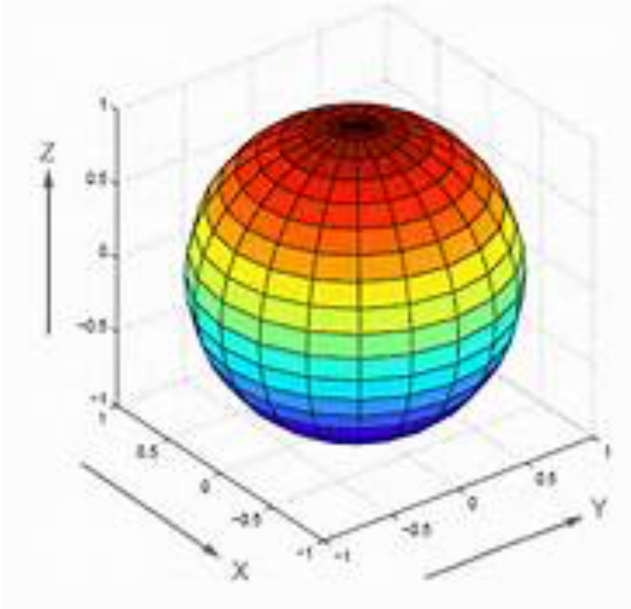
Ultimately, the desired solution produced by modern-day practitioners has been to avoid the use of gimbals entirely, and instead mounting inertial sensors directly to the vehicle such that these sensors can judge the position of the vehicle and adjust its inertial frame using accelerometers. These inertial sensors are programmed using quaternions because they do not have the same limitations as Euler angles and can fairly easily compensate for Gimbal lock. Generally, we can think of Euler angles as being the outdated, old-fashioned and crude method of rotational mechanics, while quaternions are a more flexible, versatile, and all-encompassing tool.

Comparative Analysis

Issue/criterion 1: Geometric representation

Answer: Quaternion Interpolation tends to win out over Euler angles in terms of producing more spatially accurate and understandable representations and descriptions of 3-D space. For example, in the previous space-craft example, the rotational quaternion yielded results: (0, 0, 0.707107, 0.707107), which is a direct point on the 3-dimensional Unit Sphere (a mapping of the unit quaternion).

Figure 9. Unit Quaternion Sphere (Unit sphere in 4-D)



Note: A quaternion can be defined as a point on the 4-d unit sphere, and interpolating rotations corresponds to the curves of the sphere.

This means that if there were a series of rotations, the quaternions on the curve of the sphere could be used to give each rotation a definition. This is generally used in Spherical Linear Interpolation or SLERP method which is used to find the shortest distance between two points on a sphere. The practical applications extend to mapping out the distance between two cities, such as San Francisco and London, considering the curvature of the Earth's surface.

Issue/criterion 2: Avoiding Gimbal Lock

Answer: Quaternions win over Euler angles in terms of avoiding Gimbal lock because Gimbal Lock is caused by the excessive quantity of variables and parameters present in the roll, pitch, and yaw of Euler's Rotational Theorem. Quaternions only need one defining variable. Matrix multiplication is not needed, simply a rotational quaternion with complex number parameters. Rotational angles can also be easily incorporated into a unit quaternion, whereas the matrices of Euler angles have over 9 matrix variables to deal with. Moreover, Gimbal Lock occurs because the dimensions of rotation provided by Euler angles are not homeomorphic at every point, meaning at some points, the degrees of freedom drop below 3, which is why axes get "locked". There are points where Euler Angles cannot map every change in the target space, which is three-dimensional Euclidean space.

In contrast, quaternions are topologically smooth because they are not shaped in a 3-torus like Euler angles. When mapping 3-dimensional space, the 3-sphere leaves no holes or trivialities and is therefore unable to cause Gimbal Lock.

Issue/criterion 3: Intuitive understanding

Answer: Euler angles are more simple and intuitive. They are packaged in a way such that each parameter is independent of the rest and the process of multiplying matrices is very much a linear, step-by-step operation. Furthermore, the interplay and harmony between linear co-ordinates and angular co-ordinates, like in Polar co-ordinates, makes Euler Angles very intuitive.

All translations can be described using x, y and z, and as successive and consecutive linear movements along the three perpendicular axes. Similarly, rotations can be described using three angle parameters, each a successive rotation around a perpendicular adjacent axis.

Overall, Euler angles are more intuitive and simple than Quaternions.

Conclusion

From ancient geometry and classical methods of shape-shifting to modern methods steeped in post-Euclidean theories, men have for centuries forayed into the inconceivably vast, yet discoverable 3-dimensional world. This world among many in Euclidean space holds the very people who have tried to search for its ends, and their journey to discover truth. In fact, the stranded echoes of past attempts lost in the boundlessness of Euclidean 3-D space can still be perceived today. There is, undeniably, a vast amount of uncharted territory still left to be discovered, but every exploration must end somewhere.

Every sound theory must contain the unfathomable, and cage it in a tangible set of parameters. This investigation has done just that. Initially, the exploration was centered on defining the modern understanding of Euclidean Space and the major transition that took place during Reimann's time where the archaic methods of classical geometry were largely replaced by non-Euclidean, coordinated and algebraic methods of modern geometry. This transition led to the discovery of the foundations of complex numbers, and vectors, and eventually, quaternions. As a curricular extension, quaternions are 4-dimensional vectors that denote a rotation in 3-D space using the multiplication of a unit vector with angular parameters. They are useful because they are homeomorphic, have limited restricting parameters, and satisfy the modern axioms in field theory such as associativity. They are also more useful that Euler angles because they don't cause Gimbal Lock, and produce more accurate results when mapped topologically. Hence, to answer the original research question I set out to explore: *To what extent is Quaternion geometry capable of replacing Euler angles in 3-D modeling?* Quaternion geometry is already replacing Euler angles in 3-D computer programming, inertial navigation, and graphical design because it is such a versatile and multi-purpose mathematical tool. With further research on applying vector operations to quaternions, there will be limitless potential for quaternions to be integrated into frontier fields of STEM leading world-wide research on the most modern versions of robotic and vehicle navigation, and 3-D animation in the next decade.

Word count: 3432

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