Classical Mechanics

Revision Notes

Lily Asquith

University of Sussex

December 2018

Module Information

These revision notes are designed to help you revise the material in chapters 1-14 of your textbook. There is no better way to revise than to create your own set of revision notes, so please consider using these as a basis for that.

The canvas page for this module is here.

The final grade for this course is determined by:
10% assignment 1 + 10% assignment 2 + 80% unseen examination.

In order to progress to year 2, students must achieve a pass grade of at least 40% in this module examination.

The grades received in year 1 of both BSc and MPhys degree courses carry zero weight in determining the final degree classification.

Further details on assessment may be found in the: handbook.

- Wiley+ - do the problems.
- HyperPhysics is a resource I used as an undergraduate and continue to find useful.
- Brilliant is something I was introduced to recently by an AT - looks good
- Veritassium is my favourite youTuber for physics. My other favourites are Physics Girl and Khan academy

Remember: I am an experimental particle physicist. I have no formal training in being a teacher. I am not a textbook writer. I learned this material myself in 2002 as a first year undergraduate at UCL, while raising a small child. Your level of expertise at the end of this module should be pretty much equivalent to mine, because you won't learn it again. And one day, you may find yourself teaching it.

Vectors

Components ✧ Dot Product ✧ Cross Product

Book / Wiley+ : Chapter 3

Components

A vector $\underline{\mathbf{a}}$ has both magnitude a and direction $\hat{\mathbf{a}}$, and can be expressed in terms of **components**: a_x, a_y, a_z and **unit vectors**: $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$, as :

$$\underline{\mathbf{a}} = a_x\hat{\boldsymbol{i}} + a_y\hat{\boldsymbol{j}} + a_z\hat{\boldsymbol{k}} \quad (1.1)$$

This is **unit vector notation**.

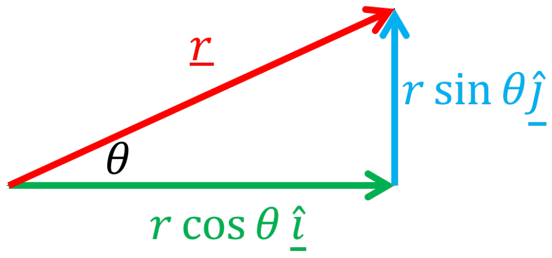


Figure 1: Components of a vector $\underline{\mathbf{r}}$ in 2D.

We can use trig to write the components in terms of angles:

The x-**component**: $a_x = a\cos\theta$

The y-**component**: $a_y = a\sin\theta$

The z-**component**: $a_z = a\sin\phi$

The **magnitude**: $|\underline{\mathbf{a}}| = a = \sqrt{a_x^2 + a_y^2 + a_z^2}$

The **angle** in (x, y) plane: $\theta = \arctan\frac{a_y}{a_x}$

The **angle**in (y, z) plane: $\phi = \arctan\frac{a_z}{a_y}$

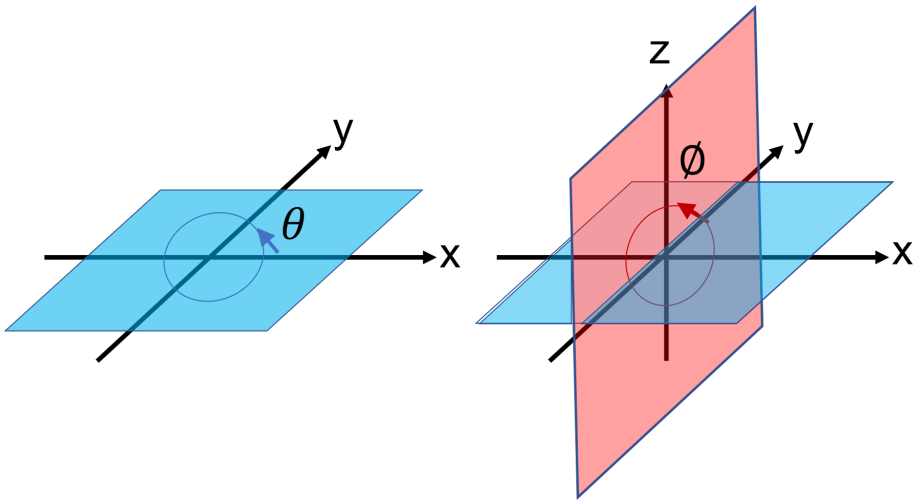


Figure 2: Visualising 3D

Dot Product

The dot product between two vectors is written

$\underline{\mathbf{a}} \cdot \underline{\mathbf{b}} = ab\cos\gamma$ **1.2**

where γ is the angle between $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$.

If $\gamma = 90^\circ \rightarrow \cos\gamma = 0 \rightarrow \underline{\mathbf{a}} \cdot \underline{\mathbf{b}} = 0$:

The dot product of vectors at right angles is zero

If $\gamma = 0^\circ \rightarrow \cos\gamma = 1 \rightarrow \underline{\mathbf{a}} \cdot \underline{\mathbf{b}} = ab$: the dot product of parallel vectors is maximum

If $\gamma = 180^\circ \rightarrow \cos\gamma = -1 \rightarrow \underline{\mathbf{a}} \cdot \underline{\mathbf{b}} = -ab$: the dot product of anti-parallel vectors is negative maximum

If we are given the vectors, we do dot products in components like this:
$\underline{\mathbf{a}} = a_x\hat{\boldsymbol{i}} + a_y\hat{\boldsymbol{j}} + a_z\hat{\boldsymbol{k}}$
$\underline{\mathbf{b}} = b_x\hat{\boldsymbol{i}} + b_y\hat{\boldsymbol{j}} + b_z\hat{\boldsymbol{k}}$

$\underline{\mathbf{a}} \cdot \underline{\mathbf{b}} = (a_x)(b_x) + (a_y)(b_y) + (a_z)(b_z)$ **1.3**

Numerical example:
$\underline{\mathbf{a}} = 3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$
$\underline{\mathbf{b}} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$

$\underline{\mathbf{a}} \cdot \underline{\mathbf{b}} = (3)(1) + (-4)(2) + (0)(1) = -3$

Note that the result of a dot product is a scalar (a number) not a vector.

We can find the angle between two vectors with $\underline{\mathbf{a}} \cdot \underline{\mathbf{b}}/ab = \cos\gamma$, where for the magnitudes a and b in the denominator we use:
$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$ and $b = \sqrt{a_x^2 + a_y^2 + a_z^2}$

Dot products are used in lots of mechanics problems, for example Work $W = \underline{\mathbf{F}} \cdot \underline{\mathbf{d}}$.

Cross Product

The cross product between two vectors is written:

$$\underline{\mathbf{a}} \times \underline{\mathbf{b}} = ab \sin\lambda\, \hat{\boldsymbol{n}} \quad (1.4)$$

where λ is the angle between $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ and

the resultant direction $\hat{\boldsymbol{n}}$ is perpendicular to both $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$.

If we are given the vectors, we do cross products in components like this:

$\underline{\mathbf{a}} = a_x\hat{\boldsymbol{i}} + a_y\hat{\boldsymbol{j}} + a_z\hat{\boldsymbol{k}}$

$\underline{\mathbf{b}} = b_x\hat{\boldsymbol{i}} + b_y\hat{\boldsymbol{j}} + b_z\hat{\boldsymbol{k}}$

$$\underline{\mathbf{a}} \times \underline{\mathbf{b}} = ((a_y)(b_z) - (a_z)(b_y))\,\hat{\boldsymbol{i}} + ((a_z)(b_x) - (a_x)(b_z))\,\hat{\boldsymbol{j}} + ((a_x)(b_y) - (a_y)(b_x))\,\hat{\boldsymbol{k}}. \quad (1.5)$$

It can be helpful to write this out in matrix format when calculating cross products:

$$\underline{\mathbf{a}} \times \underline{\mathbf{b}} = \begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix}\hat{\boldsymbol{i}} + \begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix}\hat{\boldsymbol{j}} + \begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix}\hat{\boldsymbol{k}}$$

Remember to go **cyclicly**: don't make the mistake of writing the $\hat{\boldsymbol{j}}$ component as $((a_x)(b_z) - (a_z)(b_x))\,\hat{\boldsymbol{j}}$ - this will give you the wrong answer.

Note that the result $\underline{\mathbf{c}}$ of a cross product is a vector.

Check if we got the cross product right : take the dot product of the resultant vector with either $\underline{\mathbf{a}}$ or $\underline{\mathbf{b}}$ - the result must be zero because they are perpendicular.

$$\text{If } \underline{\mathbf{c}} = \underline{\mathbf{a}} \times \underline{\mathbf{b}}, \text{ then } \underline{\mathbf{c}} \cdot \underline{\mathbf{a}} = 0 \text{ and } \underline{\mathbf{c}} \cdot \underline{\mathbf{b}} = 0 \quad (1.6)$$

Cross products are useful for rotational mechanics problems, as the cross product is used in the definition of torque $\underline{\tau} = \underline{\mathbf{r}} \times \underline{\mathbf{F}}$ and angular momentum $\underline{\mathbf{L}} = \underline{\mathbf{r}} \times \underline{\mathbf{p}}$

Sample Problems

1. Write an expression for a displacement vector $\underline{\mathbf{r}}$ which is in the x, y plane, has length 1.9 cm, and is at an angle 71° from the x-axis.

2. Vector $\underline{\alpha}$, which is directed along an x-axis, is to be added to vector $\underline{\beta}$, which has a magnitude of 7 m. The sum is a third vector that is directed along the y-axis, with a magnitude that is 3 times that of $\underline{\alpha}$. What is that magnitude of $\underline{\alpha}$?

3. A vector product $\underline{\mathbf{P}} = a\underline{\mathbf{B}} \times \underline{\mathbf{C}}$, where $a = 2$, $\underline{\mathbf{B}} = 2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}$ and $\underline{\mathbf{C}} = 4\hat{\boldsymbol{i}} - 20\hat{\boldsymbol{j}} + 12\hat{\boldsymbol{k}}$.

What is $\underline{\mathbf{C}}$ in unit vector notation if $C_x = C_y$?

4. $k = 9$, $\underline{\mathbf{A}} = 2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}$, $\underline{\mathbf{B}} = 4\hat{\boldsymbol{i}} - 20\hat{\boldsymbol{j}} + 12\hat{\boldsymbol{k}}$, and $\underline{\mathbf{C}} = \hat{\boldsymbol{i}} - 10\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$.

What is $k\underline{\mathbf{A}} \cdot (\underline{\mathbf{B}} \times \underline{\mathbf{C}})$?

Equations of motion

Derivations ✧ Rearrangements

Book / Wiley+ : Chapter 2.4

Derivations

The first equation of motion for translational motion is derived starting from the definition for :

$$\underline{\mathbf{a}} = \frac{d}{dt}\underline{\mathbf{v}} \rightarrow \int \underline{\mathbf{a}}dt = \int \frac{d}{dt}\underline{\mathbf{v}}dt \rightarrow \underline{\mathbf{a}}t + \underline{\mathbf{c}} = \underline{\mathbf{v}}$$

Note that we can write $\int \underline{\mathbf{a}}dt = \underline{\mathbf{a}}t$ because a is constant: not a function of time.

We find the constant of integration $\underline{\mathbf{c}}$ by setting $t = 0 \rightarrow \underline{\mathbf{c}} = \underline{\mathbf{v}}_0$.

This gives us $\underline{\mathbf{a}}t + \underline{\mathbf{v}}_0 = \underline{\mathbf{v}}$, which is often reorganised as

$\underline{\mathbf{v}} = \underline{\mathbf{u}} + \underline{\mathbf{a}}t$ **2.1** where we are now using $\underline{\mathbf{v}}_0 \rightarrow \underline{\mathbf{u}}$ for the **initial velocity**.

This equation contains no $\underline{\mathbf{s}}$, so we use it when $\underline{\mathbf{s}}$ is unknown and unwanted

The second equation of motion is derived starting from the definition for velocity:

$$\underline{\mathbf{v}} = \frac{d}{dt}\underline{\mathbf{s}} \rightarrow \int \underline{\mathbf{v}}dt = \int \frac{d}{dt}\underline{\mathbf{s}}dt$$

Now we have to substitute for $\underline{\mathbf{v}} = \underline{\mathbf{u}} + \underline{\mathbf{a}}t$ (suvat 1) in the LHS because unlike $\underline{\mathbf{a}}$ (which has to be constant for these equations to work) **$\underline{\mathbf{v}}$ can be a function of time**:

$$\int (\underline{\mathbf{u}} + \underline{\mathbf{a}}t)dt = \int \frac{d}{dt}\underline{\mathbf{s}}dt \rightarrow \underline{\mathbf{u}}t + \tfrac{1}{2}\underline{\mathbf{a}}t^2 + \underline{\mathbf{c}} = \underline{\mathbf{s}}$$

We find the constant of integration $\underline{\mathbf{c}}$ by setting $t = 0 \rightarrow \underline{\mathbf{c}} = \underline{\mathbf{s}}_0$

This gives us $\underline{\mathbf{u}}t + \frac{1}{2}\underline{\mathbf{a}}t^2 + \underline{\mathbf{s}}_0 = \underline{\mathbf{s}}$, which is often reorganised as

$\underline{\mathbf{s}} = \underline{\mathbf{u}}t + \frac{1}{2}\underline{\mathbf{a}}t^2$ **2.2** where we are now using $\underline{\mathbf{s}}_0 = 0$ for the initial position.

This equation contains no $\underline{\mathbf{v}}$, so we use it when $\underline{\mathbf{v}}$ is unknown and unwanted

Rearrangements

The other SUVAT equations are found from the first two, and are defined in terms of missing variables as follows:

Eliminate a by rearranging 2.1:

$a = \frac{v-u}{t}$

Substitute into 2.2:

$s = ut + \frac{1}{2}[\frac{v-u}{t}]t^2 = ut + \frac{1}{2}(v-u)t$

$s = \frac{1}{2}(v+u)t$ **2.3**. contains no a, so we use it when a is unknown

Eliminate t by rearranging 2.1:

$t = \frac{v-u}{a}$

Substitute into 2.2:

$s = u[\frac{v-u}{a}] + \frac{1}{2}a[\frac{v-u}{a}]^2$

$s = \frac{1}{2a}(v^2 - u^2)$ **2.4** contains no t, so we use it when t is unknown

Eliminate u by rearranging 2.1:

$u = v - at$

Substitute into 2.2:

$s = [v - at]t + \frac{1}{2}at^2 = vt - at^2 + \frac{1}{2}at^2$

$s = vt - \frac{1}{2}at^2$ **2.5** contains no u, so we use it when u is unknown

Note that you can rearrange these equations to find the variable you are looking for in terms of your known variables.

Problems

1.The position of a particle moving along an x-axis is given by $x = 12t^2 - 2t^3$, where x is in meters and t is in seconds. Determine:
(a) the position,

(b) the velocity, and

(c) the acceleration of the particle at $t = 4$ s

2. (a) If the maximum acceleration that is tolerable for passengers in an underground train is 1.34ms^{-2} and stations are located 806 m apart, what is the maximum speed a train can attain between stations?

(b) What is the travel time between stations?

(c) If a subway train stops for 20 s at each station, what is the maximum average speed of the train, from one start-up to the next?

3.An object falls a distance h from rest. If it travels $0.5h$ in the last 1.00 s, find
(a) the time and

(b) the height of its fall.

(c) Explain the physically unacceptable solution of the quadratic equation in t that you obtain.

4. A rock is thrown vertically upward from ground level at time $t = 0$. At $t = 1.5$ s it passes the top of a tall tower, and 1.0 s later it reaches its maximum height. What is the height of the tower?

Newton's Laws

N1 ✧ F=ma ✧ $\underline{\mathbf{F}}_{ab} = -\underline{\mathbf{F}}_{ba}$

Book / Wiley+ : Chapter 5.1, 5.3

Newton's First Law

N1: No external forces = constant velocity 3.1

Newton's first law states that an object in motion with constant velocity will remain in motion with that velocity unless an external force acts on it. The constant velocity can be zero. This seems like stating the bleeding obvious, but there are subtle ways in which we can encounter some cognitive dissonance with mechanics problems later in the course, and it helps us to come back to this.

An *external* force is one from outside the body/system for which we are claiming N1 holds. If a force can cause acceleration of a body, it is *external* to that body.

I cannot pull myself up by my hair. This is because my hand is part of the same system as my hair. I feel the pull, but I do not move. The pulling force is internal.

If we define a system as (me plus my skateboard), then I can completely disregard the forces *between* me and my skateboard (the weight of me down on the board balanced by the normal force up from the board on me). We would do this if we are solving a problem where me and my skateboard remain attached at all times.

If we define the system as (me), and I am on a skateboard, then we have to take into account the force exerted on me by the skateboard. We would do this if we are solving a problem where I become detached from my skateboard (including if I slide along it).

If a body/system changes direction from that of a *straight line*, then there must be an external force acting.

Newton's Second Law

N2: $\underline{\mathbf{F}} = m\underline{\mathbf{a}}$ **3.2**

Newton's second law states that the acceleration of an object is dependent on the net force applied to it and the mass of the object.
$\underline{\mathbf{F}}$ is the vector sum of all external forces acting on the object which has mass m.

$\underline{\mathbf{a}}$ is the resulting acceleration of the object.

We always break the vector sum of the forces down into components, like this:

$$\underline{\mathbf{F}} = \underline{\mathbf{F_x}} + \underline{\mathbf{F_y}} + \underline{\mathbf{F_z}} = F_x\hat{\boldsymbol{i}} + F_y\hat{\boldsymbol{j}} + F_z\hat{\boldsymbol{k}} = ma_x\hat{\boldsymbol{i}} + ma_y\hat{\boldsymbol{j}} + ma_z\hat{\boldsymbol{k}}$$

So if I have for example three forces acting on an object:

$$\underline{\mathbf{F_1}} = F_{1x}\hat{\boldsymbol{i}} + F_{1y}\hat{\boldsymbol{j}}; \qquad \underline{\mathbf{F_2}} = F_{2y}\hat{\boldsymbol{j}}; \qquad \underline{\mathbf{F_3}} = F_{3x}\hat{\boldsymbol{i}} + F_{3y}\hat{\boldsymbol{j}} + F_{3z}\hat{\boldsymbol{k}}$$

The vector sum of these forces
$\underline{\mathbf{F}} = \underline{\mathbf{F_1}} + \underline{\mathbf{F_2}} + \underline{\mathbf{F_3}} = (F_{1x} + F_{3x})\hat{\boldsymbol{i}} + (F_{1y} + F_{2y} + F_{3y})\hat{\boldsymbol{j}} + (F_{3z})\hat{\boldsymbol{k}}$

$\underline{\mathbf{F_x}} = (F_{1x} + F_{3x})\hat{\boldsymbol{i}}$ - there is no contribution in direction $\hat{\boldsymbol{i}}$ from $\underline{\mathbf{F_2}}$

$\underline{\mathbf{F_y}} = (F_{1y} + F_{2y} + F_{3y})\hat{\boldsymbol{j}}$ - all of the three forces have a component in $\hat{\boldsymbol{j}}$

$\underline{\mathbf{F_z}} = (F_{3z})\hat{\boldsymbol{k}}$ - only $\underline{\mathbf{F_3}}$ has a component in $\hat{\boldsymbol{k}}$

The acceleration of the body/system on which the forces are acting is then:
$\underline{\mathbf{a}} = \underline{\mathbf{F}}/m$

Newton's Third Law

N3: $\underline{\mathbf{F}}_{\mathbf{A:B}} = -\underline{\mathbf{F}}_{\mathbf{B:A}}$ **3.3**

Newton's third law states that the force acting on object B due to object A is identical in magnitude and opposite in direction to the force acting on object A due to object B:

The gravitational force acting on the moon (subscript L for Luna) due to the earth (subscript T for Terra) is $\underline{\mathbf{F}}_{T:L} = m_L\underline{\mathbf{a}}_L$; this results in an acceleration of the moon that is inversely proportional to the mass of the moon: $\underline{\mathbf{a}}_L = \underline{\mathbf{F}}_{T:L}/m_L$.

The gravitational force acting on the earth due to the moon is $\underline{\mathbf{F}}_{\mathbf{L:T}} = m_T\underline{\mathbf{a}}_T$ this results in an acceleration of the earth that is inversely proportional to the mass of the earth: $\underline{\mathbf{a}}_T = \underline{\mathbf{F}}_{L:T}/m_T$.

The **magnitude** of $\underline{\mathbf{F}}_{L:T}$ is identical to the magnitude of $\underline{\mathbf{F}}_{T:L}$, so we can write:

$F_{L:T} = F_{T:L} \rightarrow m_T a_T = m_L a_L$.

$\boxed{\text{\Large m}\ \text{\small a} = \text{\small m}\ \text{\Large a}}$.

Our intuition that the force on the moon is greater is because what we are observing is the *acceleration* of the moon, which is very noticeable because the mass of the moon is so much smaller than the mass of the earth.

Problems

1. While two forces act on it, a particle is to move at the constant velocity $\underline{\mathbf{v}} = (3ms^{-1})\hat{\boldsymbol{i}} - (2ms^{-1})\hat{\boldsymbol{j}}$. One of the forces is $\underline{\mathbf{F}} = (1N)\hat{\boldsymbol{i}} + (3N)\hat{\boldsymbol{j}} - (6N)\hat{\boldsymbol{k}}$. What is the other force?

2. A car traveling at 40 mph hits a curb. A passenger in the car moves forward a distance of 65 cm (with respect to the road) while being brought to rest by an inflated air bag. What magnitude of force (assumed constant) acts on the passenger's upper torso, which has a mass of 35 kg?

3.Holding on to a towrope moving parallel to a frictionless ski slope, a 50 kg skier is pulled up the slope, which is at an angle of 8.0° with the horizontal. What is the magnitude of the force on the skier from the rope when the initial speed of the skier is $u = 2\text{ms}^{-1}$ and

(a) the skier's speed is constant ?

(b) the skier's speed increases at a rate of 0.1 ms^{-2} ?

Forces and the work they do

Work ✧ Gravity ✧ Friction ✧ Drag ✧ Springs

Book / Wiley+ : Chapter 5.2, 6.1, 6.2,

Work

The work done by a force is equal to the dot product between the force $\underline{\mathbf{F}}$ and the resulting displacement $\underline{\mathbf{s}}$

$$W = \underline{\mathbf{F}} \cdot \underline{\mathbf{s}} = Fs \sin\theta \quad (4.1)$$

If a force acting on a body results in zero displacement of that body, then the force has done no work on the body.

When the force $\underline{\mathbf{F}}$ and displacement $\underline{\mathbf{s}}$ are in the same direction (parallel) the angle between them $\theta = 0$ and therefore $\sin\theta = 1$ and W=Fs.

Also note that when the force applied is at a right angle to the displacement (orthogonal) the angle between them $\theta = 90^\circ$ and therefore $\sin\theta = 0$ and $W = 0$. This makes sense - we cannot move an object in direction $\hat{\boldsymbol{i}}$ if the force applied has no $\hat{\boldsymbol{i}}$ component.

What if we apply a force in the direction $\hat{\boldsymbol{i}}$ and the displacement is in the opposite direction $-\hat{\boldsymbol{i}}$? You may well say 'oh come on, that is not physically possible' and I would agree. If we used our formula for work in this impossible situation, we would have $\theta = 180^\circ$, so $\sin\theta = -1$, so W = -Fd. The work is negative. Or in other words, the crate has done work on us!

Gravity

The gravitational pull of the earth on a body with mass m can be written in terms of 3.2:

$$F_g = ma_g \quad (4.2)$$

where a_g is the resulting acceleration of the body. This acceleration is denoted g and is measured to have an average value of

$$g = 9.81ms^{-2} \quad (4.3)$$

at the earth's surface.

The gravitational force between me and the earth does work, **if it results in a displacement**. When I am standing stationary on the earth surface, F_g is not doing any work. If I fall out of a window and land on the earth's surface 1.30 m below, then F_g has done work. I can say that I have done work on the earth, and the earth has done work on me.

The **work done on me by the Earth** is equal to $F_g d = mgd$ where d is the distance I have travelled - 1.30 m in this case. So the work done on me by the earth is :
$\mathrm{W} = mgd = (70.0kg)(9.81ms^2)(1.30m) = 892.71Nm = \mathbf{893\ Nm}$

The **work done on the Earth by me** is -**893 Nm**. This is interesting because it allows us to figure out the distance the earth has moved due to me and the earth (subscript T for Terra) coming together over a distance of 1.3 m:

$W = m_T gd = -893Nm$

$d = -893Nm/(m_T(9.81ms^{-2})$

Plugging in $m_T = 5.972 \times 10^{24}$ kg we get displacement of Earth:

$d = 1.52 \times 10^{-23}m$, or 15.2 *yoctometres*.

Pretty tiny distance!

Friction

Friction is a bag of frogs: we will try to keep it as simple as possible as we are not engineers.

The frictional force available between two bodies is due to the microscopic imperfections of their surfaces, and electromagnetic forces between the molecules within them. We sum up these effects by using a handy constant μ, which is the coefficient of friction between two bodies, and has measured values between (0,1).

Friction is only present when two bodies are in contact. Two bodies are *usually* in contact due to the gravitational attraction between them. So we express the frictional forces in terms of F_g.

The maximum frictional force available between two bodies is given by $\boxed{F_f = \mu F_g}$ **4.4** : the combination of the material properties μ and the force between them F_g.

Because of the complex way that materials behave, we need to break the F_f and the resulting ma into two different scenarios:
1. The behaviour (*ie* the resulting acceleration) when the body is **not sliding**: $F_s = \mu_s F_g$ where μ_s is the coefficient of **static friction**.
2. The behaviour when the body is **sliding**: $F_k = \mu_k F_g$ where μ_k is the coefficient of **kinetic friction**.

A rolling object does not slide. So the correct μ to use for rolling without sliding is μ_s.

When we apply a force to an object in direction $\hat{\boldsymbol{i}}$, the frictional force is directed in the opposite direction -$\hat{\boldsymbol{i}}$.

The static frictional force will continue to increase with (and against) the applied force, until it reaches its maximum value given by $F_f = \mu_s F_g$. If the applied force exceeds this maximum available frictional forces, the object will begin to slide.

Drag

Drag is like friction, but for a body moving in a fluid. In this course we will think about drag in terms of **air resistance** on a moving object.

The drag force on an object moving through air is is due to the **density of the air** ρ, the **cross-sectional area the object presents in its direction of motion** A, the **objects speed** v, and various complicated things including the shape and material of the object, which we will put into a **dimensionless constant** C_D (like we did for μ). This is written:

$$F_D = \frac{1}{2} C_D \rho A v^2 \quad \textbf{4.5}$$

Like the frictional force, the drag force opposes the direction of motion. This has a nice consequence when we consider a body of mass m falling to earth; as the object accelerates downwards due to F_g, the drag force will counteract this acceleration. The net force is given by:

$$\underline{\mathbf{F}} = m\underline{\mathbf{a}} = -F_g\hat{\boldsymbol{k}} + F_D\hat{\boldsymbol{k}} = (-mg + \frac{1}{2}C_D\rho A v^2)\hat{\boldsymbol{k}}$$

When the drag force upwards equals the gravitational force downwards, the body will stop accelerating and continue moving downwards at constant velocity:

$$0 = (-mg + \frac{1}{2}C_D\rho A v^2)\hat{\boldsymbol{k}}$$

$$mg = \frac{1}{2}C_D\rho A v^2\hat{\boldsymbol{k}}$$

$$v = \sqrt{\frac{2mg}{C_D\rho A}} \quad \textbf{4.6}$$

The value of v is known as the terminal velocity, because it is the velocity at which the net acceleration stops (terminates).

Springs

The spring force is written $\underline{\mathbf{F}} = -k\underline{\mathbf{s}}$ **4.7**: this holds for springs that obey **Hooke's law**. This force comes from the fact that a spring (of the right material) will oscillate around its equilibrium position forever in the absence of external forces.

Springs are interesting because the spring force is a **conservative force**, like the gravitational force. This means that **the work done in moving an object from point A to point B does not depend on the path taken to get there**.

$W = \int_{s_i}^{s_f} \underline{\mathbf{F}} \cdot \underline{\mathbf{ds}} = -\frac{1}{2}k(s_i^2 - s_f^2) = \frac{1}{2}ks_f^2$: we have chosen $s_i = 0$ to simplify the problem.

Often we write this as $W = \frac{1}{2}kx^2$: this is the one dimensional form.

Note that this is a scalar, as it is the result of a scalar (dot) product between two vectors, $\underline{\mathbf{F}}$ and displacement $\underline{\mathbf{ds}}$.

If a mass is doing work on the spring (**stretching**), we say the work done on the spring is positive, so the **work done by the spring is negative**.

If the spring is doing work on a mass (**contracting**), then we say that the **work done by the spring is positive**, so the work done on the mass is negative.

Springs are also a nice way to introduce potential energy: **all conservative forces have an associated potential energy**.

$PE = \int_{s_i}^{s_f} \underline{\mathbf{F}} \cdot \underline{\mathbf{ds}} = -\frac{1}{2}k(s_i^2 - s_f^2) = -\frac{1}{2}ks_i^2$: we have chosen $s_f = 0$ to simplify the problem.

Problems

1. If I push a crate with mass 1.00 kg a distance of 5.00 m by applying a force of 10.0 N, and the coefficient of kinetic friction between floor and crate is 0.500, how much work I have done on the crate?:

2. Explain why skydivers fall in the spread-eagled position using the equation for the drag force. Which factors in the drag force equation can change mid-flight? Which factors will definitely change mid-flight?

3.When you are trying to shift a heavy sofa, is it more effort to (a) get it sliding or (b) keep it sliding? What does this imply for the values of μ_k and μ_s?

4.Why do we choose $s_i = 0$ for work and $s_f = 0$ for PE ?

Frames of Reference

Frames ✧ Relative motion ✧ Centre of Mass

Book / Wiley+ : Chapter 4.6, 4.7, 9.1

Frames of Reference

We live on the Earth, and so our natural frame of reference is to treat the Earth as stationary, and consider everything on the Earth to be moving around on its 'horizontal' surface.

In this fantasy, **we are in an inertial (non-accelerating) frame of reference, so we can use Newton's laws**.

An astronaut looking down from the ISS will immediately have this fantasy shattered. To her, the Earth is clearly approximately spherical and rotating around its axis (there must be a centripetal acceleration!)

An astronomer finds that the Earth is also orbiting the Sun, which in turn seems to be orbiting something else (the centre of the Milky way), and so on...

Whaa! Must we abandon Newton's laws? No.

We can use Newton's laws (approximately) inside our chosen frame of reference, we can label 'Earth + Me'.

The astronaut can use Newton's laws (approximately) in her chosen frame of reference, we can label 'ISS + Earth'.

The astronomer can use Newton's laws (approximately) in his chosen frame of reference, which he will define according to what he is trying to measure, for example 'Earth+Sun', or 'Solar system + Milky Way'

We must simply make sure we

define a frame of reference, and then ignore anything outside it.

The correct way to define a reference frame is to find the centre of mass position of the frame, and to treat the bodies within the frame as moving relative to the frame's centre of mass.

In many problems based on Earth, we tend not to do this:

If we think of the Earth as stationary, we can use Newton's laws to describe motion with respect to the earth.

The Earth's mass is so huge compared to everyday objects, that the centre of mass of the system is almost identical to the centre of mass position of the earth.

Relative Motion

I am sitting on an aeroplane moving with $\underline{\mathbf{v}}_{plane}$ when an emotional support peacock struts down the aisle towards me at a velocity of $\underline{\mathbf{v}}_{peacock}$. What is the peacock's velocity?

This is not a question that can be answered.

The question mentions two frames of reference: (Earth+Plane) and (Plane+Peacock), and does not specify which frame it wants the answer in. Let's clarify the question:

I am sitting on an aeroplane moving with $\underline{\mathbf{v}}_{plane:earth}$ **relative to earth** when an emotional support peacock struts down the aisle towards me at a velocity of $\underline{\mathbf{v}}_{peacock:plane}$. What is the peacock's velocity **relative to earth** $\underline{\mathbf{v}}_{peacock:earth}$?

Once we have cleared up the question, the answer is easy:

$\underline{\mathbf{v}}_{Peacock+Earth} = \underline{\mathbf{v}}_{Peacock+Plane} + \underline{\mathbf{v}}_{Plane+Earth}$

$$\boxed{\underline{\mathbf{v}}_{A:C} = \underline{\mathbf{v}}_{A:B} + \underline{\mathbf{v}}_{B:C}} \quad \mathbf{5.1}$$

In this course we consider only frames that move at constant velocity relative to each other. In our example here, this means that the (Plane+Peacock) frame must be moving at constant velocity with respect to the (Earth+Plane) frame, or more simply put, the plane's velocity is constant. To find the relative acceleration, we can differentiate the relative velocity with respect to time:

$\frac{d}{dt}\underline{\mathbf{v}}_{A:C} = \frac{d}{dt}\underline{\mathbf{v}}_{A:B} + \frac{d}{dt}\underline{\mathbf{v}}_{B:C}$ and $\frac{d}{dt}\underline{\mathbf{v}}_{B:C}$ is zero, so:

$$\boxed{\underline{\mathbf{a}}_{A:C} = \underline{\mathbf{a}}_{A:B}} \quad \mathbf{5.2}$$

Note that this is only true if there is no relative acceleration between the two frames of reference.

Centre Of Mass Position

All bodies have a **centre of mass** position (com), which we denote x_{com}. For a perfectly uniform sphere, the com is at the centre of the sphere.

For a system of two particles, we can write this as:

$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \quad \textbf{5.3}$$

As usual, we can choose our origin $x_0 = 0$ to be anywhere we like. It makes things simple for us if we choose to place the origin x_0 at the position of one of the particles, for example by setting $x_1 = x_0 = 0$. Then:

$$x_{com} = \frac{m_2 x_2}{m_1 + m_2} \quad \textbf{5.4}$$

We can extend this procedure to any number of particles we wish:

$$x_{com} = \frac{1}{M} \sum_i m_i x_i \quad \textbf{5.5} \quad \rightarrow \quad \underline{\mathbf{s}}_{com} = \frac{1}{M} \sum_i m_i \underline{\mathbf{s}}_i \quad \textbf{5.6}$$

We can do exactly the same thing as for $\underline{\mathbf{s}}_{com}$ to write $\underline{\mathbf{v}}_{com}$ for two particles:

$$\underline{\mathbf{v}}_{com} = \frac{m_1 \underline{\mathbf{v}}_1 + m_2 \underline{\mathbf{v}}_2}{m_1 + m_2} \quad \textbf{5.7}$$

And for multiple particles:

$$\underline{\mathbf{v}}_{com} = \frac{1}{M} \sum_i m_i \underline{\mathbf{v}}_i \quad \textbf{5.8}$$

Problems

1. What is the centre of mass position (relative to Earth's centre of mass) of a system comprising Earth + an apple of mass 100 g?

2. A boat is traveling upstream in the positive direction of an x-axis at 3.0ms^{-1} with respect to the water of a river. The water is flowing at 5.0ms^{-1} with respect to the ground. What are the (a) magnitude and (b) direction of the boat's velocity with respect to the ground? A child on the boat walks from front to rear at 0.4 ms^{-1} with respect to the boat. What are the (c) magnitude and (d) direction of the child's velocity with respect to the ground?

3. After flying for 15 min at an angle of 20° south of east, in a wind blowing at 35 kmh^{-1} relative to the ground, a plane is over a town that is due north of the starting point. What is the speed of the plane relative to the air?

Energy

Kinetic and Potential Energy ✧ Work & Power ✧ Conservation of Energy

Book / Wiley+ : Chapter 7.1, 7.2, 7.6, 8.1, 8.2, 8.5

Kinetic & Potential Energy

Kinetic Energy is present for bodies in motion

$KE = \frac{1}{2}mv^2$ **6.1**

Kinetic energy is measured in Joules $\mathrm{J} = \mathrm{kgm^2s^{-2}}$

Note that KE is a scalar: it is a property of a body that has no direction

Note that a body that is not moving has zero kinetic energy. An interesting question arising from this is Einstein's $E = mc^2$, which looks a little like $KE = \frac{1}{2}mv^2$

Something puzzling to think about: a particle of light (a photon) has zero mass, so does it have zero KE? Not examined!

Potential Energy is associated only with conservative forces

Gravitational Potential Energy (vertical):

$PE = \int_i^f Fdx = \int_i^f (-mg)dx = -(mgx_f - mgx_i) = mgx_i = mgh$ because we define the final position as at 'ground zero'

$PE = mgh$ **6.2**

Elastic Potential Energy (1D):

$PE = \int_i^f Fdx = \int_i^f (-kx)dx = \frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2 = \frac{1}{2}k(\Delta x)^2$

$PE = \frac{1}{2}k(\Delta x)^2$ **6.3**

Note that later on, in Gravitation, we will show that the gravitational PE can be written in the same form as the elastic PE

Energy and Work, Power

Work Energy theorem:

$$W = KE_f - KE_i = \Delta KE \quad \textbf{6.4}$$

$$W = PE_i - PE_f = -\Delta PE \quad \textbf{6.5}$$

Work as a transaction:

It can be useful to think of KE and PE as objects for trade, and work as currency. If I want to sell some KE, I am paid in work, and I can then use that work to purchase some PE.

Power is the rate of change of work with time $P = \frac{dW}{dt}$ **6.6**

If we think about work as the force applied over a distance:
$P = \frac{d}{dt} \int \underline{\mathbf{F}} \cdot \underline{\mathbf{s}} = \underline{\mathbf{F}} \cdot \frac{d}{dt}\underline{\mathbf{s}}$, thus:

$$P = \underline{\mathbf{F}} \cdot \underline{\mathbf{v}} \quad \textbf{6.7}$$

Note that power, like work and energy, is a scalar.

Conservation of Energy

The total energy of a closed system is conserved:

$$[KE + PE]_i + W_{in} = [KE + PE]_f + W_{out} \quad \textbf{6.8}$$

Let's imagine an initial situation, in which the system is made up two bodies: A and B.The first thing we must do is define the frame in which we are working. I am going to choose A to be at rest and B moving relative to A.

Initially: B has $KE_i = \frac{1}{2}mu^2$ and $PE = mgh$. So if the mass of B is 2 kg and the initial position of B relative to A is $h = 10m$ and the initial speed of B relative to A is 5ms^{-1}, we have:

$KE_i = \frac{1}{2}(2kg)(5ms^{-1})^2 = 25J$ and $PE_i = (2kg)(10ms^{-1})(10m) = 200J$

B is accelerating towards A with acceleration g, so eventually they will 'collide'.

No external forces are doing any work on this system, so $W_in = 0$. If there were an atmosphere that B had to move through, then there would be $W_in = F_D h = \frac{1}{2}C_D A\rho v^2 h$

Finally: B has $KE_i = \frac{1}{2}mv^2 = 0$ (B has come to a stop, so $v = 0$) and $PE = 0$ (because we define $PE = 0$ at 'ground zero').

Is there any work done by the system? Yes, there must be, because energy cannot just disappear!

$KE_i + PE_i + W_{in} = KE_f + PE_f + W_{out}$

$25J + 200J + 0 = 0 + 0 + W_{out} \rightarrow W_{out} = 225J$

This work done by the system could be a transfer of energy into the material of A and B, for example a crater on B, the break up of A, masses of heat produced (as radiation). There is no atmosphere, so there can be no energy carried off as sound in this case!

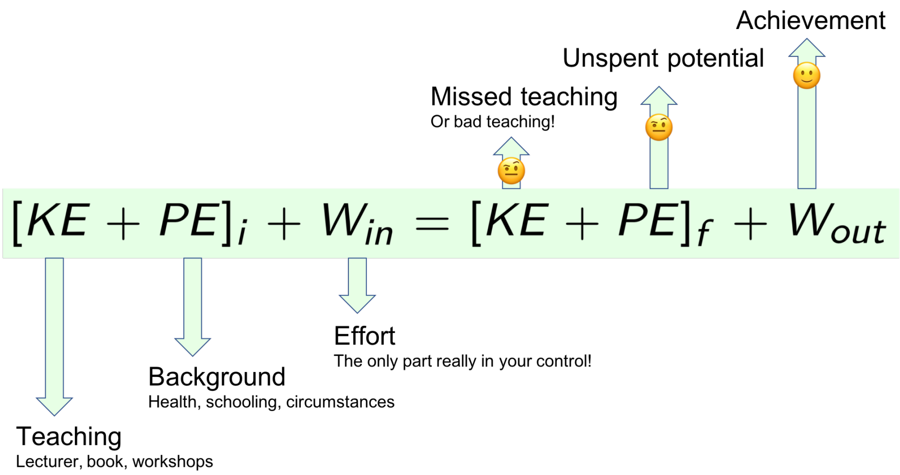


Figure 3: Conservation of Energy and Studentship

Sample Problems

1. Starting from the definition of power as $P = \underline{\mathbf{F}} \cdot \underline{\mathbf{v}}$, can you prove that work is equal to the change in KE?

2. A 4.00 kg block is pulled up a frictionless inclined plane by a 50.0 N force that is parallel to the plane, starting from rest. The normal force on the block from the plane has magnitude 13.41 N. What is the block's speed when its displacement up the ramp is 3.00 m?

3. A spring with a spring constant of 15 Ncm^{-1} has a cage attached to its free end. (a) How much work does the spring force do on the cage when the spring is stretched from its relaxed length by 5.60 mm? (b) How much additional work is done by the spring force when the spring is stretched by an additional 5.60 mm?

Momentum

Conservation of Momentum ✧ Collisions ✧ Impulse

Book / Wiley+ : Chapter 9.3, 9.4, 9.5, 9.6, 9.7, 9.8

Conservation of momentum

We define the momentum of a particle as $\boxed{\underline{\mathbf{p}} = m\underline{\mathbf{v}}}$ 7.1.

We can relate this to force: $F = ma = m\frac{dv}{dt} = \frac{dp}{dt} \rightarrow \boxed{\underline{\mathbf{F}} = \frac{d\underline{\mathbf{p}}}{dt}}$ 7.2

Newton's third law states that $\underline{\mathbf{F}}_{A:B} = -\underline{\mathbf{F}}_{B:A}$; let's see what this means for momentum:

$$m_B a_B = -m_A a_A \rightarrow m_B \frac{dv_B}{dt} = -m_A \frac{dv_A}{dt} \rightarrow m_B \frac{v_B - u_B}{\Delta t} = -m_A \frac{v_A - u_A}{\Delta t}$$

We can reorganise this as :

$$m_B(v_B - u_B) = m_A(u_A - v_A) \rightarrow m_B v_B + m_A v_A = m_A u_A + m_B u_B \rightarrow$$
$$[p_A + p_B]_{final} = [p_A + p_B]_{initial}$$

What this means is if we sum over the initial momenta of all particles in a closed system, this will equal the sum over the final momenta of all particles in that system.

$\boxed{\sum \underline{\mathbf{p}}_i = \sum \underline{\mathbf{p}}_f}$ 7.3

The total momentum of a system is conserved

Collisions

When one body collides with another, various things could happen. As usual, we will think about the extremes:

Key question: is there any work done by external forces during the collision?

If there is a noise, any breaking up of the bodies, any heat production etc, then we must conclude that some KE is transferred to work.

$\boxed{KE \text{ not conserved : Inelastic Collision}}$

If there is no transfer of KE into work, we conclude that KE is conserved, and we have an elastic collision.

$\boxed{KE \text{ conserved : Elastic Collision}}$

Note that elastic collisions don't exist in the real world.

Note that $\boxed{\text{momentum is always conserved in both elastic and inelastic collisions}}$.

Collisions

For Elastic Collisions:

Conservation of $\underline{\mathbf{p}}$: $m_A v_A + m_B v_B = m_A u_A + m_B u_B$

Conservation of KE : $m_A v_A^2 + m_B v_B^2 = m_A u_A^2 + m_B u_B^2$

We want to write these in common terms so we can divide one by the other:

Conservation of $\underline{\mathbf{p}}$: $m_A(u_A - v_A) = -m_B(u_B - v_B)$

Conservation of KE : $m_A(u_A^2 - v_A^2) = -m_B(u_B^2 - v_B^2)$

: $m_A(u_A + v_A)(u_A - v_A) = -m_B(u_B + v_B)(u_B - v_B)$

So now we can divide through to get:

$v_A = \frac{m_A - m_B}{m_A + m_B} u_A + \frac{2m_B}{m_A + m_B} u_B$

For Inelastic Collisions:

Conservation of $\underline{\mathbf{p}}$: $m_A v_A + m_B v_B = m_A u_A + m_B u_B$

Conservation of KE : No, because KE is not conserved in inelastic collisions. We turn some into noise / heat / etc

$v_A = u_A + \frac{m_B}{m_A}(u_B - v_B)$

$v_B = u_B + \frac{m_A}{m_B}(u_A - v_A)$

If a collision is completely inelastic, it means that the colliding bodies stick together or become one composite object.

Examples of completely inelastic collisions:
An asteroid collides with the moon, and imbeds itself completely into the moon **with no debris**: asteroid + moon ⇒ (asteroid+moon)

I catch a ball: hand + ball ⇒ (hand+ball)

Inelastic Collisions : '2 become 1' (Spice Girls)

Impulse

Impulse J is **the change in momentum**, or equivalently, the force applied multiplied by the length of time it is applied for.

$$J = \int F dt = \int \frac{dp}{dt} dt = p_f - p_i \quad \textbf{7.4}$$

If you bend your knees upon impact in a fall, you extend the time of collision dt and lessen the impact force $F = \frac{dp}{dt}$.

A boxer moves away from a punch, extending the time of impact and lessening the force.

Cars are made to collapse upon impact, extending the time of collision and lessening the impact force

Sample Problems

1. Algebra! Starting from:

Conservation of energy: $m_A(u_A + v_A)(u_A - v_A) = -m_B(u_B + v_B)(u_B - v_B)$ and

Conservation of momentum: $m_A(u_A - v_A) = -m_B(u_B - v_B)$

Show that :

$v_A = \frac{m_A - m_B}{m_A + m_B} u_A + \frac{2m_B}{m_A + m_B} u_B$

and

$v_B = \frac{2m_A}{m_A + m_B} u_B + \frac{m_B - m_1}{m_A + m_B} u_A$

2. An asteroid collides with the moon and breaks up into fragments on impact. Why is this not classified as a completely inelastic collision?

3. A 5.20 g bullet moving at 700 ms^{-1} strikes a wooden block at rest on a frictionless surface. The bullet emerges, traveling in the same direction with its speed reduced to 350 ms^{-1}. (a) What is the resulting speed of the block? (b) What is the speed of the bullet-block centre of mass?

Projectiles

Polar Coordinates ✧ Range

Book / Wiley+ : Chapter 4.4

Polar Coordinates

We can write any of the kinematic vector quantities ($\underline{\mathbf{s}}, \underline{\mathbf{u}}, \underline{\mathbf{v}}, \underline{\mathbf{a}}$) in terms of angles and components:

$\underline{\mathbf{s}} = s_x\hat{\boldsymbol{i}} + s_y\hat{\boldsymbol{j}} = s\cos\theta\hat{\boldsymbol{i}} + s\sin\theta\hat{\boldsymbol{j}}$

$\underline{\mathbf{u}} = u_x\hat{\boldsymbol{i}} + u_y\hat{\boldsymbol{j}} = u\cos\theta\hat{\boldsymbol{i}} + u\sin\theta\hat{\boldsymbol{j}}$

$\underline{\mathbf{v}} = v_x\hat{\boldsymbol{i}} + v_y\hat{\boldsymbol{j}} = v\cos\theta\hat{\boldsymbol{i}} + v\sin\theta\hat{\boldsymbol{j}}$

$\underline{\mathbf{a}} = a_x\hat{\boldsymbol{i}} + a_y\hat{\boldsymbol{j}} = a\cos\theta\hat{\boldsymbol{i}} + a\sin\theta\hat{\boldsymbol{j}}$

We can reformat the equations of motion in a form which is useful for solving projectile problems.

Horizontal components:

$v_x = u_x + a_x t = u_x$ there are no forces acting in the x-direction: $a_x = 0$

$\boxed{v_x = u\cos\theta}$ **8.1**

$s_x = u_x t + \frac{1}{2}a_x t^2 = u_x t$

$\boxed{s_x = u\cos\theta t}$ **8.2**

Vertical components:

$v_y = u_y + a_y t = u_y - gt$: there is one force acting in the y-direction: $a_y = -g$

$\boxed{v_y = u\cos\theta - gt}$ **8.3**

$s_y = u_y t + \frac{1}{2}a_y t^2 = u_y t - \frac{1}{2}gt^2$

$\boxed{s_y = u\cos\theta t - \frac{1}{2}gt^2}$ **8.4**

Range

Horizontal Range:

For the horizontal range, we are looking for the value of s_x when s_y returns to zero, so:

$s_y = u \sin\theta t - \frac{1}{2}gt^2 = 0$

We have gained enough information to get rid of the pesky t^2

$u \sin\theta t = \frac{1}{2}gt^2$

$u \sin\theta = \frac{1}{2}gt$

Use our favourite substitution for t:

$t = s_x/u\cos\theta \rightarrow u\sin\theta = \frac{1}{2}g(s_x/u\cos\theta) \rightarrow s_x = \frac{2}{g}u^2\sin\theta\cos\theta$

Then we can use $2\sin\theta\cos\theta = \sin 2\theta$

$s_x = \frac{u^2}{g}\sin 2\theta$ **8.5**

This tells us that the maximum value of s_x, *ie* the maximum horizontal range, is when $\sin 2\theta = 1 \rightarrow 2\theta = 90^\circ$

The horizontal range is maximised when the launch angle $\theta = 45^\circ$

Vertical Range:

We can write the equation of a projectile path $y = f(x)$: this means that we want s_y as a function of s_y, not a function of t.

We do not have or want t, so we rearrange suvat2,x : $s_x = u\cos\theta t$ in terms of t:

$t = s_x/u\cos\theta$

Plug it into suvat2,y and simplify:

$s_y = u\sin\theta t - \frac{1}{2}gt^2$

$s_y = u\sin\theta(s_x/u\cos\theta) - \frac{1}{2}g(s_x/u\cos\theta)^2$

$s_y = (\tan\theta)s_x - (\frac{1}{2}g\frac{1}{(u\cos\theta)^2})s_x^2$ **8.6**

Thus the equation for vertical displacement s_y is a quadratic function in s_x, of the form $y = ax + bx^2$, where a and b are constants.

This is a parabolic path.

Sample Problems

1. A pebble is catapulted from Brighton pier at time $t = 0$, with an initial velocity of magnitude $u = 3\text{ms}^{-1}$ at an angle of $\theta = 40.0^\circ$ above the horizontal. What is the pebble's displacement $\underline{\mathbf{s}}$ from the catapult site at $t = 1.5$ s?

2. A tennis player serves the ball at 18.9 ms^{-1}, with the centre of the ball leaving the racquet horizontally at a vertical distance 2.37 m above the ground. The net is 12 m away and 0.90 m high. By what vertical distance does the ball clear the net?

Course material:

Book / Wiley+ : Chapter 4.5, 6.3

Circles

If we approximate a wedge of circle to a triangle we can write:

[opp] = [hyp] $\sin\theta$

Let's rewrite that in terms of our usual circle notation:

$\underline{\mathbf{s}} = (r + dr)\sin\underline{\theta}$

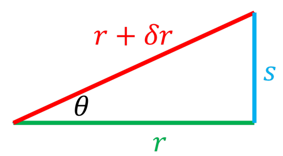


Figure 4: A wedge of a circle approximated to a triangle

$\boxed{\text{The direction of the vector } \underline{\theta} \text{ is defined by the right hand rule.}}$

Now if we take this towards the limit where θ (magnitude) is very very scriptsize, we can use two approximations:

1. $r + dr \rightarrow r$

2. $\sin\theta \rightarrow \theta$ (this is called the scriptsize angle approximation)

Such that:

$\boxed{\underline{\mathbf{s}} = r\underline{\theta}}$ 9.1

And similarly:

$\underline{\mathbf{v}} = \frac{d}{dt}\underline{\mathbf{s}} = r\frac{d}{dt}\underline{\theta} \rightarrow$ $\boxed{\underline{\mathbf{v}} = r\underline{\omega}}$ 9.2 $\boxed{\underline{\omega} \text{ is the angular velocity}}$

$\underline{\mathbf{a}} = \frac{d}{dt}\underline{\mathbf{v}} = r\frac{d}{dt}\underline{\omega} \rightarrow$ $\boxed{\underline{\mathbf{a}} = r\underline{\alpha}}$ 9.3 $\boxed{\underline{\alpha} \text{ is the angular acceleration}}$

Figure 5: The Right hand rule

Centripetal Acceleration

We can write down the position vector for a particle moving in a circle

$$\underline{\mathbf{r_c}} = r\cos\theta\hat{\boldsymbol{i}} + r\sin\theta\hat{\boldsymbol{j}} \quad \mathbf{9.4}$$

Differentiating with respect to time gives us the velocity:

$\frac{d}{dt}\underline{\mathbf{r_c}} = \frac{d}{dt}(r\cos\theta)\hat{\boldsymbol{i}} + \frac{d}{dt}(r\sin\theta)\hat{\boldsymbol{j}}$

The radius r of a circle is constant so we only differentiate the $\cos\theta$ and $\sin\theta$ pieces:

$\frac{d}{dt}(\cos\theta) = (-\sin\theta)(\frac{d\theta}{dt}) = (-\sin\theta)(\omega)$
$\frac{d}{dt}(\sin\theta) = (\cos\theta)(\frac{d\theta}{dt}) = (\cos\theta)(\omega)$

$$\underline{\mathbf{v_c}} = \omega\left((-r\sin\theta)\hat{\boldsymbol{i}} + (r\cos\theta)\hat{\boldsymbol{j}}\right) \quad \mathbf{9.5}$$

Differentiating velocity with respect to time gives us the acceleration:

$\underline{\mathbf{a_c}} = \frac{d}{dt}\underline{\mathbf{v_c}} = r(-\cos\theta)(\omega)(\omega)\hat{\boldsymbol{i}} + r(-\sin\theta)(\omega)(\omega)\hat{\boldsymbol{j}}$

Which simplifies to

$$\underline{\mathbf{a_c}} = -\omega^2\left(r\cos\theta\hat{\boldsymbol{i}} + r\sin\theta\hat{\boldsymbol{j}}\right) \quad \mathbf{9.6}$$

$\underline{\mathbf{a_c}} = -\omega^2\underline{\mathbf{r_c}}$

This is the **centripetal acceleration**, in the direction of the centre of the circle, which is always present when there is circular motion.
Key formulae are :

$$\underline{\mathbf{a_c}} = -\omega^2 r\hat{\boldsymbol{r}} = -\frac{v^2}{r}\hat{\boldsymbol{r}} \quad \mathbf{9.7}$$

And the period, which is the time taken for a single revolution can be usefully written:

$\omega = \frac{d}{dt}\theta = \frac{\Delta\theta}{\Delta t}$ - the instantaneous angular speed is equal to the average angular speed, because the angular speed ω is constant

$\omega = \frac{2\pi}{T}$ - for one revolution, $\Delta\theta = 2\pi$ and $\Delta t = T$: T is the period. Thus:

$$T = \frac{2\pi}{\omega} = \frac{2\pi r}{v} \quad \mathbf{9.8}$$

Centripetal Force

The centripetal acceleration exists for uniform circular motion; although the angular velocity ω is constant (therefore the angular acceleration $\alpha = 0$) and the magnitude of the transverse velocity v is constant (therefore the transverse acceleration $a = 0$), the **direction of the transverse velocity is changing with time**.

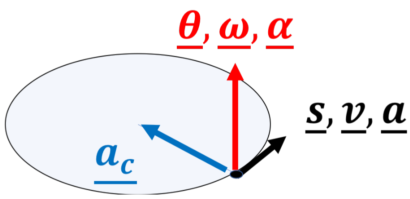


Figure 6: Three kinds of acceleration

Newton's first law states that an object in motion on a path will continue on that path with constant velocity (which could be zero) unless acted upon by an external force.

So $\boxed{\text{all objects in circular motion must be experiencing an external force}}$.

Examples:
A stone on a rope, swinging in a circle at constant ω - the centripetal acceleration is provided by the tension in the rope.

A car driving on a circular racetrack with constant ω - the centripetal acceleration is provided by the static frictional force between track and tyres.

A student sitting at their desk on earth, which is rotating with a constant ω - the centripetal acceleration is provided by gravity minus the normal force.

A moon in orbit around the earth with a constant ω - the centripetal acceleration is provided by gravity.

Sample Problems

1. A child places a picnic basket on the outer rim of a merry-go-round that has a radius of 4.6 m and revolves once every 60 s. (a) What is the speed of a point on that rim? (b) What is the lowest value of the coefficient of static friction between basket and merry-go-round that allows the basket to stay on the ride?

2. A grandmother drives a car over the top of a hill, the cross section of which can be approximated by a circle of radius 300 m. What is the greatest speed at which she can drive without the car leaving the road at the top of the hill?

Rotation

Course material:

Book / Wiley+ : Chapter 10,11

Torque

Up to now we have worked with forces in terms of their components: $\underline{\mathbf{F}} = F_x\hat{\boldsymbol{i}} + F_y\hat{\boldsymbol{j}} + F_z\hat{\boldsymbol{k}}$. This is great for forces that result in linear (*translational*) acceleration.

If I apply a force to a body that is restricted in terms of its ability to accelerate in a linear direction, then I have to think again about where the resulting acceleration is...

A specific example of this is applying a force to something that is constrained to move in a circle, for example a door on a hinge, or a bolt on a nut.
I notice that in this situation, the magnitude and direction of the applied force are not the only thing that matters; it also matters *how far from the hinge I apply the force*. We refer to the distance between the point at which the force is applied and the hinge or pivot as r : the lever arm

The resulting acceleration of e.g. the door is now contained in the torque

$$\underline{\tau} = \underline{\mathbf{r}} \times \underline{\mathbf{F}} = rF\sin\theta \quad (10.1)$$

θ is the angle between the lever arm vector $\underline{\mathbf{r}}$ and the force vector $\underline{\mathbf{F}}$

Note that if the force is applied perpendicular to the lever arm, then $\tau = rF$ is maximised.

Angular Momentum

Just as we can write down the resultant torque as a cross product between the lever arm and the applied force, we can express:

Angular momentum $\boxed{\underline{\mathbf{L}} = \underline{\mathbf{r}} \times \underline{\mathbf{p}} = rp\sin\theta}$ **10.2**

θ is the angle between the lever arm vector $\underline{\mathbf{r}}$ and the momentum vector $\underline{\mathbf{p}}$

Angular momentum $\underline{\mathbf{L}}$ is a vector in the same direction as $\underline{\theta}$, $\underline{\omega}$, $\underline{\alpha}$, and $\underline{\tau}$

Recall the rules for a cross product: If $\underline{\mathbf{C}} = \underline{\mathbf{A}} \times \underline{\mathbf{B}}$, then the direction of $\underline{\mathbf{C}}$ is perpendicular to both $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$.

Note that $\underline{\mathbf{A}} \times \underline{\mathbf{B}} = -\underline{\mathbf{B}} \times \underline{\mathbf{A}}$: we must have $\underline{\mathbf{r}}$ as the first term in the cross product because we are defining the positive direction as anticlockwise, and so we treat $\underline{\mathbf{r}}$ as our x-direction and $\underline{\mathbf{p}}$ (or $\underline{\mathbf{F}}$, in the case of $\underline{\tau} = \underline{\mathbf{r}} \times \underline{\mathbf{F}}$) as our y-direction.

Moment of Inertia

When we applying a force / momentum to a body to cause rotation rather than moving it linearly, we must always consider the lever arm : r

The moment of inertia for a point mass is $I = mr^2$ 10.3

For any other body, it helps us to write the more general form
$I = \int dI = \int r^2 dm$ 10.4

Two things are important in determining the moment of inertia of a body:

1. **What shape is the body?**
2. **Where is the axis of rotation?**

We can **look up** the moment of inertia for an object with a certain topology (e.g. rod, disc, sphere) around a certain axis, then we can use the

Parallel axis theorem $I = I_{com} + mr^2$ 10.5

to get the moment of inertia around any other axis that is parallel to the original one.
For example, the moment of inertia for a rod of length L about an axis at its centre of mass is $I = \frac{1}{12}ML^2$

To find the moment of inertia of the same rod about an axis at one end we write: $I = (I_{com}) + (Mh^2) = \frac{1}{12}ML^2 + M(\frac{L}{2})^2 = \frac{1}{3}ML^2$
Note that we have put $h = \frac{L}{2}$ above because h **is the distance we have shifted the axis**: half the length of the rod.

Moment of Inertia thoughts - not examined

The moment of inertia is a bit of a mouthful. I think of it as the 'angular mass'.
The mass of a body is related to how much effort it takes to move it: think about the role of m in $\underline{\mathbf{F}} = m\underline{\mathbf{a}}$, and $KE = \frac{1}{2}mv^2$. A nice way to think about where the definition of moment of inertia (spoiler : $I \sim mr^2$) comes from is by thinking about the work done in circular motion.

1. The work done in moving a body along ds is $W = \int_i^f Fds = \int_i^f Frd\theta$

By rearranging our definition of angular velocity ω, we can write: $d\theta = \omega dt$

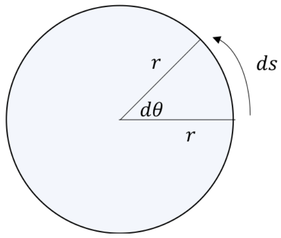
Substitute in for $d\theta$: $W = \int_i^f Fr\omega dt$, so $\frac{d}{dt}W = Fr\omega$

2. We can also write the work done as $W = \Delta KE = \frac{1}{2}mv^2$ (letting $u = 0$)

Substitue in for $v = r\omega$: $W = \Delta KE = \frac{1}{2}mr^2\omega^2$, so
$\frac{d}{dt}W = \frac{1}{2}mr^2\frac{d}{dt}(\omega^2) = \frac{1}{2}mr^2(\omega\alpha + \alpha\omega) = mr^2\omega\alpha$

Equating our two expressions for $\frac{d}{dt}W$ we have $Fr\omega = mr^2\omega\alpha$ and thus:

$\boxed{Fr = mr^2\alpha \Rightarrow \tau = mr^2\alpha, \text{ analogous to } F = ma}$



It makes sense then to define mr^2 as the angular equivalent of mass, or the **moment of inertia**. Voila.

Not examined!

Derivations of moment of inertia for ring, disk, cylinder:

$I = \int_M r^2 \delta m$

Compare these:

RING

DISK

RING:

$dI = r^2 \delta m$

all elements δm have same r

so $I = MR^2$

→ just like for point mass

DISK: $\delta I = r^2 \delta m$

$\delta m = \rho \, dA$

$A = \pi r^2$, so $\frac{dA}{dr} = 2\pi r$

→ $dA = 2\pi r \, \delta R$

$\delta I = r^2 \rho \, 2\pi r \, \delta R$

$I = \int dI = 2\pi\rho \int_0^R r^3 \, \delta R$

$= 2\pi\rho \frac{R^4}{4}$ and $\rho = \frac{M}{A} = \frac{M}{\pi R^2}$

so $I = 2\pi M R^4 / 4\pi R^2 = \frac{1}{2} MR^2$

$I = \int_M r^2 \delta m$

SOLID CYLINDER

SOLID CYLINDER: $\rho = \frac{M}{V} = \frac{M}{\pi R^2 h}$

$dI = r^2 \delta M$

$\delta M = \rho \, dV$

$V = \text{Area} \times \text{height} = \pi r^2 h$

$\frac{dV}{dr} = 2\pi r h$ so $dV = 2\pi r h \, \delta r$

$dI = r^2 \rho \, 2\pi r h \, \delta r$

$I = \rho \, 2\pi h \int_0^R r^3 \, \delta r$

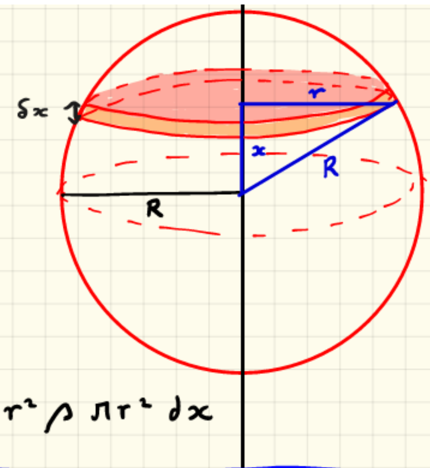
$= \frac{M \, 2\pi h}{\pi R^2 h} \frac{R^4}{4} = \frac{2MR^2}{4} = \frac{1}{2} MR^2$

FORMULA FOR $I_{DISK} = I_{CYLINDER}$

→ extent along axis is irrelevant to I

Not examined!

Derivation of moment of inertia for sphere :



We know $I_{DISK} = \frac{1}{2} MR^2$

so $\delta I = \frac{1}{2} r^2 \delta m$

→ now summing over disks, not point masses.

$\delta m = \rho \, dV$ where $\rho = M / \frac{4}{3}\pi R^3$

$dV = \pi r^2 \, dx$

$\rightarrow \delta I = \frac{1}{2} r^2 \delta m = \frac{1}{2} r^2 \rho \, dV = \frac{1}{2} r^2 \rho \pi r^2 \, dx$

$\delta I = \frac{\rho \pi}{2} r^4 \, dx$ $\boxed{\text{TRICKY ? write } r^2 = R^2 - x^2}$

$r^4 = (R^2 - x^2)^2 = R^4 + x^4 - 2R^2x^2$

$dI = \frac{\rho\pi}{2} \left(R^4 + x^4 - 2R^2x^2 \right) dx$

$dI = \frac{\rho\pi}{2} \left(R^4 + x^4 - 2R^2x^2 \right) dx$

$$I = \int dI = \frac{\rho\pi}{2} \int_{x=-R}^{R} R^4 + x^4 - 2R^2x^2 \; dx$$

$$= \frac{\rho\pi}{2} \left[R^4 x + \frac{x^5}{5} - \frac{2}{3} R^2 x^3 \right]_{x=-R}^{R}$$

at $x = R$: $R^5 + \frac{R^5}{5} - \frac{2}{3} \frac{R^5}{\cancel{4}}$ (1)

at $x = -R$: $-R^5 - \frac{R^5}{5} + \frac{2}{3} \frac{R^5}{\cancel{4}}$ (2)

SORRY!

$(1) - (2) = 2R^5 + \frac{2R^5}{5} - \frac{4}{3} \frac{R^5}{\cancel{4}}$

and $\rho = \frac{M}{\frac{4}{3}\pi R^3}$ so $I = \frac{M\pi}{2\pi R^3} \frac{3}{4} \frac{R^5}{15} \left(30 + 6 - 20 \right) = M \frac{3}{8} \frac{R^2}{15} 16 = \frac{2MR^2}{5}$

Linking the rotational variables

Just as the linear momentum is written $\underline{\mathbf{p}} = m\underline{\mathbf{v}}$, the angular momentum can be written $\boxed{\underline{\mathbf{L}} = I\underline{\omega}}$ **10.6**

Here we have made the two replacements:

1. $m \Rightarrow I$: the mass m used in linear motion is replaced with the moment of inertia I
2. $\underline{\mathbf{v}} \Rightarrow \underline{\omega}$: the velocity $\underline{\mathbf{v}}$ is replaced with the angular velocity $\underline{\omega}$

Just as the linear force is written $\underline{\mathbf{F}} = m\underline{\mathbf{a}}$, the angular force $\underline{\tau}$ can be written $\boxed{\underline{\tau} = I\underline{\alpha}}$ **10.7**

Here we have made the two replacements:

1. $m \Rightarrow I$: as above
2. $\underline{\mathbf{a}} \Rightarrow \underline{\alpha}$: the acceleration $\underline{\mathbf{a}}$ is replaced with the angular angular acceleration $\underline{\alpha}$

Translational		Rotational		Relationship
Displacement	$\underline{\mathbf{s}}$	Angular displacement	$\underline{\theta}$	$\underline{\theta} = \frac{s}{r}\hat{\boldsymbol{\theta}}$
Velocity	$\underline{\mathbf{v}}$	Angular velocity	$\underline{\omega}$	$\underline{\omega} = \frac{v}{r}\hat{\boldsymbol{\theta}}$
Acceleration	$\underline{\mathbf{a}}$	Angular acceleration	$\underline{\alpha}$	$\underline{\alpha} = \frac{a}{r}\hat{\boldsymbol{\theta}}$
Mass	m	Moment of Inertia	I	$I = \int_M r^2 dm$
Force	$\underline{\mathbf{F}}$	Torque	$\underline{\tau}$	$\underline{\tau} = I\underline{\alpha} = \underline{\mathbf{r}} \times \underline{\mathbf{F}}$
Momentum	$\underline{\mathbf{p}}$	Angular momentum	$\underline{\mathbf{L}}$	$\underline{\mathbf{L}} = I\underline{\omega} = \underline{\mathbf{r}} \times \underline{\mathbf{p}}$

Recall: $\underline{\theta}$ is 'upwards' for anticlockwise motion, and 'downwards' for clockwise motion.

Also recall: $\underline{\mathbf{A}} \times \underline{\mathbf{B}} = -(\underline{\mathbf{B}} \times \underline{\mathbf{A}})$

Sample Problems

1. Screws are generally 'right-handed'. DIY fans may remember this as 'righty tighty, lefty loosey' where 'righty' corresponds to clockwise motion of the screw and 'lefty' is anticlockwise. Why do car manufacturers sometimes use 'left-handed' screws on the left side (UK passenger side) wheels of a car?

2. I am standing at a point defined by $(x = 0, y = 0)$. I notice a particle at position vector $(\underline{\mathbf{x}} = 1m\hat{\boldsymbol{i}}, \underline{\mathbf{y}} = 2m\hat{\boldsymbol{j}})$. It is moving with velocity $\underline{\mathbf{v}} = 3ms^{-1}\hat{\boldsymbol{i}}$ and I know its mass is 4 g.
(a) what is the particle's momentum? (b) what is the particle's angular momentum?

3. I have a solid, homegenous blob with mass $m = 14$ kg and I am told that its moment of inertia about an axis through its centre of mass is 53 kgm^2. What is its moment of inertia about a parallel axis located a distance $h = 3.5$ cm from the original axis? What would be the difference if my blob was a uniform sphere?

Equilibrium

Book / Wiley+ : Chapter 12.1,12.2

Static Equilibrium

The conditions for static equilibrium are:

$\boxed{\sum F_x = 0}$ and $\boxed{\sum F_y = 0}$ and $\boxed{\sum F_z = 0}$ and $\boxed{\sum \tau = 0}$

For systems that are not moving, we can use these conditions to help determine the forces at work.

Note that when we sum the torques, we can use any position we like as our pivot point, and we will get the same result.

Note also that we must be consistent in our definition of positive and negative directions for torque: we choose anticlockwise as positive.

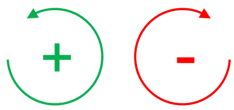
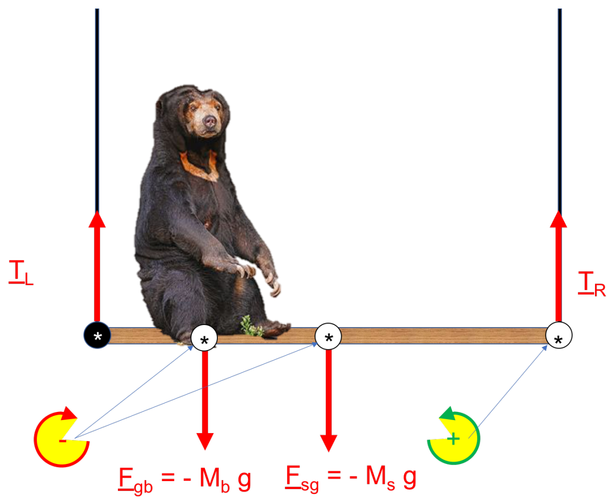


Figure 7: Always do a little diagram like this on your FBDs with torque, to remind you where to put your negative signs.

A worked example is provided on the next page

Worked Example

A sun bear of mass $m_b = 175$ kg is sitting $\frac{1}{4}$ of the way from the left end of a horizontal plank swing of mass $m_s = 14$ kg and length $D = 4$ m supported by a vertical cable at each end, and is not moving. What is the tension in the cables?
Here there are four forces present: the gravitational forces F_{gb} and F_{gs}, and the tension in the cables, T_L and T_R.



$\sum F_x = 0$: this doesn't help us, because there are no horizontal forces present

$\sum F_y = 0$: this does help us: $\sum F_y = T_L + T_R - (F_{sg} + F_{bg}) = 0$ so
$T_L + T_R = F_{sg} + F_{bg} = (175kg + 14kg)(9.81ms^{-2})$

$\sum \tau = 0$: this also helps us:

$\sum(\tau)_L = (-F_{bg})(\frac{D}{4}) + (-F_{sg})(\frac{D}{2}) + (+T_R)(D) = 0$

This yields: $(m_s g)(\frac{D}{2}) + (m_b g)(\frac{D}{4}) = (T_R)(D) \Rightarrow g(\frac{m_s}{2} + \frac{m_b}{4}) = T_R$

If we choose to sum the torques about the right hand side we **must redraw our torgue directions about each point**:

$\sum(\tau)_R = (-T_L)(D) + (+F_{bg})(\frac{3D}{4}) + (+F_{sg})(\frac{D}{2}) = 0$

This yields: $(m_s g)(\frac{D}{2}) + (m_b g)(\frac{3D}{4}) = (T_L)(D) \Rightarrow g(\frac{m_s}{2} + \frac{3m_b}{4}) = T_L$

Sample Problems

1. An archer's bow is drawn at its midpoint until the tension in the string is equal to the force exerted by the archer. What is the angle between the two halves of the string?

2. A door of mass 27 kg has a height of 2.1 m along a y-axis that extends vertically upward and a width of 0.91 m along an x-axis that extends outward from the hinged edge of the door. A hinge 0.30 m from the top and a hinge 0.30 m from the bottom each support half the door's mass. In unit-vector notation, what are the forces on the door at (a) the top hinge and (b) the bottom hinge?

3. A 1.2 kg bird sits at the midpoint on a wire with negligible mass, causing the wire to sag by 4.0 cm. The wire was initially horizontal, with a length of 25.0 m. What is the tension in the wire?

Elasticity

Book / Wiley+ : Chapter 12.3

Types of Elasticity

Stress : $\frac{F}{A}$ - is measured in Pa = $\mathrm{kgm^{-1}s^{-2}}$. Like pressure, but for solids.

We write $\boxed{\text{Stress} = \text{Modulus} \times \text{Strain}}$

We consider three types of stress in this module:

Stress	**Modulus**	**Strain**
Tension / compression	Young's modulus Y	$\frac{dL}{L}$
Shear	Shear modulus S	$\frac{dx}{L}$
Hydraulic	Bulk modulus B	$\frac{dV}{V}$

For problems involving linear tension / stretching / compression we use $\boxed{\text{Tensile stress} = Y\frac{dL}{L}}$

For problems involving shear we use $\boxed{\text{Shear stress} = S\frac{dx}{L}}$

For problems involving hydraulic stress we use $\boxed{\text{Hydraulic stress} = B\frac{dV}{V}}$

Sample Problems

1. An insect is caught at the midpoint of a spider-web thread. The thread breaks under a stress of 48.20×10^8 Nm^{-2} and a strain of 2.00. Initially, it was horizontal and had a length of 2.00 cm and a cross-sectional area of 8×10^{-12} m^2 . As the thread was stretched under the weight of the insect, its volume remained constant. If the weight of the insect puts the thread on the verge of breaking, what is the insect's mass? (A spider's web is built to break if a potentially harmful insect, such as a bumble bee, becomes snared in the web.)

2. The leaning Tower of Pisa is 59.1 m high and 7.44 m in diameter. The top of the tower is displaced 4.01 m from the vertical. Treat the tower as a uniform, circular cylinder. (a) What additional displacement, measured at the top, would bring the tower to the verge of toppling? (b) What angle would the tower then make with the vertical?

Fluids

Course material:

Book / Wiley+ : Chapter 14

Lectures : 17-18

Workshop : 9

Pressure at Depth

For fluids, it is useful to think in terms of pressure and density rather than force and mass:

Pressure $\underline{\mathbf{p}} = \frac{F}{A}$: force F perpendicular to area, over area A.

Density $\rho = \frac{m}{V}$: mass M over volume V

Note that pressure is a scalar: it is defined as the force perpendicular to the area in a plane of our choosing. Pressure is the same in every direction.

Density is also a scalar, more obviously, as neither mass nor volume are vector quantities.

Pressure in a fluid varies with vertical height or depth h, and we define it relative to a baseline pressure p_0 (often the atmospheric pressure at sea level)

$$p = p_0 + \rho g h \quad \textbf{13.1}$$

Archimedes' Principle

The $\boxed{\text{Buoyant force}}$ on a body is equal to the mass of fluid displaced by the body m_f times the acceleration due to gravity g:

$F_B = m_f g$ **13.2**

Note that the buoyant force is not determined by the mass of the body m, but the mass of the fluid displaced by the body m_f

In the special case where a body is **floating**:

$\sum \underline{\mathbf{F}}_y = 0 \Rightarrow \underline{\mathbf{F}}_B - \underline{\mathbf{F}}_g = 0 \Rightarrow \underline{\mathbf{F}}_B = \underline{\mathbf{F}}_g \Rightarrow \boxed{m_f = m}$ **The mass of fluid displaced by the body is equal to the mass of the body; the body has the same density as the fluid**

The **apparent weight** of a body in a fluid is given by $\sum \underline{\mathbf{F}}_y = \underline{\mathbf{F}}_B - \underline{\mathbf{F}}_g$.

In the special case where a body is **sinking**:

$\sum \underline{\mathbf{F}}_y < 0 \Rightarrow \underline{\mathbf{F}}_B - \underline{\mathbf{F}}_g < 0 \Rightarrow \underline{\mathbf{F}}_B < \underline{\mathbf{F}}_g \Rightarrow \boxed{m_f < m}$ **The mass of fluid displaced by the body is less than the mass of the body; the fluid is less dense than the body**

Pascal's Principle

A change in the pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid and to the walls of the containing vessel.

We can use this property of fluids to do some interesting things, such as the hydraulic lever:

$p_{in} = p_{out} \Rightarrow \frac{F_{in}}{A_{in}} = \frac{F_{out}}{A_{out}} \Rightarrow \boxed{F_{out} = F_{in}\frac{A_{out}}{A_{in}}}$ **13.3**

The output force F_{out} is greater than the input force F_{in} if $\frac{A_{out}}{A_{in}} > 1$.

This can seem counterintuitive: a scriptsize force applied over a scriptsize area results in a large force over a large area. It can help to think about work:

The work we put in is equal to the work we get out: $W_{in} = W_{out}$, so

$F_{in}.s_{in} = F_{out}.s_{out} \Rightarrow \boxed{F_{out} = F_{in}\frac{s_{in}}{s_{out}}}$ **13.4** (s is displacement)

The output force F_{out} is greater than the input force if $\frac{s_{in}}{s_{out}} > 1$

Now this seems more intuitive: $\frac{A_{out}}{A_{in}} = \frac{s_{in}}{s_{out}} \Rightarrow A_{out}s_{out} = A_{in}s_{in} \Rightarrow V_{out} = V_{in}$

The volume of fluid shifted is constant.
Note that F_{in} and F_{out} are in opposing directions; you can make sense of this vectorially by considering $\theta(A_{out}, A_{in}) = 180°$

Ideal Fluids & Continuity Equation

Ideal fluids are defined as follows:

1. Steady (laminar) flow: the speed of fluid passing any particular point is constant.
2. Incompressible: the density of the fluid is fixed - you cannot fit more of it into the same volume by applying pressure.
3. Non-viscous: there are no frictional effects at work
4. Non-rotating: the fluid can be circulating, but the body of fluid is not rotating around a centre-of-mass

For ideal fluids in motion, the volume of fluid shifted in a certain amount of time is constant.

$$\frac{dV_{in}}{dt} = \frac{dV_{out}}{dt} \quad \textbf{13.5}$$

The volume is defined as before as $V = As$: the cross-sectional area times the displacement (or length)

We can write $\frac{dA_{in}s_{in}}{dt} = \frac{dA_{out}s_{out}}{dt} \Rightarrow A_{in}\frac{ds_{in}}{dt} = A_{out}\frac{ds_{out}}{dt}$

$$A_{out}|\underline{\mathbf{v}}_{out}| = A_{in}|\underline{\mathbf{v}}_{in}| \quad \textbf{13.6}$$

The speed of the fluid is $|\underline{\mathbf{v}}|$.

This can also be written as $Av = \text{constant}$: the volume flow rate Av (cross-sectional area times speed) is constant.

Bernoulli

Bernoulli's equation takes into account three 'energy density' properties of a fluid, and realises that they are constant using conservation of energy arguments.

$\boxed{p + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}}$ **13.7**

The second term is $\frac{1}{2}\rho v^2$: this looks like the kinetic energy formula $KE = \frac{1}{2}mv^2$, with $m \rightarrow \rho$, *ie* $m \rightarrow \frac{1}{V}m$

The third term is $\rho g h$: this looks like the potential energy formula $PE = mgh$, again with $m \rightarrow \rho$, *ie* $m \rightarrow \frac{1}{V}m$

The first term is pressure p: we know this is $\frac{F}{A} = \frac{ma}{A}$ - if we divide this by volume V in analogy with the other two terms, we have $\frac{maV}{A} = mas = Fs$ - this looks a lot like the elastic potential energy formula $PE = \int Fdx$

We can therefore understand Bernoulli's equation as
(Elastic potential energy density + Kinetic energy density + Gravitational potential energy density) = constant

Note that any of the three terms can change, but their sum is constant.

An interesting consequence of Bernoulli's equation is that, **if we consider horizontal flow** ($\rho g h$ = constant, because h=constant) we can write:

$\boxed{p + \frac{1}{2}\rho v^2 = \text{constant}}$ **13.8** (for horizontal flow)

This indicates that when the pressure of a fluid drops, the kinetic energy density of the fluid must increase. This means that

$\boxed{\text{Increased pressure} \leftrightarrow \text{decreased velocity}}$.

This is weird because pressure is force/area, so

$\boxed{\text{Increased (Force/Area)} \leftrightarrow \text{decreased velocity}}$

This does not mean that a decrease in velocity comes from an *increase in force* but an *increase in force/area*. This can happen if the force is decreased, but **the area is decreased more**.

It can be helpful to think of the inverse relation between pressure and velocity as due to particles having less chance to bounce off the wall of a container (lower pressure) when they are moving quickly (higher velocity)

Sample Problems

1. You inflate the front tyres on your car to 28 psi. Later, you measure your blood pressure, obtaining a reading of 120/80, the readings being in mmHg. In metric countries (which is to say, most of the world), these pressures are customarily reported in kilopascals . In kilopascals, what are (a) your tyre pressure and (b) your blood pressure?

2. Two identical cylindrical vessels with their bases at the same level each contain a liquid of density 1.3×10^3 kgm^{-3}. The area of each base is 4.00 cm, but in one vessel the liquid height is 0.854 m and in the other it is 1.560 m. Find the work done by the gravitational force in equalizing the levels when the two vessels are connected.

3. Water is moving with a speed of 5.0 ms^{-1} through a pipe with a cross-sectional area of 4.0 cm^2. The water gradually descends 10 m as the pipe cross-sectional area increases to 8.0 cm^2. (a) What is the speed at the lower level? (b) If the pressure at the upper level is 1.5×10^5 Pa, what is the pressure at the lower level?

Gravitation

Book / Wiley+ : Chapter 13

Newton's Law of Gravitation

Newton's law of gravitation states that the gravitational force **between two bodies** is given by:

$$F_G = \frac{m_1 m_2 G}{r^2} \quad \textbf{14.1}$$

Where $G = 1.67 \times 10^{-11} Nm^2 kg^{-2}$ is the *gravitational constant*, m_1 and m_2 are the masses of the two bodies, and r is the separation between the two bodies.

Points to note here:

The gravitational force is an Inverse square law force, proportional to $\frac{1}{r^2}$ - the force decreases as the square of the distance.

The magnitude of the force on the body with m_1 due to the body with m_2 is equal to the magnitude of the force on the body with m_2 due to the body with m_1. Recall Newton's third law $\underline{\mathbf{F_{A:B}}} = -\underline{\mathbf{F_{B:A}}}$

When one body has a much scriptsizeer mass than another, e.g. $m_1 << m_2$, it can seem counterintuitive that the magnitude of the force on m_1 is equal to the magnitude of the force on m_2. We can understand this in terms of Newton's second law $F = ma \Rightarrow m_1 a_1 = m_2 a_2 \Rightarrow \frac{m_1}{m_2} = \frac{a_2}{a_1}$: a much scriptsizeer m_1 means a much larger a_1: it is this acceleration that we observe as being much larger, which we tend to confuse with a larger force being present.

The Principle of Superposition

Newton's law is used for pairs of bodies, but what if we wish to take into account the force on a body A from several other bodies? We sum those forces:

$\underline{\mathbf{F}}_A = \sum_i \underline{\mathbf{F}}_{i:A}$ **14.2**

For example, I would like to know the net gravitational force on an asteroid passing through the space between Earth and Mars, at an equal distance from the centre of mass of each of them.

(mass m_a) due to Earth is:

$\underline{\mathbf{F}}_{e:a} = \frac{Gm_a M_e}{r^2_{e:a}} \hat{\boldsymbol{n}}$

The gravitational force on the asteroid due to Mars (opposite direction) is :

$\underline{\mathbf{F}}_{m:a} = -\frac{Gm_a M_m}{r^2_{m:a}} \hat{\boldsymbol{n}}$

Because we are given that $r_{m:a} = r_{e:a} = r$, we can write
$\underline{\mathbf{F}}_a = \underline{\mathbf{F}}_{e:a} + \underline{\mathbf{F}}_{m:a} = \frac{Gm_a(M_e - M_m)}{r^2} \hat{\boldsymbol{n}}$

Looking up the masses of earth and mars I find
$M_e - M_m = 5.97 \times 10^{24} kg - 6.39 \times 10^{23} kg = 5.33 \times 10^{24} kg$

This is positive, so the net gravitational force is in the direction $\underline{\mathbf{n}}$ - which is the direction we defined as towards Earth's centre: the asteroid will curve towards the more massive planet: Earth.

Note that we made this problem a bit simpler by saying that the asteroid was equidistant from the centres of Mars and Earth, $r_{m:a} = r_{e:a}$. If the question had stated the asteroid was equidistant from the *surfaces* of the two planets, we would have a different answer, because Mars and Earth have different radii.

The Shell Theorem

The gravitational force on a particle that is outside a uniform spherical shell can, by Newton's shell theory, be calculated by assuming that the shell's mass is concentrated at the shell's centre.

A shell of mass m can be modelled as a point mass at the shell's centre.

A uniform shell of matter exerts no net gravitational force on a particle located inside it.

The gravitational force inside a shell of mass m is zero, because the vector sum of forces cancels.

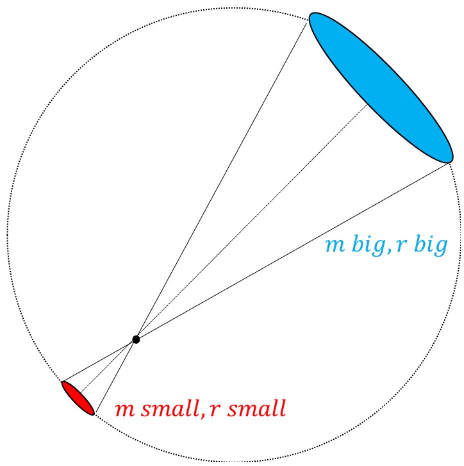


Figure 8: Inside a sphere the forces from opposing cones cancel

Near Earth's Surface

Until now we have always used the value of the acceleration near Earth's surface as $g = 9.81$ ms^{-2}This is an **approximation**.

Approximation part 1: fixed radius

$F_g = mg = \frac{mMG}{r^2}$ where we take M as Earth's mass, r as Earth's *average* radius.

$g = \frac{MG}{r^2}$ **14.3**

Earth is not a perfect sphere - the value of g is larger for scriptsizeer r (for example a deep valley) than for a larger r (for example up a mountain). **We weigh less (mg is less) up a mountain.**

Approximation 2: mass

If we go beneath the Earth's surface, we can no longer use M as the mass of the entire Earth, because some of the Earth's mass is above us.

We must therefore write down M as $M = \rho V = \rho\frac{4\pi r^3}{3}$

We then write $g = \frac{MG}{r^2} = ((\rho)(\frac{4\pi}{3}r^3))\frac{G}{r^2} = (\rho G\frac{4\pi}{3})r$ - this is nice because if we assume constant density, we have $F_g = mg = \left(\frac{4\pi}{3}m\rho G\right)r = \kappa r$: the gravitational force can be written down in the same form as Hooke's law $F = kx$. Happy days!

Near Earth's Surface

On the previous page we talked about two approximations that go into our use of a fixed value $g = 9.81ms^{-2}$:

1.Treating Earth as a perfect sphere with a fixed radius, and

2. Ignoring the possibility of *eg* entering into the Earth, such that some of its mass is above us (and therefore ignored - discussed on next page)

We make two more approximations:

3. The Earth's density is constant: we will not dwell on this, but from the same argument as (2), we can see how this affects our value for g.

4. The Earth is rotating around its centre of mass. We will dwell on this a little because it is interesting.

Recall that for any rotating body, we must consider an additional acceleration, the centripetal acceleration $a_c = \frac{mv^2}{r}$.

If the net force on me due to Earth is the centripetal force, then it is this clear that the gravitiational force downwards is not in fact balance perfectly by the normal force upwards:

$$\sum F = \frac{mv^2}{r} = F_g - N \neq 0 \quad \text{14.4}$$

We would weigh less if the Earth were not rotating $N = F_g + \frac{mv^2}{r}$

Gravitational Energy

From Newton's law of gravitation

$F_G = GmMr^{-2}$ and $\Delta PE = W = \int F.ds$ we can write

$PE_G = \int GmMr^{-2}dr = GmM \int r^{-2}dr = -GmMr^{-1}$

$PE = -\frac{mMG}{r}$ **14.5**

We can use this to determine the escape speed of a body: the minimum speed that will cause it to move upward forever, theoretically coming to rest only at infinity, rather than undergoing the projectile motion we usually observe.

For the body (projectile) to escape, the gravitational PE must be balanced or exceeded by the projectile's KE:

$PE + KE = 0 \Rightarrow \frac{1}{2}mv^2 = \frac{mMG}{r} \Rightarrow v = \sqrt{\frac{2MG}{r}}$

$v = \sqrt{\frac{2MG}{r}}$ **14.6**

Kepler's Laws

1. Law of Orbits: All planets move in elliptical orbits with the Sun at one focus.

2. Law of Areas: A line joining any planet to the Sun sweeps out equal areas in equal time intervals. (This statement is equivalent to conservation of angular momentum.)

3. Law of Periods: The square of the period T of any planet is proportional to the cube of the semimajor axis a of its orbit.

For circular orbits with radius r, the law of periods results in this very useful formula:

$$T^2 = \frac{4\pi^2}{GM} r^3 \quad \textbf{14.7}$$

The Reduced Mass

We can define the **relative motion** between two bodies 1 and 2 as follows:

$\underline{\mathbf{F}}_{1:2} = m_1\underline{\mathbf{a}}_1$ and $\underline{\mathbf{F}}_{2:1} = m_2\underline{\mathbf{a}}_2$ and $\underline{\mathbf{F}}_{1:2} = -\underline{\mathbf{F}}_{2:1}$

So, we can write $m_1\underline{\mathbf{a}}_1 = -m_2\underline{\mathbf{a}}_2 \Rightarrow \underline{\mathbf{a}}_2 = -\frac{m_1}{m_2}\underline{\mathbf{a}}_1$

Therefore the relative acceleration between them is
$\underline{\mathbf{a}}_{rel} = \underline{\mathbf{a}}_1 - \underline{\mathbf{a}}_2 = \underline{\mathbf{a}}_1\left(1 + \frac{m_2}{m_1}\right)$

Now it is useful to rearrange this to look like $\underline{\mathbf{a}}_{rel} = \frac{force}{mass}$

$\underline{\mathbf{a}}_{rel} = \frac{\underline{\mathbf{a}}_1 m_1}{\mu}$ where $\boxed{\mu = \frac{m_1 m_2}{m_1 + m_2}}$ **14.8** is the **reduced mass**

Problems

1. The Sun and Earth each exert a gravitational force on the Moon. What is the ratio of these two forces? (The average Sun-Moon distance is equal to the Sun-Earth distance.)

2. A uniform solid sphere of radius R produces a gravitational acceleration of a_g on its surface. At what distance from the sphere's center are there points (a) inside and (b) outside the sphere where the gravitational acceleration is $a_g/3$?

3.A satellite is put in a circular orbit about Earth with a radius equal to one-half the radius of the Moon's orbit. What is its period of revolution in lunar months? (A lunar month is the period of revolution of the Moon.)

Exam Tips Part 1

Why?

Before the exam, try to stay calm. Remember to breathe [pdf link].

Refer to the FAQs [pdf link]

Your examination result will go towards your final degree result - even though first year courses have zero weighting.

What to expect?

A 2 hour exam - the date I have is Thursday 10-Jan-2019 14.00 - please stay informed, check emails.

Section A : 10 question worth 2 marks each

Section B : 2 question worth 15 marks each

There are 3 section B questions - if you attempt them all I will mark them all, and take the best two marks.

You must get at least 40% in order to pass a module and proceed to year 2. If not, retakes are in August.

How to do section A questions

Section A questions look like this (example from last year):

A.4 A car of mass 300kg is driving around an unbanked racetrack with radius of curvature 200m. The track has $\mu_s = 0.5$ and $\mu_k = 0.4$. What is the maximum speed the car can go at without skidding sideways?

Note that this question, like all Section A questions, is worth 2 marks. That translates to 4% of your final mark, or 10% of the pass mark.

The marking scheme is predefined in painstaking detail, usually broken down into single or half marks.

For this question I assigned the 2 marks as:

- 0.5 marks for writing that frictional force is $F_f = \mu_s mg$
- 0.5 marks for writing the centripetal force is $F_c = \frac{mv^2}{r}$
- 0.5 marks for writing $v = \sqrt{\mu_s gr}$
- 0.5 marks for writing $v = 31.3ms^{-1}$

It is in your interest to write each of these pieces down clearly.

How to do section A questions

If you only give me a numerical answer:

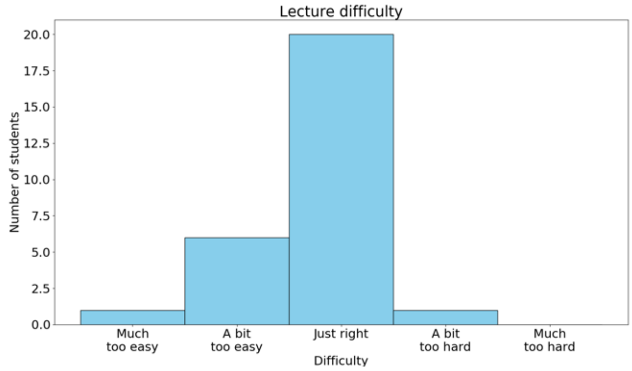
- If you simply give your answer as $v = 31.3ms^{-1}$ I will give you **2 marks**
- If you have a brain spasm and press the wrong key on the calculator for the final step, I would have to give you **0 marks**, which would make me very sad.

If you write out your steps:

- If you get $v = 31.3ms^{-1}$ I will give you **2 marks**
- If you have a brain spasm and press the wrong key on the calculator for the final step, which happens all the time in an exam situation, but have the first three steps, I will be able to give you **1.5 marks**.
- If you leave your answer as $v = \sqrt{\mu gr}$, you would also get **1.5 marks**, because you cannot reach this point without knowing the first two steps - evidence enough for me.
- If you remember the formulae for F_c and F_f, and write these down, but forget you need to equate them, I will give you **1 mark**
- If you use μ_k instead of μ_s, but get F_c correct and end up with $v = \sqrt{\mu_k gr}$ and the wrong numerical answer, I give you **1 mark**
- If all your brain will give you is $F_c = \frac{mv^2}{r}$, or $F_f = \mu mg$ I will give you **0.5 marks**

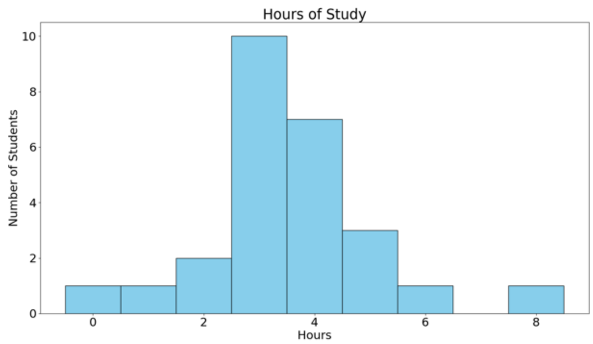
Your Feedback

Mid-term Feedback



My analysis of this is that we are at about the right level, but I should not be afraid of giving some additional tricky stuff out.

I conclude from this that the workshops are extremely popular!



My conclusion here is that you are not doing enough self study. I will try to help you with this by making more suggested problems. The recommendation for self-study time is 8 hours per week, which should be made up from workshops, assignments, and book problems. Try to do all the book problems some of them are really hard.

Feedback on lectures

How might the lectures be improved?

Working through problems is time consuming and sometimes you make mistakes do this in pre-written steps to avoid long time & errors. - I will save my real life work-through examples for workshops and office hours, and try a more power point-reveal method for lectures, which will allow us to move more quickly (giving more time to depth).

More structure / solid explanations / better organised / Examples of how to apply what we learn in lectures. - I think that by structure you are asking for a clearer breakdown of how each topic we go through fits into the big picture, and for a way to measure your progress through the course. This makes total sense I would definitely want the same, and hadnt realised I wasnt providing this outside my own head.

Dont leave questions unanswered until next time. - I suspect this might actually be a very polite comment about me saying I will come back to something next time and then completely forgetting, which is v. frustrating sorry. I am aware I do this sometimes. I wont do it any more!

Hand drawn notes make it difficult to revise. - I agree. I have yet to find a way to combine a non-dry lecture with useful revision notes. Because of this, I will provide clear typed revision notes for each topic.

What do you like about the lectures?

Solving problems and being given examples, use of both the ipad and the blackboard, covering lots of material, informality, being given problems to do in your own time

Feedback on workshops

How might the workshops be improved?

Less time waiting for help - Im afraid this is what it is we have three ATs. I am reluctant to ask them to spend less time with each student/group who has a question, as I think this would make it difficult for them to continue being so clear and helpful. Hopefully staggering the difficulty of the questions will make this issue a bit less of a problem.
More questions sure!

Different levels of difficulty for questions Yes. I will implement this from W7

Indicate difficulty of questions Yes. I will implement this from W7

Release questions ahead of time Sure! I usually put them up on canvas on the Monday, but could do so earlier.

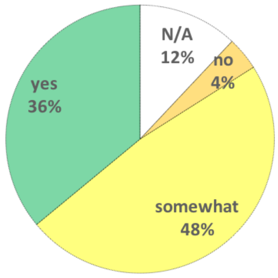
Release solutions promptly Thanks to those who pointed out that I had not linked the workshop solutions to the canvas page this is now solved and I will ensure solutions are available right after the workshops.

What do you like about the workshops?

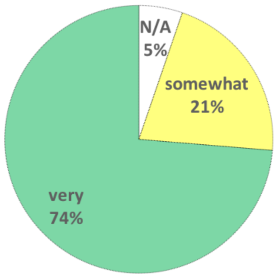
Group working; Good, challenging questions; Clear, approachable and helpful TAs; Problems interesting and have answers supplied after; Forces me to do hard questions!

End-term Feedback

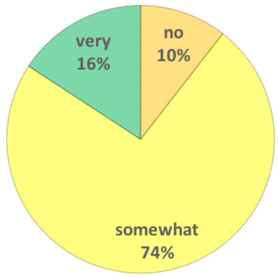
WAS MIDTERM FEEDBACK TAKEN ON?



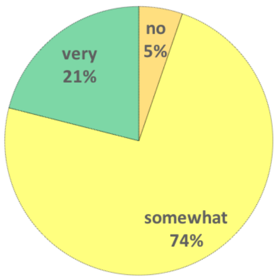
WERE WORKSHOPS USEFUL?



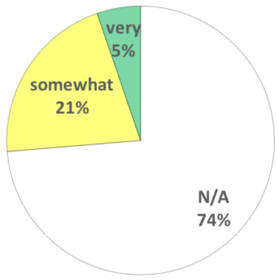
WERE REVISION NOTES USEFUL?



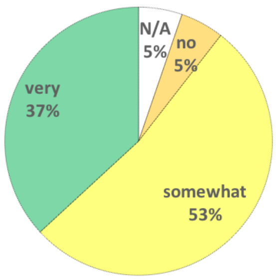
WERE LECTURES USEFUL?



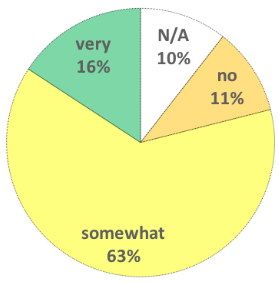
WERE RECORDINGS USEFUL?



WAS WILEY+ USEFUL?



WAS THE TEXTBOOK USEFUL?



End-term Feedback

Your favourite topics were things you had seen before, easy concepts, useful formulae, challenging concepts.
Vectors, Equations of motion, Newtons laws, Energy, Momentum, Work, Rotation, Fluids, Gravitation, Revision lectures

Least favourite topics were confusing, conceptually difficult, hard, tricky, boring, brand new.
Vectors, Energy, Projectiles, Circular Motion, Rotation, Elasticity, Fluids

It's difficult to summarise this feedback as, unsuprisingly, everyone likes different aspects of the module for different reasons. **Unpopular topics**: Projectiles, Circular Motion, Elasticity; **Popular topics**: Equations of motion, Newtons laws, Momentum, Work, Gravitation, Revision.

Keep doing it: Enthusiasm, friendly, approachable, Clear explanations, Use of ipad, Recaps, Use of board, Going through examples/ problems, Diagrams, Student participation

Try a different way: dont leave unanswered questions, check your slides for typos, more basic examples /varying difficulty, use the mic every time, use bigger font / more slides, a practise assignment, spend less time on recaps, write down assumptions for workshop problems, more dog