

Quantization of Charge, Light, and Energy

The idea that all matter is composed of tiny particles, or atoms, dates to the speculations of the Greek philosopher Democritus¹ and his teacher Leucippus in about 450 B.C. However, little attempt was made to correlate such speculations with observations of the physical world until the seventeenth century. Pierre Gassendi, in the mid-seventeenth century, and Robert Hooke, somewhat later, attempted to explain states of matter and the transitions between them with a model of tiny, indestructible solid objects flying in all directions. But it was Avogadro's hypothesis, advanced in 1811, that all gases at a given temperature contain the same number of molecules per unit volume, that led to great success in the interpretation of chemical reactions and to development of kinetic theory in about 1900. Avogadro's hypothesis made possible quantitative understanding of many bulk properties of matter and led to general (though not unanimous) acceptance of the molecular theory of matter. Thus, matter is not continuous, as it appears, but is *quantized* (i.e., discrete) on the microscopic scale. Scientists of the day understood that the small size of the atom prevented the discreteness of matter from being readily observable.

In this chapter, we will study how three additional great quantization discoveries were made: (1) electric charge, (2) light energy, and (3) energy of oscillating mechanical systems. The quantization of electric charge was not particularly surprising to scientists in 1900; it was quite analogous to the quantization of mass. However, the quantization of light energy and mechanical energy, which are of central importance in modern physics, were revolutionary ideas.

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3-1 Quantization of Electric Charge

Early Measurements of e and e/m

The first estimates of the order of magnitude of the electric charges found in atoms were obtained from Faraday's law. The work of Michael Faraday (1791–1867) in the early to mid-1800s stands out even today for its vision, experimental ingenuity, and thoroughness. The story of this self-educated blacksmith's son who rose from errand boy and

bookbinder's apprentice to become the director of the distinguished Royal Institution of London and the foremost experimental investigator of his time is a fascinating one. One aspect of his work concerned the study of the conduction of electricity in weakly conducting solutions. His discovery that the same quantity of electricity, F , now called the *faraday* and equal to about 96,500 C, always decomposes one gram-ionic weight, that is, Avogadro's number N_A , of monovalent ions leads to the reasonable conclusion that each monovalent ion carries the same electric charge, e , and therefore

$$F = N_A e \quad 3-1$$

Equation 3-1 is called Faraday's law of electrolysis. While F was readily measurable, neither N_A nor e could be experimentally determined at the time. Faraday was aware of this but could not determine either quantity. Even so, it seemed logical to expect that electric charge, like matter, was not continuous, but consisted of particles of some discrete minimum charge. In 1874, G. J. Stoney² used an estimate of N_A from kinetic theory to compute the value of e from Equation 3-1 to be about 10^{-20} C; however, direct measurement of the value of e had to await an ingenious experiment conducted by R. A. Millikan a third of a century later.

Meanwhile, Pieter Zeeman, in 1896, obtained the first evidence for the existence of atomic particles with a specific charge-to-mass ratio by looking at the changes in the discrete *spectral lines* emitted by atoms when they were placed in a strong magnetic field. He discovered that the individual spectral lines split into three very closely spaced lines of slightly different frequencies when the atoms were placed in the magnetic field. (This phenomenon is called the *Zeeman effect* and will be discussed further in Chapter 7.) Classical electromagnetic theory relates the slight differences in the frequencies of adjacent lines to the charge-to-mass ratio of the oscillating charges producing the light.



From his measurements of the splitting, Zeeman calculated q/m to be about 1.6×10^{11} C/kg, which compares favorably with the presently accepted value, 1.759×10^{11} C/kg (see Appendix D). From the polarization of the spectral lines, Zeeman concluded that the oscillating particles were negatively charged.

Discovery of the Electron: J. J. Thomson's Experiment

The year following Zeeman's work, J. J. Thomson³ measured the q/m value for the so-called cathode rays that were produced in electrical discharges in gases and pointed out that, if their charge was Faraday's charge e as determined by Stoney, then their mass was only a small fraction of the mass of a hydrogen atom. Two years earlier J. Perrin had collected cathode rays on an electrometer and found them to carry a negative electric charge.⁴ Thus, with his measurement of q/m for the cathode rays, Thomson had, in fact, discovered the *electron*. That direct measurement of e/m of electrons by J. J. Thomson in 1897, a little over a century ago, can be justly considered to mark the beginning of our understanding of atomic structure.



Measurement of e/m When a uniform magnetic field of strength B is established perpendicular to the direction of motion of charged particles, the particles move in a circular path. The radius R of the path can be obtained from Newton's second law by setting the magnetic force quB equal to the mass m times the centripetal acceleration u^2/R , where u is the speed of the particles:

$$quB = \frac{mu^2}{R} \quad \text{or} \quad R = \frac{mu}{qB} \quad \text{and} \quad \frac{q}{m} = \frac{u}{RB} \quad 3-2$$

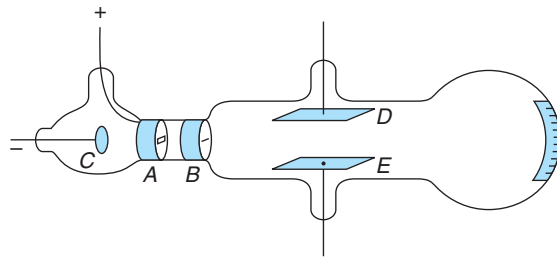


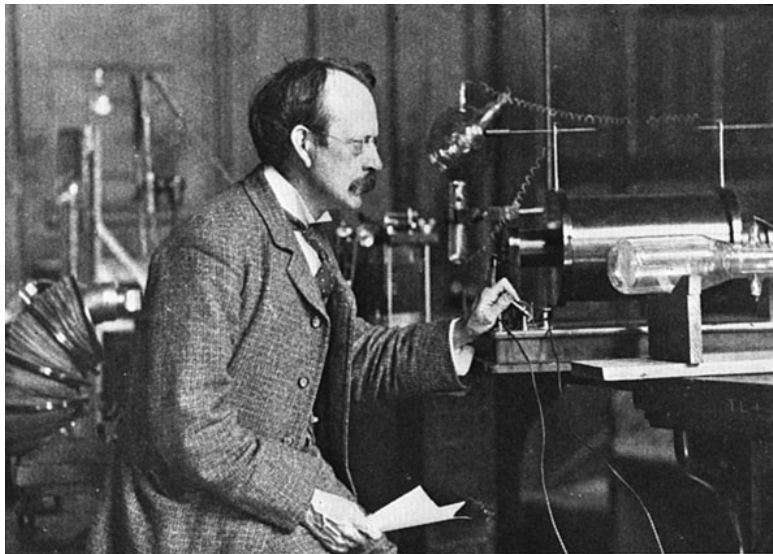
Figure 3-1 J. J. Thomson's tube for measuring e/m . Electrons from the cathode C pass through the slits at A and B and strike a phosphorescent screen. The beam can be deflected by an electric field between the plates D and E or by a magnetic field (not shown) whose direction is perpendicular to the electric field between D and E . From the deflections measured on a scale on the tube at the screen, e/m can be determined. [From J. J. Thomson, "Cathode Rays," *Philosophical Magazine* (5), **44**, 293 (1897).]

Thomson performed two e/m experiments of somewhat different designs. The second, more reproducible of the two has become known as the *J. J. Thomson experiment* (Figure 3-1). In this experiment he adjusted perpendicular B and \mathcal{E} fields so that the particles were undeflected. This allowed him to determine the speed of the electrons by equating the magnitudes of the magnetic and electric forces and then to compute e/m ($\equiv q/m$) from Equation 3-2:

$$quB = q\mathcal{E} \quad \text{or} \quad u = \frac{\mathcal{E}}{B} \quad 3-3$$

Thomson's experiment was remarkable in that he measured e/m for a subatomic particle using only a voltmeter, an ammeter, and a measuring rod, obtaining the result 0.7×10^{11} C/kg. Present-day particle physicists routinely use the modern equivalent of Thomson's experiment to measure the momenta of elementary particles.

Thomson's technique for controlling the direction of the electron beam with "crossed" electric and magnetic fields was subsequently applied in the development of cathode ray tubes used in oscilloscopes and the picture tubes of television receivers.



J. J. Thomson in his laboratory. He is facing the screen end of an e/m tube; an older cathode ray tube is visible in front of his left shoulder. [Courtesy of Cavendish Laboratory.]

Thomson repeated the experiment with different gases in the tube and different metals for cathodes and always obtained the same value for e/m within his experimental uncertainty, thus showing that these particles are common to all metals. The agreement of these results with Zeeman's led to the unmistakable conclusion that these particles—called *corpuscles* by Thomson and later called *electrons* by Lorentz—which have one unit of negative charge e and a mass about 2000 times less than the mass of the lightest known atom were constituents of *all* atoms.

Questions

1. One advantage of Thomson's evidence over others' (such as Faraday's or Zeeman's) was its directness. Another was that it was not just a statistical inference. How does the Thomson experiment show that e/m is the same for a large number of the particles?
2. Thomson noted that his values for e/m were about 2000 times larger than those for the lightest known ion, that of hydrogen. Could he tell from his data whether this was the result of the electron having either a greater charge or a smaller mass than the hydrogen ion?

Measuring the Electric Charge: Millikan's Experiment

The fact that Thomson's e/m measurements always yielded the same results regardless of the materials used for the cathodes or the kind of gas in the tube was a persuasive argument that the electrons all carried one unit e of negative electric charge. Thomson initiated a series of experiments to determine the value of e . The first of these experiments, which turned out to be very difficult to do with high precision, was carried out by his student J. S. E. Townsend. The idea was simple: a small (but visible) cloud of identical water droplets, each carrying a single charge e , was observed to drift downward in response to the gravitational force. The total charge on the cloud $Q = Ne$ was measured, as were the mass of the cloud and the radius of a single drop. Finding the radius allowed calculation of N , the total number of drops in the cloud, and, hence, the value of e .

The accuracy of Thomson's method was limited by the uncertain rate of evaporation of the cloud. In addition, the assumption that each droplet contained a single charge could not be verified. R. A. Millikan tried to eliminate the evaporation problem by using a field strong enough to hold the top surface of the cloud stationary so that he could observe the rate of evaporation and correct for it. That, too, turned out to be very difficult, but then he made a discovery of enormous importance, one that allowed him to measure directly the charge of a single electron! Millikan described his discovery in the following words:

It was not found possible to balance the cloud as had been originally planned, but it was found possible to do something much better: namely, to hold individual charged drops suspended by the field for periods varying from 30 to 60 seconds. I have never actually timed drops which lasted more than 45 seconds, although I have several times observed drops which in my judgment lasted considerably longer than this. The drops which it was found possible to balance by an electric field always carried multiple charges, and the difficulty experienced in balancing such drops was less than had been anticipated.⁵

The discovery that he could see individual droplets and that droplets suspended in a vertical electric field sometimes suddenly moved upward or downward, evidently because they had picked up a positive or negative ion, led to the possibility of observing the charge of a single ion. In 1909, Millikan began a series of experiments that not only showed that charges occurred in integer multiples of an elementary unit e , but measured the value of e to about 1 part in 1000. To eliminate evaporation, he used oil drops sprayed into dry air between the plates of a capacitor (Figure 3-2). These drops were already charged by the spraying process, i.e., by friction in the spray nozzle, and during the course of the observation they picked up or lost additional charges. By switching the direction of the electric field between the plates, a drop could be moved up or down and observed for several hours. When the charge on a drop changed, the velocity of the drop with the field “on” changed also. Assuming only that the terminal velocity of the drop was proportional to the force acting on it (this assumption was carefully checked experimentally), Millikan’s oil drop experiment⁶ gave conclusive evidence that electric charges always occur in integer multiples of a fundamental unit e , whose value he determined to be 1.601×10^{-19} C. The currently accepted value⁷ is $1.60217653 \times 10^{-19}$ C. An expanded discussion of Millikan’s experiment is included in the Classical Concept Review.

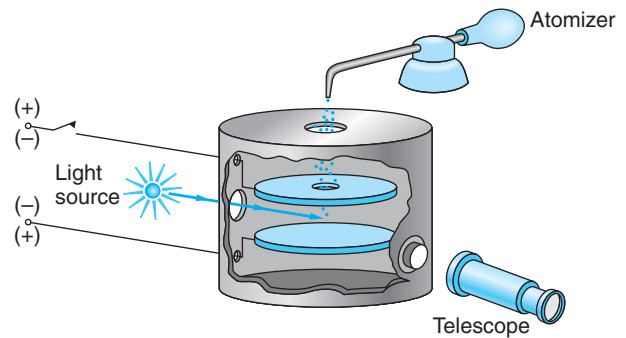


Figure 3-2 Schematic diagram of Millikan’s oil drop experiment. The drops are sprayed from an atomizer and pick up a static charge, a few falling through the hole in the top plate. Their fall due to gravity and their rise due to the electric field between the capacitor plates can be observed with the telescope. From measurements of the rise and fall times, the electric charge on a drop can be calculated. The charge on a drop could be changed by exposure to x rays from a source (not shown) mounted opposite the light source.

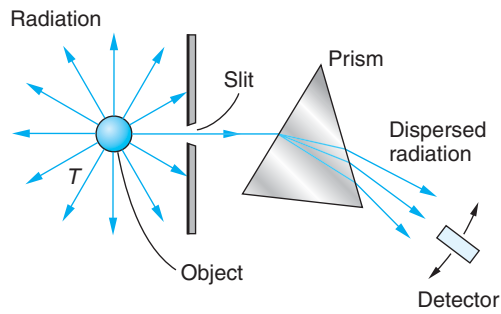


3-2 Blackbody Radiation

The first clue to the quantum nature of radiation came from the study of thermal radiation emitted by opaque bodies. When radiation falls on a opaque body, part of it is reflected and the rest is absorbed. Light-colored bodies reflect most of the visible radiation incident on them, whereas dark bodies absorb most of it. The absorption part of the process can be described briefly as follows. The radiation absorbed by the body increases the kinetic energy of the constituent atoms, which oscillate about their equilibrium positions. Since the average translational kinetic energy of the atoms determines the temperature of the body, the absorbed energy causes the temperature to rise. However, the atoms contain charges (the electrons), and they are accelerated by the oscillations. Consequently, as required by electromagnetic theory, the atoms emit electromagnetic radiation, which reduces the kinetic energy of the oscillations and tends to reduce the temperature. When the rate of absorption equals the rate of emission, the temperature is constant, and we say that the body is in thermal equilibrium with its surroundings. A good absorber of radiation is therefore also a good emitter.

The electromagnetic radiation emitted under these circumstances is called *thermal radiation*. At ordinary temperatures (below about 600°C) the thermal radiation emitted by a body is not visible; most of the energy is concentrated in wavelengths much longer than those of visible light. As a body is heated, the quantity of thermal radiation emitted increases, and the energy radiated extends to shorter and shorter wavelengths. At about $600^{\circ}\text{--}700^{\circ}\text{C}$ there is enough energy in the visible spectrum so that the body glows and becomes a dull red, and at higher temperatures it becomes bright red or even “white hot.”

Figure 3-3 Radiation emitted by the object at temperature T that passes through the slit is dispersed according to its wavelengths. The prism shown would be an appropriate device for that part of the emitted radiation in the visible region. In other spectral regions other types of devices or wavelength-sensitive detectors would be used.



A body that absorbs *all* radiation incident on it is called an *ideal blackbody*. In 1879 Josef Stefan found an empirical relation between the power radiated by an ideal blackbody and the temperature:

$$R = \sigma T^4 \quad 3-4$$

where R is the power radiated per unit area, T is the absolute temperature, and $\sigma = 5.6703 \times 10^{-8} \text{ W/m}^2\text{K}^4$ is a constant called Stefan's constant. This result was also derived on the basis of classical thermodynamics by Ludwig Boltzmann about five years later, and Equation 3-4 is now called the Stefan-Boltzmann law. Note that the power per unit area radiated by a blackbody depends only on the temperature and not on any other characteristic of the object, such as its color or the material of which it is composed. Note, too, that R tells us the *rate* at which energy is emitted by the object. For example, doubling the absolute temperature of an object, e.g., a star, increases the energy flow out of the object by a factor of $2^4 = 16$. An object at room temperature (300°C) will double the rate at which it radiates energy as a result of a temperature increase of only 57°C . Thus, the Stefan-Boltzmann law has an enormous effect on the establishment of thermal equilibrium in physical systems.

Objects that are not ideal blackbodies radiate energy per unit area at a rate less than that of a blackbody at the same temperature. For those objects the rate does depend on properties in addition to the temperature, such as color and the composition of the surface. The effects of those dependencies are combined into a factor called the *emissivity* ϵ which multiplies the right side of Equation 3-4. The values of ϵ , which is itself temperature dependent, are always less than unity.

Like the total radiated power R , the *spectral distribution* of the radiation emitted by a blackbody is found empirically to depend *only* on the absolute temperature T . The spectral distribution is determined experimentally as illustrated schematically in Figure 3-3. With $R(\lambda) d\lambda$ the power emitted per unit area with wavelength between λ and $\lambda + d\lambda$, Figure 3-4 shows the measured spectral distribution function $R(\lambda)$ versus λ for several values of T ranging from 1000 K to 6000 K.

The $R(\lambda)$ curves in Figure 3-4 are quite remarkable in several respects. One is that the wavelength at which the distribution has its maximum value varies inversely with the temperature:

$$\lambda_m \propto \frac{1}{T}$$

or

$$\lambda_m T = \text{constant} = 2.898 \times 10^{-3} \text{ m} \cdot \text{K} \quad 3-5$$

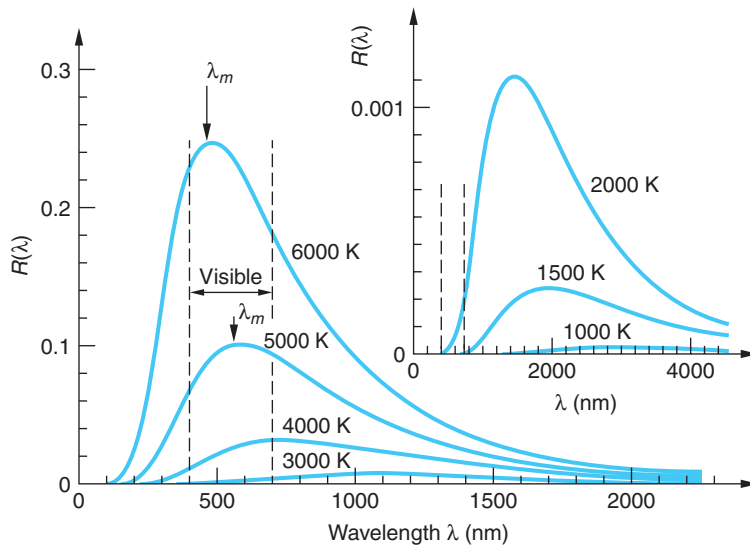


Figure 3-4 Spectral distribution function $R(\lambda)$ measured at different temperatures. The $R(\lambda)$ axis is in arbitrary units for comparison only. Notice the range of λ in the visible spectrum. The Sun emits radiation very close to that of a blackbody at 5800 K. λ_m is indicated for the 5000-K and 6000-K curves.

This result is known as Wien's displacement law. It was obtained by Wien in 1893. Examples 3-1 and 3-2 illustrate its application.

EXAMPLE 3-1 **How Big Is a Star?** Measurement of the wavelength at which the spectral distribution $R(\lambda)$ from a certain star is maximum indicates that the star's surface temperature is 3000 K. If the star is also found to radiate 100 times the power P_{\odot} radiated by the Sun, how big is the star? (The symbol \odot = Sun.) The Sun's surface temperature is 5800 K.

SOLUTION

If the Sun and the star both radiate as blackbodies (astronomers nearly always make that assumption, based on, among other things, the fact that the solar spectrum is very nearly that of an ideal blackbody), their surface temperatures from Equation 3-5 are 5800 K and 3000 K, respectively. Measurement also indicates that $P_{\text{star}} = 100 P_{\odot}$. Thus, from Equation 3-4 we have

$$R_{\text{star}} = \frac{P_{\text{star}}}{(\text{area})_{\text{star}}} = \frac{100 P_{\odot}}{4\pi r_{\text{star}}^2} = \sigma T_{\text{star}}^4$$

and

$$R_{\odot} = \frac{P_{\odot}}{(\text{area})_{\odot}} = \frac{P_{\odot}}{4\pi r_{\odot}^2} = \sigma T_{\odot}^4$$

So we have

$$\begin{aligned} r_{\text{star}}^2 &= 100 r_{\odot}^2 \left(\frac{T_{\odot}}{T_{\text{star}}} \right)^4 \\ r_{\text{star}} &= 10 r_{\odot} \left(\frac{T_{\odot}}{T_{\text{star}}} \right)^2 = 10 \left(\frac{5800}{3000} \right)^2 r_{\odot} \\ r_{\text{star}} &= 37.4 r_{\odot} \end{aligned}$$

Since $r_{\odot} = 6.96 \times 10^8$ m, this star has a radius of about 2.6×10^{10} m, or about half the radius of the orbit of Mercury. This star is a red giant (see Chapter 13).

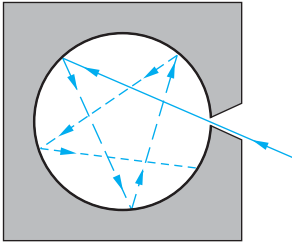


Figure 3-5 A small hole in the wall of a cavity approximating an ideal blackbody. Radiation entering the hole has little chance of leaving before it is completely absorbed within the cavity.

Rayleigh-Jeans Equation

The calculation of the distribution function $R(\lambda)$ involves the calculation of the energy density of electromagnetic waves in a cavity. Materials such as black velvet or lamp-black come close to being ideal blackbodies, but the best practical realization of an ideal blackbody is a small hole leading into a cavity (such as a keyhole in a closet door; Figure 3-5). Radiation incident on the hole has little chance of being reflected back out of the hole before it is absorbed by the walls of the cavity. The power radiated *out* of the hole is proportional to the total energy density U (the energy per unit volume of the radiation in the cavity). The proportionality constant can be shown to be $c/4$, where c is the speed of light.⁸

$$R = \frac{1}{4}cU \quad 3-6$$

Similarly, the spectral distribution of the power emitted from the hole is proportional to the spectral distribution of the energy density in the cavity. If $u(\lambda) d\lambda$ is the fraction of the energy per unit volume in the cavity in the range $d\lambda$, then $u(\lambda)$ and $R(\lambda)$ are related by

$$R(\lambda) = \frac{1}{4}cu(\lambda) \quad 3-7$$

The energy density distribution function $u(\lambda)$ can be calculated from classical physics in a straightforward way. The method involves finding the number of modes of oscillation of the electromagnetic field in the cavity with wavelengths in the interval $d\lambda$ and multiplying by the average energy per mode. The result is that the number of modes of oscillation per unit volume, $n(\lambda)$, is independent of the shape of the cavity and is given by

$$n(\lambda) = 8\pi\lambda^{-4} \quad 3-8$$

According to classical kinetic theory, the average energy per mode of oscillation is kT , the same as for a one-dimensional harmonic oscillator, where k is the Boltzmann constant. Classical theory thus predicts for the energy density distribution function

$$u(\lambda) = kT n(\lambda) = 8\pi kT \lambda^{-4} \quad 3-9$$

This prediction, initially derived by Lord Rayleigh,⁹ is called the *Rayleigh-Jeans equation*. It is illustrated in Figure 3-6.

At very long wavelengths the Rayleigh-Jeans equation agrees with the experimentally determined spectral distribution, but at short wavelengths this equation predicts that $u(\lambda)$ becomes large, approaching infinity as $\lambda \rightarrow 0$, whereas experiment shows (see Figure 3-4) that the distribution actually approaches zero as $\lambda \rightarrow 0$. This enormous disagreement between the experimental measurement of $u(\lambda)$ and the prediction of the fundamental laws of classical physics at short wavelengths was called the *ultraviolet catastrophe*. The word *catastrophe* was not used lightly; Equation 3-9 implies that

$$\int_0^{\infty} u(\lambda) d\lambda \longrightarrow \infty \quad 3-10$$

That is, every object would have an infinite energy density, which observation assures us is not true.

Planck's Law

In 1900 the German physicist Max Planck¹⁰ announced that by making somewhat strange assumptions, he could derive a function $u(\lambda)$ that agreed with the experimental data. He first found an empirical function that fit the data and then searched for a way to modify the usual calculation so as to predict his empirical formula. We can see the type of modification needed if we note that, for any cavity, the shorter the wavelength, the more standing waves (modes) will be possible. Therefore, as $\lambda \rightarrow 0$ the number of modes of oscillation approaches infinity, as evidenced in Equation 3-8. In order for the energy density distribution function $u(\lambda)$ to approach zero, we expect the average energy per mode to depend on the wavelength λ and approach zero as λ approaches zero, rather than be equal to the value kT predicted by classical theory.

Parenthetically, we should note that those working on the ultraviolet catastrophe at the time—and there were many besides Planck—had no a priori way of knowing whether the number of modes $n(\lambda)$ or the average energy per mode kT (or both) was the source of the problem. Both were correct classically. Many attempts were made to rederive each so as to solve the problem. As it turned out, it was the average energy per mode (that is, kinetic theory) that was at fault.

Classically, the electromagnetic waves in the cavity are produced by accelerated electric charges in the walls of the cavity vibrating as simple harmonic oscillators. Recall that the radiation emitted by such an oscillator has the same frequency as the oscillation itself. The average energy for a one-dimensional simple harmonic oscillator is calculated classically from the energy distribution function, which in turn is found from the Maxwell-Boltzmann distribution function. That energy distribution function has the form (see Chapter 8)

$$f(E) = Ae^{-E/kT} \quad 3-11$$

where A is a constant and $f(E)$ is the fraction of the oscillators with energy equal to E . The average energy \bar{E} is then found, as is any weighted average, from

$$\bar{E} = \int_0^\infty E f(E) dE = \int_0^\infty EAe^{-E/kT} dE \quad 3-12$$

with the result $\bar{E} = kT$, as was used by Rayleigh and others.

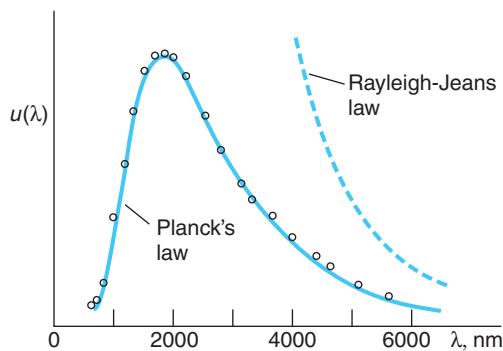


Figure 3-6 Comparison of Planck's law and the Rayleigh-Jeans equation with experimental data at $T = 1600$ K obtained by W. W. Coblentz in about 1915. The $u(\lambda)$ axis is linear. [Adapted from F. K. Richmyer, E. H. Kennard, and J. N. Cooper, *Introduction to Modern Physics*, 6th ed., McGraw-Hill, New York (1969), by permission.]

Planck found that he could derive his empirical formula by calculating the average energy \bar{E} assuming that the energy of the oscillating charges, and hence the radiation that they emitted, was a discrete variable; i.e., that it could take on only the values $0, \epsilon, 2\epsilon, \dots, n\epsilon$ where n is an integer; and further, that E was proportional to the frequency of the oscillators and, hence, to that of the radiation. Planck therefore wrote the energy as

$$E_n = n\epsilon = nhf \quad n = 0, 1, 2, \dots \quad 3-13$$

where the proportionality constant h is now called *Planck's constant*. The Maxwell-Boltzmann distribution (Equation 3-11) then becomes

$$f_n = Ae^{-E_n/kT} = Ae^{-n\epsilon/kT} \quad 3-14$$

where A is determined by the normalization condition that the sum of all fractions f_n must, of course, equal 1, i.e.,

$$\sum_{n=0}^{\infty} f_n = A \sum_{n=0}^{\infty} e^{-n\epsilon/kT} = 1 \quad 3-15$$

The average energy of an oscillator is then given by the discrete-sum equivalent of Equation 3-12:

$$\bar{E} = \sum_{n=0}^{\infty} E_n f_n = \sum_{n=0}^{\infty} E_n A e^{-E_n/kT} \quad 3-16$$

Calculating the sums in Equations 3-15 and 3-16 (see Problem 3-58) yields the result

$$\bar{E} = \frac{\epsilon}{e^{\epsilon/kT} - 1} = \frac{hf}{e^{hf/kT} - 1} = \frac{hc/\lambda}{e^{hc/\lambda kT} - 1} \quad 3-17$$

Multiplying this result by the number of oscillators per unit volume in the interval $d\lambda$ given by Equation 3-8, we obtain for the energy density distribution function of the radiation in the cavity

$$u(\lambda) = \frac{8\pi hc \lambda^{-5}}{e^{hc/\lambda kT} - 1} \quad 3-18$$

This function, called *Planck's law*, is sketched in Figure 3-6. It is clear from the figure that the result fits the data quite well.

For very large λ , the exponential in Equation 3-18 can be expanded using $e^x \approx 1 + x + \dots$ for $x \ll 1$, where $x = hc/\lambda kT$. Then

$$e^{hc/\lambda kT} - 1 \approx \frac{hc}{\lambda kT}$$

and

$$u(\lambda) \rightarrow 8\pi \lambda^{-4} kT$$

which is the Rayleigh-Jeans formula. For short wavelengths, we can neglect the 1 in the denominator of Equation 3-18, and we have

$$u(\lambda) \rightarrow 8\pi hc\lambda^{-5} e^{-hc/\lambda kT} \rightarrow 0$$

as $\lambda \rightarrow 0$. The value of the constant in Wein's displacement law also follows from Planck's law, as you will show in Problem 3-23.

The value of Planck's constant, h , can be determined by fitting the function given by Equation 3-18 to the experimental data, although direct measurement (see Section 3-3) is better but more difficult. The presently accepted value is

$$h = 6.626 \times 10^{-34} \text{ J} \cdot \text{s} = 4.136 \times 10^{-15} \text{ eV} \cdot \text{s} \quad \mathbf{3-19}$$

Planck tried at length to reconcile his treatment with classical physics but was unable to do so. The fundamental importance of the quantization assumption implied by Equation 3-13 was suspected by Planck and others but was not generally appreciated until 1905. In that year Einstein applied the same ideas to explain the photoelectric effect and suggested that, rather than being merely a mysterious property of the oscillators in the cavity walls and blackbody radiation, quantization was a fundamental characteristic of light energy.

EXAMPLE 3-2 Peak of the Solar Spectrum The surface temperature of the Sun is about 5800 K, and measurements of the Sun's spectral distribution show that it radiates very nearly like a blackbody, deviating mainly at very short wavelengths. Assuming that the Sun radiates like an ideal blackbody, at what wavelength does the peak of the solar spectrum occur?

SOLUTION

1. The wavelength at the peak, or maximum intensity, of an ideal blackbody is given by Equation 3-5:

$$\lambda_m T = \text{constant} = 2.898 \times 10^{-3} \text{ m} \cdot \text{K}$$

2. Rearranging and substituting the Sun's surface temperature yield

$$\begin{aligned} \lambda_m &= (2.898 \times 10^{-3} \text{ m} \cdot \text{K})/T = \frac{2.898 \times 10^{-3} \text{ m} \cdot \text{K}}{5800 \text{ K}} \\ &= \frac{2.898 \times 10^6 \text{ nm} \cdot \text{K}}{5800 \text{ K}} = 499.7 \text{ nm} \end{aligned}$$

where $1 \text{ nm} = 10^{-9} \text{ m}$.

Remarks: This value is near the middle of the visible spectrum.

EXAMPLE 3-3 Average Energy of an Oscillator What is the average energy \bar{E} of an oscillator that has a frequency given by $hf = kT$ according to Planck's calculation?

SOLUTION

$$\bar{E} = \frac{\epsilon}{e^{\epsilon/kT} - 1} = \frac{kT}{e^1 - 1} = 0.582 kT$$

Remarks: Recall that according to classical theory, $\bar{E} = kT$ regardless of the frequency.

The electromagnetic spectrum emitted by incandescent bulbs is a common example of blackbody radiation, the amount of visible light being dependent on the temperature of the filament. Another application is the pyrometer, a device that measures the temperature of a glowing object, such as molten metal in a steel mill.

EXAMPLE 3-4 Stefan-Boltzmann from Planck Show that the total energy density in a blackbody cavity is proportional to T^4 in accordance with the Stefan-Boltzmann law.

SOLUTION

The total energy density is obtained from the distribution function (Equation 3-18) by integrating over all wavelengths:

$$U = \int_0^{\infty} u(\lambda) d\lambda = \int_0^{\infty} \frac{8\pi hc \lambda^{-5}}{e^{hc/\lambda kT} - 1} d\lambda$$

Define the dimensionless variable $x = hc/\lambda kT$. Then $dx = -(hc/\lambda^2 kT) d\lambda$ or $d\lambda = -\lambda^2(kT/hc) dx$. Then

$$U = - \int_0^{\infty} \frac{8\pi hc \lambda^{-3}}{e^x - 1} \left(\frac{kT}{hc} \right) dx = 8\pi hc \left(\frac{kT}{hc} \right)^4 \int_0^{\infty} \frac{x^3}{e^x - 1} dx$$

Since the integral is now dimensionless, this shows that U is proportional to T^4 . The value of the integral can be obtained from tables; it is $\pi^4/15$. Then $U = (8\pi^5 k^4/15 h^3 c^3) T^4$. This result can be combined with Equations 3-4 and 3-6 to express Stefan's constant in terms of π , k , h , and c (see Problem 3-13).

A dramatic example of an application of Planck's law on the current frontier of physics is in tests of the Big Bang theory of the formation and present expansion of the universe. Current cosmological theory holds that the universe originated in an extremely high-temperature explosion of space, one consequence of which was to fill the infant universe with radiation whose spectral distribution must surely have been that of an ideal blackbody. Since that time, the universe has expanded to its present size and cooled to its present temperature T_{now} . However, it should still be filled with radiation whose spectral distribution should be that characteristic of a blackbody at T_{now} .

In 1965, Arno Penzias and Robert Wilson discovered radiation of wavelength 7.35 cm reaching Earth with the same intensity from all directions in space. It was soon recognized that this radiation could be a remnant of the Big Bang fireball, and measurements were subsequently made at other wavelengths in order to construct an experimental energy density $u(\lambda)$ versus λ graph. The most recent data from the Cosmic Background Explorer (COBE) satellite, shown in Figure 3-7, and by the

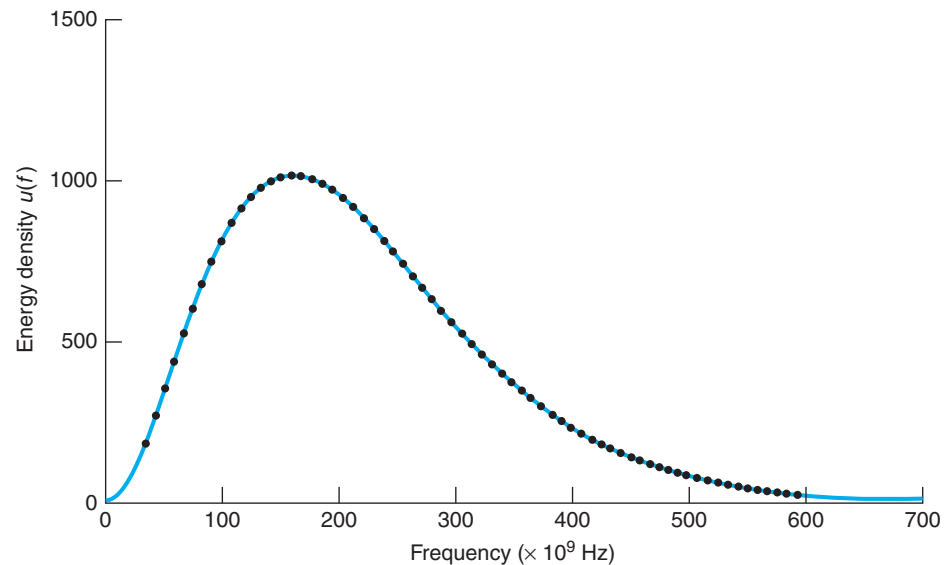


Figure 3-7 The energy density spectral distribution of the cosmic microwave background radiation. The solid line is Planck's law with $T = 2.725$ K. These measurements (the black dots) were made by the COBE satellite.

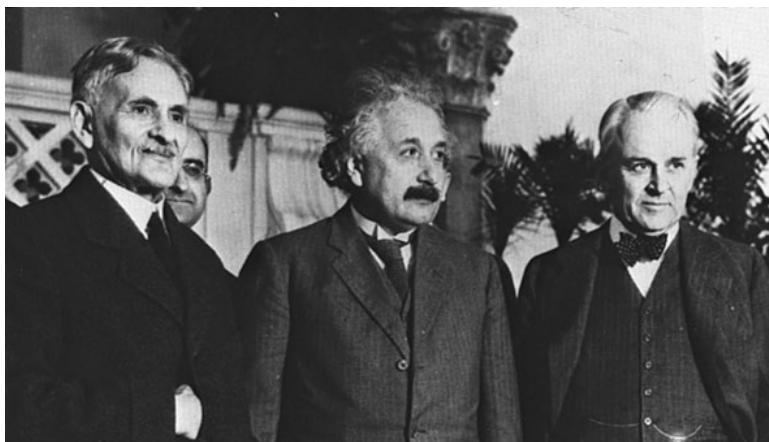
Wilkinson Microwave Anisotropy Probe (WMAP) have established the temperature of the background radiation field at 2.725 ± 0.001 K. The excellent agreement of the data with Planck's equation, indeed, the best fit that has ever been measured, is considered to be very strong support for the Big Bang theory (see Chapter 13).

3-3 The Photoelectric Effect

It is one of the ironies in the history of science that in the famous experiment of Heinrich Hertz¹¹ in 1887, in which he produced and detected electromagnetic waves, thus confirming Maxwell's wave theory of light, he also discovered the photoelectric effect, which led directly to the particle description of light. Hertz was using a spark gap in a tuned circuit to generate the waves and another similar circuit to detect them. He noticed accidentally that when the light from the generating gap was shielded from the receiving gap, the receiving gap had to be made shorter in order for the spark to jump the gap. Light from any spark that fell on the terminals of the gap facilitated the passage of the sparks. He described the discovery with these words:

In a series of experiments on the effects of resonance between very rapid electric oscillations that I had carried out and recently published, two electric sparks were produced by the same discharge of an induction coil, and therefore simultaneously. One of these sparks, spark *B*, was the discharge spark of the induction coil, and served to excite the primary oscillation. I occasionally enclosed spark *B* in a dark case so as to make observations more easily, and in so doing I observed that the maximum spark length became decidedly smaller inside the case than it was before.¹²

The unexpected discovery of the photoelectric effect annoyed Hertz because it interfered with his primary research, but he recognized its importance immediately and interrupted his other work for six months in order to study it in detail. His results, published later that year, were then extended by others. It was found that negative particles were emitted from a clean surface when exposed to light. P. Lenard in 1900 deflected them in a magnetic field and found that they had a charge-to-mass ratio of the same magnitude as that measured by Thomson for cathode rays: the particles being emitted were electrons.



Albert A. Michelson, Albert Einstein, and Robert A. Millikan at a meeting in Pasadena, California, in 1931. [AP/Wide World Photos.]

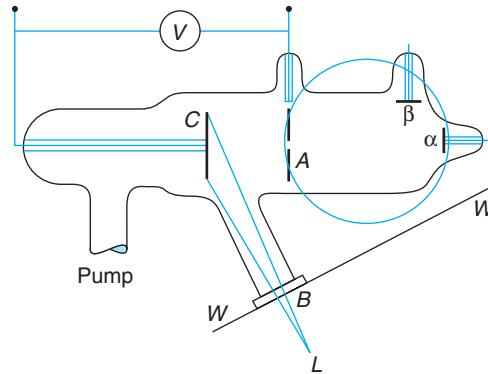


Figure 3-8 Schematic diagram of the apparatus used by P. Lenard to demonstrate the photoelectric effect and to show that the particles emitted in the process were electrons. Light from the source L strikes the cathode C . Photoelectrons going through the hole in anode A are recorded by the electrometer connected to α . A magnetic field, indicated by the circular pole piece, could deflect the particles to a second electrometer connected to β , making possible the establishment of the sign of the charges and their e/m ratio. [P. Lenard, *Annalen der Physik*, 2, 359 (1900).]

Figure 3-8 shows a schematic diagram of the basic apparatus used by Lenard. When light L is incident on a clean metal surface (cathode C), electrons are emitted. If some of these electrons that reach the anode A pass through the small hole, a current results in the external electrometer circuit connected to α . The number of the emitted electrons reaching the anode can be increased or decreased by making the anode positive or negative with respect to the cathode. Letting V be the potential difference between the cathode and anode, Figure 3-9a shows the current versus V for two values of the intensity of light incident on the cathode. When V is positive, the electrons are attracted to the anode. At sufficiently large V all the emitted electrons reach the anode and the current reaches its maximum value. Lenard observed that the maximum current was proportional to the light intensity, an expected result since doubling the energy per unit time incident on the cathode should double the number of electrons emitted. Intensities too low to provide the electrons with the energy necessary to escape from the metal should result in no emission of electrons. However, in contrast with the classical expectation, there was no minimum intensity below which the current was absent. When V is negative, the electrons are repelled from the anode. Then only electrons with initial kinetic energy $mv^2/2$ greater than $e|V|$ can reach the anode. From Figure 3-9a we see that if V is less than $-V_0$, no electrons reach the anode. The potential V_0 is called the *stopping potential*. It is related to the maximum kinetic energy of the emitted electrons by

$$\left(\frac{1}{2}mv^2\right) = eV_0 \quad 3-20$$

The experimental result, illustrated by Figure 3-9a, that V_0 is independent of the incident light intensity was surprising. Apparently, increasing the rate of energy falling on the cathode does not increase the maximum kinetic energy of the emitted electrons, contrary to classical expectations. In 1905 Einstein offered an explanation of this result in a remarkable paper in the same volume of *Annalen der Physik* that contained his papers on special relativity and Brownian motion.

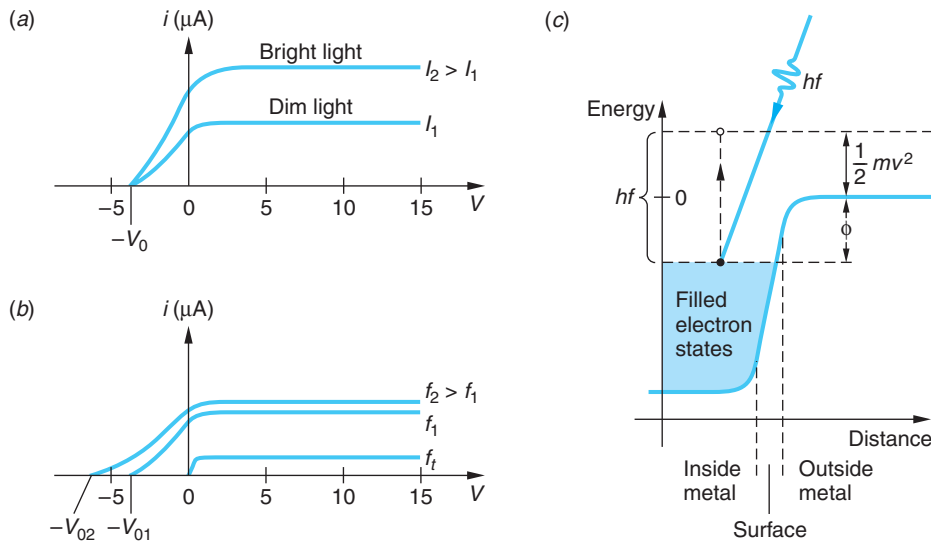


Figure 3-9 (a) Photocurrent i versus anode voltage V for light of frequency f with two intensities I_1 and I_2 , where $I_2 > I_1$. The stopping voltage V_0 is the same for both. (b) For constant I , Einstein's explanation of the photoelectric effect indicates that the magnitude of the stopping voltage should be greater for f_2 than f_1 , as observed, and that there should be a threshold frequency f_t below which no photoelectrons were seen, also in agreement with experiment. (c) Electron potential energy curve across the metal surface. An electron with the highest energy in the metal absorbs a photon of energy hf . Conservation of energy requires that its kinetic energy after leaving the surface be $hf - \phi$.

Einstein assumed that the *energy quantization used by Planck in solving the blackbody radiation problem was, in fact, a universal characteristic of light*. Rather than being distributed evenly in the space through which it propagated, light energy consisted of discrete quanta, each of energy hf . When one of these quanta, called a *photon*, penetrates the surface of the cathode, all of its energy may be absorbed completely by a single electron. If ϕ is the energy necessary to remove an electron from the surface (ϕ is called the *work function* and is a characteristic of the metal), the maximum kinetic energy of an electron leaving the surface will be $hf - \phi$ as a consequence of energy conservation; see Figure 3-9c. (Some electrons will have less than this amount because of energy lost in traversing the metal.) Thus, the stopping potential should be given by

$$eV_0 = \left(\frac{1}{2}mv^2 \right)_{\max} = hf - \phi \quad 3-21$$

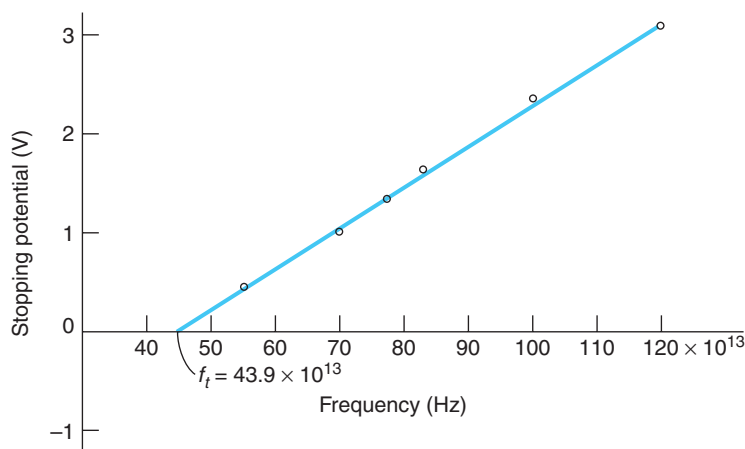
Equation 3-21 is referred to as the photoelectric effect equation. As Einstein noted,

If the derived formula is correct, then V_0 , when represented in Cartesian coordinates as a function of the frequency of the incident light, must be a straight line whose slope is independent of the nature of the emitting substance.¹³

As can be seen from Equation 3-21, the slope of V_0 versus f should equal h/e . At the time of this prediction there was no evidence that Planck's constant had anything to do with the photoelectric effect. There was also no evidence for the dependence of

Among the many applications of the photoelectric effect is the photomultiplier, a device that makes possible the accurate measurement of the energy of the light absorbed by a photosensitive surface. The SNO and Kamiokande neutrino observatories (see Chapter 12) use thousands of photomultipliers. Hundreds more are being deployed inside the Antarctic ice cap in the Ice Cube particle physics experiment.

Figure 3-10 Millikan's data for stopping potential versus frequency for the photoelectric effect. The data fall on a straight line with slope h/e , as predicted by Einstein a decade before the experiment. The intercept on the stopping potential axis is $-\phi/e$. [R. A. Millikan, *Physical Review*, 7, 362 (1915).]



the stopping potential V_0 on the frequency. Careful experiments by Millikan, reported in 1914 and in more detail in 1916, showed that Equation 3-21 was correct and that measurements of h from it agreed with the value obtained by Planck. A plot taken from this work is shown in Figure 3-10.

The minimum, or threshold, frequency for the photoelectric effect, labeled f_t in this plot and in Figure 3-9b, and the corresponding threshold wavelength λ_t are related to the work function ϕ by setting $V_0 = 0$ in Equation 3-21:

$$\phi = hf_t = \frac{hc}{\lambda_t} \quad 3-22$$

Photons of frequencies lower than f_t (and therefore having wavelengths greater than λ_t) do not have enough energy to eject an electron from the metal. Work functions for metals are typically on the order of a few electron volts. The work functions for several elements are given in Table 3-1.

Table 3-1 Photoelectric work functions

Element	Work function (eV)
Na	2.28
C	4.81
Cd	4.07
Al	4.08
Ag	4.73
Pt	6.35
Mg	3.68
Ni	5.01
Se	5.11
Pb	4.14

EXAMPLE 3-5 Photoelectric Effect in Potassium The threshold frequency of potassium is 558 nm. What is the work function for potassium? What is the stopping potential when light of 400 nm is incident on potassium?

SOLUTION

1. Both questions can be answered with the aid of Equation 3-21:

$$eV_0 = \left(\frac{1}{2}mv^2 \right)_{\max} = hf - \phi$$

$$V_0 = \frac{hf}{e} - \frac{\phi}{e}$$

2. At the threshold wavelength the photoelectrons have just enough energy to overcome the work function barrier, so $\left(\frac{1}{2}mv^2 \right)_{\max} = 0$, hence $V_0 = 0$, and

$$\frac{\phi}{e} = \frac{hf_{te}}{e} = \frac{hc}{e\lambda_t}$$

$$= \frac{1240 \text{ eV} \cdot \text{nm}}{558 \text{ nm}} = 2.22 \text{ eV}$$

3. When 400-nm light is used, V_0 is given by Equation 3-21:

$$V_0 = \frac{hf}{e} - \frac{\phi}{e} = \frac{hc}{e\lambda} - \frac{\phi}{e}$$

$$= \frac{1240 \text{ eV} \cdot \text{nm}}{400 \text{ nm}} - 2.22 \text{ eV}$$

$$= 3.10 \text{ eV} - 2.22 \text{ eV} = 0.88 \text{ V}$$

Another interesting feature of the photoelectric effect that is contrary to classical physics but is easily explained by the photon hypothesis is the lack of any time lag between the turning on of the light source and the appearance of photoelectrons. Classically, the incident energy is distributed uniformly over the illuminated surface; the time required for an area the size of an atom to acquire enough energy to allow the emission of an electron can be calculated from the intensity (power per unit area) of the incident radiation. Experimentally, the incident intensity can be adjusted so that the calculated time lag is several minutes or even hours. But no time lag is ever observed. The photon explanation of this result is that although the rate at which photons are incident on the metal is very small when the intensity is low, *each* photon has enough energy to eject an electron, and there is some chance that a photon will be absorbed immediately. The classical calculation gives the correct *average* number of photons absorbed per unit time.

EXAMPLE 3-6 Classical Time Lag Light of wavelength 400 nm and intensity 10^{-2} W/m^2 is incident on potassium. Estimate the time lag for the emission of photoelectrons expected classically.

SOLUTION

According to Example 3-5, the work function for potassium is 2.22 eV. If we assume $r = 10^{-10} \text{ m}$ to be the typical radius of an atom, the total energy falling on the atom in time t is

$$E = (10^{-2} \text{ W/m}^2)(\pi r^2)t = (10^{-2} \text{ W/m}^2)(\pi 10^{-20} \text{ m}^2)t$$

$$= (3.14 \times 10^{-22} \text{ J/s})t$$

Setting this energy equal to 2.22 eV gives

$$(3.14 \times 10^{-22} \text{ J/s})t = (2.22 \text{ eV})(1.60 \times 10^{-19} \text{ J/eV})$$

$$t = \frac{(2.22 \text{ eV})(1.60 \times 10^{-19} \text{ J/eV})}{(3.14 \times 10^{-22} \text{ J/s})} = 1.13 \times 10^3 \text{ s} = 18.8 \text{ min}$$

According to the classical prediction, no atom would be expected to emit an electron until 18.8 min after the light source was turned on. According to the photon model of light, each photon has enough energy to eject an electron immediately. Because of the low intensity, there are few photons incident per second, so the chance of any particular atom absorbing a photon and emitting an electron in any given time interval is small. However, there are so many atoms in the cathode that some emit electrons immediately.

EXAMPLE 3-7 Incident Photon Intensity In Example 3-6, how many photons are incident per second per square meter?

SOLUTION

The energy of each photon is

$$E = hf = hc/\lambda = (1240 \text{ eV} \cdot \text{nm})/(400 \text{ nm}) = (3.10 \text{ eV})(1.60 \times 10^{-19} \text{ J/eV})$$

$$= 4.96 \times 10^{-19} \text{ J}$$

Since the incident intensity is $10^{-2} \text{ W/m}^2 = 10^{-2} \text{ J/s} \cdot \text{m}^2$, the number of photons per second per square meter is

$$N = \frac{10^{-2} \text{ J/s} \cdot \text{m}^2}{4.96 \times 10^{-19} \text{ J/photon}}$$

$$= 2.02 \times 10^{16} \text{ photons/s} \cdot \text{m}^2$$

This is, of course, a lot of photons, not a few; however, the number n per atom at the surface is quite small. $n = 2.02 \times 10^{16} \text{ photons/s} \cdot \text{m}^2 \times \pi(10^{-10})^2 \text{ m}^2/\text{atom} = 6.3 \times 10^{-4} \text{ photons/s} \cdot \text{atom}$, or about 1 photon for every 1000 atoms.

Questions

- How is the result that the maximum photoelectric current is proportional to the intensity explained in the photon model of light?
- What experimental features of the photoelectric effect can be explained by classical physics? What features cannot?

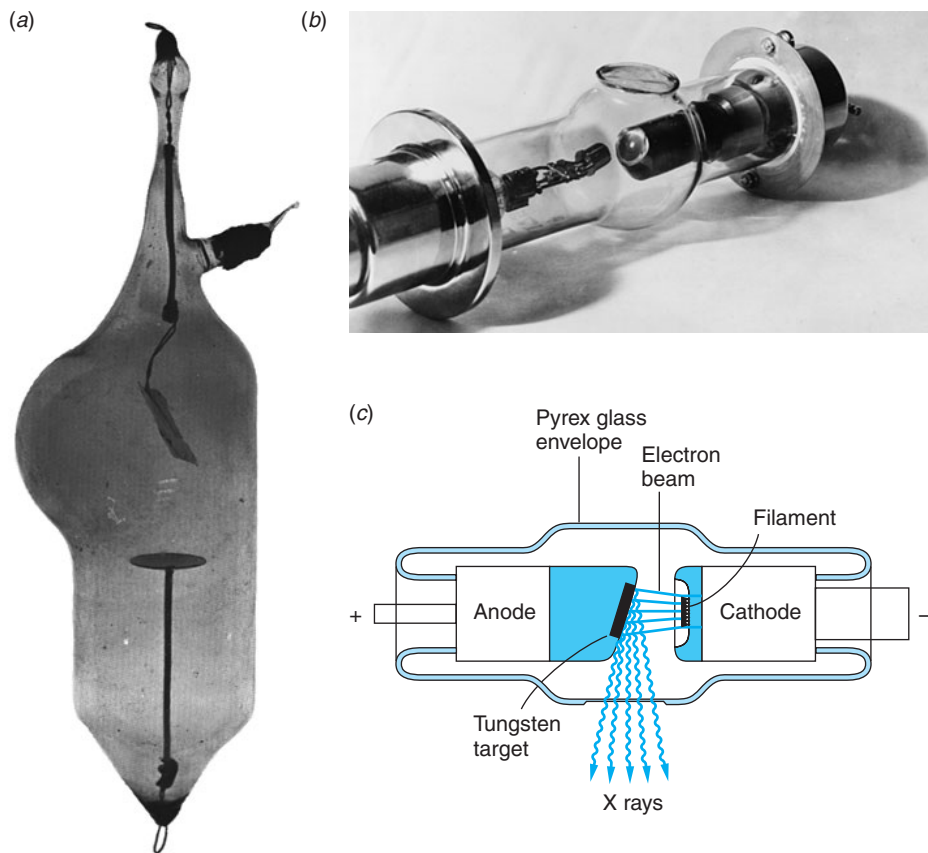
The photoemission of electrons has developed into a significant technique for investigating the detailed structure of molecules and solids, making possible discoveries far beyond anything that Hertz may have imagined. The use of x-ray sources (see Section 3-4) and precision detectors has made possible precise determination of valence electron configurations in chemical compounds, leading to detailed understanding of chemical bonding and the differences between the bulk and surface atoms of solids. Photoelectric-effect microscopes will show the chemical situation of each element in a specimen, a prospect of intriguing and crucial importance in molecular biology and microelectronics. And they are all based on a discovery that annoyed Hertz—at first.

3-4 X Rays and the Compton Effect

Further evidence of the correctness of the photon concept was furnished by Arthur H. Compton, who measured the scattering of x rays by free electrons and, by his analysis of the data, resolved the last lingering doubts regarding special relativity (see Chapter 1). Before we examine Compton scattering in detail, we will briefly describe some of the early work with x rays since it provides a good conceptual understanding of x-ray spectra and scattering.

X Rays

The German physicist Wilhelm K. Roentgen discovered x rays in 1895 when he was working with a cathode-ray tube. Coming five years before Planck's explanation of the blackbody emission spectrum, Roentgen's discovery turned out to be the first significant development in quantum physics. He found that "rays" originating from the point where the cathode rays (electrons) hit the glass tube, or a target within the tube, could pass through materials opaque to light and activate a fluorescent screen or photographic film. He investigated this phenomenon extensively and found that all materials are transparent to these rays to some degree and that the transparency decreases with increasing density. This fact led to the medical use of x rays within months after the publication of Roentgen's first paper.¹⁴



(a) Early x-ray tube. [Courtesy of Cavendish Laboratory.] (b) X-ray tubes became more compact over time. This tube was a design typical of the mid-twentieth century. [Courtesy of Schenectady Museum, Hall of Electrical History, Schenectady, NY.] (c) Diagram of the components of a modern x-ray tube. Design technology has advanced enormously, enabling very high operating voltages, beam currents, and x-ray intensities, but essential elements of the tubes remain unchanged.

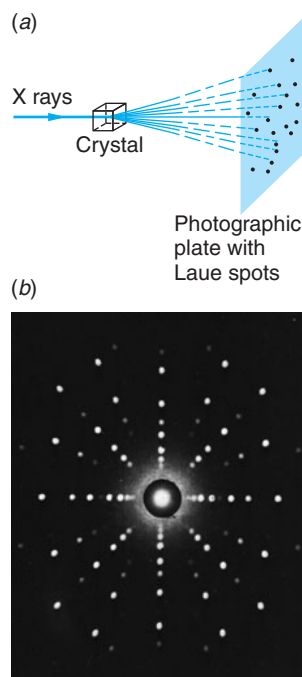


Figure 3-11 (a) Schematic sketch of a Laue experiment. The crystal acts as a three-dimensional grating, which diffracts the x-ray beam and produces a regular array of spots, called a *Laue pattern*, on photographic film or an x-ray-sensitive charge-coupled device (CCD) detector. (b) Laue x-ray diffraction pattern using a niobium boride crystal and 20-keV molybdenum x rays. [General Electric Company.]



An x ray of Mrs. Roentgen's hand taken by Roentgen shortly after his discovery.

Roentgen was unable to deflect these rays in a magnetic field, nor was he able to observe refraction or the interference phenomena associated with waves. He thus gave the rays the somewhat mysterious name of x rays. Since classical electromagnetic theory predicts that accelerated charges will radiate electromagnetic waves, it is natural to expect that x rays are electromagnetic waves produced by the acceleration of the electrons when they are deflected and stopped by the atoms of a target. Such radiation is called *bremsstrahlung*, German for “braking radiation.” The slight diffraction broadening of an x-ray beam after passing through slits a few thousandths of a millimeter wide indicated the wavelength of x rays to be of the order of $10^{-10} \text{ m} = 0.1 \text{ nm}$. In 1912 Max von Laue suggested that since the wavelengths of x rays were of the same order of magnitude as the spacing of atoms in a crystal, the regular array of atoms in a crystal might act as a three-dimensional grating for the diffraction of x rays. Experiments (Figure 3-11) soon confirmed that x rays are a form of electromagnetic radiation with wavelengths in the range of about 0.01 to 0.10 nm and that atoms in crystals are arranged in regular arrays.

W. L. Bragg, in 1912, proposed a simple and convenient way of analyzing the diffraction of x rays by crystals.¹⁵ He examined the interference of x rays due to scattering from various sets of parallel planes of atoms, now called *Bragg planes*. Two sets of Bragg planes are illustrated in Figure 3-12 for NaCl, which has a cubic structure called *face-centered cubic*. Consider Figure 3-13. Waves scattered from the two successive atoms within a plane will be in phase and thus interfere constructively, independent of the wavelength, if the scattering angle equals the incident angle. (This condition is the same as for reflection.) Waves scattered at equal angles from atoms in two

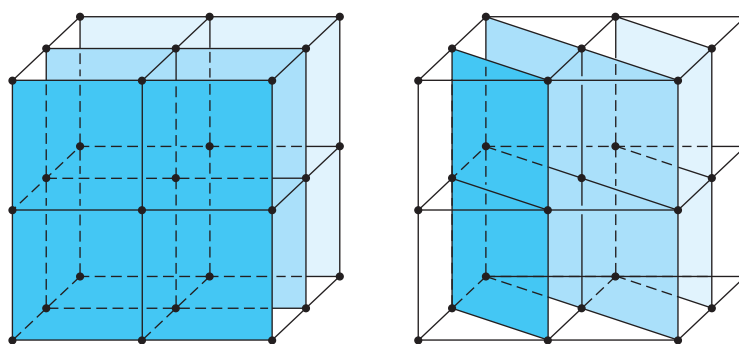


Figure 3-12 A crystal of NaCl showing two sets of Bragg planes.

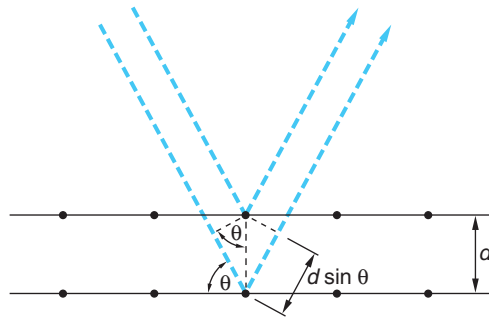


Figure 3-13 Bragg scattering from two successive planes. The waves from the two atoms shown have a path length difference of $2d \sin \theta$. They will be in phase if the Bragg condition $2d \sin \theta = m\lambda$ is met.

different planes will be in phase (constructive interference) only if the difference in path length is an integral number of wavelengths. From Figure 3-13 we see that this condition is satisfied if

$$2d \sin \theta = m\lambda \quad \text{where } m = \text{an integer} \quad \mathbf{3-23}$$

Equation 3-23 is called the *Bragg condition*.

Measurements of the spectral distribution of the intensity of x rays as a function of the wavelength using an experimental arrangement such as that shown in Figure 3-14 produces the x-ray spectrum and, for classical physics, some surprises. Figure 3-15a shows two typical x-ray spectra produced by accelerating electrons through two voltages V and bombarding a tungsten target mounted on the anode of the tube. In this figure $I(\lambda)$ is the intensity emitted within the wavelength interval $d\lambda$ for each value of λ . Figure 3-15b shows the short wavelength lines produced with a molybdenum target and 35-keV electrons. Three features of the spectra are of immediate interest, only one of which could be explained by classical physics. (1) The spectrum consists of a series of sharp lines, called the *characteristic spectrum*, superimposed on (2) the continuous

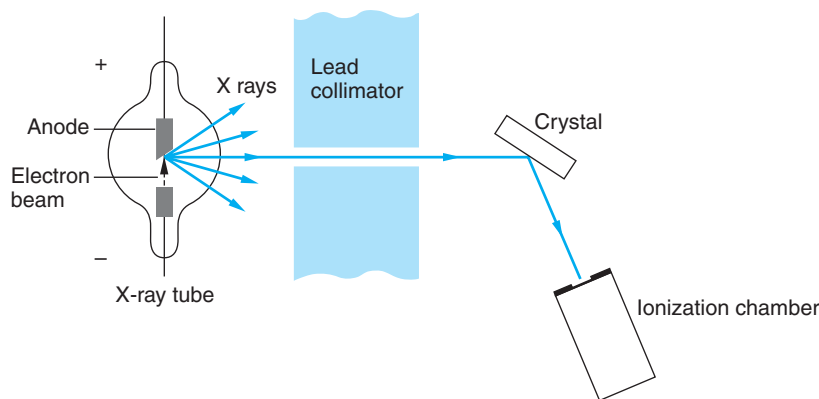


Figure 3-14 Schematic diagram of a Bragg crystal spectrometer. A collimated x-ray beam is incident on a crystal and scattered into an ionization chamber. The crystal and ionization chamber can be rotated to keep the angles of incidence and scattering equal as both are varied. By measuring the ionization in the chamber as a function of angle, the spectrum of the x rays can be determined using the Bragg condition $2d \sin \theta = m\lambda$, where d is the separation of the Bragg planes in the crystal. If the wavelength λ is known, the spacing d can be determined.

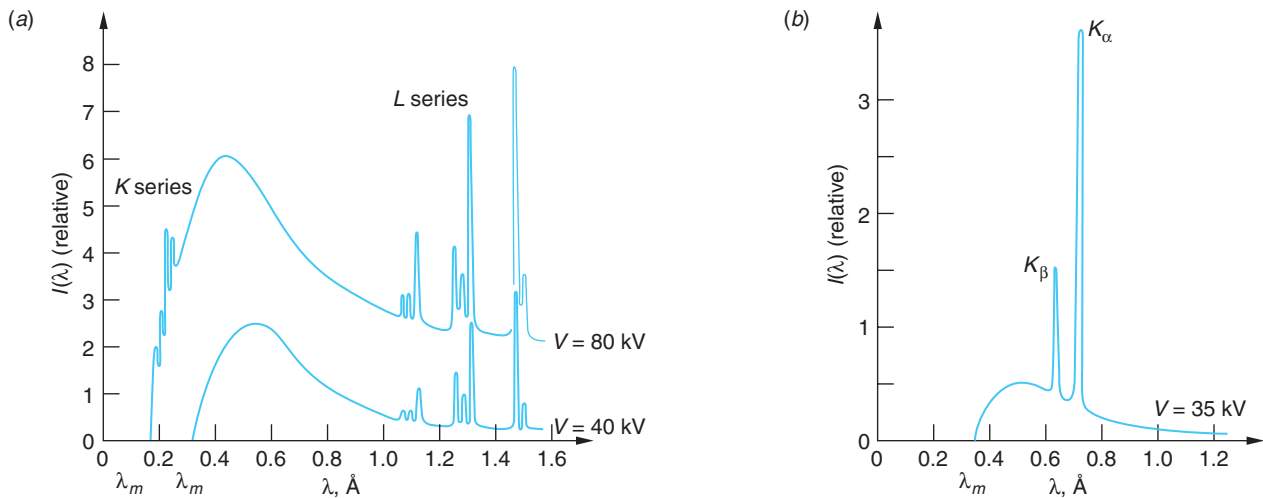


Figure 3-15 (a) X-ray spectra from tungsten at two accelerating voltages and (b) from molybdenum at one. The names of the line series (K and L) are historical and explained in Chapter 4. The L -series lines for molybdenum (not shown) are at about 0.5 nm . The cutoff wavelength λ_m is independent of the target element and is related to the voltage on the x-ray tube V by $\lambda_m = hc/eV$. The wavelengths of the lines are characteristic of the element.

bremsstrahlung spectrum. The line spectrum is characteristic of the target material and varies from element to element. (3) The continuous spectrum has a sharp cutoff wavelength, λ_m , which is independent of the target material but depends on the energy of the bombarding electrons. If the voltage on the x-ray tube is V volts, the cutoff wavelength is found empirically to be given by

$$\lambda_m = \frac{1.24 \times 10^3}{V} \text{ nm} \quad 3-24$$

Equation 3-24 is called the *Duane-Hunt rule*, after its discoverers. It was pointed out rather quickly by Einstein that x-ray production by electron bombardment was an inverse photoelectric effect and that Equation 3-21 should apply. The Duane-Hunt λ_m simply corresponds to a photon with the maximum energy of the electrons, that is, the photon emitted when the electron loses all of its kinetic energy in a single collision. Since the kinetic energy of the electrons in an x-ray tube is $20,000 \text{ eV}$ or higher, the work function ϕ (a few eV) is negligible by comparison. That is, Equation 3-21 becomes $eV \approx hf = hc/\lambda_m$ or $\lambda_m = hc/eV = 1.2407 \times 10^{-6} V^{-1} \text{ m} = 1.24 \times 10^3 V^{-1} \text{ nm}$. Thus, the Duane-Hunt rule is explained by Planck's quantum hypothesis. (Notice that the value of λ_m can be used to determine h/e .)

The continuous spectrum was understood as the result of the acceleration (i.e., “braking”) of the bombarding electrons in the strong electric fields of the target atoms. Maxwell's equation predicted the continuous radiation. The real problem for classical physics was the sharp lines. The wavelengths of the sharp lines were a function of the target element, the set for each element being always the same. But the sharp lines never appeared if V was such that λ_m was larger than the particular line, as can be seen from Figure 3-15a, where the shortest-wavelength group disappears when V is reduced from 80 keV to 40 keV so that λ_m becomes larger. The origin of the sharp lines was a mystery that had to await the discovery of the nuclear atom. We will explain them in Chapter 4.

Well-known applications of x rays are medical and dental x rays (both diagnostic and treatment) and industrial x-ray inspection of welds and castings. Perhaps not so well known is the use of x rays in determining the structure of crystals, identifying black holes in the cosmos, and “seeing” the folded shapes of proteins in biological materials.

Compton Effect

It had been observed that scattered x rays were “softer” than those in the incident beam, that is, were absorbed more readily. Compton¹⁶ pointed out that if the scattering process were considered a “collision” between a photon of energy hf_1 (and momentum hf_1/c) and an electron, the recoiling electron would absorb part of the incident photon’s energy. The energy hf_2 of the scattered photon would therefore be less than the incident one and thus of lower frequency f_2 and momentum hf_2/c . (The fact that electromagnetic radiation of energy E carried momentum E/c was known from classical theory and from the experiments of Nichols and Hull in 1903. This relation is also consistent with the relativistic expression $E^2 = p^2c^2 + (mc^2)^2$ for a particle with zero rest energy.) Compton applied the laws of conservation of momentum and energy in their relativistic form (see Chapter 2) to the collision of a photon with an isolated electron to obtain the change in the wavelength $\lambda_2 - \lambda_1$ of the photon as a function of the scattering angle θ . The result, called *Compton’s equation* and derived in a More section on the home page, is

$$\lambda_2 - \lambda_1 = \frac{h}{mc}(1 - \cos \theta) \quad 3-25$$

The change in wavelength is thus predicted to be independent of the original wavelength. The quantity h/mc has the dimensions of length and is called the *Compton wavelength of the electron*. Its value is

$$\lambda_c = \frac{h}{mc} = \frac{hc}{mc^2} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{5.11 \times 10^5 \text{ eV}} = 0.00243 \text{ nm}$$

Because $\lambda_2 - \lambda_1$ is small, it is difficult to observe unless λ_1 is very small so that the fractional change $(\lambda_2 - \lambda_1)/\lambda_1$ is appreciable. For this reason the Compton effect is generally only observed for x rays and gamma radiation.

Compton verified his result experimentally using the characteristic x-ray line of wavelength 0.0711 nm from molybdenum for the incident monochromatic photons and scattering these photons from electrons in graphite. The wavelength of the scattered photons was measured using a Bragg crystal spectrometer. His experimental arrangement is shown in Figure 3-16; Figure 3-17 shows his results. The first peak at

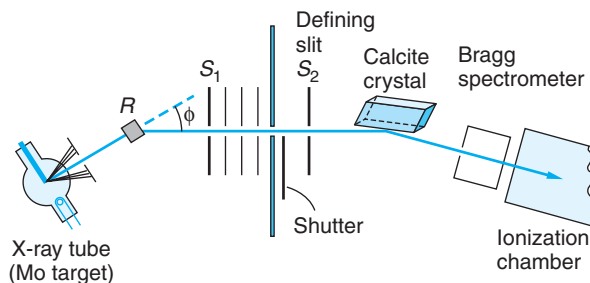


Figure 3-16 Schematic sketch of Compton’s apparatus. X rays from the tube strike the carbon block R and are scattered into a Bragg-type crystal spectrometer. In this diagram, the scattering angle is 30° . The beam was defined by slits S_1 and S_2 . Although the entire spectrum is being scattered by R , the spectrometer scanned the region around the K_α line of molybdenum.

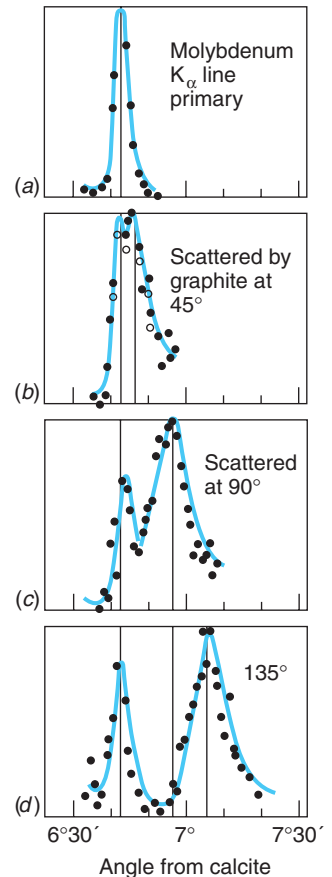


Figure 3-17 Intensity versus wavelength for Compton scattering at several angles. The left peak in each case results from photons of the original wavelength that are scattered by tightly bound electrons, which have an effective mass equal to that of the atom. The separation in wavelength of the peaks is given by Equation 3-25. The horizontal scale used by the Compton “angle from calcite” refers to the calcite analyzing crystal in Figure 3-16.

Arthur Compton. After discovering the Compton effect, he became a world traveler seeking an explanation for cosmic rays. He ultimately showed that their intensity varied with latitude, indicating an interaction with Earth's magnetic field, and thus proved that they are charged particles. [Courtesy of American Institute of Physics, Niels Bohr Library.]



each scattering angle corresponds to scattering with no shift in the wavelength due to scattering by the inner electrons of carbon. Since these are tightly bound to the atom, it is the entire atom that recoils rather than the individual electrons. The expected shift in this case is given by Equation 3-25, with m being the mass of the atom, which is about 10^4 times that of the electron; thus, this shift is negligible. The variation of $\Delta\lambda = \lambda_2 - \lambda_1$ with θ was found to be that predicted by Equation 3-25.

We have seen in this section and the preceding two sections that the interaction of electromagnetic radiation with matter is a discrete interaction that occurs at the atomic level. It is perhaps curious that after so many years of debate about the nature of light, we now find that we must have both a particle (i.e., quantum) theory to describe in detail the energy exchange between electromagnetic radiation and matter and a wave theory to describe the interference and diffraction of electromagnetic radiation. We will discuss this so-called wave-particle duality in more detail in Chapter 5.



More

Derivation of Compton's Equation, applying conservation of energy and momentum to the relativistic collision of a photon and an electron, is included on the home page: www.whfreeman.com/tiplermodernphysics5e. See also Equations 3-26 and 3-27 and Figure 3-18 here.

Questions

5. Why is it extremely difficult to observe the Compton effect using visible light?
6. Why is the Compton effect unimportant in the transmission of television and radio waves? How many Compton scatterings would a typical FM signal have before its wavelengths were shifted by 0.01 percent?

EXAMPLE 3-8 X Rays from TV The acceleration voltage of the electrons in a typical television picture tube is 25 keV. What is the minimum wavelength x ray produced when these electrons strike the inner front surface of the tube?

SOLUTION

From Equation 3-24, we have

$$\lambda_m = \frac{1.24 \times 10^3}{V} \text{ nm} = \frac{1.24 \times 10^3}{25,000} = 0.050 \text{ nm}$$

These x rays penetrate matter very effectively. Manufacturers provide essential shields to protect against the hazard.

EXAMPLE 3-9 Compton Effect In a particular Compton scattering experiment it is found that the incident wavelength λ_1 is shifted by 1.5 percent when the scattering angle $\theta = 120^\circ$. (a) What is the value of λ_1 ? (b) What will be the wavelength λ_2 of the shifted photon when the scattering angle is 75° ?

SOLUTION

1. For question (a), the value of λ_1 is found from Equation 3-25:

$$\begin{aligned} \lambda_2 - \lambda_1 &= \Delta\lambda = \frac{h}{mc}(1 - \cos\theta) \\ &= 0.00243(1 - \cos 120^\circ) \end{aligned}$$

2. That the scattered wavelength λ_2 is shifted by 1.5 percent from λ_1 means that

$$\frac{\Delta\lambda}{\lambda_1} = 0.015$$

3. Combining these yields:

$$\begin{aligned} \lambda_1 &= \frac{\Delta\lambda}{0.015} = \frac{0.00243(1 - \cos 120^\circ)}{0.015} \\ &= 0.243 \text{ nm} \end{aligned}$$

4. Question (b) is also solved with the aid of Equation 3-25, rearranged as

$$\lambda_2 = \lambda_1 + 0.00243(1 - \cos\theta)$$

5. Substituting $\theta = 75^\circ$ and λ_1 from above yields

$$\begin{aligned} \lambda_2 &= 0.243 + 0.00243(1 - \cos 75^\circ) \\ &= 0.243 + 0.002 \\ &= 0.245 \text{ nm} \end{aligned}$$

A Final Comment

In this chapter, together with Section 2-4 of the previous chapter, we have introduced and discussed at some length the three primary ways by which photons interact with matter: (1) the photoelectric effect, (2) the Compton effect, and (3) pair production. As we proceed with our explorations of modern physics throughout the remainder of the book, we will have many occasions to apply what we have learned here to aid in our understanding of myriad phenomena, ranging from atomic structure to the fusion “furnaces” of stars.

Summary

TOPIC	RELEVANT EQUATIONS AND REMARKS
1. J. J. Thomson's experiment	Thomson's measurements with cathode rays showed that the same particle (the electron), with e/m about 2000 times that of ionized hydrogen, exists in all elements.
2. Quantization of electric charge	$e = 1.60217653 \times 10^{-19} \text{ C}$
3. Blackbody radiation	
Stefan-Boltzmann law	$R = \sigma T^4$ 3-4
Wein's displacement law	$\lambda_m T = 2.898 \times 10^{-3} \text{ m} \cdot \text{K}$ 3-5
Planck's radiation law	$u(\lambda) = \frac{8\pi hc \lambda^{-5}}{e^{hc/\lambda kT} - 1}$ 3-18
Planck's constant	$h = 6.626 \times 10^{-34} \text{ J} \cdot \text{s}$ 3-19
4. Photoelectric effect	$eV_0 = hf - \phi$ 3-21
5. Compton effect	$\lambda_2 - \lambda_1 = \frac{h}{mc}(1 - \cos \theta)$ 3-25
6. Photon-matter interaction	The (1) photoelectric effect, (2) Compton effect, and (3) pair production are the three ways of interaction.

General References

The following general references are written at a level appropriate for readers of this book.

Millikan, R. A., *Electrons (+ and -) Protons, Photons, Neutrons, Mesotrons, and Cosmic Rays*, 2d ed., University of Chicago Press, Chicago, 1947. This book on modern physics by one of the great experimentalists of his time contains fascinating, detailed descriptions of Millikan's oil drop experiment and his verification of the Einstein photoelectric-effect equation.

Mohr, P. J., and B. N. Taylor, "The Fundamental Physical Constants," *Physics Today* (August 2004). Also available at <http://www.physicstoday.org/guide/fundcont.html>.

Richtmyer, F. K., E. H. Kennard, and J. N. Cooper, *Introduction to Modern Physics*, 6th ed., McGraw-Hill, New York, 1969. This excellent text was originally published in 1928, intended as a survey course for graduate students.

Shamos, M. H. (ed.), *Great Experiments in Physics*, Holt, Rinehart, and Winston, New York, 1962. This book contains 25 original papers and extensive editorial comment. Of particular interest for this chapter are papers by Faraday, Hertz, Roentgen, J. J. Thomson, Einstein (photoelectric effect), Millikan, Planck, and Compton.

Thomson, G. P., *J. J. Thomson, Discoverer of the Electron*, Doubleday/Anchor, Garden City, NY, 1964. An interesting study of J. J. Thomson by his son, also a physicist.

Virtual Laboratory (PEARL), Physics Academic Software, North Carolina State University, Raleigh, 1996. Computer simulation software allows the user to analyze blackbody radiation emitted over a wide range of temperatures and investigate the Compton effect in detail.

Weart, S. R., *Selected Papers of Great American Physicists*, American Institute of Physics, New York, 1976. The bi-centennial commemorative volume of the American Physical Society.

Notes

1. Democritus (about 470 B.C. to about 380 B.C.). Among his other modern-sounding ideas were the suggestions that the Milky Way is a vast conglomeration of stars and that the Moon, like Earth, has mountains and valleys.

2. G. J. Stoney (1826–1911). An Irish physicist who first called the fundamental unit of charge the electron. After Thomson discovered the particle that carried the charge, the name was transferred from the quantity of charge to the particle itself by Lorentz.

3. Joseph J. Thomson (1856–1940). English physicist and director for more than 30 years of the Cavendish Laboratory, the first laboratory in the world established expressly for research in physics. He was awarded the Nobel Prize in 1906 for his work on the electron. Seven of his research assistants also won Nobel Prizes.

4. Much early confusion existed about the nature of cathode rays due to the failure of Heinrich Hertz in 1883 to observe any deflection of the rays in an electric field. The failure was later found to be the result of ionization of the gas in the tube; the ions quickly neutralized the charges on the deflecting plates so that there was actually no electric field between the plates. With better vacuum technology in 1897, Thomson was able to work at lower pressure and observe electrostatic deflection.

5. R. A. Millikan, *Philosophical Magazine* (6), **19**, 209 (1910). Millikan, who held the first physics Ph.D. awarded by Columbia University, was one of the most accomplished experimentalists of his time. He received the Nobel Prize in 1923 for the measurement of the electron's charge. Also among his many contributions, he coined the phrase *cosmic rays* to describe radiation produced in outer space.

6. R. A. Millikan, *Physical Review*, **32**, 349 (1911).

7. Mohr, P. J., and B. N. Taylor, "The Fundamental Physical Constants," *Physics Today* (August 2004).

8. See pp. 135–137 of F. K. Richtmyer, E. H. Kennard, and J. N. Cooper (1969).

9. John W. S. Rayleigh (1842–1919). English physicist, almost invariably referred to by the title he inherited from his father. He was Maxwell's successor and Thomson's predecessor as director of the Cavendish Laboratory.

10. Max K. E. L. Planck (1858–1947). Most of his career was spent at the University of Berlin. In his later years his renown in the world of science was probably second only to that of Einstein.

11. Heinrich R. Hertz (1857–1894), German physicist, student of Helmholtz. He was the discoverer of electromagnetic "radio" waves, later developed for practical communication by Marconi.

12. H. Hertz, *Annalen der Physik*, **31**, 983 (1887).

13. A. Einstein, *Annalen der Physik*, **17**, 144 (1905).

14. A translation of this paper can be found in E. C. Watson, *American Journal of Physics*, **13**, 284 (1945), and in Shamos (1962). Roentgen (1845–1923) was honored in 1901 with the first Nobel Prize in Physics for his discovery of x rays.

15. William Lawrence Bragg (1890–1971), Australian-English physicist. An infant prodigy, his work on x-ray diffraction performed with his father, William Henry Bragg (1862–1942), earned for them both the Nobel Prize in Physics in 1915, the only father-son team to be so honored thus far. In 1938 W. L. Bragg became director of the Cavendish Laboratory, succeeding Rutherford.

16. Arthur H. Compton (1892–1962), American physicist. It was Compton who suggested the name *photon* for the light quantum. His discovery and explanation of the Compton effect earned him a share of the Nobel Prize in Physics in 1927.

Problems

Level I

Section 3-1 Quantization of Electric Charge

3-1. A beam of charged particles consisting of protons, electrons, deuterons, and singly ionized helium atoms and H_2 molecules all pass through a velocity selector, all emerging with speeds of 2.5×10^6 m/s. The beam then enters a region of uniform magnetic field $B = 0.40$ T directed perpendicular to their velocity. Compute the radius of curvature of the path of each type of particle.

3-2. Consider Thomson's experiment with the electric field turned "off." If the electrons enter a region of uniform magnetic field B and length l , show that the electrons are deflected through an angle $\theta \approx e\ell B/mu\ell$ for small values of θ . (Assume that the electrons are moving at non-relativistic speeds.)

3-3. Equation 3-3 suggests how a velocity selector for particles or mixtures of different particles all having the same charge can be made. Suppose you wish to make a velocity selector that allows undeflected passage for electrons whose kinetic energy is 5.0×10^4 eV. The electric field available to you is 2.0×10^5 V/m. What magnetic field will be needed?

3-4. A cosmic ray proton approaches Earth vertically at the equator, where the horizontal component of Earth's magnetic field is 3.5×10^{-5} T. If the proton is moving at 3.0×10^6 m/s, what is the ratio of the magnetic force to the gravitational force on the proton?

3-5. An electron of kinetic energy 45 keV moves in a circular orbit perpendicular to a magnetic field of 0.325 T. (a) Compute the radius of the orbit. (b) Find the period and frequency of the motion.

3-6. If electrons have kinetic energy of 2000 eV, find (a) their speed, (b) the time needed to traverse a distance of 5 cm between plates *D* and *E* in Figure 3-1, and (c) the vertical component of their velocity after passing between the plates if the electric field is 3.33×10^3 V/m.

3-7. In J. J. Thomson's first method (see Problem 3-44), the heat capacity of the beam stopper was about 5×10^{-3} cal/°C and the temperature increase was about 2°C. How many 2000-eV electrons struck the beam stopper?

3-8. On drop #16, Millikan measured the following total charges, among others, at different times:

$$\begin{array}{lll} 25.41 \times 10^{-19} \text{ C} & 17.47 \times 10^{-19} \text{ C} & 12.70 \times 10^{-19} \text{ C} \\ 20.64 \times 10^{-19} \text{ C} & 19.06 \times 10^{-19} \text{ C} & 14.29 \times 10^{-19} \text{ C} \end{array}$$

What value of the fundamental quantized charge e do these numbers imply?

3-9. Show that the electric field needed to make the rise time of the oil drop equal to its field-free fall time is $\mathcal{E} = 2mg/q$.

3-10. One variation of the Millikan oil drop apparatus arranges the electric field horizontally, rather than vertically, giving charged droplets acceleration in the horizontal direction. The result is that the droplet falls in a straight line that makes an angle θ with the vertical. Show that

$$\sin \theta = q\mathcal{E}/bv'_t$$

where v'_t is the terminal speed along the angled path.

3-11. A charged oil droplet falls 5.0 mm in 20.0 s at terminal speed in the absence of an electric field. The specific gravity of air is 1.35×10^{-3} and that of oil is 0.75. The viscosity of air is 1.80×10^{-5} N·s/m². (a) What are the mass and radius of the drop? (b) If the droplet carries two units of electric charge and is in an electric field of 2.5×10^5 V/m, what is the ratio of the electric force to the gravitational force on the droplet?

Section 3-2 Blackbody Radiation

3-12. Find λ_m for blackbody radiation at (a) $T = 3$ K, (b) $T = 300$ K, and (c) $T = 3000$ K.

3-13. Use the result of Example 3-4 and Equations 3-4 and 3-6 to express Stefan's constant in terms of h , c , and k . Using the known values of these constants, calculate Stefan's constant.

3-14. Show that Planck's law, Equation 3-18, expressed in terms of the frequency f , is

$$u(f) = \frac{8\pi f^2}{c^3} \frac{hf}{e^{hf/kT} - 1}$$

3-15. As noted in the chapter, the cosmic microwave background radiation fits the Planck equation for a blackbody at 2.7 K. (a) What is the wavelength at the maximum intensity of the spectrum of the background radiation? (b) What is the frequency of the radiation at the maximum? (c) What is the total power incident on Earth from the background radiation?

3-16. Find the temperature of a blackbody if its spectrum has its peak at (a) $\lambda_m = 700$ nm (visible), (b) $\lambda_m = 3$ cm (microwave region), and (c) $\lambda_m = 3$ m (FM radio waves).

3-17. If the absolute temperature of a blackbody is doubled, by what factor is the total emitted power increased?

3-18. Calculate the average energy \bar{E} per mode of oscillation for (a) a long wavelength $\lambda = 10 hc/kT$, (b) a short wavelength $\lambda = 0.1 hc/kT$, and compare your results with the classical prediction kT (see Equation 3-9). (The classical value comes from the equipartition theorem discussed in Chapter 8.)

- 3-19.** A particular radiating cavity has the maximum of its spectra distribution of radiated power at a wavelength of $27.0\ \mu\text{m}$ (in the infrared region of the spectrum). The temperature is then changed so that the total power radiated by the cavity doubles. (a) Compute the new temperature. (b) At what wavelength does the new spectral distribution have its maximum value?
- 3-20.** A certain very bright star has an effective surface temperature of $20,000\ \text{K}$. (a) Assuming that it radiates as a blackbody, what is the wavelength at which $u(\lambda)$ is maximum? (b) In what part of the electromagnetic spectrum does the maximum lie?
- 3-21.** The energy reaching Earth from the Sun at the top of the atmosphere is $1.36 \times 10^3\ \text{W/m}^2$, called the *solar constant*. Assuming that Earth radiates like a blackbody at uniform temperature, what do you conclude is the equilibrium temperature of Earth?
- 3-22.** A 40-W incandescent bulb radiates from a tungsten filament operating at $3300\ \text{K}$. Assuming that the bulb radiates like a blackbody, (a) what are the frequency f_m and the wavelength λ_m at the maximum of the spectral distribution? (b) If f_m is a good approximation of the average frequency of the photons emitted by the bulb, about how many photons is the bulb radiating per second? (c) If you are looking at the bulb from $5\ \text{m}$ away, how many photons enter your eye per second? (The diameter of your pupil is about $5.0\ \text{mm}$.)
- 3-23.** Use Planck's law, Equation 3-18, to derive the constant in Wein's law, Equation 3-5.

Section 3-3 The Photoelectric Effect

- 3-24.** The wavelengths of visible light range from about $380\ \text{nm}$ to about $750\ \text{nm}$. (a) What is the range of photon energies (in eV) in visible light? (b) A typical FM radio station's broadcast frequency is about $100\ \text{MHz}$. What is the energy of an FM photon of the frequency?
- 3-25.** The orbiting space shuttle moves around Earth well above 99 percent of the atmosphere, yet it still accumulates an electric charge on its skin due, in part, to the loss of electrons caused by the photoelectric effect with sunlight. Suppose the skin of the shuttle is coated with Ni, which has a relatively large work function $\phi = 4.87\ \text{eV}$ at the temperatures encountered in orbit. (a) What is the maximum wavelength in the solar spectrum that can result in the emission of photoelectrons from the shuttle's skin? (b) What is the maximum fraction of the total power falling on the shuttle that could potentially produce photoelectrons?
- 3-26.** The work function for cesium is $1.9\ \text{eV}$, the lowest of any metal. (a) Find the threshold frequency and wavelength for the photoelectric effect. Find the stopping potential if the wavelength of the incident light is (b) $300\ \text{nm}$ and (c) $400\ \text{nm}$.
- 3-27.** (a) If 5 percent of the power of a 100-W bulb is radiated in the visible spectrum, how many visible photons are radiated per second? (b) If the bulb is a point source radiating equally in all directions, what is the flux of photons (number per unit time per unit area) at a distance of $2\ \text{m}$?
- 3-28.** The work function of molybdenum is $4.22\ \text{eV}$. (a) What is the threshold frequency for the photoelectric effect in molybdenum? (b) Will yellow light of wavelength $560\ \text{nm}$ cause ejection of photoelectrons from molybdenum? Prove your answer.
- 3-29.** The NaCl molecule has a bond energy of $4.26\ \text{eV}$; that is, this energy must be supplied in order to dissociate the molecule into neutral Na and Cl atoms (see Chapter 9). (a) What are the minimum frequency and maximum wavelength of the photon necessary to dissociate the molecule? (b) In what part of the electromagnetic spectrum is this photon?
- 3-30.** A photoelectric experiment with cesium yields stopping potentials for $\lambda = 435.8\ \text{nm}$ and $\lambda = 546.1\ \text{nm}$ to be $0.95\ \text{V}$ and $0.38\ \text{V}$, respectively. Using these data only, find the threshold frequency and work function for cesium and the value of h .
- 3-31.** Under optimum conditions, the eye will perceive a flash if about 60 photons arrive at the cornea. How much energy is this in joules if the wavelength of the light is $550\ \text{nm}$?
- 3-32.** The longest wavelength of light that will cause emission of electrons from cesium is $653\ \text{nm}$. (a) Compute the work function for cesium. (b) If light of $300\ \text{nm}$ (ultraviolet) were to shine on cesium, what would be the energy of the ejected electrons?

Section 3-4 X Rays and the Compton Effect

- 3-33.** Use Compton's equation (Equation 3-25) to compute the value of $\Delta\lambda$ in Figure 3-17d. To what percent shift in the wavelength does this correspond?

3-34. X-ray tubes currently used by dentists often have accelerating voltages of 80 kV. What is the minimum wavelength of the x rays they produce?

3-35. Find the momentum of a photon in eV/c and in $\text{kg} \cdot \text{m/s}$ if the wavelength is (a) 400 nm, (b) $1 \text{ \AA} = 0.1 \text{ nm}$, (c) 3 cm, and (d) 2 nm.

3-36. Gamma rays emitted by radioactive nuclei also exhibit measurable Compton scattering. Suppose a 0.511-MeV photon from a positron-electron annihilation scatters at 110° from a free electron. What are the energies of the scattered photon and the recoiling electron? Relative to the initial direction of the 0.511-MeV photon, what is the direction of the recoiling electron's velocity vector?

3-37. The wavelength of Compton-scattered photons is measured at $\theta = 90^\circ$. If $\Delta\lambda/\lambda$ is to be 1 percent, what should the wavelength of the incident photon be?

3-38. Compton used photons of wavelength 0.0711 nm. (a) What is the energy of these photons? (b) What is the wavelength of the photons scattered at $\theta = 180^\circ$? (c) What is the energy of the photons scattered at $\theta = 180^\circ$? (d) What is the recoil energy of the electrons if $\theta = 180^\circ$?

3-39. Compute $\Delta\lambda$ for photons scattered at 120° from (a) free protons, (b) free electrons, and (c) N_2 molecules in air.

3-40. Compton's equation (Equation 3-25) indicates that a graph of λ_2 versus $(1 - \cos \theta)$ should be a straight line whose slope h/mc allows a determination of h . Given that the wavelength of λ_1 in Figure 3-17 is 0.0711 nm, compute λ_2 for each scattering angle in the figure and graph the results versus $(1 - \cos \theta)$. What is the slope of the line?

3-41. (a) Compute the Compton wavelength of an electron and a proton. (b) What is the energy of a photon whose wavelength is equal to the Compton wavelength of (1) the electron and (2) the proton?

Level II

3-42. When light of wavelength 450 nm is incident on potassium, photoelectrons with stopping potential of 0.52 V are emitted. If the wavelength of the incident light is changed to 300 nm, the stopping potential is 1.90 V. Using *only* these numbers together with the values of the speed of light and the electron charge, (a) find the work function of potassium and (b) compute a value for Planck's constant.

3-43. Assuming that the difference between Thomson's calculated e/m in his second experiment (Figure 3-19) and the currently accepted value was due entirely to his neglecting the horizontal component of Earth's magnetic field outside the deflection plates, what value for that component does the difference imply? (Thomson's data: $B = 5.5 \times 10^{-4} \text{ T}$, $\mathcal{E} = 1.5 \times 10^4 \text{ V/m}$, $x_1 = 5 \text{ cm}$, $y_2/x_2 = 8/110$.)

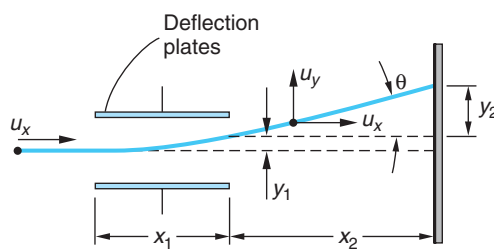


Figure 3-19 Deflection of the electron beam in Thomson's apparatus. The deflection plates are D and E in Figure 3-1. Deflection is shown with magnetic field off and the top plate positive. The magnetic field is applied perpendicular to the plane of the diagram and directed into the page.

3-44. In his first e/m experiment, Thomson determined the speed of electrons accelerated through a potential ΔV by collecting them in an insulated beam stopper and measuring both the total collected charge Q and the temperature rise ΔT of the beam stopper. (a) Show that with those measurements he could obtain an expression for e/m in terms of the speed of the electrons and the directly measured quantities. (b) Show that the expression obtained in (a) together with the result of Problem 3-2 enabled Thomson to compute e/m in terms of directly measured quantities.

3-45. Data for stopping potential versus wavelength for the photoelectric effect using sodium are

λ , nm	200	300	400	500	600
V_0 , V	4.20	2.06	1.05	0.41	0.03

Plot these data in such a way as to be able to obtain (a) the work function, (b) the threshold frequency, and (c) the ratio h/e .

3-46. Prove that the photoelectric effect cannot occur with a completely free electron, i.e., one not bound to an atom. (*Hint:* Consider the reference frame in which the total momentum of the electron and the incident photon is zero.)

3-47. When a beam of monochromatic x rays is incident on a particular NaCl crystal, Bragg reflection in the first order (i.e., with $m = 1$) occurs at $\theta = 20^\circ$. The value of $d = 0.28$ nm. What is the minimum voltage at which the x-ray tube can be operating?

3-48. A 100-W beam of light is shone onto a blackbody of mass 2×10^{-3} kg for 10^4 s. The blackbody is initially at rest in a frictionless space. (a) Compute the total energy and momentum absorbed by the blackbody from the light beam, (b) calculate the blackbody's velocity at the end of the period of illumination, and (c) compute the final kinetic energy of the blackbody. Why is the latter less than the total energy of the absorbed photons?

3-49. Show that the maximum kinetic energy E_k , called the Compton edge, that a recoiling electron can carry away from a Compton scattering event is given by

$$E_k = \frac{hf}{1 + mc^2/2hf} = \frac{2E_\gamma^2}{2E_\gamma + mc^2}$$

3-50. The x-ray spectrometer on board a satellite measures the wavelength at the maximum intensity emitted by a particular star to be $\lambda_m = 82.8$ nm. Assuming that the star radiates like a blackbody, (a) compute the star's surface temperature. (b) What is the ratio of the intensity radiated at $\lambda = 70$ nm and at $\lambda = 100$ nm to that radiated at λ_m .

3-51. Determine the fraction of the energy radiated by the Sun in the visible region of the spectrum (350 nm to 700 nm). Assume that the Sun's surface temperature is 5800 K.

3-52. Millikan's data for the photoelectric effect in lithium are shown in the table:

Incident λ (nm)	253.5	312.5	365.0	404.7	433.9
Stopping voltage V_0 (V)	2.57	1.67	1.09	0.73	0.55

(a) Graph the data and determine the work function for lithium. (b) Find the value of Planck's constant directly from the graph in (a). (c) The work function for lead is 4.14 eV. Which, if any, of the wavelengths in the table would not cause emission of photoelectrons from lead?

Level III

3-53. This problem is to derive the Wein displacement law, Equation 3-5. (a) Show that the energy density distribution function can be written $u = C\lambda^{-5}(e^{a/\lambda} - 1)^{-1}$, where C is a constant and $a = hc/kT$. (b) Show that the value of λ for which $du/d\lambda = 0$ satisfies the equation $5\lambda(1 - e^{-a/\lambda}) = a$. (c) This equation can be solved with a calculator by the trial-and-error method. Try $\lambda = \alpha a$ for various values of α until λ/a is determined to four significant figures. (d) Show that your solution in (c) implies $\lambda_m T = \text{constant}$ and calculate the value of the constant.

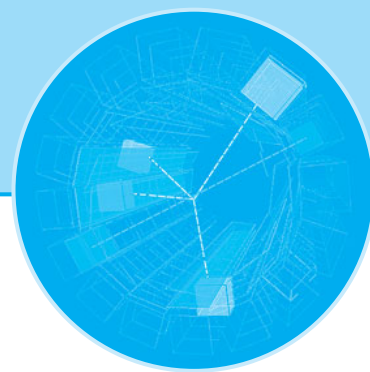
3-54. This problem is one of *estimating* the time lag (expected classically, but not observed) for the photoelectric effect. Assume that a point light source emits $1 \text{ W} = 1 \text{ J/s}$ of light energy. (a) Assuming uniform radiation in all directions, find the light intensity in $\text{eV/s} \cdot \text{m}^2$ at a distance of 1 m from the light source. (b) Assuming some reasonable size for an atom, find the energy per unit time incident on the atom for this intensity. (c) If the work function is 2 eV, how long does it take for this much energy to be absorbed, assuming that all of the energy hitting the atom is absorbed?

3-55. A photon can be absorbed by a system that can have internal energy. Assume that a 15-MeV photon is absorbed by a carbon nucleus initially at rest. The recoil momentum of the carbon nucleus must be $15 \text{ MeV}/c$. (a) Calculate the kinetic energy of the carbon nucleus. What is the internal energy of the nucleus? (b) The carbon nucleus comes to rest and then loses its internal energy by emitting a photon. What is the energy of the photon?

3-56. The maximum kinetic energy given to the electron in a Compton scattering event plays a role in the measurement of gamma-ray spectra using scintillation detectors. The maximum is referred to as the *Compton edge*. Suppose that the Compton edge in a particular experiment is found to be 520 keV. What were the wavelength and energy of the incident gamma rays?

3-57. An electron accelerated to 50 keV in an x-ray tube has two successive collisions in being brought to rest in the target, emitting two bremsstrahlung photons in the process. The second photon emitted has a wavelength 0.095 nm longer than the first. (a) What are the wavelengths of the two photons? (b) What was the energy of the electron after emission of the first photon?

3-58. Derive Equation 3-17 from Equations 3-15 and 3-16.



The Nuclear Atom

Among his many experiments, Newton found that sunlight passing through a small opening in a window shutter could be refracted by a glass prism so that it would fall on a screen. The white sunlight thus refracted was spread into a rainbow-colored band—a spectrum. He had discovered *dispersion*, and his experimental arrangement was the prototype of the modern *spectroscope* (Figure 4-1a). When, 150 years later, Fraunhofer¹ dispersed sunlight using an experimental setup similar to that shown in Figure 4-1b to test prisms made of glasses that he had developed, he found that the solar spectrum was crossed by more than 600 narrow, or sharp, dark lines.² Soon after, a number of scientists observed sharp *bright* lines in the spectra of light emitted by flames, arcs, and sparks. *Spectroscopy* quickly became an important area of research.

It soon became clear that chemical elements and compounds emit three general types of spectra. *Continuous* spectra, emitted mainly by incandescent solids, show no lines at all, bright or dark, in spectroscopes of the highest possible resolving power. *Band* spectra consist of very closely packed groups of lines that appear to be continuous in instruments of low resolving power. These are emitted when small pieces of solid materials are placed in the source flame or electrodes. The *line* spectra mentioned above arise when the source contains unbound chemical elements. The lines and bands turned out to be characteristic of individual elements and chemical compounds when excited under specific conditions. Indeed, the spectra could be (and are today) used as a highly sensitive test for the presence of elements and compounds.

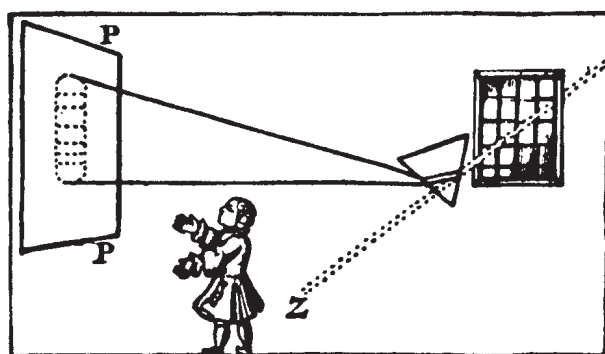
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Voltaire's depiction of Newton's discovery of dispersion. [*Elémens de la Philosophie de Newton*, Amsterdam, 1738.]

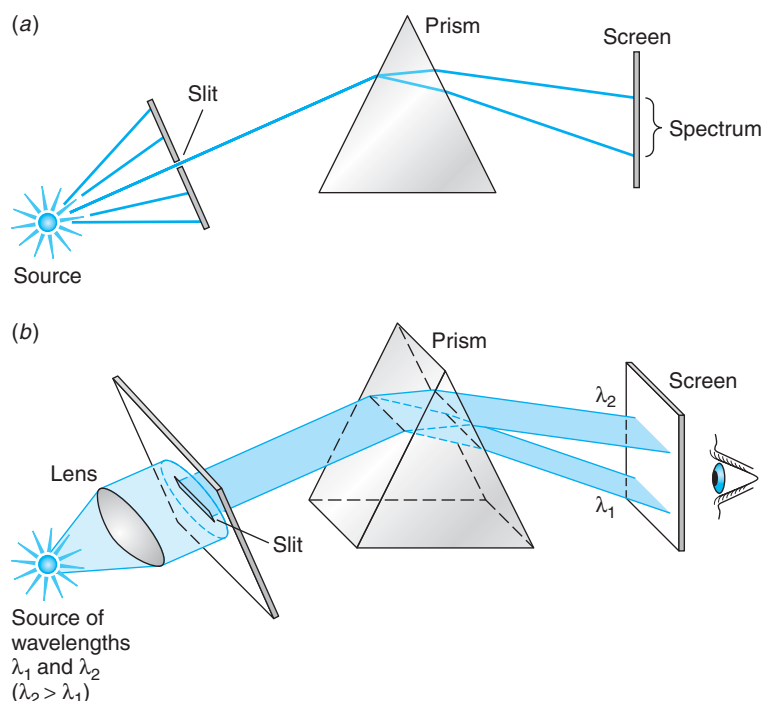


Figure 4-1 (a) Light from the source passes through a small hole or a narrow slit before falling on the prism. The purpose of the slit is to ensure that all the incident light strikes the prism face at the same angle so that the dispersion by the prism causes the various frequencies that may be present to strike the screen at different places with minimum overlap. (b) The source emits only two wavelengths, $\lambda_2 > \lambda_1$. The source is located at the focal point of the lens so that parallel light passes through the narrow slit, projecting a narrow line onto the face of the prism. Ordinary dispersion in the prism bends the shorter wavelength through the larger total angle, separating the two wavelengths at the screen. In this arrangement each wavelength appears on the screen (or on film replacing the screen) as a narrow line, which is an image of the slit. Such a spectrum was dubbed a “line spectrum” for that reason. Prisms have been almost entirely replaced in modern spectroscopes by diffraction gratings, which have much higher resolving power.

Line spectra raised an enormous theoretical problem: although classical physics could account for the existence of a continuous spectrum (if not its detailed shape, as we saw with blackbodies), it could in no way explain *why* sharp lines and bands should exist. Explaining the origin of the sharp lines and accounting for the primary features of the spectrum of hydrogen, the simplest element, was a major success of the so-called old quantum theory begun by Planck and Einstein and will be the main topic in this chapter. Full explanation of the lines and bands requires the later, more sophisticated quantum theory, which we will begin studying in Chapter 5.

4-1 Atomic Spectra

The characteristic radiation emitted by atoms of individual elements in a flame or in a gas excited by an electrical discharge was the subject of vigorous study during the late nineteenth and early twentieth centuries. When viewed or photographed through a spectroscope, this radiation appears as a set of discrete lines, each of a particular

color or wavelength; the positions and intensities of the lines are characteristic of the element. The wavelengths of these lines could be determined with great precision, and much effort went into finding and interpreting regularities in the spectra. A major breakthrough was made in 1885 by a Swiss schoolteacher, Johann Balmer, who found that the lines in the visible and near ultraviolet spectrum of hydrogen could be represented by the empirical formula

$$\lambda_n = 364.6 \frac{n^2}{n^2 - 4} \text{ nm} \quad 4-1$$

where n is a variable integer that takes on the values $n = 3, 4, 5, \dots$. Figure 4-2a shows the set of spectral lines of hydrogen (now known as the *Balmer series*) whose wavelengths are given by Balmer's formula. For example, the wavelength of the H_α line could be found by letting $n = 3$ in Equation 4-1 (try it!), and other integers each predicted a line that was found in the spectrum. Balmer suggested that his formula might be a special case of a more-general expression applicable to the spectra of other elements when ionized to a single electron, i.e., hydrogen-like elements.

The uniqueness of the line spectra of the elements has enabled astronomers to determine the composition of stars, chemists to identify unknown compounds, and theme parks and entertainers to have laser shows.

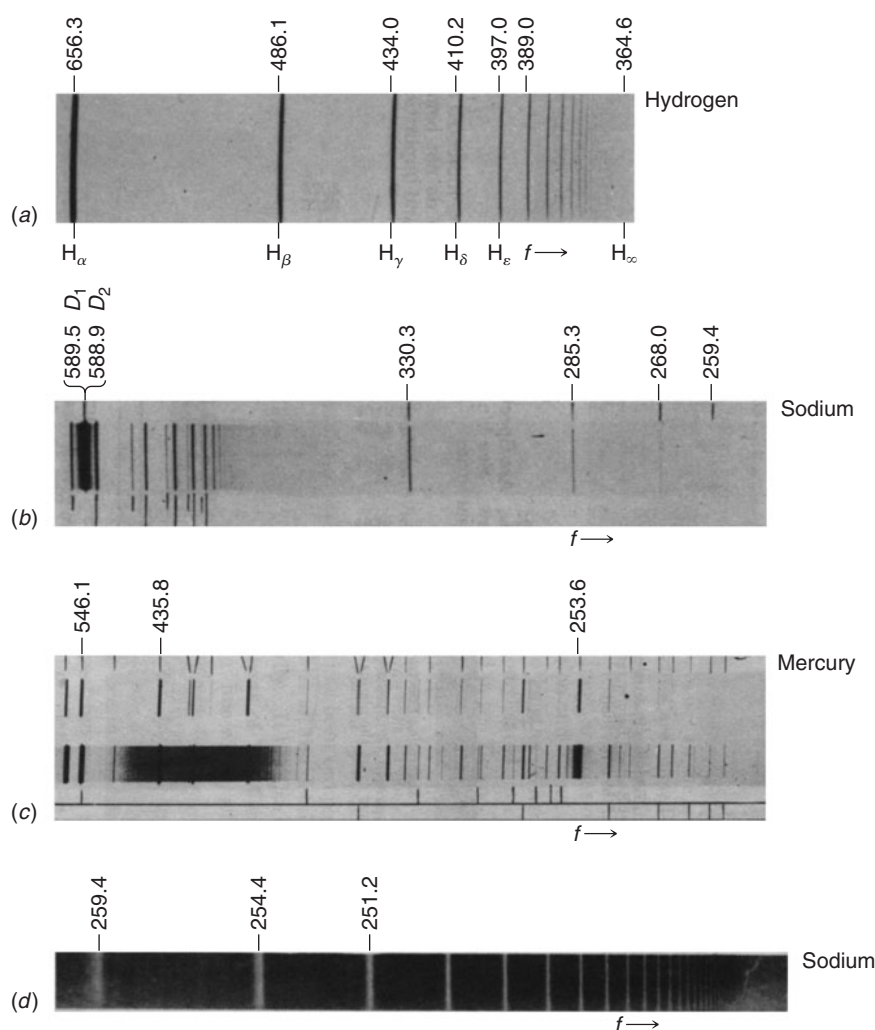


Figure 4-2 (a) Emission line spectrum of hydrogen in the visible and near ultraviolet. The lines appear dark because the spectrum was photographed; hence, the bright lines are exposed (dark) areas on the film. The names of the first five lines are shown, as is the point beyond which no lines appear, H_∞ , called the *limit* of the series. (b) Part of the emission spectrum of sodium. The two very close bright lines at 589 nm are the D_1 and D_2 lines. They are the principal radiation from sodium street lighting. (c) Part of the emission spectrum of mercury. (d) Part of the dark line (absorption) spectrum of sodium. White light shining through sodium vapor is absorbed at certain wavelengths, resulting in no exposure of the film at those points. Note that the line at 259.4 nm is visible here in both the bright and dark line spectra. Note, too, that frequency increases toward the right, wavelength toward the left in the four spectra shown.

Such an expression, found independently by J. R. Rydberg and W. Ritz and thus called the *Rydberg-Ritz formula*, gives the reciprocal wavelength³ as

$$\frac{1}{\lambda_{mn}} = R \left(\frac{1}{m^2} - \frac{1}{n^2} \right) \quad \text{for } n > m \quad 4-2$$

where m and n are integers and R , the *Rydberg constant*, is the same for all series of spectral lines of the same element and varies only slightly, and in a regular way, from element to element. For hydrogen, the value of R is $R_H = 1.096776 \times 10^7 \text{ m}^{-1}$. For very heavy elements, R approaches the value of $R_\infty = 1.097373 \times 10^7 \text{ m}^{-1}$. Such empirical expressions were successful in predicting other series of spectral lines, such as other hydrogen lines outside the visible region.

EXAMPLE 4-1 Hydrogen Spectral Series The hydrogen Balmer series reciprocal wavelengths are those given by Equation 4-2, with $m = 2$ and $n = 3, 4, 5, \dots$. For example, the first line of the series, H_α , would be for $m = 2, n = 3$:

$$\frac{1}{\lambda_{23}} = R \left(\frac{1}{2^2} - \frac{1}{3^2} \right) = \frac{5}{36} R = 1.523 \times 10^6 \text{ m}^{-1}$$

or

$$\lambda_{23} = 656.5 \text{ nm}$$

Other series of hydrogen spectral lines were found for $m = 1$ (by Theodore Lyman) and $m = 3$ (by Friedrich Paschen). Compute the wavelengths of the first lines of the Lyman and Paschen series.

SOLUTION

For the Lyman series ($m = 1$), the first line is for $m = 1, n = 2$:

$$\frac{1}{\lambda_{12}} = R \left(\frac{1}{1^2} - \frac{1}{2^2} \right) = \frac{3}{4} R = 8.22 \times 10^6 \text{ m}^{-1}$$

$$\lambda_{12} = 121.6 \text{ nm} \quad (\text{in the ultraviolet})$$

For the Paschen series ($m = 3$), the first line is for $m = 3, n = 4$:

$$\frac{1}{\lambda_{34}} = R \left(\frac{1}{3^2} - \frac{1}{4^2} \right) = \frac{7}{144} R = 5.332 \times 10^5 \text{ m}^{-1}$$

$$\lambda_{34} = 1876 \text{ nm} \quad (\text{in the infrared})$$

All of the lines predicted by the Rydberg-Ritz formula for the Lyman and Paschen series are found experimentally. Note that no lines are predicted to lie beyond $\lambda_\infty = 1/R = 91.2 \text{ nm}$ for the Lyman series and $\lambda_\infty = 9/R = 820.6 \text{ nm}$ for the Paschen series and none are found by experiments.

4-2 Rutherford's Nuclear Model

Many attempts were made to construct a model of the atom that yielded the Balmer and Rydberg-Ritz formulas. It was known that an atom was about 10^{-10} m in diameter (see Problem 4-6), that it contained electrons much lighter than the atom (see Section 3-1), and that it was electrically neutral. The most popular model was J. J. Thomson's model, already quite successful in explaining chemical reactions. Thomson

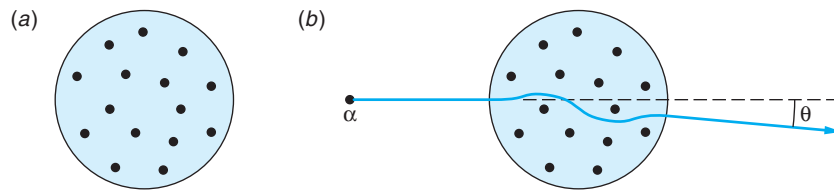


Figure 4-3 Thomson's model of the atom: (a) A sphere of positive charge with electrons embedded in it so that the net charge would normally be zero. The atom shown would have been phosphorus. (b) An α particle scattered by such an atom would have a scattering angle θ much smaller than 1° .

attempted various models consisting of electrons embedded in a fluid that contained most of the mass of the atom and had enough positive charge to make the atom electrically neutral. (See Figure 4-3a.) He then searched for configurations that were stable and had normal modes of vibration corresponding to the known frequencies of the spectral lines. One difficulty with all such models was that electrostatic forces alone cannot produce stable equilibrium. Thus, the charges were required to move and, if they stayed within the atom, to accelerate; however, the acceleration would result in continuous emission of radiation, which is not observed. Despite elaborate mathematical calculations, Thomson was unable to obtain from his model a set of frequencies of vibration that corresponded with the frequencies of observed spectra.

The Thomson model of the atom was replaced by one based on the results of a set of experiments conducted by Ernest Rutherford⁴ and his students H. W. Geiger and E. Marsden. Rutherford was investigating radioactivity and had shown that the radiations from uranium consisted of at least two types, which he labeled α and β . He showed, by an experiment similar to that of Thomson, that q/m for the α was half that of the proton. Suspecting that the α particles were doubly ionized helium, Rutherford and his co-workers in a classic experiment let a radioactive substance α decay in a previously evacuated chamber; then, by spectroscopy, they detected the spectral lines of ordinary helium gas in the chamber. Realizing that this energetic, massive α particle would make an excellent probe for “feeling about” within the interiors of other atoms, Rutherford began a series of experiments with this purpose.



Hans Geiger and Ernest Rutherford in their Manchester Laboratory. [Courtesy of University of Manchester.]

In these latter experiments, a narrow beam of α particles fell on a zinc sulfide screen, which emitted visible light scintillations when struck (Figure 4-4). The distribution of scintillations on the screen was observed when various thin metal foils were placed between it and the source. Most of the α particles were either undeflected or

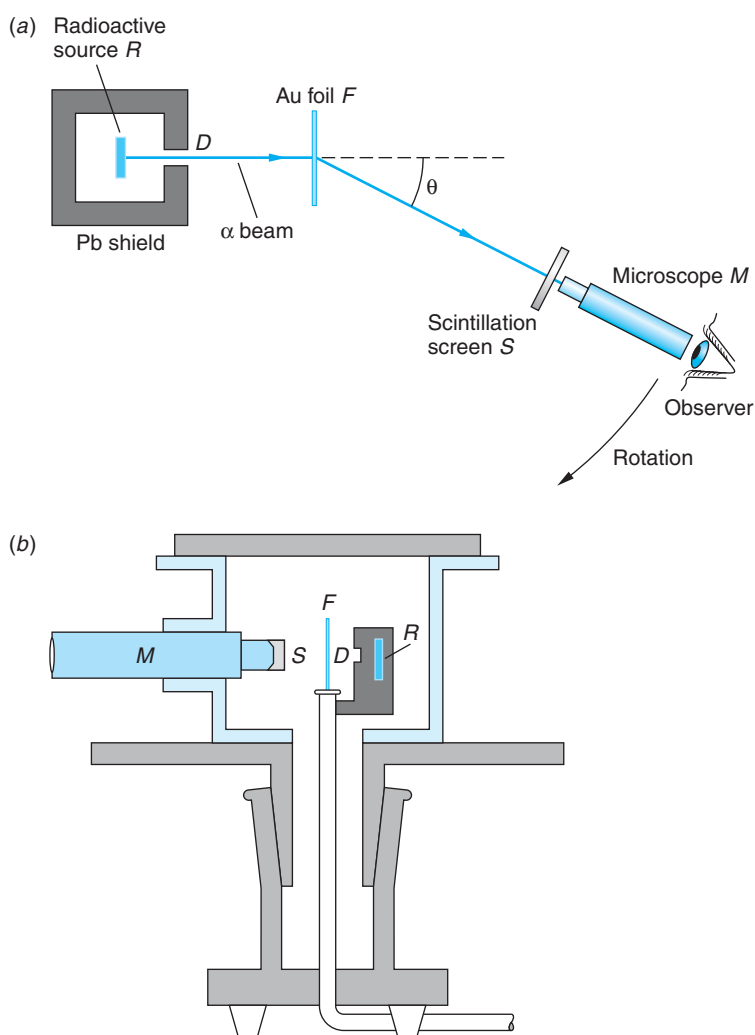


Figure 4-4 Schematic diagram of the apparatus used by Geiger and Marsden to test Rutherford's atomic model. (a) The beam of α particles is defined by the small hole D in the shield surrounding the radioactive source R of ^{214}Bi (called RaC in Rutherford's day). The α beam strikes an ultrathin gold foil F (about 2000 atoms thick), and the α particles are individually scattered through various angles. Those scattering at the angle θ shown strike a small screen S coated with a scintillator, i.e., a material that emits tiny flashes of light (scintillations) when struck by an α particle. The scintillations were viewed by the observer through a small microscope M . The scintillation screen–microscope combination could be rotated about the center of the foil. The region traversed by the α beam is evacuated. The experiment consisted of counting the number of scintillations as a function of θ . (b) A diagram of the actual apparatus as it appeared in Geiger and Marsden's paper describing the results. The letter key is the same as in (a). [Part (b) from H. Geiger and E. Marsden, *Philosophical Review*, 25, 507 (1913).]

deflected through very small angles of the order of 1° . Quite unexpectedly, however, a few α particles were deflected through angles as large as 90° or more. If the atom consisted of a positively charged sphere of radius 10^{-10} m, containing electrons as in the Thomson model, only a very small deflection could result from a single encounter between an α particle and an atom, even if the α particle penetrated into the atom. Indeed, calculations showed that the Thomson atomic model could not possibly account for the number of large-angle scatterings that Rutherford saw. The unexpected scatterings at large angles were described by Rutherford with these words:

It was quite the most incredible event that ever happened to me in my life. It was as incredible as if you fired a 15-inch shell at a piece of tissue paper and it came back and hit you.

Rutherford's Scattering Theory and the Nuclear Atom

The question is, then, Why would one obtain the large-angle scattering that Rutherford saw? The trouble with the Thomson atom is that it is too “soft”—the maximum force experienced by the α is too weak to give a large deflection. If the positive charge of the atom is concentrated in a more compact region, however, a much larger force will occur at near impacts. Rutherford concluded that the large-angle scattering obtained experimentally could result only from a single encounter of the α particle with a massive charge confined to a volume much smaller than that of the whole atom. Assuming this “nucleus” to be a point charge, he calculated the expected angular distribution for the scattered α particles. His predictions of the dependence of scattering probability on angle, nuclear charge, and kinetic energy were completely verified in a series of experiments carried out in his laboratory by Geiger and Marsden.

We will not go through Rutherford's derivation in detail, but merely outline the assumptions and conclusions. Figure 4-5 shows the geometry of an α particle being scattered by a nucleus, which we take to be a point charge Q at the origin. Initially, the α particle approaches with speed v along a line a distance b from a parallel line COA through the origin. The force on the α particle is $F = kq_\alpha Q/r^2$, given by

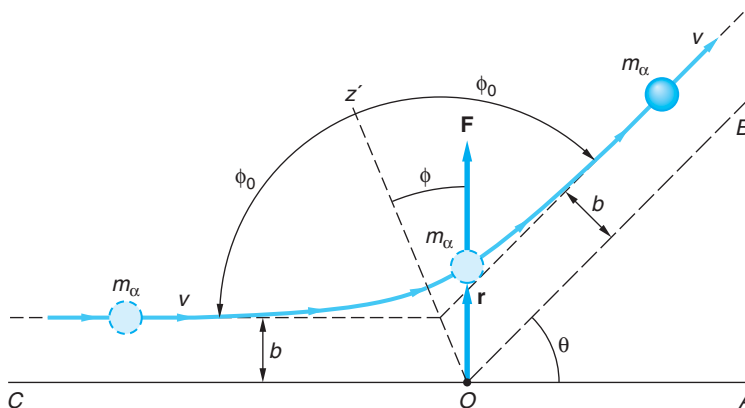


Figure 4-5 Rutherford scattering geometry. The nucleus is assumed to be a point charge Q at the origin O . At any distance r the α particle experiences a repulsive force $kq_\alpha Q/r^2$. The α particle travels along a hyperbolic path that is initially parallel to line COA a distance b from it and finally parallel to line OB , which makes an angle θ with OA . The scattering angle θ can be related to the impact parameter b by classical mechanics.

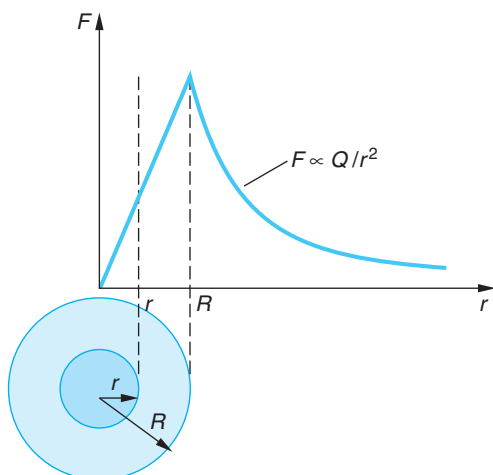


Figure 4-6 Force on a point charge versus distance r from the center of a uniformly charged sphere of radius R . Outside the sphere the force is proportional to Q/r^2 , where Q is the total charge. Inside the sphere, the force is proportional to $q'/r^2 = QrR/R^3$, where $q' = Q(r/R)^3$ is the charge within a sphere of radius r . The maximum force occurs at $r = R$.

Coulomb's law (Figure 4-6). After scattering, when the α particle is again far from the nucleus, it is moving with the same speed v parallel to the line OB , which makes an angle θ with line COA . (Since the potential energy is again zero, the final speed must be equal to the initial speed by conservation of energy, assuming, as Rutherford did, that the massive nucleus remains fixed during the scattering.) The distance b is called the *impact parameter* and the angle θ , the *scattering angle*. The path of the α particle can be shown to be a hyperbola, and the scattering angle θ can be related to the impact parameter b from the laws of classical mechanics. The result is

$$b = \frac{kq_{\alpha}Q}{m_{\alpha}v^2} \cot \frac{\theta}{2} \quad 4-3$$

Of course, it is not possible to choose or know the impact parameter for any particular α particle, but when one recalls the values of the cotangent between 0° and 90° , all such particles with impact parameters less than or equal to a particular b will be scattered through an angle θ greater than or equal to that given by Equation 4-3; i.e., the smaller the impact parameter, the larger the scattering angle (Figure 4-7). Let the intensity of the incident α particle beam be I_0 particles per second per unit area. The number per second scattered by one nucleus through angles greater than θ equals the number per second that have impact parameters less than $b(\theta)$. This number is $\pi b^2 I_0$.

The quantity πb^2 , which has the dimensions of an area, is called the *cross section* σ for scattering through angles greater than θ . The cross section σ is thus defined as the number scattered per nucleus per unit time divided by the incident intensity.

The particle-scattering technique devised by Rutherford to "look" at atoms now has wide application throughout physics. Scattering of high-energy electrons from protons and neutrons provided our first experimental hint of the existence of quarks. Rutherford back-scattering spectroscopy is widely used as a highly sensitive surface analysis technique.

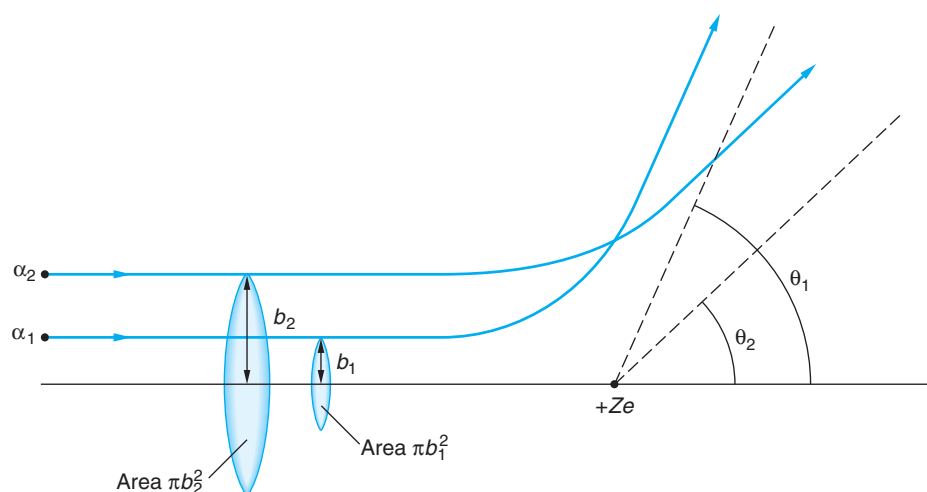


Figure 4-7 Two α particles with equal kinetic energies approach the positive charge $Q = +Ze$ with impact parameters b_1 and b_2 , where $b_1 < b_2$. According to Equation 4-3, the angle θ_1 through which α_1 is scattered will be larger than θ_2 . In general, all α particles with impact parameters smaller than a particular value of b will have scattering angles larger than the corresponding value of θ from Equation 4-3. The area πb^2 is called the cross section for scattering with angles greater than θ .

The total number of particles scattered per second is obtained by multiplying $\pi b^2 I_0$ by the number of nuclei in the scattering foil (this assumes the foil to be thin enough to make the chance of overlap negligible). Let n be the number of nuclei per unit volume:

$$n = \frac{\rho(\text{g/cm}^3)N_A(\text{atoms/mol})}{M(\text{g/mol})} = \frac{\rho N_A}{M} \frac{\text{atoms}}{\text{cm}^3} \quad 4-4$$

For a foil of thickness t , the total number of nuclei “seen” by the beam is nAt , where A is the area of the beam (Figure 4-8). The total number scattered per second through angles greater than θ is thus $\pi b^2 I_0 nAt$. If we divide this by the number of α particles incident per second $I_0 A$, we get the fraction f scattered through angles greater than θ :

$$f = \pi b^2 nt \quad 4-5$$

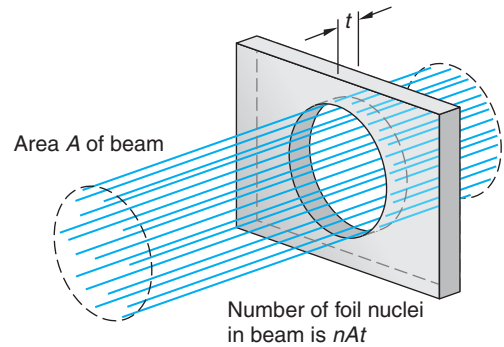


Figure 4-8 The total number of nuclei of foil atoms in the area covered by the beam is nAt , where n is the number of foil atoms per unit volume, A is the area of the beam, and t is the thickness of the foil.

EXAMPLE 4-2 Scattered Fraction f Calculate the fraction of an incident beam of α particles of kinetic energy 5 MeV that Geiger and Marsden expected to see for $\theta \geq 90^\circ$ from a gold foil ($Z = 79$) 10^{-6} m thick.

SOLUTION

- The fraction f is related to the impact parameter b , the number density of nuclei n , and the thickness t by Equation 4-5: $f = \pi b^2 nt$
- The particle density n is given by Equation 4-4:
$$n = \frac{\rho N_A}{M} = \frac{(19.3 \text{ g/cm}^3)(6.02 \times 10^{23} \text{ atoms/mol})}{197 \text{ gm/mol}} = 5.90 \times 10^{22} \text{ atoms/cm}^3 = 5.90 \times 10^{28} \text{ atoms/m}^3$$
- The impact parameter b is related to θ by Equation 4-3:
$$b = \frac{kq_\alpha Q}{m_\alpha v^2} \cot \frac{\theta}{2} = \frac{(2)(79)ke^2}{2K_\alpha} \cot \frac{90^\circ}{2} = \frac{(2)(79)(1.44 \text{ eV} \cdot \text{nm})}{(2)(5 \times 10^6 \text{ eV})} = 2.28 \times 10^{-5} \text{ nm} = 2.28 \times 10^{-14} \text{ m}$$
- Substituting these into Equation 4-5 yields f :
$$f = \pi(2.28 \times 10^{-14} \text{ m})^2 \left(5.9 \times 10^{28} \frac{\text{atoms}}{\text{m}^3} \right) (10^{-6} \text{ m}) = 9.6 \times 10^{-5} \approx 10^{-4}$$

Remarks: This outcome is in good agreement with Geiger and Marsden's measurement of about 1 in 8000 in their first trial. Thus, the nuclear model is in good agreement with their results.

On the strength of the good agreement between the nuclear atomic model and the measured fraction of the incident α particles scattered at angles $\theta \geq 90^\circ$, Rutherford derived an expression, based on the nuclear model, for the number of α particles ΔN that would be scattered at any angle θ . That number, which also depends on the atomic

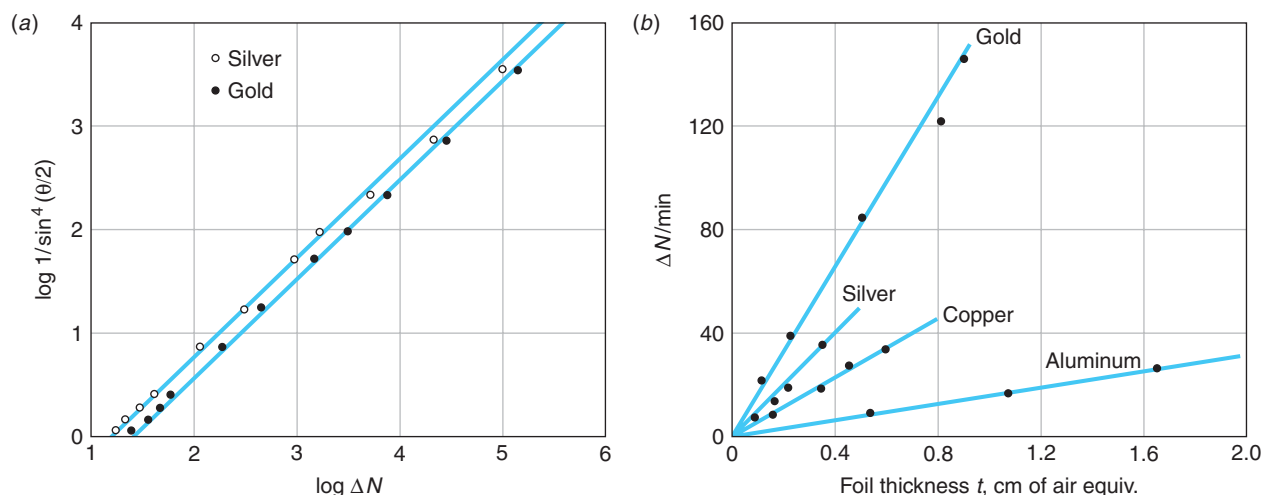


Figure 4-9 (a) Geiger and Marsden's data for α scattering from thin gold and silver foils. The graph is a log-log plot to show the data over several orders of magnitude. Note that scattering angle increases downward along the vertical axis. (b) Geiger and Marsden also measured the dependence of ΔN on t predicted by Equation 4-6 for foils made from a wide range of elements, this being an equally critical test. Results for four of the elements used are shown.

number Z and thickness t of the scattering foil, on the intensity I_0 of the incident α particles and their kinetic energy E_k , and on the geometry of the detector (A_{sc} is the detector area and r is the foil-detector distance), is given by

$$\Delta N = \left(\frac{I_0 A_{sc} n t}{r^2} \right) \left(\frac{k Z e^2}{2 E_k} \right)^2 \frac{1}{\sin^4 \frac{\theta}{2}} \quad 4-6$$

Within the uncertainties of their experiments, which involved visually observing several hundred thousand α particles, Geiger and Marsden verified every one of the predictions of Rutherford's formula over four orders of magnitude of ΔN . The excellent agreement of their data with Equation 4-6 firmly established the nuclear atomic model as the correct basis for further studies of atomic and nuclear phenomena. (See Figure 4-9.)



More

Rutherford's derivation of Equation 4-6 was based on his atomic model and the well-known Coulomb scattering process of charged particles. *Rutherford's Prediction and Geiger and Marsden's Results* are described on the home page: www.whfreeman.com/tiplermodernphysics5e. See also Equations 4-7 through 4-10 here, as well as Figures 4-10 through 4-12.

The Size of the Nucleus

The fact that the force law is shown to be correct, confirming Rutherford's model, does not imply that the nucleus is a mathematical point charge, however. The force law would be the same even if the nucleus were a ball of charge of some radius R_0 as long as the α particle did not penetrate the ball. (See Figures 4-6 and 4-13.) For a given scattering angle, the distance of closest approach of the α particle to the nucleus can be calculated from the geometry of the collision. For the largest angle, near 180° , the collision is nearly "head-on." The corresponding distance of closest approach r_d is thus an experimental

upper limit on the size of the target nucleus. We can calculate the distance of closest approach for a head-on collision r_d by noting that conservation of energy requires the potential energy at this distance to equal the original kinetic energy:

$$\begin{aligned} (V + E_k)_{\text{large } r} &= (V + E_k)_{r_d} \\ \left(0 + \frac{1}{2}m_\alpha v^2\right)_{\text{large } r} &= \left(\frac{kq_\alpha Q}{r_d} + 0\right)_{r_d} \\ \frac{1}{2}m_\alpha v^2 &= \frac{kq_\alpha Q}{r_d} \end{aligned}$$

or

$$r_d = \frac{kq_\alpha Q}{\frac{1}{2}m_\alpha v^2}$$

4-11

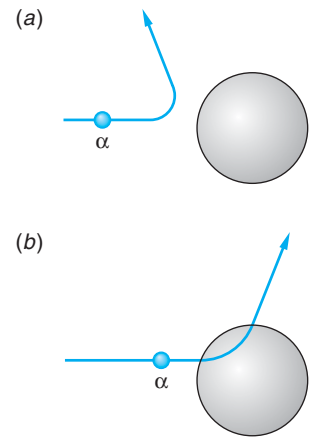


Figure 4-13 (a) If the α particle does not penetrate the nuclear charge, the nucleus can be considered a point charge located at the center. (b) If the particle has enough energy to penetrate the nucleus, the Rutherford scattering law does not hold but would require modification to account for that portion of the nuclear charge “behind” the penetrating α particle.

For the case of 7.7-MeV α particles, the distance of closest approach for a head-on collision is

$$r_d = \frac{(2)(79)(1.44 \text{ eV} \cdot \text{nm})}{7.7 \times 10^6 \text{ eV}} = 3 \times 10^{-5} \text{ nm} = 3 \times 10^{-14} \text{ m}$$

For other collisions, the distance of closest approach is somewhat greater, but for α particles scattered at large angles it is of the same order of magnitude. The excellent agreement of Geiger and Marsden's data at large angles with the prediction of Equation 4-6 thus indicates that the radius of the gold nucleus is no larger than about $3 \times 10^{-14} \text{ m}$. If higher-energy particles could be used, the distance of closest approach would be smaller, and as the energy of the α particles increased, we might expect that eventually the particles would penetrate the nucleus. Since, in that event, the force law is no longer $F = kq_\alpha Q/r^2$, the data would not agree with the point-nucleus calculation. Rutherford did not have higher-energy α particles available, but he could reduce the distance of closest approach by using targets of lower atomic numbers.⁹ For the case of aluminum, with $Z = 13$, when the most energetic α particles that he had available (7.7 MeV from ^{214}Bi) scattered at large angles, they did not follow the predictions of Equation 4-6. However, when the kinetic energy of the particles was reduced by passing the beam through thin mica sheets of various thicknesses, the data again followed the prediction of Equation 4-6. Rutherford's data are shown in Figure 4-14. The value of r_d (calculated from Equation 4-11) at which the data begin to deviate from the prediction can be thought of as the surface of the nucleus. From these data, Rutherford estimated the radius of the aluminum nucleus to be about $1.0 \times 10^{-14} \text{ m}$. (The radius of the Al nucleus is actually about $3.6 \times 10^{-15} \text{ m}$. See Chapter 11.)

A unit of length convenient for describing nuclear sizes is the fermi, or femtometer (fm), defined as $1 \text{ fm} = 10^{-15} \text{ m}$. As we will see in Chapter 11, the nuclear radius varies from about 1 to 10 fm from the lightest to the heaviest atoms.

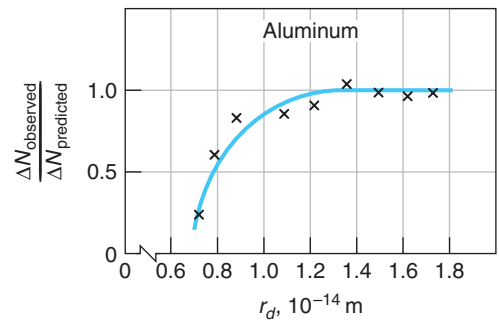


Figure 4-14 Data from Rutherford's group showing observed α scattering at a large fixed angle versus values of r_d computed from Equation 4-11 for various kinetic energies.

EXAMPLE 4-3 Rutherford Scattering at Angle θ In a particular experiment, α particles from ^{226}Ra are scattered at $\theta = 45^\circ$ from a silver foil and 450 particles are counted each minute at the scintillation detector. If everything is kept the same except that the detector is moved to observe particles scattered at 90° , how many will be counted per minute?

SOLUTION

Using Equation 4-6, we have that $\Delta N = 450$ when $\theta = 45^\circ$, but we don't have any of the other parameters available. Letting all of the quantities in the parenthesis equal a constant C , we have that

$$\Delta N = 450 = C \sin^{-4} \frac{45^\circ}{2}$$

or

$$C = 450 \sin^4 \left(\frac{45^\circ}{2} \right)$$

When the detector is moved to $\theta = 90^\circ$, the value of C is unchanged, so

$$\begin{aligned} \Delta N &= C \sin^{-4} \left(\frac{90^\circ}{2} \right) = 450 \sin^4 \left(\frac{45^\circ}{2} \right) \sin^{-4} \left(\frac{90^\circ}{2} \right) \\ &= 38.6 \approx 39 \text{ particles/min} \end{aligned}$$

EXAMPLE 4-4 Alpha Scattering A beam of α particles with $E_k = 6.0$ MeV impinges on a silver foil $1.0 \mu\text{m}$ thick. The beam current is 1.0 nA. How many α particles will be counted by a small scintillation detector of area equal to 5 mm^2 located 2.0 cm from the foil at an angle of 75° ? (For silver $Z = 47$, $\rho = 10.5 \text{ gm/cm}^3$, and $M = 108$.)

SOLUTION

- The number counted ΔN is given by Equation 4-6:

$$\Delta N = \left(\frac{I_0 A_{\text{sc}} n t}{r^2} \right) \left(\frac{k Z e^2}{2 E_k} \right)^2 \frac{1}{\sin^4 \frac{\theta}{2}}$$
- Since each α particle has $q_\alpha = 2e$, I_0 is:

$$\begin{aligned} I_0 &= (1.0 \times 10^{-9} \text{ A})(2 \times 1.60 \times 10^{-19} \text{ C}/\alpha)^{-1} \\ &= 3.12 \times 10^9 \alpha/\text{s} \end{aligned}$$
- The kinetic energy of each α is

$$\begin{aligned} E_k &= (6.0 \text{ MeV})(1.60 \times 10^{-13} \text{ J/MeV}) \\ &= 9.60 \times 10^{-13} \text{ J} \end{aligned}$$
- For silver, n is given by

$$\begin{aligned} n &= \rho N_A / M \\ &= \frac{(10.5 \text{ g/cm}^3)(6.02 \times 10^{23} \text{ atoms/mol})}{108 \text{ g/mol}} \\ &= 5.85 \times 10^{22} \text{ atoms/cm}^3 = 5.85 \times 10^{28} \text{ atoms/m}^3 \end{aligned}$$
- Substituting the given values and computed results into Equation 4-6 gives ΔN :

$$\begin{aligned} \Delta N &= \frac{(3.12 \times 10^9 \alpha/\text{s})(5 \times 10^{-6} \text{ m}^2)(5.85 \times 10^{28} \text{ atoms/m}^3)(10^{-6} \text{ m})}{(2 \times 10^{-2})^2 \sin^4 (75^\circ/2)} \\ &\quad \times \left[\frac{(9 \times 10^9)(47)(1.60 \times 10^{-19})^2}{(2)(9.60 \times 10^{-13})} \right]^2 \\ &= 528 \alpha/\text{s} \end{aligned}$$

EXAMPLE 4-5 Radius of the Au Nucleus The radius of the gold (Au) nucleus has been measured by high-energy electron scattering as 6.6 fm . What kinetic energy α particles would Rutherford have needed so that for 180° scattering, the α particle would just reach the nuclear surface before reversing direction?

SOLUTION

From Equation 4-11, we have

$$\begin{aligned}\frac{1}{2}mv^2 &= \frac{kq_\alpha Q}{r_d} = \frac{(9 \times 10^9)(2)(79)(1.60 \times 10^{-19})^2}{6.6 \times 10^{-15}} \\ &= 5.52 \times 10^{-12} \text{ J} = 34.5 \text{ MeV}\end{aligned}$$

Alpha particles of such energy are not emitted by naturally radioactive materials and so were not accessible to Rutherford. Thus, he could not have performed an experiment for Au equivalent to that for Al illustrated by Figure 4-14.

Questions

1. Why can't the impact parameter for a particular α particle be chosen?
2. Why is it necessary to use a very thin target foil?
3. Why could Rutherford place a lower limit on the radius of the Al nucleus but not on the Au nucleus?
4. How could you use the data in Figure 4-9a to determine the charge on a silver nucleus relative to that on a gold nucleus?
5. How would you expect the data (not the curve) to change in Figure 4-9 if the foil were so thick that an appreciable number of gold nuclei were hidden from the beam by being in the "shadow" of the other gold nuclei?

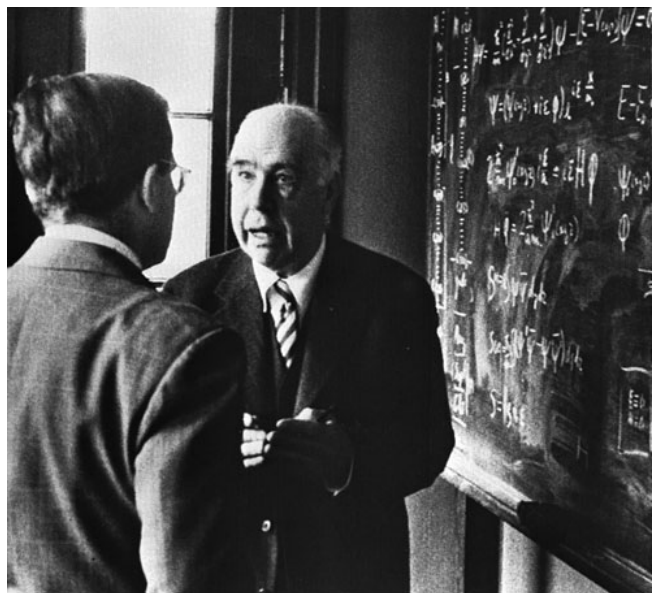
4-3 The Bohr Model of the Hydrogen Atom

In 1913, the Danish physicist Niels H. D. Bohr¹⁰ proposed a model of the hydrogen atom that combined the work of Planck, Einstein, and Rutherford and was remarkably successful in predicting the observed spectrum of hydrogen. The Rutherford model assigned charge and mass to the nucleus but was silent regarding the distribution of the charge and mass of the electrons. Bohr, who had been working in Rutherford's laboratory during the experiments of Geiger and Marsden, made the assumption that the electron in the hydrogen atom moved in an orbit about the positive nucleus, bound by the electrostatic attraction of the nucleus. Classical mechanics allows circular or elliptical orbits in this system, just as in the case of the planets orbiting the Sun. For simplicity, Bohr chose to consider circular orbits.

Such a model is mechanically stable because the Coulomb potential $V = -kZe^2/r$ provides the centripetal force

$$F = \frac{kZe^2}{r^2} = \frac{mv^2}{r} \quad 4-12$$

necessary for the electron to move in a circle of radius r at speed v , but it is electrically unstable because the electron is always accelerating toward the center of the circle.



Niels Bohr explains a point in front of the blackboard (1956).
[American Institute of Physics, Niels Bohr Library, Margrethe Bohr Collection.]

The laws of electrodynamics predict that such an accelerating charge will radiate light of frequency f equal to that of the periodic motion, which in this case is the frequency of revolution. Thus, classically,

$$f = \frac{v}{2\pi r} = \left(\frac{kZe^2}{rm} \right)^{1/2} \frac{1}{2\pi r} = \left(\frac{kZe^2}{4\pi^2 m} \right)^{1/2} \frac{1}{r^{3/2}} \sim \frac{1}{r^{3/2}} \quad 4-13$$

The total energy of the electron is the sum of the kinetic and the potential energies:

$$E = \frac{1}{2}mv^2 + \left(-\frac{kZe^2}{r} \right)$$

From Equation 4-12, we see that $\frac{1}{2}mv^2 = kZe^2/2r$ (a result that holds for circular motion in any inverse-square force field), so the total energy can be written as

$$E = \frac{kZe^2}{2r} - \frac{kZe^2}{r} = -\frac{kZe^2}{2r} \sim -\frac{1}{r} \quad 4-14$$

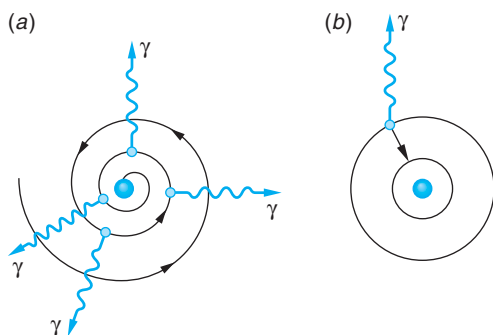
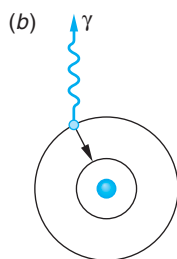


Figure 4-15 (a) In the classical orbital model, the electron orbits about the nucleus and spirals into the center because of the energy radiated. (b) In the Bohr model, the electron orbits without radiating until it jumps to another allowed radius of lower energy, at which time radiation is emitted.



Thus, classical physics predicts that, as energy is lost to radiation, the electron's orbit will become smaller and smaller while the frequency of the emitted radiation will become higher and higher, further increasing the rate at which energy is lost and ending when the electron reaches the nucleus. (See Figure 4-15a.) The time required for the electron to spiral into the nucleus can be calculated from classical mechanics and electrodynamics; it turns out to be less than a microsecond. Thus, at first sight, this model predicts that the atom will radiate a continuous spectrum (since the frequency of revolution changes continuously as the electron spirals in) and will collapse after a very short time, a result that fortunately does not occur. Unless excited by some external means, atoms do not radiate at all, and when excited atoms do radiate, a line spectrum is emitted, not a continuous one.

Bohr “solved” these formidable difficulties with two decidedly nonclassical postulates. His first postulate was that *electrons could move in certain orbits without radiating*. He called these orbits *stationary states*. His second postulate was to assume that *the atom radiates when the electron makes a transition from one stationary state to another* (Figure 4-15b) and that *the frequency f of the emitted radiation is not the frequency of motion in either stable orbit but is related to the energies of the orbits by Planck's theory*

$$hf = E_i - E_f \quad 4-15$$

where h is Planck's constant and E_i and E_f are the energies of the initial and final states. The second assumption, which is equivalent to that of energy conservation with the emission of a photon, is crucial because it deviated from classical theory, which requires the frequency of radiation to be that of the motion of the charged particle. Equation 4-15 is referred to as the *Bohr frequency condition*.

In order to determine the energies of the allowed, nonradiating orbits, Bohr made a third assumption, now known as the *correspondence principle*, which had profound implications:

In the limit of large orbits and large energies, quantum calculations must agree with classical calculations.

Thus, the correspondence principle says that whatever modifications of classical physics are made to describe matter at the submicroscopic level, when the results are extended to the macroscopic world, they must agree with those from the classical laws of physics that have been so abundantly verified in the everyday world. While Bohr's detailed model of the hydrogen atom has been supplanted by modern quantum theory, which we will discuss in later chapters, his frequency condition (Equation 4-15) and the correspondence principle remain as essential features of the new theory.

In his first paper,¹¹ in 1913, Bohr pointed out that his results implied that the angular momentum of the electron in the hydrogen atom can take on only values that are integral multiples of Planck's constant divided by 2π , in agreement with a discovery made a year earlier by J. W. Nicholson. That is, *angular momentum is quantized; it can assume only the values $nh/2\pi$, where n is an integer*. Rather than follow the intricacies of Bohr's derivation, we will use the fundamental conclusion of angular momentum quantization to find his expression for the observed spectra. The development that follows applies not only to hydrogen, but to any atom of nuclear charge $+Ze$ with a single orbital electron—e.g., singly ionized helium He^+ or doubly ionized lithium Li^{2+} .

If the nuclear charge is $+Ze$ and the electron charge $-e$, we have noted (Equation 4-12) that the centripetal force necessary to move the electron in a circular orbit is provided by the Coulomb force kZe^2/r^2 . Solving Equation 4-12 for the speed of the orbiting electron yields

$$v = \left(\frac{kZe^2}{mr} \right)^{1/2} \quad 4-16$$

Bohr's quantization of the angular momentum L is

$$L = mvr = \frac{nh}{2\pi} = n\hbar \quad n = 1, 2, 3, \dots \quad 4-17$$

where the integer n is called a *quantum number* and $\hbar = h/2\pi$. (The constant \hbar , read “h-bar,” is often more convenient to use than h itself, just as the angular frequency $\omega = 2\pi f$ is often more convenient than the frequency f .) Combining Equations 4-16 and 4-17 allows us to write for the circular orbits

$$r = \frac{n\hbar}{mv} = \frac{n\hbar}{m} \left(\frac{rm}{kZe^2} \right)^{1/2}$$

Squaring this relation gives

$$r^2 = \frac{n^2\hbar^2}{m^2} \left(\frac{rm}{kZe^2} \right)$$

and canceling common quantities yields

$$r_n = \frac{n^2\hbar^2}{mkZe^2} = \frac{n^2a_0}{Z} \quad 4-18$$

where

$$a_0 = \frac{\hbar^2}{mke^2} = 0.529 \text{ \AA} = 0.0529 \text{ nm} \quad 4-19$$

is called the *Bohr radius*. The \AA , a unit commonly used in the early days of spectroscopy, equals 10^{-10} m or 10^{-1} nm . Thus, we find that the stationary orbits of Bohr's first postulate have quantized radii, denoted in Equation 4-18 by the subscript on r_n .

Notice that the Bohr radius a_0 for hydrogen ($Z = 1$) corresponds to the orbit radius with $n = 1$, the smallest Bohr orbit possible for the electron in a hydrogen atom. Since $r_n \sim Z^{-1}$, the Bohr orbits for single-electron atoms with $Z > 1$ are closer to the nucleus than the corresponding ones for hydrogen.

The total energy of the electron (Equation 4-14) then becomes, upon substitution of r_n from Equation 4-18,

$$\begin{aligned} E_n &= -\frac{kZe^2}{2r_n} = -\frac{kZe^2}{2} \left(\frac{mkZe^2}{n^2\hbar^2} \right) \\ E_n &= -\frac{mk^2Z^2e^4}{2\hbar^2n^2} = -E_0 \frac{Z^2}{n^2} \quad n = 1, 2, 3, \dots \end{aligned} \quad 4-20$$

where $E_0 = mk^2e^4/2\hbar^2$. Thus, the energy of the electron is also quantized; i.e., the stationary states correspond to specific values of the total energy. This means that energies E_i and E_f that appear in the frequency condition of Bohr's second postulate must be from the allowed set E_n , and Equation 4-15 becomes

$$hf = E_{n_i} - E_{n_f} = -E_0 \frac{Z^2}{n_i^2} - \left(-E_0 \frac{Z^2}{n_f^2} \right)$$

or

$$f = \frac{E_0Z^2}{h} \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right) \quad 4-21$$

which can be written in the form of the Rydberg-Ritz equation (Equation 4-2) by substituting $f = c/\lambda$ and dividing by c to obtain

$$\frac{1}{\lambda} = \frac{E_0Z^2}{hc} \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right)$$

or

$$\frac{1}{\lambda} = Z^2R \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right) \quad 4-22$$

where

$$R = \frac{E_0}{hc} = \frac{mk^2e^4}{4\pi c\hbar^3} \quad 4-23$$

is Bohr's prediction for the value of the Rydberg constant.

Using the values of m , e , c , and \hbar known in 1913, Bohr calculated R and found his result to agree (within the limits of uncertainties of the constants) with the value obtained from spectroscopy, $1.097 \times 10^7 \text{ m}^{-1}$. Bohr noted in his original paper that this equation might be valuable in determining the best values for the constants e , m , and \hbar because of the extreme precision possible in measuring R . This has indeed turned out to be the case.

The possible values of the energy of the hydrogen atom predicted by Bohr's model are given by Equation 4-20 with $Z = 1$:

$$E_n = -\frac{mk^2e^4}{2\hbar^2n^2} = -\frac{E_0}{n^2} \quad 4-24$$

where

$$E_0 = \frac{mk^2e^4}{2\hbar^2} = 2.18 \times 10^{-18} \text{ J} = 13.6 \text{ eV}$$

is the magnitude of E_n with $n = 1$. $E_1 (= -E_0)$ is called the *ground state*. It is convenient to plot these allowed energies of the stationary states as in Figure 4-16. Such a plot is called an *energy-level diagram*. Various series of transitions between the stationary states are indicated in this diagram by vertical arrows drawn between the levels. The frequency of light emitted in one of these transitions is the energy difference divided by h according to Bohr's frequency condition, Equation 4-15. The energy required to remove the electron from the atom, 13.6 eV, is called the *ionization energy*, or *binding energy*, of the electron.

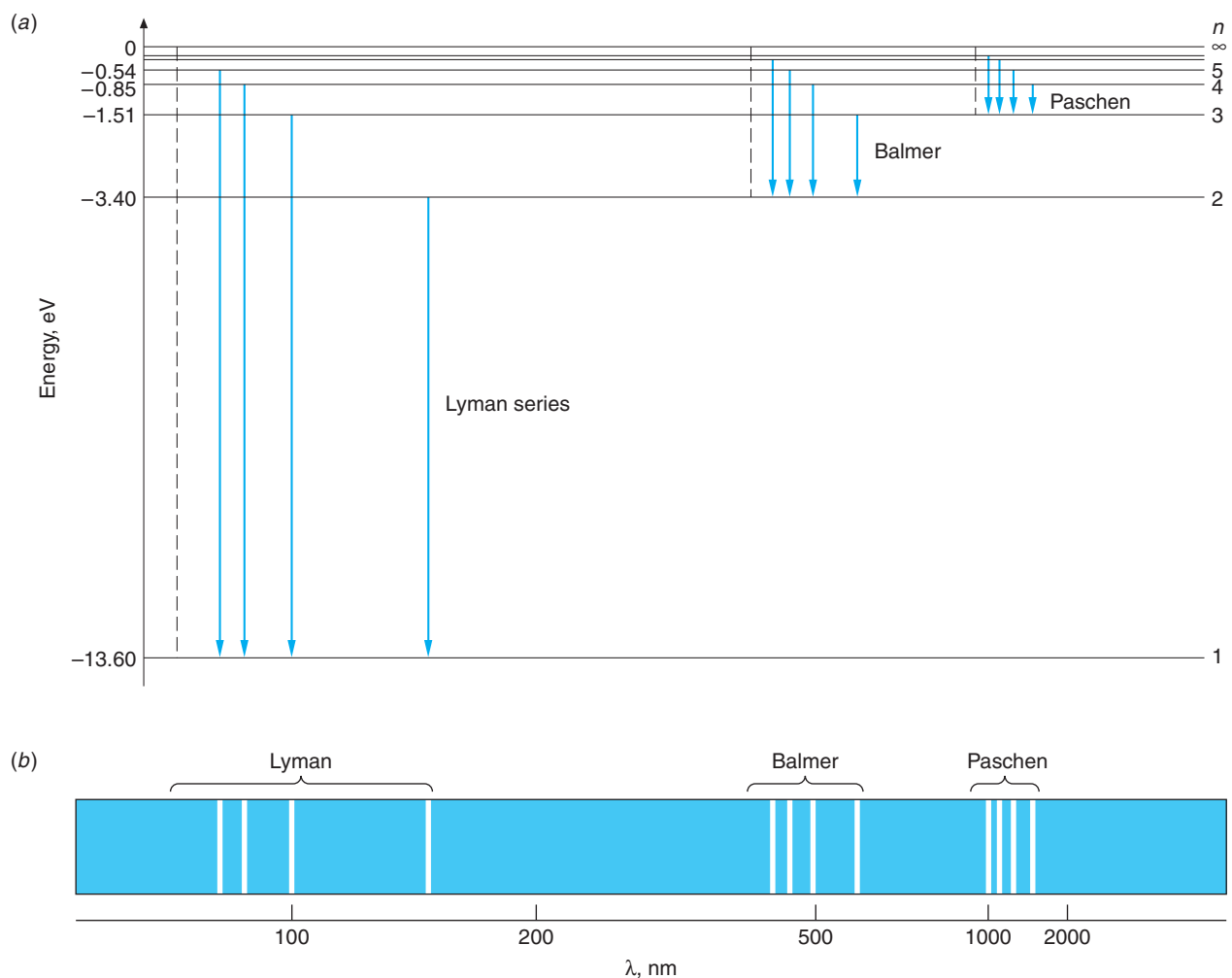


Figure 4-16 Energy-level diagram for hydrogen showing the seven lowest stationary states and the four lowest energy transitions each for the Lyman, Balmer, and Paschen series. There are an infinite number of levels. Their energies are given by $E_n = -13.6/n^2$ eV, where n is an integer. The dashed line shown for each series is the *series limit*, corresponding to the energy that would be radiated by an electron at rest far from the nucleus ($n \rightarrow \infty$) in a transition to the state with $n = n_f$ for that series. The horizontal spacing between the transitions shown for each series is proportional to the wavelength spacing between the lines of the spectrum. (b) The spectral lines corresponding to the transitions shown for the three series. Notice the regularities within each series, particularly the short-wavelength limit and the successively smaller separation between adjacent lines as the limit is approached. The wavelength scale in the diagram is not linear.

A bit different sort of application, the Bohr/Rutherford model of the nuclear atom and electron orbits is the picture that, for millions of people, provides their link to the world of the atom and subatomic phenomena.

At the time Bohr's paper was published, there were two spectral series known for hydrogen: the Balmer series, corresponding to $n_f = 2$, $n_i = 3, 4, 5, \dots$, and a series named after its discoverer, Paschen (1908), corresponding to $n_f = 3$, $n_i = 4, 5, 6, \dots$. Equation 4-22 indicates that other series should exist for different values of n_f . In 1916 Lyman found the series corresponding to $n_f = 1$, and in 1922 and 1924 F. S. Brackett and A. H. Pfund, respectively, found series corresponding to $n_f = 4$ and $n_f = 5$. As can be easily determined by computing the wavelengths for these series, only the Balmer series lies primarily in the visible portion of the electromagnetic spectrum. The Lyman series is in the ultraviolet, the others in the infrared.

EXAMPLE 4-6 Wavelength of the H_β Line Compute the wavelength of the H_β spectral line, i.e., the second line of the Balmer series predicted by Bohr's model. The H_β line is emitted in the transition from $n_i = 4$ to $n_f = 2$.

SOLUTION

1. *Method 1:* The wavelength is given by Equation 4-22 with $Z = 1$:

$$\frac{1}{\lambda} = R \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right)$$

2. Substituting $R = 1.097 \times 10^7 \text{ m}^{-1}$ and the values of n_i and n_f :

$$\frac{1}{\lambda} = (1.097 \times 10^7 \text{ m}^{-1}) \left(\frac{1}{2^2} - \frac{1}{4^2} \right)$$

or

$$\lambda = 4.86 \times 10^{-7} = 486 \text{ nm}$$

3. *Method 2:* The wavelength may also be computed from Equation 4-15:

$$hf = hc/\lambda = E_i - E_f$$

or

$$\frac{1}{\lambda} = \frac{1}{hc} (E_i - E_f)$$

4. The values of E_i and E_f are given by Equation 4-24:

$$E_i = -\left(\frac{13.6 \text{ eV}}{n_i^2} \right) = -\left(\frac{13.6 \text{ eV}}{4^2} \right) = -0.85 \text{ eV}$$

$$E_f = -\left(\frac{13.6 \text{ eV}}{n_f^2} \right) = -\left(\frac{13.6 \text{ eV}}{2^2} \right) = -3.4 \text{ eV}$$

5. Substituting these into Equation 4-15 yields

$$\begin{aligned} \frac{1}{\lambda} &= \frac{[-0.85 \text{ eV} - (-3.4 \text{ eV})](1.60 \times 10^{-19} \text{ J/eV})}{(6.63 \times 10^{-34} \text{ J} \cdot \text{s})(3.00 \times 10^8 \text{ m/s})} \\ &= 2.051 \times 10^6 \text{ m}^{-1} \end{aligned}$$

or

$$\lambda = 4.87 \times 10^{-7} \text{ m} = 487 \text{ nm}$$

Remarks: The difference in the two results is due to rounding of the Rydberg constant to three decimal places.

Reduced Mass Correction

The assumption by Bohr that the nucleus is fixed is equivalent to the assumption that it has infinite mass. In fact, the Rydberg constant in Equation 4-23 is normally written a R_∞ , as we will do henceforth. If the nucleus has mass M , its kinetic energy will be $\frac{1}{2}Mv^2 = p^2/2M$, where $p = Mv$ is the momentum. If we assume that the total momentum of the atom is zero, conservation of momentum requires that the momenta of the nucleus and electron be equal in magnitude. The total kinetic energy is then

$$E_k = \frac{p^2}{2M} + \frac{p^2}{2m} = \frac{M+m}{2mM} p^2 = \frac{p^2}{2\mu}$$

where

$$\mu = \frac{mM}{m+M} = \frac{m}{1+m/M} \quad 4-25$$

This is slightly different from the kinetic energy of the electron because μ , called the *reduced mass*, is slightly different from the electron mass. The results derived above for a nucleus of infinite mass can be applied directly for the case of a nucleus of mass M if we replace the electron mass in the equations by reduced mass μ , defined by Equation 4-25. (The validity of this procedure is proved in most intermediate and advanced mechanics books.) The Rydberg constant (Equation 4-23) is then written

$$R = \frac{\mu k^2 e^4}{4\pi c \hbar^3} = \frac{mk^2 e^4}{4\pi c \hbar^3} \left(\frac{1}{1+m/M} \right) = R_\infty \left(\frac{1}{1+m/M} \right) \quad 4-26$$

This correction amounts to only 1 part in 2000 for the case of hydrogen and to even less for other nuclei; however, the predicted variation in the Rydberg constant from atom to atom is precisely that which is observed. For example, the spectrum of a singly ionized helium atom, which has one remaining electron, is just that predicted by Equation 4-22 and 4-26 with $Z = 2$ and the proper helium mass. The current value for the Rydberg constant R_∞ from precision spectroscopic measurements¹² is

$$R_\infty = 1.0973732 \times 10^7 \text{ m}^{-1} = 1.0973732 \times 10^{-2} \text{ nm}^{-1} \quad 4-27$$

Urey¹³ used the reduced mass correction to the spectral lines of the Balmer series to discover (in 1931) a second form of hydrogen whose atoms had twice the mass of ordinary hydrogen. The heavy form was called *deuterium*. The two forms, atoms with the same Z but different masses, are called *isotopes*.

EXAMPLE 4-7 Rydberg Constants for H and He⁺ Compute the Rydberg constants for H and He⁺ applying the reduced mass correction. ($m = 9.1094 \times 10^{-31}$ kg, $m_p = 1.6726 \times 10^{-27}$ kg, $m_\alpha = 5.0078 \times 10^{-27}$ kg)

SOLUTION

For hydrogen:

$$\begin{aligned} R_H &= R_\infty \left(\frac{1}{1+m/M_H} \right) = R_\infty \left(\frac{1}{1+9.1094 \times 10^{-31}/1.6726 \times 10^{-27}} \right) \\ &= 1.09677 \times 10^7 \text{ m}^{-1} \end{aligned}$$

For helium: Since M in the reduced mass correction is the mass of the nucleus, for this calculation we use M equal to the α particle mass.

$$R_{\text{He}} = R_{\infty} \left(\frac{1}{1 + m/M_{\text{H}}} \right) = R_{\infty} \left(\frac{1}{1 + 9.1094 \times 10^{-31} / 5.0078 \times 10^{-27}} \right) \\ = 1.09752 \times 10^7 \text{ m}^{-1}$$

Thus, the two Rydberg constants differ by about 0.04 percent.

Correspondence Principle

According to the correspondence principle, which applies also to modern quantum mechanics, when the energy levels are closely spaced, quantization should have little effect; classical and quantum calculations should give the same results. From the energy-level diagram of Figure 4-16, we see that the energy levels are close together when the quantum number n is large. This leads us to a slightly different statement of Bohr's correspondence principle: *In the region of very large quantum numbers (n in this case) quantum calculation and classical calculation must yield the same results.* To see that the Bohr model of the hydrogen atom does indeed obey the correspondence principle, let us compare the frequency of a transition between level $n_i = n$ and level $n_f = n - 1$ for large n with the classical frequency, which is the frequency of revolution of the electron. From Equation 4-22 we have

$$f = \frac{c}{\lambda} = \frac{Z^2 m k^2 e^4}{4\pi\hbar^3} \left[\frac{1}{(n-1)^2} - \frac{1}{n^2} \right] = \frac{Z^2 m k^2 e^4}{4\pi\hbar^3} \frac{2n-1}{n^2(n-1)^2}$$

For large n we can neglect the ones subtracted from n and $2n$ to obtain

$$f = \frac{Z^2 m k^2 e^4}{4\pi\hbar^3} \frac{2}{n^3} = \frac{Z^2 m k^2 e^4}{2\pi\hbar^3 n^3} \quad 4-28$$

The classical frequency of revolution of the electron is (see Equation 4-13)

$$f_{\text{rev}} = \frac{v}{2\pi r}$$

Using $v = n\hbar/mr$ from Equation 4-17 and $r = n^2\hbar^2/mkZe^2$ from Equation 4-18, we obtain

$$f_{\text{rev}} = \frac{(n\hbar/mr)}{2\pi r} = \frac{n\hbar}{2\pi m r^2} = \frac{n\hbar}{2\pi m (n^2\hbar^2/mkZe^2)^2} \\ f_{\text{rev}} = \frac{m^2 k^2 Z^2 e^4 n\hbar}{2\pi m n^4 \hbar^4} = \frac{mk^2 Z^2 e^4}{2\pi\hbar^3 n^3} \quad 4-29$$

which is the same as Equation 4-28.

Fine Structure Constant

The demonstration of the correspondence principle for large n in the preceding paragraph was for $\Delta n = n_i - n_f = 1$; however, we have seen (see Figure 4-16) that transitions occur in the hydrogen atom for $\Delta n \geq 1$ when n is small, and such transitions should occur for large n too. If we allow $\Delta n = 2, 3, \dots$ for large values of n , then the frequencies of the emitted radiation would be, according to Bohr's model, integer multiples of the frequency given in Equation 4-28. In that event, Equations 4-28 and 4-29 would not agree. This disagreement can be avoided by allowing elliptical orbits.¹⁴ A result of Newtonian mechanics, familiar from planetary motion, is that in an inverse-square force

field, the energy of an orbiting particle depends only on the major axis of the ellipse and not on its eccentricity. There is consequently no change in the energy at all unless the force differs from inverse square or unless Newtonian mechanics is modified. A. Sommerfeld considered the effect of special relativity on the mass of the electron in the Bohr model in an effort to explain the observed *fine structure* of the hydrogen spectral lines.¹⁵ Since the relativistic corrections should be of the order of v^2/c^2 (see Chapter 2), it is likely that a highly eccentric orbit would have a larger correction because v becomes greater as the electron moves nearer the nucleus. The Sommerfeld calculations are quite complicated, but we can estimate the order of magnitude of the effect of special relativity by calculating v/c for the first Bohr orbit in hydrogen. For $n = 1$, we have from Equation 4-17 that $mvr_1 = \hbar$. Then, using $r_1 = a_0 = \hbar^2/mke^2$, we have



$$v = \frac{\hbar}{mr_1} = \frac{\hbar}{m(\hbar^2/mke^2)} = \frac{ke^2}{\hbar}$$

and

$$\frac{v}{c} = \frac{ke^2}{\hbar c} = \frac{1.44 \text{ eV} \cdot \text{nm}}{197.3 \text{ eV} \cdot \text{nm}} \approx \frac{1}{137} = \alpha \quad 4-30$$

where we have used another convenient combination

$$\hbar c = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{2\pi} = 197.3 \text{ eV} \cdot \text{nm} \quad 4-31$$

The dimensionless quantity $ke^2/\hbar c = \alpha$ is called the *fine-structure constant* because of its first appearance in Sommerfeld's theory, but as we will see, it has much more fundamental importance.

Though v^2/c^2 is very small, an effect of this magnitude is observable. In Sommerfeld's theory, the fine structure of the hydrogen spectrum is explained in the following way. For each allowed circular orbit of radius r_n and energy E_n , a set of n elliptical orbits is possible of equal major axes but different eccentricities. Since the velocity of a particle in an elliptical orbit depends on the eccentricity, so then will the mass and momentum, and therefore the different ellipses for a given n will have slightly different energies. Thus, the energy radiated when the electron changes orbit depends slightly on the eccentricities of the initial and final orbits as well as on their major axes. The splitting of the energy levels for a given n is called *fine-structure splitting*, and its value turns out to be of the order of $v^2/c^2 = \alpha^2$, just as Sommerfeld predicted. However, the agreement of Sommerfeld's prediction with the observed fine-structure splitting was quite accidental and led to considerable confusion in the early days of quantum theory. Although he had used the relativistic mass and momentum, he computed the energy using classical mechanics, leading to a correction much larger than that actually due only to relativistic effects. As we will see in Chapter 7, fine structure is associated with a completely nonclassical property of the electron called *spin*.

A lasting contribution of Sommerfeld's effort was the introduction of the fine-structure constant $\alpha = ke^2/\hbar c \approx 1/137$. With it we can write the Bohr radius a_0 and the quantized energies of the Bohr model in a particularly elegant form. Equations 4-24 and 4-19 for hydrogen become

$$E_n = -\frac{mk^2e^4}{2\hbar^2n^2} \frac{c^2}{c^2} = -\frac{mc^2}{2}\alpha^2 \frac{1}{n^2} \quad 4-32$$

$$a_0 = \frac{\hbar^2}{mke^2} \frac{c}{c} = \frac{\hbar}{mc} \frac{1}{\alpha} \quad 4-33$$

Since α is a dimensionless number formed of universal constants, *all* observers will measure the same value for it and find that energies and dimensions of atomic systems are proportional to α^2 and $1/\alpha$, respectively. We will return to the implications of this intriguing fact later in the book.



EXPLORING Giant Atoms

Giant atoms called *Rydberg atoms*, long understood to be a theoretical possibility and first detected in interstellar space in 1965, are now being produced and studied in the laboratory. Rydberg atoms are huge! They are atoms that have one of the valence electrons in a state with a very large quantum number n . (See Figure 4-17.) Notice in Equation 4-18 that the radius of the electron orbit $r_n = n^2 a_0 / Z \propto n^2$ and n can be any positive integer, so the diameter of a hydrogen atom (or any other atom, for that matter) could be very large, a millimeter or even a meter! What keeps such giant atoms from being common is that the energy difference between adjacent allowed energy states is extremely small when n is large and the allowed states are very near the $E_\infty = 0$ level where ionization occurs, because $E_n \propto 1/n^2$. For example, if $n = 1000$, the diameter of a hydrogen atom would be $r_{1000} = 0.1$ mm, but both E_{1000} and the difference in energy $\Delta E = E_{1001} - E_{1000}$ are about 10^{-5} eV! This energy is far below the average energy of thermal motion at ordinary temperatures (about 0.025 eV), so random collisions would quickly ionize an atom whose electron happened to get excited to a level with n equal to 20 or so with r still only about 10^{-8} m.

The advent of precisely tunable dye lasers in the 1970s made it possible to nudge electrons carefully into orbits with larger and larger n values. The largest Rydberg atoms made so far, typically using sodium or potassium, are 10,000 times the diameter of ordinary atoms, about 20 μm across or the size of a fine grain of sand, and exist for several milliseconds inside vacuum chambers. For hydrogen, this corresponds to quantum number $n \approx 600$. An electron moving so far from the nucleus is bound by a minuscule force. It also moves rather slowly since the classical period of $T = 1/f \propto n^3$ and follows an elliptical orbit. These characteristics of very large n orbits provide several intriguing possibilities. For example, very small electric fields might be studied, making possible the tracking of chemical reactions that proceed too quickly to be followed otherwise. More dramatic is the possibility of directly testing Bohr's correspondence principle by directly observing the slow (since $v \propto 1/n$) movement of the electron around the large n orbits—the transition from quantum mechanics to classical mechanics. Computer simulations of the classical motion of a Rydberg electron “wave” (see Chapter 5) in orbit around a nucleus are aiding the design of experiments to observe the correspondence principle.



Figure 4-17 A lithium ($Z = 3$) Rydberg atom. The outer electron occupies a small volume and follows a nearly classical orbit with a large value of n . The two inner electrons are not shown.

Questions

6. If the electron moves in an orbit of greater radius, does its total energy increase or decrease? Does its kinetic energy increase or decrease?
7. What is the energy of the shortest-wavelength photon that can be emitted by the hydrogen atom?
8. How would you characterize the motion and location of an electron with $E = 0$ and $n \rightarrow \infty$ in Figure 4-16?

4-4 X-Ray Spectra

The extension of the Bohr theory to atoms more complicated than hydrogen proved difficult. Quantitative calculations of the energy levels of atoms of more than one electron could not be made from the model, even for helium, the next element in the periodic table. However, experiments by H. Moseley in 1913 and J. Franck and G. Hertz in 1914 strongly supported the general Bohr-Rutherford picture of the atom as a positively charged core surrounded by electrons that moved in quantized energy states relatively far from the core. Moseley's analysis of x-ray spectra will be discussed in this section, and the Franck-Hertz measurement of the transmission of electrons through gases will be discussed in the chapter's concluding section.

Using the methods of crystal spectrometry that had just been developed by W. H. Bragg and W. L. Bragg, Moseley¹⁶ measured the wavelengths of the characteristic x-ray line spectra for about 40 different target elements. (Typical x-ray spectra are shown in Figure 3-15.) He noted that the x-ray line spectra varied in a regular way from element to element, unlike the irregular variations of optical spectra. He surmised that this regular variation occurred because characteristic x-ray spectra were due to transitions involving the innermost electrons of the atoms. (See Figure 4-18.) Because the inner electrons are



Henry G.-J. Moseley.
[Courtesy of University of Manchester.]

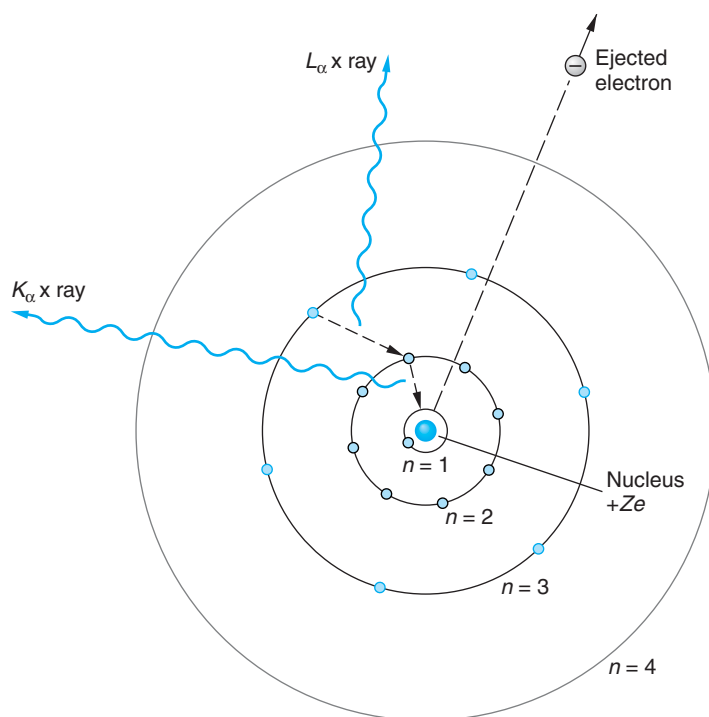


Figure 4-18 A stylized picture of the Bohr circular orbits for $n = 1, 2, 3$, and 4. The radii $r_n \sim n^2$. In a high- Z element (elements with $Z \geq 12$ emit x rays), electrons are distributed over all the orbits shown. If an electron in the $n = 1$ orbit is knocked from the atom, e.g., by being hit by a fast electron accelerated by the voltage across an x-ray tube, the vacancy thus produced is filled by an electron of higher energy (i.e., $n = 2$ or higher). The difference in energy between the two orbits is emitted as a photon, according to the Bohr frequency condition, whose wavelength will be in the x-ray region of the spectrum if Z is large enough.

shielded from the outermost electrons by those in intermediate orbits, their energies do not depend on the complex interactions of the outer electrons, which are responsible for the complicated optical spectra. Furthermore, the inner electrons are well shielded from the interatomic forces that are responsible for the binding of atoms in solids.

According to the Bohr theory (published earlier the same year, 1913), the energy of an electron in the first Bohr orbit is proportional to the square of the nuclear charge (see Equation 4-20). Moseley reasoned that the energy, and therefore the frequency, of a characteristic x-ray photon should vary as the square of the atomic number of the target element. He therefore plotted the square root of the frequency of a particular characteristic line in the x-ray spectrum of various target elements versus the atomic number Z of the element. Such a plot, now called a *Moseley plot*, is shown in Figure 4-19.

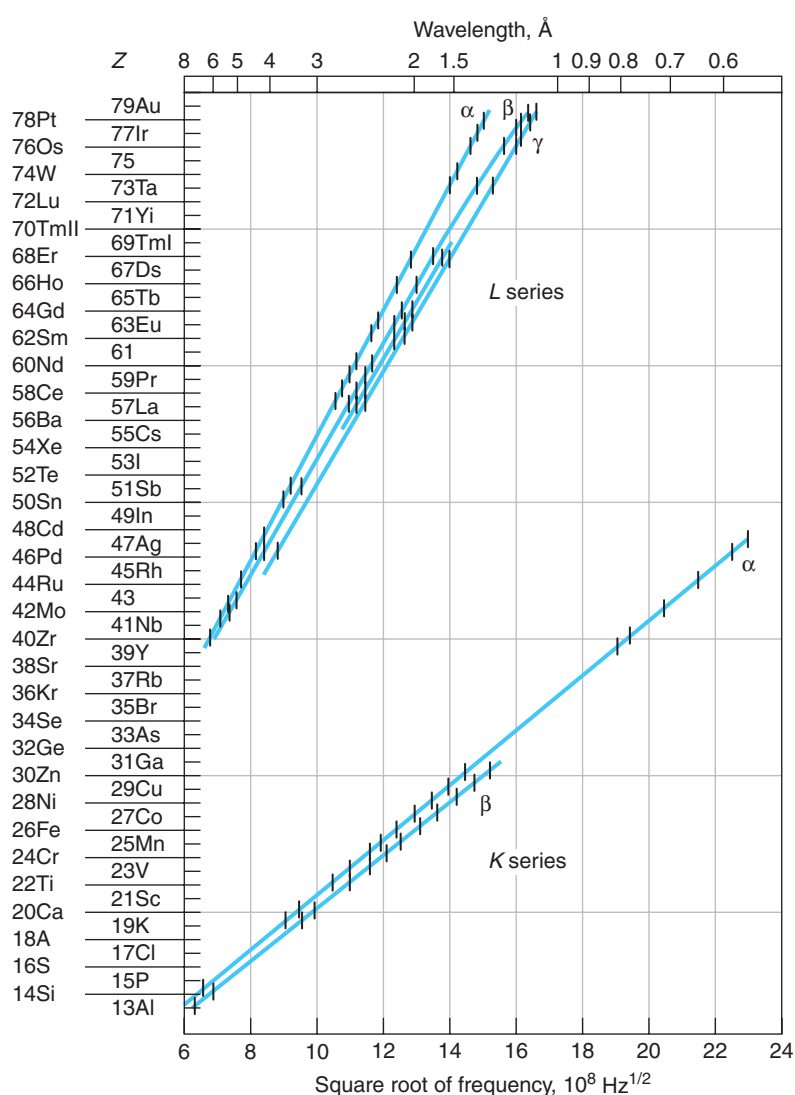


Figure 4-19 Moseley's plots of the square root of frequency versus Z for characteristic x rays. When an atom is bombarded by high-energy electrons, an inner atomic electron is sometimes knocked out, leaving a vacancy in the inner shell. The K -series x rays are produced by atomic transitions to vacancies in the $n = 1$ (K) shell, whereas the L series is produced by transitions to the vacancies in the $n = 2$ (L) shell. [From H. Moseley, *Philosophical Magazine* (6), 27, 713 (1914).]

These curves can be fitted by the empirical equation

$$f^{1/2} = A_n(Z - b) \quad 4-34$$

where A_n and b are constants for each characteristic x-ray line. One family of lines, called the *K series*, has $b = 1$ and slightly different values of A_n for each line in the graph. The other family shown in Figure 4-19, called the *L series*,¹⁷ could be fitted by Equation 4-34 with $b = 7.4$.

If the bombarding electron in the x-ray tube causes ejection of an electron from the innermost orbit ($n = 1$) in a target atom completely out of the atom, photons will be emitted corresponding to transitions of electrons in other orbits ($n = 2, 3, \dots$) to fill the vacancy in the $n = 1$ orbit. (See Figure 4-18.) (Since these lines are called the *K series*, the $n = 1$ orbit came to be called the *K shell*.) The lowest-frequency line corresponds to the lowest-energy transition ($n = 2 \rightarrow n = 1$). This line is called the K_α line. The transition $n = 3 \rightarrow n = 1$ is called the K_β line. It is of higher energy, and hence higher frequency, than the K_α line. A vacancy created in the $n = 2$ orbit by emission of a K_α x ray may then be filled by an electron of higher energy, e.g., one in the $n = 3$ orbit, resulting in the emission of a line in the *L series*, and so on. The multiple *L* lines in the Moseley plot (Figure 4-19) are due in part to the fact that there turn out to be small differences in the energies of electrons with a given n that are not predicted by the Bohr model. Moseley's work gave the first indication of these differences, but the explanation will have to await our discussion of more advanced quantum theory in Chapter 7.

Using the Bohr relation for a one-electron atom (Equation 4-21) with $n_f = 1$ and using $(Z - 1)$ in place of Z , we obtain for the frequencies of the *K series*

$$f = \frac{mk^2e^4}{4\pi\hbar^3}(Z - 1)^2\left(\frac{1}{1^2} - \frac{1}{n^2}\right) = cR_\infty(Z - 1)^2\left(1 - \frac{1}{n^2}\right) \quad 4-35$$

where R_∞ is the Rydberg constant. Comparing this with Equation 4-34, we see that A_n is given by

$$A_n^2 = cR_\infty\left(1 - \frac{1}{n^2}\right) \quad 4-36$$

The wavelengths of the lines in the *K series* are then given by

$$\lambda = \frac{c}{f} = \frac{c}{A_n^2(Z - 1)^2} = \frac{1}{R_\infty(Z - 1)^2\left(1 - \frac{1}{n^2}\right)} \quad 4-37$$

EXAMPLE 4-8 K_α for Molybdenum Calculate the wavelength of the K_α line of molybdenum ($Z = 42$), and compare the result with the value $\lambda = 0.0721$ nm measured by Moseley and with the spectrum in Figure 3-15b.

SOLUTION

Using $n = 2$, $R_\infty = 1.097 \times 10^7 \text{ m}^{-1}$, and $Z = 42$, we obtain

$$\lambda = \left[(1.097 \times 10^7 \text{ m}^{-1})(41)^2 \left(1 - \frac{1}{4}\right) \right]^{-1} = 7.23 \times 10^{-11} \text{ m} = 0.0723 \text{ nm}$$

This value is within 0.3 percent of Moseley's measurement and agrees well with that in Figure 3-15b.

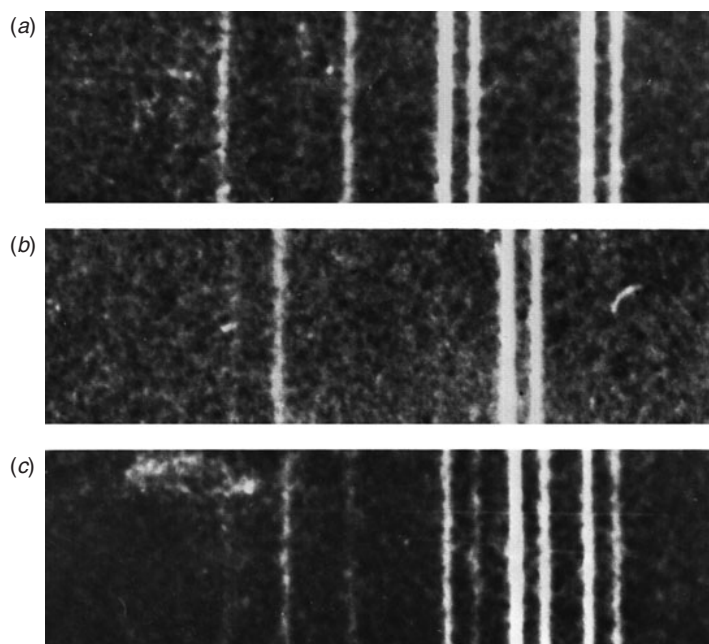
The fact that f is proportional to $(Z - 1)^2$ rather than to Z is explained by the partial shielding of the nuclear charge by the other electron remaining in the K shell as “seen” by electrons in the $n = 2$ (L) shell.¹⁸ Using this reasoning, Moseley concluded that, since $b = 7.4$ for the L series, these lines involved electrons farther from the nucleus, which “saw” the nuclear charge shielded by more inner electrons. Assuming that the L series was due to transitions to the $n = 2$ shell, we see that the frequencies for this series are given by

$$f = cR_{\infty} \left(\frac{1}{2^2} - \frac{1}{n^2} \right) (Z - 7.4)^2 \quad 4-38$$

where $n = 3, 4, 5, \dots$

Before Moseley’s work, the atomic number was merely the place number of the element in Mendeleev’s periodic table of the elements arranged by weight. The experiments of Geiger and Marsden showed that the nuclear charge was approximately $A/2$, while x-ray scattering experiments by C. G. Barkla showed that the number of electrons in an atom was also approximately $A/2$. These two experiments are consistent since the atom as a whole must be electrically neutral. However, several discrepancies were found in the periodic table as arranged by weight. For example, the 18th element in order of weight is potassium (39.102), and the 19th is argon (39.948). Arrangement by weight, however, puts potassium in the column with the inert gases and argon with the active metals, the reverse of their known chemical properties. Moseley showed that for these elements to fall on the line $f^{1/2}$ versus Z , argon had to have $Z = 18$ and potassium $Z = 19$. Arranging the elements by the Z number obtained from the Moseley plot rather than by weight, gave a periodic chart in complete agreement with the chemical properties. Moseley also pointed out that there were gaps in the periodic table at $Z = 43, 61$, and 75 , indicating the presence of undiscovered elements. All have subsequently been found. Figure 4-20 illustrates the discovery of promethium ($Z = 61$).

Figure 4-20 Characteristic x-ray spectra. (a) Part of the spectra of neodymium ($Z = 60$) and samarium ($Z = 62$). The two pairs of bright lines are the K_{α} and K_{β} lines. (b) Part of the spectrum of the artificially element promethium ($Z = 61$). This element was first positively identified in 1945 at the Clinton Laboratory (now Oak Ridge). Its K_{α} and K_{β} lines fall between those of neodymium and samarium, just as Moseley predicted. (c) Part of the spectra of all three of the elements neodymium, promethium, and samarium. [Courtesy of J. A. Swartout, Oak Ridge National Laboratory.]



Auger Electrons

The process of producing x rays necessarily results in the ionization of the atom since an inner electron is ejected. The vacancy created is filled by an outer electron, producing the x rays studied by Moseley. In 1923 Pierre Auger discovered that, as an alternative to x-ray emission, the atom may eject a third electron from a higher-energy outer shell via a radiationless process called the *Auger effect*. In the Auger (pronounced oh-zhay) process, the energy difference $\Delta E = E_2 - E_1$ that could have resulted in the emission of a K_α x ray is removed from the atom by the third electron, e.g., one in the $n = 3$ shell. Since the magnitude of $E_3 < \Delta E$, the $n = 3$ electron would leave the atom with a characteristic kinetic energy $\Delta E - |E_3|$, which is determined by the stationary-state energies of the particular atom.¹⁹ Thus, each element has a characteristic Auger electron spectrum. (See Figure 4-21a.) Measurement of the Auger electrons provides a simple and highly sensitive tool for identifying impurities on clean surfaces in electron microscope systems and investigating electron energy shifts associated with molecular bonding. (See Figure 4-21b.)

Question

9. Why did Moseley plot $f^{1/2}$ versus Z rather than f versus Z ?

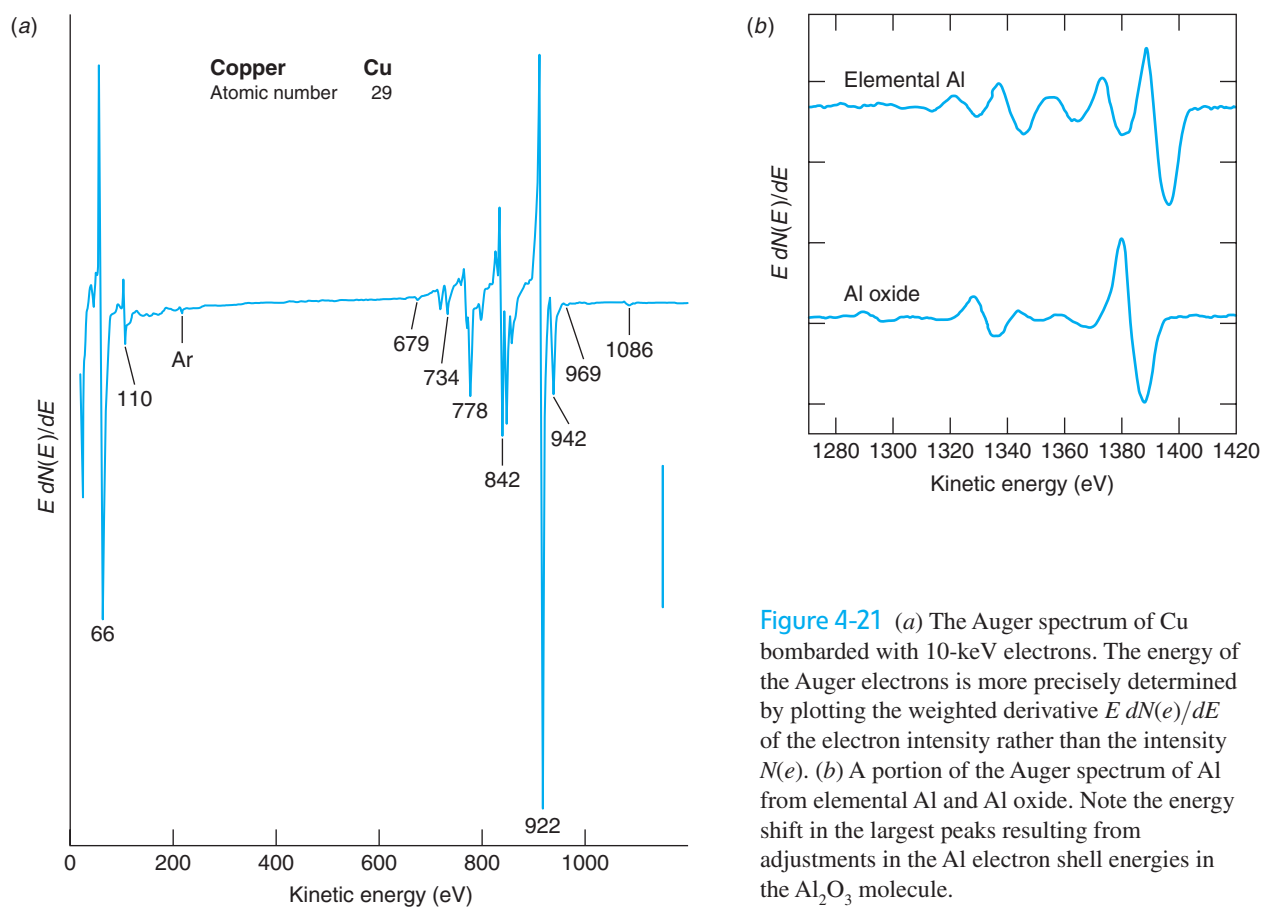


Figure 4-21 (a) The Auger spectrum of Cu bombarded with 10-keV electrons. The energy of the Auger electrons is more precisely determined by plotting the weighted derivative $E dN(e)/dE$ of the electron intensity rather than the intensity $N(e)$. (b) A portion of the Auger spectrum of Al from elemental Al and Al oxide. Note the energy shift in the largest peaks resulting from adjustments in the Al electron shell energies in the Al_2O_3 molecule.

4-5 The Franck-Hertz Experiment

We conclude this chapter with discussion of an important experiment that provided strong support for the quantization of atomic energies, thus helping to pave the way for modern quantum mechanics. While investigating the inelastic scattering of electrons, J. Franck and G. Hertz²⁰ made a discovery that confirmed *by direct measurement* Bohr's hypothesis of energy quantization in atoms. First done in 1914, it is now a standard undergraduate laboratory experiment. Figure 4-22a is a schematic diagram of the apparatus. A small heater heats the cathode. Electrons are ejected from the heated cathode and accelerated toward a grid, which is at a positive potential V_0 relative to the cathode. Some electrons pass through the grid and reach the plate P , which is at a slightly lower potential $V_p = V_0 - \Delta V$. The tube is filled with a low-pressure gas of the element being investigated (mercury vapor, in Franck and Hertz's experiment). The experiment involves measuring the plate current as a function of V_0 . As V_0 is increased from 0, the current increases until a critical value (about 4.9 V for Hg) is reached, at which point the current suddenly decreases. As V_0 is increased further, the current rises again.

The explanation of this result is a bit easier to visualize if we think for the moment of a tube filled with hydrogen atoms instead of mercury. (See Figure 4-22b.) Electrons accelerated by V_0 that collide with hydrogen electrons cannot transfer energy to the latter unless they have acquired kinetic energy $eV_0 = E_2 - E_1 = 10.2$ eV since the hydrogen electron according to Bohr's model cannot occupy states with energies intermediate between E_1 and E_2 . Such a collision will thus be elastic; i.e., the incident electron's kinetic energy will be unchanged by the collision, and thus it can overcome the potential ΔV and contribute to the current I . However, if $eV_0 \geq 10.2$ eV,

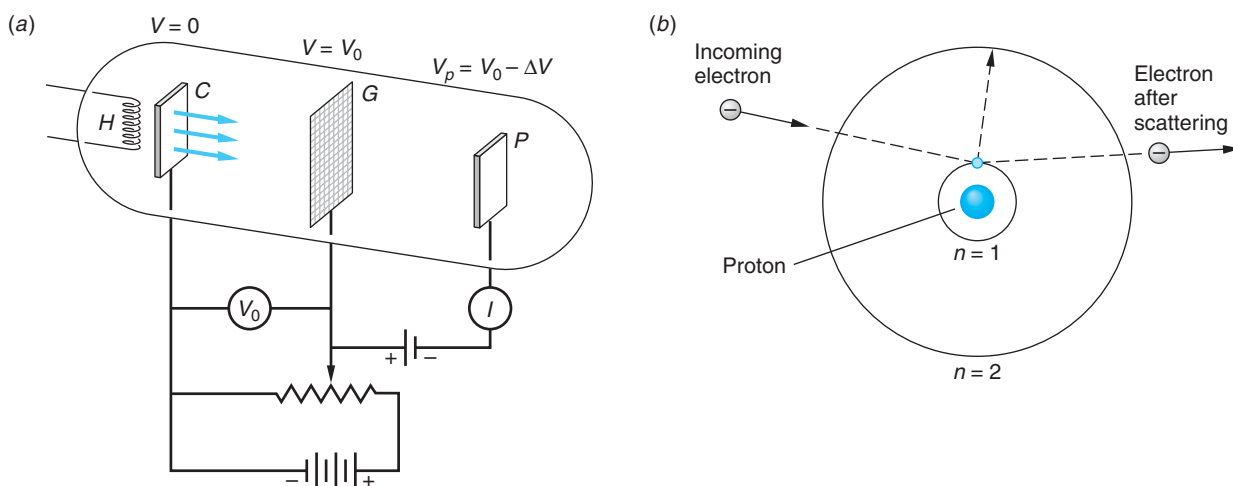


Figure 4-22 (a) Schematic diagram of the Franck-Hertz experiment. Electrons ejected from the heated cathode C at zero potential are drawn to the positive grid G . Those passing through the holes in the grid can reach the plate P and thereby contribute to the current I if they have sufficient kinetic energy to overcome the small back potential ΔV . The tube contains a low-pressure gas of the element being studied. (b) Results for hydrogen. If the incoming electron does not have sufficient energy to transfer $\Delta E = E_2 - E_1$ to the hydrogen electron in the $n = 1$ orbit (ground state), then the scattering will be elastic. If the incoming electron does have at least ΔE kinetic energy, then an inelastic collision can occur in which ΔE is transferred to the $n = 1$ electron, moving it to the $n = 2$ orbit. The excited electron will typically return to the ground state very quickly, emitting a photon of energy ΔE .

then the incoming electron can transfer 10.2 eV to the hydrogen electron in the *ground state* ($n = 1$ orbit), putting it into the $n = 2$ orbit (called the *first excited state*). The incoming electron's energy is thus reduced by 10.2 eV; it has been inelastically scattered. With insufficient energy to overcome the small retarding potential ΔV , the incoming electrons can no longer contribute to the plate current I , and I drops sharply.

The situation with Hg in the tube is more complicated since Hg has 80 electrons. Although Bohr's theory cannot predict their individual energies, we still expect the energy to be quantized with a ground state, first excited state, and so on, for the atom. Thus, the explanation of the observed 4.9-V critical potential for Hg is that the first excited state is about 4.9 eV above the lowest level (ground state). Electrons with energy less than this cannot lose energy to the Hg atoms, but electrons with energy greater than 4.9 eV can have inelastic collisions and lose 4.9 eV. If this happens near the grid, these electrons cannot gain enough energy to overcome the small back voltage ΔV and reach the plate; the current therefore decreases. If this explanation is correct, the Hg atoms that are excited to an energy level of 4.9 eV above the ground state should return to the ground state by emitting light of wavelength

$$\lambda = \frac{c}{f} = \frac{hc}{hf} = \frac{hc}{eV_0} = 253 \text{ nm}$$

There is indeed a line of this wavelength in the mercury spectrum. When the tube is viewed with a spectroscope, this line is seen when V_0 is greater than 4.9 eV, while no lines are seen when V_0 is less than this amount. For further increases in V_0 , additional sharp decreases in the current are observed, corresponding either to excitation of other levels in Hg (e.g., the second excited state of Hg is at 6.7 eV above the ground state) or to multiple excitations of the first excited state, i.e., due to an electron losing 4.9 eV more than once. In the usual setup, multiple excitations of the first level are observed and dips are seen every 4.9 V.²¹ The probability of observing such multiple first-level excitations, or excitations of other levels, depends on the detailed variation of the potential of the tube. For example, a second decrease in the current at $V_0 = 2 \times 4.9 = 9.8 \text{ V}$ results when electrons have inelastic collisions with Hg atoms about halfway between the cathode and grid (see Figure 4-22a). They are reaccelerated, reaching 4.9 eV again in the vicinity of the grid. A plot of the data of Franck and Hertz is shown in Figure 4-23.

The Franck-Hertz experiment was an important confirmation of the idea that discrete optical spectra were due to the existence in atoms of discrete energy levels that could be excited by nonoptical methods. It is particularly gratifying to be able to detect the existence of discrete energy levels directly by measurements using only voltmeters and ammeters.

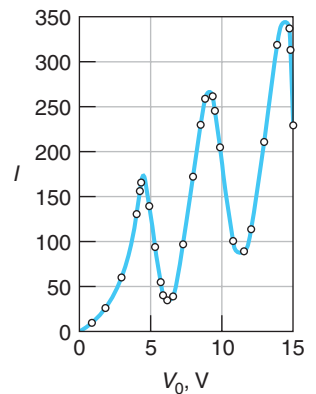
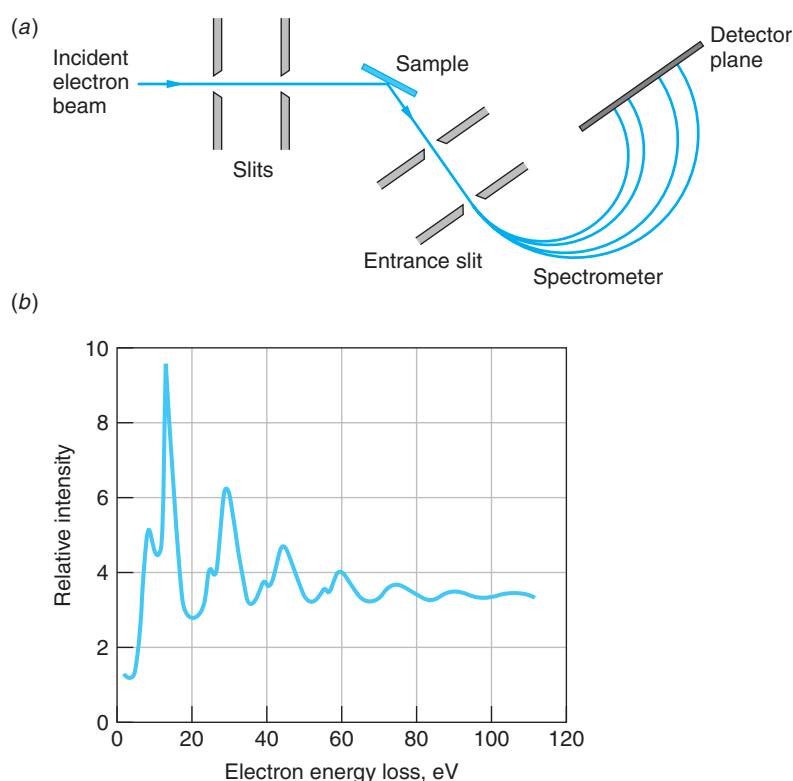


Figure 4-23 Current versus accelerating voltage in the Franck-Hertz experiment. The current decreases because many electrons lose energy due to inelastic collisions with mercury atoms in the tube and therefore cannot overcome the small back potential indicated in Figure 4-21a. The regular spacing of the peaks in this curve indicates that only a certain quantity of energy, 4.9 eV, can be lost to the mercury atoms. This interpretation is confirmed by the observation of radiation of photon energy 4.9 eV emitted by the mercury atoms, when V_0 is greater than this energy. [From J. Franck and G. Hertz, *Verband Deutscher Physikalischer Gesellschaften*, **16**, 457 (1914).]

Electron Energy Loss Spectroscopy

The Franck-Hertz experiment was the precursor of a highly sensitive technique for measuring the quantized energy states of atoms in both gases and solids. The technique, called *electron energy loss spectroscopy (EELS)*, is particularly useful in solids, where it makes possible measurement of the energy of certain types of lattice vibrations and other processes. It works like this. Suppose that the electrons in an incident beam all have energy E_{inc} . They collide with the atoms of a material, causing them to undergo some process (e.g., vibration, lattice rearrangement, electron excitation) that requires energy E_1 . Then, if a beam electron initiates a single such process, it will exit the material with energy $E_{\text{inc}} - E_1$ —i.e., it has been inelastically scattered.

Figure 4-24 Energy-loss spectrum measurement. (a) A well-defined electron beam impinges upon the sample. Electrons inelastically scattered at a convenient angle enter the slit of the magnetic spectrometer, whose B field is directed out of the paper, and turn through radii R determined by their energy $E_{\text{inc}} - E_1$ via Equation 3-2 written in the form $R = [2m(E_{\text{inc}} - E_1)]^{1/2}/eB$. (b) An energy-loss spectrum for a thin Al film. [From C. J. Powell and J. B. Swan, *Physical Review*, **115**, 869 (1954).]



The exit energy can be measured very accurately with, e.g., a magnetic spectrometer designed for electrons.²² Figure 4-24a illustrates a typical experimental arrangement for measuring an energy-loss spectrum.

As an example of its application, if an incident beam of electrons with $E_{\text{inc}} = 2 \text{ keV}$ is reflected from a thin Al film, the scattered electron energies measured in the magnetic spectrometer result in the energy-loss spectrum shown in Figure 4-24b, which directly represents the quantized energy levels of the target material. The loss peaks in this particular spectrum are due to the excitation of harmonic vibrations in the thin film sample, as well as some surface vibrations. The technique is also used to measure the vibrational energies of impurity atoms that may be absorbed on the surface and, with higher incident electron energies, to measure energy losses at the atomic inner levels, thus yielding information about bonding and other characteristics of absorbed atoms. Inelastic scattering techniques, including those using particles in addition to electrons, provide very powerful means for probing the energy structure of atomic, molecular, and nuclear systems. We will have occasion to refer to them many times throughout the rest of the book.



More

Here and in Chapter 3 we have discussed many phenomena that were “explained” by various ad hoc quantum assumptions. A *Critique of Bohr Theory and the “Old Quantum Mechanics”* contrasts some of its successes with some of its failures on the Web page: whfreeman.com/modphysics5e.

Summary

TOPIC	RELEVANT EQUATIONS AND REMARKS
1. Atomic spectra	$\frac{1}{\lambda_{mn}} = R \left(\frac{1}{m^2} - \frac{1}{n^2} \right) \quad n > m \quad 4-2$ <p>This empirical equation computes the correct wavelengths of observed spectral lines. The Rydberg constant R varies in a regular way from element to element.</p>
2. Rutherford scattering	<p>Impact parameter</p> $b = \frac{kq_{\alpha}Q}{m_{\alpha}v^2} \cot \frac{\theta}{2} \quad 4-3$ <p>Scattered fraction f</p> $f = \pi b^2 n t \quad 4-5$ <p>for a scattering foil with n nuclei/unit volume and thickness t</p> <p>Number of scattered alphas observed</p> $\Delta N = \left(\frac{I_0 A_{\text{sc}} n t}{r^2} \right) \left(\frac{kZe^2}{2E_k} \right)^2 \frac{1}{\sin^4 \frac{\theta}{2}} \quad 4-6$ <p>Size of nucleus</p> $r_d = \frac{kq_{\alpha}Q}{\frac{1}{2}m_{\alpha}v^2} \quad 4-11$
3. Bohr model	<p>Bohr's postulates</p> <ol style="list-style-type: none"> Electrons occupy only certain nonradiating, stable, circular orbits selected by quantization of the angular momentum L. $L = mvr = \frac{nh}{2\pi} = n\hbar \quad \text{for integer } n \quad 4-17$ <ol style="list-style-type: none"> Radiation of frequency f occurs when the electron jumps from an allowed orbit of energy E_i to one of lower energy E_f. f is given by the frequency condition $hf = E_i - E_f \quad 4-15$ <p>Correspondence principle</p> <p>In the region of very large quantum numbers classical and quantum calculations must yield the same results.</p> <p>Bohr radius</p> $a_0 = \frac{\hbar^2}{mke^2} = \frac{\hbar}{mc\alpha} = 0.0529 \text{ nm} \quad 4-19$ <p>Allowed energies</p> $E_n = -\frac{Z^2 E_0}{n^2} \quad \text{for } n = 1, 2, 3, \dots \quad 4-20$ <p>where $E_0 = mk^2 e^2 / 2\hbar^2 = 13.6 \text{ eV}$</p> <p>Reduced mass</p> $\mu = \frac{mM}{m + M} \quad 4-25$ <p>Fine-structure constant</p> $\alpha = \frac{ke^2}{\hbar c} \approx 1/137 \quad 4-30$

TOPIC	RELEVANT EQUATIONS AND REMARKS	
4. X-ray spectra Moseley equation	$f^{1/2} = A_n(Z - b)$	4-34
5. Franck-Hertz experiment	Supported Bohr's theory by verifying the quantization of atomic energies in absorption.	

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- Virtual Spectroscope, Physics Academic Software, North Carolina State University, Raleigh, 2003. Several sources can be viewed with a spectroscope to display the corresponding spectral lines.
- Visual Quantum Mechanics, Kansas State University, Manhattan, 1996. The atomic spectra component of this software provides an interactive construction of the energy levels for several elements, including hydrogen and helium.

Notes

1. Joseph von Fraunhofer (1787–1826), German physicist. Although he was not the first to see the dark lines in the solar spectrum that bear his name (Wollaston had seen seven, 12 years earlier), he systematically measured their wavelengths, named the prominent ones, and showed that they always occurred at the same wavelength even if the sunlight were reflected from the moon or a planet.

2. To date, more than 10,000 Fraunhofer lines have been found in the solar spectrum.

3. Although experimentalists preferred to express their measurements in terms of wavelengths, it had been shown that the many empirical formulas being constructed to explain the observed regularities in the line spectra could be expressed in simpler form if the reciprocal wavelength, called the *wave number* and equal to the number of waves per unit length, was used instead. Since $c = f\lambda$, this was equivalent to expressing the formulas in terms of the frequency.

4. Ernest Rutherford (1871–1937), English physicist, an exceptional experimentalist and a student of J. J. Thomson. He was an early researcher in the field of radioactivity and received the Nobel Prize in 1908 for his work in the transmutation of

elements. He bemoaned the fact that his prize was awarded in chemistry, not in physics, as work with the elements was considered chemistry in those days. He was Thomson's successor as director of the Cavendish Laboratory.

5. Alpha particles, like all charged particles, lose energy by exciting and ionizing the molecules of the materials through which they are moving. The energy lost per unit path length ($-dE/dx$) is a function of the ionization potential of the molecules, the atomic number of the atoms, and the energy of the α particles. It can be computed (with some effort) and is relatively simple to measure experimentally.

6. Notice that $2\pi \sin \theta d\theta = d\Omega$, the differential solid angle subtended at the scattering nucleus by the surface in Figure 4-11. Since the cross section $\sigma = \pi b^2$, then $d\sigma = 2\pi b db$ and Equation 4-9 can be rewritten as

$$\frac{d\sigma}{d\Omega} = \left(\frac{kZe^2}{m_a v^2} \right) \frac{1}{\sin^4(\theta/2)}$$

$d\sigma/d\Omega$ is called the *differential cross section*.

7. H. Geiger and E. Marsden, *Philosophical Magazine* (6), 25, 605 (1913).

8. The value of Z could not be measured directly in this experiment; however, relative values for different foil materials could be found and all materials heavier than aluminum had Z approximately equal to half the atomic weight.

9. This also introduces a deviation from the predicted ΔN associated with Rutherford's assumption that the nuclear mass was much larger than the α particle mass. For lighter-atomic-weight elements that assumption is not valid. Correction for the nuclear mass effect can be made, however, and the data in Figure 4-9b reflect the correction.

10. Niels H. D. Bohr (1885–1962), Danish physicist and first-rate soccer player. He went to the Cavendish Laboratory to work with J. J. Thomson after receiving his Ph.D.; however, Thomson is reported to have been impatient with Bohr's soft, accented English. Happily, the occasion of Thomson's annual birthday banquet brought Bohr in contact with Rutherford, whom he promptly followed to the latter's laboratory at Manchester, where he learned of the nuclear atom. A giant of twentieth-century physics, Bohr was awarded the Nobel Prize in 1922 for his explanation of the hydrogen spectrum. On a visit to the United States in 1939, he brought the news that the fission of uranium atoms had been observed. The story of his life makes absolutely fascinating reading.

11. N. Bohr, *Philosophical Magazine* (6), **26**, 1 (1913).

12. P. J. Mohr and B. N. Taylor, "The Fundamental Physical Constants," *Physics Today* (August 2004). Only 8 of the 14 current significant figures are given in Equation 4-27. The relative uncertainty in the value is about 1 part in 10^{12} !

13. Harold C. Urey (1893–1981), American chemist. His work opened the way for the use of isotopic tracers in biological systems. He was recognized with the Nobel Prize in 1934.

14. The basic reason that elliptical orbits solve this problem is that the frequency of the radiation emitted classically depends on the acceleration of the charge. The acceleration is constant for a circular orbit but varies for elliptical orbits, being dependent on the instantaneous distance from the focus. The energy of a particle in a circular orbit of radius r is the same as that of a particle in an elliptical orbit with a semimajor axis of r , so one would expect the only allowed elliptical orbits to be those whose semimajor axis was equal to an allowed Bohr circular orbit radius.

15. Viewed with spectrographs of high resolution, the spectral lines of hydrogen in Figure 4-2a—and, indeed, most spectral lines of all elements—are found to consist of very closely

spaced sets of lines, i.e., fine structure. We will discuss this topic in detail in Chapter 7.

16. Henry G.-J. Moseley (1887–1915), English physicist, considered by some the most brilliant of Rutherford's students. He would surely have been awarded the Nobel Prize had he not been killed in action in World War I. His father was a naturalist on the expedition of the HMS *Challenger*, the first vessel ever devoted to the exploration of the oceans.

17. The identifiers L and K were assigned by the English physicist C. G. Barkla, the discoverer of the characteristic x-ray lines, for which he received the Nobel Prize in 1917. He discovered two sets of x-ray lines for each of several elements, the *longer* wavelength of which he called the L series, the other the K series. The identifiers stuck and were subsequently used to label the atomic electron shells.

18. That the remaining K electron should result in $b = 1$, i.e., shielding of exactly $1e$, is perhaps a surprise. Actually it was a happy accident. It is the combined effect of the remaining K electron and the penetration of the electron waves of the outer L electrons that resulted in making $b = 1$, as we will see in Chapter 7.

19. Since in multielectron atoms the energies of the stationary states depend in part on the number of electrons in the atom (see Chapter 7), the energies E_n for a given atom change slightly when it is singly ionized, as in the production of characteristic x-ray lines, or doubly ionized, as in the Auger effect.

20. James Franck (1882–1964), German-American physicist; Gustav L. Hertz (1887–1975), German physicist. Franck won an Iron Cross as a soldier in World War I and later worked on the Manhattan Project. Hertz was a nephew of Heinrich Hertz, discoverer of the photoelectric effect. For their work on the inelastic scattering of electrons, Franck and Hertz shared the 1925 Nobel Prize in Physics.

21. We should note at this point that there is an energy state in the Hg atom at about 4.6 eV, slightly lower than the one found by Franck and Hertz. However, transitions from the ground state to the 4.6-eV level are not observed, and their absence is in accord with the prediction of more advanced quantum mechanics, as we will see in Chapter 7.

22. Since q/m for electrons is much larger than for ionized atoms, the radius for an electron magnetic spectrometer need not be as large as for a mass spectrometer, even for electron energies of several keV. (See Equation 3-2.)

Problems

Level I

Section 4-1 Atomic Spectra

4-1. Compute the wavelength and frequency of the series limit for the Lyman, Balmer, and Paschen spectral series of hydrogen.

4-2. The wavelength of a particular line in the Balmer series is measured to be 379.1 nm. What transition does it correspond to?

4-3. An astronomer finds a new absorption line with $\lambda = 164.1$ nm in the ultraviolet region of the Sun's continuous spectrum. He attributes the line to hydrogen's Lyman series. Is he right? Justify your answer.

4-4. The series of hydrogen spectral lines with $m = 4$ is called Brackett's series. Compute the wavelengths of the first four lines of Brackett's series.

4-5. In a sample that contains hydrogen, among other things, four spectral lines are found in the infrared with wavelengths 7460 nm, 4654 nm, 4103 nm, and 3741 nm. Which one does not belong to a hydrogen spectral series?

Section 4-2 Rutherford's Nuclear Model

4-6. A gold foil of thickness $2.0\ \mu\text{m}$ is used in a Rutherford experiment to scatter α particles with energy 7.0 MeV. (a) What fraction of the particles will be scattered at angles greater than 90° ? (b) What fraction will be scattered at angles between 45° and 75° ? (c) Use N_A , ρ , and M for gold to compute the approximate radius of a gold atom. (For gold, $\rho = 19.3\ \text{g/cm}^3$ and $M = 197\ \text{g/mol}$.)

4-7. (a) What is the ratio of the number of particles per unit area on the screen scattered at 10° to those at 1° ? (b) What is the ratio of those scattered at 30° to those at 1° ?

4-8. For α particles of 7.7 MeV (those used by Geiger and Marsden), what impact parameter will result in a deflection of 2° for a thin gold foil?

4-9. What will be the distance of closest approach r_d to a gold nucleus for an α particle of 5.0 MeV? 7.7 MeV? 12 MeV?

4-10. What energy α particle would be needed to just reach the surface of an Al nucleus if its radius is 4 fm?

4-11. If a particle is deflected by 0.01° in each collision, about how many collisions would be necessary to produce an rms deflection of 10° ? (Use the result from the one-dimensional random walk problem in statistics stating that the rms deflection equals the magnitude of the individual deflections times the square root of the number of deflections.) Compare this result with the number of atomic layers in a gold foil of thickness $10^{-6}\ \text{m}$, assuming that the thickness of each atom is $0.1\ \text{nm} = 10^{-10}\ \text{m}$.

4-12. Consider the foil and α particle energy in Problem 4-6. Suppose that 1000 of those particles suffer a deflection of more than 25° . (a) How many of these are deflected by more than 45° ? (b) How many are deflected between 25° and 45° ? (c) How many are deflected between 75° and 90° ?

Section 4-3 The Bohr Model of the Hydrogen Atom

4-13. The radius of the $n = 1$ orbit in the hydrogen atom is $a_0 = 0.053\ \text{nm}$. (a) Compute the radius of the $n = 6$ orbit. (b) Compute the radius of the $n = 6$ orbit in singly ionized helium (He^+), which is hydrogen-like, i.e., it has only a single electron outside the nucleus.

4-14. Show that Equation 4-19 for the radius of the first Bohr orbit and Equation 4-20 for the magnitude of the lowest energy for the hydrogen atom can be written as

$$a_0 = \frac{hc}{\alpha mc^2} = \frac{\lambda_c}{2\pi\alpha} \quad E_1 = \frac{1}{2}\alpha^2 mc^2$$

where $\lambda_c = h/mc$ is the Compton wavelength of the electron and $\alpha = ke^2/hc$ is the fine-structure constant. Use these expressions to check the numerical values of the constants a_0 and E_1 .

4-15. Calculate the three longest wavelengths in the Lyman series ($n_f = 1$) in nm and indicate their position on a horizontal linear scale. Indicate the series limit (shortest wavelength) on this scale. Are any of these lines in the visible spectrum?

4-16. If the angular momentum of Earth in its motion around the Sun were quantized like a hydrogen electron according to Equation 4-17, what would Earth's quantum number be? How much energy would be released in a transition to the next lowest level? Would that energy release (presumably as a gravity wave) be detectable? What would be the radius of that orbit? (The radius of Earth's orbit is $1.50 \times 10^{11}\ \text{m}$.)

4-17. On average, a hydrogen atom will exist in an excited state for about $10^{-8}\ \text{sec}$ before making a transition to a lower energy state. About how many revolutions does an electron in the $n = 2$ state make in $10^{-8}\ \text{sec}$?

4-18. An atom in an excited state will on average undergo a transition to a state of lower energy in about 10^{-8} seconds. If the electron in a doubly ionized lithium atom (Li^{+2} , which is hydrogenlike) is placed in the $n = 4$ state, about how many revolutions around the nucleus does it make before undergoing a transition to a lower energy state?

4-19. It is possible for a muon to be captured by a proton to form a muonic atom. A muon is identical to an electron except for its mass, which is $105.7 \text{ MeV}/c^2$. (a) Calculate the radius of the first Bohr orbit of a muonic atom. (b) Calculate the magnitude of the lowest energy state. (c) What is the shortest wavelength in the Lyman series for this atom?

4-20. In the lithium atom ($Z = 3$) two electrons are in the $n = 1$ orbit and the third is in the $n = 2$ orbit. (Only two are allowed in the $n = 1$ orbit because of the exclusion principle, which will be discussed in Chapter 7.) The interaction of the inner electrons with the outer one can be approximated by writing the energy of the outer electron as

$$E = -Z'^2(E_1/n^2)$$

where $E_1 = 13.6 \text{ eV}$, $n = 2$, and Z' is the effective nuclear charge, which is less than 3 because of the screening effect of the two inner electrons. Using the measured ionization energy of 5.39 eV , calculate Z' .

4-21. Draw to careful scale an energy-level diagram for hydrogen for levels with $n = 1, 2, 3, 4, \infty$. Show the following on the diagram: (a) the limit of the Lyman series, (b) the H_β line, (c) the transition between the state whose binding energy (= energy needed to remove the electron from the atom) is 1.51 eV and the state whose excitation energy is 10.2 eV , and (d) the longest wavelength line of the Paschen series.

4-22. A hydrogen atom at rest in the laboratory emits the Lyman α radiation. (a) Compute the recoil kinetic energy of the atom. (b) What fraction of the excitation energy of the $n = 2$ state is carried by the recoiling atom? (*Hint:* Use conservation of momentum.)

4-23. (a) Draw accurately to scale and label completely a partial energy-level diagram for C^{5+} , including at minimum the energy levels for $n = 1, 2, 3, 4, 5$, and ∞ . (b) Compute the wavelength of the spectral line resulting from the $n = 3$ to the $n = 2$ transition, the $\text{C}^{5+}H_\alpha$ line. (c) In what part of the EM spectrum does this line lie?

4-24. The electron-positron pair that was discussed in Chapter 2 can form a hydrogenlike system called *positronium*. Calculate (a) the energies of the three lowest states and (b) the wavelength of the Lyman α and β lines. (Detection of those lines is a “signature” of positronium formation.)

4-25. With the aid of tunable lasers, Rydberg atoms of sodium have been produced with $n \approx 100$. The resulting atomic diameter would correspond in hydrogen to $n \approx 600$. (a) What would be the diameter of a hydrogen atom whose electron is in the $n \approx 600$ orbit? (b) What would be the speed of the electron in that orbit? (c) How does the result in (b) compare with the speed in the $n \approx 1$ orbit?

Section 4-4 X-Ray Spectra

4-26. (a) Calculate the next two longest wavelengths in the K series (after the K_α line) of molybdenum. (b) What is the wavelength of the shortest wavelength in this series?

4-27. The wavelength of the K_α x-ray line for an element is measured to be 0.0794 nm . What is the element?

4-28. Moseley pointed out that elements with atomic numbers 43, 61, and 75 should exist and (at that time) had not been found. (a) Using Figure 4-19, what frequencies would Moseley’s graphical data have predicted for the K_α x ray for each of these elements? (b) Compute the wavelengths for these lines predicted by Equation 4-37.

4-29. What is the approximate radius of the $n = 1$ orbit of gold ($Z = 79$)? Compare this with the radius of the gold nucleus, about 7.1 fm .

4-30. An electron in the K shell of Fe is ejected by a high-energy electron in the target of an x-ray tube. The resulting hole in the $n = 1$ shell could be filled by an electron from the $n = 2$ shell, the L shell; however, instead of emitting the characteristic Fe K_α x ray, the atom ejects an Auger electron from the $n = 2$ shell. Using Bohr theory, compute the energy of the Auger electron.

4-31. In a particular x-ray tube, an electron approaches the target moving at $2.25 \times 10^8 \text{ m/s}$. It slows down on being deflected by a nucleus of the target, emitting a photon of energy 32.5 keV . Ignoring the nuclear recoil, but not relativity, compute the final speed of the electron.

4-32. (a) Compute the energy of an electron in the $n = 1$ (K shell) of tungsten, using $Z - 1$ for the effective nuclear charge. (b) The experimental result for this energy is 69.5 keV . Assume that the effective nuclear charge is $Z - \sigma$, where σ is called the screening constant, and calculate σ from the experimental result.

4-33. Construct a Moseley plot similar to Figure 4-19 for the K_β x rays of the elements listed below (the x-ray energies are given in keV):

Al 1.56	Ar 3.19	Sc 4.46	Fe 7.06
Ge 10.98	Kr 14.10	Zr 17.66	Ba 36.35

Determine the slope of your plot, and compare it with the K_β line in Figure 4-19.

Section 4-5 The Franck-Hertz Experiment

4-34. Suppose that, in a Franck-Hertz experiment, electrons of energy up to 13.0 eV can be produced in the tube. If the tube contained atomic hydrogen, (a) what is the shortest-wavelength spectral line that could be emitted from the tube? (b) List all of the hydrogen lines that can be emitted by this tube.

4-35. Using the data in Figure 4-24b and a good ruler, draw a carefully scaled energy-level diagram covering the range from 0 eV to 60 eV for the vibrational states of this solid. What approximate energy is typical of the transitions between adjacent levels corresponding to the larger of each pair of peaks?

4-36. The transition from the first excited state to the ground state in potassium results in the emission of a photon with $\lambda = 770$ nm. If potassium vapor is used in a Franck-Hertz experiment, at what voltage would you expect to see the first decrease in current?

4-37. If we could somehow fill a Franck-Hertz tube with positronium, what cathode-grid voltage would be needed to reach the second current decrease in the positronium equivalent of Figure 4-23? (See Problem 4-24.)

4-38. Electrons in the Franck-Hertz tube can also have elastic collisions with the Hg atoms. If such a collision is a head-on, what fraction of its initial kinetic energy will an electron lose, assuming the Hg atom to be at rest? If the collision is not head-on, will the fractional loss be greater or less than this?

Level II

4-39. Derive Equation 4-8 along the lines indicated in the paragraph that immediately precedes it.

4-40. Geiger and Marsden used α particles with 7.7-MeV kinetic energy and found that when they were scattered from thin gold foil, the number observed to be scattered at all angles agreed with Rutherford's formula. Use this fact to compute an upper limit on the radius of the gold nucleus.

4-41. (a) The current i due to a charge q moving in a circle with frequency f_{rev} is qf_{rev} . Find the current due to the electron in the first Bohr orbit. (b) The magnetic moment of a current loop is iA , where A is the area of the loop. Find the magnetic moment of the electron in the first Bohr orbit in units $\text{A}\cdot\text{m}^2$. This magnetic moment is called a *Bohr magneton*.

4-42. Use a spreadsheet to calculate the wavelengths (in nm) of the first five spectral lines of the Lyman, Balmer, Paschen, and Brackett series of hydrogen. Show the positions of these lines on a linear scale and indicate which ones lie in the visible.

4-43. Show that a small change in the reduced mass of the electron produces a small change in a spectral line given by $\Delta\lambda/\lambda = \Delta\mu/\mu$. Use this to calculate the difference $\Delta\lambda$ in the Balmer red line $\lambda = 656.3$ nm between hydrogen and deuterium, which has a nucleus with twice the mass of hydrogen.

4-44. Consider the Franck-Hertz experiment with Hg vapor in the tube and the voltage between the cathode and the grid equal to 4.0 V, i.e., not enough for the electrons to excite the Hg atom's first excited state. Therefore, the electron-Hg atom collisions are elastic. (a) If the kinetic energy of the electrons is E_k , show that the maximum kinetic energy that a recoiling Hg atom can have is approximately $4mE_k/M$, where M is the Hg atom mass. (b) What is the approximate maximum kinetic energy that can be lost by an electron with $E_k = 2.5$ eV?

4-45. The Li^{2+} ion is essentially identical to the H atom in Bohr's theory, aside from the effect of the different nuclear charges and masses. (a) What transitions in Li^{2+} will yield emission lines whose wavelengths are very nearly equal to the first two lines of the Lyman series in hydrogen?

(b) Calculate the difference between the wavelength of the Lyman α line of hydrogen and the emission line from Li^{2+} that has very nearly the same wavelength.

4-46. In an α scattering experiment, the area of the α particle detector is 0.50 cm^2 . The detector is located 10 cm from a $1.0\text{-}\mu\text{m}$ -thick silver foil. The incident beam carries a current of 1.0 nA, and the energy of each α particle is 6.0 MeV. How many α particles will be counted per second by the detector at (a) $\theta = 60^\circ$? (b) $\theta = 120^\circ$?

4-47. The K_α , L_α , and M_α x rays are emitted in the $n = 2 \rightarrow n = 1$, $n = 3 \rightarrow n = 2$, and $n = 4 \rightarrow n = 3$ transitions, respectively. For calcium ($Z = 20$) the energies of these transitions are 3.69 keV, 0.341 keV, and 0.024 keV, respectively. Suppose that energetic photons impinging on a calcium surface cause ejection of an electron from the K shell of the surface atoms. Compute the energies of the Auger electrons that may be emitted from the L , M , and N shells ($n = 2, 3$, and 4) of the sample atoms, in addition to the characteristic x rays.

4-48. Figure 3-15b shows the K_α and K_β characteristic x rays emitted by a molybdenum (Mo) target in an x-ray tube whose accelerating potential is 35 kV. The wavelengths are $K_\alpha = 0.071 \text{ nm}$ and $K_\beta = 0.063 \text{ nm}$. (a) Compute the corresponding energies of these photons. (b) Suppose we wish to prepare a beam consisting primarily of K_α x rays by passing the molybdenum x rays through a material that absorbs K_β x rays more strongly than K_α x rays by photoelectric effect on K -shell electrons of the material. Which of the materials listed in the accompanying table with their K -shell binding energies would you choose? Explain your answer.

Element	Zr	Nb	Mo	Tc	Ru
Z	40	41	42	43	44
E_K (keV)	18.00	18.99	20.00	21.04	22.12

Level III

4-49. A small shot of negligible radius hits a stationary smooth, hard sphere of radius R , making an angle β with the normal to the sphere, as shown in Figure 4-25. It is reflected at an equal angle to the normal. The scattering angle is $\theta = 180^\circ - 2\beta$, as shown. (a) Show by the geometry of the figure that the impact parameter b is related to θ by $b = R \cos \frac{1}{2}\theta$. (b) If the incoming intensity of the shot is I_0 particles/s \cdot area, how many are scattered through angles greater than θ ? (c) Show that the cross section for scattering through angles greater than 0° is πR^2 . (d) Discuss the implication of the fact that the Rutherford cross section for scattering through angles greater than 0° is infinite.

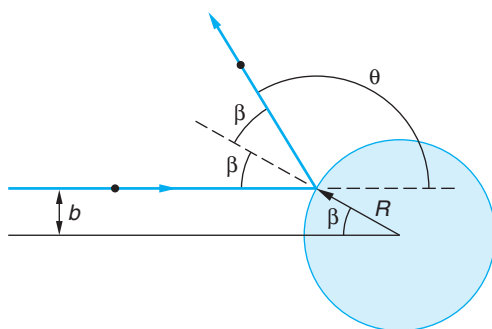


Figure 4-25 Small particle scattered by a hard sphere of radius R .

4-50. Singly ionized helium He^+ is hydrogenlike. (a) Construct a carefully scaled energy-level diagram for He^+ similar to that in Figure 4-16, showing the levels for $n = 1, 2, 3, 4, 5$, and ∞ . (b) What is the ionization energy of He^+ ? (c) Compute the difference in wavelength between each of the first two lines of the Lyman series of hydrogen and the first two lines of the He^+ Balmer series. Be sure to include the reduced mass correction for both atoms. (d) Show that for every spectral line of hydrogen, He^+ has a spectral line of very nearly the same wavelength. (Mass of $\text{He}^+ = 6.65 \times 10^{-27} \text{ kg}$.)

4-51. Listed in the table are the L_{α} x-ray wavelengths for several elements. Construct a Moseley plot from these data. Compare the slope with the appropriate one in Figure 4-19. Determine and interpret the intercept on your graph, using a suitably modified version of Equation 4-35.

Element	P	Ca	Co	Kr	Mo	I
Z	15	20	27	36	42	53
Wavelength (nm)	10.41	4.05	1.79	0.73	0.51	0.33

4-52. In this problem you are to obtain the Bohr results for the energy levels in hydrogen without using the quantization condition of Equation 4-17. In order to relate Equation 4-14 to the Balmer-Ritz formula, assume that the radii of allowed orbits are given by $r_n = n^2 r_0$, where n is an integer and r_0 is a constant to be determined. (a) Show that the frequency of radiation for a transition to $n_f = n - 1$ is given by $f \approx kZe^2/hr_0 n^3$ for large n . (b) Show that the frequency of revolution is given by

$$f_{\text{rev}}^2 = \frac{kZe^2}{4\pi^2 m r_0^3 n^6}$$

(c) Use the correspondence principle to determine r_0 and compare with Equation 4-19.

4-53. Calculate the energies and speeds of electrons in circular Bohr orbits in a hydrogenlike atom using the relativistic expressions for kinetic energy and momentum.

4-54. (a) Write a computer program for your personal computer or programmable calculator that will provide you with the spectral series of H-like atoms. Inputs to be included are n_i , n_f , Z , and the nuclear mass M . Outputs are to be the wavelengths and frequencies of the first six lines and the series limit for the specified n_f , Z , and M . Include the reduced mass correction. (b) Use the program to compute the wavelengths and frequencies of the Balmer series. (c) Pick an $n_f > 100$, name the series the [your name] series, and use your program to compute the wavelengths and frequencies of the first three lines and the limit.

4-55. Figure 4-26 shows an energy loss spectrum for He measured in an apparatus such as that shown in Figure 4-24a. Use the spectrum to construct and draw carefully to scale an energy-level diagram for He.

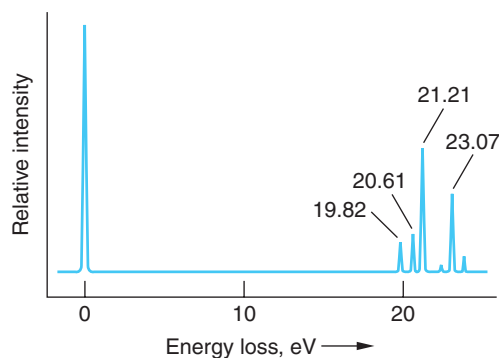
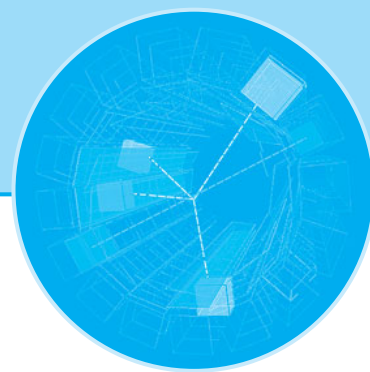


Figure 4-26 Energy-loss spectrum of helium. Incident electron energy was 34 eV. The elastically scattered electrons cause the peak at 0 eV.

4-56. If electric charge did not exist and electrons were bound to protons by the gravitational force to form hydrogen, derive the corresponding expressions for a_0 and E_n and compute the energy and frequency of the H_{α} line and the limit of the Balmer series. Compare these with the corresponding quantities for “real” hydrogen.

4-57. A sample of hydrogen atoms are all in the $n = 5$ state. If all the atoms return to the ground state, how many different photon energies will be emitted, assuming all possible transitions occur? If there are 500 atoms in the sample and assuming that from any state all possible downward transitions are equally probable, what is the total number of photons that will be emitted when all of the atoms have returned to the ground state?

4-58. Consider muonic atoms (see Problem 4-19). (a) Draw a correctly scaled and labeled partial energy level diagram including levels with $n = 1, 2, 3, 4, 5$, and ∞ for muonic hydrogen. (b) Compute the radius of the $n = 1$ muon orbit in muonic H, He^{1+} , Al^{12+} , and Au^{78+} . (c) Compare the results in (b) with the radii of these nuclei. (d) Compute the wavelength of the photon emitted in the $n = 2$ to $n = 1$ transition for each of these muonic atoms.



The Wavelike Properties of Particles

In 1924, a French graduate student, Louis de Broglie,¹ proposed in his doctoral dissertation that the dual—i.e., wave-particle—behavior that was by then known to exist for radiation was also a characteristic of matter, in particular, electrons. This suggestion was highly speculative, since there was yet no experimental evidence whatsoever for any wave aspects of electrons or any other particles. What had led him to this seemingly strange idea? It was a “bolt out of the blue,” like Einstein’s “happy thought” that led to the principle of equivalence (see Chapter 2). De Broglie described it with these words:

After the end of World War I, I gave a great deal of thought to the theory of quanta and to the wave-particle dualism. . . . It was then that I had a sudden inspiration. Einstein’s wave-particle dualism was an absolutely general phenomenon extending to all physical nature.²

Since the visible universe consists entirely of matter and radiation, de Broglie’s hypothesis is a fundamental statement about the grand symmetry of nature. (There is currently strong observational evidence that ordinary matter makes up only about 4 percent of the visible universe. About 22 percent is some unknown form of invisible “dark matter,” and approximately 74 percent consists of some sort of equally mysterious “dark energy.” See Chapter 13.)

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5-1 The de Broglie Hypothesis

De Broglie stated his proposal mathematically with the following equations for the frequency and wavelength of the electron waves, which are referred to as the *de Broglie relations*:

$$f = \frac{E}{h} \quad 5-1$$

$$\lambda = \frac{h}{p} \quad 5-2$$

where E is the total energy, p is the momentum, and λ is called the *de Broglie wavelength* of the particle. For photons, these same equations result directly from

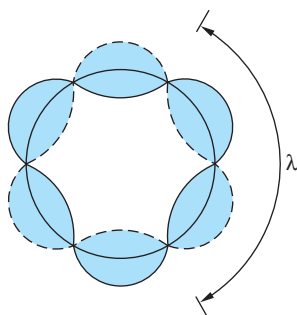


Figure 5-1 Standing waves around the circumference of a circle. In this case the circle is 3λ in circumference. If the vibrator were, for example, a steel ring that had been suitably tapped with a hammer, the shape of the ring would oscillate between the extreme positions represented by the solid and broken lines.

Einstein's quantization of radiation $E = hf$ and Equation 2-31 for a particle of zero rest energy $E = pc$ as follows:

$$E = pc = hf = \frac{hc}{\lambda}$$

By a more indirect approach using relativistic mechanics, de Broglie was able to demonstrate that Equations 5-1 and 5-2 also apply to particles with mass. He then pointed out that these equations lead to a physical interpretation of Bohr's quantization of the angular momentum of the electron in hydrogenlike atoms, namely, that the quantization is equivalent to a standing-wave condition. (See Figure 5-1.) We have

$$mvr = n\hbar = \frac{nh}{2\pi} \quad \text{for } n = \text{integer}$$

$$2\pi r = \frac{nh}{mv} = \frac{nh}{p} = n\lambda = \text{circumference of orbit} \quad \mathbf{5-3}$$

The idea of explaining discrete energy states in matter by standing waves thus seemed quite promising.

De Broglie's ideas were expanded and developed into a complete theory by Erwin Schrödinger late in 1925. In 1927, C. J. Davisson and L. H. Germer verified the de Broglie hypothesis directly by observing interference patterns, a characteristic of waves, with electron beams. We will discuss both Schrödinger's theory and the Davisson-Germer experiment in later sections, but first we have to ask ourselves why wavelike behavior of matter had not been observed before de Broglie's work. We can understand why if we first recall that the wave properties of light were not noticed either until apertures or slits with dimensions of the order of the wavelength of light could be obtained. This is because the wave nature of light is not evident in experiments where the primary dimensions of the apparatus are large compared with the wavelength of the light used. For example, if A represents the diameter of a lens or the width of a slit, then diffraction effects³ (a manifestation of wave properties) are limited to angles θ around the forward direction ($\theta = 0^\circ$) where $\sin \theta = \lambda/A$. In geometric (ray) optics $\lambda/A \rightarrow 0$, so $\theta \approx \sin \theta \rightarrow 0$, too. However, if a characteristic



Louis V. de Broglie, who first suggested that electrons might have wave properties.
[Courtesy of Culver Pictures.]

dimension of the apparatus becomes of the order of (or smaller than) λ , the wavelength of light passing through the system, then $\lambda/A \rightarrow 1$. In that event $\theta \approx \lambda/A$ is readily observable, and the wavelike properties of light become apparent. Because Planck's constant is so small, the wavelength given by Equation 5-2 is extremely small for any macroscopic object. This point is among those illustrated in the following section.

5-2 Measurements of Particle Wavelengths

Although we now have diffraction systems of nuclear dimensions, the smallest-scale systems to which de Broglie's contemporaries had access were the spacings between the planes of atoms in crystalline solids, about 0.1 nm. This means that even for an extremely small macroscopic particle, such as a grain of dust ($m \approx 0.1$ mg) moving through air with the average kinetic energy of the atmospheric gas molecules, the smallest diffraction systems available would have resulted in diffraction angles τ only of the order of 10^{-10} radian, far below the limit of experimental detectability. The small magnitude of Planck's constant ensures that λ will be smaller than any readily accessible aperture, placing diffraction beyond the limits of experimental observation. For objects whose momenta are larger than that of the dust particle, the possibility of observing *particle*, or *matter waves*, is even less, as the following example illustrates.

EXAMPLE 5-1 De Broglie Wavelength of a Ping-Pong Ball What is the de Broglie wavelength of a Ping-Pong ball of mass 2.0 g after it is slammed across the table with speed 5 m/s?

SOLUTION

$$\begin{aligned}\lambda &= \frac{h}{mv} = \frac{6.63 \times 10^{-34} \text{ J} \cdot \text{s}}{(2.0 \times 10^{-3} \text{ kg})(5 \text{ m/s})} \\ &= 6.6 \times 10^{-32} \text{ m} = 6.6 \times 10^{-23} \text{ nm}\end{aligned}$$

This is 17 orders of magnitude smaller than typical nuclear dimensions, far below the dimensions of any possible aperture.

The case is different for low-energy electrons, as de Broglie himself realized. At his *soutenance de thèse* (defense of the thesis), de Broglie was asked by Perrin⁴ how his hypothesis could be verified, to which he replied that perhaps passing particles, such as electrons, through very small slits would reveal the waves. Consider an electron that has been accelerated through V_0 volts. Its kinetic energy (nonrelativistic) is then

$$E = \frac{p^2}{2m} = eV_0$$

Solving for p and substituting into Equation 5-2,

$$\lambda = \frac{h}{p} = \frac{hc}{pc} = \frac{hc}{(2mc^2 eV_0)^{1/2}}$$

Using $hc = 1.24 \times 10^3 \text{ eV} \cdot \text{nm}$ and $mc^2 = 0.511 \times 10^6 \text{ eV}$, we obtain

$$\lambda = \frac{1.226}{V_0^{1/2}} \text{ nm} \quad \text{for} \quad eV_0 \sim mc^2 \quad \mathbf{5-4}$$

The following example computes an electron de Broglie wavelength, giving a measure of just how small the slit must be.

EXAMPLE 5-2 De Broglie Wavelength of a Slow Electron Compute the de Broglie wavelength of an electron whose kinetic energy is 10 eV.

SOLUTION

1. The de Broglie wavelength is given by Equation 5-2:

$$\lambda = \frac{h}{p}$$

2. *Method 1:* Since a 10-eV electron is nonrelativistic, we can use the classical relation connecting the momentum and the kinetic energy:

$$E_k = \frac{p^2}{2m}$$

or

$$\begin{aligned} p &= \sqrt{2mE_k} \\ &= \sqrt{(2)(9.11 \times 10^{-31} \text{ kg})(10 \text{ eV})(1.60 \times 10^{-19} \text{ J/eV})} \\ &= 1.71 \times 10^{-24} \text{ kg} \cdot \text{m/s} \end{aligned}$$

3. Substituting this result into Equation 5-2:

$$\begin{aligned} \lambda &= \frac{6.63 \times 10^{-34} \text{ J} \cdot \text{s}}{1.71 \times 10^{-24} \text{ kg} \cdot \text{m/s}} \\ &= 3.88 \times 10^{-10} \text{ m} = 0.39 \text{ nm} \end{aligned}$$

4. *Method 2:* The electron's wavelength can also be computed from Equation 5-4 with $V_0 = 10 \text{ V}$:

$$\begin{aligned} \lambda &= \frac{1.226}{V^{1/2}} = \frac{1.226}{\sqrt{10}} \\ &= 0.39 \text{ nm} \end{aligned}$$

Remarks: Though this wavelength is small, it is just the order of magnitude of the size of an atom and of the spacing of atoms in a crystal.

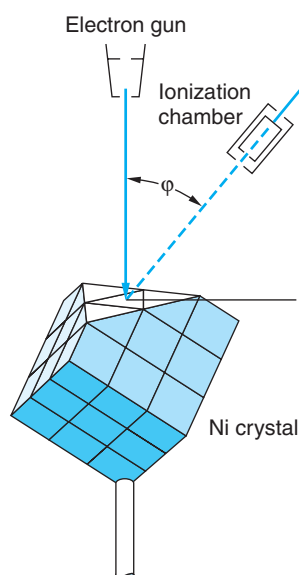


Figure 5-2 The Davisson-Germer experiment. Low-energy electrons scattered at angle ϕ from a nickel crystal are detected in an ionization chamber. The kinetic energy of the electrons could be varied by changing the accelerating voltage on the electron gun.

The Davisson-Germer Experiment

In a brief note in the August 14, 1925, issue of the journal *Naturwissenschaften*, Walter Elsasser, at the time a student of J. Franck's (of the Franck-Hertz experiment), proposed that the wave effects of low-velocity electrons might be detected by scattering them from single crystals. The first such measurements of the wavelengths of electrons were made in 1927 by C. J. Davisson⁵ and L. H. Germer, who were studying electron reflection from a nickel target at Bell Telephone Laboratories, unaware of either Elsasser's suggestion or de Broglie's work. After heating their target to remove an oxide coating that had accumulated during an accidental break in their vacuum system, they found that the scattered electron intensity as a function of the scattering angle showed maxima and minima. The surface atoms of their nickel target had, in the process of cooling, formed relatively large single crystals, and they were observing electron diffraction. Recognizing the importance of their accidental discovery, they then prepared a target consisting of a single crystal of nickel and extensively investigated the scattering of electrons from it. Figure 5-2 illustrates their experimental arrangement. Their data for 54-eV electrons, shown in Figure 5-3, indicate a strong maximum of scattering at $\phi = 50^\circ$. Consider the scattering from a set of Bragg

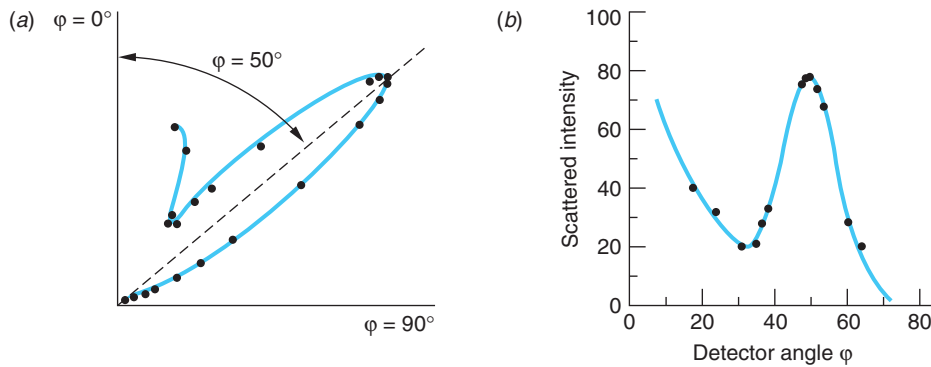


Figure 5-3 Scattered intensity vs. detector angle for 54-eV electrons. (a) Polar plot of the data. The intensity at each angle is indicated by the distance of the point from the origin. Scattering angle ϕ is plotted clockwise starting at the vertical axes. (b) The same data plotted on a Cartesian graph. The intensity scales are arbitrary but the same on both graphs. In each plot there is maximum intensity at $\phi = 50^\circ$, as predicted for Bragg scattering of waves having wavelength $\lambda = h/p$. [From *Nobel Prize Lectures: Physics* (Amsterdam and New York: Elsevier, © Nobel Foundation, 1964).]

planes, as shown in Figure 5-4. The Bragg condition for constructive interference is $n\lambda = 2d \sin \theta = 2d \cos \alpha$. The spacing of the Bragg planes d is related to the spacing of the atoms D by $d = D \sin \alpha$; thus

$$n\lambda = 2D \sin \alpha \cos \alpha = D \sin 2\alpha$$

or

$$n\lambda = D \sin \phi \quad 5-5$$

where $\phi = 2\alpha$ is the scattering angle.

The spacing D for Ni is known from x-ray diffraction to be 0.215 nm. The wavelength calculated from Equation 5-5 for the peak observed at $\phi = 50^\circ$ by Davisson and Germer is, for $n = 1$,

$$\lambda = 0.215 \sin 50^\circ = 0.165 \text{ nm}$$

The value calculated from the de Broglie relation for 54-eV electrons is

$$\lambda = \frac{1.226}{(54)^{1/2}} = 0.167 \text{ nm}$$

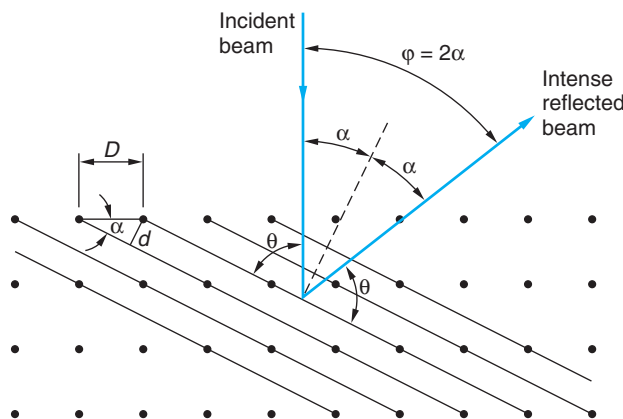


Figure 5-4 Scattering of electrons by a crystal. Electron waves are strongly scattered if the Bragg condition $n\lambda = 2d \sin \theta$ is met. This is equivalent to the condition $n\lambda = D \sin \phi$.

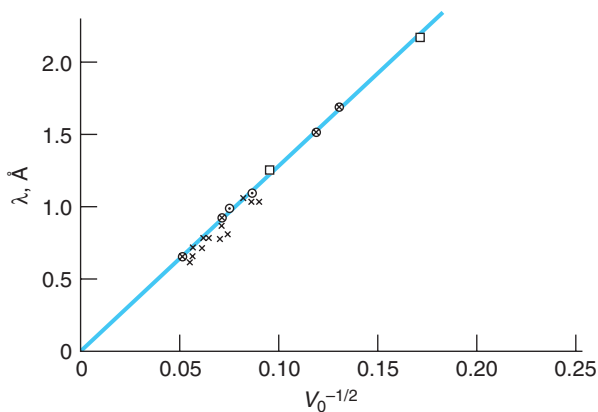


Figure 5-5 Test of the de Broglie formula $\lambda = h/p$. The wavelength is computed from a plot of the diffraction data plotted against $V_0^{-1/2}$, where V_0 is the accelerating voltage. The straight line is $1.226V_0^{-1/2}$ nm as predicted from $\lambda = h(2mE)^{-1/2}$. These are the data referred to in the quotation from Davisson's Nobel lecture. (× From observations with diffraction apparatus; ⊗ same, particularly reliable; □ same, grazing beams. ⊙ From observations with reflection apparatus.) [From *Nobel Prize Lectures: Physics* (Amsterdam and New York: Elsevier, © Nobel Foundation, 1964).]

The agreement with the experimental observation is excellent! With this spectacular result Davisson and Germer then conducted a systematic study to test the de Broglie relation using electrons up to about 400 eV and various experimental arrangements. Figure 5-5 shows a plot of measured wavelengths versus $V_0^{-1/2}$. The wavelengths measured by diffraction are slightly lower than the theoretical predictions because the refraction of the electron waves at the crystal surface has been neglected. We have seen from the photoelectric effect that it takes work of the order of several eV to remove an electron from a metal. Electrons entering a metal thus gain kinetic energy; therefore, their de Broglie wavelength is slightly less inside the crystal.⁶

A subtle point must be made here. Notice that the wavelength in Equation 5-5 depends only on D , the interatomic spacing of the crystal, whereas our derivation of that equation included the interplane spacing as well. The fact that the structure of the crystal really is essential shows up when the energy is varied, as was done in collecting the data for Figure 5-5. Equation 5-5 suggests that a change in λ , resulting from a change in the energy, would mean only that the diffraction maximum would occur at some other value of φ such that the equation remains satisfied. However, as can be seen from examination of Figure 5-4, the value of φ is determined by α , the angle

of the planes determined by the crystal structure. Thus, if there are no crystal planes making an angle $\alpha = \varphi/2$ with the surface, then setting the detector at $\varphi = \sin^{-1}(\lambda/D)$ will not result in constructive interference and strong reflection for that value of λ , even though Equation 5-5 is satisfied. This is neatly illustrated by Figure 5-6, which shows a series of polar graphs (like Figure 5-3a) for electrons of energies from 36 eV through 68 eV. The building to a strong reflection at $\varphi = 50^\circ$ is evident for $V_0 = 54$ V, as we have already seen. But Equation 5-5 by itself would also lead us to expect, for example, a strong reflection at $\varphi = 64^\circ$ when $V_0 = 40$ V, which obviously does not occur.

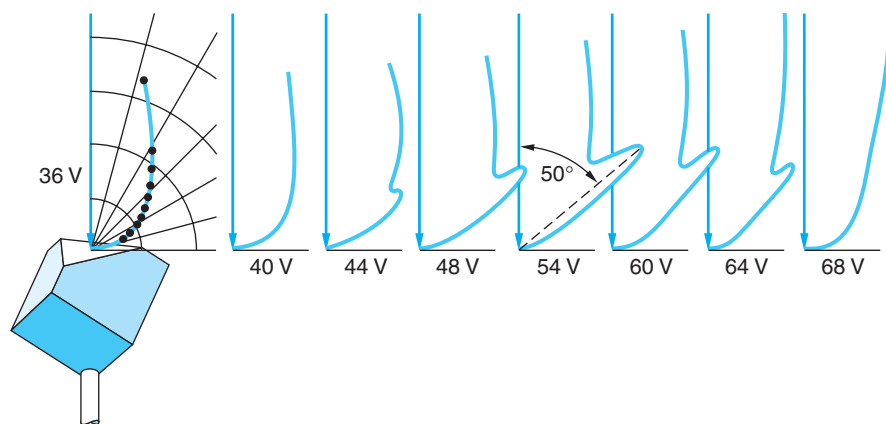


Figure 5-6 A series of polar graphs of Davisson and Germer's data at electron accelerating potentials from 36 V to 68 V. Note the development of the peak at $\varphi = 50^\circ$ to a maximum when $V_0 = 54$ V.



Clinton J. Davisson (left) and Lester H. Germer at Bell Laboratories, where electron diffraction was first observed. [Bell Telephone Laboratories, Inc.]

In order to show the dependence of the diffraction on the inner atomic layers, Davisson and Germer kept the detector angle φ fixed and varied the accelerating voltage rather than search for the correct angle for a given λ . Writing Equation 5-5 as

$$\lambda = \frac{D \sin \varphi}{n} = \frac{D \sin (2\alpha)}{n} \quad 5-6$$

and noting that $\lambda \propto V_0^{-1/2}$, a graph of intensity versus $V_0^{1/2} (\propto 1/\lambda)$ for a given angle φ should yield (1) a series of equally spaced peaks corresponding to successive values of the integer n , if $\alpha = \varphi/2$ is an existing angle for atomic planes, or (2) no diffraction peaks if $\varphi/2$ is not such an angle. Their measurements verified the dependence upon the interplane spacing, the agreement with the prediction being about ± 1 percent. Figure 5-7 illustrates the results for $\varphi = 50^\circ$. Thus, Davisson and Germer showed conclusively that particles with mass moving at speeds $v \ll c$ do indeed have wavelike properties, as de Broglie had proposed.

The diffraction pattern formed by high-energy electron waves scattered from nuclei provides a means by which nuclear radii and the internal distribution of the nuclear charge (the protons) are measured. See Chapter 11.

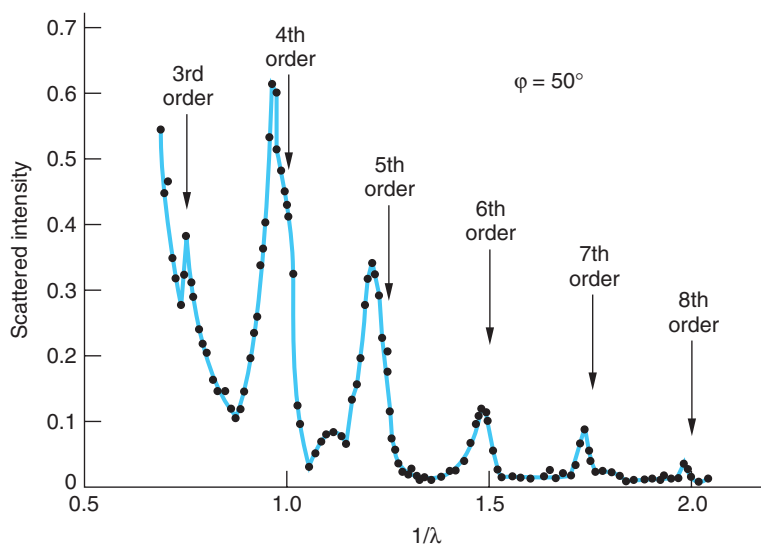


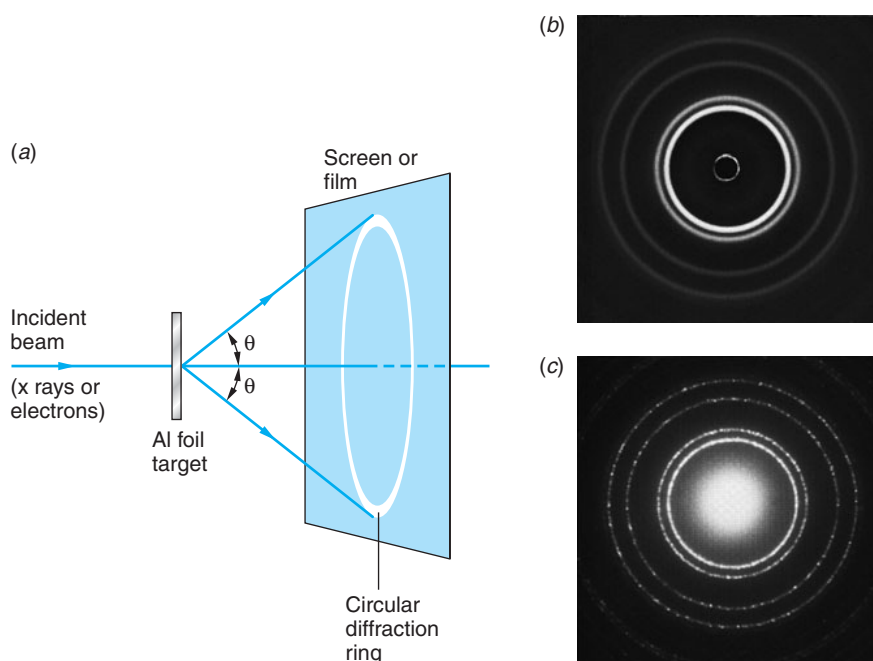
Figure 5-7 Variation of the scattered electron intensity with wavelength for constant φ . The incident beam in this case was 10° from the normal, the resulting refraction causing the measured peaks to be slightly shifted from the positions computed from Equation 5-5, as explained in note 6. [After C. J. Davisson and L. H. Germer, *Proceedings of the National Academy of Sciences*, **14**, 619 (1928).]

Here is Davisson's account of the connection between de Broglie's predictions and their experimental verification:

Perhaps no idea in physics has received so rapid or so intensive development as this one. De Broglie himself was in the van of this development, but the chief contributions were made by the older and more experienced Schrödinger. It would be pleasant to tell you that no sooner had Elsasser's suggestion appeared than the experiments were begun in New York which resulted in a demonstration of electron diffraction—pleasanter still to say that the work was begun the day after copies of de Broglie's thesis reached America. The true story contains less of perspicacity and more of chance. . . It was discovered, purely by accident, that the intensity of elastic scattering [of electrons] varies with the orientations of the scattering crystals. Out of this grew, quite naturally, an investigation of elastic scattering by a single crystal of predetermined orientation. . . Thus the New York experiment was not, at its inception, a test of wave theory. Only in the summer of 1926, after I had discussed the investigation in England with Richardson, Born, Franck and others, did it take on this character.⁷

A demonstration of the wave nature of relativistic electrons was provided in the same year by G. P. Thomson, who observed the transmission of electrons with energies in the range of 10 to 40 keV through thin metallic foils (G. P. Thomson, the son of J. J. Thomson, shared the Nobel Prize in 1937 with Davisson). The experimental arrangement (Figure 5-8a) was similar to that used to obtain Laue patterns with x rays (see Figure 3-11). Because the metal foil consists of many tiny crystals randomly oriented, the diffraction pattern consists of concentric rings. If a crystal is oriented at an angle θ with the incident beam, where θ satisfies the Bragg condition, this crystal will strongly scatter at an equal angle θ ; thus there will be a scattered beam making an angle 2θ with the incident beam. Figure 5-8b and c show the similarities in patterns produced by x rays and electron waves.

Figure 5-8 (a) Schematic arrangement used for producing a diffraction pattern from a polycrystalline aluminum target. (b) Diffraction pattern produced by x rays of wavelength 0.071 nm and an aluminum foil target. (c) Diffraction pattern produced by 600-eV electrons (de Broglie wavelength of about 0.05 nm) and an aluminum foil target. The pattern has been enlarged by 1.6 times to facilitate comparison with (b). [Courtesy of Film Studio, Education Development Center.]



Diffraction of Other Particles

The wave properties of neutral atoms and molecules were first demonstrated by O. Stern and I. Estermann in 1930 with beams of helium atoms and hydrogen molecules diffracted from a lithium fluoride crystal. Since the particles are neutral, there is no possibility of accelerating them with electrostatic potentials. The energy of the molecules was that of their average thermal motion, about 0.03 eV, which implies a de Broglie wavelength of about 0.10 nm for these molecules, according to Equation 5-2. Because of their low energy, the scattering occurs just from the array of atoms on the surface of the crystal, in contrast to Davisson and Germer's experiment. Figure 5-9 illustrates the geometry of the surface scattering, the experimental arrangement, and the results. Figure 5-9c indicates clearly the diffraction of He atom waves.

Since then, diffraction of other atoms, of protons, and of neutrons has been observed (see Figures 5-10, 5-11, and 5-12 on page 194). In all cases the measured wavelengths agree with de Broglie's prediction. There is thus no doubt that all matter has wavelike, as well as particlelike, properties, in symmetry with electromagnetic radiation.

The diffraction patterns formed by helium atom waves are used to study impurities and defects on the surfaces of crystals. Being a noble gas, helium does not react chemically with molecules on the surface nor "stick" to the surface.

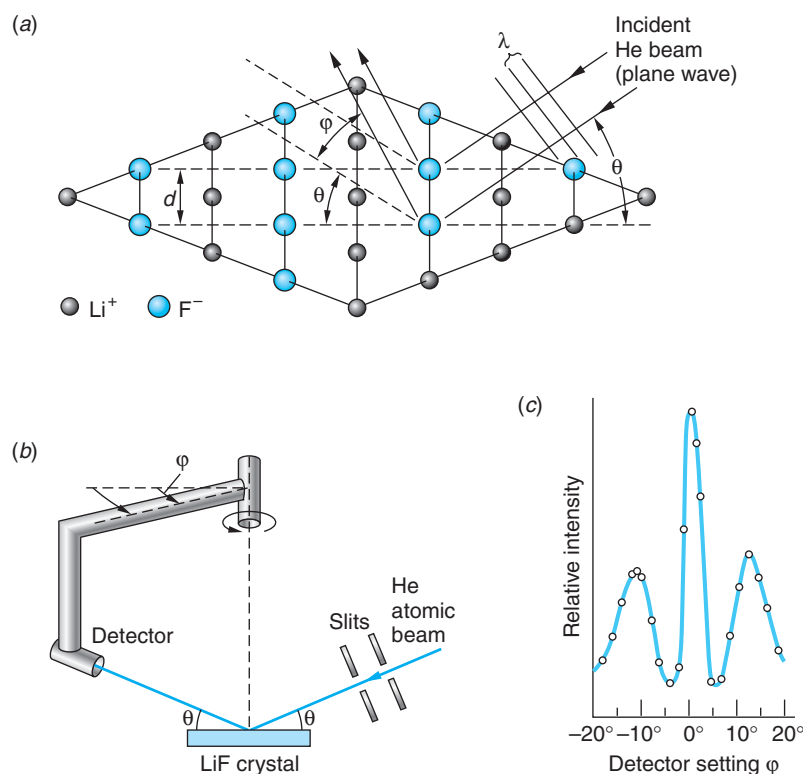


Figure 5-9 (a) He atoms impinge upon the surface of the LiF crystal at angle θ ($\theta = 18.5^\circ$ in Estermann and Stern's experiment). The reflected beam also makes the same angle θ with the surface but is also scattered at azimuthal angles ϕ relative to an axis perpendicular to the surface. (b) The detector views the surface at angle θ but can scan through the angle ϕ . (c) At angle ϕ where the path difference ($d \sin \phi$) between adjacent "rays" is $n\lambda$, constructive interference, i.e., a diffraction peak, occurs. The $n = 1$ peaks occur on either side of the $n = 0$ maximum.

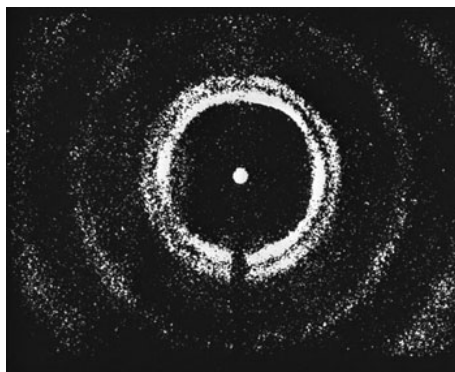


Figure 5-10 Diffraction pattern produced by 0.0568-eV neutrons (de Broglie wavelength of 0.120 nm) and a target of polycrystalline copper. Note the similarity in the patterns produced by x rays, electrons, and neutrons. [Courtesy of C. G. Shull.]

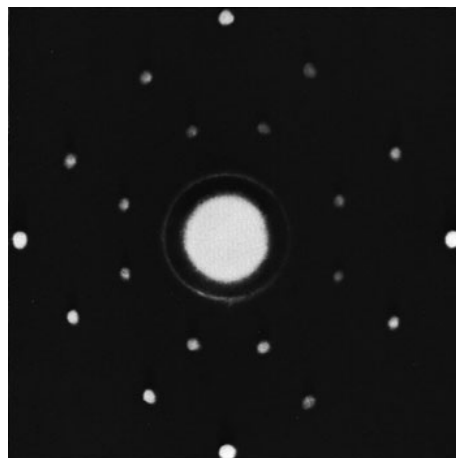


Figure 5-11 Neutron Laue pattern of NaCl. Compare this with the x-ray Laue pattern in Figure 3-14. [Courtesy of E. O. Wollan and C. G. Shull.]

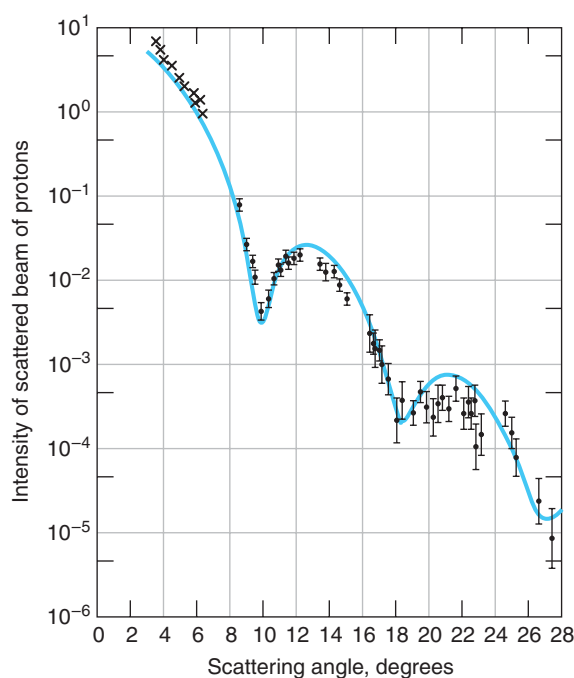


Figure 5-12 Nuclei provide scatterers whose dimensions are of the order of 10^{-15} m. Here the diffraction of 1-GeV protons from oxygen nuclei result in a pattern similar to that of a single slit.

An Easy Way to Determine de Broglie Wavelengths

It is frequently helpful to know the de Broglie wavelength for particles with a specific kinetic energy. For low energies where relativistic effects can be ignored, the equation leading to Equation 5-4 can be rewritten in terms of the kinetic energy as follows:

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE_k}} \quad 5-7$$

To find the equivalent expression that covers both relativistic and nonrelativistic speeds, we begin with the relativistic equation relating the total energy to the momentum:

$$E^2 = (pc)^2 + (mc^2)^2 \quad 5-31$$

Writing E_0 for the rest energy mc^2 of the particle for convenience, this becomes

$$E^2 = (pc)^2 + E_0^2 \quad 5-8$$

Since the total energy $E = E_0 + E_k$, Equation 5-8 becomes

$$(E_0 + E_k)^2 = (pc)^2 + E_0^2$$

that, when solved for p , yields

$$p = \frac{(2E_0E_k + E_k^2)^{1/2}}{c}$$

from which Equation 5-2 gives

$$\lambda = \frac{hc}{(2E_0E_k + E_k^2)^{1/2}} \quad 5-9$$

This can be written in a particularly useful way applicable to any particle of any energy by dividing the numerator and denominator by the rest energy $E_0 = mc^2$ as follows:

$$\lambda = \frac{hc/mc^2}{(2E_0E_k + E_k^2)^{1/2}/E_0} = \frac{h/mc}{[2(E_k/E_0) + (E_k/E_0)^2]^{1/2}}$$

Recognizing h/mc as the Compton wavelength λ_c of the particle of mass m (see Section 3-4), we have that, for any particle,

$$\lambda/\lambda_c = \frac{1}{[2(E_k/E_0) + (E_k/E_0)^2]^{1/2}} \quad 5-10$$

A log-log graph of λ/λ_c versus E_k/E_0 is shown in Figure 5-13. It has two sections of nearly constant slope, one for $E_k \ll mc^2$ and the other for $E_k \gg mc^2$, connected by a curved portion lying roughly between $0.1 < E_k/E_0 < 10$. The following example illustrates the use of Figure 5-13.

EXAMPLE 5-3 The de Broglie Wavelength of a Cosmic Ray Proton Detectors on board a satellite measure the kinetic energy of a cosmic ray proton to be 150 GeV. What is the proton's de Broglie wavelength, as read from Figure 5-13?

SOLUTION

The rest energy of the proton is $mc^2 = 0.938$ GeV and the proton's mass is 1.67×10^{-27} kg. Thus, the ratio E_k/E_0 is

$$\frac{E_k}{E_0} = \frac{150 \text{ GeV}}{0.938 \text{ GeV}} = 160$$

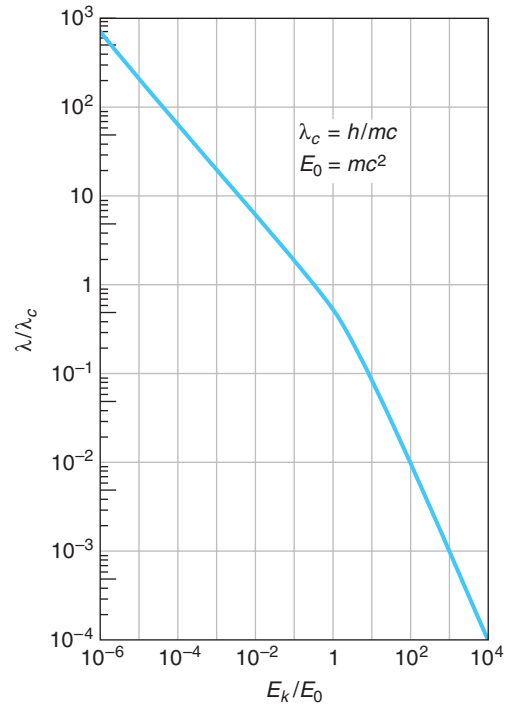


Figure 5-13 The de Broglie wavelength λ expressed in units of the Compton wavelength λ_c for a particle of mass m versus the kinetic energy of the particle E_k expressed in units of its rest energy $E_0 = mc^2$. For protons and neutrons $E_0 = 0.938$ GeV and $\lambda_c = 1.32$ fm. For electrons $E_0 = 0.511$ MeV and $\lambda_c = 0.00234$ nm.

This value on the curve corresponds to about 6×10^{-3} on the λ/λ_c axis. The Compton wavelength of the proton is

$$\lambda_c = \frac{h}{mc} = \frac{6.63 \times 10^{-34} \text{ J} \cdot \text{s}}{(1.67 \times 10^{-27} \text{ kg})(3 \times 10^8 \text{ m/s})} = 1.32 \times 10^{-15} \text{ m}$$

and we have then for the particle's de Broglie wavelength

$$\lambda = (6 \times 10^{-3})(1.32 \times 10^{-15} \text{ m}) = 7.9 \times 10^{-18} \text{ m} = 7.9 \times 10^{-3} \text{ fm}$$

Questions

1. Since the electrons used by Davisson and Germer were low energy, they penetrated only a few atomic layers into the crystal, so it is rather surprising that the effects of the inner layers shows so clearly. What feature of the diffraction is most affected by the relatively shallow penetration?
2. How might the frequency of de Broglie waves be measured?
3. Why is it not reasonable to do crystallographic studies with protons?

5-3 Wave Packets

In any discussion of waves the question arises, “What’s waving?” For some waves the answer is clear: for waves on the ocean, it is the water that “waves”; for sound waves in air, it is the molecules that make up the air; for light, it is the **E** and the **B**. So what is waving for matter waves? For matter waves as for light waves, there is no “ether.” As will be developed in this section and the next, for matter it is the *probability of finding the particle* that waves.

Classical waves are solutions of the classical *wave equation*

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \quad 5-11$$

Important among classical waves is the *harmonic wave* of amplitude y_0 , frequency f , and period T , traveling in the $+x$ direction as written here:

$$y(x, t) = y_0 \cos(kx - \omega t) = y_0 \cos 2\pi \left(\frac{x}{\lambda} - \frac{t}{T} \right) = y_0 \cos \frac{2\pi}{\lambda} (x - vt) \quad 5-12$$

where the *angular frequency* ω and the *wave number*⁸ k are defined by

$$\omega = 2\pi f = \frac{2\pi}{T} \quad 5-13a$$

and

$$k = \frac{2\pi}{\lambda} \quad 5-13b$$

and the velocity v of the wave, the so-called *wave* or *phase velocity* v_p , is given by

$$v_p = f\lambda \quad 5-14$$

A familiar wave phenomenon that cannot be described by a single harmonic wave is a pulse, such as the flip of one end of a long string (Figure 5-14), a sudden noise, or the brief opening of a shutter in front of a light source. The main characteristic of a pulse is localization in time and space. A single harmonic wave is not localized in either time or space. The description of a pulse can be obtained by the superposition of a group of harmonic waves of different frequencies and wavelengths. Such a group is called a *wave packet*. The mathematics of representing arbitrarily shaped pulses by sums of sine or cosine functions involves Fourier series and Fourier integrals. We will illustrate the phenomenon of wave packets by considering some simple and somewhat artificial examples and discussing the general properties qualitatively. Wave groups are particularly important because a wave description of a particle must include the important property of localization.

Consider a simple group consisting of only two waves of equal amplitude and nearly equal frequencies and wavelengths. Such a group occurs in the phenomenon of beats and is described in most introductory textbooks. The quantities k , ω , and v are related to one another via Equations 5-13 and 5-14. Let the wave numbers be k_1 and k_2 , the angular frequencies ω_1 and ω_2 , and the speeds v_1 and v_2 . The sum of the two waves is

$$y(x, t) = y_0 \cos(k_1 x - \omega_1 t) + y_0 \cos(k_2 x - \omega_2 t)$$

which, with the use of a bit of trigonometry, becomes

$$y(x, t) = 2y_0 \cos\left(\frac{\Delta k}{2}x - \frac{\Delta \omega}{2}t\right) \cos\left(\frac{k_1 + k_2}{2}x - \frac{\omega_1 + \omega_2}{2}t\right)$$

where $\Delta k = k_2 - k_1$ and $\Delta \omega = \omega_2 - \omega_1$. Since the two waves have nearly equal values of k and ω , we will write $\bar{k} = (k_1 + k_2)/2$ and $\bar{\omega} = (\omega_1 + \omega_2)/2$ for the mean values. The sum is then

$$y(x, t) = 2y_0 \cos\left(\frac{1}{2}\Delta kx - \frac{1}{2}\Delta \omega t\right) \cos(\bar{k}x - \bar{\omega}t) \quad \mathbf{5-15}$$

Figure 5-15 shows a sketch of $y(x, t_0)$ versus x at some time t_0 . The dashed curve is the envelope of the group of two waves, given by the first cosine term in Equation 5-15. The wave within the envelope moves with the speed $\bar{\omega}/\bar{k}$, the phase velocity v_p due to the second cosine term. If we write the first (amplitude-modulating) term as $\cos\{\frac{1}{2}\Delta k[x - (\Delta \omega/\Delta k)t]\}$, we see that the envelope moves with speed $\Delta \omega/\Delta k$. The speed of the envelope is called the *group velocity* v_g .

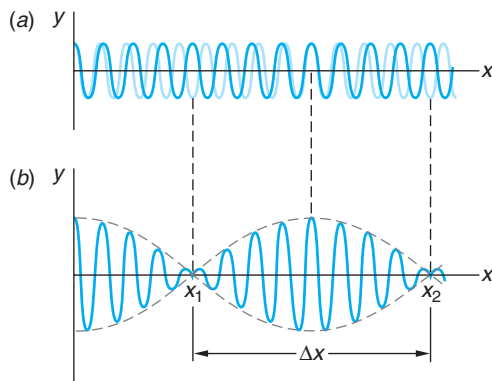


Figure 5-15 Two waves of slightly different wavelength and frequency produce beats. (a) Shows $y(x)$ at a given instant for each of the two waves. The waves are in phase at the origin, but because of the difference in wavelength, they become out of phase and then in phase again. (b) The sum of these waves. The spatial extent of the group Δx is inversely proportional to the difference in wave numbers Δk , where k is related to the wavelength by $k = 2\pi/\lambda$. Identical figures are obtained if y is plotted versus time t at a fixed point x . In that case the extent in time Δt is inversely proportional to the frequency difference $\Delta \omega$.

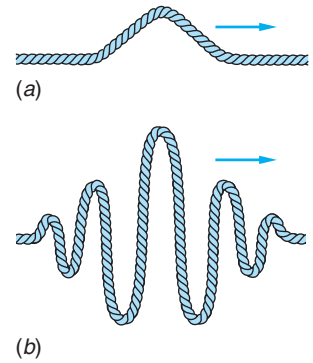


Figure 5-14 (a) Wave pulse moving along a string. A pulse has a beginning and an end; i.e., it is localized, unlike a pure harmonic wave, which goes on forever in space and time. (b) A wave packet formed by the superposition of harmonic waves.

A more general wave packet can be constructed if, instead of adding just two sinusoidal waves as in Figure 5-15, we superpose a larger, finite number with slightly different wavelengths and different amplitudes. For example, Figure 5-16a illustrates the superposing of seven cosines with wavelengths from $\lambda_9 = 1/9$ to $\lambda_{15} = 1/15$ (wave numbers from $k_9 = 18\pi$ to $k_{15} = 30\pi$) at time t_0 . The waves are all in phase at $x = 0$ and again at $x = \pm 12, x = \pm 24, \dots$. Their sum $y(x, t_0) = \sum_{i=9}^{15} y_i(x, t_0)$

oscillates with maxima at those values of x , decreasing and increasing at other values as a result of the changing phases of the waves (see Figure 5-16b). Now, if we superpose an infinite number of waves from the same range of wavelengths and wave numbers as in Figure 5-16 with infinitesimally different values of k , the central group around $x = 0$ will be essentially the same as in that figure. However, the additional groups will no longer be present since there is now no length along the x axis into which an exactly integral number of all of the infinite number of component waves can fit. Thus, we have formed a single wave packet throughout this (one-dimensional) space.

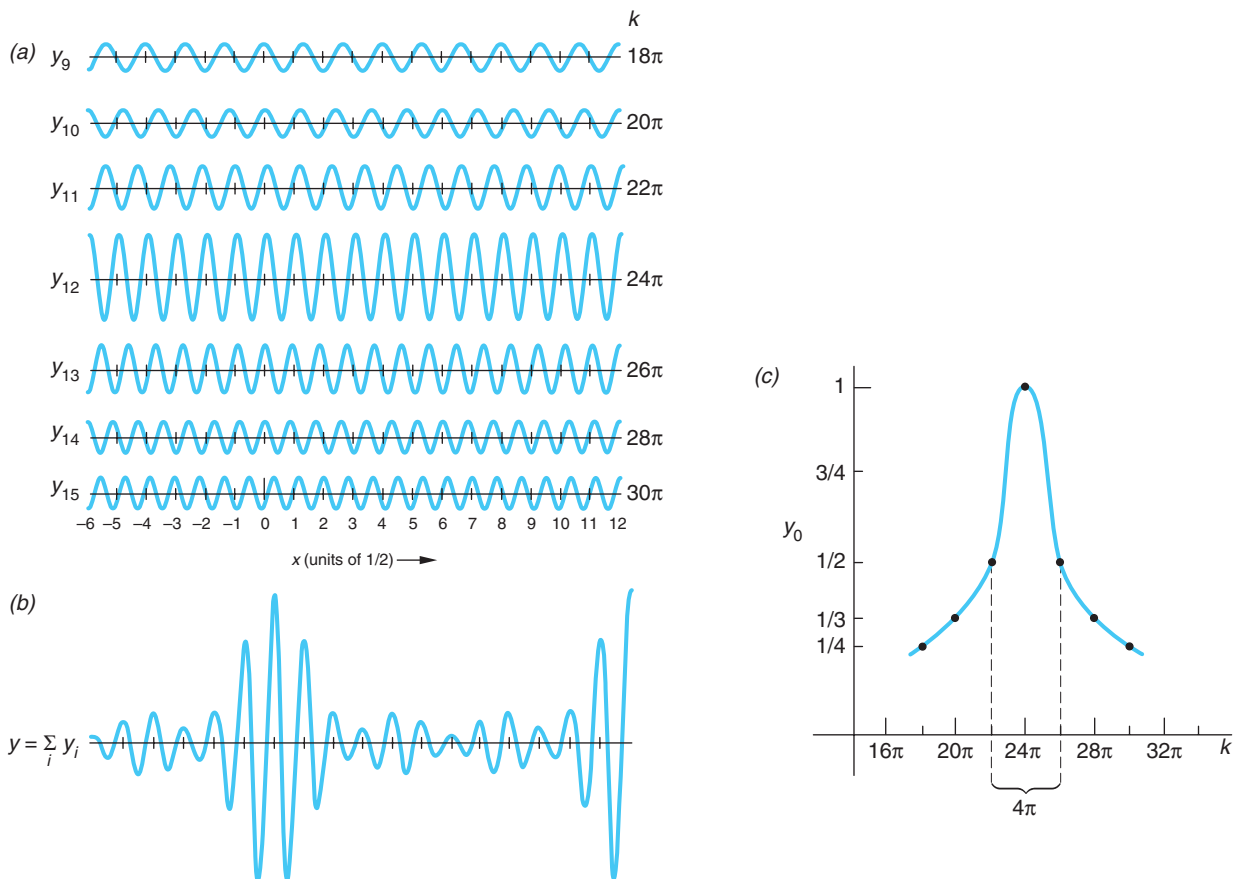


Figure 5-16 (a) Superposition of seven sinusoids $y_k(x, t) = y_{0k} \cos(kx - \omega t)$ with uniformly spaced wave numbers ranging from $k = (2\pi)9$ to $k = (2\pi)15$ with $t = 0$. The maximum amplitude is 1 at the center of the range ($k = (2\pi)12$), decreasing to $1/2, 1/3$, and $1/4$, respectively, for the waves on each side of the central wave. (b) The sum $y(x, 0) = \sum_{i=9}^{15} y_i(x, 0)$ is maximum at $x = 0$ with additional maxima equally spaced along the $\pm x$ axis. (c) Amplitudes of the sinusoids y_i versus wave number k .

This packet moves at the group velocity $v_g = d\omega/dk$. The mathematics needed to demonstrate the above involves use of the Fourier integral described in the Classical Concept Review.

The phase velocities of the individual harmonic waves are given by Equation 5-14:

$$v_p = f\lambda = \left(\frac{\omega}{2\pi}\right)\left(\frac{2\pi}{k}\right) = \frac{\omega}{k}$$

Writing this as $\omega = kv_p$, the relation between the group and phase velocities is given by Equation 5-16:

$$v_g = \frac{d\omega}{dk} = v_p + k \frac{dv_p}{dk} \quad 5-16$$

If the phase velocity is the same for all frequencies and wavelengths, then $dv_p/dk = 0$, and the group velocity is the same as the phase velocity. A medium for which the phase velocity is the same for all frequencies is said to be *nondispersive*. Examples are waves on a perfectly flexible string, sound waves in air, and electromagnetic waves in a vacuum. An important characteristic of a nondispersive medium is that, since all the harmonic waves making up a packet move with the same speed, the packet maintains its shape as it moves; thus, it does not change its shape in time. Conversely, if the phase velocity is different for different frequencies, the shape of the pulse will change as it travels. In that case the group velocity and phase velocity are not the same. Such a medium is called a *dispersive* medium; examples are water waves, waves on a wire that is not perfectly flexible, light waves in a medium such as glass or water in which the index of refraction has a slight dependence on frequency, and electron waves. It is the speed of the packet, the group velocity v_g , that is normally seen by an observer.

Classical Uncertainty Relations

Notice that the width of the group⁹ Δx of the superposition $y(x, t_0)$ in Figure 5-16b is just a bit larger than $1/12$. Similarly, the graph of the amplitude of these waves versus k has width $\Delta k = 4\pi$, which is a bit more than 12 (Figure 5-16c), so we see that

$$\Delta k \Delta x \sim 1 \quad 5-17$$

By a similar analysis, we would also conclude that

$$\Delta \omega \Delta t \sim 1 \quad 5-18$$

The range of wavelengths or frequencies of the harmonic waves needed to form a wave packet depends on the extent in space and duration in time of the pulse. In general, if the extent in space Δx is to be small, the range Δk of wave numbers must be large. Similarly, if the duration in time Δt is small, the range of frequencies $\Delta \omega$ must be large. We have written these as order-of-magnitude equations because the exact value of the products $\Delta k \Delta x$ and $\Delta \omega \Delta t$ depends on how these ranges are defined, as well as on the particular shape of the packets. Equation 5-18 is sometimes known as the *response time–bandwidth* relation, expressing the result that a circuit component such as an amplifier must have a large bandwidth ($\Delta \omega$) if it is to be able to respond to signals of short duration.



The classical uncertainty relations define the range of signal frequencies to which all kinds of communications equipment and computer systems must respond, from cell phones to supercomputers.

There is a slight variation of Equation 5-17 that is also helpful in interpreting the relation between Δx and Δk . Differentiating the wave number in Equation 5-13b yields

$$dk = \frac{-2\pi d\lambda}{\lambda^2} \quad 5-19$$

Replacing the differentials by small intervals and concerning ourselves only with magnitudes, Equation 5-19 becomes

$$\Delta k = \frac{2\pi\Delta\lambda}{\lambda^2}$$

which when substituted into Equation 5-17 gives

$$\Delta x \Delta \lambda \approx \frac{\lambda^2}{2\pi} \quad 5-20$$

Equation 5-20 says that the product of the spatial extent of a classical wave Δx and the uncertainty (or “error”) in the determination of its wavelength $\Delta \lambda$ will always be of the order of $\lambda^2/2\pi$. The following brief examples will illustrate the meaning of Equations 5-17 and 5-18, often referred to as the *classical uncertainty relations*, and Equation 5-20.

EXAMPLE 5-4 $\Delta \lambda$ for Ocean Waves Standing in the middle of a 20-m-long pier, you notice that at any given instant there are 15 wave crests between the two ends of the pier. Estimate the minimum uncertainty in the wavelength that could be computed from this information.

SOLUTION

1. The minimum uncertainty $\Delta \lambda$ in the wavelength is given by Equation 5-20:

$$\Delta x \Delta \lambda = \frac{\lambda^2}{2\pi}$$

2. The wavelength λ of the waves is

$$\lambda = \frac{20 \text{ m}}{15 \text{ waves}} = 1.3 \text{ m}$$

3. The spatial extent of the waves used for this calculation is:

$$\Delta x = 20 \text{ m}$$

4. Solving Equation 5-20 for $\Delta \lambda$ and substituting these values gives

$$\begin{aligned} \Delta \lambda &= \frac{\lambda^2}{2\pi\Delta x} = \frac{(1.3)^2}{2\pi \times 20} \\ &= 0.013 \text{ m} \\ \Delta \lambda &\approx 0.01 \text{ m} = 1 \text{ cm} \end{aligned}$$

Remarks: This is the minimum uncertainty. Any error that may exist in the measurement of the number of wave crests and the length of the pier would add further uncertainty to the determination of λ .

EXAMPLE 5-5 Frequency Control The frequency of the alternating voltage produced at electric generating stations is carefully maintained at 60.00 Hz. The frequency is monitored on a digital frequency meter in the control room. For how long must the frequency be measured and how often can the display be updated if the reading is to be accurate to within 0.01 Hz?

SOLUTION

Since $\omega = 2\pi f$, then $\Delta\omega = 2\pi\Delta f = 2\pi(0.01)$ rad/s and

$$\Delta t \sim 1/\Delta\omega = 1/2\pi(0.01)$$

$$\Delta t \sim 16 \text{ s}$$

Thus, the frequency must be measured for about 16 s if the reading is to be accurate to 0.01 Hz and the display cannot be updated more often than once every 16s.

Questions

4. Which is more important for communication, the group velocity or the phase velocity?
5. What are Δx and Δk for a purely harmonic wave of a single frequency and wavelength?

Particle Wave Packets

The quantity analogous to the displacement $y(x, t)$ for waves on a string, to the pressure $P(x, t)$ for a sound wave, or to the electric field $\mathcal{E}(x, t)$ for electromagnetic waves is called the *wave function for particles* and is usually designated $\Psi(x, t)$. It is $\Psi(x, t)$ that we will relate to the probability of finding the particle and, as we alerted you earlier, it is the probability that waves. Consider, for example, an electron wave consisting of a single frequency and wavelength; we could represent such a wave by any of the following, exactly as we did the classical wave: $\Psi(x, t) = A \cos(kx - \omega t)$, $\Psi(x, t) = A \sin(kx - \omega t)$, or $\Psi(x, t) = Ae^{i(kx - \omega t)}$.

The phase velocity for this wave is given by

$$v_p = f\lambda = (E/h)(h/p) = E/p$$

where we have used the de Broglie relations for the wavelength and frequency. Using the nonrelativistic expression for the energy of a particle moving in free space, i.e., no potential energy) with no forces acting upon it,

$$E = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

we see that the phase velocity is

$$v_p = E/p = (p^2/2m)/p = p/2m = v/2$$

i.e., the phase velocity of the wave is half the velocity of an electron with momentum p . The phase velocity does *not* equal the particle velocity. Moreover, a wave of a single frequency and wavelength is not localized but is spread throughout space, which makes it difficult to see how the particle and wave properties of the electron could be related.

An application of phase and particle speeds by nature: produce a wave on a still pond (or in a bathtub) and watch the wavelets that make up the wave appear to “climb over” the wave crest at twice the speed of the wave.

Thus, for the electron to have the particle property of being localized, the matter waves of the electron must also be limited in spatial extent—i.e., realistically, $\Psi(x, t)$ must be a wave packet containing many more than one wave number k and frequency ω . It is the wave packet $\Psi(x, t)$ that we expect to move at a group velocity equal to the particle velocity, which we will show below is indeed the case. The particle, if observed, we will expect to find somewhere within the spatial extent of the wave packet $\Psi(x, t)$, precisely where within that space being the subject of the next section.

To illustrate the equality of the group velocity v_g and the particle velocity v , it is convenient to express de Broglie’s relations in a slightly different form. Writing Equation 5-1 as follows,

$$E = hf = h\omega/2\pi \quad \text{or} \quad E = \hbar\omega \quad 5-21$$

and Equation 5-2 as

$$p = \frac{h}{\lambda} = \frac{h}{2\pi/k} = \frac{hk}{2\pi} \quad \text{or} \quad p = \hbar k \quad 5-22$$

The group velocity is then given by

$$v_g = d\omega/dk = (dE/\hbar)/(dp/\hbar) = dE/dp$$

Again using the nonrelativistic expression $E = p^2/2m$, we have that

$$v_g = dE/dp = p/m = v$$

and *the wave packet $\Psi(x, t)$ moves with the velocity of the electron*. This was, in fact, one of de Broglie’s reasons for choosing Equations 5-1 and 5-2. (De Broglie used the relativistic expression relating energy and momentum, which also leads to the equality of the group velocity and particle velocity.)

5-4 The Probabilistic Interpretation of the Wave Function

Let us consider in more detail the relation between the wave function $\Psi(x, t)$ and the location of the electron. We can get a hint about this relation from the case of light. The wave equation that governs light is Equation 5-11, with $y = \mathcal{E}$, the electric field, as the wave function. The energy per unit volume in a light wave is proportional to \mathcal{E}^2 , but the energy in a light wave is quantized in units of hf for each photon. We expect, therefore, that the number of photons in a unit volume is proportional to \mathcal{E}^2 , a connection first pointed out by Einstein.

Consider the famous double-slit interference experiment (Figure 5-17). The pattern observed on the screen is determined by the interference of the waves from the slits. At a point on the screen where the wave from one slit is 180° out of phase with that from the other, the resultant electric field is zero; there is no light energy at this point, and this point on the screen is dark. If we reduce the intensity to a very low value, we can still observe the interference pattern if we replace the ordinary screen by a scintillation screen or a two-dimensional array of tiny photon detectors (e.g., a CCD camera) and wait a sufficient length of time.

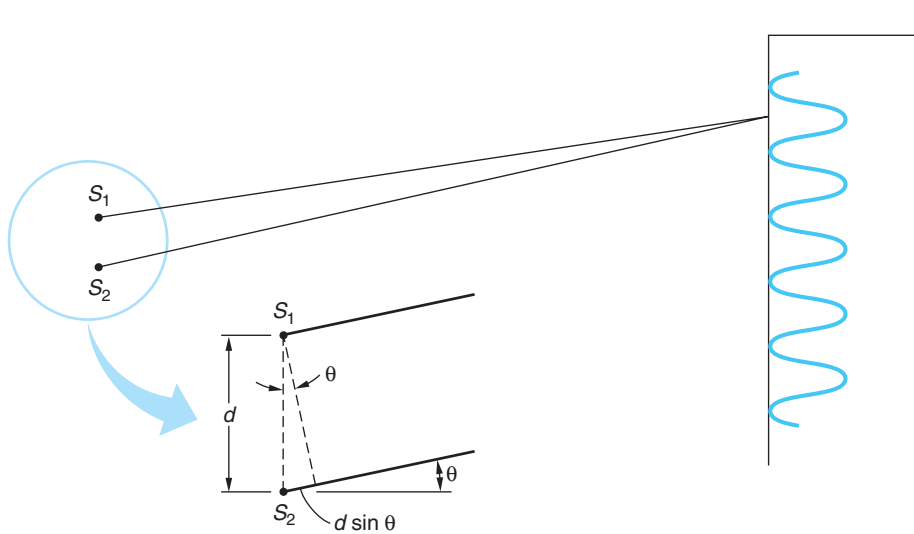


Figure 5-17 Two-source interference pattern. If the sources are coherent and in phase, the waves from the sources interfere constructively at points for which the path difference ($d \sin \theta$) is an integral number of wavelengths.

The interaction of light with the detector or scintillator is a quantum phenomenon. If we illuminate the scintillators or detectors for only a very short time with a low-intensity source, we do not see merely a weaker version of the high-intensity pattern; we see, instead, “dots” caused by the interactions of individual photons (Figure 5-18). At points where the waves from the slits interfere destructively, there are no dots, and at points where the waves interfere constructively, there are many dots. However,

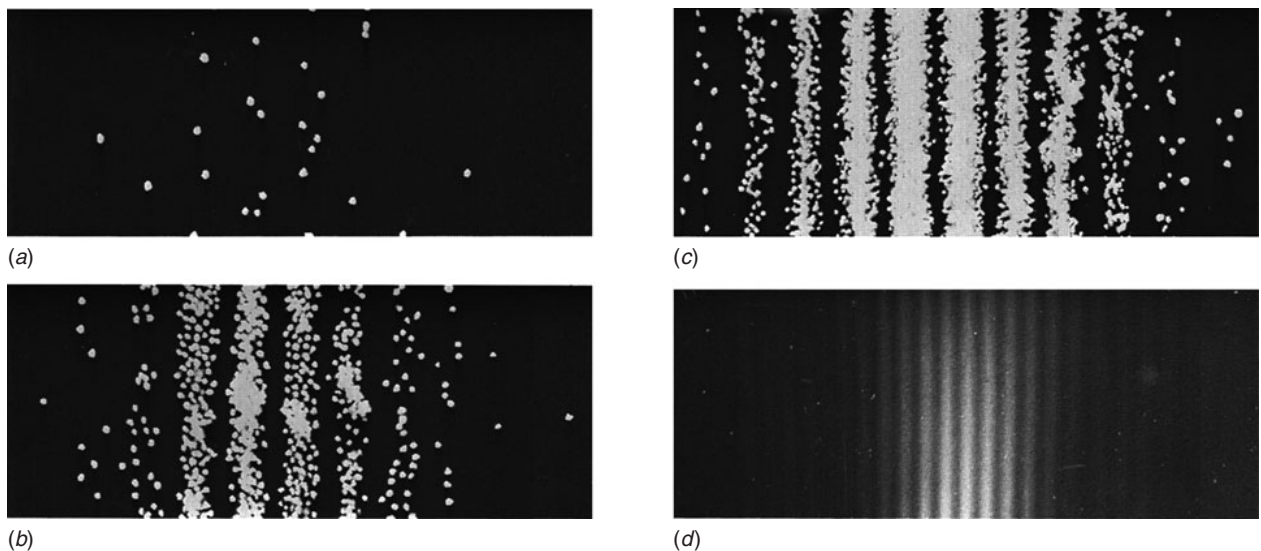


Figure 5-18 Growth of two-slit interference pattern. The photo (d) is an actual two-slit electron interference pattern in which the film was exposed to millions of electrons. The pattern is identical to that usually obtained with photons. If the film were to be observed at various stages, such as after being struck by 28 electrons, then after about 1000 electrons, and again after about 10,000 electrons, the patterns of individually exposed grains would be similar to those shown in (a), (b), and (c) except that the exposed dots would be smaller than the dots drawn here. Note that there are no dots in the region of the interference minima. The probability of any point of the film being exposed is determined by wave theory, whether the film is exposed by electrons or photons. [Parts (a), (b), and (c) from E. R. Huggins, *Physics 1*, © by W. A. Benjamin, Inc., Menlo Park, California. Photo (d) courtesy of C. Jonsson.]

when the exposure is short and the source weak, random fluctuations from the average predictions of the wave theory are clearly evident. If the exposure is long enough that many photons reach the detector, the fluctuations average out and the quantum nature of light is not noticed. The interference pattern depends only on the total number of photons interacting with the detector and not on the rate. *Even when the intensity is so low that only one photon at a time reaches the detector, the wave theory predicts the correct average pattern.* For low intensities, we therefore interpret \mathcal{E}^2 as proportional to the *probability* of detecting a photon in a unit volume of space. At points on the detector where \mathcal{E}^2 is zero, photons are never observed, whereas they are most likely to be observed at points where \mathcal{E}^2 is large.

It is not necessary to use light waves to produce an interference pattern. Such patterns can be produced with electrons and other particles as well. In the wave theory of electrons the de Broglie wave of a *single* electron is described by a wave function Ψ . The amplitude of Ψ at any point is related to the probability of finding the particle at that point. In analogy with the foregoing interpretation of \mathcal{E}^2 , *the quantity $|\Psi|^2$ is proportional to the probability of detecting an electron in a unit volume*, where $|\Psi|^2 \equiv \Psi^*\Psi$, the function Ψ^* being the complex conjugate of Ψ . In one dimension, $|\Psi|^2 dx$ is the probability of an electron being in the interval dx .¹⁰ (See Figure 5-19.) If we call this probability $P(x)dx$, where $P(x)$ is the probability distribution function, we have

$$P(x)dx = |\Psi|^2 dx \quad 5-23$$

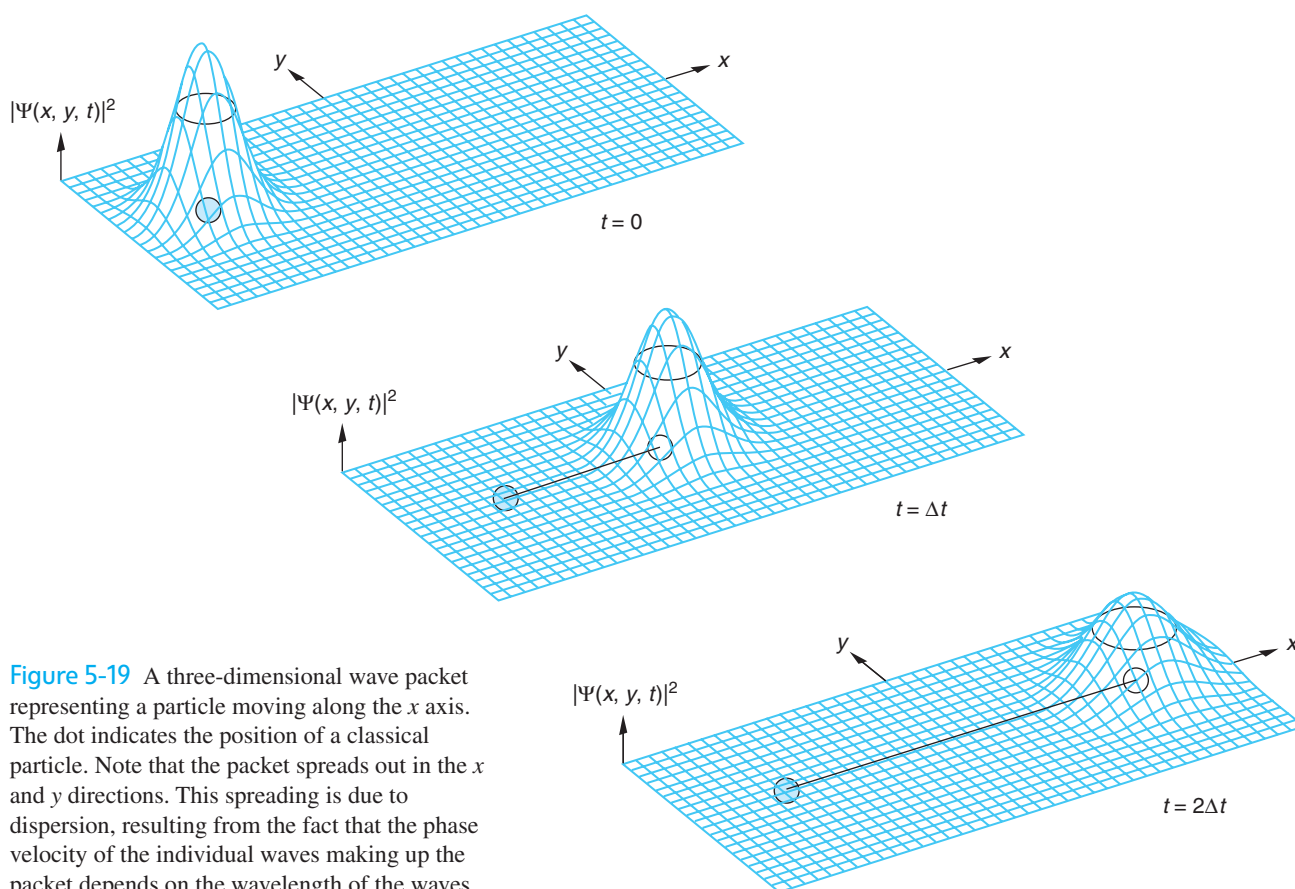


Figure 5-19 A three-dimensional wave packet representing a particle moving along the x axis. The dot indicates the position of a classical particle. Note that the packet spreads out in the x and y directions. This spreading is due to dispersion, resulting from the fact that the phase velocity of the individual waves making up the packet depends on the wavelength of the waves.

In the next chapter we will more thoroughly discuss the amplitudes of matter waves associated with particles, in particular developing the mathematical system for computing the amplitudes and probabilities in various situations. The uneasiness that you may feel at this point regarding the fact that we have not given a precise physical interpretation to the amplitude of the de Broglie matter wave can be attributed in part to the complex nature of the wave amplitude; i.e., it is in general a complex function with a real part and an imaginary part, the latter proportional to $i = (-1)^{1/2}$. We cannot directly measure or physically interpret complex numbers in our world of real numbers. However, as we will see, defining the probability in terms of $|\Psi|^2$, which is always real, presents no difficulty in its physical interpretation. Thus, even though the amplitudes of the wave functions Ψ have no simple meaning, the waves themselves behave just as classical waves do, exhibiting the wave characteristics of reflection, refraction, interference, and diffraction and obeying the principles of superposition.

5-5 The Uncertainty Principle

The uncertainty relations for classical wave packets (Equations 5-17 and 5-18) have very important matter wave analogs.

Consider a wave packet $\Psi(x, t)$ representing an electron. The most probable position of the electron is the value of x for which $|\Psi(x, t)|^2$ is a maximum. Since $|\Psi(x, t)|^2$ is proportional to the probability that the electron is located at x and $|\Psi(x, t)|^2$ is nonzero for a range of values of x , there is an *uncertainty* in the position of the electron (see Figure 5-19). This means that if we make a number of position measurements on identical electrons—electrons with the same wave function—we will not always obtain the same result. In fact, the distribution function for the results of such measurements will be given by $|\Psi(x, t)|^2$. If the wave packet is very narrow, the uncertainty in position will be small. However, a narrow wave packet must contain a wide range of wave numbers k . Since the momentum is related to the wave number by $p = \hbar k$, a wide range of k values means a wide range of momentum values. We have seen that for all wave packets the ranges Δx and Δk are related by

$$\Delta k \Delta x \sim 1 \quad 5-17$$

Similarly, a packet that is localized in time Δt must contain a range of frequencies $\Delta \omega$, where the ranges are related by

$$\Delta \omega \Delta t \sim 1 \quad 5-18$$

Equations 5-17 and 5-18 are inherent properties of waves. If we multiply these equations by \hbar and use $p = \hbar k$ and $E = \hbar \omega$, we obtain

$$\Delta x \Delta p \sim \hbar \quad 5-24$$

and

$$\Delta E \Delta t \sim \hbar \quad 5-25$$

Equations 5-24 and 5-25 provide a statement of the *uncertainty principle* first enunciated in 1927 by Werner K. Heisenberg.¹¹ Equation 5-24 expresses the fact that the distribution functions for position and momentum cannot both be made arbitrarily narrow simultaneously (see Figure 5-16); thus, measurements of position and momentum will have similar uncertainties, which are related by Equation 5-24. Of course, because of

Heisenberg's uncertainty principle is the key to the existence of *virtual particles* that hold the nuclei together (see Chapter 11) and is the root of quantum fluctuations that may have been the origin of the Big Bang (see Chapter 13).

inaccurate measurements, the product of Δx and Δp can be, and usually is, much larger than \hbar . The lower limit is not due to any technical problem in the design of measuring equipment that might be solved at some later time; it is instead due to the wave and particle nature of both matter and light.

If we define precisely what we mean by the uncertainty in the measurements of position and momentum, we can give a precise statement of the uncertainty principle. For example, if σ_x is the standard deviation for measurements of position and σ_k is the standard deviation for measurements of the wave number, the product $\sigma_x \sigma_k$ has its minimum value of 1/2 when the distribution functions are Gaussian. If we define Δx and Δp to be the standard deviations, the minimum value of their product is $\hbar/2$. Thus,

$$\Delta x \Delta p \geq \frac{1}{2} \hbar \quad 5-26$$

Similarly,

$$\Delta E \Delta t \geq \frac{1}{2} \hbar \quad 5-27$$

Question

6. Does the uncertainty principle say that the momentum of a particle can never be precisely known?



EXPLORING

The Gamma-Ray Microscope

Let us see how one might attempt to make a measurement so accurate as to violate the uncertainty principle. A common way to measure the position of an object such as an electron is to look at it with light, i.e., scatter light from it and observe the diffraction pattern. The momentum can be obtained by looking at it again a short time later and computing what velocity it must have had the instant before the light scattered from it. Because of diffraction effects, we cannot hope to make measurements of length (position) that are smaller than the wavelength of the light used, so we will use the shortest-wavelength light that can be obtained, gamma rays. (There is, in principle, no limit to how short the wavelength of electromagnetic radiation can be.) We also know that light carries momentum and energy, so that when it scatters off the electron, the motion of the electron will be disturbed, affecting the momentum. We must, therefore, use the minimum intensity possible so as to disturb the electron as little as possible. Reducing the intensity decreases the number of photons, but we must scatter at least one photon to observe the electron. The minimum possible intensity, then, is that corresponding to one photon. The scattering of a photon by a free electron is, of course, a Compton scattering, which was discussed in Section 3-4. The momentum of the photon is $hf/c = h/\lambda$. The smaller λ that is used to measure the position, the more the photon will disturb the electron, but we can correct for that with a Compton-effect analysis, provided only that we know the photon's momentum and the scattering angles of the event.

Figure 5-20 illustrates the problem. (This illustration was first given as a gedanken-experiment, or thought experiment, by Heisenberg. Since a single photon doesn't form a diffraction pattern, think of the diffraction pattern as being built up by photons from

many identical scattering experiments.) The position of the electron is to be determined by viewing it through a microscope. We will assume that only one photon is used. We can take for the uncertainty in position the minimum separation distance for which two objects can be resolved; this is¹²

$$\Delta x = \frac{\lambda}{2 \sin \theta}$$

where θ is the half angle subtended by the lens aperture, as shown in Figure 5-20*a* and *b*.

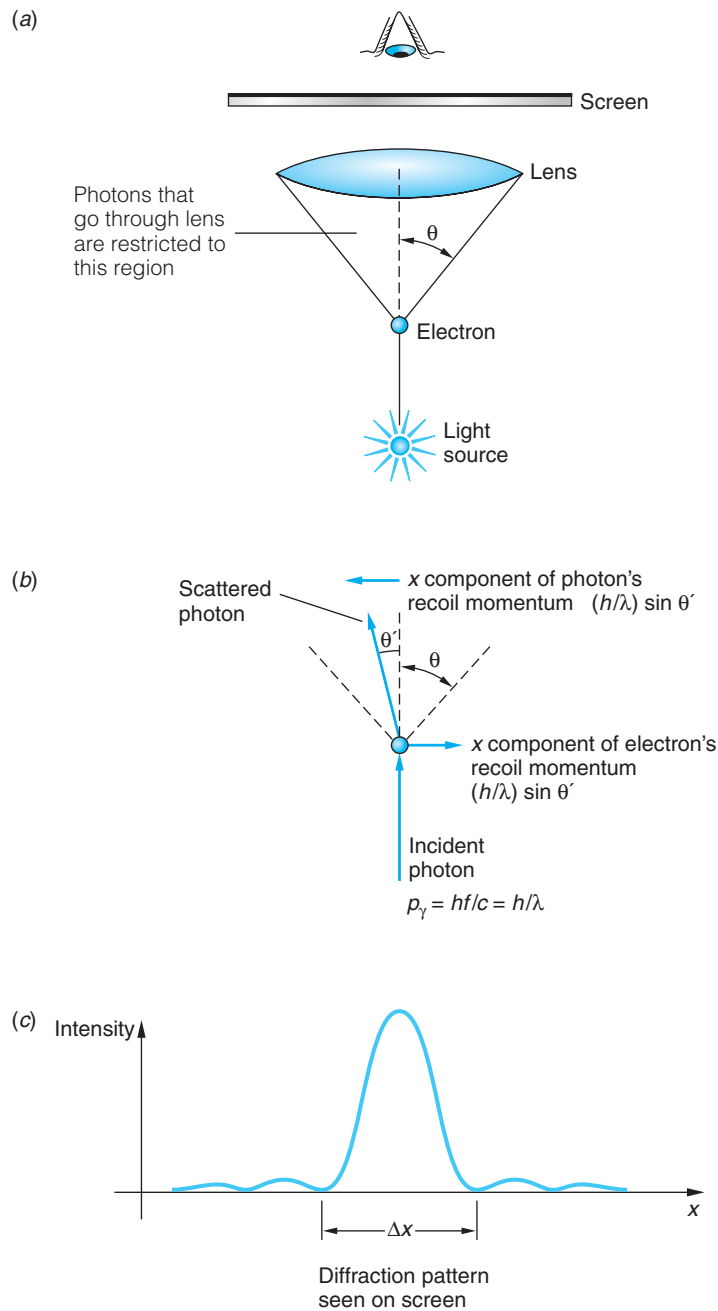


Figure 5-20 “Seeing an electron” with a gamma-ray microscope. (b) Because of the size of the lens, the momentum of the scattered photon is uncertain by $\Delta p_x \approx p \sin \theta = h \sin \theta / \lambda$. Thus the recoil momentum of the electron is also uncertain by at least this amount. (c) The position of the electron cannot be resolved better than the width of the central maximum of the diffraction pattern $\Delta x \approx \lambda / \sin \theta$. The product of the uncertainties $\Delta p_x \Delta x$ is therefore of the order of Planck’s constant h .

Let us assume that the x component of momentum of the incoming photon is known precisely from a previous measurement. To reach the screen and contribute to the diffraction pattern in Figure 5-20c, the scattered photon need only go through the lens aperture. Thus, the scattered photon can have any x component of momentum from 0 to $p_x = p \sin \theta$, where p is the total momentum of the scattered photon. By conservation of momentum, the uncertainty in the momentum of the electron after the scattering must be greater than or equal to that of the scattered photon (it would be equal, of course, if the electron's initial momentum were known precisely); therefore, we write

$$\Delta p_x \geq p \sin \theta = \frac{h}{\lambda} \sin \theta$$

and

$$\Delta x \Delta p_x \geq \frac{\lambda}{2 \sin \theta} \frac{h \sin \theta}{\lambda} = \frac{1}{2} h$$

Thus, even though the electron prior to our observation may have had a definite position and momentum, our observation has unavoidably introduced an uncertainty in the measured values of those quantities. This illustrates the essential point of the uncertainty principle—that this product of uncertainties cannot be less than about h in principle, that is, even in an ideal situation. If electrons rather than photons were used to locate the object, the analysis would not change since the relation $\lambda = h/p$ is the same for both.

5-6 Some Consequences of the Uncertainty Principle

In the next chapter we will see that the Schrödinger wave equation provides a straightforward method of solving problems in atomic physics. However, the solution of the Schrödinger equation is often laborious and difficult. Much semiquantitative information about the behavior of atomic systems can be obtained from the uncertainty principle alone without a detailed solution of the problem. The general approach used in applying the uncertainty principle to such systems will first be illustrated by considering a particle moving in a box with rigid walls. We then use that analysis in several numerical examples and as a basis for discussing some additional consequences.

Minimum Energy of a Particle in a Box

An important consequence of the uncertainty principle is that a particle confined to a finite space cannot have zero kinetic energy. Let us consider the case of a one-dimensional “box” of length L . If we know that the particle is in the box, Δx is not larger than L . This implies that Δp is at least \hbar/L . (Since we are interested in orders of magnitude, we will ignore the $1/2$ in the minimum uncertainty product. In general, distributions are not Gaussian anyway, so $\Delta p \Delta x$ will be larger than $\hbar/2$.) Let us take the standard deviation as a measure of Δp ,

$$(\Delta p)^2 = (p - \bar{p})_{\text{av}}^2 = (p^2 - 2p\bar{p} + \bar{p}^2)_{\text{av}} = \overline{p^2} - \bar{p}^2$$

If the box is symmetric, \bar{p} will be zero since the particle moves to the left as often as to the right. Then

$$(\Delta p)^2 = \overline{p^2} \geq \left(\frac{\hbar}{L}\right)^2$$

and the average kinetic energy is

$$\overline{E} = \frac{\overline{p^2}}{2m} \geq \frac{\hbar^2}{2mL^2} \quad 5-28$$

Thus, we see that the uncertainty principle indicates that the minimum energy of a particle (*any* particle) in a “box” (*any* kind of “box”) cannot be zero. This minimum energy given by Equation 5-28 for a particle in a one-dimensional box is called the *zero point energy*.

EXAMPLE 5-6 A Macroscopic Particle in a Box Consider a small but macroscopic particle of mass $m = 10^{-6}$ g confined to a one-dimensional box with $L = 10^{-6}$ m, e.g., a tiny bead on a very short wire. Compute the bead’s minimum kinetic energy and the corresponding speed.

SOLUTION

1. The minimum kinetic energy is given by Equation 5-28:

$$\begin{aligned} \overline{E} &= \frac{\hbar^2}{2mL^2} = \frac{(1.055 \times 10^{-34} \text{ J} \cdot \text{s})^2}{(2)(10^{-9} \text{ kg})(10^{-6} \text{ m})^2} \\ &= 5.57 \times 10^{-48} \text{ J} \\ &= 3.47 \times 10^{-29} \text{ eV} \end{aligned}$$

2. The speed corresponding to this kinetic energy is:

$$\begin{aligned} v &= \sqrt{\frac{2\overline{E}}{m}} = \sqrt{\frac{2(5.57 \times 10^{-48} \text{ J})}{10^{-9} \text{ kg}}} \\ &= 1.06 \times 10^{-19} \text{ m/s} \end{aligned}$$

Remarks: We can see from this calculation that the minimum kinetic energy implied by the uncertainty principle is certainly not observable for macroscopic objects even as small as 10^{-6} g.

EXAMPLE 5-7 An Electron in an Atomic Box If the particle in a one-dimensional box of length $L = 0.1$ nm (about the diameter of an atom) is an electron, what will be its zero-point energy?

SOLUTION

Again using Equation 5-28, we find that

$$E = \frac{(\hbar c)^2}{2mc^2L^2} = \frac{(197.3 \text{ eV} \cdot \text{nm})^2}{2(0.511 \times 10^6 \text{ eV})(0.1 \text{ nm})^2} = 3.81 \text{ eV}$$

This is the correct order of magnitude for the kinetic energy of an electron in an atom.

Size of the Hydrogen Atom

The energy of an electron of momentum p a distance r from a proton is

$$E = \frac{p^2}{2m} - \frac{ke^2}{r}$$

If we take for the order of magnitude of the position uncertainty $\Delta x = r$, we have

$$(\Delta p)^2 = \overline{p^2} \geq \frac{\hbar^2}{r^2}$$

The energy is then

$$E = \frac{\hbar^2}{2mr^2} - \frac{ke^2}{r}$$

There is a radius r_m at which E is minimum. Setting $dE/dr = 0$ yields r_m and E_m :

$$r_m = \frac{\hbar^2}{ke^2m} = a_0 = 0.0529 \text{ nm}$$

and

$$E_m = -\frac{k^2e^4m}{2\hbar^2} = -13.6 \text{ eV}$$

The fact that r_m came out to be exactly the radius of the first Bohr orbit is due to the judicious choice of $\Delta x = r$ rather than $2r$ or $r/2$, which are just as reasonable. It should be clear, however, that any reasonable choice for Δx gives the correct order of magnitude of the size of an atom.

Widths of Spectral Lines

Equation 5-27 implies that the energy of a system cannot be measured exactly unless an infinite amount of time is available for the measurement. If an atom is in an excited state, it does not remain in that state indefinitely but makes transitions to lower energy states until it reaches the ground state. The decay of an excited state is a statistical process.

We can take the mean time for decay τ , called the *lifetime*, to be a measure of the time available to determine the energy of the state. For atomic transitions, τ is of the order of 10^{-8} s. The uncertainty in the energy corresponding to this time is

$$\Delta E \geq \frac{\hbar}{\tau} = \frac{6.58 \times 10^{-16} \text{ eV} \cdot \text{s}}{10^{-8} \text{ s}} \approx 10^{-7} \text{ eV}$$

This uncertainty in energy causes a spread $\Delta\lambda$ in the wavelength of the light emitted. For transitions to the ground state, which has a perfectly certain energy E_0 because of its infinite lifetime, the percentage spread in wavelength can be calculated from

$$\begin{aligned} E - E_0 &= \frac{hc}{\lambda} \\ dE &= -hc \frac{d\lambda}{\lambda^2} \\ |\Delta E| &\approx hc \frac{|\Delta\lambda|}{\lambda^2} \end{aligned}$$

thus,

$$\frac{\Delta\lambda}{\lambda} \approx \frac{\Delta E}{E - E_0}$$

The energy width $\Delta E = \hbar/\tau$ is called the *natural line width* and is represented by Γ_0 . Other effects that cause broadening of spectral lines are the Doppler effect, the recoil of the emitting atom, and atomic collisions. For optical spectra in the eV energy range, the Doppler width D is about 10^{-6} eV at room temperature, i.e., roughly 10 times the natural width Γ_0 , and the recoil width is negligible. For nuclear transitions in the MeV range, both the Doppler width and the recoil width are of the order of eV, much larger than the natural line width. We will see in Chapter 11 that in some special cases of atoms in solids at low temperatures, the Doppler and recoil widths are essentially zero and the width of the spectral line is just the natural width. This effect, called the *Mössbauer effect* after its discoverer, is extremely important since it provides photons of well-defined energy that are useful in experiments demanding extreme precision. For example, the 14.4-keV photon from ^{57}Fe has a natural width of the order of 10^{-11} of its energy.

Questions

7. What happens to the zero-point energy of a particle in a one-dimensional box as the length of the box $L \rightarrow \infty$?
8. Why is the uncertainty principle not apparent for macroscopic objects?

EXAMPLE 5-8 Emission of a Photon Most excited atomic states decay, i.e., emit a photon, within about $\tau = 10^{-8}$ s following excitation. What is the minimum uncertainty in the (1) energy and (2) frequency of the emitted photon?

SOLUTION

1. The minimum energy uncertainty is the natural line width $\Gamma_0 = \hbar/\tau$; therefore,

$$\Gamma_0 = \frac{6.63 \times 10^{-34} \text{ J} \cdot \text{s}}{2\pi \times 10^{-8} \text{ s}} = \frac{4.14 \times 10^{-15} \text{ eV} \cdot \text{s}}{2\pi \times 10^{-8} \text{ s}} = 6.6 \times 10^{-8} \text{ eV}$$

2. From de Broglie's relation $E = \hbar\omega$ we have

$$\Delta E = \hbar\Delta\omega = \hbar(2\pi\Delta f) = h\Delta f$$

so that Equation 5-27 can be written as

$$\Delta E\Delta t = h\Delta f\Delta t \geq \hbar$$

and the minimum uncertainty in the frequency becomes

$$\Delta f \geq \frac{1}{2\pi\Delta t} = \frac{1}{2\pi \times 10^{-8}} \\ \Delta f \geq 1.6 \times 10^7 \text{ Hz}$$

Remarks: The frequency of photons in the visible region of the spectrum is of the order of 10^{14} Hz.

5-7 Wave-Particle Duality

We have seen that electrons, which were once thought of as simply particles, exhibit the wave properties of diffraction and interference. In earlier chapters we saw that light, which we previously had thought of as a wave, also has particle properties in its interaction with matter, as in the photoelectric effect or the Compton effect. All phenomena—electrons, atoms, light, sound—have both particle and wave characteristics. It is sometimes said that an electron, for example, behaves both as a wave and a particle. This may seem confusing since, in classical physics, the concepts of waves and particles are mutually exclusive. A *classical particle* behaves like a pellet or BB shot from an air-powered rifle. It can be localized and scattered, it exchanges energy suddenly in a lump, and it obeys the laws of conservation of energy and momentum in collisions, but it does *not* exhibit interference and diffraction. A *classical wave* behaves like a water wave. It exhibits diffraction and interference patterns and has its energy spread out continuously in space and time, not quantized in lumps. Nothing, it was thought, could be both a classical particle and a classical wave.

We now see that the classical concepts do not adequately describe either waves or particles. Both matter and radiation have both particle and wave aspects. When emission and absorption are being studied, it is the particle aspects that are dominant. When matter and radiation propagate through space, wave aspects dominate. Notice that emission and absorption are events characterized by exchange of energy and discrete locations. For example, light strikes the retina of your eye and a photon is absorbed, transferring its energy to a particular rod or cone: an observation has occurred. This illustrates the point that *observations* of matter and radiation are described in terms of the particle aspects. On the other hand, predicting the intensity distribution of the light on your retina involves consideration of the amplitudes of waves that have propagated through space and been diffracted at the pupil. Thus, *predictions*, i.e., a priori statements about what may be observed, are described in terms of the wave aspects. Let's elaborate on this just a bit.

Every phenomenon is describable by a wave function that is the solution of a wave equation. The wave function for light is the electric field $\mathcal{E}(x, t)$ (in one space dimension), which is the solution of a wave equation like Equation 5-11. We have called the wave function for an electron $\Psi(x, t)$. We will study the wave equation of which Ψ is the solution, called the *Schrödinger equation*, in the next chapter. The magnitude squared of the wave function gives the probability per unit volume that the electron, if looked for, will be found in a given volume or region. The wave function exhibits the classical wave properties of interference and diffraction. In order to predict where an electron, or other particle, is likely to be, we must find the wave function by methods similar to those of classical wave theory. When the electron (or light) interacts and exchanges energy and momentum, the wave function is changed by the interaction. The interaction can be described by classical particle theory, as is done in the Compton effect. There are times when classical particle theory and classical wave theory give the same results. *If the wavelength is much smaller than any object or aperture, particle theory can be used as well as wave theory to describe wave propagation because diffraction and interference effects are too small to be observed.* Common examples are geometrical optics, which is really a particle theory, and the motion of baseballs and jet aircraft. If one is interested only in time averages of energy and momentum exchange, the wave theory works as well as the particle theory. For example, the wave theory of light correctly predicts that the total electron current in the photoelectric effect is proportional to the intensity of the light.



More

That matter can exhibit wavelike characteristics as well as particlelike behavior can be a difficult concept to understand. A wonderfully clear discussion of wave-particle duality was given by R. P. Feynman, and we have used it as the basis of our explanation on the home page of the *Two-Slit Interference Pattern* for electrons: whfreeman.com/tiplermodern/physics5e/. See also Figures 5-21 and 5-22 and Equation 5-29 here.

Summary

TOPIC	RELEVANT EQUATIONS AND REMARKS	
1. De Broglie relations	$f = E/h$	5-1
	$\lambda = h/p$	5-2
	Electrons and all other particles exhibit the wave properties of interference and diffraction	
2. Detecting electron waves		
Davisson and Germer	Showed that electron waves diffracted from a single Ni crystal according to Bragg's equation	
	$n\lambda = D \sin \varphi$	5-5
3. Wave packets		
Wave equation	$\frac{d^2y}{dx^2} = \frac{1}{v^2} \frac{d^2y}{dt^2}$	5-11
Uncertainty relations	$\Delta k \Delta x \sim 1$	5-17
	$\Delta \omega \Delta t \sim 1$	5-18
Wave speed	$v_p = f\lambda = \omega/k$	
Group (packet) speed	$v_g = \frac{d\omega}{dk} = v_p + k \frac{dv_p}{dk}$	5-16
Matter waves	The wave packet moves with the particle speed; i.e., the particle speed is the group speed v_g .	
4. Probabilistic interpretation	The magnitude square of the wave function is proportional to the probability of observing a particle in the region dx at x and t .	
	$P(x)dx = \Psi ^2 dx$	5-23
5. Heisenberg uncertainty principle	$\Delta x \Delta p \geq \frac{1}{2} \hbar$	5-26
	$\Delta E \Delta t \geq \frac{1}{2} \hbar$	5-27
	where each of the uncertainties is defined to be the standard deviation.	

TOPIC	RELEVANT EQUATIONS AND REMARKS
Particle in a box	$\bar{E} \geq \frac{\hbar^2}{2mL^2}$ <p>The minimum energy of any particle in any “box” cannot be zero.</p>
Energy of H atom	The Heisenberg principle predicts $E_{\min} = -13.6 \text{ eV}$ in agreement with the Bohr model.

General References

The following general references are written at a level appropriate for the readers of this book.

De Broglie, L., *Matter and Light: The New Physics*, Dover, New York, 1939. In this collection of studies is de Broglie’s lecture on the occasion of receiving the Nobel Prize, in which he describes his reasoning that led to the prediction of the wave nature of matter.

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Feynman, R. P., R. B. Leighton, and M. Sands, *Lectures on Physics*, Addison-Wesley, Reading, MA, 1965.

Fowles, G. R., *Introduction to Modern Optics*, Holt, Rinehart & Winston, New York, 1968.

Hecht, E., *Optics*, 2d ed., Addison-Wesley, Reading, MA, 1987.
Jenkins, F. A., and H. E. White, *Fundamentals of Optics*, 4th ed., McGraw-Hill, New York, 1976.

Mehra, J., and H. Rechenberg, *The Historical Development of Quantum Theory*, Vol. 1, Springer-Verlag, New York, 1982.

Resnick, R., and D. Halliday, *Basic Concepts in Relativity and Early Quantum Theory*, 2d ed., Wiley, New York, 1992.

Tipler, P. A., and G. Mosca, *Physics for Scientists and Engineers*, 6th ed., W. H. Freeman and Co., New York, 2008. Chapters 15 and 16 include a complete discussion of classical waves.

Notes

1. Louis V. P. R. de Broglie (1892–1987), French physicist. Originally trained in history, he became interested in science after serving as a radio engineer in the French army (assigned to the Eiffel Tower) and through the work of his physicist brother Maurice. The subject of his doctoral dissertation received unusual attention because his professor, Paul Langevin (who discovered the principle on which sonar is based), brought it to the attention of Einstein, who described de Broglie’s hypothesis to Lorentz as “the first feeble ray of light to illuminate . . . the worst of our physical riddles.” He received the Nobel Prize in Physics in 1929, the first person so honored for work done for a doctoral thesis.

2. L. de Broglie, *New Perspectives in Physics*, Basic Books, New York, 1962.

3. See, e.g., Tipler, *Physics for Scientists and Engineers*, 5th ed. (New York: W. H. Freeman and Co., 2008), Section 35-5.

4. Jean-Baptiste Perrin (1870–1942), French physicist. He was the first to show that cathode rays are actually charged particles, setting the stage for J. J. Thomson’s measurement of their q/m ratio. He was also the first to measure the approximate size of atoms and molecules and determined Avogadro’s number. He received the Nobel Prize in Physics for that work in 1926.

5. Clinton J. Davisson (1881–1958), American physicist. He shared the 1937 Nobel Prize in Physics with G. P. Thomson for demonstrating the diffraction of particles. Davisson’s Nobel Prize was the first ever awarded for work done somewhere other than at an academic institution. Germer was one of Davisson’s assistants at Bell Telephone Laboratory.

6. Matter (electron) waves, like other waves, change their direction in passing from one medium (e.g., Ni crystal) into another (e.g., vacuum) in the manner described by Snell’s law and the indices of refraction of the two media. For normal incidence Equation 5-5 is not affected, but for other incident angles it is altered a bit, and that change has not been taken into account in either Figure 5-6 or 5-7.

7. *Nobel Prize Lectures: Physics* (Amsterdam and New York: Elsevier, 1964).

8. In spectroscopy, the quantity $k = \lambda^{-1}$ is called the *wave number*. In the theory of waves, the term *wave number* is used for $k = 2\pi/\lambda$.

9. Following convention, the “width” is defined as the full width of the pulse or envelope measured at half the maximum amplitude.

10. This interpretation of $|\Psi|^2$ was first developed by German physicist Max Born (1882–1970). One of his positions early in his career was at the University of Berlin, where he was to relieve Planck of his teaching duties. Born received the Nobel Prize in Physics in 1954, in part for his interpretation of $|\Psi|^2$.

11. Werner K. Heisenberg (1901–1976), German physicist. After obtaining his Ph.D. under Sommerfeld, he served as an assistant to Born and to Bohr. He was the director of research for Germany's atomic bomb project during World War II. His work on quantum theory earned him the Nobel Prize in Physics in 1932.

12. The resolving power of a microscope is discussed in some detail in Jenkins and White, *Fundamentals of Optics*,

4th ed. (New York: McGraw-Hill, 1976), pp. 332–334. The expression for Δx used here is determined by Rayleigh's criterion, which states that two points are just resolved if the central maximum of the diffraction pattern from one falls at the first minimum of the diffraction pattern of the other.

13. Richard P. Feynman (1918–1988), American physicist. This discussion is based upon one in his classic text *Lectures on Physics* (Reading, MA: Addison-Wesley, 1965). He shared the 1965 Nobel Prize in Physics for his development of quantum electrodynamics (QED). It was Feynman who, while a member of the commission on the *Challenger* disaster, pointed out that the booster stage O-rings were at fault. A genuine legend in American physics, he was also an accomplished bongo drummer and safecracker.

Problems

Level I

Section 5-1 The de Broglie Hypothesis

5-1. (a) What is the de Broglie wavelength of a 1-g mass moving at a speed of 1 m per year?
(b) What should be the speed of such a mass if its de Broglie wavelength is to be 1 cm?

5-2. If the kinetic energy of a particle is much greater than its rest energy, the relativistic approximation $E \approx pc$ holds. Use this approximation to find the de Broglie wavelength of an electron of energy 100 MeV.

5-3. Electrons in an electron microscope are accelerated from rest through a potential difference V_0 so that their de Broglie wavelength is 0.04 nm. What is V_0 ?

5-4. Compute the de Broglie wavelengths of (a) an electron, (b) a proton, and (c) an alpha particle of 4.5-keV kinetic energy.

5-5. According to statistical mechanics, the average kinetic energy of a particle at temperature T is $3kT/2$, where k is the Boltzmann constant. What is the average de Broglie wavelength of nitrogen molecules at room temperature?

5-6. Find the de Broglie wavelength of a neutron of kinetic energy 0.02 eV (this is of the order of magnitude of kT at room temperature).

5-7. A free proton moves back and forth between rigid walls separated by a distance $L = 0.01$ nm.
(a) If the proton is represented by a one-dimensional standing de Broglie wave with a node at each wall, show that the allowed values of the de Broglie wavelength are given by $\lambda = 2L/n$ where n is a positive integer.
(b) Derive a general expression for the allowed kinetic energy of the proton and compute the values for $n = 1$ and 2.

5-8. What must be the kinetic energy of an electron if the ratio of its de Broglie wavelength to its Compton wavelength is (a) 10^2 , (b) 0.2, and (c) 10^{-3} ?

5-9. Compute the wavelength of a cosmic-ray proton whose kinetic energy is (a) 2 GeV and (b) 200 GeV.

Section 5-2 Measurements of Particle Wavelengths

5-10. What is the Bragg scattering angle ϕ for electrons scattered from a nickel crystal if their energy is (a) 75 eV, (b) 100 eV?

5-11. Compute the kinetic energy of a proton whose de Broglie wavelength is 0.25 nm. If a beam of such protons is reflected from a calcite crystal with crystal plane spacing of 0.304 nm, at what angle will the first-order Bragg maximum occur?

5-12. (a) The scattering angle for 50-eV electrons from MgO is 55.6° . What is the crystal spacing D ? (b) What would be the scattering angle for 100-eV electrons?

5-13. A certain crystal has a set of planes spaced 0.30 nm apart. A beam of neutrons strikes the crystal at normal incidence and the first maximum of the diffraction pattern occurs at $\varphi = 42^\circ$. What are the de Broglie wavelength and kinetic energy of the neutrons?

5-14. Show that in Davisson and Germer's experiment with 54-eV electrons using the $D = 0.215$ nm planes, diffraction peaks with $n = 2$ and higher are not possible.

5-15. A beam of electrons with kinetic energy 350 eV is incident normal to the surface of a KCl crystal that has been cut so that the spacing D between adjacent atoms in the planes parallel to the surface is 0.315 nm. Calculate the angle φ at which diffraction peaks will occur for all orders possible.

Section 5-3 Wave Packets

5-16. Information is transmitted along a cable in the form of short electric pulses at 100,000 pulses/s. (a) What is the longest duration of the pulses such that they do not overlap? (b) What is the range of frequencies to which the receiving equipment must respond for this duration?

5-17. Two harmonic waves travel simultaneously along a long wire. Their wave functions are $y_1 = 0.002 \cos(8.0x - 400t)$ and $y_2 = 0.002 \cos(7.6x - 380t)$, where y and x are in meters and t in seconds. (a) Write the wave function for the resultant wave in the form of Equation 5-15. (b) What is the phase velocity of the resultant wave? (c) What is the group velocity? (d) Calculate the range Δx between successive zeros of the group and relate it to Δk .

5-18. (a) Starting from Equation 5-16, show that the group velocity can also be expressed as

$$v_g = v_p - \lambda(dv_p/d\lambda)$$

(b) The phase velocity of each wavelength of white light moving through ordinary glass is a function of the wavelength; i.e., glass is a dispersive medium. What is the general dependence of v_p on λ in glass? Is $dv_p/d\lambda$ positive or negative?

5-19. A radar transmitter used to measure the speed of pitched baseballs emits pulses of 2.0-cm wavelength that are $0.25 \mu\text{s}$ in duration. (a) What is the length of the wave packet produced? (b) To what frequency should the receiver be tuned? (c) What must be the minimum bandwidth of the receiver?

5-20. A certain standard tuning fork vibrates at 880 Hz. If the tuning fork is tapped, causing it to vibrate, then stopped a quarter of a second later, what is the approximate range of frequencies contained in the sound pulse that reached your ear?

5-21. If a phone line is capable of transmitting a range of frequencies $\Delta f = 5000$ Hz, what is the approximate duration of the shortest pulse that can be transmitted over the line?

5-22. (a) You are given the task of constructing a double slit experiment for 5-eV electrons. If you wish the first minimum of the diffraction pattern to occur at 5° , what must be the separation of the slits? (b) How far from the slits must the detector plane be located if the first minima on each side of the central maximum are to be separated by 1 cm?

Section 5-4 The Probabilistic Interpretation of the Wave Function

5-23. A 100-g rigid sphere of radius 1 cm has a kinetic energy of 2 J and is confined to move in a force-free region between two rigid walls separated by 50 cm. (a) What is the probability of finding the center of the sphere exactly midway between the two walls? (b) What is the probability of finding the center of the sphere between the 24.9- and 25.1-cm marks?

5-24. A particle moving in one dimension between rigid walls separated by a distance L has the wave function $\Psi(x) = A \sin(\pi x/L)$. Since the particle must remain between the walls, what must be the value of A ?

5-25. The wave function describing a state of an electron confined to move along the x axis is given at time zero by

$$\Psi(x, 0) = Ae^{-x^2/4\sigma^2}$$

Find the probability of finding the electron in a region dx centered at (a) $x = 0$, (b) $x = \sigma$, and (c) $x = 2\sigma$. (d) Where is the electron most likely to be found?

Section 5-5 The Uncertainty Principle

5-26. A tuning fork of frequency f_0 vibrates for a time Δt and sends out a waveform that looks like that in Figure 5-23. This wave function is similar to a harmonic wave except that it is confined to a time Δt and space $\Delta x = v\Delta t$, where v is the phase velocity. Let N be the approximate number of cycles of vibration. We can measure the frequency by counting the cycles and dividing by Δt . (a) The number of cycles is uncertain by approximately ± 1 cycle. Explain why (see the figure). What uncertainty does this introduce in the determination of the frequency f ? (b) Write an expression for the wave number k in terms of Δx and N . Show that the uncertainty in N of ± 1 leads to an uncertainty in k of $\Delta k = 2\pi/\Delta x$.

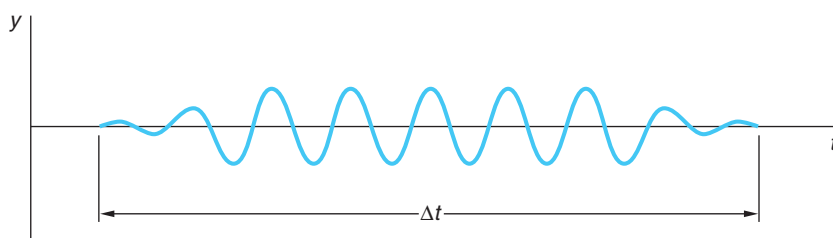


Figure 5-23

5-27. If an excited state of an atom is known to have a lifetime of 10^{-7} s, what is the uncertainty in the energy of photons emitted by such atoms in the spontaneous decay to the ground state?

5-28. A ladybug 5 mm in diameter with a mass of 1.0 mg being viewed through a low-power magnifier with a calibrated reticule is observed to be stationary with an uncertainty of 10^{-2} mm. How fast might the ladybug actually be walking?

5-29. ^{222}Rn decays by the emission of an α particle with a lifetime of 3.823 days. The kinetic energy of the α particle is measured to be 5.490 MeV. What is the uncertainty in this energy? Describe in one sentence how the finite lifetime of the excited state of the radon nucleus translates into an energy uncertainty for the emitted α particle.

5-30. If the uncertainty in the position of a wave packet representing the state of a quantum-system particle is equal to its de Broglie wavelength, how does the uncertainty in momentum compare with the value of the momentum of the particle?

5-31. In one of G. Gamow's Mr. Tompkins tales, the hero visits a "quantum jungle" where \hbar is very large. Suppose that you are in such a place where $\hbar = 50 \text{ J} \cdot \text{s}$. A cheetah runs past you a few meters away. The cheetah is 2 m long from nose to tail tip and its mass is 30 kg. It is moving at 40 m/s. What is the uncertainty in the location of the "midpoint" of the cheetah? Describe in one sentence how the cheetah would look different to you than when \hbar has its actual value.

5-32. In order to locate a particle, e.g., an electron, to within 5×10^{-12} m using electromagnetic waves ("light"), the wavelength must be at least this small. Calculate the momentum and energy of a photon with $\lambda = 5 \times 10^{-12}$ m. If the particle is an electron with $\Delta x = 5 \times 10^{-12}$ m, what is the corresponding uncertainty in its momentum?

5-33. The decay of excited states in atoms and nuclei often leave the system in another, albeit lower-energy, excited state. (a) One example is the decay between two excited states of the nucleus of ^{48}Ti . The upper state has a lifetime of 1.4 ps, the lower state 3.0 ps. What is the fractional uncertainty $\Delta E/E$ in the energy of 1.3117-MeV gamma rays connecting the two states?

(a) Another example is the H_α line of the hydrogen Balmer series. In this case the lifetime of both states is about the same, 10^{-8} s. What is the uncertainty in the energy of the H_α photon?

5-34. Laser pulses of femtosecond duration can be produced but for such brief pulses it makes no sense to speak of the pulse's color. To see this, compute the time duration of a laser pulse whose range of frequencies covers the entire visible spectrum (4.0×10^{14} Hz to 7.5×10^{14} Hz).

Section 5-6 Some Consequences of the Uncertainty Principle

5-35. A neutron has a kinetic energy of 10 MeV. What size object is necessary to observe neutron diffraction effects? Is there anything in nature of this size that could serve as a target to demonstrate the wave nature of 10-MeV neutrons?

5-36. Protons and neutrons in nuclei are bound to the nucleus by exchanging pions (π mesons) with each other (see Chapter 11). This is possible to do without violating energy conservation provided the pion is reabsorbed within a time consistent with the Heisenberg uncertainty relations. Consider the emission reaction $p \rightarrow p + \pi$ where $m_\pi = 135 \text{ MeV}/c^2$. (a) Ignoring kinetic energy, by how much is energy conservation violated in this reaction? (b) Within what time interval must the pion be reabsorbed in order to avoid violation of energy conservation?

5-37. Show that the relation $\Delta p_s \Delta s > \hbar$ can be written $\Delta L \Delta \phi > \hbar$ for a particle moving in a circle about the z axis, where p_s is the linear momentum tangential to the circle, s is the arc length, and L is the angular momentum. How well can the angular position of the electron be specified in the Bohr atom?

5-38. An excited state of a certain nucleus has a half-life of 0.85 ns. Taking this to be the uncertainty Δt for emission of a photon, calculate the uncertainty in the frequency Δf , using Equation 5-25. If $\lambda = 0.01 \text{ nm}$, find $\Delta f/f$.

5-39. The lifetimes of so-called resonance particles cannot be measured directly but is computed from the energy width (or uncertainty) of the scattering cross section versus energy graph (see Chapter 12). For example, the scattering of a pion (π meson) and a proton can produce a short-lived Δ resonance particle with a mass of $1685 \text{ MeV}/c^2$ and an energy width of 250 MeV as shown in Figure 5-24: $\pi + p \rightarrow \Delta$. Compute the lifetime of the Δ .

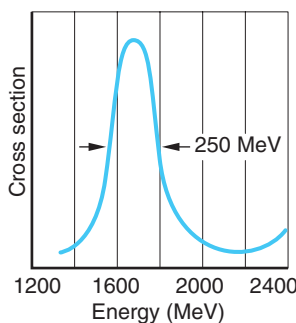


Figure 5-24

Section 5-7 Wave-Particle Duality

There are no problems for this section.

Level II

5-40. Neutrons and protons in atomic nuclei are confined within a region whose diameter is about 10^{-15} m. (a) At any given instant, how fast might an individual proton or neutron be moving? (b) What is the approximate kinetic energy of a neutron that is localized to within such a region? (c) What would be the corresponding energy of an electron localized to within such a region?

5-41. Using the relativistic expression $E^2 = p^2 c^2 + m^2 c^4$, (a) show that the phase velocity of an electron wave is greater than c . (b) Show that the group velocity of an electron wave equals the particle velocity of the electron.

5-42. Show that if y_1 and y_2 are solutions of Equation 5-11, the function $y_3 = C_1 y_1 + C_2 y_2$ is also a solution for any values of the constants C_1 and C_2 .

5-43. The London “Bobby” whistle has a frequency of 2500 Hz. If such a whistle is given a 3.0-s blast, (a) what is the uncertainty in the frequency? (b) How long is the wave train of this blast? (c) What would be the uncertainty in measuring the wavelength of this blast? (d) What is the wavelength of this blast?

5-44. A particle of mass m moves in a one-dimensional box of length L . (Take the potential energy of the particle in the box to be zero so that its total energy is its kinetic energy

$p^2/2m$.) Its energy is quantized by the standing-wave condition $n(\lambda/2) = L$, where λ is the de Broglie wavelength of the particle and n is an integer. (a) Show that the allowed energies are given by $E_n = n^2 E_1$ where $E_1 = h^2/8mL^2$. (b) Evaluate E_n for an electron in a box of size $L = 0.1$ nm and make an energy-level diagram for the state from $n = 1$ to $n = 5$. Use Bohr's second postulate $f = \Delta E/h$ to calculate the wavelength of electromagnetic radiation emitted when the electron makes a transition from (c) $n = 2$ to $n = 1$, (d) $n = 3$ to $n = 2$, and (e) $n = 5$ to $n = 1$.

5-45. (a) Use the results of Problem 5-44 to find the energy of the ground state ($n = 1$) and the first two excited states of a proton in a one-dimensional box of length $L = 10^{-15}$ m = 1 fm. (These are of the order of magnitude of nuclear energies.) Calculate the wavelength of electromagnetic radiation emitted when the proton makes a transition from (b) $n = 2$ to $n = 1$, (c) $n = 3$ to $n = 2$, and (d) $n = 3$ to $n = 1$.

5-46. (a) Suppose that a particle of mass m is constrained to move in a one-dimensional space between two infinitely high barriers located A apart. Using the uncertainty principle, find an expression for the zero-point (minimum) energy of the particle. (b) Using your result from (a), compute the minimum energy of an electron in such a space if $A = 10^{-10}$ m and $A = 1$ cm. (c) Calculate the minimum energy for a 100-mg bead moving on a thin wire between two stops located 2 cm apart.

5-47. A proton and a 10-g bullet each move with a speed of 500 m/s, measured with an uncertainty of 0.01 percent. If measurements of their respective positions are made simultaneous with the speed measurements, what is the minimum uncertainty possible in the position measurements?

Level III

5-48. Show that Equation 5-11 is satisfied by $y = f(\varphi)$, where $\varphi = x - vt$ for any function f .

5-49. An electron and a positron are moving toward each other with equal speeds of 3×10^6 m/s. The two particles annihilate each other and produce two photons of equal energy. (a) What were the de Broglie wavelengths of the electron and positron? Find the (b) energy, (c) momentum, and (d) wavelength of each photon.

5-50. It is possible for some fundamental particles to “violate” conservation of energy by creating and quickly reabsorbing another particle. For example, a proton can emit a π^+ according to $p = n + \pi^+$ where the n represents a neutron. The π^+ has a mass of $140 \text{ MeV}/c^2$. The reabsorption must occur within a time Δt consistent with the uncertainty principle. (a) Considering the example shown, by how much ΔE is energy conservation violated? (Ignore kinetic energy.) (b) For how long Δt can the π^+ exist? (c) Assuming that the π^+ is moving at nearly the speed of light, how far from the nucleus could it get in the time Δt ? (As we will discuss in Chapter 11, this is the approximate range of the strong nuclear force.) (d) Assuming that as soon as one pion is reabsorbed another is emitted, how many pions would be recorded by a “nucleon camera” with a shutter speed of $1 \mu\text{s}$?

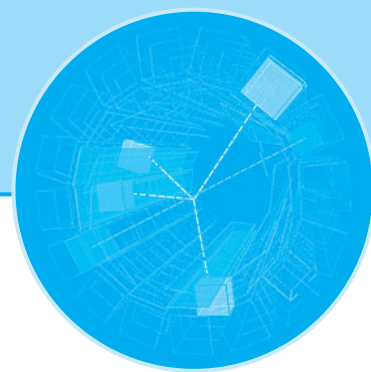
5-51. De Broglie developed Equation 5-2 initially for photons, assuming that they had a small but finite mass. His assumption was that RF waves with $\lambda = 30$ m traveled at a speed of at least 99 percent of that of visible light with $\lambda = 500$ nm. Beginning with the relativistic expression $hf = \gamma mc^2$, verify de Broglie's calculation that the upper limit of the rest mass of a photon is 10^{-44} g. (Hint: Find an expression for v/c in terms of hf and mc^2 , and then let $mc^2 \ll hf$) ($\gamma = 1/\sqrt{1 - v^2/c^2}$).

5-52. Suppose that you drop BBs onto a bull's-eye marked on the floor. According to the uncertainty principle, the BBs do not necessarily fall straight down from the release point to the center of the bull's-eye but are affected by the initial conditions. (a) If the location of the release point is uncertain by an amount Δx perpendicular to the vertical direction and the horizontal component of the speed is uncertain by Δv_x , derive an expression for the minimum spread ΔX

of impacts at the bull's-eye if it is located a distance y_0 below the release point. (b) Modify your result in (a) to include the effect on ΔX of uncertainties Δy and Δv_y at the release point.

5-53. Using the first-order Doppler-shift formula $f' = f_0(1 + v/c)$, calculate the energy shift of a 1-eV photon emitted from an iron atom moving toward you with energy $(3/2)kT$ at $T = 300$ K. Compare this Doppler line broadening with the natural line width calculated in Example 5-8. Repeat the calculation for a 1-MeV photon from a nuclear transition.

5-54. Calculate the order of magnitude of the shift in energy of a (a) 1-eV photon and (b) 1-MeV photon resulting from the recoil of an iron nucleus. Do this by first calculating the momentum of the photon and then by calculating $p^2/2m$ for the nucleus using that value of momentum. Compare with the natural line width calculated in Example 5-8.



The Schrödinger Equation

The success of the de Broglie relations in predicting the diffraction of electrons and other particles, and the realization that classical standing waves lead to a discrete set of frequencies, prompted a search for a wave theory of electrons analogous to the wave theory of light. In this electron wave theory, classical mechanics should appear as the short-wavelength limit, just as geometric optics is the short-wavelength limit of the wave theory of light. The genesis of the correct theory went something like this, according to Felix Bloch,¹ who was present at the time:

... in one of the next colloquia, Schrödinger gave a beautifully clear account of how de Broglie associated a wave with a particle and how he [i.e., de Broglie] could obtain the quantization rules ... by demanding that an integer number of waves should be fitted along a stationary orbit. When he had finished Debye² casually remarked that he thought this way of talking was rather childish ... [that to] deal properly with waves, one had to have a wave equation.

In 1926, Erwin Schrödinger³ published his now-famous wave equation, which governs the propagation of matter waves, including those of electrons. A few months earlier, Werner Heisenberg had published a seemingly different theory to explain atomic phenomena. In the Heisenberg theory, only measurable quantities appear. Dynamical quantities such as energy, position, and momentum are represented by matrices, the diagonal elements of which are the possible results of measurement. Though the Schrödinger and Heisenberg theories appear to be different, it was eventually shown by Schrödinger himself that they were equivalent, in that each could be derived from the other. The resulting theory, now called *wave mechanics* or *quantum mechanics*, has been amazingly successful. Though its principles may seem strange to us, whose experiences are limited to the macroscopic world, and though the mathematics required to solve even the simplest problem is quite involved, there seems to be no alternative to describe correctly the experimental results in atomic and nuclear physics. In this book we will confine our study to the Schrödinger theory because it is easier to learn and is a little less abstract than the Heisenberg theory. We will begin by restricting our discussion to problems in one space dimension.

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6-1 The Schrödinger Equation in One Dimension

The wave equation governing the motion of electrons and other particles with mass, which is analogous to the classical wave equation (Equation 5-11), was found by Schrödinger late in 1925 and is now known as the Schrödinger equation. Like the classical wave equation, the Schrödinger equation relates the time and space derivatives of the wave function. The reasoning followed by Schrödinger is somewhat difficult and not important for our purposes. In any case, it must be emphasized that we can't derive the Schrödinger equation just as we can't derive Newton's laws of motion. *Its validity, like that of any fundamental equation, lies in its agreement with experiment.* Just as Newton's second law is not relativistically correct, neither is Schrödinger's equation, which must ultimately yield to a relativistic wave equation. But, as you know, Newton's laws of motion are perfectly satisfactory for solving a vast array of nonrelativistic problems. So, too, will be Schrödinger's equation when applied to the equally extensive range of nonrelativistic problems in atomic, molecular, and solid-state physics. Schrödinger tried without success to develop a relativistic wave equation, a task accomplished in 1928 by Dirac.

Although it would be logical merely to postulate the Schrödinger equation, we can get some idea of what to expect by first considering the wave equation for photons, which is Equation 5-11 with speed $v = c$ and with $y(x, t)$ replaced by the electric field $\mathcal{E}(x, t)$.

$$\frac{\partial^2 \mathcal{E}}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \mathcal{E}}{\partial t^2} \quad 6-1$$

As discussed in Chapter 5, a particularly important solution of this equation is the harmonic wave function $\mathcal{E}(x, t) = \mathcal{E}_0 \cos(kx - \omega t)$. Differentiating this function twice, we obtain

$$\frac{\partial^2 \mathcal{E}}{\partial t^2} = -\omega^2 \mathcal{E}_0 \cos(kx - \omega t) = -\omega^2 \mathcal{E}(x, t)$$

and

$$\frac{\partial^2 \mathcal{E}}{\partial x^2} = -k^2 \mathcal{E}(x, t)$$

Substitution into Equation 6-1 then gives

$$-k^2 = -\frac{\omega^2}{c^2}$$

or

$$\omega = kc \quad 6-2$$

Using $\omega = E/\hbar$ and $p = \hbar k$ for electromagnetic radiation, we have

$$E = pc \quad 6-3$$

which, as we saw earlier, is the relation between the energy and momentum of a photon.



Erwin Schrödinger. [Courtesy of the Niels Bohr Library, American Institute of Physics.]

Now let us use the de Broglie relations for a particle such as an electron to find the relation between ω and k that is analogous to Equation 6-2 for photons. We can then use this relation to work backward and see how the wave equation for electrons must differ from Equation 6-1. The total energy (nonrelativistic) of a particle of mass m is

$$E = \frac{p^2}{2m} + V \quad 6-4$$

where V is the potential energy. Substituting the de Broglie relations (Equations 5-21 and 5-22) in Equation 6-4, we obtain

$$\hbar\omega = \frac{\hbar^2 k^2}{2m} + V \quad 6-5$$

This differs from Equation 6-2 for a photon because it contains the potential energy V and because the angular frequency ω does not vary linearly with k . Note that we get a factor of ω when we differentiate a harmonic wave function with respect to time and a factor of k when we differentiate with respect to position. We expect, therefore, that the wave equation that applies to electrons will relate the *first* time derivative to the *second* space derivative and will also involve the potential energy of the electron.

Finally, we require that the wave equation for electrons will be a differential equation that is linear in the wave function $\Psi(x, t)$. This ensures that, if $\Psi_1(x, t)$ and $\Psi_2(x, t)$ are both solutions of the wave equation for the same potential energy, then any arbitrary linear combination of these solutions is also a solution—i.e., $\Psi(x, t) = a_1\Psi_1(x, t) + a_2\Psi_2(x, t)$ is a solution, with a_1 and a_2 being arbitrary constants. Such a combination is called *linear* because both $\Psi_1(x, t)$ and $\Psi_2(x, t)$ appear only to the first power. Linearity guarantees that the wave functions will add together to produce constructive and destructive interference, which we have seen to be a characteristic of matter waves, as well as all other wave phenomena. Note, in particular, that (1) the linearity requirement means that *every* term in the wave equation must be linear in $\Psi(x, t)$ and (2) that *any* derivative of $\Psi(x, t)$ is linear in $\Psi(x, t)$.⁴

The Schrödinger Equation

We are now ready to postulate the Schrödinger equation for a particle of mass m . In one dimension, it has the form

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x, t)}{\partial x^2} + V(x, t) \Psi(x, t) = i\hbar \frac{\partial \Psi(x, t)}{\partial t} \quad 6-6$$

We will now show that this equation is satisfied by a harmonic wave function in the special case of a free particle, one on which no net force acts, so that the potential energy is constant, $V(x, t) = V_0$. First note that a function of the form $\cos(kx - \omega t)$ does not satisfy this equation because differentiation with respect to time changes the cosine to a sine, but the second derivative with respect to x gives back a cosine. Similar reasoning rules out the form $\sin(kx - \omega t)$. However, the exponential form of the harmonic wave function does satisfy the equation. Let

$$\begin{aligned} \Psi(x, t) &= Ae^{i(kx - \omega t)} \\ &= A[\cos(kx - \omega t) + i \sin(kx - \omega t)] \end{aligned} \quad 6-7$$

where A is a constant. Then

$$\frac{\partial \Psi}{\partial t} = -i\omega Ae^{i(kx - \omega t)} = -i\omega \Psi$$

and

$$\frac{\partial^2 \Psi}{\partial x^2} = (ik)^2 Ae^{i(kx - \omega t)} = -k^2 \Psi$$

Substituting these derivatives into the Schrödinger equation with $V(x, t) = V_0$ gives

$$\frac{-\hbar^2}{2m} (-k^2 \Psi) + V_0 \Psi = i\hbar (-i\omega) \Psi$$

or

$$\frac{\hbar^2 k^2}{2m} + V_0 = \hbar \omega$$

which is Equation 6-5.

An important difference between the Schrödinger equation and the classical wave equation is the explicit appearance⁵ of the imaginary number $i = (-1)^{1/2}$. The wave functions that satisfy the Schrödinger equation are not necessarily real, as we see from the case of the free-particle wave function of Equation 6-7. Evidently the wave function $\Psi(x, t)$ which solves the Schrödinger equation is not a directly measurable function as the classical wave function $y(x, t)$ is since measurements always yield real numbers. However, as we discussed in Section 5-4, the probability of

finding the electron in dx is certainly measurable, just as is the probability that a flipped coin will turn up heads. The probability $P(x)dx$ that the electron will be found in the volume dx was defined by Equation 5-23 to be equal to $|\Psi|^2 dx$. This probabilistic interpretation of Ψ was developed by Max Born and was recognized, over the early and formidable objections of both Schrödinger and Einstein, as the appropriate way of relating solutions of the Schrödinger equation to the results of physical measurements. The probability that an electron is in the region dx , a real number, can be measured by counting the fraction of time it is found there in a very large number of identical trials. In recognition of the complex nature of $\Psi(x, t)$, we must modify slightly the interpretation of the wave function discussed in Chapter 5 to accommodate Born's interpretation so that the probability of finding the electron in dx is real. We take for the probability

$$P(x, t) dx = \Psi^*(x, t)\Psi(x, t) dx = |\Psi(x, t)|^2 dx \quad 6-8$$

where Ψ^* , the complex conjugate of Ψ , is obtained from Ψ by replacing i with $-i$ wherever it appears.⁶ The complex nature of Ψ serves to emphasize the fact that, in reality, we should not ask or try to answer the question, "What is waving in a matter wave?" or inquire as to what medium supports the motion of a matter wave. The wave function is a computational device with utility in Schrödinger's theory of wave mechanics. Physical significance is associated not with Ψ itself, but with the product $\Psi^*\Psi = |\Psi|^2$, which is the probability distribution $P(x, t)$ or, as it is often called, the *probability density*. In keeping with the analogy with classical waves and wave functions, $\Psi(x, t)$ is also sometimes referred to as the *probability density amplitude*, or just the *probability amplitude*.

The probability of finding the electron in dx at x_1 or in dx at x_2 is the sum of separate probabilities, $P(x_1) dx + P(x_2) dx$. Since the electron must certainly be somewhere in space, the sum of the probabilities over all possible values of x must equal 1. That is⁷

$$\int_{-\infty}^{+\infty} \Psi^*\Psi dx = 1 \quad 6-9$$

Equation 6-9 is called the *normalization condition*. This condition plays an important role in quantum mechanics, for it places a restriction on the possible solutions of the Schrödinger equation. In particular, the wave function $\Psi(x, t)$ must approach zero sufficiently fast as $x \rightarrow \pm\infty$ so that the integral in Equation 6-9 remains finite. If it does not, then the probability becomes unbounded. As we will see in Section 6-3, it is this restriction together with boundary conditions imposed at finite values of x that leads to energy quantization for bound particles.

In the chapters that follow, we are going to be concerned with solutions to the Schrödinger equation for a wide range of real physical systems, but in what follows in this chapter, our intent is to illustrate a few of the techniques of solving the equation and to discover the various, often surprising properties of the solutions. To this end we will focus our attention on one-dimensional problems, as noted earlier, and use some potential energy functions with unrealistic physical characteristics, e.g., infinitely rigid walls, which will enable us to illustrate various properties of the solutions without obscuring the discussion with overly complex mathematics.

Separation of the Time and Space Dependencies of $\Psi(x, t)$

Schrödinger's first application of his wave equation was to problems such as the hydrogen atom (Bohr's work) and the simple harmonic oscillator (Planck's work), in which he showed that the energy quantization in those systems can be explained naturally in terms of standing waves. We referred to these in Chapter 4 as stationary states, meaning they did not change with time. Such states are also called *eigenstates*. For such problems that also have potential energy functions that are independent of time, the space and time dependence of the wave function can be separated, leading to a greatly simplified form of the Schrödinger equation.⁸ The separation is accomplished by first assuming that $\Psi(x, t)$ can be written as a product of two functions, one of x and one of t , as

$$\Psi(x, t) = \psi(x)\phi(t) \quad 6-10$$

If Equation 6-10 turns out to be incorrect, we will find that out soon enough, but it turns out that *if* the potential function is *not* an explicit function of time, i.e., if the potential is given by $V(x)$, our assumption turns out to be valid. That this is true can be seen as follows:

Substituting $\Psi(x, t)$ from Equation 6-10 into the general, time-dependent Schrödinger equation (Equation 6-6) yields

$$\frac{-\hbar^2}{2m} \frac{\partial^2 \psi(x)\phi(t)}{\partial x^2} + V(x)\psi(x)\phi(t) = i\hbar \frac{\partial \psi(x)\phi(t)}{\partial t} \quad 6-11$$

which is

$$\frac{-\hbar^2}{2m} \phi(t) \frac{d^2 \psi(x)}{dx^2} + V(x)\psi(x)\phi(t) = i\hbar \psi(x) \frac{d\phi(t)}{dt} \quad 6-12$$

where the derivatives are now ordinary rather than partial ones. Dividing Equation 6-12 by Ψ in the assumed product form $\psi\phi$ gives

$$\frac{-\hbar^2}{2m} \frac{1}{\psi(x)} \frac{d^2 \psi(x)}{dx^2} + V(x) = i\hbar \frac{1}{\phi(t)} \frac{d\phi(t)}{dt} \quad 6-13$$

Notice that each side of Equation 6-13 is a function of only one of the independent variables x and t . This means that, for example, changes in t cannot affect the value of the left side of Equation 6-13, and changes in x cannot affect the right side. Thus, both sides of the equation must be equal to the same constant C , called the *separation constant*, and we see that the assumption of Equation 6-10 is valid—the variables have been separated. We have thus replaced a partial differential equation containing two independent variables, Equation 6-6, with two ordinary differential equations each a function of only one of the independent variables:

$$\frac{-\hbar^2}{2m} \frac{1}{\psi(x)} \frac{d^2 \psi(x)}{dx^2} + V(x) = C \quad 6-14$$

$$i\hbar \frac{1}{\phi(t)} \frac{d\phi(t)}{dt} = C \quad 6-15$$

Let us solve Equation 6-15 first. The reason for doing so is twofold: (1) Equation 6-15 does not contain the potential $V(x)$; consequently, the time-dependent part $\phi(t)$ of *all* solutions $\Psi(x, t)$ to the Schrödinger equation will have the same form when the potential is not an explicit function of time, so we only have to do this once. (2) The separation constant C has particular significance that we want to discover before we tackle Equation 6-14. Writing Equation 6-15 as

$$\frac{d\phi(t)}{\phi(t)} = \frac{C}{i\hbar} dt = -\frac{iC}{\hbar} dt \quad 6-16$$

The general solution of Equation 6-16 is

$$\phi(t) = e^{-iCt/\hbar} \quad 6-17a$$

which can also be written as

$$\phi(t) = e^{-iCt/\hbar} = \cos\left(\frac{Ct}{\hbar}\right) - i \sin\left(\frac{Ct}{\hbar}\right) = \cos\left(2\pi \frac{Ct}{h}\right) - i \sin\left(2\pi \frac{Ct}{h}\right) \quad 6-17b$$

Thus, we see that $\phi(t)$, which describes the time variation of $\Psi(x, t)$, is an oscillatory function with frequency $f = C/h$. However, according to the de Broglie relation (Equation 5-1), the frequency of the wave represented by $\Psi(x, t)$ is $f = E/h$; therefore, we conclude that the separation constant $C = E$, the total energy of the particle, and we have

$$\phi(t) = e^{-iEt/\hbar} \quad 6-17c$$

for all solutions to Equation 6-6 involving time-independent potentials. Equation 6-14 then becomes, on multiplication by $\psi(x)$,

$$\frac{-\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x) \quad 6-18$$

Equation 6-18 is referred to as the *time-independent Schrödinger equation*.

The time-independent Schrödinger equation in one dimension is an ordinary differential equation in one variable x and is therefore much easier to handle than the general form of Equation 6-6. The normalization condition of Equation 6-9 can be expressed in terms of $\psi(x)$, since the time dependence of the absolute square of the wave function cancels. We have

$$\Psi^*(x, t)\Psi(x, t) = \psi^*(x)e^{+iEt/\hbar}\psi(x)e^{-iEt/\hbar} = \psi^*(x)\psi(x) \quad 6-19$$

and Equation 6-9 then becomes

$$\int_{-\infty}^{+\infty} \psi^*(x)\psi(x)dx = 1 \quad 6-20$$

Conditions for Acceptable Wave Functions

The form of the wave function $\psi(x)$ that satisfies Equation 6-18 depends on the form of the potential energy function $V(x)$. In the next few sections we will study some simple but important problems in which $V(x)$ is specified. Our example potentials will be approximations to real physical potentials, simplified to make calculations easier.

In some cases, the slope of the potential energy may be discontinuous, e.g., $V(x)$ may have one form in one region of space and another form in an adjacent region. (This is a useful mathematical approximation to real situations in which $V(x)$ varies rapidly over a small region of space, such as at the surface boundary of a metal.) The procedure in such cases is to solve the Schrödinger equation separately in each region of space and then require that the solutions join smoothly at the point of discontinuity.

Since the probability of finding a particle cannot vary discontinuously from point to point, the wave function $\psi(x)$ must be continuous.⁹ Since the Schrödinger equation involves the second derivative $d^2\psi/dx^2 = \psi''$, the first derivative ψ' (which is the slope) must also be continuous. That is, the graph of $\psi(x)$ versus x must be smooth. (In a special case in which the potential energy becomes infinite, this restriction is relaxed. Since no particle can have infinite potential energy, $\psi(x)$ must be zero in regions where $V(x)$ is infinite. Then, at the boundary of such a region, ψ' may be discontinuous.)

If either $\psi(x)$ or $d\psi/dx$ were not finite or not single valued, the same would be true of $\Psi(x, t)$ and $d\Psi/dx$. As we will shortly see, the predictions of wave mechanics regarding the results of measurements involve both of those quantities and would thus not necessarily predict finite or definite values for real physical quantities. Such results would not be acceptable since measurable quantities, such as angular momentum and position, are never infinite or multiple valued. A final restriction on the form of the wave function $\psi(x)$ is that in order to obey the normalization condition, $\psi(x)$ must approach zero sufficiently fast as $x \rightarrow \pm \infty$ so that normalization is preserved. For future reference, we may summarize the conditions that the wave function $\psi(x)$ must meet in order to be acceptable as follows:

1. $\psi(x)$ must exist and satisfy the Schrödinger equation.
2. $\psi(x)$ and $d\psi/dx$ must be continuous.
3. $\psi(x)$ and $d\psi/dx$ must be finite.
4. $\psi(x)$ and $d\psi/dx$ must be single valued.
5. $\psi(x) \rightarrow 0$ fast enough as $x \rightarrow \pm \infty$ so that the normalization integral, Equation 6-20, remains bounded.

Note that, given Equation 6-10, the acceptability conditions above ultimately apply to $\Psi(x, t)$.

Questions

1. Like the classical wave equation, the Schrödinger equation is linear. Why is this important?
2. There is no factor $i = (-1)^{1/2}$ in Equation 6-18. Does this mean that $\psi(x)$ must be real?
3. Why must the electric field $\mathcal{E}(x, t)$ be real? Is it possible to find a nonreal wave function that satisfies the classical wave equation?
4. Describe how the de Broglie hypothesis enters into the Schrödinger wave equation.
5. What would be the effect on the Schrödinger equation of adding a constant rest energy for a particle with mass to the total energy E in the de Broglie relation $f = E/h$?
6. Describe in words what is meant by normalization of the wave function.