Please upload your solution to Problem 3 to canvas for marking after the workshop.

Problem 1

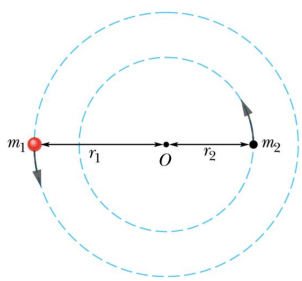
The radius R_h and mass M_h of a black hole are related by $R_h = 2GM_h/c^2$, where c is the speed of light. Assume that the gravitational acceleration a_g of an object at a distance $r_o = 1.001R_h$ from the centre of a black hole is given by $a_g = GM/r^2$ (it is, for large black holes).
(a) In terms of M_h, find a_g at r_o.
(b) If an astronaut of height 1.70 m is at r_o with her feet down, what is the difference in gravitational acceleration between her head and feet?

Problem 2

Zero, a hypothetical planet, has a mass of 5.0×10^{23} kg, a radius of 3.0×10^6 m, and no atmosphere. A 10 kg space probe is to be launched vertically from its surface. If the probe is launched with an initial energy of 5.0×10^7 J, what will be its kinetic energy when it is 4.0×10^6 m from the centre of Zero?

Problem 3

Observations of the light from a certain star indicate that it is part of a binary (two-star) system. This visible star has orbital speed $v = 270$ kms^{-1}, orbital period $T = 1.70$ days, and approximate mass $m_1 = 6M_s$, where M_s is the Sun's mass, 1.99×10^{30} kg. Assume that the visible star and its companion star, which is dark and unseen, are both in circular orbits, as in the figure shown below. What integer multiple of M_s gives the approximate mass m_2 of the dark star?



Problem 4

A satellite orbits a planet of unknown mass in a circle of radius 2.0×10^7 m. The magnitude of the gravitational force on the satellite from the planet is $F = 80$ N. What is the kinetic energy of the satellite in this orbit?

Want more practice?

Further problems on Newton's Gravitation: Chapter 13.1-13.3
Further problems on Gravitational PE: Chapter 13.4-13.5
Further problems on Kepler's Laws & Orbits: Chapter 13.6-13.7

Problem 1

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(a) In terms of M_h, find a_g at r_o.
(b) If an astronaut of height 1.70 m is at r_o with her feet down, what is the difference in gravitational acceleration between her head and feet?

$c = 3 \times 10^8 \ \mathrm{ms^{-1}}$

$G = 6.67 \times 10^{-11} \ \mathrm{m^3\,kg^{-1}\,s^{-2}}$

$$a_g = \frac{GM}{r^2} = \frac{GM_h}{r_o^2} = \frac{GM_h}{(1.001R_h)^2} = \frac{GM_h}{\left[(1.001)(2)GM_h\left(\frac{1}{c^2}\right)\right]^2} = \frac{1}{GM_h}\frac{c^4}{(2.002)^2}$$

a) $\boxed{a_g = (3.02 \times 10^{43} \ \mathrm{kg\,ms^{-2}})M_h^{-1}}$

b) $$\frac{d a_g}{dr} = \frac{d}{dr} GMr^{-2} = GM(-2r^{-3})$$

$$\therefore da_g = -2GM\frac{1}{\left(2.002GM_h\left(\frac{1}{c^2}\right)\right)^3}dr = \frac{-2}{(GM_h)^2}\frac{c^6}{(2.002)^3}dr$$

and $dr = 1.70\mathrm{m}$ gives $da_g = (-6.94 \times 10^{70} \ \mathrm{kg\,ms^{-2}})M_h^{-2}$

ANALYSIS

a_g at $r_o \sim (3 \times 10^{43} \ \mathrm{kg\,ms^{-2}})M_h^{-1}$: More massive → less a_g !

$da_g \sim (-7 \times 10^{70} \ \mathrm{kg\,ms^{-2}})M_h^{-2}$: there is M^2 in the denominator which means the change in a_g with r_o is a "SECOND-ORDER" effect.

If we let $M_h = 3 \times 10^{42}$ kg, a_g at $r_o \approx 10 \ \mathrm{ms^{-2}}$.

And $$da_g \approx \frac{7 \times 10^{70}}{(3 \times 10^{42})^2} \approx \frac{7}{9} \times 10^{(70-84)} \approx 0.77.. \times 10^{-14} \ \mathrm{ms^{-2}}$$

VERY SMALL DIFFERENCE !

GRAVITATIONAL PE

Problem 2

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$E_{TOT,i} = E_{TOT,f}$ $\quad G = 6.67 \times 10^{-11}\ m^3\,kg^{-1}\,s^{-2}$

$(KE + PE)_i = (KE + PE)_f$

PE of the planet + probe system: $PE = -\dfrac{G M_Z M_P}{r_{ZP}}$

Initially,

$r_{ZP} = 3 \times 10^{6}\ m$: probe on surface

So, $$PE_i = -\frac{(6.67 \times 10^{-11}\ m^3\,kg^{-1}\,s^{-2})(5 \times 10^{23}\ kg)(10\ kg)}{(3 \times 10^{6}\ m)}$$

$$= -1.11 \times 10^{8}\ m^2\,kg\,s^{-2}$$

$PE_i = -1.1 \times 10^{8}\ J$

and initial $KE_i = 5 \times 10^{7}\ J$,

$KE_i = 5 \times 10^{7}\ J$.

Finally (at 4×10^{6} m from centre of zero)

$PE_f = \dfrac{3}{4} PE_i = -8.34 \times 10^{7}\ m^2\,kg\,s^{-2}$ $\quad PE_f = -8.3 \times 10^{7}\ J$

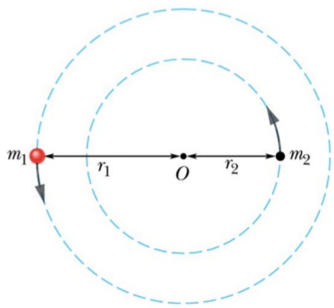
$$KE_f = PE_i + KE_i - PE_f = (-11 + 5 + 8.3) \times 10^{7}\ J$$

$$\boxed{KE_f = 2.3 \times 10^{7}\ J}$$

Problem 3

KEPLER

Observations of the light from a certain star indicate that it is part of a binary (two-star) system. This visible star has orbital speed $v = 270\ \mathrm{kms^{-1}}$, orbital period $T = 1.70$ days, and approximate mass $m_1 = 6M_s$, where M_s is the Sun's mass, 1.99×10^{30} kg. Assume that the visible star and its companion star, which is dark and unseen, are both in circular orbits, as in the figure shown below. What integer multiple of M_s gives the approximate mass m_2 of the dark star?



$$F_{12} = \frac{G M_1 M_2}{r_{12}^2} = \frac{M_1 V_1^2}{r_1}$$

HINT! FIND C.O.M. REL. TO VISIBLE STAR.

$$r_{COM} = \frac{M_1 r_1 + M_2 r_2}{M_1 + M_2} = \frac{M_2 r}{M_1 + M_2} \quad \therefore r = \frac{r_{COM}(M_1 + M_2)}{M_2}$$

We can also find r_{COM} from V, T:

$$V_{com} = \frac{2\pi}{T} r_{com} \quad \therefore r_{com} = \frac{V_{com} T}{2\pi} \quad \therefore r = \frac{vT}{2\pi} \frac{(M_1 + M_2)}{M_2}.$$

So, $$F_{12} = \frac{G \cancel{M_1} M_2 \, 4\pi^{\cancel{2}} M_2^2}{V^2 T^{\cancel{2}} (M_1 + M_2)^2} = \frac{\cancel{M_1} V^{\cancel{2}} \, 2\cancel{\pi}}{\cancel{V}\cancel{T}}$$

$$\frac{M_2^3}{(M_1 + M_2)^2} = \frac{2 v^3 T}{4\pi G} \quad \text{and} \quad M_1 = 6 M_s$$

So write $M_2 = K M_s$ and solve for k:

LHS: $$\frac{(K M_s)^3}{(6 + K)^2 M_s^2} = \frac{K^3}{(6+K)^2} M_s$$

RHS: $$\frac{2 v^3 T}{4\pi G} = 6.90 \times 10^{30}\ \mathrm{kg} = 3.47\ M_s$$

$$\frac{K^3}{(6+K)^2} - 3.47 = 0 \quad \text{gives} \quad K_{real} = 9.3, \quad \boxed{M_2 = 9.3\ M_s}$$