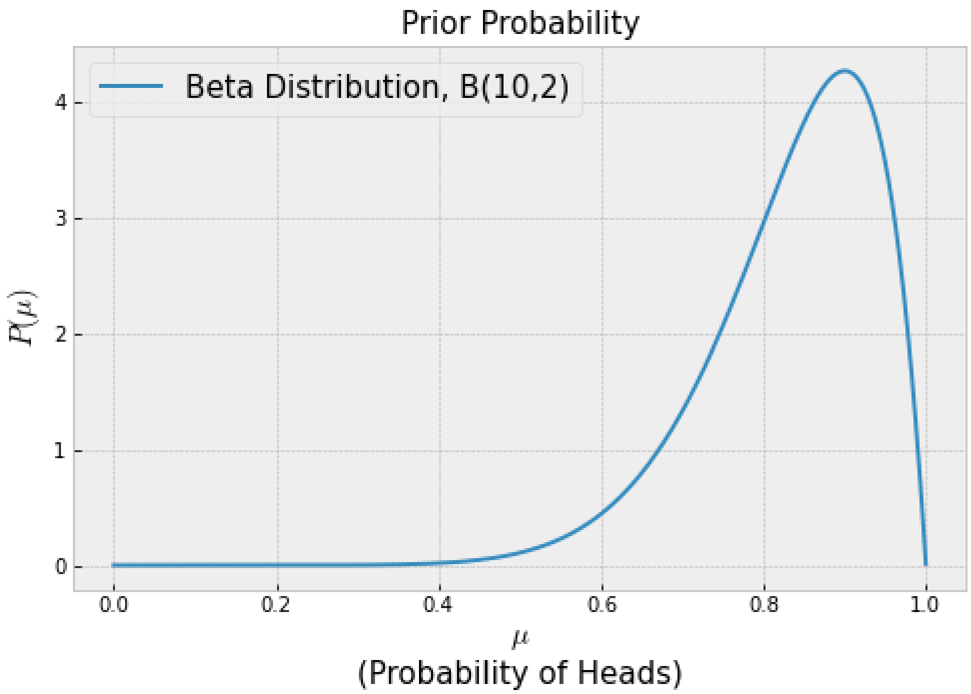
Lecture 12 - Conjugate Priors; Online Update

In [2]:

```
import numpy as np
import scipy.stats as stats
import math
from scipy.stats import multivariate_normal
import textwrap

import matplotlib.pyplot as plt
%matplotlib inline
plt.style.use('bmh')
```

In [8]:

```
trueMU = 0.5 # 0.5 for a fair coin
Nflips = 10
a = 10
b = 2
x = np.arange(0,1,0.0001)
Beta = (math.gamma(a+b)/(math.gamma(a)*math.gamma(b)))*x**(a-1)*(1-x)**(b-1)

plt.figure(figsize=(8,5))
plt.plot(x, Beta, label='Beta Distribution, B('+str(a)+','+str(b)+')')
plt.legend(loc='best',fontsize=15)
plt.xlabel('$\mu$\n(Probability of Heads)',size=15)
plt.ylabel('$P(\mu)$',size=15);
plt.title('Prior Probability', size=15)
plt.show();

Outcomes = []
for i in range(Nflips):
    Outcomes += [stats.bernoulli(trueMU).rvs(1)[0]]
    print(Outcomes)
    print('MLE aka Frequentist Probability of Heads = ', np.sum(Outcomes)/len(Outcomes)
    print('MAP aka Bayesian Probability of Heads = ', (np.sum(Outcomes)+a-1)/(len(Outco
    input('Press enter to flip the coin again...\n')
```




```
[1]
MLE aka Frequentist Probability of Heads =  1.0
MAP aka Bayesian Probability of Heads =  0.9090909090909091
Press enter to flip the coin again...

[1, 0]
MLE aka Frequentist Probability of Heads =  0.5
MAP aka Bayesian Probability of Heads =  0.8333333333333334
Press enter to flip the coin again...

[1, 0, 1]
MLE aka Frequentist Probability of Heads =  0.6666666666666666
MAP aka Bayesian Probability of Heads =  0.8461538461538461
Press enter to flip the coin again...

[1, 0, 1, 1]
MLE aka Frequentist Probability of Heads =  0.75
MAP aka Bayesian Probability of Heads =  0.8571428571428571
Press enter to flip the coin again...

[1, 0, 1, 1, 1]
MLE aka Frequentist Probability of Heads =  0.8
MAP aka Bayesian Probability of Heads =  0.8666666666666667
Press enter to flip the coin again...

[1, 0, 1, 1, 1, 0]
MLE aka Frequentist Probability of Heads =  0.6666666666666666
MAP aka Bayesian Probability of Heads =  0.8125
Press enter to flip the coin again...

[1, 0, 1, 1, 1, 0, 0]
MLE aka Frequentist Probability of Heads =  0.5714285714285714
MAP aka Bayesian Probability of Heads =  0.7647058823529411
Press enter to flip the coin again...

[1, 0, 1, 1, 1, 0, 0, 0]
MLE aka Frequentist Probability of Heads =  0.5
MAP aka Bayesian Probability of Heads =  0.7222222222222222
```

```
Press enter to flip the coin again...

[1, 0, 1, 1, 1, 0, 0, 0, 0]
MLE aka Frequentist Probability of Heads =  0.4444444444444444
MAP aka Bayesian Probability of Heads =  0.6842105263157895
Press enter to flip the coin again...

[1, 0, 1, 1, 1, 0, 0, 0, 0, 1]
MLE aka Frequentist Probability of Heads =  0.5
MAP aka Bayesian Probability of Heads =  0.7
Press enter to flip the coin again...
```

In [ ]:

Maximum Likelihood Estimation (MLE)

(Frequentist approach)

$$\arg_{\mathrm{w}} \max \mathrm{P}(\mathrm{t}|\mathrm{w})$$

In **Maximum Likelihood Estimation** we *find the set of parameters* that **maximize** the data likelihood $\mathrm{P}(\mathrm{t}|\mathrm{w})$. We find the *optimal* set of parameters under some assumed distribution such that the data is most likely.

- MLE focuses on maximizing the data likelihood, which *usually* provides a pretty good estimate
- A common trick to maximize the data likelihood is to maximize the log likelihood
- MLE is purely data driven
- MLE works best when we have lots and lots of data
- MLE will likely overfit when we have small amounts of data or, at least, becomes unreliable
- It estimates relative frequency for our model parameters. Therefore it needs incredibly large amounts of data (infinite!) to estimate the true likelihood parameters
    - This is a problem when we want to make inferences and/or predictions outside the range of what the training data has learned

Maximum A Posteriori (MAP)

(Bayesian approach)

$$\begin{aligned} &\arg_{\mathrm{w}} \max \mathrm{P}(\mathrm{t}|\mathrm{w})\mathrm{P}(\mathrm{w}) && (1) \\ \propto\ &\arg_{\mathrm{w}} \max \mathrm{P}(\mathrm{w}|\mathrm{t}) && (2) \end{aligned}$$

In **Maximum A Posteriori** we *find the set of parameters* that **maximize** the the posterior probability $\mathrm{P}(\mathrm{w}|\mathrm{t})$. We find the *optimal* set of parameters under some assumed distribution such that the parameters are most likely to have been drawn off of.

- MAP focuses on maximizing the posterior probability - data likelihood with a prior
- A common trick to maximize the posterior probability is to maximize the log likelihood
- MAP is data driven
- MAP is mostly driven by the prior beliefs
- MAP works great with small amounts of data *if* our prior was chosen well
- We need to assume and select a distribution for our prior beliefs
  - A wrong choice of prior distribution can impact negatively our model estimation
- When we have lots and lots of data, the data likelihood will take over and the posterior will depend less and less on the prior

Conjugate Priors

- If the posterior probability and the prior probability have the same (parametric) form, they are said to have a **conjugate prior** relationship.
  - For example, consider the data likelihood $P(t|\mu) \sim \mathcal{N}(\mu, \sigma^2)$ and the prior distribution $P(\mu) \sim \mathcal{N}(\mu_0, \sigma_0^2)$. The posterior probability will also be Gaussian distributed

$$P(\mu|t) \sim \mathcal{N}\left(\frac{\sum_{i=1}^{N} x_i\sigma_0^2 + \mu_0\sigma^2}{N\sigma_0^2 + \sigma^2}, \left(\frac{N}{\sigma^2} + \frac{1}{\sigma_0^2}\right)^{-1}\right)$$

- There are many conjugate prior relationships, e.g., Bernoulli-Beta, Gausian-Gaussian, Gaussian-Inverse Wishart, Multinomial-Dirichlet.

Conjugate prior relationships play an important role for **online updating** of the prior distribution.

- In an online model estimation scenario, where the posterior has the same form as the prior, we can use the posterior as our new prior. This new prior is now **data-informed** and will update its parameters based on (1) our initial choice, and (2) the data.

Conjugate Prior Relationship: Gaussian-Gaussian

For a D-dimensional Gaussian data likelihood with mean μ and covariance βI and a prior distribution with mean μ_0 and covariance Σ_0

$$P(t|w) \sim \mathcal{N}(\mu, \beta I) \tag{3}$$
$$P(w) \sim \mathcal{N}(\mu_0, \Sigma_0) \tag{4}$$

The posterior distribution

$$P(w|t) \sim \mathcal{N}(\mu_N, \Sigma_N) \tag{5}$$
$$\mu_N = \Sigma_N (\Sigma_0^{-1}\mu_0 + \beta X^T t) \tag{6}$$
$$\Sigma_N^{-1} = \Sigma_0^{-1} + \beta X^T X \tag{7}$$

where X is the feature matrix of size $N \times M$.

- What happens with different values of β and Σ_0?

To simplify, let's assume the covariance of prior is **isotropic**, that is, it is a diagonal matrix with the same value along the diagonal, $\Sigma_0 = \alpha^{-1}I$. And, let also $\mu_0 = [0, 0]$, thus

$$\mu_N = \beta \Sigma_N X^T t$$

and

$$\Sigma_N = (\alpha^{-1}I + \beta X^T X)^{-1}$$

To be continued...