# Online Appendix

## D. Arbitrary Information Technologies

In this section, we characterize the predictions attainable as one ranges over all information technologies. In particular, we do not require the information technology to be flexible or monotone. We also show it is without loss to require the technology to be flexible, and costs to be weakly monotone.[19] Formally, a cost function $C_i$ is **weakly monotone** if less informative experiment are weakly cheaper to acquire: if $\xi_i, \xi_i' \in \mathcal{E}_i$ are such that $\xi_i \succsim \xi_i'$, then $C_i(\xi_i) \geq C_i(\xi_i')$.

**Proposition 4.** *Fix a base game $\mathcal{G}$. An information technology $\mathcal{T}$ exists that induces the outcome-value pair $(p, v)$ in an equilibrium of $(\mathcal{G}, \mathcal{T})$ if and only if*

*(i) $p$ is a BCE, and*

*(ii) for every $i \in I$, $v_i \in [\underline{v}_i(p), \bar{v}_i(p)]$.*

*In addition, for every player $i$, one can choose $\mathcal{E}_i$ flexible and $C_i$ weakly monotone.*

*Proof.* "If." Let $(p, v)$ be an outcome-value pair such that $p$ is a BCE and, for every $i \in I$, $v_i \in [\underline{v}_i(p), \bar{v}_i(p)]$. Since $p$ is a BCE, by Bergemann and Morris (2016) there exist an information structure $\mathcal{S} = (Z, \zeta, (X_i, \xi_i)_{i \in I})$ and a profile of action plans $\sigma = (\sigma_i)_{i \in I}$ such that $p$ is the outcome of $(\xi, \sigma)$, and for every player $i$, $\sigma_i$ maximizes

$$\sum_{a,x,z,\theta} u_i(a, \theta) \left( \sigma_i'(a_i|x_i) \xi_i(x_i|z, \theta) \prod_{j \neq i} \sigma_j(a_j|x_j) \xi_j(x_j|z, \theta) \right) \zeta(z|\theta) \pi(\theta) \tag{27}$$

over all $\sigma_i' \in \Sigma_i$. To ease notation, denote the quantity in (27) by $u_i(\xi_i', \sigma_i', \xi_{-i}, \sigma_{-i})$.

For every player $i$, let $\mathcal{E}_i = \{\xi_i' : \xi_i \succeq \xi_i'\}$. In addition, take $\lambda_i \in [0, 1]$ such that

$$v_i = \lambda_i \underline{v}_i(p) + (1 - \lambda_i) \bar{v}_i(p).$$

For every $\xi_i' \in \mathcal{E}_i$, define $C_i(\xi_i') = \lambda_i (\max_{\sigma_i'} u_i(\xi_i', \sigma_i', \xi_{-i}, \sigma_{-i}) - \underline{v}_i(p))$. Notice that $\mathcal{E}_i$ is flexible and $C_i$ is weakly monotone.

[19] For an analogous result in single-agent settings, see Caplin and Dean (2015).

It follows from (27) that $C_i(\xi_i) = \lambda_i\left(\bar{v}_i(p) - \underline{v}_i(p)\right)$, which in turn implies that $u_i(\xi,\sigma) - C_i(\xi_i) = v_i$. We also see that for every $\xi_i' \in \mathcal{E}_i$,

$$\begin{aligned}
u_i(\xi,\sigma) - C_i(\xi_i) &= \max_{\sigma_i'} u_i(\xi,\sigma_i',\sigma_{-i}) - C_i(\xi_i) \\
&= \lambda_i \underline{v}_i(p) + (1-\lambda_i)\max_{\sigma_i'} u_i(\xi,\sigma_i',\sigma_{-i}) \\
&\geq \lambda_i \underline{v}_i(p) + (1-\lambda_i)\max_{\sigma_i'} u_i(\xi_i',\xi_{-i},\sigma_i',\sigma_{-i}) \\
&= \max_{\sigma_i'} u_i(\xi_i',\xi_{-i},\sigma_i',\sigma_{-i}) - C_i(\xi_i'),
\end{aligned}$$

where the first equality follows from (27) and the weak inequality from $\xi_i \succeq \xi_i'$. We conclude $(\xi,\sigma)$ is an equilibrium of $(\mathcal{G},\mathcal{T})$ with $\mathcal{T} = (Z,\zeta,(X_i,\mathcal{E}_i,C_i)_{i\in I})$; in addition, $(p,v)$ is the outcome-value pair corresponding to $(\xi,\sigma)$.

"Only if." Let $(p,v)$ be the outcome-value pair of an equilibrium $(\xi,\sigma)$ of an information acquisition game $(\mathcal{G},\mathcal{T})$, with $\mathcal{T} = (Z,\zeta,(X_i,\mathcal{E}_i,C_i)_{i\in I})$. Define the information structure $\mathcal{S} = (Z,\zeta,(X_i,\xi_i)_{i\in I})$. Since $(\xi,\sigma)$ is an equilibrium of $(\mathcal{G},\mathcal{T})$, $\sigma$ is an equilibrium of $(\mathcal{G},\mathcal{S})$. By Bergemann and Morris (2016), $p$ is a BCE.

For every player $i$, $C_i(\xi_i) \geq 0$, which implies that $v_i \leq \bar{v}_i(p)$. In addition, by hypothesis there exists an experiment $\xi_i'$ such that $C_i(\xi_i') = 0$. Thus, since $(\xi_i,\sigma_i)$ is a best response to $(\xi_{-i},\sigma_{-i})$, we have that

$$v_i \geq \max_{\sigma_i'} u_i(\xi_i',\xi_{-i},\sigma_i',\sigma_{-i}) \geq \underline{v}_i(p).$$

We conclude that $v_i \in [\bar{v}_i(p), \underline{v}_i(p)]$. □

# E. Examples Where Separation Binds

In this section we present a few simple examples in which the separation constraint has substantial bite, that is, in which the sBCE set is not dense in the BCE set. In all the examples that follow, the sBCE set is nowhere dense is the BCE set. As Theorem 3 predicts, if the sBCE set is not dense in the BCE set, it must be nowhere dense. To ease the exposition, we assume the payoff state is degenerate (i.e., $\Theta$ is a singleton), and we omit it.

The simplest example in which the separation constraint has stark effects is the scenario in which the players' utilities are constant: $u_i(a) = u_i(b)$ for all $i \in I$ and $a, b \in A$. In this case, the BCE set is the entire simplex $\Delta(A)$. On the other hand, a BCE is separated if and only if the players' actions are independent. Thus, the sBCE set can be identified with $\prod_{i \in I} \Delta(A_i)$, the set of mixed-action profiles.

The presence of weakly dominated actions is a factor that may put a wedge between BCE and sBCE. For example, consider the following $2 \times 2$ game:

| | $a_2$ | $b_2$ |
|---|---|---|
| $a_1$ | $2, 2$ | $2, 2$ |
| $b_1$ | $3, 1$ | $0, 0$ |

This game can be seen as the reduced normal form of a Battle of the Sexes with Outside Option.[20] Note that $a_2$ weakly dominates $b_2$.

It is easy to see that the BCEs are all the outcomes $p$ such that $p(b_1, b_2) = 0$ and $p(a_1, a_2) \leq 2p(a_1, b_2)$. However, a BCE $p$ is separated if and only if $p(b_1, a_2) \in \{0, 1\}$. To see why, first consider the case in which $p(b_1, a_2) = \{0, 1\}$. Then, since player 1's action is deterministic (either $p(a_1) = 1$ or $p(b_1) = 1$), the separation constraint is trivially satisfied for both players. Conversely, suppose that $p(b_1, a_2) \in (0, 1)$. In this case, player 2 takes both actions with positive probability, and they induce different beliefs about player 1's action: $p_{a_2}(b_1) > 0 = p_{b_2}(b_1)$. Since $a_2$ weakly dominates $b_2$, we have $a_2 \in BR(p_{a_2}) \cap BR(p_{b_2})$. Hence, the separation constraint is not satisfied.

One should not overstate the relationship between weakly dominated actions and separation. As the next example highlights, the separation constraint can have a substantial impact even if no action is weakly dominated:

| | $a_2$ | $b_2$ | $c_2$ |
|---|---|---|---|
| $a_1$ | $8, 8$ | $3, 7$ | $2, 6$ |
| $b_1$ | $7, 3$ | $5, 1$ | $0, 5$ |
| $c_1$ | $6, 2$ | $1, 4$ | $4, 0$ |

The game, which is a variation of Myerson (1997, Figure 6), has no weakly dominated

[20] In the BoS with OO we have in mind, player 1 first chooses between *Out* and *In*. Given *Out*, each player obtains a payoff of 2. Given *In*, the players participate in a coordination game in which they simultaneously choose between a *Bach* concert and a *Stravinsky* concert. If they coordinate on *Bach*, player 1 gets 3 and player 2 gets 1; if they coordinate on *Stravinsky*, player 1 gets 1 and player 2 gets 3; if they mis-coordinate, they both obtain 0.

action. It has one pure Nash equilibrium and one mixed Nash equilibrium:

$$(a_1, a_2) \quad \text{and} \quad \left(\frac{1}{2}b_1 + \frac{1}{2}c_1, \frac{1}{2}b_2 + \frac{1}{2}c_2\right).$$

The BCEs are the convex combinations of the two Nash equilibria: for $t \in [0, 1]$,

$$p^t = t(a_1, a_2) + (1-t)\left(\frac{1}{2}b_1 + \frac{1}{2}c_1, \frac{1}{2}b_2 + \frac{1}{2}c_2\right).$$

The game has only two separated BCE, namely, the two Nash equilibria. Indeed, for every $t \in (0, 1)$ and every player $i$, the action recommendations $a_i$ and $b_i$ (or $c_i$) induce distinct posterior beliefs about the action of the opponent: $p^t_{a_i}(a_j) = 1$, while $p^t_{b_i}(b_j) = p^t_{b_i}(c_j) = 1/2$. Yet, $a_i$ is best response to the belief induced by $b_i$:

$$\frac{1}{2}u_i(a_i, b_j) + \frac{1}{2}u_i(a_i, c_j) = \frac{5}{2} = \frac{1}{2}u_i(b_i, b_j) + \frac{1}{2}u_i(b_i, c_j).$$

# F. Strict BCE: Single-Agent Settings

A BCE $p$ is **strict** if all $i \in I$, $a_i \in \operatorname{supp}_i(p)$, and $b_i \in A_i$ with $b_i \neq a_i$,

$$\sum_{a_{-i},\theta} \left(u_i(a_i, a_{-i}, \theta) - u_i(b_i, a_{-i}, \theta)\right) p(a_i, a_{-i}, \theta) > 0.$$

In the main text, discussing Theorem 2, we mentioned the following result:

**Proposition 5.** *Let $I = \{i\}$ be a singleton. For generic $u_i$, the set of strict BCE is dense in the BCE set.*

We expect the result to be known in the literature. However, we could not find a good reference. Thus, next we provide a self-contained proof. The proof relies on two lemmas on dominated actions. A mixed action $\alpha_i \in \Delta(A_i)$ **weakly dominates** a pure action $a_i \in A_i$ if $\sum_{b_i} u_i(b_i, a_{-i}, \theta)\alpha_i(b_i) \geq u_i(a_i, a_{-i}, \theta)$. for all $a_{-i} \in A_{-i}$ and $\theta \in \Theta$. The next result provides a characterization of weakly dominated actions:[21]

**Lemma 8.** *The following statements are equivalent:*

*(i) There is no belief $\mu_{a_i} \in \Delta(A_{-i} \times \Theta)$ for which $a_i$ is the unique best response.*

[21] See any textbook on statistical decision theory for closely related results on admissibility.

*(ii) There is a mixed action $\alpha_i \in \Delta(A_i \setminus \{a_i\})$ that weakly dominates $a_i$.*

*Proof.* Condition (i) can be rewritten as

$$\max_{\mu_i \in \Delta(A_{-i} \times \Theta)} \min_{b_i \in A_i \setminus \{a_i\}} \sum_{a_{-i}, \theta} (u_i(a_i, a_{-i}, \theta) - u_i(b_i, a_{-i}, \theta))\mu_i(a_{-i}, \theta) \leq 0.$$

Equivalently,

$$\max_{\mu_i \in \Delta(A_{-i} \times \Theta)} \min_{\alpha_i \in \Delta(A_i \setminus \{a_i\})} \sum_{a_{-i}, \theta} (u_i(a_i, a_{-i}, \theta) - u_i(b_i, a_{-i}, \theta))\mu_i(a_{-i}, \theta)\alpha_i(b_i) \leq 0.$$

By the minimax theorem (e.g., Rockafellar, 1970, Corollary 37.3.2), the above inequality holds if and only if

$$\min_{\alpha_i \in \Delta(A_i \setminus \{a_i\})} \max_{\mu_i \in \Delta(A_{-i} \times \Theta)} \sum_{a_{-i}, \theta} (u_i(a_i, a_{-i}, \theta) - u_i(b_i, a_{-i}, \theta))\mu_i(a_{-i}, \theta)\alpha_i(b_i) \leq 0.$$

Equivalently,

$$\min_{\alpha_i \in \Delta(A_i \setminus \{a_i\})} \max_{a_{-i}, \theta} \sum_{a_{-i}, \theta} (u_i(a_i, a_{-i}, \theta) - u_i(b_i, a_{-i}, \theta))\alpha_i(b_i) \leq 0.$$

which is another way of expressing condition (ii). □

A mixed action $\alpha_i \in \Delta(A_i)$ **strictly dominates** a pure action $a_i \in A_i$ if for all $a_{-i} \in A_{-i}$ and $\theta \in \Theta$, $\sum_{b_i} u_i(b_i, a_{-i}, \theta)\alpha_i(b_i) > u_i(a_i, a_{-i}, \theta)$. The next result shows that generically, weakly dominated actions are strictly dominated.

**Lemma 9.** *Let $I = \{i\}$ be a singleton. For generic $u_i$, if an action $a_i$ is weakly dominated by some mixed action $\alpha_i \in \Delta(A_i \setminus \{a_i\})$, then it is strictly dominated by some mixed action $\beta_i \in \Delta(A_i)$.*

*Proof.* Let $a_i$ be an action that is weakly dominated by a mixed action $\alpha_i \in \Delta(A_i \setminus \{a_i\})$. Let $A_i'$ be the support of $\alpha_i$, and let $\Theta'$ be set of states $\theta$ for which

$$u_i(a_i, \theta) = \sum_{b_i} u_i(b_i, \theta)\alpha_i(b_i). \tag{28}$$

Let $m$ be the cardinality of $A_i'$, and let $n$ be the cardinality of $\Theta'$. We consider the $m \times n$ matrix $M \in \mathbb{R}^{A_i' \times \Theta'}$ given by $M(b_i, \theta) = u_i(a_i, \theta) - u_i(b_i, \theta)$. For generic $u_i$, the matrix $M$ has full rank. By (28), the rows of $M$ are linearly dependent. Thus, the rank of $M$ must be $n$, the number of columns. We obtain that the row space of $M$ has dimension $n$. Hence, we can find $\beta_i \in \mathbb{R}^{A'_i}$ such that for every $\theta \in \Theta'$ $[\sum_{b_i}(u_i(a_i,\theta) - u(b_i,\theta))\beta_i(b_i) < 0$. For every $t > 0$, we define $\alpha_i^t \in \mathbb{R}^{A'_i}$ by

$$\alpha_i^t(b_i) = \frac{\alpha_i(b_i) + t\beta_i(b_i)}{\sum_{c_i} \alpha_i(c_i) + t\beta_i(c_i)}.$$

For $t$ sufficiently small, $\alpha_i^t$ is a mixed action that strictly dominates $a_i$. □

We are now ready to prove the proposition on strict BCE.

*Proof of Proposition 5.* Let $A_i^*$ be the set of actions that are not strictly dominated. Since $u_i$ is generic, it follows from Lemma 9 that each $a_i \in A_i^*$ is not weakly dominated by a mixed action $\alpha_i \in \Delta(A_i \setminus \{a_i\})$. By Lemma 8, there is a belief $\mu_{a_i} \in \Delta(\Theta)$ for which $a_i$ is the unique best response.

Since $\pi$ has full support, we can find $\nu \in \Delta(\Theta)$ and for every $a_i \in A_i^*$, $t_{a_i} \in (0,1)$—with $\sum_{a_i \in A_i^*} t_{a_i} \leq 1$—such that

$$\pi = \sum_{a_i \in A_i^*} t_{a_i}\mu_{a_i} + \left(1 - \sum_{a_i \in A_i^*} t_{a_i}\right)\nu.$$

Let $a_i^*$ be a best response to $\nu$; necessarily, $a_i^* \in A_i^*$. Define the outcome $p \in \Delta_\pi(A_i \times \Theta)$ as follows:

$$p(a_i,\theta) = \begin{cases} t_{a_i}\mu_{a_i}(\theta) & \text{if } a_i \in A_i^* \setminus \{a_i^*\}, \\ t_{a_i^*}\mu_{a_i^*}(\theta) + \left(1 - \sum_{a_i \in A_i^*} t_{a_i}\right)\nu(\theta) & \text{if } a_i = a_i^*, \\ 0 & \text{otherwise.} \end{cases}$$

The outcome $p$ is a strict BCE. Moreover, if $q$ is a BCE, then $\text{supp}_i(q) \subseteq A_i^* = \text{supp}_i(p)$. Thus, $\{sq + (1-s)p : s \in (0,1) \text{ and } q \in BCE\}$ is a subset of the set of strict BCE, and it is dense in the BCE set. We conclude that (for generic $u_i$) the set of strict BCE is dense in the BCE set. □

# G. Proofs for Section 6

## G.1. Proof of Proposition 1

First, we show that focusing on symmetric outcomes is without loss for welfare analysis in symmetric games (the assumption of binary actions has no role in this result).

*Claim* 7. For every BCE $p$, there is a symmetric BCE $q$ such that $\bar{w}(q) = \bar{w}(p)$ and $\underline{w}(q) \leq \underline{w}(p)$.

*Proof.* Fix a BCE $p$. Let $\Phi$ be the set of permutations of $I$. For every permutation $\phi \in \Phi$, we define the outcome $p_\phi$ by $p_\phi(a, \theta) = p(a_\phi, \theta)$. Note that player $i$ in $p_\phi$ behaves as player $j = \phi^{-1}(i)$ in $p$. One can verify that $p_\phi$ because $p$ is a BCE and the game is symmetric.

We define the outcome $q$ by $q = \frac{1}{|\Phi|}\sum_{\phi\in\Phi} p_\phi$, where $|\Phi|$ is the cardinality of $\Phi$. As noted above, each $p_\phi$ is a BCE. Since the BCE set is convex, $q$ is a BCE.

The outcome $q$ is symmetric. Indeed, $\Phi = \{\psi^{-1} \circ \phi : \phi \in \Phi\}$ for every permutation $\psi \in \Phi$. We deduce that

$$\begin{aligned} q(a_\psi, \theta) &= \frac{1}{|\Phi|}\sum_{\phi\in\Phi} p_\phi(a_\psi, \theta) = \frac{1}{|\Phi|}\sum_{\phi\in\Phi} p_{(\psi^{-1}\circ\phi)}(a_\psi, \theta) \\ &= \frac{1}{|\Phi|}\sum_{\phi\in\Phi} p(a_\phi, \theta) = \frac{1}{|\Phi|}\sum_{\phi\in\Phi} p_\phi(a, \theta) = q(a, \theta). \end{aligned}$$

Hence, $q$ is symmetric.

To conclude the proof, we observe $\bar{w}(q) = \frac{1}{|\Phi|}\sum_{\phi\in\Phi} \bar{w}(p_\phi) = \frac{1}{|\Phi|}\sum_{\phi\in\Phi} \bar{w}(p) = \bar{w}(p)$, where the first equality holds because $\bar{w}(p_\phi)$ is affine in $p_\phi$, and the second equality because the game is symmetric. Finally, note that $\underline{w}(q) \leq \frac{1}{|\Phi|}\sum_{\phi\in\Phi} \underline{w}(p_\phi) = \frac{1}{|\Phi|}\sum_{\phi\in\Phi} \underline{w}(p) = \underline{w}(p)$, where the first inequality holds because $\underline{w}(p_\phi)$ is convex in $p_\phi$, and the second equality because the game is symmetric. $\square$

By Claim 7, $\bar{w}$ is the value of the optimization problem

$$\min_{p\in BCE^{sy}} \bar{w}(p), \tag{29}$$

and $\underline{w}$ is the value of the optimization problem

$$\min_{p \in BCE^{sy}} \underline{w}(p). \tag{30}$$

We consider also the following optimization problem:

$$\min_{p \in \Delta_{\pi}^{sy}(A \times \Theta)} \underline{w}(p), \tag{31}$$

We proceed by successive claims.

*Claim* 8. The following conditions are equivalent: (i) $\underline{w} < \bar{w}$, and (ii) $\underline{v}_i(p) < \bar{v}_i(p)$ for all players $i$ and optimal solutions $p$ of (30).

*Proof.* We first prove that (i) implies (ii). Suppose $\underline{w} < \bar{w}$ and let $p$ be an optimal solution of (30). Then, $\underline{w}(p) = \underline{w} < \bar{w} \leq \bar{w}(p)$. The inequality $\underline{w}(p) < \bar{w}(p)$ implies that $\underline{v}_i(p) < \bar{v}_i(p)$ for some player $i$. Since the game is symmetric and $p$ is symmetric, $\underline{v}_i(p) < \bar{v}_i(p)$ for all players $i$.

We now prove that (ii) implies (i). Suppose $\underline{v}_i(p) < \bar{v}_i(p)$ for all players $i$ and optimal solutions $p$ of (30). Let $p \in BCE$ be an optimal solution of (29). If $p$ is also an optimal solution of (30), then $\underline{w}(p) = \sum_i \underline{v}_i(p) < \sum_i \bar{v}_i(p) = \bar{w}(p)$ by hypothesis; thus, $\underline{w} < \bar{w}$. If instead $p$ is not an optimal solution of (30), then $\underline{w} < \underline{w}(p) \leq \bar{w}(p) = \bar{w}$; thus, $\underline{w} < \bar{w}$. □

*Claim* 9. For every $p \in BCE$ and $i \in I$, the following conditions are equivalent: (i) $\underline{v}_i(p) < \bar{v}_i(p)$, and (ii) $a_i \in \text{supp}_i(p)$ and $BR(p_{a_i}) = \{a_i\}$ for all $a_i \in A_i$.

*Proof.* Condition (i) holds if and only if player $i$ is strictly better by following the action recommendation of the mediator rather then best responding ex ante. In other terms, player $i$ has no action $a_i$ such that for all $b_i \in \text{supp}_i(p)$, $a_i \in BR(p_{b_i})$. Given that $A_i$ has two elements, this is equivalent to condition (ii). □

*Claim* 10. The following conditions are equivalent: (i) all optimal solutions of (30) satisfy (5), and (ii) all optimal solutions of (31) satisfy (5).

*Proof.* First we show that (i) implies (ii). Let $p$ be an optimal solution of (30) and let $q$ be an optimal solution of (31). For every $t \in [0, 1]$, define $p^t = (1 - t)p + tq$. Furthermore, set $s = \max\{t : p^t \in BCE^{sy}\}$. Note that $s$ is well defined: the set $BCE^{sy}$ is closed and $p^0 = p \in BCE^{sy}$.

We observe that $p^s$ is an optimal solution of (30): since $\underline{w}(p^t)$ is convex in $t$, $\underline{w}(p^s) \leq (1-s)\underline{w}(p) + s\underline{w}(q) \leq \underline{w}(p) = \underline{w}$. Thus, $p^s$ must satisfy (5). But this implies that $s = 1$; otherwise, one could find $\epsilon > 0$ sufficiently small so that $p^{s+\epsilon} \in BCE^{sy}$, contradicting the definition of $p^s$. This implies that $q = p^s$ satisfies (5).

Now we show that (ii) implies (i). Let $p$ be an optimal solution of (30) and let $q$ be an optimal solution of (31). Since $q$ satisfies (5), $q$ is a BCE. Thus, $q$ is an optimal solution of (30). This implies that $p$ is an optimal solution of (31), and therefore satisfies (5). $\square$

By combining the three claims above, we obtain Proposition 1.

## G.2. Proof of Claim 1

We begin with a result that establishes a necessary condition for an outcome to solve the relaxed program from Proposition 1. To state the result, let $U_i(a_i, p)$ be player $i$'s payoff if she always takes action $a_i$ while $(a_{-i}, \theta)$ is distributed according to $p$:

$$U_i(a_i, p) = \sum_{b_i, a_{-i}, \theta} u_i(a_i, a_{-i}, \theta) p(b_i, a_{-i}, \theta).$$

Note that for all $p \in \Delta_\pi^{sy}(A \times \Theta)$ and $i \in I$, $\underline{w}(p) = n \max\{U_i(0,p), U_i(1,p)\}$. Thus,

$$\operatorname*{argmin}_{p \in \Delta_\pi^{sy}(A\times\Theta)} \underline{w}(p) = \operatorname*{argmin}_{p \in \Delta_\pi^{sy}(A\times\Theta)} \max\{U_i(0,p), U_i(1,p)\}.$$

*Claim* 11. Every $p^* \in \operatorname{argmin}_{p \in \Delta_\pi^{sy}(A\times\Theta)} \underline{w}(p)$ has $U_i(0,p^*) = U_i(1,p^*)$ for all $i \in I$.

*Proof.* We prove the contrapositive: if $p^* \in \Delta_\pi^{sy}(A\times\Theta)$ has $U_i(0,p^*) \neq U_i(1,p^*)$, then $p^* \notin \operatorname{argmin}_{p \in \Delta_\pi^{sy}(A\times\Theta)} \underline{w}(p)$.

We first consider the case in which $U_i(0,p^*) > U_i(1,p^*)$. Let $q$ be the outcome where all investors always attack. Observe that $q \in \operatorname{argmin}_{p \in \Delta_\pi^{sy}(A\times\Theta)} U_i(0,p)$, and that every $r \in \operatorname{argmin}_{p \in \Delta_\pi^{sy}(A\times\Theta)} U_i(0,p)$ has the speculative attack succeeding with probability one. Hence, every such $r$ has $U_i(0,r) < U_i(1,r)$, which implies that $p^* \notin \operatorname{argmin}_{p \in \Delta_\pi^{sy}(A\times\Theta)} U_i(0,p)$. We deduce that $U_i(0,q) < U_i(0,p^*)$.

For every $\epsilon \in (0,1)$, we define $q^\epsilon = \epsilon q + (1-\epsilon)p^* \in \Delta_\pi^{sy}(A\times\Theta)$. Using the inequality $U_i(0,q) < U_i(0,p^*)$, we obtain that for all $\epsilon > 0$ small enough,

$$\underline{w}(q^\epsilon) = nU_i(0,p^\epsilon) = n\left(\epsilon U_i(0,q) + (1-\epsilon)U_i(0,p^*)\right) < nU_i(0,p^*) = \underline{w}(p^*).$$

We conclude that $p^* \notin \operatorname{argmin}_{p\in\Delta_\pi^{sy}(A\times\Theta)} \underline{w}(p)$.

The argument for the case $U_i(0,p^*) < U_i(1,p^*)$ is similar, but with $q$ being replaced by the outcome where no one ever speculates. $\square$

Thanks to Claim 11, to determine $\operatorname{argmin}_{p\in\Delta_\pi^{sy}(A\times\Theta)} \underline{w}(p)$, we can study the following "simpler" optimization problem:

$$\min_{p\in\Delta_\pi^{sy}(A\times\Theta)} U_i(0,p) \quad \text{s.t.} \quad U_i(0,p) = U_i(1,p). \tag{32}$$

*Claim* 12. An outcome $p \in \Delta_\pi^{sy}(A\times\Theta)$ is an optimal solution of (32) if and only if

$$p\left(\sum_{j\neq i} a_j = \theta - 1\right) = 0 \quad \text{and} \quad p\left(\sum_{j\neq i} a_j \geq \theta\right) = \frac{k}{1+x}.$$

*Proof.* Simple algebra shows that every $p$ that satisfies $U_i(1,p) = U_i(0,p)$ must yield

$$U_i(0,p) = -xp\left(\sum_{j\neq i} a_j \geq \theta\right) = -x\left[\frac{k - p\left(\sum_{j\neq i} a_j = \theta-1\right)}{1+x}\right].$$

Therefore, we get that (32) is the same as

$$\min_{p\in\Delta_\pi^{sy}(A\times\Theta)} \quad p\left(\sum_{j\neq i} a_j = \theta - 1\right) \quad \text{s.t.} \quad p\left(\sum_{j\neq i} a_j \geq \theta\right) = \frac{k - p\left(\sum_{j\neq i} a_j = \theta-1\right)}{1+x}.$$

Hence, to complete the proof, we only need to be sure that there is $p \in \Delta_\pi^{sy}(A\times\Theta)$ such that

$$p\left(\sum_{j\neq i} a_j = \theta - 1\right) = 0 \quad \text{and} \quad p\left(\sum_{j\neq i} a_j \geq \theta\right) = \frac{k}{1+x}.$$

Such an outcome is easy to construct: with probability $k/1+x$, all players attack; with the remaining probability, no player attacks. $\square$

The following result connects what we have just found with the conditions in the statement of Claim 1.

*Claim* 13. For an outcome $p \in \Delta_\pi(A\times\Theta)$, the following conditions are equivalent:

(i) For all players $i$ and payoff states $\theta$,

$$p\left(\sum_{j\neq i} a_j = \theta - 1\right) = 0, \tag{33}$$

$$p\left(\sum_{j\neq i} a_j \geq \theta\right) = \frac{k}{1+x}. \tag{34}$$

(ii) For all payoff states $\theta$,

$$p\left(\theta - 1 \leq \sum_i a_i \leq \theta\right) = 0, \tag{35}$$

$$p\left(\sum_i a_i > \theta\right) = \frac{k}{1+x}. \tag{36}$$

*Proof.* First we show that (i) implies (ii). Since $\max \Theta < n$,

$$p\left(\sum_i a_i = \theta - 1\right) = p\left(\sum_i a_i = \theta - 1, \text{ and } a_i = 0 \text{ for some } i\right).$$

Thus, $p\left(\sum_i a_i = \theta - 1\right) \leq \sum_i p\left(\sum_{j\neq i} a_j = \theta - 1, \text{ and } a_i = 0\right) = 0$, where the last equality follows from (33). Moreover, since $\min \Theta > 0$,

$$p\left(\sum_i a_i = \theta\right) = p\left(\sum_i a_i = \theta, \text{ and } a_i = 1 \text{ for some } i\right).$$

Thus, $p\left(\sum_i a_i = \theta\right) \leq \sum_i p\left(\sum_{j\neq i} a_j = \theta - 1, \text{ and } a_i = 1\right) = 0$, where the last equality follows from (33). We conclude that (35) holds.

To prove (36), notice that $p\left(\sum_i a_i > \theta\right) = p\left(\sum_i a_i \geq \theta\right)$, because we have just verified that $p\left(\sum_i a_i = \theta\right) = 0$. Then, fixing some player $i^*$,

$$\begin{aligned} p\left(\sum_i a_i \geq \theta\right) &= p\left(\sum_{i\neq i^*} a_i \geq \theta, \text{ and } a_{i^*} = 0\right) + p\left(\sum_{i\neq i^*} a_i \geq \theta - 1, \text{ and } a_{i*} = 1\right) \\ &= p\left(\sum_{i\neq i^*} a_i \geq \theta, \text{ and } a_{i^*} = 0\right) + p\left(\sum_{i\neq i^*} a_i \geq \theta, \text{ and } a_{i^*} = 1\right) \\ &= p\left(\sum_{i\neq i^*} a_i \geq \theta\right) = \frac{k}{1+x}, \end{aligned}$$

where the second equality holds by (33), and the last equality by (34). We deduce (36). This completes the proof that (i) implies (ii).

Now we show that (ii) implies (i). Observe that

$$p\left(\sum_{j\neq i} a_j = \theta - 1\right) = p\left(\sum_{j} a_j = \theta - 1, \text{ and } a_i = 0\right) + p\left(\sum_{j} a_j = \theta, \text{ and } a_i = 1\right).$$

By (35), the right-hand side is equal to zero: we deduce (33). We obtain (34) from the following chain of equalitites:

$$\begin{aligned} p\left(\sum_{j\neq i} a_j \geq \theta\right) &= p\left(\sum_{j} a_j \geq \theta, \text{ and } a_i = 0\right) + p\left(\sum_{j} a_j > \theta, \text{ and } a_i = 1\right) \\ &= p\left(\sum_{j} a_j > \theta, \text{ and } a_i = 0\right) + p\left(\sum_{j} a_j > \theta, \text{ and } a_i = 1\right) \\ &= p\left(\sum_{j} a_j > \theta\right) = \frac{k}{1+x}, \end{aligned}$$

where the second equality follows from (35), and the last equality from (36). This completes the proof that (ii) implies (i). □

Combining the three results above, we obtain Claim 1.

## G.3. Proof of Claim 2

First, we obtain necessary and sufficient conditions for $\underline{w} < \bar{w}$ in the regime change game for an arbitrary number of states.

*Claim* 14. The inequality $\underline{w} < \bar{w}$ holds if and only if all symmetric outcomes $p$ that satisfy (6) and (7), also satisfy

$$p_{a_i=1}\left(\sum_{j} a_j \geq \theta\right) > \frac{k}{1+x}, \tag{37}$$

where $p_{a_i=1}$ is the conditional probability of $(a_{-i}, \theta)$ given $a_i = 1$.

*Proof.* By Proposition 1, the inequality $\underline{w} < \bar{w}$ holds if and only if all optimal solutions of $\min_{p\in\Delta_\pi^{sy}(A\times\Theta)} \underline{w}(p)$ satisfy (5). By Claim 1, the latter condition is equivalent to the following statement: all symmetric outcomes $p$ that satisfy (6) and (7), also satisfy

(5). Next we verify that, for all symmetric outcomes $p$ that satisfy (6) and (7), the conditions (5) and (37) are equivalent.

Let $p$ be a symmetric outcome that satisfy (6) and (7). First, note (7) implies the attack succeeds with a probability strictly between 0 and 1, and so players must both attack and and not attack with positive probability due to symmetry. Hence $\text{supp}_i(p) = \{0, 1\} = A_i$.

Given $\text{supp}_i(p) = \{0, 1\}$, $i$'s obedience constraints are strict when

$$p_{a_i=1}\left(\sum_{j\neq i} a_j \geq \theta - 1\right) - k > -xp_{a_i=1}\left(\sum_{j\neq i} a_j \geq \theta\right), \tag{38}$$

$$p_{a_i=0}\left(\sum_{j\neq i} a_j \geq \theta - 1\right) - k < -xp_{a_i=0}\left(\sum_{j\neq i} a_j \geq \theta\right). \tag{39}$$

By (6)—see also Claim 13—

$$p_{a_i=1}\left(\sum_{j\neq i} a_j \geq \theta - 1\right) = p_{a_i=1}\left(\sum_{j\neq i} a_j \geq \theta\right),$$

and

$$p_{a_i=0}\left(\sum_{j\neq i} a_j \geq \theta - 1\right) = p_{a_i=0}\left(\sum_{j\neq i} a_j \geq \theta\right).$$

Thus, (38) and (39) hold if and only if

$$p_{a_i=1}\left(\sum_{j\neq i} a_j \geq \theta\right) > \frac{k}{1+x} > p_{a_i=0}\left(\sum_{j\neq i} a_j \geq \theta\right).$$

By (6) and (7)—see also Claim 13— $p\left(\sum_{j\neq i} a_j \geq \theta\right) = \frac{k}{1+x}$. Thus, by the law of total probability, (38) and (39) hold if and only if

$$p_{a_i=1}\left(\sum_{j} a_j \geq \theta\right) = p_{a_i=1}\left(\sum_{j\neq i} a_j \geq \theta - 1\right) = p_{a_i=1}\left(\sum_{j\neq i} a_j \geq \theta\right) > \frac{k}{1+x}.$$

Overall, we conclude that, for all symmetric outcomes $p$ that satisfy (6) and (7), the conditions (5) and (37) are equivalent. □

Next we refine the characterization $\underline{w} < \bar{w}$ obtained in Claim 14. To state this refinement, denote the CDF of $\theta$ by $F(\theta) := \sum_{\theta' \leq \theta} \pi(\theta)$. Define also the cutoff $\theta^*$ by

$$\theta^* = \min \left\{ \theta \in \Theta : F(\theta) \geq \frac{k}{1+x} \right\}.$$

*Claim* 15. The inequality $\bar{w} > \underline{w}$ holds if and only if

$$F(\theta^*) \left(\theta^* - \mathbb{E}[\theta | \theta \leq \theta^*]\right) < \frac{k}{1+x} \left( 3 - \frac{3k}{1+x} + \theta^* - \mathbb{E}[\theta] \right). \tag{40}$$

*Proof.* By Claim 14, $\underline{w} < \bar{w}$ is equivalent to

$$\begin{aligned} \frac{k}{1+x} < \min_{p \in \Delta_\pi^{sy}(A \times \Theta)} & p_{a_i=1} \left( \sum_{j \neq i} a_j \geq \theta - 1 \right) \\ \text{s.t.} \quad & (6) \text{ and } (7). \end{aligned} \tag{41}$$

Hence, showing (40) and (41) are equivalent is sufficient. To show this equivalence, we first characterize the unique solution to the program on the right hand side of (41). This solution gives the value of the program, which we then compare to $k/(1+x)$.

We begin with an alternative way of representing symmetric outcomes. This representation is based on the observation that an outcome $p \in \Delta_\pi(A \times \Theta)$ is symmetric if and only if, conditional on the state, all action profiles with the same number of attackers have the same probability. Consequently, $p \in \Delta_\pi^{sy}(A \times \Theta)$ if and only if there is $Q : \Theta \to \Delta(\{0, \ldots, n\})$ such that

$$p(a, \theta) = \binom{n}{\sum_j a_j} Q \left( \sum_j a_j \middle| \theta \right) \pi(\theta),$$

where $\binom{n}{\sum_j a_j}$ is the binomial coefficient. Thus, one can write

$$p(a_i = 1) = \sum_\theta \pi(\theta) \sum_{m=1}^{n} \frac{m}{n} Q(m|\theta).$$

Moreover, condition (6) is equivalent to $Q(\theta - 1|\theta) = Q(\theta|\theta) = 0$. Therefore,

$$p \left( \sum_{j \neq i} a_j \geq \theta - 1 \text{ and } a_i = 1 \right) = \sum_\theta \pi(\theta) \sum_{m \geq \theta + 1} \frac{m}{n} Q(m|\theta),$$

and

$$p\left(\sum_j a_j \geq \theta\right) = \sum_\theta \pi(\theta) \sum_{m \geq \theta+1} Q(m|\theta).$$

Hence, letting

$$f(Q) = \frac{\sum_\theta \pi(\theta) \sum_{m \geq \theta+1} mQ(m|\theta)}{\sum_\theta \pi(\theta) \sum_{m=1}^{n} mQ(m|\theta)},$$

we can write the program on the right hand side of (40) as

$$\begin{aligned} \min_{Q:\Theta \to \Delta(\{0,\ldots,n\})} & f(Q) \\ \text{s.t.} \quad & \sum_\theta \pi(\theta) \sum_{m \geq \theta+1} Q(m|\theta) = \frac{k}{1+x}, \\ & Q(\theta-1|\theta) = Q(\theta|\theta) = 0 \text{ for all } \theta. \end{aligned} \tag{42}$$

Since the constraint set is compact and the objective continuous, the above program admits a solution, $Q^*$. We now use perturbation-based arguments to show $Q^*$ must satisfy a few properties:

1. $Q^*(m|\theta) = 0$ whenever $m \notin \{\theta-2, \theta+1\}$: if $Q^*(m|\theta) > 0$ for $m > \theta+1$ (resp., $m < \theta-2$), one can reduce the objective without violating the constraints by moving $\epsilon > 0$ mass from $Q^*(m|\theta)$ to $Q^*(\theta+1|\theta)$ (resp., $Q^*(\theta-2|\theta)$).

2. If $Q^*(\theta+1|\theta) > 0$, then $Q^*(\theta'+1|\theta') = 1$ for all $\theta' < \theta$: For a contradiction, suppose $Q^*(\theta+1|\theta) > 0$, but $Q^*(\theta'+1|\theta') < 1$ for some $\theta' < \theta$. For every $\epsilon > 0$, define the following perturbation $Q^\epsilon$ of $Q$:

$$Q^\epsilon(m|\hat{\theta}) = \begin{cases} Q^*(\theta+1|\theta) - \epsilon & \text{if } m = \theta+1, \hat{\theta} = \theta, \\ Q^*(\theta-2|\theta) + \epsilon & \text{if } m = \theta-2, \hat{\theta} = \theta, \\ Q^*(\theta'+1|\theta') + \epsilon\frac{\pi(\theta)}{\pi(\theta')} & \text{if } m = \theta'+1, \hat{\theta} = \theta', \\ Q^*(\theta'-2|\theta') - \epsilon\frac{\pi(\theta)}{\pi(\theta')} & \text{if } m = \theta'-2, \hat{\theta} = \theta', \\ Q^*(m|\hat{\theta}) & \text{otherwise.} \end{cases}$$

The contradiction assumption means $Q^\epsilon$ is feasible for all sufficiently small $\epsilon > 0$. Direct computation shows

$$\lim_{\epsilon \searrow 0} \frac{1}{\epsilon}\left(f(Q^\epsilon) - f(Q^*)\right) = \frac{\pi(\theta)(\theta'-\theta)}{\sum_{\hat{\theta}} \pi(\hat{\theta}) \sum_{m=1}^{n} mQ^*(m|\hat{\theta})} < 0,$$

contradicting the optimality of $Q$.

3. If $Q^*(\theta-2|\theta) > 0$, then $Q^*(\theta'-2|\theta') = 1$ for all $\theta' > \theta$: For a contradiction, suppose $Q^*(\theta-2|\theta) > 0$, but $Q^*(\theta'-2|\theta') < 1$ for some $\theta' > \theta$. For every $\epsilon > 0$, define the following perturbation $Q^\epsilon$ of $Q$:

$$Q^\epsilon(m|\hat{\theta}) = \begin{cases} Q^*(\theta-2|\theta) - \epsilon & \text{if } m = \theta-2, \hat{\theta} = \theta, \\ Q^*(\theta+1|\theta) + \epsilon & \text{if } m = \theta+1, \hat{\theta} = \theta, \\ Q^*(\theta'-2|\theta') + \epsilon\frac{\pi(\theta)}{\pi(\theta')} & \text{if } m = \theta'-2, \hat{\theta} = \theta', \\ Q^*(\theta'+1|\theta') - \epsilon\frac{\pi(\theta)}{\pi(\theta')} & \text{if } m = \theta'+1, \hat{\theta} = \theta', \\ Q^*(m|\hat{\theta}) & \text{otherwise.} \end{cases}$$

The contradiction assumption means $Q^\epsilon$ is feasible for all sufficiently small $\epsilon > 0$. Direct computation shows

$$\lim_{\epsilon \searrow 0} \frac{1}{\epsilon}\left(f(Q^\epsilon) - f(Q^*)\right) = \frac{\pi(\theta)(\theta - \theta')}{\sum_{\hat{\theta}} \pi(\hat{\theta}) \sum_{m=1}^n mQ(m|\hat{\theta})} < 0,$$

contradicting the optimality of $Q^*$.

The above conditions imply the optimal $Q^*$ admits a cutoff $\tilde{\theta}$ such that $Q^*(\theta+1|\theta) = 1$ for all $\theta < \tilde{\theta}$, $Q^*(\theta-2|\theta) = 1$ for all $\theta > \tilde{\theta}$, and $Q^*(\{\tilde{\theta}+1, \tilde{\theta}-2\}|\tilde{\theta}) = 1$. Then, the constraint

$$\sum_\theta \pi(\theta) \sum_{m \geq \theta+1} Q(m|\theta) = \frac{k}{1+x}$$

pins down the optimum: we must have $\tilde{\theta} = \theta^*$, and

$$Q^*(\theta^*+1|\theta^*) = \frac{1}{\pi(\theta^*)}\left(\frac{k}{1+x} - F(\theta^*-1)\right).$$

Therefore, the inequality (41) becomes

$$\begin{aligned} \frac{k}{1+x} < f(Q^*) &= \frac{\sum_\theta \pi(\theta) \sum_{m \geq \theta+1} mQ^*(m|\theta)}{\sum_\theta \pi(\theta) \sum_{m=1}^n mQ^*(m|\theta)} \\ &= \frac{F(\theta^*)\mathbb{E}[\theta+1|\theta \leq \theta^*] - \left(F(\theta^*) - \frac{k}{1+x}\right)(\theta^*+1)}{\mathbb{E}[\theta] + \frac{k}{1+x} - 2\left(1 - \frac{k}{1+x}\right)}. \end{aligned}$$

Rearranging the above equation gives (40). $\square$

Finally, we prove Claim 2 by specializing Claim 15 to two states.

*Proof of Claim 2.* We begin the proof by explicitly stating the implication of (40) for the binary case. In particular, we show $\underline{w} < \bar{w}$ if and only if one of the following two conditions hold:

(i) $\frac{k}{1+x} > \pi(\underline{\theta})$ and $\frac{k}{1+x} > \frac{1}{3}\pi(\underline{\theta})(\bar{\theta} - \underline{\theta})$.

(ii) $\frac{k}{1+x} \leq \pi(\underline{\theta})$ and $\frac{k}{1+x} < 1 - \frac{1}{3}\pi(\bar{\theta})(\bar{\theta} - \underline{\theta})$.

To prove the above, we consider two cases, depending on the value of $\theta^*$:

- *Case 1:* $k/(1+x) > \pi(\underline{\theta})$. Then $\theta^* = \bar{\theta}$, and the inequality (40) specializes to

$$\bar{\theta} - \mathbb{E}[\theta] < \frac{k}{1+x}\left(3 - \frac{3k}{1+x} + \bar{\theta} - \mathbb{E}[\theta]\right).$$

  Substituting $\bar{\theta} - \mathbb{E}[\theta] = \pi(\underline{\theta})(\bar{\theta} - \underline{\theta})$ and rearranging gives

$$\left(1 - \frac{k}{1+x}\right)\pi(\underline{\theta})(\bar{\theta} - \underline{\theta}) < 3\frac{k}{1+x}\left(1 - \frac{k}{1+x}\right),$$

  which is equivalent to

$$\frac{1}{3}\pi(\underline{\theta})(\bar{\theta} - \underline{\theta}) < \frac{k}{1+x}.$$

  Thus, we have established (i) is sufficient for $\underline{w} < \bar{w}$, and necessary if $\pi(\underline{\theta}) \geq \frac{k}{1+x}$.

- *Case 2:* Suppose now $k/(1+x) \leq \pi(\underline{\theta})$. Then $\theta^* = \underline{\theta}$. Thus, the inequality (40) is now

$$0 < \frac{k}{1+x}\left(3 - \frac{3k}{1+x} + \underline{\theta} - \mathbb{E}[\theta]\right).$$

  Note $\underline{\theta} - \mathbb{E}[\theta] = -\pi(\bar{\theta})(\bar{\theta} - \underline{\theta})$. Therefore, the above inequality is equivalent to

$$\frac{k}{1+x} < 1 - \frac{1}{3}\pi(\bar{\theta})(\bar{\theta} - \underline{\theta}).$$

  Hence, (ii) is sufficient for $\underline{w} < \bar{w}$, and necessary if $\pi(\underline{\theta}) \leq \frac{k}{1+x}$.

Next, we argue that a violation of one of the claim's conditions implies that either (i) or (ii) above hold. Suppose first $\bar{\theta} - \underline{\theta} < 3$. In this case, $\frac{1}{3}\pi(\underline{\theta})(\bar{\theta} - \underline{\theta}) < \pi(\underline{\theta})$, and so (i) holds whenever $\frac{k}{1+x} > \pi(\underline{\theta})$. If $\frac{k}{1+x} \leq \pi(\underline{\theta})$, then (ii) holds, because

$$1 - \frac{1}{3}\pi(\bar{\theta})(\bar{\theta} - \underline{\theta}) > 1 - \pi(\bar{\theta}) = \pi(\underline{\theta}) \geq \frac{k}{1+x}.$$

Suppose now $\bar{\theta} - \underline{\theta} \geq 3$, but (9) fails. Then one of the following inequality chains must hold: either

$$\frac{k}{1+x} > \frac{1}{3}(\bar{\theta} - \underline{\theta})(1 - \pi(\bar{\theta})) = \frac{1}{3}(\bar{\theta} - \underline{\theta})\pi(\underline{\theta}) \geq \pi(\underline{\theta}),$$

or

$$\frac{k}{1+x} < 1 - \frac{1}{3}(\bar{\theta} - \underline{\theta})\pi(\bar{\theta}) \leq 1 - \pi(\bar{\theta}) = \pi(\underline{\theta}).$$

Either way, $\underline{w} < \bar{w}$ holds: the first inequality chain implies (i), whereas the second inequality chain implies (ii).

To conclude the proof, we show that the claim's condition must hold if neither (i) nor (ii) hold. Suppose first that $\frac{k}{1+x} > \pi(\underline{\theta})$, but (i) fails. Then

$$\frac{1}{3}(\bar{\theta} - \underline{\theta})(1 - \pi(\bar{\theta})) = \frac{1}{3}(\bar{\theta} - \underline{\theta})\pi(\underline{\theta}) \geq \frac{k}{1+x} > \pi(\underline{\theta}),$$

meaning $\bar{\theta} - \underline{\theta} \geq 3$, and the right inequality in (9) holds. For the left inequality, note that

$$1 - \frac{1}{3}(\bar{\theta} - \underline{\theta})\pi(\bar{\theta}) \leq 1 - \pi(\bar{\theta}) = \pi(\underline{\theta}) < \frac{k}{1+x}.$$

Suppose now $\frac{k}{1+x} \leq \pi(\underline{\theta})$, but (ii) fails. Then,

$$1 - \pi(\bar{\theta}) \geq \frac{k}{1+x} \geq 1 - \frac{1}{3}(\bar{\theta} - \underline{\theta})\pi(\bar{\theta}).$$

The right inequality above delivers the left inequality in (9). Moreover, the implied inequality between the left most expression and the right most expression implies

$$\frac{1}{3}(\bar{\theta} - \underline{\theta})\pi(\bar{\theta}) \geq \pi(\bar{\theta}),$$

and so $\bar{\theta} - \underline{\theta} \geq 3$. Finally, to get the right inequality in (9), notice that

$$\frac{k}{1+x} \leq 1 - \pi(\bar{\theta}) \leq \frac{1}{3}(\bar{\theta} - \underline{\theta})(1 - \pi(\bar{\theta})),$$

where the last inequality holds because $\bar{\theta} - \underline{\theta} \geq 3$. $\square$

# H. Proofs for Section 7

## H.1. Main Results

In this section, we prove Proposition 2 and Proposition 3. As a first step, we prove a basic lemma about best responses. In what follows, for $p \in \Delta(A \times \Theta)$, $a_i \in \operatorname{supp}_i(p)$, and $b_i \in A_i$, take $u_i\left(b_i, p_{a_i}\right) \in \mathbb{R}$ to be

$$u_i\left(b_i, p_{a_i}\right) = \sum_{a_{-i},\theta} u_i(b_i, a_{-i}, \theta) p_{a_i}(a_{-i}, \theta).$$

**Lemma 10.** *For every $t \in (0,1)$, $p, q \in BCE$, $i \in I$, and $a_i \in \operatorname{supp}_i(p)$,*

$$BR\left((tp + (1-t)q)_{a_i}\right) \subseteq BR\left(p_{a_i}\right).$$

*Proof.* Take $b_i \in BR\left((tp + (1-t)q)_{a_i}\right)$. If $a_i \notin \operatorname{supp}_i(q)$, then $(tp + (1-t)q)_{a_i} = p_{a_i}$, which immediately implies the desired result.

Suppose now that $a_i \in \operatorname{supp}_i(q)$. Since $p, q \in BCE$, we have

$$u_i(a_i, p_{a_i}) \geq u_i(b_i, p_{a_i}) \quad \text{and} \quad u_i(a_i, q_{a_i}) \geq u_i(b_i, q_{a_i}).$$

Simple algebra shows that there exists $s \in (0,1)$ such that

$$(tp + (1-t)q)_{a_i} = sp_{a_i} + (1-s)q_{a_i}.$$

Since $b_i \in BR\left((tp + (1-t)q)_{a_i}\right)$, we obtain that

$$\begin{aligned} su_i(b_i, p_{a_i}) + (1-s)u_i(b_i, q_{a_i}) &= u_i(b_i, sp_{a_i} + (1-s)q_{a_i}) \\ &\geq u_i(a_i, sp_{a_i} + (1-s)q_{a_i}) \\ &= su_i(a_i, p_{a_i}) + (1-s)u_i(a_i, q_{a_i}). \end{aligned}$$

We conclude that $u_i(a_i, p_{a_i}) = u_i(b_i, p_{a_i})$ and $u_i(a_i, q_{a_i}) = u_i(b_i, q_{a_i})$. It follows from $p \in BCE$ that $b_i \in BR\left(p_{a_i}\right)$. $\square$

Next, we show that taking convex combinations of BCEs usually preserve the set of action recommendations that lead to different beliefs.

**Lemma 11.** *For every $p, q \in \Delta(A \times \Theta)$, $i \in I$, and $a_i, b_i \in \text{supp}_i(p)$ with $p_{a_i} \neq p_{b_i}$, there are at most two $t \in (0,1)$ such that*

$$(tp + (1-t)q)_{a_i} = (tp + (1-t)q)_{b_i}. \tag{43}$$

*Proof.* Note that $t \in (0,1)$ is a solution of (43) if and only if for every $a_{-i} \in A_{-i}$ and $\theta \in \Theta$,

$$\begin{aligned} &(tp(a_i, a_{-i}, \theta) + (1-t)q(a_i, a_{-i}, \theta))\,(tp(b_i) + (1-t)q(b_i)) \\ =&(tp(b_i, a_{-i}, \theta) + (1-t)q(b_i, a_{-i}, \theta))\,(tp(a_i) + (1-t)q(a_i))\,. \end{aligned} \tag{44}$$

Each equation (44) is polynomial in $t$, with degree at most two. Since $p_{a_i} \neq p_{b_i}$, at least one such polynomial equation does not have degree zero and, therefore, has at most two solutions. We deduce that (43) has at most two solutions for $t \in (0,1)$. $\square$

Our next goal is to show that minimally mixed BCEs are the norm rather than the exception. As an intermediate step, we first show the set of minimally mixed BCEs is non-empty.

**Lemma 12.** *A minimally mixed BCE exists.*

*Proof.* For every $p \in BCE$, define the set

$$X(p) = \bigcup_i \left\{(a_i, b_i) : a_i, b_i \in \text{supp}_i(p) \text{ and } p_{a_i} \neq p_{b_i}\right\}.$$

Note that $p \in BCE$ is minimally mixed if and only if it has maximal support and for every $q \in BCE$, $X(q) \subseteq X(p)$.

Since the set $A \times \Theta$ is finite and $BCE$ is a convex set, we can find a maximal support $p \in BCE$ such that for every maximal-support $q \in BCE$, the cardinality of $X(p)$ is larger than the cardinality of $X(q)$.

We now show that $p$ is $BCE$-minimally mixed. Fix an arbitrary $q \in BCE$. For every $t \in (0,1)$, define $p^t = tp + (1-t)q$, which is a BCE because $BCE$ is convex. Since $p$ has maximal support, the same is true for $p^t$. Thus, the cardinality of $X(p)$ is larger than the cardinality of $X(p^t)$. By Lemma 11, we can find $t \in (0,1)$ such that $X(p) \subseteq X(p^t)$ and $X(q) \subseteq X(p^t)$. This shows that $X(q) \subseteq X(p)$; otherwise, the cardinality of $X(p^t)$ would be strictly larger than the cardinality of $X(p)$. We conclude that $p$ is minimally mixed. $\square$

We now show that the minimally mixed BCEs includes most BCEs in a precise sense.

**Lemma 13.** *The set of minimally mixed BCEs is open and dense in the set of BCEs.*

*Proof.* Let $P_M$ denote the set of minimally mixed BCEs. We first argue that $P_M$ is open in $BCE$. Towards this goal, note the following sets are open in $BCE$ for every $i \in I$ and $a_i, b_i \in A_i$:

$$\{p \in BCE : p(a_i) > 0\}, \quad \text{and} \quad \{p \in BCE : p(a_i)p(b_i) > 0 \text{ and } p_{a_i} \neq p_{b_i}\}.$$

Since $A$ is finite, we obtain that $P_M$ equals the intersection of a finite number of open subsets of $P_M$. It follows $P_M$ is open in $BCE$.

To see $P_M$ is dense in $BCE$, fix some $q \in BCE$. Take $p$ to be a minimally mixed BCE, which exists by Lemma 12. For every $t \in (0,1)$, define $p^t = tp + (1-t)q$. Because $p$ has maximal support, the same is true for $p^t$ for all $t \in (0,1)$. Moreover, by Lemma 11, a finite set $T \subseteq (0,1)$ exists such that for all $t \in (0,1) \setminus T$, $i \in I$, and $a_i, b_i \in \operatorname{supp}_i(p)$,

$$p_{a_i} \neq p_{b_i} \quad \text{implies} \quad p^t_{a_i} \neq p^t_{b_i}.$$

Thus, $p^t$ is a minimally mixed BCE for all $t \in (0,1) \setminus T$. Thus, $q$ is a limit point of $\{p^t : t \in (0,1) \setminus T\}$, which implies it is a limit point of $P_M$. $\square$

We are now ready to prove Proposition 2 and Proposition 3.

*Proof of Proposition 2.* That (i) implies (ii) follows from Lemma 13.

We now show (ii) implies (iii). Let $q$ be a minimally mixed sBCE. Fix any $p \in BCE$, $i \in I$ and $a_i, b_i \in \operatorname{supp}_i(p)$ such that $p_{a_i} \neq p_{b_i}$. Since $q$ is minimally mixed, $a_i, b_i \in \operatorname{supp}_i(q)$ (because $q$ has maximal support) and $q_{a_i} \neq q_{b_i}$. Thus,

$$\varnothing = BR(q_{a_i}) \cap BR(q_{b_i}) \supseteq J(a_i) \cap J(b_i),$$

where we use the separation constraint, and then the fact that $J(c_i) = \cap_{\tilde{p} \in P} BR(\tilde{p}_{c_i})$ for all $c_i \in A_i$. We conclude (ii) implies (iii).

Finally, we argue (iii) implies (i). Fix any $p \in BCE$. Because $A$ is finite and $BCE$ is convex, it follows from Lemma 10 that we can find $q \in BCE$ such that $q$ has maximal support and

$$BR(q_{a_i}) = J(q_{a_i}) \tag{45}$$

for all $i \in I$ and $a_i \in \text{supp}_i(q)$.

For $t \in (0,1)$, let $p^t = tp + (1-t)q$. We claim that $p^t \in sBCE$. That $p^t \in BCE$ follows from convexity of the BCE set. To see $p^t$ is a sBCE, take any $i \in I$ and $a_i, b_i \in \text{supp}_i(p^t)$ such that $p^t_{a_i} \neq p^t_{b_i}$. Since $q$ has maximal support, $a_i, b_i \in \text{supp}_i(q)$. Then,

$$BR(p^t_{a_i}) \cap BR(p^t_{b_i}) \subseteq BR(q_{a_i}) \cap BR(q_{b_i}) = J(a_i) \cap J(b_i) = \varnothing,$$

where first we use Lemma 10, then (45), and finally Proposition 2-(iii). We conclude $p^t \in sBCE$ for all $t \in (0,1)$. Proposition 2-(i) then follows from $p = \lim_{t\to 1} p^t$. ◻

*Proof of Proposition 3.* It is enough to prove that if $sBCE$ is not nowhere dense in $BCE$, then it is dense in $BCE$. Suppose $sBCE$ is dense in some non-empty set $\tilde{P} \subseteq BCE$ that is open in $BCE$. Let $P_M$ the set of minimally mixed BCEs. Note $\tilde{P} \cap P_M$ is open (in $BCE$) and non-empty by Lemma 13. But $sBCE$ is dense in $\tilde{P}$, and so $sBCE \cap (\tilde{P} \cap P_M)$ must also be non-empty. Thus, we have found a minimally mixed sBCE. That $sBCE$ is dense in $BCE$ then follows from Proposition 2. ◻

## H.2. Checking for Equal Beliefs

To check the conditions of Proposition 2, knowing which actions induce different beliefs for some BCE is useful. In this section, we prove a result that shows how to find actions that lead to different beliefs in a closed convex set of outcomes $P \subseteq \Delta(A \times \Theta)$.[22]

For a player $i$, say an action $a_i$ is **$P$-coherent** if a $p \in P$ exists with $p(a_i) > 0$.[23] Let $\mathbf{0}$ be the all-zeros vector in $\mathbb{R}^{A_{-i}\times\Theta}$; in what follows, we use the convention that $p_{a_i} = \mathbf{0}$ for every $p \in \Delta(A \times \Theta)$ and $a_i \in A_i$ such that $p(a_i) = 0$. We say an outcome $p \in P$ has **$P$-maximal support** if the support of every other $q \in P$ is contained by the support of $p$.

[22] Neither the obedience nor the separation constraint play any role in this section.

[23] Our notion of $P$-coherent is inspired by the notion of coherence in Nau and McCardle's "Coherent behavior in noncooperative games" (Journal of Economic Theory, vol. 50, pp. 424-444, 1990).

**Proposition 6.** *Fix a player $i$ and two $P$-coherent actions $a_i, b_i \in A_i$. Then every $p \in P$ with $a_i, b_i \in \operatorname{supp}_i(p)$ has $p_{a_i} = p_{b_i}$ if and only if one of the following two conditions hold:*

*(i) A $\mu \in \Delta(A_{-i} \times \Theta)$ exists such that for all $p \in \operatorname{ext}(P)$, $\{p_{a_i}, p_{b_i}\} \subseteq \{\mu, \boldsymbol{0}\}$.*

*(ii) A constant $\lambda > 0$ exists such that for all $p \in \operatorname{ext}(P)$, $p(a_i)p_{a_i} = \lambda p(b_i)p_{b_i}$.*

Thus, to know whether a pair of actions leads to the same beliefs in all outcomes in $P$, it is enough to check the extreme points of $P$ for one of two properties. The first property states these actions induce the same beliefs in all of the set's extreme points. The second property requires the likelihood ratio for these actions to be constant across all these extreme points.

To prove the proposition, we need the following lemma.

**Lemma 14.** *Fix a player $i$ and two actions $a_i, b_i \in A_i$. Let $p, q \in \Delta(A \times \Theta)$ such that $\{a_i, b_i\} \subseteq \operatorname{supp}_i(p) \cup \operatorname{supp}_i(q)$. Suppose $r_{a_i} = r_{b_i}$ for all $r \in \{p, q\}$ with $\{a_i, b_i\} \subseteq \operatorname{supp}_i(r)$. If $(tp + (1-t)q)_{a_i} = (tp + (1-t)q)_{b_i}$ for some $t \in (0, 1)$, then one of the following two conditions hold:*

*(i) A $\mu \in \Delta(A_{-i} \times \Theta)$ exists such that for all $r \in \{p, q\}$, $\{r_{a_i}, r_{b_i}\} \subseteq \{\mu, \boldsymbol{0}\}$.*

*(ii) A constant $\lambda > 0$ exists such that for all $r \in \{p, q\}$, $r(a_i) = \lambda r(b_i)$.*

*Proof.* Let $p^t := tp + (1-t)q$. We proceed by contradiction: we assume that Lemma 14-(i) and Lemma 14-(ii) both fail and show that $p^t_{a_i} \neq p^t_{b_i}$.

We begin by noting that one can rewrite the condition that $r_{a_i} = r_{b_i}$ for all $r \in \{p, q\}$ with $\{a_i, b_i\} \subseteq \operatorname{supp}_i(r)$ as

$$r(a_i)r(b_i)r_{a_i} = r(a_i)r(b_i)r_{b_i} \text{ for all } r \in \{p, q\}. \tag{46}$$

Because $\operatorname{supp}_i(p^t) = \operatorname{supp}_i(p) \cup \operatorname{supp}_i(q)$ and $\{a_i, b_i\} \subseteq \operatorname{supp}_i(p) \cup \operatorname{supp}_i(q)$, we have $\{a_i, b_i\} \subseteq \operatorname{supp}_i(p^t)$. Thus, applying Bayes rule, we obtain that $p^t_{a_i} = p^t_{b_i}$ if and only if for every $a_{-i} \in A_{-i}$ and $\theta \in \Theta$, one has

$$p^t(a_i)p^t(b_i, a_{-i}, \theta) - p^t(b_i)p^t(a_i, a_{-i}, \theta) = 0.$$

Expanding the left hand side of the above equation by substituting in the definition of $p^t$, rearranging terms as a polynomial in $t$, and using (46), delivers that the above display equation is equivalent to

$$(t-t^2)\Big[p(a_i)q(b_i,a_{-i},\theta)+q(a_i)p(b_i,a_{-i},\theta)-q(b_i)p(a_i,a_{-i},\theta)-p(b_i)q(a_i,a_{-i},\theta)\Big]=0.$$

Since $t\in(0,1)$, we get that $p^t_{a_i}=p^t_{b_i}$ if and only if for every $a_{-i}\in A_{-i}$ and $\theta\in\Theta$, one has

$$p(a_i)q(b_i,a_{-i},\theta)+q(a_i)p(b_i,a_{-i},\theta)-q(b_i)p(a_i,a_{-i},\theta)-p(b_i)q(a_i,a_{-i},\theta)=0.$$

Writing the above in vector notation delivers that $p^t_{a_i}=p^t_{b_i}$ is equivalent to

$$p(a_i)q(b_i)q_{b_i}+q(a_i)p(b_i)p_{b_i}-q(b_i)p(a_i)p_{a_i}-p(b_i)q(a_i)q_{a_i}=\mathbf{0}. \tag{47}$$

We now divide the proof into cases. Consider first the case in which $\{a_i,b_i\}\subseteq \mathrm{supp}_i(p)\cap\mathrm{supp}_i(q)$. In this case, (46) implies $p_{a_i}=p_{b_i}$ and $q_{a_i}=q_{b_i}$, and so we get that

$$\begin{aligned}p(a_i)q(b_i)q_{b_i}+q(a_i)p(b_i)p_{b_i}-q(b_i)p(a_i)p_{a_i}-p(b_i)q(a_i)q_{a_i}&=\\ =(p(a_i)q(b_i)-p(b_i)q(a_i))(q_{a_i}-p_{a_i})&\neq\mathbf{0},\end{aligned}$$

where the inequality follows from failure of Lemma 14-(i) and Lemma 14-(ii). We conclude (47) fails.

Consider now the case in which $\{a_i,b_i\}\nsubseteq \mathrm{supp}_i(p)\cap\mathrm{supp}_i(q)$. Because Lemma 14-(ii) fails, we can assume $p(a_i)=0<p(b_i)$ without loss of generality. Since the lemma assume $a_i\in\mathrm{supp}_i(p)\cup\mathrm{supp}_i(q)$, it follows $q(a_i)>0$. Therefore, we can use failure of Lemma 14-(ii) to deduce that $p_{b_i}\neq q_{a_i}$. Using these facts, we obtain that

$$p(a_i)q(b_i)q_{b_i}+q(a_i)p(b_i)p_{b_i}-q(b_i)p(a_i)p_{a_i}-p(b_i)q(a_i)q_{a_i}=q(a_i)p(b_i)(p_{b_i}-q_{a_i})\neq\mathbf{0}.$$

It follows that (47) fails. □

We are now ready to prove Proposition 6. The "if" portion is straightforward; the "only if" portion uses Lemma 14.

*Proof of Proposition 6.* We first prove the "if" portion. Let $p\in P$ and $a_i,b_i\in\mathrm{supp}_i(p)$. Let $t^1,\dots,t^n>0$ and $p^1,\dots,p^n\in\mathrm{ext}(P)$ such that $p=\sum_{m=1}^n t^m p^m$.

Simple algebra shows that for all $c_i \in \text{supp}_i(p)$

$$p_{c_i} = \sum_{m=1}^{n} \frac{t^m p^m(c_i)}{\sum_{l=1}^{n} t^l p^l(c_i)} p^m_{c_i}.$$

If Proposition 6-(i) holds, then

$$\begin{aligned} p_{a_i} = \sum_{m=1}^{n} \frac{t^m p^m(a_i)}{\sum_{l=1}^{n} t^l p^l(a_i)} p^m_{a_i} &= \sum_{m=1}^{n} \frac{t^m p^m(a_i)}{\sum_{l=1}^{n} t^l p^l(a_i)} \mu \\ &= \mu \\ &= \sum_{m=1}^{n} \frac{t^m p^m(b_i)}{\sum_{l=1}^{n} t^l p^l(b_i)} \mu = \sum_{m=1}^{n} \frac{t^m p^m(b_i)}{\sum_{l=1}^{n} t^l p^l(b_i)} p^m_{b_i} = p_{b_i}. \end{aligned}$$

Suppose now Proposition 6-(ii) holds. For every $m$, $p^m(a_i)p^m_{a_i} = \lambda p^m(b_i)p^m_{b_i}$ implies $p^m(a_i) = \lambda p^m(b_i)$ and $p^m_{a_i} = p^m_{b_i}$. Thus,

$$p_{a_i} = \sum_{m=1}^{n} \frac{t^m p^m(a_i)}{\sum_{l=1}^{n} t^l p^l(a_i)} p^m_{a_i} = \sum_{m=1}^{n} \frac{t^m \lambda p^m(b_i)}{\sum_{l=1}^{n} t^l \lambda p^l(b_i)} p^m_{b_i} = \sum_{m=1}^{n} \frac{t^m p^m(b_i)}{\sum_{l=1}^{n} t^l p^l(b_i)} p^m_{b_i} = p_{b_i}.$$

This concludes the proof of the proposition's "if" portion.

We now show the proposition's "only if" portion. We proceed by contradiction: we assume that Proposition 6-(i) and Proposition 6-(ii) both fail and show that there exists $p \in P$ such that $a_i, b_i \in \text{supp}_i(p)$ and $p_{a_i} \neq p_{b_i}$. As we are done if $p_{a_i} \neq p_{b_i}$ for some $p \in \text{ext}(P)$ with $a_i, b_i \in \text{supp}_i(p)$, assume $p_{a_i} = p_{b_i}$ holds for all such $p$.

Since Proposition 6-(i) fails, and $a_i$ and $b_i$ are $P$-coherent, there exist $p, q \in \text{ext}(P)$ such that $p(a_i) > 0$, $q(b_i) > 0$, and $p_{a_i} \neq q_{b_i}$. As we are done if $(0.5p + 0.5q)_{a_i} \neq (0.5p + 0.5q)_{b_i}$, assume $(0.5p + 0.5q)_{a_i} = (0.5p + 0.5q)_{b_i}$. Since $p_{a_i} \neq q_{b_i}$, Lemma 14-(i) fails. Thus, Lemma 14-(ii) must hold: there exist $\lambda > 0$ such that $p(a_i) = \lambda p(b_i)$ and $q(a_i) = \lambda q(b_i)$; in particular, $p(b_i) > 0$ and $q(a_i) > 0$.

Since Proposition 6-(ii) fails, there must exist $r \in \text{ext}(P)$ such that $r(a_i) \neq \lambda r(b_i)$; in particular, $r(a_i) > 0$ or $r(b_i) > 0$. Let $c_i \in \{a_i, b_i\}$ such that $r(c_i) > 0$. Since $p_{a_i} \neq q_{b_i}$, either $r_{c_i} \neq p_{a_i}$, or $r_{c_i} \neq p_{b_i}$, or both. Thus, by Lemma 14, either $(0.5p + 0.5r)_{a_i} \neq (0.5p + 0.5r)_{c_i}$, or $(0.5q + 0.5r)_{b_i} \neq (0.5q + 0.5r)_{c_i}$, or both. In any case, we have found $p \in P$ such that $a_i, b_i \in \text{supp}_i(p)$ and $p_{a_i} \neq p_{b_i}$. $\square$