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# https://towardsdatascience.com/fast-fourier-transform-937926e591cb
# Hsu H.P. - Schaum's outline of signals and systems-McGraw Hill (2011).pdf
# page 324
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Discrete Fourier Transform

The Discrete Fourier Transform (DTF) can be written as follows.

$$x[k] = \sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi kn}{N}}$$

Discrete Fourier transform $O\left(N^2\right)$

Fast Fourier transform $O\left(N log_2 N\right)$

Fast Fourier Transform Algorithm

Suppose, we separated the Fourier Transform into even and odd indexed sub-sequences.

$$\begin{cases} n = 2r & \text{if } even \\ n = 2r+1 & \text{if } odd \end{cases}$$

where $r = 1, 2, ..., \frac{N}{2} - 1$

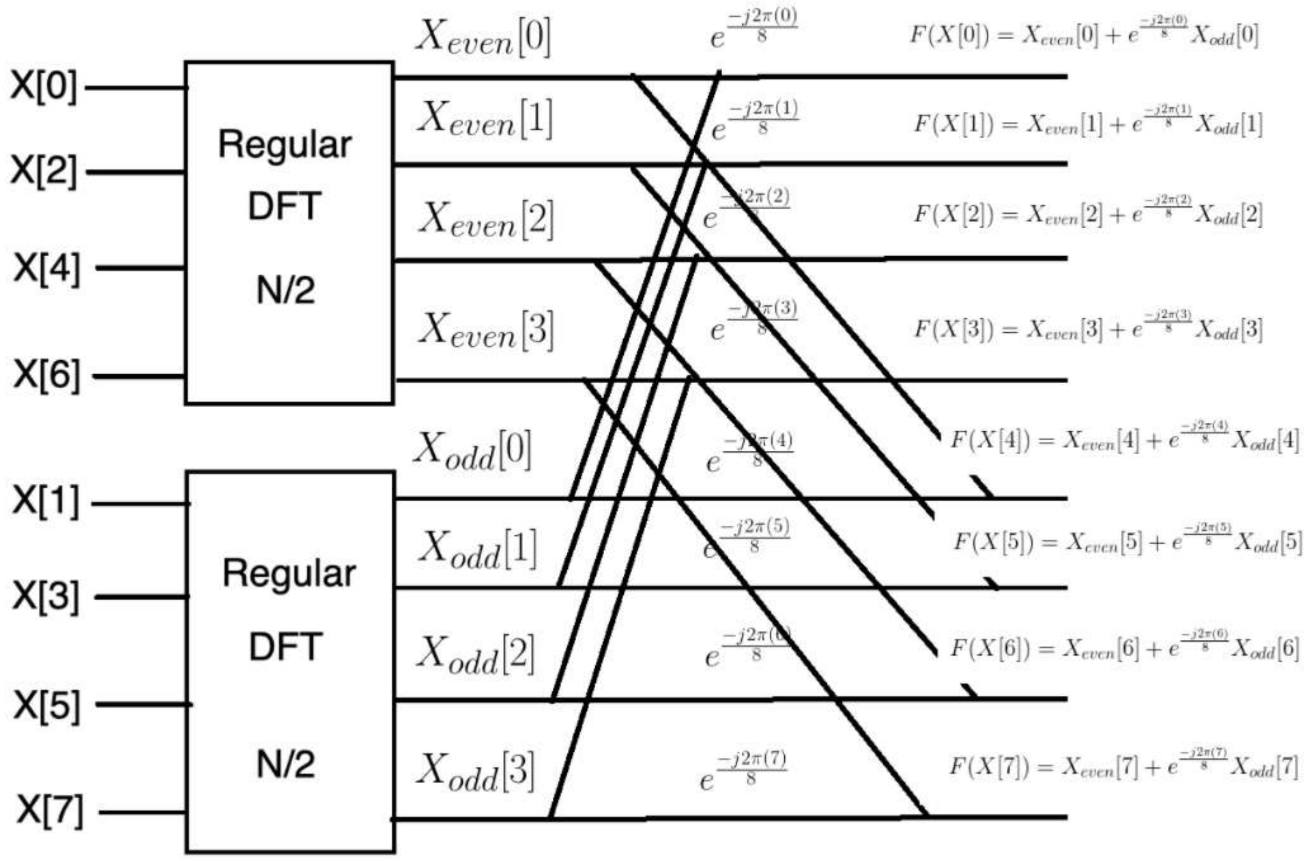
$$x[k] = \sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi kn}{N}}$$

$$x[k] = \sum_{r=0}^{\frac{N}{2}-1} x[2r] e^{\frac{-j2\pi k(2r)}{N}} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1] e^{\frac{-j2\pi k(2r+1)}{N}}$$

$$x[k] = \sum_{r=0}^{\frac{N}{2}-1} x[2r] e^{\frac{-j2\pi k(2r)}{N}} + e^{\frac{-j2\pi k}{N}} \sum_{r=0}^{\frac{N}{2}-1} x[2r+1] e^{\frac{-j2\pi k(2r)}{N}}$$

$$x[k] = \sum_{r=0}^{\frac{N}{2}-1} x[2r] e^{\frac{-j2\pi k(r)}{N/2}} + e^{\frac{-j2\pi k}{N}} \sum_{r=0}^{\frac{N}{2}-1} x[2r+1] e^{\frac{-j2\pi k(r)}{N/2}}$$

$$x[k] = x_{even}[k] + e^{\frac{-j2\pi k}{N}} x_{odd}[k]$$



DFT on N/2 elements

2 DFT

$$x[k] = x_{even}[k] \; + \; e^{\frac{-j2\pi k}{N}} x_{odd}[k]$$

$$2 \times \left(\frac{N}{2}\right)^2 + N$$

$$= \; 2 \times \frac{N^2}{4} + N$$

$$= \; \frac{N^2}{2} + N$$

$$O\left(\frac{N^2}{2} + N\right) \sim O(N^2)$$

What if we keep splitting?

$$\frac{N}{2} \longrightarrow 2\left(\frac{N}{2}\right)^2 + N = \frac{N^2}{2} + N$$

$$\frac{N}{4} \longrightarrow 2\left(2\left(\frac{N}{4}\right)^2 + \frac{N}{2}\right) + N = \frac{N^2}{4} + 2N$$

$$\frac{N}{8} \longrightarrow 2\left(2\left(2\left(\frac{N}{8}\right)^2 + \frac{N}{4}\right) + \frac{N}{2}\right) + N = \frac{N^2}{8} + 3N$$

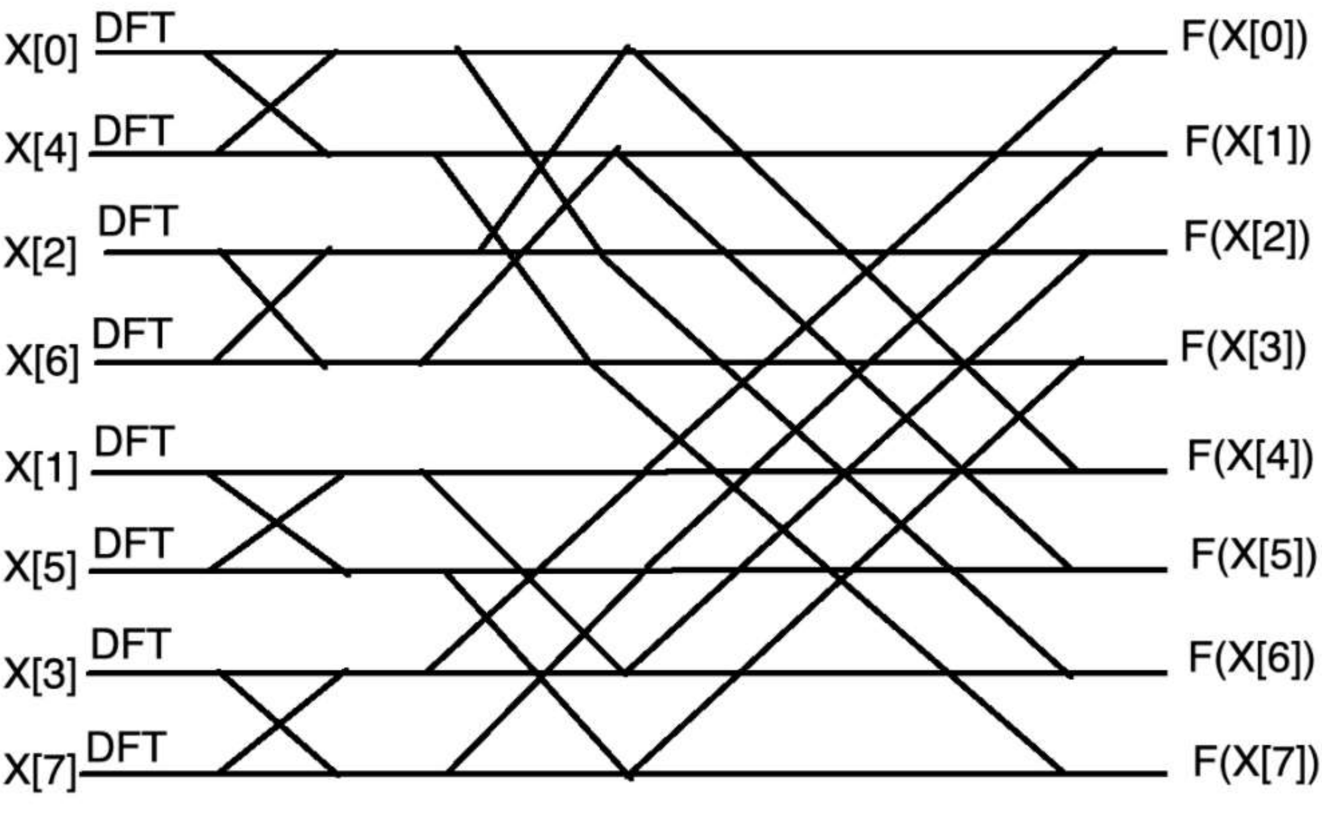
$$\vdots$$

$$\frac{N}{2^P} \longrightarrow \frac{N^2}{2^P} + PN = \frac{N^2}{N} + (\log_2 N)N = N + (\log_2 N)N$$

$$\sim O(N + Nlog_2N) \sim O(Nlog_2N)$$

Split a maximum of $P = \log_2 N$ times

N = 8 , p = 3



(*b*) Let $W_{n+1,k+1}$ denote the entry in the $(n+1)$st row and $(k+1)$st column of the $\mathbf{W}_4$ matrix. Then, from Eq. (6.207)

$$W_{n+1,k+1} = W_4^{nk} = e^{-j(2\pi/4)nk} = e^{-j(\pi/2)nk} = (-j)^{nk} \tag{6.211}$$

and we have

$$\mathbf{W}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \qquad \mathbf{W}_4^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \tag{6.212}$$