APPROCHE MULTI-ÉCHELLE DE LA FORMATION ET L'ÉVOLUTION DES AMAS D'ÉTOILE

Julien Dorval

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INTRODUCTION

Foreword

Looking at the four fundamental forces, gravity is probably the one that we, as a species, take the most for granted. Of course, few of us stop and meditate on the strong and nuclear forces on a daily basis, but we never experience their direct effect. We do not feel the strong nuclear force tying together the protons inside our bodies, neither do we feel the weak nuclear interaction inducing our potassium atoms to decay into calcium. The electromagnetic force is more present in our mind on a daily basis. Even more so since the arrival of the *fée électricité* in our lives and the advent of her child, the electronic age. Even though some manifestations of the electromagnetic force, such as sunlight, are taken for granted, humans keep a sense of wonder about electromagnetism. Magnets, lightning, electromagnetism *feels* magical, as humans have only understood it for a few generations.

What about gravity ? Gravity is part of our mental landscape, we experience its direct effects all the time. If we drop something, it falls, if we throw something, it curves back to the ground, we know this, instinctively. The absence of gravity feels much stranger as our brains evolved under the influence of this fundamental force. Thus we rarely reflect on it.

However, it is by no means the least interesting of forces.

Gravity is the Great Herder, the maker of galaxies, the creator of stars and author of planets. It brings matter together.

0.1 From Aristotle to GPU computing : an history of physics and gravity

0.1.1 Motion

For two thousand years, Aristote physics dominated european philosophy. Rocks fell to the ground because they wanted to join their element, objects in the sky were attached to eternal rotating crystal spheres, and motion was either natural or violent, the latter needing a continuous force to exist. As the importance of projectiles grew in middle-age warfare, some improvement were made to explain trajectories, such as the impetus, a "contained source of motion" imprinted to a projectile by the thrower. Introduced by Philopon in the 6th century and relayed by Avicenne in the 11th century, it was properly formalized by Jean Buridan in the 14th century in his "Questions on Aristotle's Metaphysics". Buridan's impetus had a lot in common with momentum, in that it was proportionnal to mass and velocity, but could be circular, as shown by this description of celestial motion from Buridan:

"God, when He created the world, moved each of the celestial orbs as He pleased, and in moving them he impressed in them impetuses which moved them without his having to move them any more...And those impetuses which he impressed in the celestial bodies were not decreased or corrupted afterwards, because there was no inclination of the celestial bodies for other movements. Nor was there resistance which would be corruptive or repressive of that impetus." (Clagett, 1959)

Despite the conceptual mistake of a circular momentum, Buridan, with this text, is the first to include the motion of celestial bodies in the same framework used for everyday, terrestrial motion. The impetus is not a good model, but it is a model for everything in the universe. No more eternal crystal spheres, everything in the universe must obey the same laws. Scientific revolutions do not happen in a vacuum: Buridan and others paved the way for the intellectual landslide of the 16th and 17th century.

0.1.2 Geocentrism and heliocentrism

While the concept of motion was slowly being refined, our vision of the universe was undergoing some faster changes. The dominant system in Europe since 150AD was the Ptolemaic geocentric model: the Sun and planets went around the Earth, following convoluted trajectories made of circles within circles called epicycles. Though complex, this system was consistent with Aristotle principles of celestial spheres and was accurate to a reasonable extent. Some alternate geocentric models were proposed by arab astronomers, such as Nasir ad-Din at-Tusi and Ibn al-Shatir, as well as rejected attempts to heliocentric models.

Nicolaus Copernicus studied astronomy in Cracow and Bologna, under the influence of hard critics of the ptolemaic system. Strangely, this criticism was not fueled by observations, but by astrology. Astronomy and astrology were closely intertwined, and the chaotic structure of the ptolemaic system made astrological considerations complicated (Barker et al., 2014). In a quest for consistency and simplicity, Copernicus proposed his heliocentric system, published in *De revolutionibus orbium coelestium* in 1543, the year of his death, in which all planets went around the sun, in the correct order. However, clinging to circular orbits, Copernicus had to preserve ptolemaic workarounds such as epicycles.

The astronomical evidence was, at the time, paradoxically against him. The apparent size changes of planets could not be measured yet, as well as stars parallaxes, contradicting heliocentrism. The idea of a moving Earth implied some effect on falling bodies (known today as coriolis effect) but was just not measurable at the time. Building on this apparent counter-evidence and on the work of indian astronomer Nilakantha Somayaji, Tycho Brahe, the most renowned astronomer of his time, proposed an alternative model known as the Tychonic system in the late 16th century (Ramasubramanian, 1998). Brahe maintained the Earth as the center of the universe, circled by the sun, orbited by all other planets. The system was very efficient and was quickly adopted by the Church as in compliance with the Holy Scriptures.

However, the seed of heliocentrism was planted in european scientific minds. The idea exalted the impetuous Giordano Bruno, who pushed the decentralization of Earth to the extreme, claiming stars were other suns, harboring other planets, which themselves could sustain intelligent life. For this, his rejection of catholic dogma and his vehement refusal of retractation, Bruno was burned at the stake on the Campo de Fiori in 1600. Bruno, the fiery dialectist, despised geometry and believed the mind alone could unravel any mystery. The contrast could not be greater with the next mind on our way to the scientific revolution.

Johanes Kepler believed in geometry, in consistency and in observations. Ardent supporter of copernicism, he convinced Tycho Brahe to grant him access to his astronomical data, unsurpassed at the time. Focusing on the motion of Mars, Kepler, through trial and error, found out the planet was moving around the Sun following an ellipse. He formulated his first two laws of planetary motions. Further exploration led him to the third law. The three laws of Kepler were formulated, initiating the mathematisation of astronomy, and with it of all physics.

0.1.3 The Starry Messenger

The father of modern astronomy, and precursor of modern science, Galileo Galilei was born in Pisa in 1564. For the first part of his scientific career, Galileo got famous for his lectures on mechanics and motion. Building on Buridan and Oresme's ideas, he expressed the mathematical form of free fall motion $d = \frac{gt^2}{2}$. Galileo also formulated what was essentially the future first law of motion from Newton.

In 1609, his passion for scientific instruments led Galileo to build his own "dutch perspective glass", or telescope, a pioneering optical device from the netherlands. Once pointed at the sky, the device triggered an avalanche of observations who would forever bury the aristotelitian view of perfect and unchanged heavens. Moving Jupiter satellites, Moon craters and mountains, millions of stars in the Milky Way, these were consigned into *Sidereus Nuncius* (Starry messenger), the first scientific publication of astronomical observations (Galileo, 1610).

Strong advocate of copernicism, but lacking proper evidence, Galileo caused a large controversy with his *Dialogue Concerning the Two Chief World Systems* published in 1632, a pamphlet against the ptolemaic system, presenting (arguably unintentionnaly) one of its advocates as a simpleton. Despite his friendship with the pope, he had to retract his work and reject copernicism. Galileo spent the rest of his life on house arrest. Observationnal evidence at the time still on the side of geocentrism, but the extent of the backslash against Galileo showed the febrility of a Church having absorbed Ptolemy and Aristotle principle into its doctrine, in a time where the debate was shifting from theology to physics and observations.

The relativity of motion is often attributed to Galileo, as he includes it in its controversial pamphlet, stating that a traveller inside a ship sailing smoothly would not be able to tell he's moving. Thus, people could be standing on a moving Earth without feeling it. However, this thought experiment was nothing new at the time and had been a recurring theme of mechanical philosophy since Buridan. Oresme, Copernicus and Bruno had been building on the idea, expanding and improving it, developing over the centuries an implicit understanding of inertia, until Bruno actually gives it a name: *virtù*. Galileo may have met Bruno himself, and had surely been influenced by his writings (De Angelis & Santo, 2015). Galileo's formulation was clearer, and part of a larger understanding of motion, introducing the concept of reference frame. After Copernicus decentralized the Earth, Galileo decentralized human subjectivity itself, setting the scene for the revolution to come.

0.1.4 On the shoulders of giants

Isaac Newton is without a doubt the father of modern mathematical science. Admitted in Cambridge in 1661, Newton supplemented the -still- official aristotelitian teaching with more modern authors: Copernicus, Galileo, Kepler, and most of all, Descartes. The french philospher had a profound impact on the young student, rooting his love for mathematics and deductive reasoning. However, while Descartes showed disdain for experimentation, Newton was an acute observer of the natural world.

In 1666, while in is mother's farm, having been forced out of Cambridge by the Plague, Newton began his reflexion on the motion of celestial bodies. He derived from Kepler's law that the Sun had to exert an inverse squared distance attraction on the planets. Extending the concept to the Earth, moon, and a famous apple, Newton found a way to verify his hypothesis, using data from Galileo mechanical studies on the strength of Earth attraction. The wrong estimate of Earth radius he used at the time introduced a discrepancy which put the young man off his *gravitas* studies for 18 years.

Edmond Halley, astronomer and friend of Newton, having heard of Newton inverse squared law, urged him in 1684 to communicate his work the Royal Society. With a new accurate measure of Earth radius and confronted to a concurrent claim to his law from Robert Hooke (Kramer, 1982), Newton capitulated to Halley's eager enthousiasm and communicated his work in the famous *Philosophi Naturalis Principia Mathematica* (Newton, 1687). Published at Halley's own expense, the Principia shook all of Europe. Newton had invented Calculus (in parallel of Leibniz) and applied it to derive the universal law of Gravitation.

$$F = G\frac{m_1.m_2}{r^2} \tag{1}$$

Where:

F Gravitational attraction between object 1 and object 2
G Gravitational constant, $6.67408.10^{-11} m^3 kg^{-1} s^{-2}$ (Pavese, 2015)
m_i Masses of object 1 and 2
r Distance between object 1 and 2

Though Newton was part of continuous line of geniuses and innovative minds building from each others, as he puts it "If I have seen further it is by standing on the shoulders of giants" (Maury, 1992), his input was truly revolutionnary. He made large advances in optics and mathematics, and created a consistent mathematical framework to compute motions, essentially founding modern science and sowing the seeds of the industrial revolution. This framework is summed up by Newton's three laws of motion (from recent translation Cohen I. B. 1999):

> Law I : Every body persists in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed.
>
> Law II: The alteration of motion is ever proportional to the motive force impress'd; and is made in the direction of the right line in which that force is impress'd.
>
> Law III: To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

The second law can be mathematically formulated in more modern terms:

$$\sum \mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{2}$$

Meaning the sum of all forces $\mathbf{F}$ applied to an object is equal to the time derivative of its momentum $\mathbf{p} = m.\mathbf{v}$.

0.1.5 The n=3 body problem

As the Enlightenment brought a scientific revolution in many fields, I will now limit the discussion to the development of celestial mechanics, while acknowledging input from other fields.

While the two-body problem had been solved by Newton and expanded by Bernoulli in 1710 (Barrow-Green, 1997), in the 18th century the three-body problem remained the object of much investigation and development. A general solution for the Earth-Moon-Sun system would have had applications on nautical astronomy and trans-continental navigation. Extended analytical work by d'Alembert, Clairaut, Euler and Lagrange led to the development of early families of approximate solutions or exact solutions to special cases.

From 1773 to 1793, Joseph-Louis Lagrange, helped by his invention of Lagrangian mechanics, would make a lot of advances on the three-body problem. He introduced the concept of potential and discovered libration points (later known as Lagrange points). In the same time, Pierre-Simon de Laplace proved the stability of the solar system using a newly developped perturbation theory. The solar system dynamics were being unraveled, with finely tuned perturbation computation, but the general three-body problem remained unsolved.

In 1888, Henri Poincar, greatest mathematician of his time, submitted an entry to a contest organized by the King of Sweden Oskar II. The goal was to determine a usable solution to the n-body problem, for any given n. While Poincar does not submit a complete solution, he wins the contest by presenting an in-depth exploration of the phase-space of the restricted three-body problem, which would later give rise to the Chaos theory ,see Yoccoz (2010). Poincar managed to prove that the three-body problem had no solution involving simple functions.

Contrary to popular belief, the three-body problem *has* a solution, it was derived by Karl F. Sundman in 1912 (Sundman, 1912). However, any attempt to obtain accurate trajectory predictions would face tremendous convergence time and is in practice unusable (Belorizky, 1930).

It is interesting to note that Elis Strömgren performed by-hand calculation of a three-body system, see Aarseth (2003); Strömgren (1909), prefiguring the advent of numerical orbit computation.

0.1.6 The n>3 body problem

> "The Sun attracts Jupiter and the other planets, Jupiter attracts its satellites and similarly the satellites act on one another."

By this sentence from the *Principia*, Newton formulates the n-body gravitational problem, an arbitrary number of massive bodies all interacting gravitationally, for the solar system. The "n>3-body" problem didn't receive a lot of attention at first, as the unruly three-body problem was on everyone's mind, and a n>3-body problem seemed abstract, the solar system example being appropriatly dealt in approximations.

In 1764, Charles Messier resolved individual stars in Messier 4, a globular cluster, hundreds of thousands of stars grouped together. Many new clusters were to be found afterwards, extending the catalog of real-life n-body systems. However, nothing was known of their kinematics, the stars were somehow suspended motionless in the sky. This was the case until the advent of Doppler spectroscopy, which allowed astronomer to measure stars velocities (Doppler, 1842). Stellar dynamics had begun.

The n>3-body problem was still inaccessible, so scientists like James Jeans and Arthur Eddington decided to take the problem from the other hand, and took advantage of the large number of stars. Inspired by Poincaré (1906), both astronomers applied the statistical theory of gas to stellar systems, founding the field of stellar dynamics (Jeans, 1916; Eddington, 1916).

An interesting experiment was conducted by Holmberg (1941) to understand the collision of two stellar systems (galaxies). With too few points to warrant a statistical approach, and before the rise of numerical integration, Holmberg modelled two galaxies with lightbulbs and photocells, measuring the attractive force with the amount of light received in each direction, taking advantage of the inverse squared fall of luminosity with distance, akin to gravity.

0.1.7 The numerical age

The first numerical N-body computations were performed by Sebastian Von Hoerner in 1959 when visiting the University of Tübingen, on a Siemens 2002, a cutting edge calculator at the time. The very first had N=4. Then, Von Hoerner, back in Heidelberg, worked his way up to 16 stars, then 25, programming and debugging on punch cards. This story was told by Von Hoerner himself in von Hoerner (2001). He very quickly realized the importance of binary stars and their impact on computations. He was also able to confirm some theoretical prediction on cluster dynamics, and found an interesting radial density profile with a center cusp (von Hoerner, 1960, 1963).

There was two ways to increase the number of stars in simulations: buy a better computer or improve the algorithm. Sverre Aarseth got invested in the second path, which would take over his scientific life. Aarseth pioneered the use of individual time-step, changing the rate at which particles positions are updated, gravitationnal softening, allowing convergence for close approach, and polynomial prediction for force calculations (Aarseth & Hoyle, 1964). As power and optimization grew, investigations expanded, such as the interaction star-gas (van Albada, 1968b) and binary formation (van Albada, 1968a).

The 1970s brought two new important optimisation methods: KS regularization of close pairs (Aarseth, 1972) and Ahmad-Cohen neighbour scheme (Ahmad & Cohen, 1973). The number of

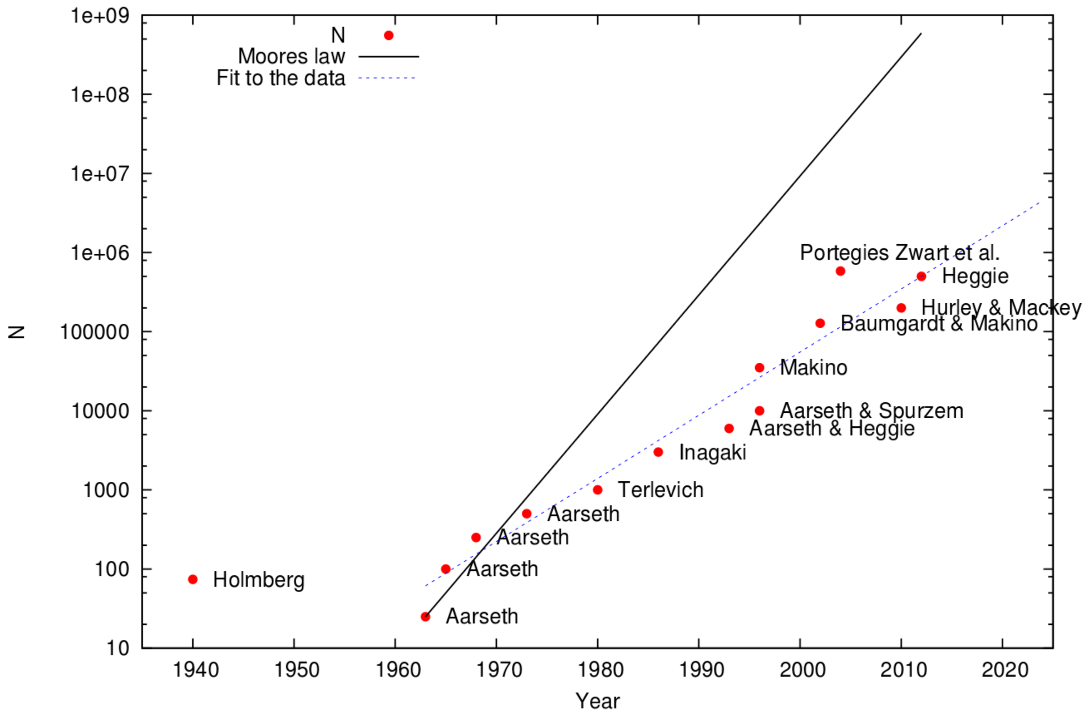

Figure 1: The evolution of the number of particles in N-body simulations. Solid line shows the Moore law. The figure was taken from Bédorf & Portegies Zwart (2012).

stars in simulations kept growing, reaching 1000 with Terlevich (1980) and materializing into the *NBODY5* integrator. At this point various methods departing from a pure collisional calculation began to emerge, such as the simplified distant interaction with the Barnes & Hut (1986) tree algorithm.

To go beyond the regular improvement of computing power with time, a group of japanese researchers, among whom Junichiro Makino, designed and built special purpose hardware for many-body problems, such as n-body gravitational interaction (Ebisuzaki et al., 1990; Ito et al., 1991). These cards vastly improved the speed of nbody simulations and were a milestone on the road to the parallelization of computing. With the force calculation directly implemented in the hardware, GRAPE dominated the field for 15 years.

The latest technological leap in Nbody simulations came from graphic cards, see Bédorf & Portegies Zwart (2012) for a more detailed historical perspective. Graphical Processing Units, or GPU, were originally designed for computer games visual rendering, applying the same transformations to a lot of pixels at the same time. These made them very efficient parallel computing machines for physics. Interest in GPU computing started to grew in the 2000s (Nyland, Harris & Prins, 2004; Elsen et al., 2006; Portegies Zwart, Belleman & Geldof, 2007) until the advent of usable GPU programming languages, like CUDA, in the late 2000s. At this point GPU were more efficient than GRAPE hardware for force calculation. Keigo Nitadori and Sverre Aarseth developped a GPU-accelerated version of the latest NBODY code, NBODY6, in 2012 (Nitadori & Aarseth, 2012).

Last year, 329 years after the publication of the *Principia*, a collisional nbody simulation of one million stars was performed with a modified version of NBODY6 running on GPU (Wang et al., 2015). Computers have made it possible for humans to study systems of incredible scales in space and time, only using the universal law of gravitation. N-body numerical integrators are the culmination of centuries of scientific development on the motion of massive bodies.

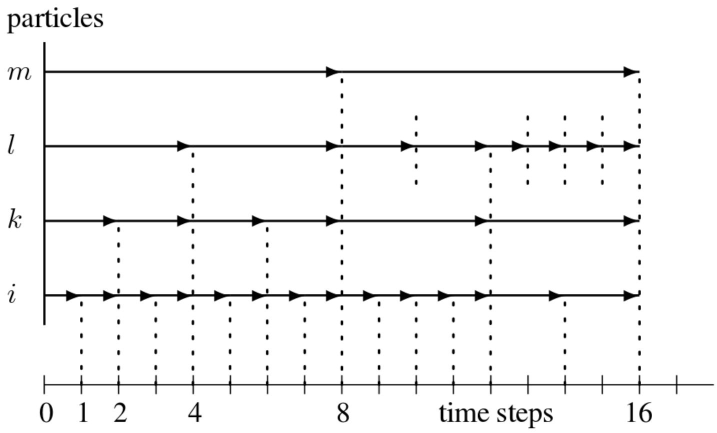

Figure 2: Illustration of block time steps on 4 particles. Particles get their positions updated for each arrow symbol, common time steps are shown as vertical dotted lines. Figure from NB6++ User Manual.

0.2 NBODY6

NBODY6 is the second youngest iteration of the NBODY family, a suite of n-body integrators created by Sverre Aarseth. It can compute the gravitational interaction between up to 128,000 stars in a collisional fashion, meaning there is no softening of the potential, at any scale. This allows for very close binaries to form and remain in the system. To achieve its impressive performances, NBODY6 relies on several optimization technique which have been developed in the 1960s and 1970s, and improved ever since. Here will be developped four major features of NBODY6, in chronological order of their implementation: block time-step, KS-regularization, Hermite scheme and Ahmad-Cohen neighbour scheme. A full description can be found in Sverre Aarseth's book (Aarseth, 2003). Inspiration for this section should be credited to the user manual of NBODY6++, written by Emil Khalisi and Rainer Spurzem.

0.2.1 Block time-step

In the first Nbody simulations, the system was integrated with an universal time-step, determined by the most accelerated star. A star in the outer regions of the cluster with a small velocity did not need to be updated that often. One of the first improvement was the introduction of individual time-step: each star is attributed its own time-step, depending on the force that is applied to it and its derivative:

$$\Delta t_i = \eta \sqrt{\frac{|\mathbf{F_i}||\mathbf{F_i^{(2)}}| + |\mathbf{F_i^{(1)}}|^2}{|\mathbf{F_i^{(1)}}||\mathbf{F_i^{(3)}}| + |\mathbf{F_i^{(2)}}|^2}} \tag{3}$$

With $\mathbf{F}_i^{(j)}$ begin the j-th derivative of the force applied to particle i and η a user-defined accuracy parameter. Such a complex formulation is the result of extensive tests and is quite robust for many special cases. Individual time-steps leads to desynchronized particles, hence the need to interpolate the positions of other particles to compute $\mathbf{F}_i$, which was achieved through fourth-order polynoms.

To limit the amount of desynchronization, block-time steps were introduced. Instead of having as many time steps as particles, one only allows quantized power of 2 of an initial time step. $\Delta t_0, \frac{\Delta t_0}{2}, \frac{\Delta t_0}{4}, \frac{\Delta t_0}{2^i}$. All time steps are then commensurate and regularly fall back on the same time steps, minimizing the amount of interpolation during the force calculations.

0.2.2 KS-regularization

0.2.3 Hermite prediction scheme

0.2.4 Ahmad-Cohen neighbour scheme

0.3 Young star clusters

CHAPTER 1

The Hubble-Lemaître fragmented model

1.1 How to build a Hubble-Lemaître model

1.1.1 Initial state

The first step to obtain a HL-fragmented model is to build an uniform sphere model. The N stars, depending on the required membership, have to be distributed randomly in space inside a certain radius, producing an uniform density. This can be achieved by sampling separately the distance to the center and the angular position of each star, in a method analog as used in Aarseth, Hénon & Wielen (1974) for a Plummer model. The distance to the center should be sampled from the function:

$$f_R(X) = R_0 X^2 \tag{1.1}$$

With R_0 the bouding radius and X a random variable following a uniform probability law between 0 and 1. A direct uniform law for the radius would overpopulate the outer regions. The angles ϕ and θ, respectively azimuthal and polar angle in the physics convention, should be sampled from:

$$f_\phi(X_1) = 2\pi X_1 \tag{1.2}$$
$$f_\theta(X_2) = \arccos{(X_2)} \tag{1.3}$$

With X_1 following a uniform probability law between 0 and 1 and X_2 between -1 and 1. The cartesian coordinates are then found:

$$x = R\sin\theta\cos\phi \tag{1.4}$$
$$y = R\sin\theta\sin\phi \tag{1.5}$$
$$z = R\cos\theta \tag{1.6}$$
$$\tag{1.7}$$

The N particles are then homogeneously distributed in space in a sphere of radius R_0. The next step is to attribute velocities. Unlike other models like the Plummer model, the velocities are here straightforward. We use the well known velocity field of neighbouring galaxies: velocities are radial from the Milky Way, larger with increasing distances, taking the form:

$$\mathbf{v} = \mathrm{H}_0\mathbf{r}, \tag{1.8}$$

with H_0 being an equivalent of the well-known Hubble parameter. For historical accuracy, I added the name of Georges Lemaître when I named my model. It has now been shown that the astronomical observations of redshifted galaxies and its interpretation as the consequence of an expanding universe predated Hubble's paper (Hubble, 1929). Georges Lemaître had published his conclusion on an expanding universe two years earlier (Lemaître, 1927). The account of this can be found in Kragh & Smith (2003); van den Bergh (2011) and Freeman et al. (2015).

An appropriate H_0 to obtain a fragmented subvirial model has to be inferior to 1.4 (see next section). The model obtained from this is then evolved through a nbody integrator, which in my case is NBODY6.

1.1.2 Fragmentation

The cluster expands, driven by the initial Hubble-Lemaître velocity field. During this expansion, poissonian fluctuation in density from the uniform model starts to grow: the part of the cluster with more mass initially attract more stars, forming clumps, clumps merge, spontaneously building substructure. These clumps will be analyzed in another section. If H_0 is well chosen, the expansion stops at some point, the initial kinetic energy has been spent and converted to potential energy: the cluster is now larger, substructured and subvirial, about to collapse. The time of the end of the expansion and the critical value of H_0 can be derived from Newton's second law applied to an expanding spherical shell of matter.

We start from a uniform sphere of radius R_0, total mass M. We consider spherical shells as mass elements, situated at distance r from the origin. As previously said, they are attributed a radial velocity following (for the shell at $r = R_0$) $\vec{v}_0 = H_0\vec{R}_0 = H_0 R_0 \vec{u}_r$. We want to follow the radial motion of the last shell of mass m, situated at R from the origin. Newton's second law gives:

$$m\frac{dv}{dt} = -\frac{GMm}{R^2} \tag{1.9}$$

By multiplying on both sides by v and integrating between a given time and $t = 0$, one finds:

$$v^2(t) - v_0^2 = 2GM\left(\frac{1}{R} - \frac{1}{R_0}\right) \tag{1.10}$$

Which becomes, by taking $\nu = v/v0$, $x = R/R0$ and $E_* = \frac{2GM}{R_0 v_0^2}$, which is a dimensionless measure of the total energy of the system:

$$\nu^2 = 1 + E_*\left(\frac{1}{x} - 1\right). \tag{1.11}$$

The evolution of the system has 3 outcomes, depending on the value of E_*:

- $E_* < 1$ The velocity is always strictly positive as the system expands ($x- > \infty$). The system is unbound.
- $E_* = 1$ The velocity approaches zero as the system expands. The expansion "stops at an infinite radius". The system is marginally bound.
- $E_* > 1$ The velocity reaches zero for a finite radius, the system is bound and will collapses back on itself once the expansion stops.

We only consider in the following the case in which $E_* < 1$. We have the expression

$$\nu = \sqrt{1 + E_*\left(\frac{1}{x} - 1\right)} \tag{1.12}$$

Taking the time derivative gives:

$$\frac{d\nu}{dt} = -\frac{E_*}{2x^2}\left[1 + E_*\left(\frac{1}{x} - 1\right)\right]^{-\frac{1}{2}}\frac{dx}{dt} \tag{1.13}$$

Combining this with (1.9), one obtains:

$$\frac{dx}{dt} = \mathrm{H}_0\sqrt{1 + E_*\left(\frac{1}{x} - 1\right)} \tag{1.14}$$

which can be rewritten, using $\tilde{\mathrm{H}_0} = \mathrm{H}_0\sqrt{E_* - 1}$ and $x_t = \frac{E_*}{E_* - 1}$

$$\frac{dx}{dt} = \tilde{\mathrm{H}_0}\sqrt{\frac{x_t}{x} - 1} \tag{1.15}$$

x_t being the extent of the maximum expansion as we assumed a bound system. The subscript t is for "turn-around". If we choose the notation $u = \frac{x}{x_t}$:

$$\sqrt{\frac{u}{u-1}}\frac{du}{dt} = \frac{\tilde{\mathrm{H}_0}}{x_t} \tag{1.16}$$

We know that x varies from 1 to x_t, thus u varies from $1/x_t$ to 1. We can then make the change of variable $u = \sin^2\theta$ and separate the variables:

$$\sqrt{\frac{\sin^2\theta}{1 - \sin^2\theta}}2\sin\theta\cos\theta d\theta = \frac{\tilde{\mathrm{H}_0}}{x_t}dt \tag{1.17}$$

which becomes after simplifications:

$$[1 - \cos(2\theta)]d\theta = \frac{\tilde{\mathrm{H}_0}}{x_t}dt. \tag{1.18}$$

We now integrate the expression from $t = 0$ to t, the time at which the expansions stops and x reaches x_t (wich implies $u_t = 1$ and $\theta_t = \pi/2$):

$$\int_{\theta_0}^{\pi/2}[1 - \cos(2\theta)]d\theta = \int_0^t \frac{\tilde{\mathrm{H}_0}}{x_t}dt \tag{1.19}$$

$$\frac{\pi}{2} - \theta_0 + \frac{\sin(2\theta_0)}{2} = \frac{\tilde{\mathrm{H}_0}}{x_t}t \tag{1.20}$$

$$\pi - 2\theta_0 + \frac{2}{\sqrt{x_t}}\sqrt{1 - \frac{1}{x_t}} = 2\frac{\tilde{\mathrm{H}_0}}{x_t}t \tag{1.21}$$

which boils down to the expression of the time at which the expansion stops:

$$t = \frac{E_*\left(\frac{\pi}{2} - \theta_0\right) + \sqrt{E_* - 1}}{\mathrm{H}_0(E_* - 1)^{-\frac{3}{2}}}. \tag{1.22}$$

Recalling the quantities:

$$E_* = \frac{2GM}{R_0 v_0^2}; \qquad x_t = \frac{E_*}{E_* - 1}; \qquad \theta_0 = \sin^{-1}\left(\frac{1}{\sqrt{x_t}}\right) \tag{1.23}$$

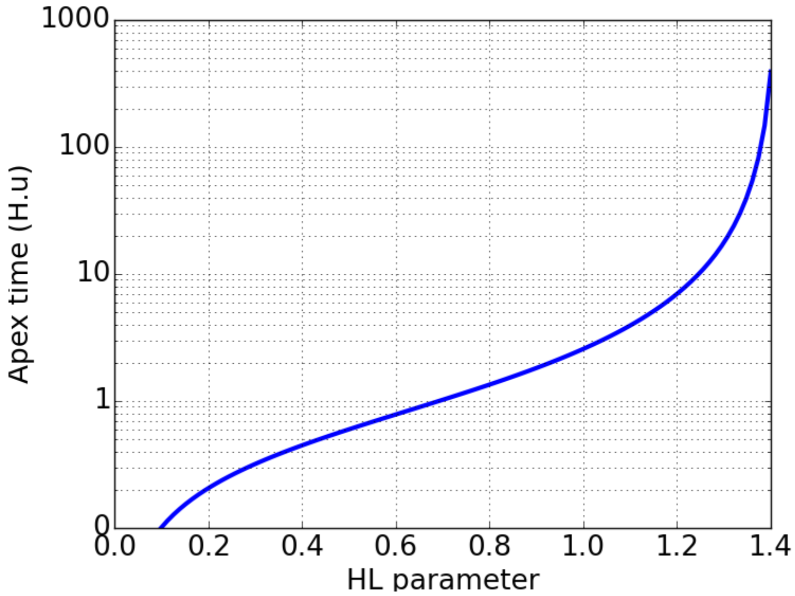

Figure 1.1:

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