

Digital Image Processing

COSC 6380/4393

Lecture – 13

Feb. 28th, 2023

Pranav Mantini

Slides from Dr. Shishir K Shah and Frank (Qingzhong) Liu,
S. Narasimhan

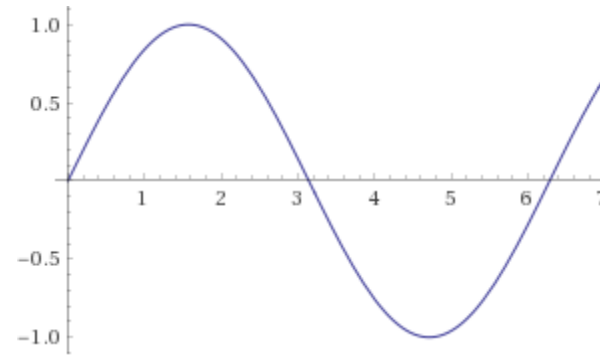
Discrete Fourier Transform (DFT)

Frequency domain analysis and Fourier Transform

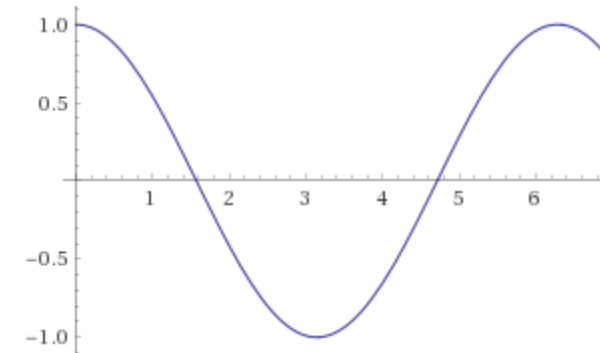
- Periodic Function

Recap: Sin and Cos

$\sin(t)$

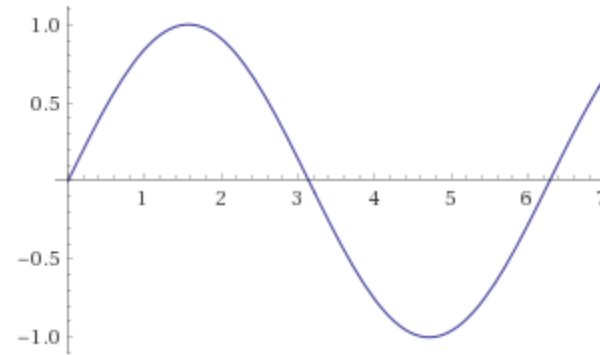


$\cos(t)$



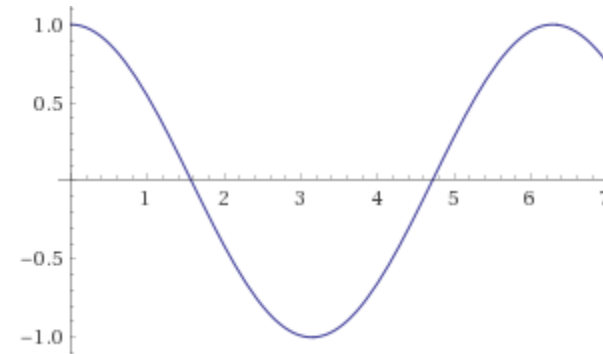
Recap: Sin and Cos

$\sin(t)$



Period = 2π

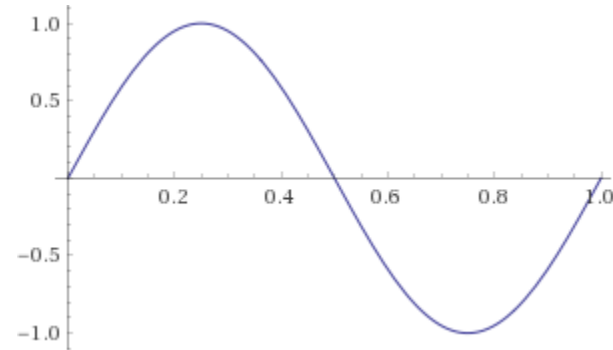
$\cos(t)$



Period = 2π

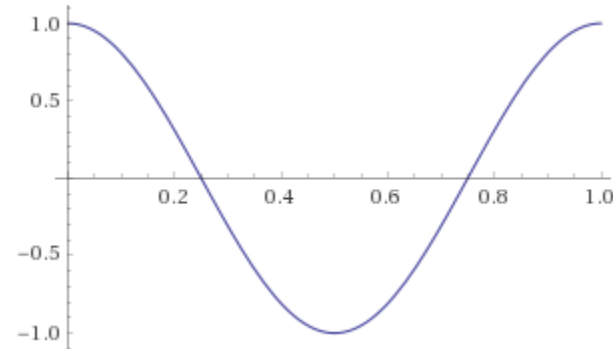
Recap: Sin and Cos

$\sin(?)$



Period = 1

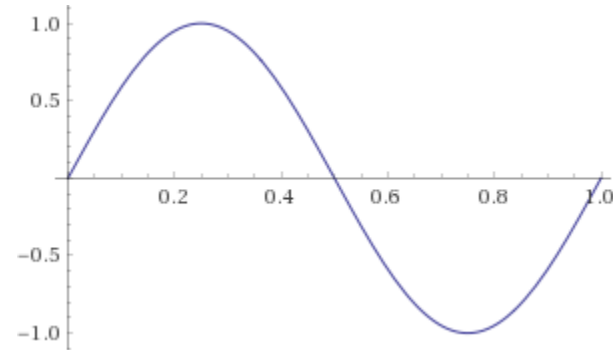
$\cos(?)$



Period = 1

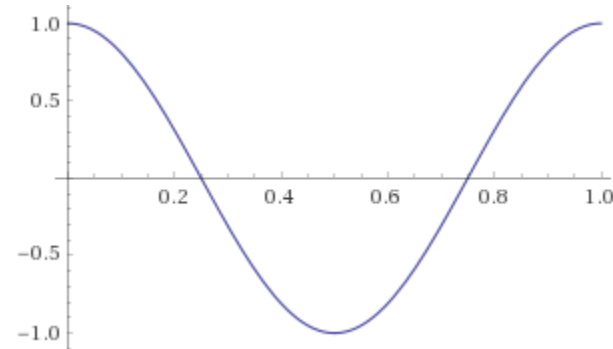
Recap: Sin and Cos

$\sin(2\pi t)$



Period = 1

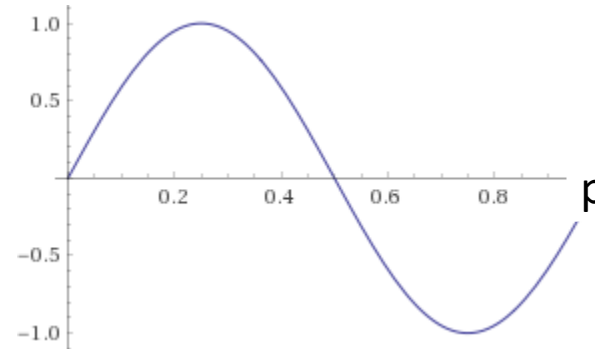
$\cos(2\pi t)$



Period = 1

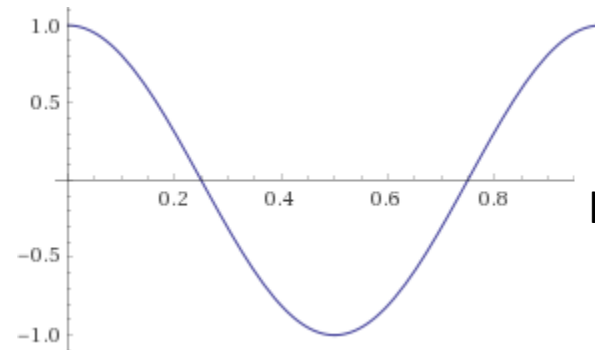
Recap: Sin and Cos

$\sin(?)$



Period = p

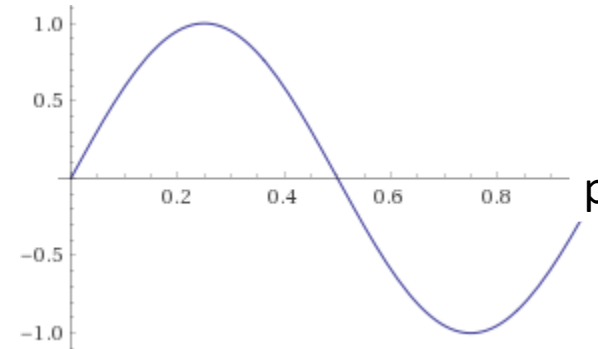
$\cos(?)$



Period = p

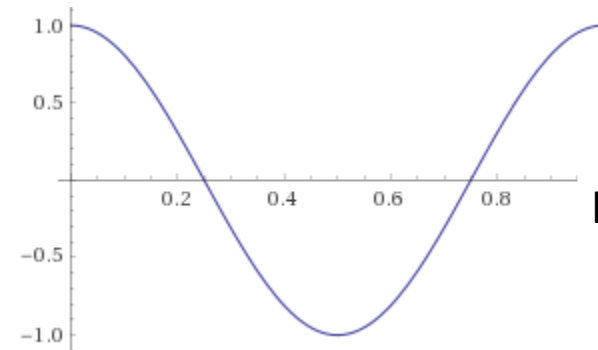
Recap: Sin and Cos

$$\sin \left(2\pi \left(\frac{1}{p} \right) t \right)$$



Period = p

$$\cos \left(2\pi \left(\frac{1}{p} \right) t \right)$$



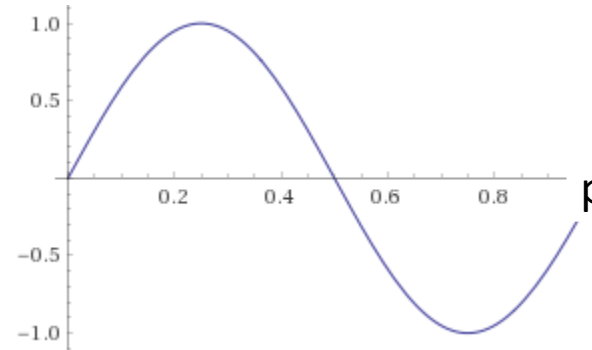
Period = p

Recap: Sin and Cos

Frequency (f) = Number of times it repeats in unit time.

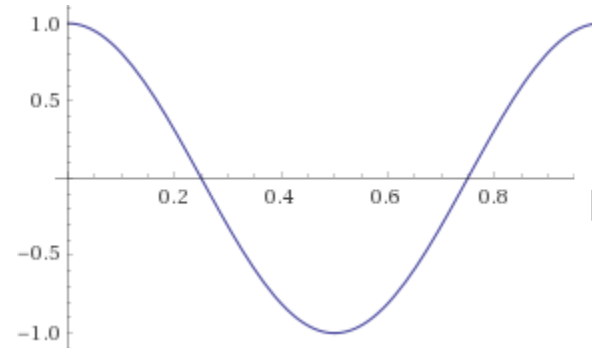
→ $fp = 1$

$$\sin \left(2\pi \left(\frac{1}{p} \right) t \right)$$



Period = p

$$\cos \left(2\pi \left(\frac{1}{p} \right) t \right)$$



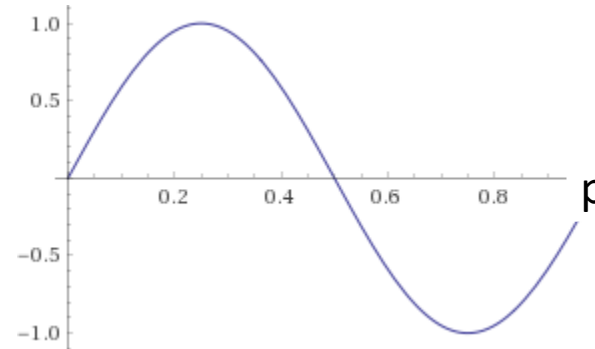
Period = p

Recap: Sin and Cos

Frequency (f) = Number of times it repeats in unit time.

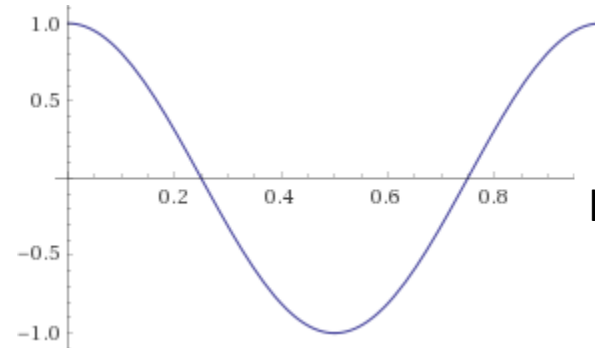
$$\rightarrow f = 1/p$$

$$\sin(2\pi f t)$$



Period = p

$$\cos(2\pi f t)$$

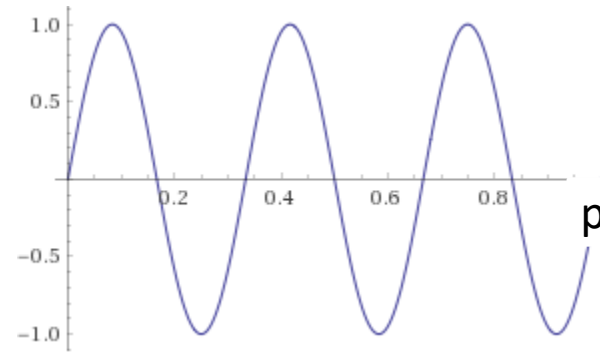


Period = p

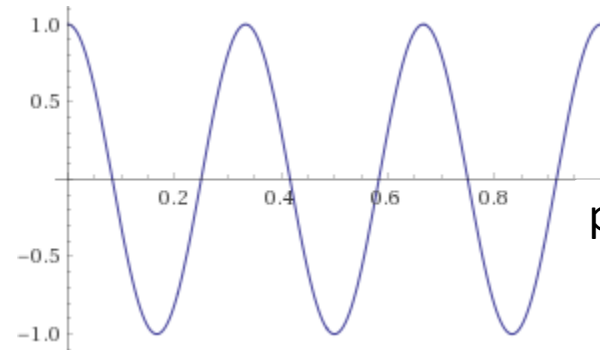
Recap: Sin and Cos

$$f = 1/p$$

sin (?)



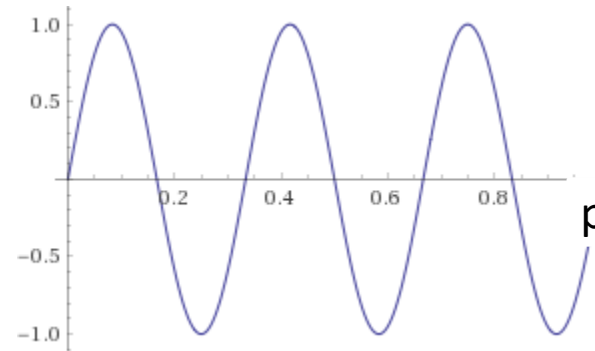
cos (?)



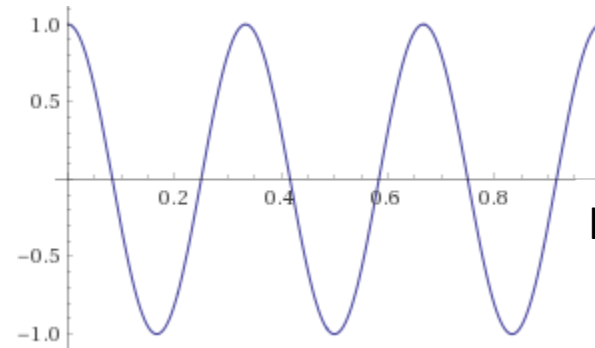
Recap: Sin and Cos

$$f = 1/p$$

$$\sin(2\pi 3f t)$$

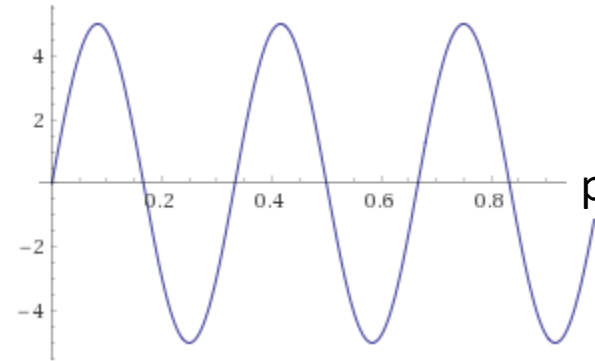


$$\cos(2\pi 3f t)$$

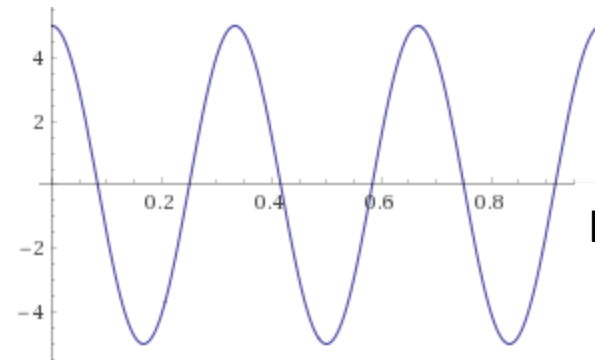


Recap: Sin and Cos

$\sin (?)$



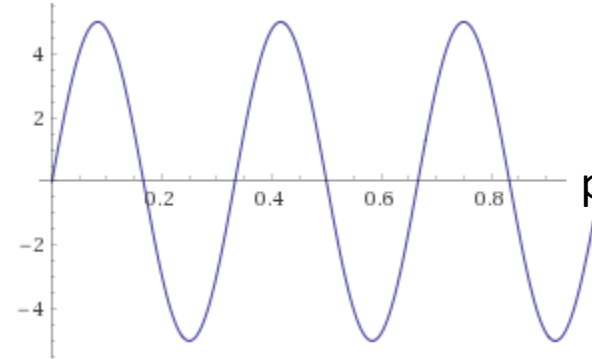
$\cos (?)$



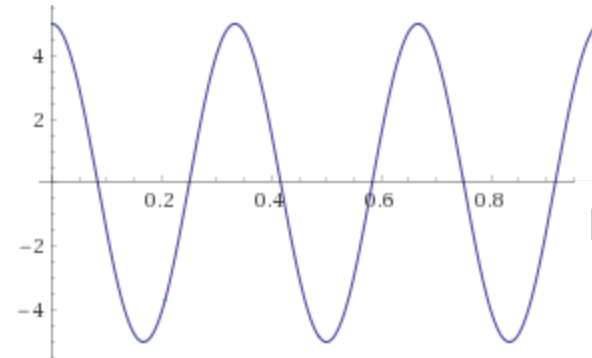
Recap: Sin and Cos

Amplitude = 5

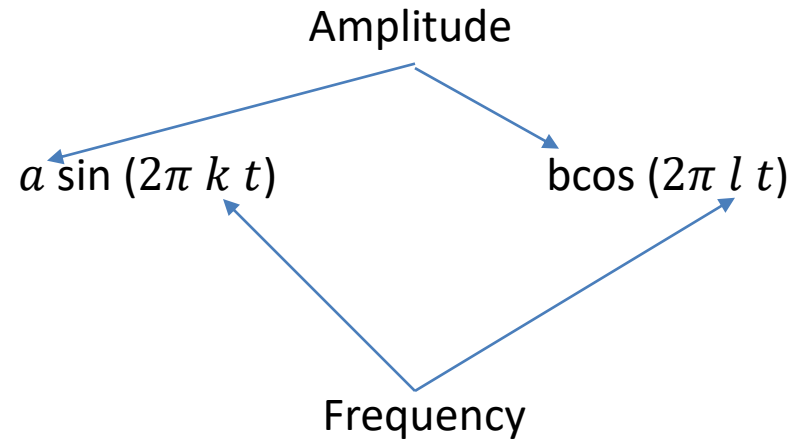
$$5\sin(2\pi 3f t)$$



$$5\cos(2\pi 3f t)$$



Recap: Sin and Cos



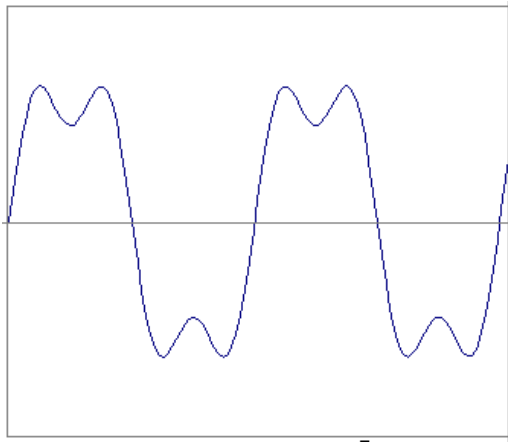
Jean Baptiste Joseph Fourier (1768-1830)

- Had crazy idea (1807): Any periodic function can be rewritten as a weighted sum of **Sines** and **Cosines** of different frequencies.
- Don't believe it?
 - Neither did Lagrange, Laplace, Poisson and other big wigs
 - Not translated into English until 1878!
- But it's true!
 - called **Fourier Series**
 - Possibly the greatest tool used in Engineering



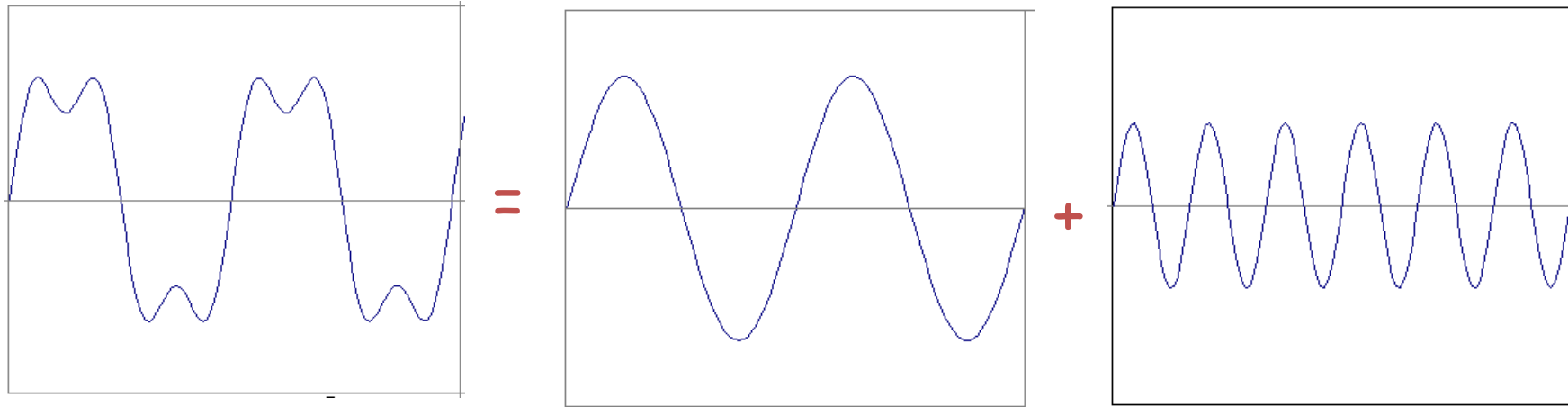
Time and Frequency

- example : $g(t)$



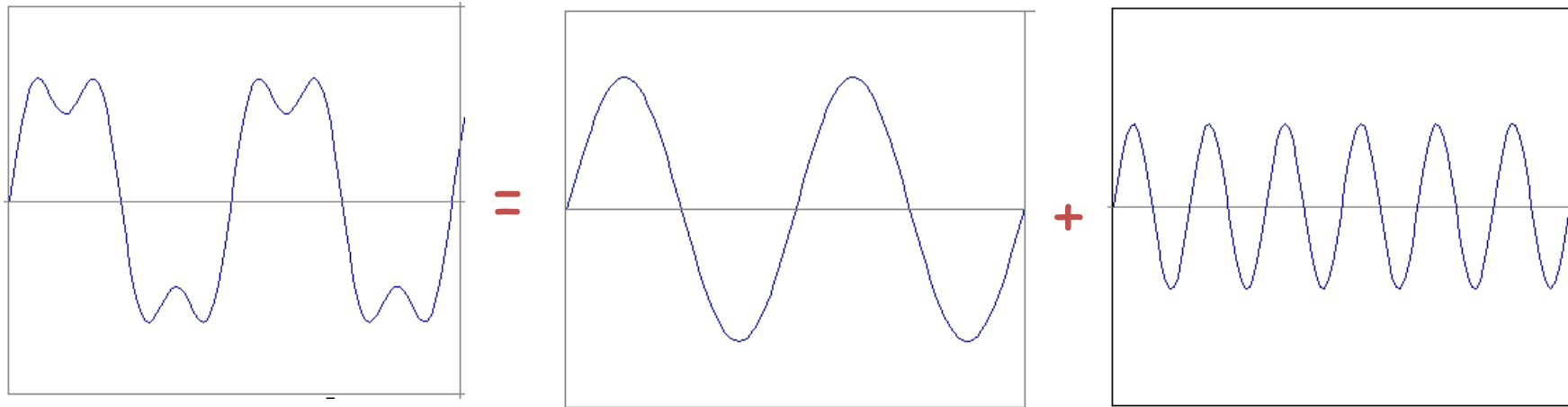
Time and Frequency

- example : $g(t)$



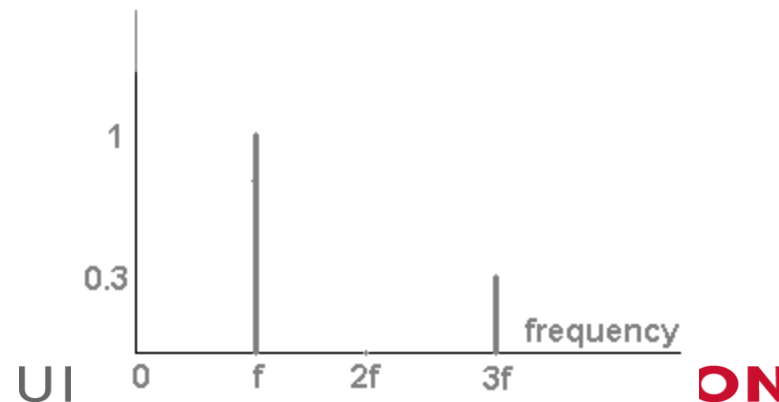
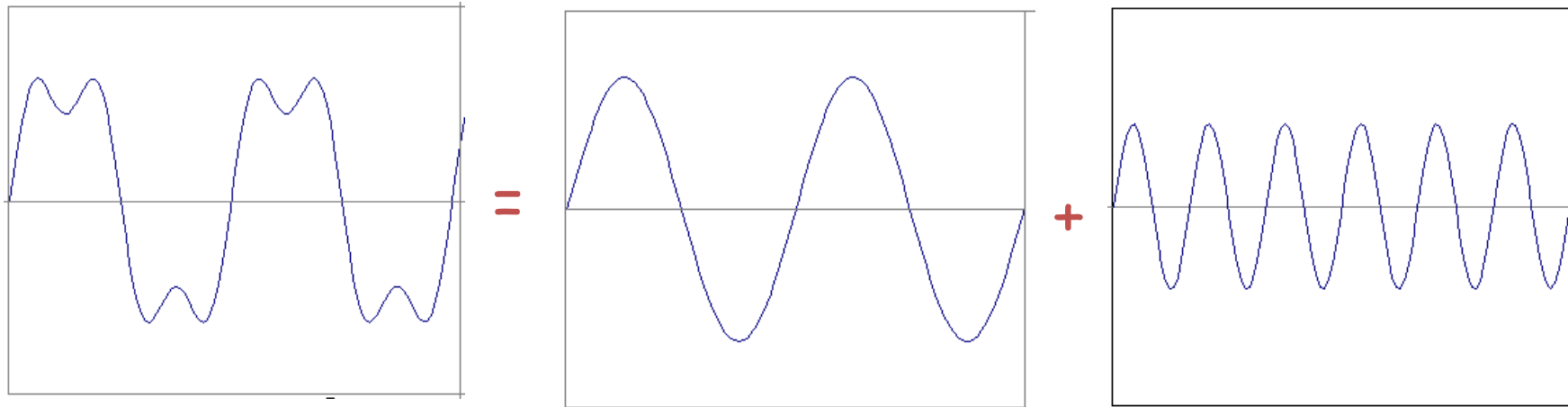
Time and Frequency

- example : $g(t) = \sin(2\pi f t) + (1/3)\sin(2\pi (3f) t)$



Frequency Spectra

- example : $g(t) = \sin(2\pi f t) + (1/3)\sin(2\pi (3f) t)$



Periodic Function

Sum of sine and cosine waves:

$$f(t)$$

Periodic Function

Sum of sine and cosine waves:

$$f(t) = a_0 + a_1 \cos(t) + a_2 \cos(2t) + \dots \\ b_1 \sin(t) + b_2 \sin(2t) + \dots$$

Recap

$$\int_0^{2\pi} \sin(mt) dt = ?$$

$$\int_0^{2\pi} \cos(mt) dt = ?$$

$$\int_0^{2\pi} \sin(mt) \cos(nt) dt = ?$$

- $\int_0^{2\pi} \sin(mt) \sin(nt) dt = ?$
($\forall m \neq n$)
- $\int_0^{2\pi} \sin(mt) \sin(nt) dt = ?$ ($m = n$)
- $\int_0^{2\pi} \cos(mt) \cos(nt) dt = ?$
($\forall m \neq n$)
- $\int_0^{2\pi} \cos(mt) \cos(nt) dt = ?$ ($m = n$)

Recap

$$\int_0^{2\pi} \sin(mt) dt = 0$$

$$\int_0^{2\pi} \cos(mt) dt = 0$$

$$\int_0^{2\pi} \sin(mt) \cos(nt) dt = 0$$

- $\int_0^{2\pi} \sin(mt) \sin(nt) dt = ?$
 $(\forall m \neq n)$
- $\int_0^{2\pi} \sin(mt) \sin(nt) dt = ? \quad (m = n)$
- $\int_0^{2\pi} \cos(mt) \cos(nt) dt = ?$
 $(\forall m \neq n)$
- $\int_0^{2\pi} \cos(mt) \cos(nt) dt = ? \quad (m = n)$

Recap

$$\int_0^{2\pi} \sin(mt) dt = 0$$

$$\int_0^{2\pi} \cos(mt) dt = 0$$

$$\int_0^{2\pi} \sin(mt) \cos(nt) dt = 0$$

Product Identities

$$\sin(x) \cos(y) = \frac{1}{2} [\sin(x+y) + \sin(x-y)]$$

$$\cos(x) \sin(y) = \frac{1}{2} [\sin(x+y) - \sin(x-y)]$$

$$\cos(x) \cos(y) = \frac{1}{2} [\cos(x-y) + \cos(x+y)]$$

$$\sin(x) \sin(y) = \frac{1}{2} [\cos(x-y) - \cos(x+y)]$$

- $\int_0^{2\pi} \sin(mt) \sin(nt) dt = ?$
($\forall m \neq n$)
- $\int_0^{2\pi} \sin(mt) \sin(nt) dt = ?$ ($m = n$)
- $\int_0^{2\pi} \cos(mt) \cos(nt) dt = ?$
($\forall m \neq n$)
- $\int_0^{2\pi} \cos(mt) \cos(nt) dt = ?$ ($m = n$)

Recap

$$\begin{aligned}\int_0^{2\pi} \sin(mt) dt &= 0 \\ \int_0^{2\pi} \cos(mt) dt &= 0 \\ \int_0^{2\pi} \sin(mt) \cos(nt) dt &= 0\end{aligned}$$

$$\begin{aligned}\int_0^{2\pi} \sin(mt) \sin(nt) dt &= 0 \\ &(\forall m \neq n) \\ \int_0^{2\pi} \sin(mt) \sin(nt) dt &= \pi (m = n) \\ \int_0^{2\pi} \cos(mt) \cos(nt) dt &= 0 \\ &(\forall m \neq n) \\ \int_0^{2\pi} \cos(mt) \cos(nt) dt &= \pi (m = n)\end{aligned}$$

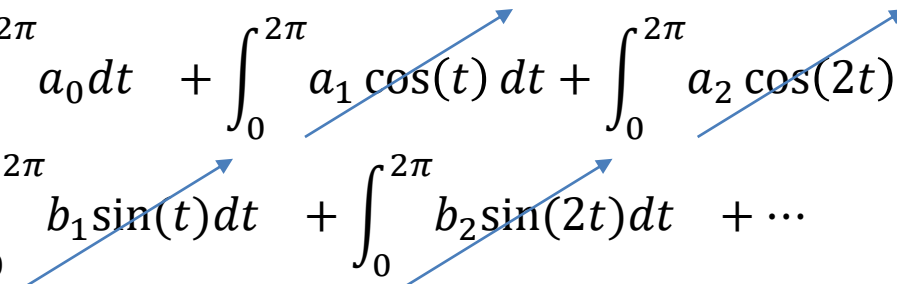
Periodic Function

Sum of sine and cosine waves:

$$f(t) = a_0 + a_1 \cos(t) + a_2 \cos(2t) + \dots \\ b_1 \sin(t) + b_2 \sin(2t) + \dots$$

Periodic Function

Sum of sine and cosine waves:

$$\int_0^{2\pi} f(t) dt = \int_0^{2\pi} a_0 dt + \int_0^{2\pi} a_1 \cos(t) dt + \int_0^{2\pi} a_2 \cos(2t) dt + \dots$$
$$\int_0^{2\pi} b_1 \sin(t) dt + \int_0^{2\pi} b_2 \sin(2t) dt + \dots$$


Periodic Function

Sum of sine and cosine waves:

$$\int_0^{2\pi} f(t) dt = \int_0^{2\pi} a_0 dt = a_0(2\pi)$$

$$\Rightarrow a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(t) dt$$

Periodic Function

Sum of sine and cosine waves:

$$\begin{aligned} f(t)\cos(nt) \\ = a_0\cos(nt) + a_1 \cos(t)\cos(nt) + a_2 \cos(2t)\cos(nt) + \cdots \\ b_1 \sin(t)\cos(nt) + b_2 \sin(2t)\cos(nt) + \cdots \end{aligned}$$

Periodic Function

Sum of sine and cosine waves:

$$\begin{aligned} & \int_0^{2\pi} f(t) \cos(nt) dt \\ &= \int_0^{2\pi} a_0 \cos(nt) dt + \int_0^{2\pi} a_1 \cos(t) \cos(nt) dt + \int_0^{2\pi} a_2 \cos(2t) \cos(nt) dt + \cdots \\ & \quad \int_0^{2\pi} b_1 \sin(t) \cos(nt) dt + \int_0^{2\pi} b_2 \sin(2t) \cos(nt) dt + \cdots \end{aligned}$$

Periodic Function

Sum of sine and cosine waves:

$$\begin{aligned}
 & \int_0^{2\pi} f(t) \cos(nt) dt \\
 &= \int_0^{2\pi} a_0 \cos(nt) dt + \int_0^{2\pi} a_1 \cos(t) \cos(nt) dt \\
 &+ \int_0^{2\pi} a_2 \cos(2t) \cos(nt) dt + \cdots \int_0^{2\pi} a_n \cos(nt) \cos(nt) dt + \cdots \\
 &\quad \int_0^{2\pi} b_1 \sin(t) \cos(nt) dt + \int_0^{2\pi} b_2 \sin(2t) \cos(nt) dt + \cdots
 \end{aligned}$$

Periodic Function

Sum of sine and cosine waves:

$$\int_0^{2\pi} f(t) \cos(nt) dt = \int_0^{2\pi} a_n \cos(nt) \cos(nt) dt$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos(nt) dt$$

$$\text{Similarly, } b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin(nt) dt$$

Periodic Function

Sum of sine and cosine waves:

$$\Rightarrow a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(t) dt$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos(nt) dt$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin(nt) dt$$