Digital Image Processing

COSC 6380/4393

Lecture – 15

Mar. 9th, 2023

Pranav Mantini

Slides from Dr. Shishir K Shah and S. Narasimhan

Discrete Fourier Transform

Spatial Domain (x) ⟶ Frequency Domain (u)

Fourier Transform

$$\boldsymbol{F(u) = \int_{-\infty}^{\infty} f(x) e^{-\sqrt{-1}ux} dx}$$

Discrete Fourier Transform

$$\boxed{F(u) = \sum_{x=-\infty}^{\infty} f(x) e^{-\sqrt{-1}ux}}$$

$$e^{-\sqrt{-1}x} = cosx - \sqrt{-1}sinx$$

Frequency Domain (u) ⟶ Spatial Domain (x)

Inverse Fourier Transform

$$\boldsymbol{f(x) = \int_{-\infty}^{\infty} F(u) e^{\sqrt{-1}ux} du}$$

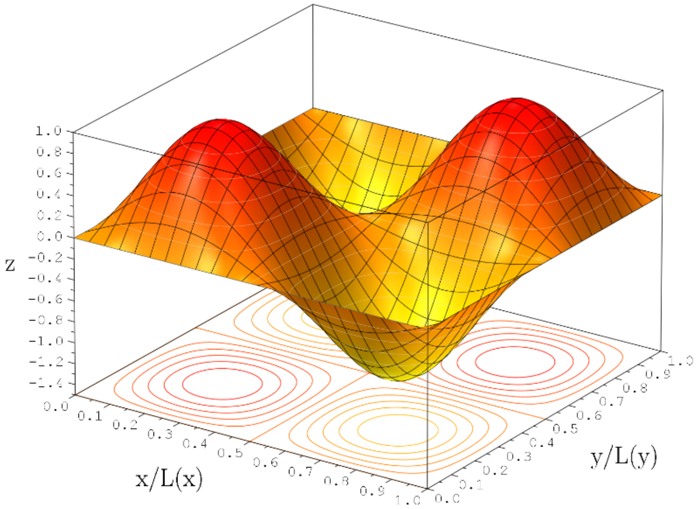
Inverse Discrete Fourier Transform

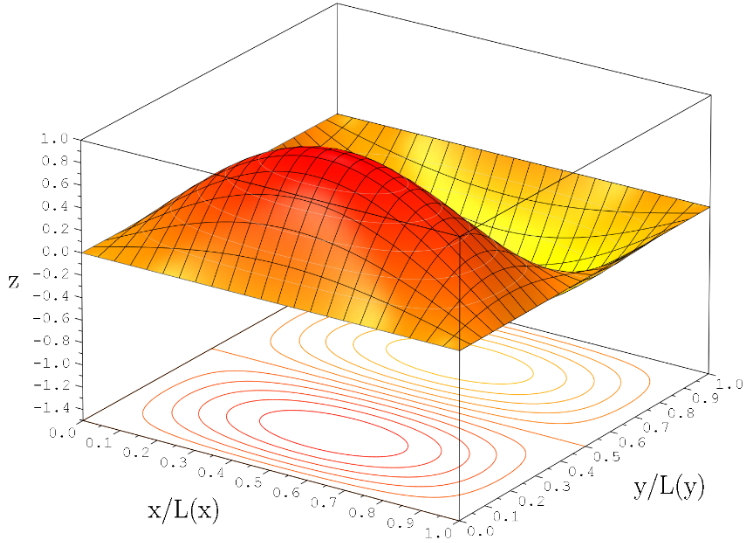
$$\boxed{f(x) = \sum_{u=-\infty}^{\infty} F(u) e^{\sqrt{-1}ux}}$$

$$e^{\sqrt{-1}x} = cosx + \sqrt{-1}sinx$$

From 1D → 2D

- One dimension (x) →frequency (u)
- Two dimensions →(i, j)
- Frequencies along (I,j) →(u,v)





Sinusoidal Images

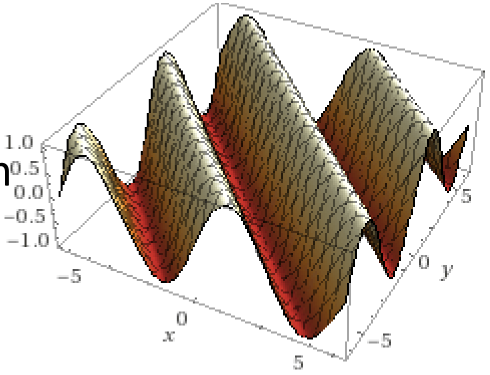
$2D\ sine\ wave \Rightarrow \sin(ui + vj)(u\ and\ v\ are\ frequencies\ along\ i\ and\ j)$

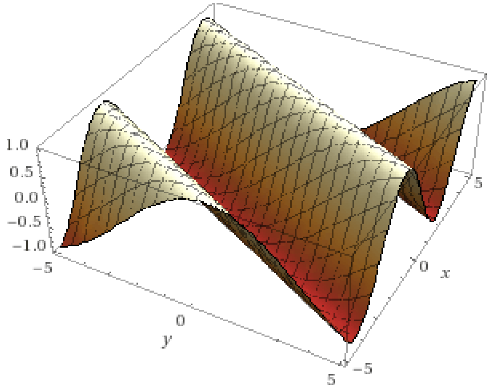
$\sin(i + j)(u = 1, v = 1)$

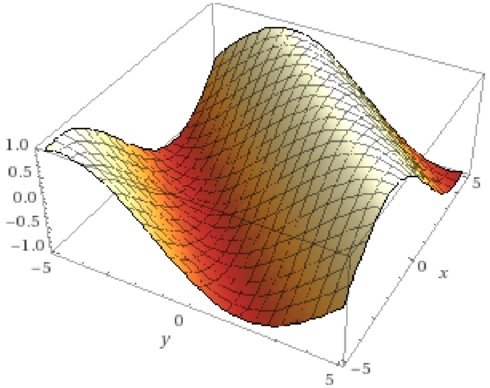
$\sin(i + 0.5j)(u = 1, v = 0.5)$

$\sin(0.5i + 0.5j)$
$u = v = 0.5$

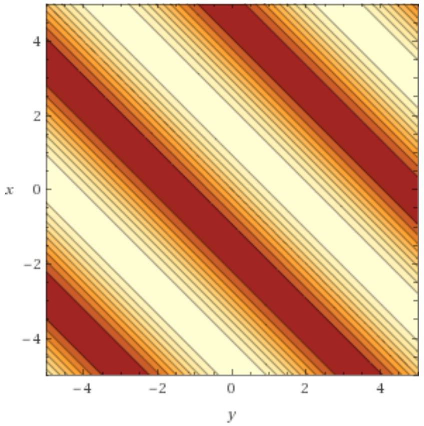
Waveform

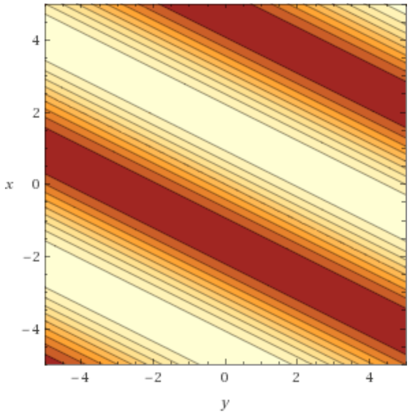


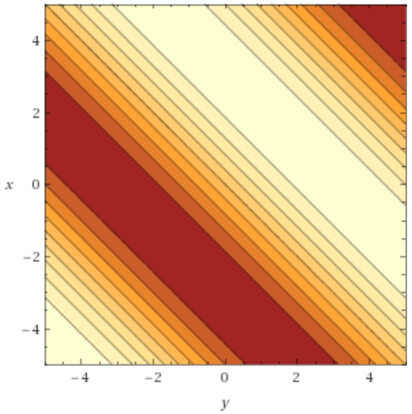




Contour plots







From Wolframalpha

2D Discrete Fourier Transform

Spatial Domain (i,*j*) ⟶ Frequency Domain (*u*,*v*)

Fourier Transform

$$\boldsymbol{F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(i,j)e^{-\sqrt{-1}(ui+vj)}\,di\,dj}$$

Discrete Fourier Transform

$$\boxed{F(u,v) = \sum_{x=-\infty}^{\infty}\sum_{y=-\infty}^{\infty} f(i,j)e^{-\sqrt{-1}(ui+vj)}}$$

Frequency Domain (*u*,*v*)⟶Spatial Domain (*i*,*j*)

Inverse Fourier Transform

$$\boldsymbol{f(i,j) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{\sqrt{-1}(ui+vj)}\,du\,dv}$$

Inverse Discrete Fourier Transform

$$\boxed{f(i,j) = \sum_{u=-\infty}^{\infty}\sum_{v=-\infty}^{\infty} F(u,v)e^{\sqrt{-1}(ui+vj)}}$$

Sinusoidal Images

- We shall make frequent discussion in this module of the **frequency content** of an image.
- First consider images having the **simplest** frequency content.
- A **digital sine image I** is an image having elements

$$I_1(i, j) = \sin\left[\frac{2\pi}{N}(ui + vj)\right] \text{ for } 0 \leq i, j \leq N\text{-}1$$

and a **digital cosine image** has elements

$$I_2(i, j) = \cos\left[\frac{2\pi}{N}(ui + vj)\right] \text{ for } 0 \leq i, j \leq N\text{-}1$$

where u and v are **integer frequencies** in the i- and j-directions (measured in cycles/image; **notice** division by N).

2D Discrete Fourier Transform

- If I is an image of size N then

Sin image $I_1(i, j) = \sin\left[\frac{2\pi}{N}(ui + vj)\right]$ for $0 \leq i, j \leq N\text{-}1$

Cos image $I_2(i, j) = \cos\left[\frac{2\pi}{N}(ui + vj)\right]$ for $0 \leq i, j \leq N\text{-}1$

- Let $\tilde{I}$ be the DFT of the I

$$\tilde{I}(u, v) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\, e^{-\sqrt{-1}\frac{2\pi}{N}(ui+vj)}$$

$$F(u, v) = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(i, j) e^{-\sqrt{-1}(ui+vj)}$$

2D Inverse Discrete Fourier Transform

- Let $\tilde{I}$ be the DFT of the I

$$I(i,j) = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{I}(u,v)\, e^{\sqrt{-1}\frac{2\pi}{N}(ui+vj)}$$

$$f(i,j) = \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} F(u,v) e^{\sqrt{-1}(ui+vj)}$$

Example

$$I = \begin{array}{|c|c|}\hline 5 & 7 \\ \hline 8 & 3 \\ \hline \end{array}$$

$$\tilde{I}(u,v) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j)\, e^{-\sqrt{-1}\frac{2\pi}{N}(ui+vj)}$$

$$\tilde{I}(0,0) = \sum_{i=0}^{2-1} \sum_{j=0}^{2-1} I(i,j)\, e^{-\sqrt{-1}\frac{2\pi}{2}(0*i+0*j)}$$

$$= \sum_{i=0}^{1} \sum_{j=0}^{1} I(i,j) = 21$$

$$\tilde{I}(0,1) = 3.+0.\sqrt{-1}$$

$$\tilde{I}(1,0) = 1.+0.\sqrt{-1} \qquad \tilde{I}(1,1) = -7.+0.\sqrt{-1}$$

23	
	-7.+0.j

Example

$$I = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array}$$

$$\tilde{I}(u,v) = \sum_{i=0}^{N-1}\sum_{j=0}^{M-1} I(i,j)\, e^{-\sqrt{-1}\left(\frac{2\pi}{N}ui+\frac{2\pi}{M}vj\right)}$$

$$\tilde{I}(0,0) = \sum_{i=0}^{2-1}\sum_{j=0}^{3-1} I(i,j)\, e^{-\sqrt{-1}\left(\frac{2\pi}{N}0*i+\frac{2\pi}{M}0*j\right)}$$

$$= \sum_{i=0}^{1}\sum_{j=0}^{2} I(i,j) = 21$$

$\tilde{I}(0,1) = -3 + 1.732051\text{j}$

$\tilde{I}(0,2) = -3 - 1.732051\text{j}$

$\tilde{I}(1,0) = -9$

$\tilde{I}(1,1) = 0 + 0\text{j}$

$\tilde{I}(1,2) = 0 + 0\text{j}$

$\tilde{I} =$

$21 + 0\sqrt{-1}$	$-3 + 1.73\sqrt{-1}$	$-3 - 1.73\sqrt{-1}$
$-9 + 0\sqrt{-1}$	$0 + 0\sqrt{-1}$	$0 + 0\sqrt{-1}$

Complex Image

Properties of DFT Matrix

- We can understand the DFT matrix better by studying some of its properties.
- Any image **I** of interest to us is composed of **real integers.**
- However, the DFT of **I** is generally **complex**.
- It can be written in the form

$$\tilde{\mathbf{I}} = \tilde{\mathbf{I}}_{real} + \sqrt{-1}\, \tilde{\mathbf{I}}_{imag}$$

where $\tilde{\mathbf{I}}_{real}$ and $\tilde{\mathbf{I}}_{imag}$ have components

$$\tilde{I}_{real}(u, v) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j) \cos\left[\frac{2\pi}{N}(ui + vj)\right]$$

$$\tilde{I}_{imag}(u, v) = -\sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j) \sin\left[\frac{2\pi}{N}(ui + vj)\right]$$

i.e.,

$$\tilde{I}(u, v) = \tilde{I}_{real}(u, v) + \sqrt{-1}\, \tilde{I}_{imag}(u, v) \text{ for } 0 \leq u, v \leq N-1$$

(These are taken directly from the original DFT equation).

Therefore $\tilde{I}$ has a **magnitude** and a **phase**.

Properties of DFT Matrix

- We can understand the DFT matrix better by studying some of its properties.
- Any image **I** of interest to us is composed of **real integers.**
- However, the DFT of **I** is generally **complex**.
- It can be written in the form

$$\tilde{\mathbf{I}} = \tilde{\mathbf{I}}_{real} + \sqrt{-1}\ \tilde{\mathbf{I}}_{imag}$$

where $\tilde{\mathbf{I}}_{real}$ and $\tilde{\mathbf{I}}_{imag}$ have components

$$\tilde{I}_{real}(u, v) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j) \cos\left[\frac{2\pi}{N}(ui + vj)\right]$$

$$\tilde{I}_{imag}(u, v) = -\sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j) \sin\left[\frac{2\pi}{N}(ui + vj)\right]$$

i.e.,

$$\tilde{I}(u, v) = \tilde{I}_{real}(u, v) + \sqrt{-1}\ \tilde{I}_{imag}(u, v) \text{ for } 0 \le u, v \le N-1$$

(These are taken directly from the original DFT equation).

Therefore $\tilde{I}$ has a **magnitude** and a **phase**.

$21 + 0\sqrt{-1}$	$-3 + 1.73\sqrt{-1}$	$-3 - 1.73\sqrt{-1}$
$-9 + 0\sqrt{-1}$	$0 + 0\sqrt{-1}$	$0 + 0\sqrt{-1}$

21	−3	−3
−9	0	0

0	1.73	− 1.73
0	0	0

Magnitude and Phase of DFT

- The **magnitude** of the DFT is the matrix

$$|\tilde{\mathbf{I}}| = \left[|\tilde{I}(u, v)| \; ; 0 \leq u, v \leq N\text{-}1 \right]$$

with elements

$$|\tilde{I}(u, v)| = \sqrt{\tilde{I}^2_{real}(u,v) + \tilde{I}^2_{imag}(u,v)}$$

which are just the magnitudes of the complex components of $\tilde{\mathbf{I}}$

21	3.46	3.46
9	0	0

- The **phase** of the DFT is the matrix

$$\angle\tilde{\mathbf{I}} = \left[\angle\tilde{I}(u, v) \; ; 0 \leq u, v \leq N\text{-}1 \right]$$

with elements

$$\angle\tilde{I}(u, v) = \tan^{-1}\left[\tilde{I}_{imag}(u, v) \,/\, \tilde{I}_{real}(u, v) \right]$$

which are just the phases of the complex components of $\tilde{\mathbf{I}}$.

- Therefore

$$\tilde{I}(u, v) = |\tilde{I}(u, v)| \exp\left\{ \sqrt{-1}\,\angle\tilde{I}(u, v) \right\}$$

Magnitude and Phase of DFT

- The **magnitude** of the DFT is the matrix

$$|\tilde{\mathbf{I}}| = \left[|\tilde{I}(u, v)| \; ; 0 \leq u, v \leq N\text{-}1\right]$$

with elements

$$|\tilde{I}(u, v)| = \sqrt{\tilde{I}^2_{real}(u,v) + \tilde{I}^2_{imag}(u,v)}$$

which are just the magnitudes of the complex components of $\tilde{\mathbf{I}}$

- The **phase** of the DFT is the matrix

$$\angle\tilde{\mathbf{I}} = \left[\angle\tilde{I}(u, v) \; ; 0 \leq u, v \leq N\text{-}1\right]$$

with elements

$$\angle\tilde{I}(u, v) = \tan^{-1}\left[\tilde{I}_{imag}(u, v) / \tilde{I}_{real}(u, v)\right]$$

0	150	-150
180	0	0

which are just the phases of the complex components of $\tilde{\mathbf{I}}$.

- Therefore

$$\tilde{I}(u, v) = |\tilde{I}(u, v)| \exp\left\{\sqrt{-1}\,\angle\tilde{I}(u, v)\right\}$$

2D Discrete Fourier Transform

- Then

$$\tilde{I}(u,v) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)\, e^{-\sqrt{-1}\frac{2\pi}{N}(ui+vj)}$$

2D Discrete Fourier Transform

- We will use the abbreviation

$$W_N = e^{-\sqrt{-1}\frac{2\pi}{N}} \Rightarrow \boldsymbol{W_N^{ui+vj}} = \boldsymbol{e^{-\sqrt{-1}\frac{2\pi}{N}(ui+vj)}}$$

- Then

$$\tilde{I}(u, v) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\, e^{-\sqrt{-1}\frac{2\pi}{N}(ui+vj)}$$

2D Discrete Fourier Transform

- We will use the abbreviation

$$W_N = e^{-\sqrt{-1}\frac{2\pi}{N}} \Rightarrow \boldsymbol{W_N^{ui+vj}} = \boldsymbol{e^{-\sqrt{-1}\frac{2\pi}{N}(ui+vj)}}$$

- Then

$$\tilde{I}(u,v) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)\, e^{-\sqrt{-1}\frac{2\pi}{N}(ui+vj)}$$

$$= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)\, \boldsymbol{W_N^{ui+vj}}$$

$$I(i,j) = \sum_{u=0}^{N-1}\sum_{u=0}^{N-1} \tilde{I}(u,v)\boldsymbol{W_N^{-(ui+vj)}}$$

$$\tilde{I}(u,v) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j)\, \boldsymbol{W}_{\boldsymbol{N}}^{\boldsymbol{ui+vj}}$$

$\tilde{I}(N - u, N - v)$

$$\tilde{I}(u, v) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\, \boldsymbol{W}_{\boldsymbol{N}}^{\boldsymbol{ui+vj}}$$

$$\tilde{I}(N - u, N - v) = \sum_{\substack{i=0\\N-1}}^{N-1} \sum_{\substack{j=0\\N-1}}^{N-1} I(i, j)\, W_N^{[(N-u)i+(N-v)j]}$$

$$\tilde{I}(u,v) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j)\, \boldsymbol{W}_{\boldsymbol{N}}^{\boldsymbol{ui+vj}}$$

$$\begin{aligned}\tilde{I}(N-u, N-v) &= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j)\, W_N^{[(N-u)i+(N-v)j]} \\ &= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j)\, W_N^{N(i+j)}\, W_N^{-(ui+vj)}\end{aligned}$$

$$\tilde{I}(u,v) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)\, \boldsymbol{W}_{\boldsymbol{N}}^{\boldsymbol{ui+vj}}$$

$$\tilde{I}(N-u, N-v) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)\, W_N^{[(N-u)i+(N-v)j]}$$

$$= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)\, W_N^{N(i+j)}\; W_N^{-(ui+vj)}$$

since

$$W_N^{N(i+j)} = e^{-\sqrt{-1}\,\frac{2\pi}{N}\cdot N(i+j)} = e^{-2\pi\sqrt{-1}\,(i+j)} = 1^{(i+j)} = 1 \text{ for any } i, j$$

and

$$W_N^{-(ui+vj)} = [W_N^{(ui+vj)}]^*.$$

$$\tilde{I}(u,v) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)\,\boldsymbol{W}_{\boldsymbol{N}}^{\boldsymbol{ui+vj}}$$

$$\begin{aligned}
\tilde{I}(N-u, N-v) &= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)\, W_N^{[(N-u)i+(N-v)j]} \\
&= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)\, W_N^{N(i+j)}\, W_N^{-(ui+vj)} \\
&= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)\left[W_N^{(ui+vj)}\right]^* = \tilde{I}^*(u,v)
\end{aligned}$$

since

$$W_N^{N(i+j)} = e^{-\sqrt{-1}\,\frac{2\pi}{N}\cdot N(i+j)} = e^{-2\pi\sqrt{-1}\,(i+j)} = 1^{(i+j)} = 1 \text{ for any } i, j$$

and

$$W_N^{-(ui+vj)} = \left[W_N^{(ui+vj)}\right]^*.$$

The DFT of an image **I** is **conjugate symmetric**:

$$\tilde{I}(u,v) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)\,\boldsymbol{W_N^{ui+vj}}$$

$$\begin{aligned}
\tilde{I}(N-u, N-v) &= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)\; W_N^{[(N-u)i+(N-v)j]} \\
&= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)\; W_N^{N(i+j)}\; W_N^{-(ui+vj)} \\
&= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)\left[W_N^{(ui+vj)}\right]^* = \tilde{I}^*(u,v)
\end{aligned}$$

since

$$W_N^{N(i+j)} = e^{-\sqrt{-1}\,\frac{2\pi}{N}\cdot N(i+j)} = e^{-2\pi\sqrt{-1}\,(i+j)} = 1^{(i+j)} = 1 \text{ for any } i, j$$

and

$$W_N^{-(ui+vj)} = \left[W_N^{(ui+vj)}\right]^*.$$

The DFT of an image **I** is **conjugate symmetric**:

$$\begin{aligned}
\tilde{I}_{real}(N-u, N-v) &= \tilde{I}_{real}(u,v)\ ;\ 0 \le u, v \le N-1 \\
\tilde{I}_{imag}(N-u, N-v) &= -\tilde{I}_{imag}(u,v)\ ;\ 0 \le u, v \le N-1
\end{aligned}$$

$$\tilde{I}(N-u, N-v) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j)\, W_N^{[(N-u)i+(N-v)j]}$$

$$= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j)\, W_N^{N(i+j)}\; W_N^{-(ui+vj)}$$

$$= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j)\, \left[W_N^{(ui+vj)}\right]^* = \tilde{I}^*(u,v)$$

since

$$W_N^{N(i+j)} = e^{-\sqrt{-1}\,\frac{2\pi}{N}\cdot N(i+j)} = e^{-2\pi\sqrt{-1}\,(i+j)} = 1^{(i+j)} = 1 \text{ for any } i, j$$

and

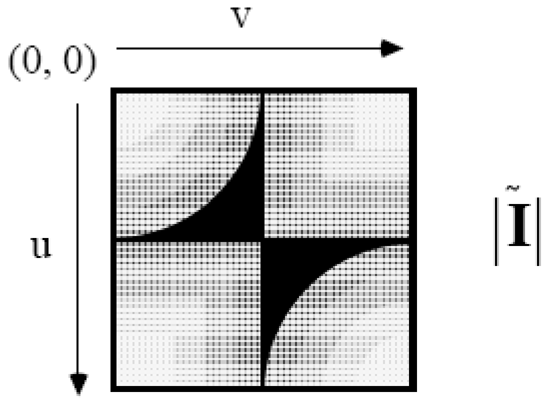
$$W_N^{-(ui+vj)} = \left[W_N^{(ui+vj)}\right]^*.$$

The DFT of an image **I** is **conjugate symmetric**:

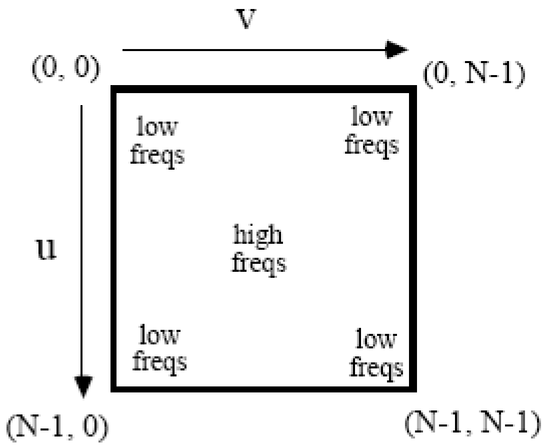
$|\tilde{I}(N-u, N-v)| = |\tilde{I}(u,v)|$ The magnitude DFT of an image **I** is **symmetric**:

Symmetry of DFT

- Depiction of the symmetry of the DFT (magnitude).



- The highest frequencies are represented near (u, v) = (N/2, N/2).



- We have defined the DFT matrix as **finite** in extent (N x N):

$$\tilde{\mathbf{I}} = \left[\tilde{I}(u, v)\ ;\ 0 \leq\ u, v \leq\ N\text{-}1\right]$$

- However, if the arguments are allowed to take values outside the range 0 ≤ u, v ≤ N-1, we find that the DFT is periodic in both the u- and v-directions, with **period N**:
- For any integers m, n

$$\tilde{I}(u{+}nN,\ v{+}mN)$$

- We have defined the DFT matrix as **finite** in extent (N x N):

$$\tilde{\mathbf{I}} = \left[\tilde{I}(u, v)\ ;\ 0 \leq\ u, v \leq\ N\text{-}1\right]$$

- However, if the arguments are allowed to take values outside the range 0 ≤ u, v ≤ N-1, we find that the DFT is periodic in both the u- and v-directions, with **period N**:
- For any integers m, n

$$\tilde{I}(u+nN, v+mN) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\ W_N^{[(u+nN)i+(v+mN)j]}$$

- We have defined the DFT matrix as **finite** in extent (N x N):

$$\tilde{\mathbf{I}} = \left[\tilde{I}(u, v) \; ; \; 0 \leq u, v \leq N\text{-}1\right]$$

- However, if the arguments are allowed to take values outside the range 0 ≤ u, v ≤ N-1, we find that the DFT is periodic in both the u- and v-directions, with **period N**:
- For any integers m, n

$$\tilde{I}(u+nN, v+mN) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\, W_N^{[(u+nN)i+(v+mN)j]}$$

$$= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\, W_N^{N(ni+mj)}\; W_N^{(ui+vj)}$$

- We have defined the DFT matrix as **finite** in extent (N x N):

$$\tilde{\mathbf{I}} = \left[\tilde{I}(u, v)\ ;\ 0 \leq\ u, v \leq\ N\text{-}1\right]$$

- However, if the arguments are allowed to take values outside the range 0 ≤ u, v ≤ N-1, we find that the DFT is periodic in both the u- and v-directions, with **period N**:
- For any integers m, n

$$\tilde{I}(u+nN, v+mN) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\, W_N^{[(u+nN)i+(v+mN)j]}$$

$$= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\, W_N^{N(ni+mj)}\ W_N^{(ui+vj)}$$

since

$$W_N^{N(ni+mj)} = e^{-\sqrt{-1}\,\frac{2\pi}{N}\,\cdot\,N(ni+mj)} = e^{-2\pi\sqrt{-1}\,(ni+mj)} = 1^{(ni+mj)} = 1$$

- We have defined the DFT matrix as **finite** in extent (N x N):

$$\tilde{\mathbf{I}} = \left[\tilde{I}(u, v)\ ;\ 0 \le u, v \le N\text{-}1\right]$$

- However, if the arguments are allowed to take values outside the range 0 ≤ u, v ≤ N-1, we find that the DFT is periodic in both the u- and v-directions, with **period N**:
- For any integers m, n

$$\begin{aligned}\tilde{I}(u+nN, v+mN) &= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\, W_N^{[(u+nN)i+(v+mN)j]} \\ &= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\, W_N^{N(ni+mj)}\ W_N^{(ui+vj)} \\ &= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\, W_N^{(ui+vj)}\end{aligned}$$

since

$$W_N^{N(ni+mj)} = e^{-\sqrt{-1}\,\frac{2\pi}{N}\cdot N(ni+mj)} = e^{-2\pi\sqrt{-1}\,(ni+mj)} = 1^{(ni+mj)} = 1$$

- We have defined the DFT matrix as **finite** in extent (N x N):

$$\tilde{\mathbf{I}} = \left[\tilde{I}(u, v)\ ;\ 0 \leq u, v \leq N\text{-}1\right]$$

- However, if the arguments are allowed to take values outside the range $0 \leq u, v \leq N\text{-}1$, we find that the DFT is periodic in both the u- and v-directions, with **period N**:
- For any integers m, n

$$\tilde{I}(u+nN, v+mN) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\, W_N^{[(u+nN)i+(v+mN)j]}$$

$$= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\, W_N^{N(ni+mj)}\, W_N^{(ui+vj)}$$

$$= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\, W_N^{(ui+vj)} = \tilde{I}(u, v)$$

since

$$W_N^{N(ni+mj)} = e^{-\sqrt{-1}\,\frac{2\pi}{N}\cdot N(ni+mj)} = e^{-2\pi\sqrt{-1}\,(ni+mj)} = 1^{(ni+mj)} = 1$$

Periodicity of DFT

- We have defined the DFT matrix as **finite** in extent (N x N):

$$\tilde{\mathbf{I}} = \left[\tilde{I}(u, v) \,;\, 0 \leq u, v \leq N\text{-}1\right]$$

- However, if the arguments are allowed to take values outside the range $0 \leq u, v \leq N\text{-}1$, we find that the DFT is periodic in both the u- and v-directions, with **period N**:
- For any integers m, n

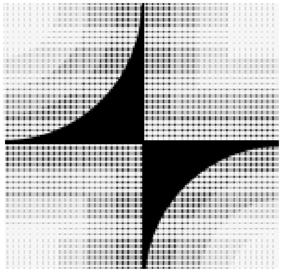
$$\begin{aligned}\tilde{I}(u+nN, v+mN) &= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i, j)\, W_N^{[(u+nN)i+(v+mN)j]} \\ &= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i, j)\, W_N^{N(ni+mj)}\, W_N^{(ui+vj)} \\ &= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i, j)\, W_N^{(ui+vj)} = \tilde{I}(u, v)\end{aligned}$$

since

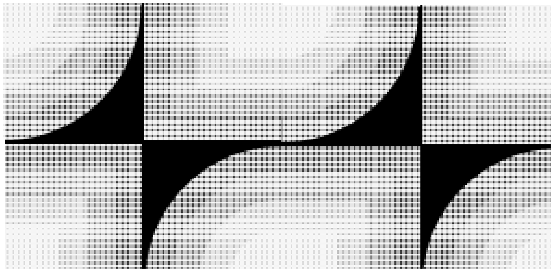
$$W_N^{N(ni+mj)} = e^{-\sqrt{-1}\,\frac{2\pi}{N}\cdot N(ni+mj)} = e^{-2\pi\sqrt{-1}\,(ni+mj)} = 1^{(ni+mj)} = 1$$

- This is called the **periodic extension** of the DFT. It is defined for all integer frequencies u, v.

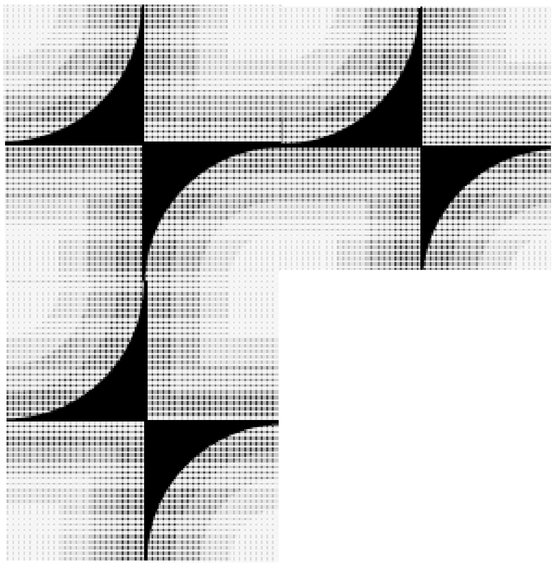
Periodic Extension of DFT



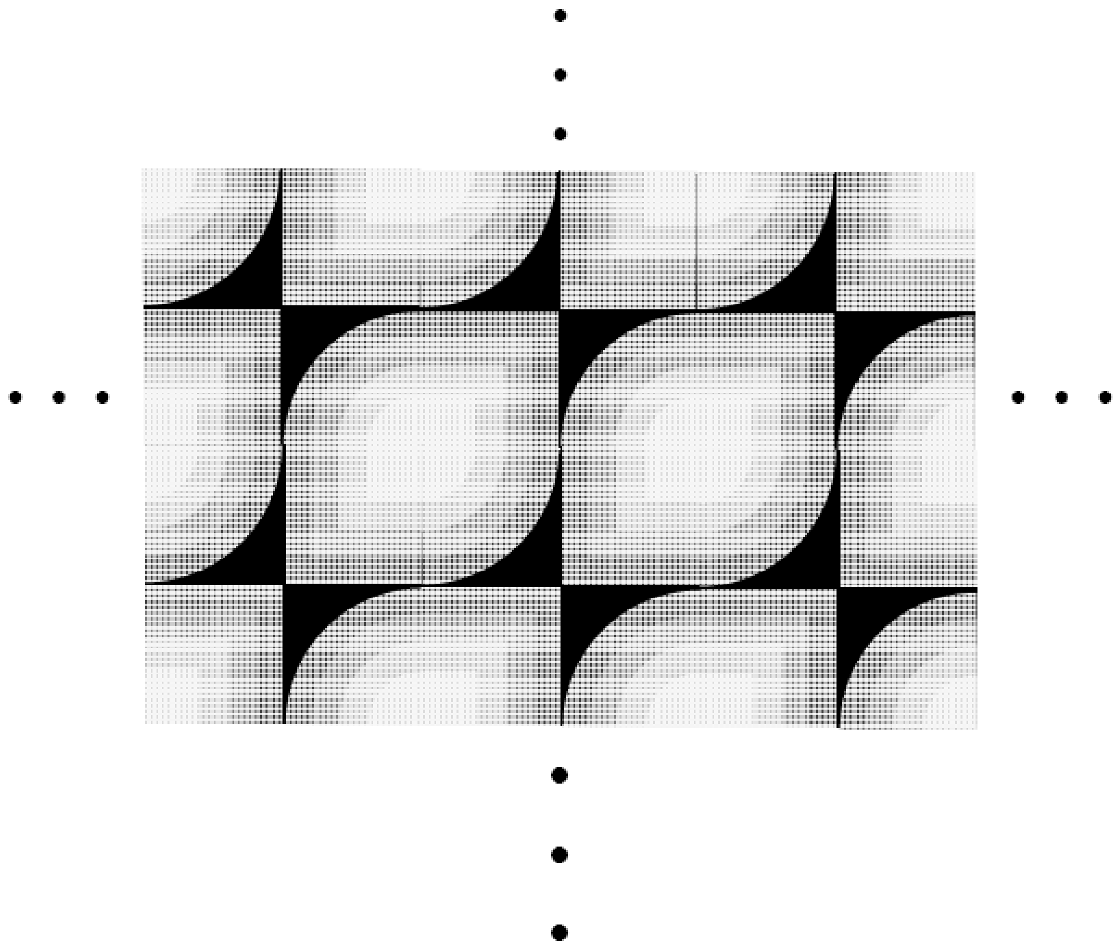
Periodic Extension of DFT



Periodic Extension of DFT



Periodic Extension of DFT



- The IDFT equation $$I(i, j) = \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{I}(u, v)\, W_N^{-(ui+vj)}$$

- Note that for any integers n, m

$$I(i+nN, j+mN)$$

- The IDFT equation

$$I(i, j) = \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{I}(u, v)\, W_N^{-(ui+vj)}$$

- Note that for any integers n, m

$$I(i+nN, j+mN) = \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{I}(u, v)\, W_N^{-[u(i+nN)+v(j+mN)]}$$

- The IDFT equation $$I(i, j) = \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{I}(u, v)\, W_N^{-(ui+vj)}$$

- Note that for any integers n, m

$$I(i+nN, j+mN) = \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{I}(u, v)\, W_N^{-[u(i+nN)+v(j+mN)]}$$

$$= \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{I}(u, v)\, W_N^{-(ui+vj)}\; W_N^{-N(nu+mv)}$$

- The IDFT equation $$I(i, j) = \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{I}(u, v)\, W_N^{-(ui+vj)}$$

- Note that for any integers n, m

$$I(i+nN, j+mN) = \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{I}(u, v)\, W_N^{-[u(i+nN)+v(j+mN)]}$$

$$= \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{I}(u, v)\, W_N^{-(ui+vj)}\; W_N^{-N(nu+mv)}$$

since

$$W_N^{-N(nu+mv)} = e^{-\sqrt{-1}\,\frac{2\pi}{N}\cdot N(nu+mv)} = e^{-2\pi\sqrt{-1}\,(nu+mv)} = 1^{(nu+mv)} = 1$$

- The IDFT equation $$I(i, j) = \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{I}(u, v)\, W_N^{-(ui+vj)}$$

- Note that for any integers n. m

$$\begin{aligned} I(i+nN, j+mN) &= \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{I}(u, v)\, W_N^{-[u(i+nN)+v(j+mN)]} \\ &= \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{I}(u, v)\, W_N^{-(ui+vj)}\ W_N^{-N(nu+mv)} \\ &= \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{I}(u, v)\, W_N^{-(ui+vj)} \end{aligned}$$

since

$$W_N^{-N(nu+mv)} = e^{-\sqrt{-1}\,\frac{2\pi}{N}\cdot N(nu+mv)} = e^{-2\pi\sqrt{-1}\,(nu+mv)} = 1^{(nu+mv)} = 1$$

- The IDFT equation $$I(i, j) = \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{I}(u, v)\, W_N^{-(ui+vj)}$$

- Note that for any integers n, m

$$\begin{aligned} I(i+nN, j+mN) &= \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{I}(u, v)\, W_N^{-[u(i+nN)+v(j+mN)]} \\ &= \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{I}(u, v)\, W_N^{-(ui+vj)}\, W_N^{-N(nu+mv)} \\ &= \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{I}(u, v)\, W_N^{-(ui+vj)} = I(i, j) \end{aligned}$$

since

$$W_N^{-N(nu+mv)} = e^{-\sqrt{-1}\,\frac{2\pi}{N} \cdot N(nu+mv)} = e^{-2\pi\sqrt{-1}\,(nu+mv)} = 1^{(nu+mv)} = 1$$

Periodic Extension of Image

- The IDFT equation $$I(i, j) = \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{I}(u, v)\, W_N^{-(ui+vj)}$$

  implies the **periodic extension of the image I** as well (with period N), simply by letting the arguments (i, j) take any integer value.
- Note that for any integers n, m

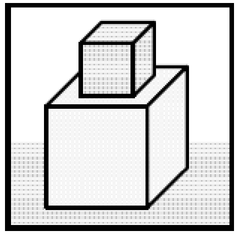
$$\begin{aligned} I(i+nN, j+mN) &= \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{I}(u, v)\, W_N^{-[u(i+nN)+v(j+mN)]} \\ &= \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{I}(u, v)\, W_N^{-(ui+vj)}\, W_N^{-N(nu+mv)} \\ &= \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{I}(u, v)\, W_N^{-(ui+vj)} = I(i, j) \end{aligned}$$

since

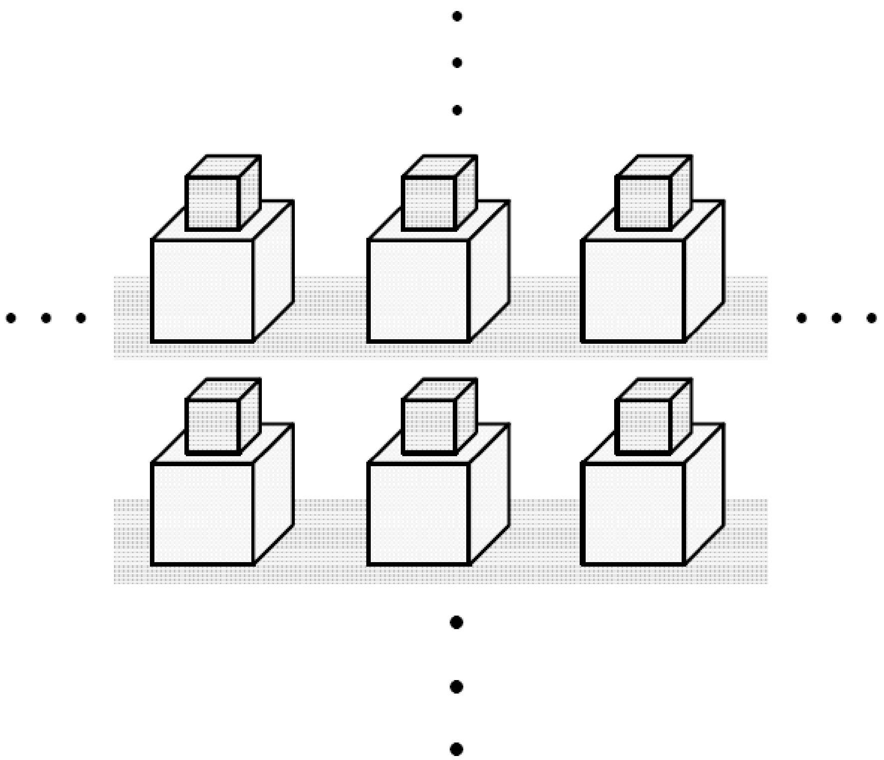
$$W_N^{-N(nu+mv)} = e^{-\sqrt{-1}\,\frac{2\pi}{N} \cdot N(nu+mv)} = e^{-2\pi\sqrt{-1}\,(nu+mv)} = 1^{(nu+mv)} = 1$$

- In a sense, the DFT **implies** that the image **I** is already periodic.
- This will be extremely important when we consider **convolution**

Periodic Extension of Image

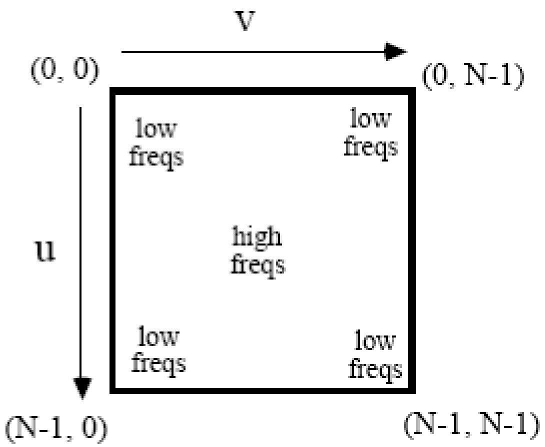


Image



Frequencies DFT

- The highest frequencies are represented near (u, v) = (N/2, N/2).



Displaying the DFT

- Usually, the DFT is displayed with its center coordinate (u, v) = (0, 0) at the center of the image.
- This way, the lower frequency information (which usually dominates an image) is clustered together near the origin at the center of the display.

Displaying the DFT

- Usually, the DFT is displayed with its center coordinate (u, v) = (0, 0) at the center of the image.
- This way, the lower frequency information (which usually dominates an image) is clustered together near the origin at the center of the display.

$$\tilde{I}\left(u - \frac{N}{2}, v - \frac{N}{2}\right)$$

Displaying the DFT

- Usually, the DFT is displayed with its center coordinate (u, v) = (0, 0) at the center of the image.
- This way, the lower frequency information (which usually dominates an image) is clustered together near the origin at the center of the display.

$$\tilde{I}(u - \tfrac{N}{2}, v - \tfrac{N}{2}) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\, W_N^{[(u-N/2)i+(v-N/2)j]}$$

Displaying the DFT

- Usually, the DFT is displayed with its center coordinate (u, v) = (0, 0) at the center of the image.
- This way, the lower frequency information (which usually dominates an image) is clustered together near the origin at the center of the display.

$$\tilde{I}(u - \tfrac{N}{2}, v - \tfrac{N}{2}) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\, W_N^{[(u-N/2)i+(v-N/2)j]}$$
$$= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\, W_N^{(ui+vj)}\ W_N^{-N(i+j)/2}$$

Displaying the DFT

- Usually, the DFT is displayed with its center coordinate (u, v) = (0, 0) at the center of the image.
- This way, the lower frequency information (which usually dominates an image) is clustered together near the origin at the center of the display.

$$\tilde{I}(u - \tfrac{N}{2}, v - \tfrac{N}{2}) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\, W_N^{[(u-N/2)i+(v-N/2)j]}$$

$$= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\, W_N^{(ui+vj)}\; W_N^{-N(i+j)/2}$$

$$W_N^{-N(i+j)/2} = e^{\sqrt{-1}\,\frac{2\pi}{N} N(i+j)/2} = e^{\sqrt{-1}\pi\,(i+j)} = (-1)^{i+j}$$

Displaying the DFT

- Usually, the DFT is displayed with its center coordinate (u, v) = (0, 0) at the center of the image.
- This way, the lower frequency information (which usually dominates an image) is clustered together near the origin at the center of the display.

$$\tilde{I}(u - \tfrac{N}{2}, v - \tfrac{N}{2}) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\, W_N^{[(u-N/2)i+(v-N/2)j]}$$

$$= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\, W_N^{(ui+vj)}\ W_N^{-N(i+j)/2}$$

$$W_N^{-N(i+j)/2} = e^{\sqrt{-1}\,\frac{2\pi}{N} N(i+j)/2} = e^{\sqrt{-1}\pi\,(i+j)} = (-1)^{i+j}$$

$$= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\, (-1)^{i+j}\, W_N^{(ui+vj)}$$

Displaying the DFT

- Usually, the DFT is displayed with its center coordinate (u, v) = (0, 0) at the center of the image.
- This way, the lower frequency information (which usually dominates an image) is clustered together near the origin at the center of the display.

$$\tilde{I}(u - \tfrac{N}{2}, v - \tfrac{N}{2}) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\ W_N^{[(u-N/2)i+(v-N/2)j]}$$

$$= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\ W_N^{(ui+vj)}\ W_N^{-N(i+j)/2}$$

$$W_N^{-N(i+j)/2} = e^{\sqrt{-1}\,\frac{2\pi}{N} N(i+j)/2} = e^{\sqrt{-1}\pi\,(i+j)} = (-1)^{i+j}$$

$$= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\,(-1)^{i+j}\ W_N^{(ui+vj)}$$

$$= \mathrm{DFT}\left[(-1)^{i+j} I(i, j)\right]$$

- This can be accomplished in practice by taking the DFT of the alternating image (for display purposes only!)

$$[(-1)^{i+j} I(i,j)\ ;\ 0 \leq\ i, j \leq\ N-1]$$

Displaying the DFT

- Usually, the DFT is displayed with its center coordinate (u, v) = (0, 0) at the center of the image.
- This way, the lower frequency information (which usually dominates an image) is clustered together near the origin at the center of the display.
- This can be accomplished in practice by taking the DFT of the alternating image (for display purposes only!)

$$[(-1)^{i+j} I(i,j) \;;\; 0 \leq i, j \leq N-1]$$

- Observe that

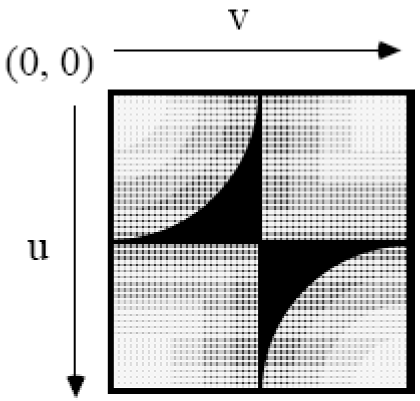
$$(-1)^{i+j} = e^{\sqrt{-1}\pi\,(i+j)} = e^{\sqrt{-1}\,\frac{2\pi}{N}\,N(i+j)/2} = W_N^{-N(i+j)/2}$$

so

$$\begin{aligned}
\mathrm{DFT}\left[(-1)^{i+j} I(i, j)\right] &= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\,(-1)^{i+j}\, W_N^{(ui+vj)} \\
&= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\, W_N^{(ui+vj)}\, W_N^{-N(i+j)/2} \\
&= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\, W_N^{[(u-N/2)i+(v-N/2)j]} \\
&= \tilde{I}(u - \tfrac{N}{2}, v - \tfrac{N}{2})
\end{aligned}$$

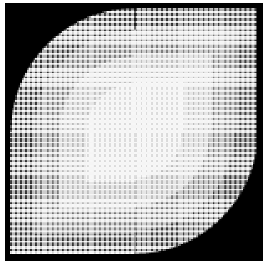
- A simple shift of the DF

Centered DFT

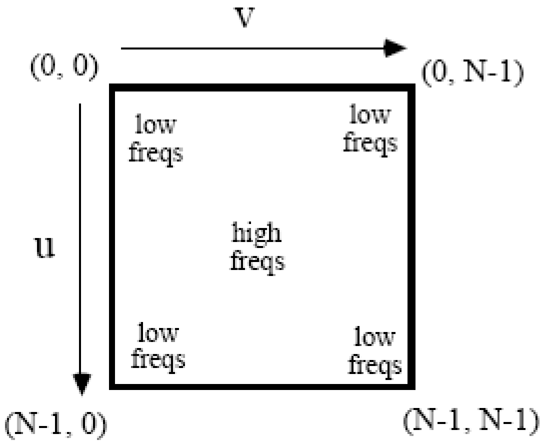


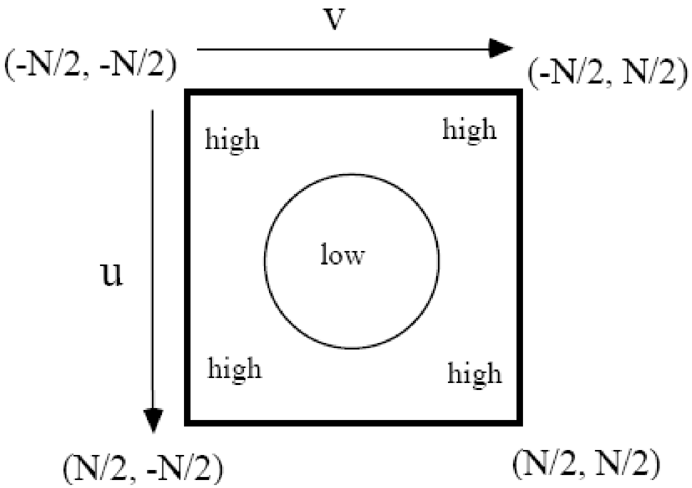
$|\tilde{\mathbf{I}}|$

Original DFT



Centered DFT





Displaying the DFT

- Since the DFT is **complex** one can display only either the magnitude or phase as an image at a time.
- Usually the phase is very difficult to interpret visually.
- To display the magnitude, usually it's best to **logarithmically compress** it:

$$\log \left[\, 1 + \left|\tilde{I}(u, v)\right| \,\right]$$

  prior to display, since (visually) the low-amplitude frequencies will be hard to see.
- Following the logarithm, it is necessary to use a linear point operation to stretch the contrast, since the log values will be very small.

The Meaning of Image Frequencies

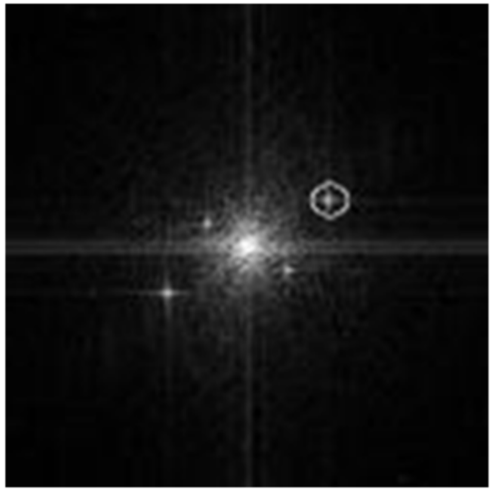
- It is sometimes easy to lose track of the meaning of the DFT and of the **frequency content** of an image in all the math.
- The DFT is precisely that - a description of the frequency content.
- By looking at the DFT or **spectrum** of an image (especially its magnitude), we can determine much about the image.

<u>Qualitative Properties of DFT</u>

- We may regard the DFT as an **image of frequency content**.
- Bright regions in the DFT "image" correspond to frequencies that have large magnitudes in the real image.
- It is very intuitive to think of the frequency content of an image in terms of its **granularity** (distribution of radial frequencies) and its **orientation**.

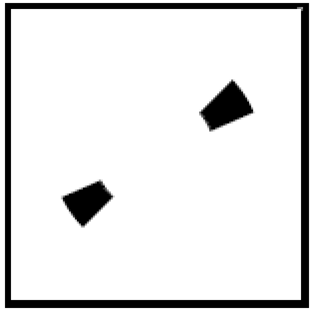
Periodic Noise removal

Periodic Noise removal



Narrowband Image

- It is also possible to produce an images that are highly granular **and** highly oriented:



- This mask was created by (pointwise) multiplying the mid-frequency mask with one of the oriented masks.

Filtered Image