Other Considerations in the Regression Model

Section 3.3

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Outline

1. Qualitative Predictors
2. Interaction Model

Categorical (Qualitative) Predictors

- Using the *mtcars* data set.
- Suppose that we wish to investigate the difference in the `mpg` based on the transmission `am`
- Here the transmission has only two categories, **automatic** and **manual**.

Dummy Variables

- Notice in `R` we have zeroes and ones for the values of `am`.
- These zeros and ones gives us a **dummy variable**, or an indicator variable.

$$x_1 = \begin{cases} 1 & \text{if } i\text{th car has a manual transmission} \\ 0 & \text{if } i\text{th car has an automatic transmission} \end{cases}$$

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- This results in the following model:

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i = \begin{cases} \beta_0 + \beta_1 + \epsilon_i & \text{if } i\text{th car has a manual transmission} \\ \beta_0 + \epsilon_i & \text{if } i\text{th car has an automatic transmission} \end{cases}$$

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- β_0 can be interpreted as the average mpg among automobiles with automatic transmission.
- $\beta_0 + \beta_1$ is the average mpg among automobiles with manual transmission.
- β_1 is the average difference in mpg between automobiles with automatic and manual transmission.

Results

```
> #Use mtcars
> summary(lm(mpg~am,data = mtcars))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   17.147      1.125  15.247 1.13e-15 ***
am             7.245      1.764   4.106 0.000285 ***
---
Signif. codes:
0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

$\hat{y} = 17.147 + 7.245\,am$

```
Residual standard error: 4.902 on 30 degrees of freedom
Multiple R-squared:  0.3598,Adjusted R-squared:  0.3385
F-statistic: 16.86 on 1 and 30 DF,  p-value: 0.000285
```

$$\hat{y} = \begin{cases} 17.147 & \text{if automatic} \\ 17.147 + 7.245 & \text{if manual} \end{cases}$$

$$\hat{y} = \begin{cases} 17.14 & \text{if automatic} \\ 24.392 & \text{if manual} \end{cases}$$

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- The average `mpg` for an automobile with automatic transmission is 17.147.

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- The average `mpg` for an automobile with automatic transmission is 17.147.
- Having a manual transmission added 7.245 to the mpg on average.

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```

$\hat{y} = 17.147 + 7.245\,am$ $\quad am = \begin{cases} 1 & \text{if manual} \\ 0 & \text{if automatic} \end{cases}$

- The average `mpg` for an automobile with automatic transmission is 17.147.
- Having a manual transmission added 7.245 to the mpg on average.
- The average `mpg` for an automobile with manual transmission is 17.147 + 7.245 = 24.392.

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- The average `mpg` for an automobile with automatic transmission is 17.147.
- Having a manual transmission added 7.245 to the mpg on average.
- The average `mpg` for an automobile with manual transmission is 17.147 + 7.245 = 24.392.
- Note could have done 1 and -1 as a code instead.

More Than Two Levels

Suppose we want use the number of cylinders as predictors. We have already indicated that this is categorical.

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i = \begin{cases} \beta_0 + \beta_1 + \epsilon_i & \text{if the } i\text{th car has 6 cylinders} \\ \beta_0 + \beta_2 + \epsilon_i & \text{if the } i\text{th car has 8 cylinders} \\ \beta_0 + \epsilon_i & \text{if the } i\text{th car has 4 cylinders} \end{cases}$$

$$x_{i1} = \begin{cases} 1 & \text{if } i\text{th car has 6 cylinders} \\ 0 & \text{if } i\text{th car does not have 6 cylinders} \end{cases}$$

$$x_{i2} = \begin{cases} 1 & \text{if } i\text{th car has 8 cylinders} \\ 0 & \text{if } i\text{th car does not have 8 cylinders} \end{cases}$$

Summary

```
> cyl.fact = as.factor(mtcars$cyl)
> summary(lm(mtcars$mpg~cyl.fact))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  26.6636     0.9718  27.437  < 2e-16 ***
cyl.fact6    -6.9208     1.5583  -4.441 0.000119 ***
cyl.fact8   -11.5636     1.2986  -8.905 8.57e-10 ***
---
Signif. codes:
0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 3.223 on 29 degrees of freedom
Multiple R-squared:  0.7325,Adjusted R-squared:  0.714
F-statistic:  39.7 on 2 and 29 DF,  p-value: 4.979e-09
```

$$\hat{y} = \begin{cases} 26.6636 & \text{if } 4\text{cyl} \\ 26.6636 - 6.9208 & \text{if } 6\text{cyl} \\ 26.6636 - 11.5636 & \text{if } 8\text{cyl} \end{cases}$$

Test for $H_0: \beta_j = 0$

Since all of the p-values are small, we reject H_0 and conclude that the number of cylinders is significant in predicting mpg.

Example from Textbook

```
library(ISLR)
summary(lm(Balance~Ethnicity,data = Credit))

Coefficients:
                   Estimate Std. Error t value Pr(>|t|)
(Intercept)          531.00      46.32  11.464   <2e-16 ***
EthnicityAsian       -18.69      65.02  -0.287    0.774
EthnicityCaucasian   -12.50      56.68  -0.221    0.826
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 460.9 on 397 degrees of freedom
Multiple R-squared:  0.0002188,Adjusted R-squared:  -0.004818
F-statistic: 0.04344 on 2 and 397 DF,  p-value: 0.9575
```

From pp 85 - 86.

$$\hat{\text{balance}} = \begin{cases} 531 & \text{if African American} \\ 531 - 18.69 & \text{if Asian} \\ 531 - 12.50 & \text{if Caucasian} \end{cases}$$

F-Test: $H_0: \beta_1 = \beta_2 = 0$ vs. H_A: At least one $\beta_j \neq 0$

p-value $= 0.9575$

Lab Questions

Use the model from the previous slide to answer the questions.

1. What is the estimated average balance for a person that is Asian?

   a) 531
   b) -18.69
   c) -12.50
   d) 512.31

2. Do we have evidence that there is a difference in balance based on ethnicity?

   a) Yes
   b) No

F-Test FRH$_0$ $H_0: \beta_1 = \beta_2 = 0$

$\therefore$ Balance is constant regardless of ethnicity.

Two Important Assumptions

1. The **additive** assumptions means that the effect of changes in a predictor X_j on the response Y is independent of the values of the other predictors.

2. The **linear** assumptions means that the change in the response Y due to a one-unit change in X_j is constants, regardless of the value of X_J.

Recall Stock Price Data: Best Subset of Predictors

additive model

```
stock2.lm <- lm(Stock_Index_Price~Interest_Rate+Unemployment_Rate,
                data = stock_price)
summary(stock2.lm)

Coefficients:
                  Estimate Std. Error t value Pr(>|t|)
(Intercept)         1798.4      899.2   2.000  0.05861 .
Interest_Rate        345.5      111.4   3.103  0.00539 **
Unemployment_Rate   -250.1      117.9  -2.121  0.04601 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 70.56 on 21 degrees of freedom
Multiple R-squared:  0.8976,Adjusted R-squared:  0.8879
F-statistic: 92.07 on 2 and 21 DF,  p-value: 4.043e-11
```

$$\widehat{Stock\_Index\_Price} = 1798.4 + 345.5 \times Interest\_Rate - 250.1 \times Unemployement\_Rate$$

Removing The Additive Assumption

- In our *stock price* data, we conclude that both *unemployment rate* and *interest rate* seem to be associated with *stock index price*.
- The linear model that we formed assumes that the effect on the stock index price of increasing one of percent of the interest rate is independent of the unemployment rate.
- For example the linear models states that the average effect on *stock index price* of a one percent increase in *interest rate* is always β_1 regardless of the *unemployment rate*.
- However, do we think that interest rate is independent of unemployment rate?
- Thus we can use an **interaction** term between interest rate and unemployment rate.
- The simplest method to construct an interaction term is to multiply two predictors together.

Model With Interaction Term

For our example we can have the model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2 + \epsilon$$

$$\begin{aligned} \mathit{Stock\_Index\_Price} &= \beta_0 + \beta_1 \times \mathit{Interest\_Rate} + \beta_2 \times \mathit{Unemployement\_Rate} \\ &+ \beta_3 \times \mathit{Interest\_Rate} \times \mathit{Unemployement\_Rate} + \epsilon \\ \\ &= \beta_0 + \beta_1 \times \mathit{Interest\_Rate} \\ &+ (\beta_2 + \beta_3 \times \mathit{Interest\_Rate}) \times \ \mathit{Unemployement\_Rate} + \epsilon \end{aligned}$$

We can interpret β_3 as the increase in the effectiveness of *unemployment rate* for a one unit increase in *interest rate* (or vice-versa).

Summary

Interaction model

Also :

```
> #Stock Price data
> #Model with Interaction term
> stock.int = lm(Stock_Index_Price~Interest_Rate*Unemployment_Rate)
> summary(stock.int)
Call:
lm(formula = Stock_Index_Price ~ Interest_Rate * Unemployment_Rate)
Residuals:
     Min       1Q   Median       3Q      Max
-156.009  -40.238   -8.873   52.131  122.073

Coefficients:
                                Estimate Std. Error t value Pr(>|t|)
(Intercept)                      2522.85    2634.04   0.958    0.350
Interest_Rate                     -32.49    1293.06  -0.025    0.980
Unemployment_Rate                -380.76     461.09  -0.826    0.419
Interest_Rate:Unemployment_Rate    68.54     233.53   0.293    0.772

Residual standard error: 72.15 on 20 degrees of freedom
Multiple R-squared:  0.8981,Adjusted R-squared:  0.8828
F-statistic: 58.74 on 3 and 20 DF,  p-value: 4.266e-10
```

Interpretation

$$
\begin{aligned}
Stock\_Index\_Price &\approx 2522.85 - 32.49 \times Interest\_Rate - 308.76 \times Unemployement\_Rate \\
&+ 68.54 \times (Interest\_Rate \times Unemployment\_Rate) \\
\\
&= 2522.85 - 32.49 \times Interest\_Rate \\
&+ (63.54 \times Interest\_Rate - 308.76) \times Unemployment\_Rate
\end{aligned}
$$

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\mathit{Stock_Index_Price} &\approx 2522.85 - 32.49 \times \mathit{Interest_Rate} - 308.76 \times \mathit{Unemployement_Rate} \\
&+ 68.54 \times (\mathit{Interest_Rate} \times \mathit{Unemployment_Rate}) \\
\\
&= 2522.85 - 32.49 \times \mathit{Interest_Rate} \\
&+ (63.54 \times \mathit{Interest_Rate} - 308.76) \times \mathit{Unemployment_Rate}
\end{aligned}$$

- Increasing the unemployment rate by 1% will increase the stock index price by $63.54 \times \mathit{Interest\_Rate} - 308.76$.

Interpretation

$$\begin{aligned}
\textit{Stock_Index_Price} &\approx 2522.85 - 32.49 \times \textit{Interest_Rate} - 308.76 \times \textit{Unemployement_Rate} \\
&+ 68.54 \times (\textit{Interest_Rate} \times \textit{Unemployment_Rate}) \\
&= 2522.85 - 32.49 \times \textit{Interest_Rate} \\
&+ (63.54 \times \textit{Interest_Rate} - 308.76) \times \textit{Unemployment_Rate}
\end{aligned}$$

- Increasing the unemployment rate by 1% will increase the stock index price by $63.54 \times \textit{Interest\_Rate} - 308.76$.
- However, notice the p-values when testing $H_0 : \beta_j = 0$. Since all are greater than 0.05, this means that at least one of these terms are not needed in the model.

Using The `step()` Function

```
> step(stock.int)
Start:  AIC=209
Stock_Index_Price ~ Interest_Rate * Unemployment_Rate

                                  Df Sum of Sq    RSS    AIC
- Interest_Rate:Unemployment_Rate  1    448.39 104559 207.11
<none>                                         104110 209.00

Step:  AIC=207.11
Stock_Index_Price ~ Interest_Rate + Unemployment_Rate

                    Df Sum of Sq    RSS    AIC
<none>                           104559 207.11
- Unemployment_Rate  1     22394 126953 209.76
- Interest_Rate      1     47932 152491 214.16

Call:
lm(formula = Stock_Index_Price ~ Interest_Rate + Unemployment_Rate)

Coefficients:
(Intercept)      Interest_Rate  Unemployment_Rate
1798.4             345.5             -250.1
```

Roller Coaster Model

- Speed = top speed of a roller coaster
- Type = 2 if steel 1 if wood
- Height = tallest point of the roller coaster
- Model:

$$\begin{aligned} \text{Speed} &\approx \beta_0 + \beta_1 \times \text{Height} + \begin{cases} \beta_2 & \text{if steel} \\ 0 & \text{if wood} \end{cases} \\ &= \beta_1 \times \text{Height} + \begin{cases} \beta_0 + \beta_2 & \text{if steel} \\ \beta_0 & \text{if wood} \end{cases} \end{aligned}$$

Summary

```
> rollercoaster$Type=factor(rollercoaster$Type)
> roller.lm = lm(Speed~Height+Type,data = rollercoaster)
> summary(roller.lm)

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 29.294434   1.563647  18.735  < 2e-16 ***
Height       0.240109   0.008658  27.733  < 2e-16 ***
Type2       -6.114395   1.486380  -4.114 7.54e-05 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 6.768 on 110 degrees of freedom
Multiple R-squared:  0.8749,Adjusted R-squared:  0.8727
F-statistic: 384.8 on 2 and 110 DF,  p-value: < 2.2e-16
```

$$\hat{speed} = \begin{cases} 29.294 + 0.240 * height & \text{if wood} \\ 29.2944 + 0.240 * height - 6.114 & \text{if steel} \end{cases}$$

Lab Questions

3. From the output is at least one of the predictors associated with the speed of the roller-coaster?

   a) Yes F test p-value small we have sig. b) No

4. From the output which predictor(s) can be used in the model, given that the other predictor is in the model?

   a) Height
   b) Type
   c) Both height and type
   d) Neither height not type

5. What is the estimate average speed for a wooden roller-coaster with height of 120 ft.?

   a) 29.29
   b) -6.11
   c) 0.24
   d) 58.09

Interaction Terms Between Quantitative and Categorical Predictors

- There maybe an interaction between height and type of roller coaster.
- With the interaction term the model becomes:

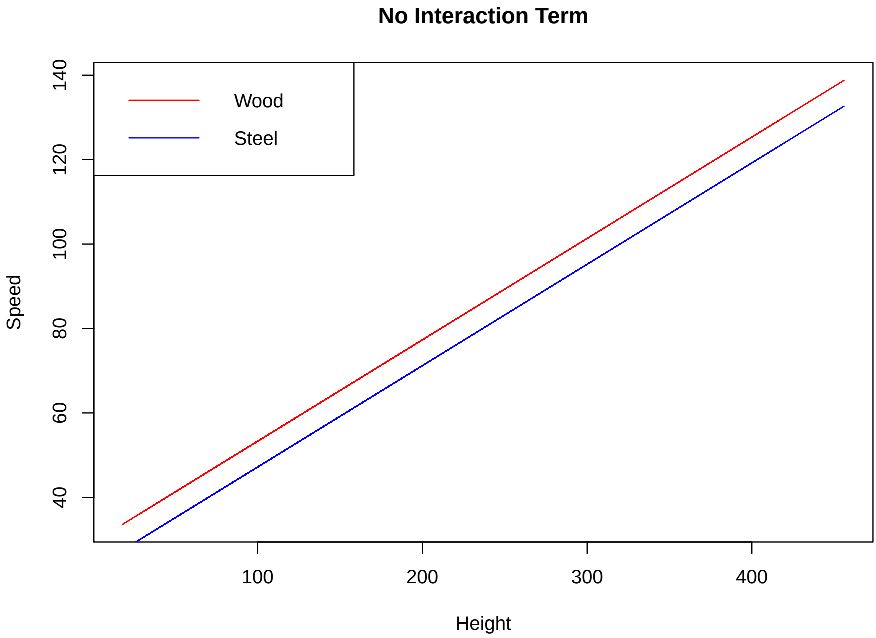
$$\begin{aligned}
\text{Speed} &\approx \beta_0 + \beta_1 \times \text{ height} + \begin{cases} \beta_2 + \beta_3 \times \text{ height}, & \text{if steel} \\ 0 & \text{if wood} \end{cases} \\
&= \begin{cases} (\beta_0 + \beta_2) + (\beta_1 + \beta_3) \times \text{ height}, & \text{if steel} \\ \beta_0 + \beta_1 \times \text{ height}, & \text{if wood} \end{cases}
\end{aligned}$$

Summary

```
> roller.int = lm(Speed~Height*Type,data = rollercoaster)
> summary(roller.int)

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)   30.35998    3.99872   7.592 1.14e-11 ***
Height         0.22985    0.03645   6.306 6.27e-09 ***
Type2         -7.25747    4.21806  -1.721   0.0882 .
Height:Type2   0.01088    0.03753   0.290   0.7726   Not significant
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
       Type2 = 0 if wood, 1 if steel
Residual standard error: 6.796 on 109 degrees of freedom
Multiple R-squared:  0.875,Adjusted R-squared:  0.8716
F-statistic: 254.4 on 3 and 109 DF,  p-value: < 2.2e-16
```

$$\hat{speed} = \begin{cases} 30.36 + 0.23 * height & \text{if wood} \\ 30.36 + 0.23 * height - 7.2575 + 0.011 * height & \text{if steel} \end{cases}$$


No Interaction Term
Wood
Steel
Speed
Height
40
60
80
100
120
140
100
200
300
400

Two Separate Regression Lines

$$\widehat{\text{speed}} = \begin{cases} 30.39 + 0.28 * \text{height} & \text{if wood} \\ 23.012 + 0.241 * \text{height} & \text{if Steel} \end{cases}$$