Digital Image Processing

COSC 6380/4393

Lecture – 14

Mar 2nd , 2023

Pranav Mantini

Slides from Dr. Shishir K Shah and Frank (Qingzhong) Liu, S. Narasimhan

Discrete Fourier Transform (DFT)

Recap: Sin and Cos

Amplitude

$a\,\sin\,(2\pi\ k\ t)$ $\qquad$ $\text{bcos}\,(2\pi\ l\ t)$

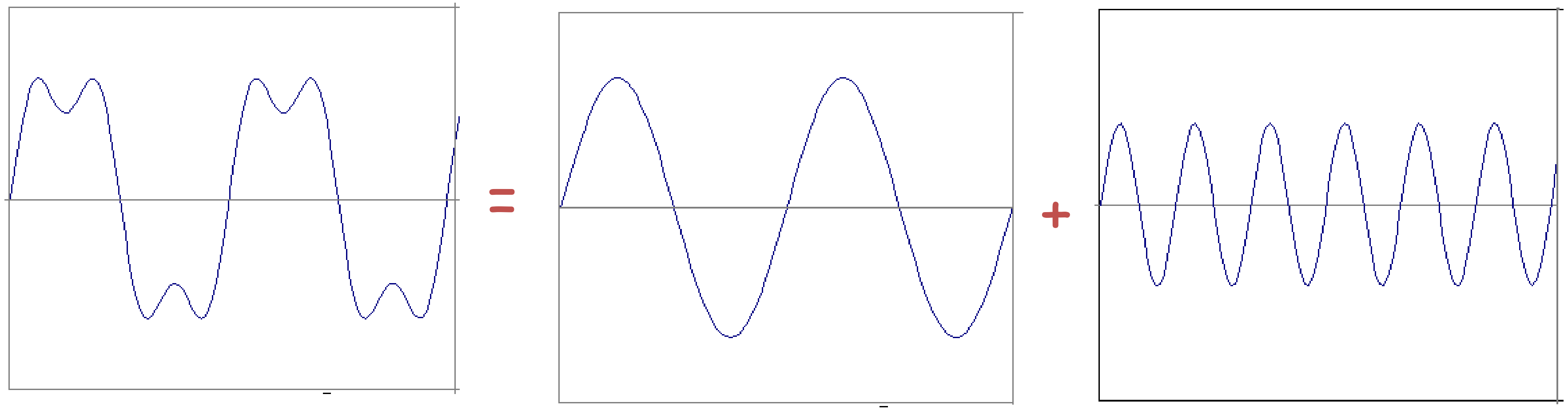
Frequency

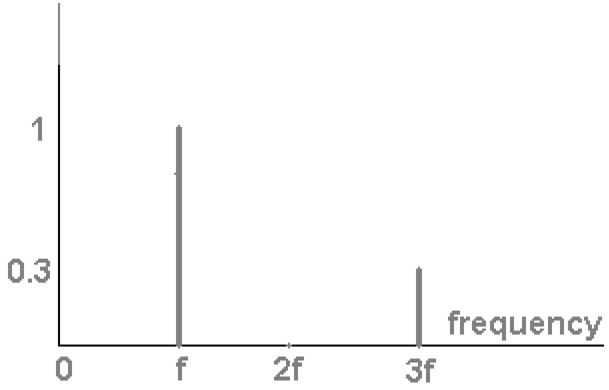
Jean Baptiste Joseph Fourier (1768-1830)

- Had crazy idea (1807): **Any** periodic function can be rewritten as a weighted sum of Sines and Cosines of different frequencies.
- Don't believe it?
  - Neither did Lagrange, Laplace, Poisson and other big wigs
  - Not translated into English until 1878!
- But it's true!
  - called Fourier Series
  - Possibly the greatest tool used in Engineering

Frequency Spectra

- example : $g(t) = \sin(2pi\, f\, t) + (1/3)\sin(2pi\,(3f)\, t)$





Periodic Function

Sum of sine and cosine waves:

$$f(t) = a_0 + a_1 \cos(t) + a_2 \cos(2t) + \cdots$$
$$b_1 \sin(t) + b_2 \sin(2t) + \cdots$$

Recap

$\int_0^{2\pi} \sin(mt)\, dt = 0$

$\int_0^{2\pi} cos(mt)\, dt = 0$

$$\int_0^{2\pi} \sin(mt)\, cos(nt)\, dt = 0$$

$$\int_0^{2\pi} \sin(mt)\, sin(nt)\, dt = 0$$

$$(\forall m \,! = n)$$

$$\int_0^{2\pi} \sin(mt)\, sin(nt)\, dt = \pi \; (m = n)$$

$$\int_0^{2\pi} \cos(mt)\, cos(nt)\, dt = 0$$

$$(\forall m \,! = n)$$

$$\int_0^{2\pi} \cos(mt)\, cos(nt)\, dt = \pi (m = n)$$

Periodic Function

Sum of sine and cosine waves:

$$f(t) = a_0 + a_1 \cos(t) + a_2 \cos(2t) + \cdots$$
$$b_1 \sin(t) + b_2 \sin(2t) + \cdots$$

Periodic Function

Sum of sine and cosine waves:

$$\int_0^{2\pi} f(t)dt = \int_0^{2\pi} a_0 dt \quad + \cancel{\int_0^{2\pi} a_1 \cos(t)\, dt} + \cancel{\int_0^{2\pi} a_2 \cos(2t)\, dt} + \cdots$$

$$\cancel{\int_0^{2\pi} b_1 \sin(t) dt} \quad + \cancel{\int_0^{2\pi} b_2 \sin(2t) dt} \quad + \cdots$$

Periodic Function

Sum of sine and cosine waves:

$$\int_0^{2\pi} f(t)dt = \int_0^{2\pi} a_0 dt = a_0(2\pi)$$

$$\Rightarrow a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt$$

Periodic Function

Sum of sine and cosine waves:

$$\int_0^{2\pi} f(t)\cos(nt)\,dt$$

$$= \int_0^{2\pi} a_0\cos(nt)\,dt + \int_0^{2\pi} a_1\cos(t)\cos(nt)\,dt + \int_0^{2\pi} a_2\cos(2t)\cos(nt)\,dt + \cdots$$

$$\int_0^{2\pi} b_1\sin(t)\cos(nt)\,dt + \int_0^{2\pi} b_2\sin(2t)\cos(nt)\,dt + \cdots$$

Periodic Function

Sum of sine and cosine waves:

$$\int_0^{2\pi} f(t)\cos(nt)\,dt$$

$$= \int_0^{2\pi} a_0\cos(nt)\,dt + \int_0^{2\pi} a_1\cos(t)\cos(nt)\,dt$$

$$+ \int_0^{2\pi} a_2\cos(2t)\cos(nt)\,dt + \cdots \int_0^{2\pi} a_n\cos(nt)\cos(nt)\,dt + \cdots$$

$$\int_0^{2\pi} b_1\sin(t)\cos(nt)\,dt + \int_0^{2\pi} b_2\sin(2t)\cos(nt)\,dt + \cdots$$

Periodic Function

Sum of sine and cosine waves:

$$\int_0^{2\pi} f(t)\cos(nt)\,dt = \int_0^{2\pi} a_n \cos(nt)\cos(nt)\,dt$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)\,dt$$

$$Similarly, b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)\,dt$$

Periodic Function

Sum of sine and cosine waves:

$$\Rightarrow a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)\,dt$$

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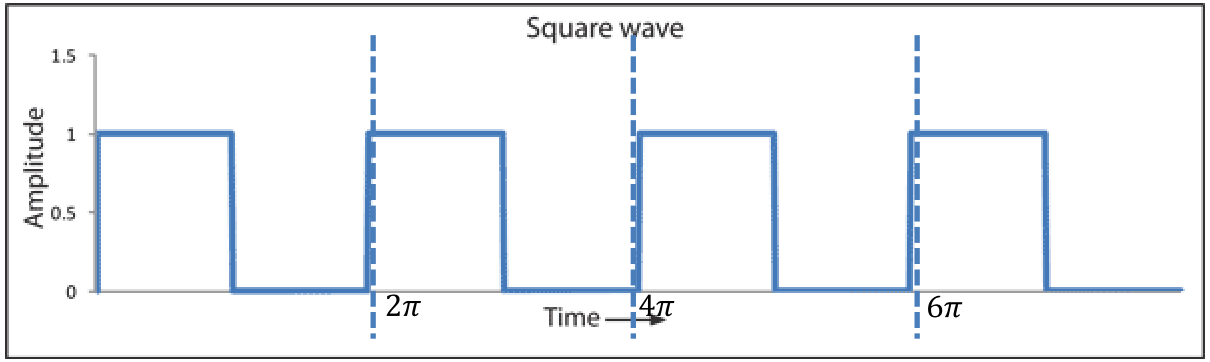
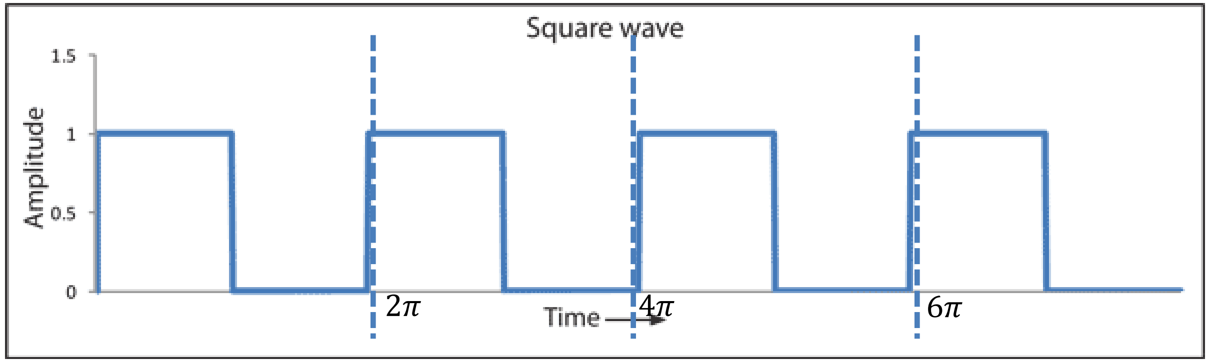
Periodic Function



Sum of sine and cosine waves:

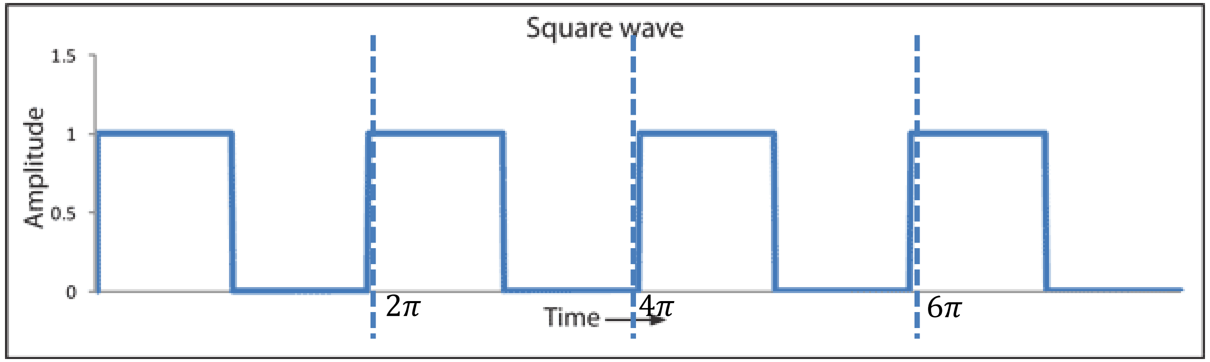
Periodic Function



Sum of sine and cosine waves:

Cosine Waves	Sine waves
$a_0 = ?$ $a_1 = ?$ $a_2 = ?$ . . . $a_n = ?$	$b_0 = ?$ $b_1 = ?$ $b_2 = ?$ . . . $b_n = ?$

Periodic Function

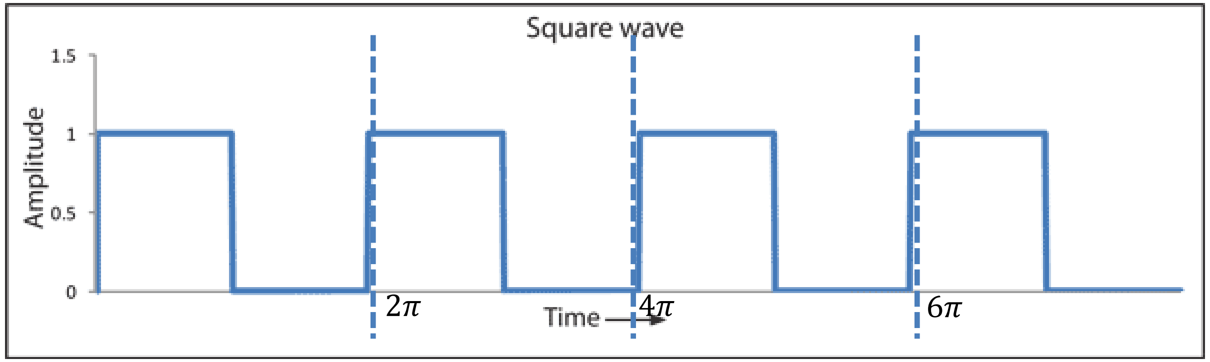


Sum of sine and cosine waves:

Periodicity: 2π

$$f(t) = \begin{cases} 1 \quad if\ 0 \le t < \pi \\ 0\ if\ \pi \le t < 2\pi \end{cases}$$

Periodic Function



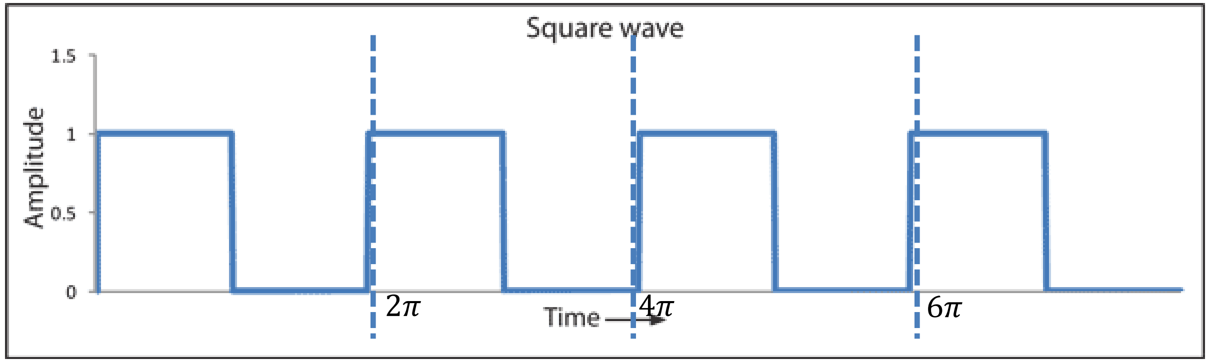
Sum of sine and cosine waves:

$$\Rightarrow a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)\,dt$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)\,dt$$

Periodic Function



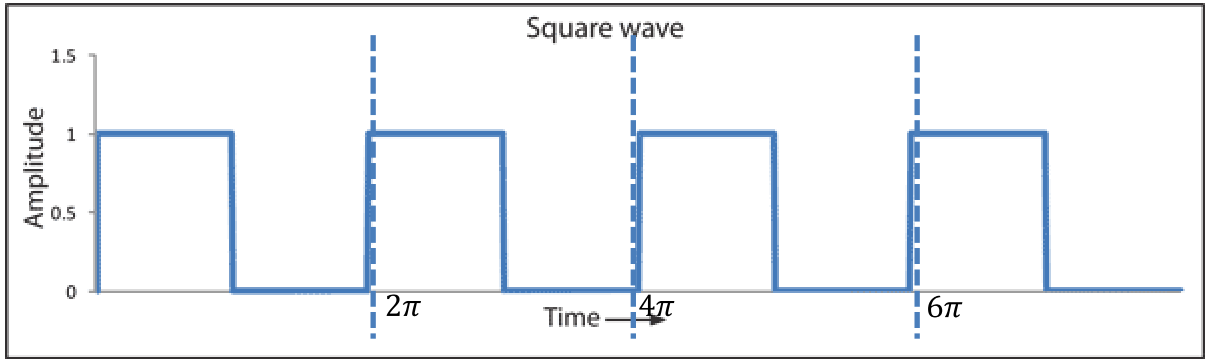
Sum of sine and cosine waves:

$$\Rightarrow a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt = 1/2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)\,dt = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)\,dt = \begin{cases} 0 \; if \; n \; is \; even \\ \frac{2}{n\pi} \; if \; n \; is \; odd \end{cases}$$

Periodic Function



Sum of sine and cosine waves:

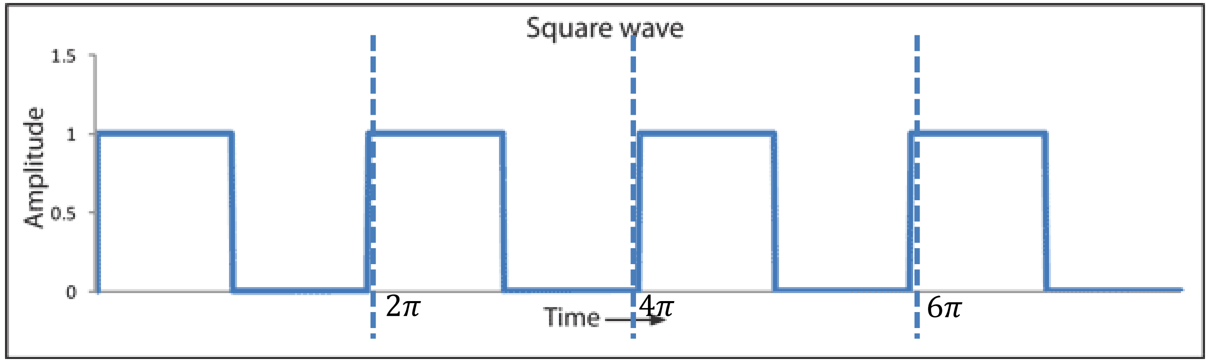
$$\Rightarrow a_0 = 1/2$$

$$a_n = 0$$

$$b_n = \begin{cases} 0 \; if \; n \; is \; even \\ \frac{2}{n\pi} \; if \; n \; is \; odd \end{cases}$$

Cosine Frequency spectra

Periodic Function



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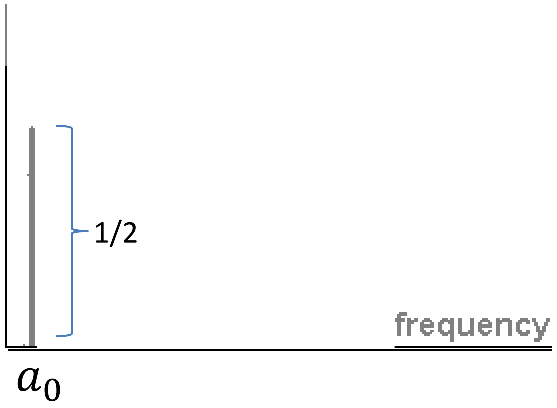
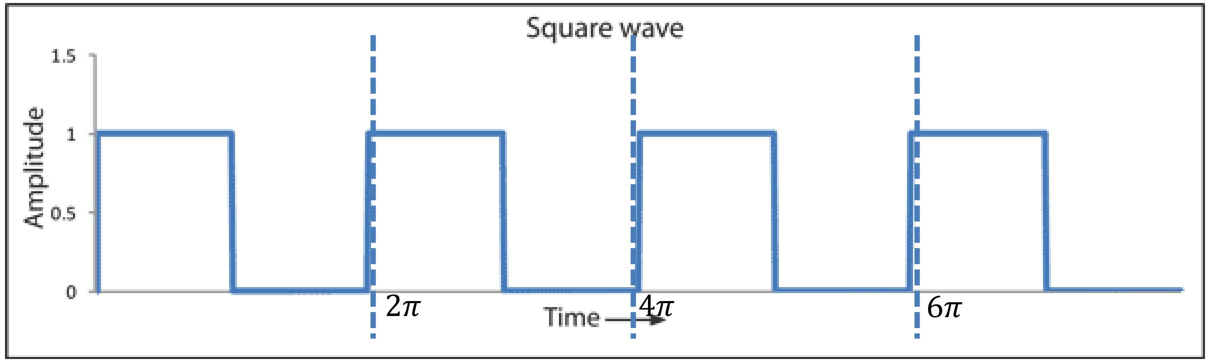


Periodic Function



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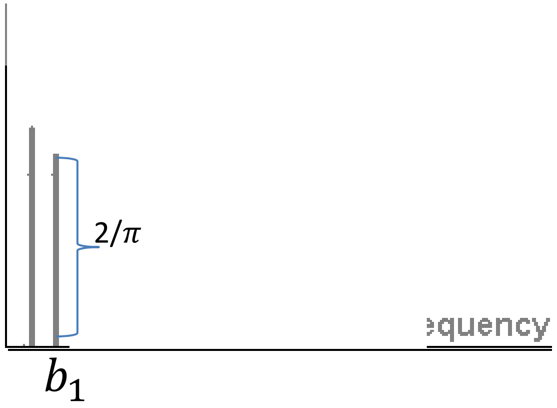
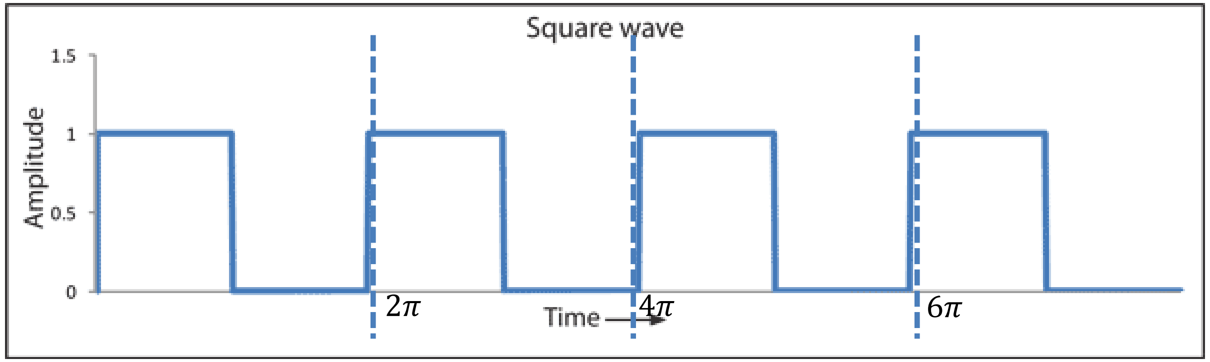
Sin Frequency spectra



Periodic Function



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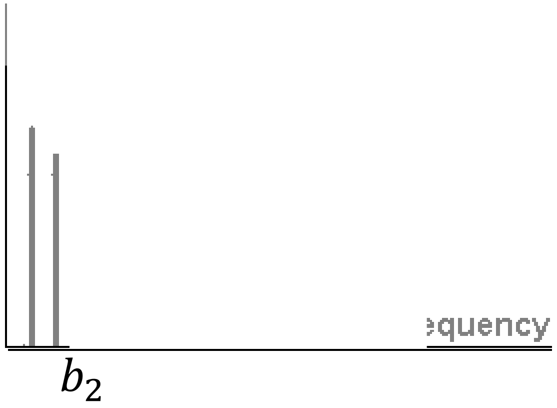
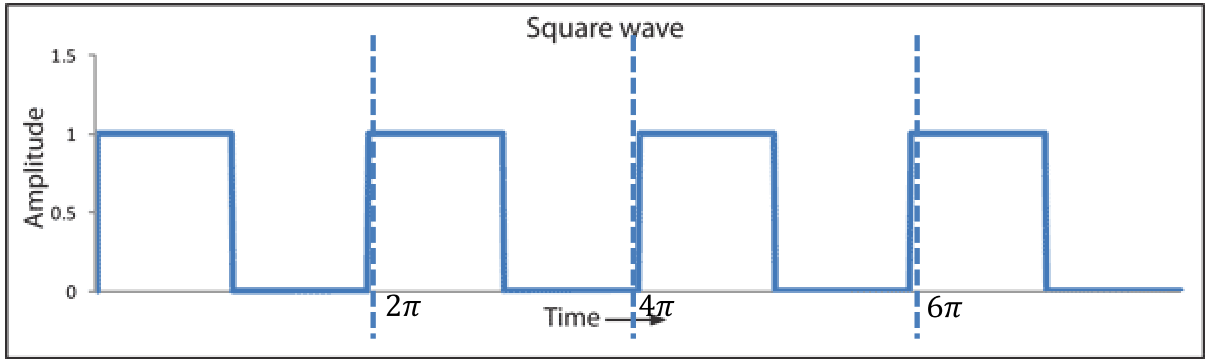


Periodic Function



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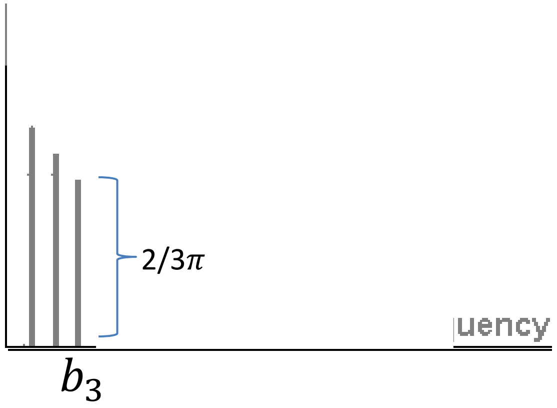
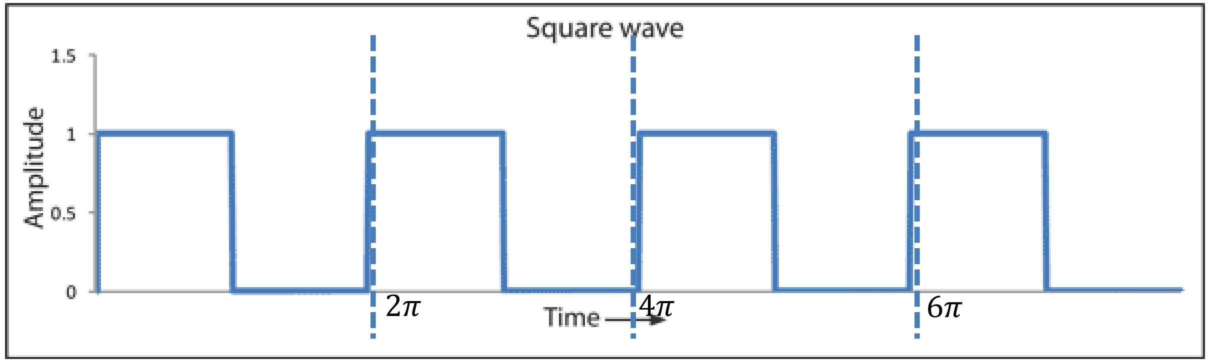
Sin Frequency spectra



Periodic Function



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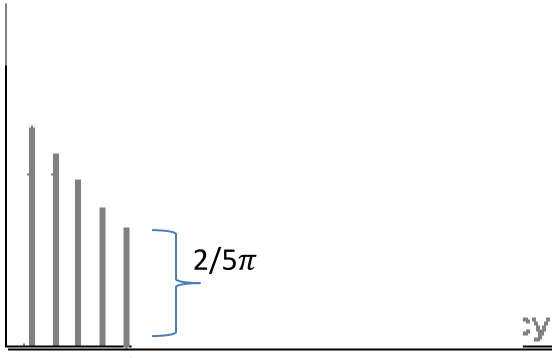
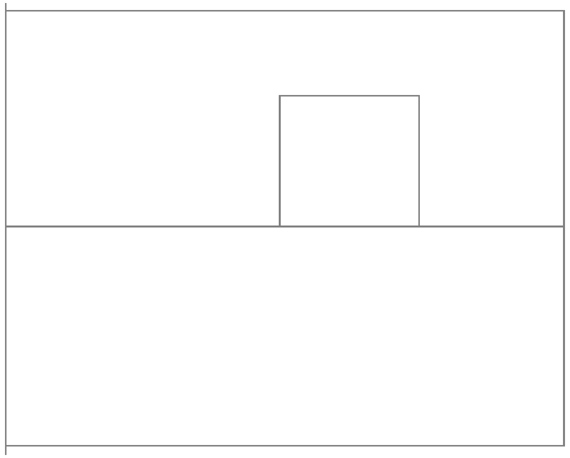
Sin Frequency spectra



Frequency Spectra



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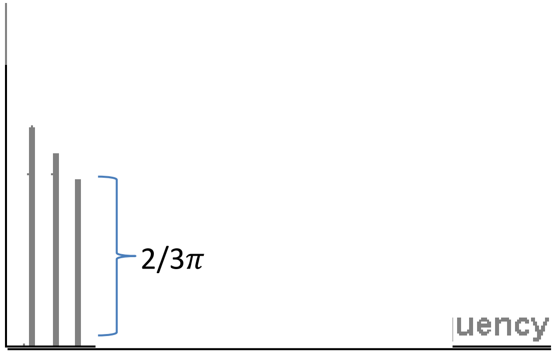

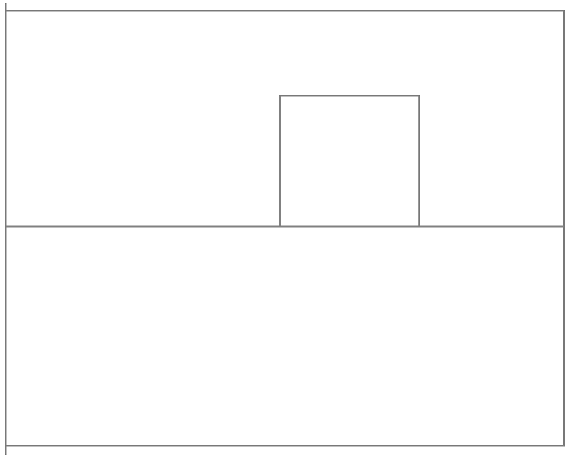

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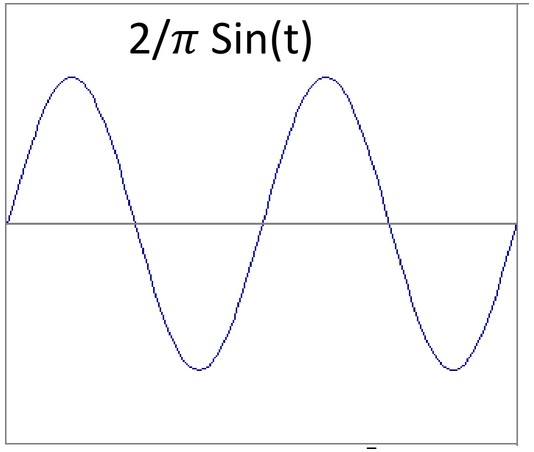

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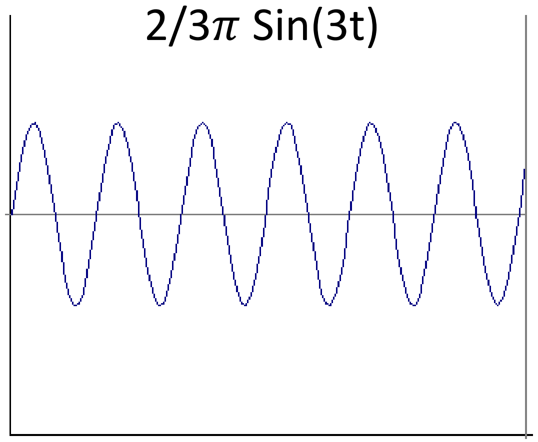

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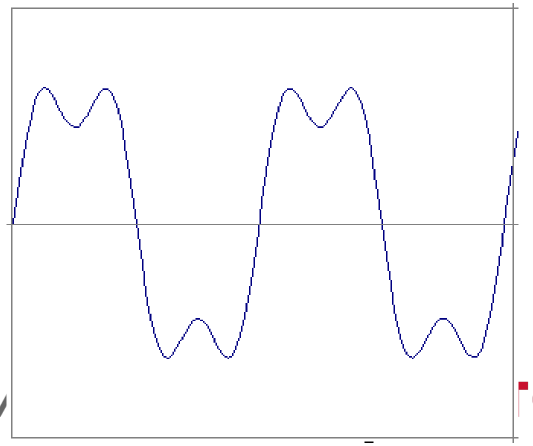

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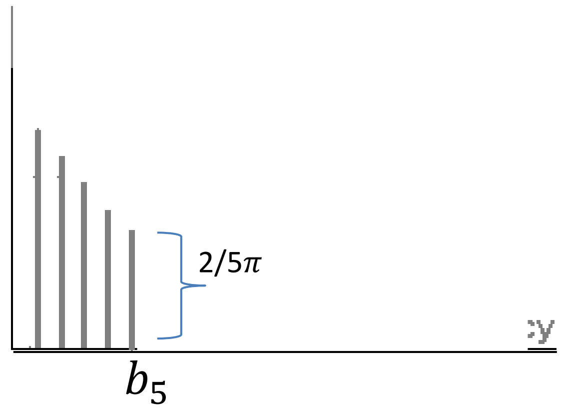
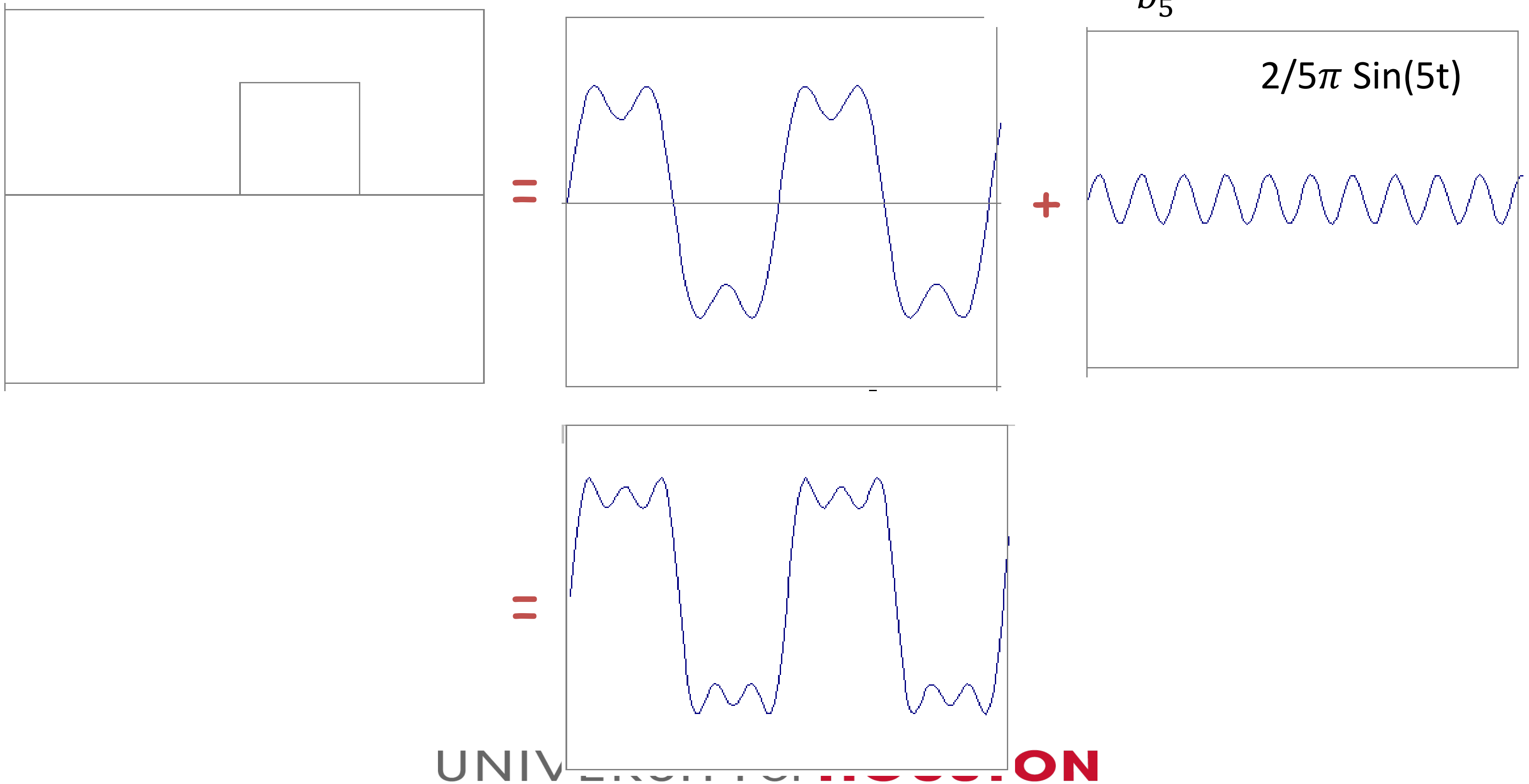
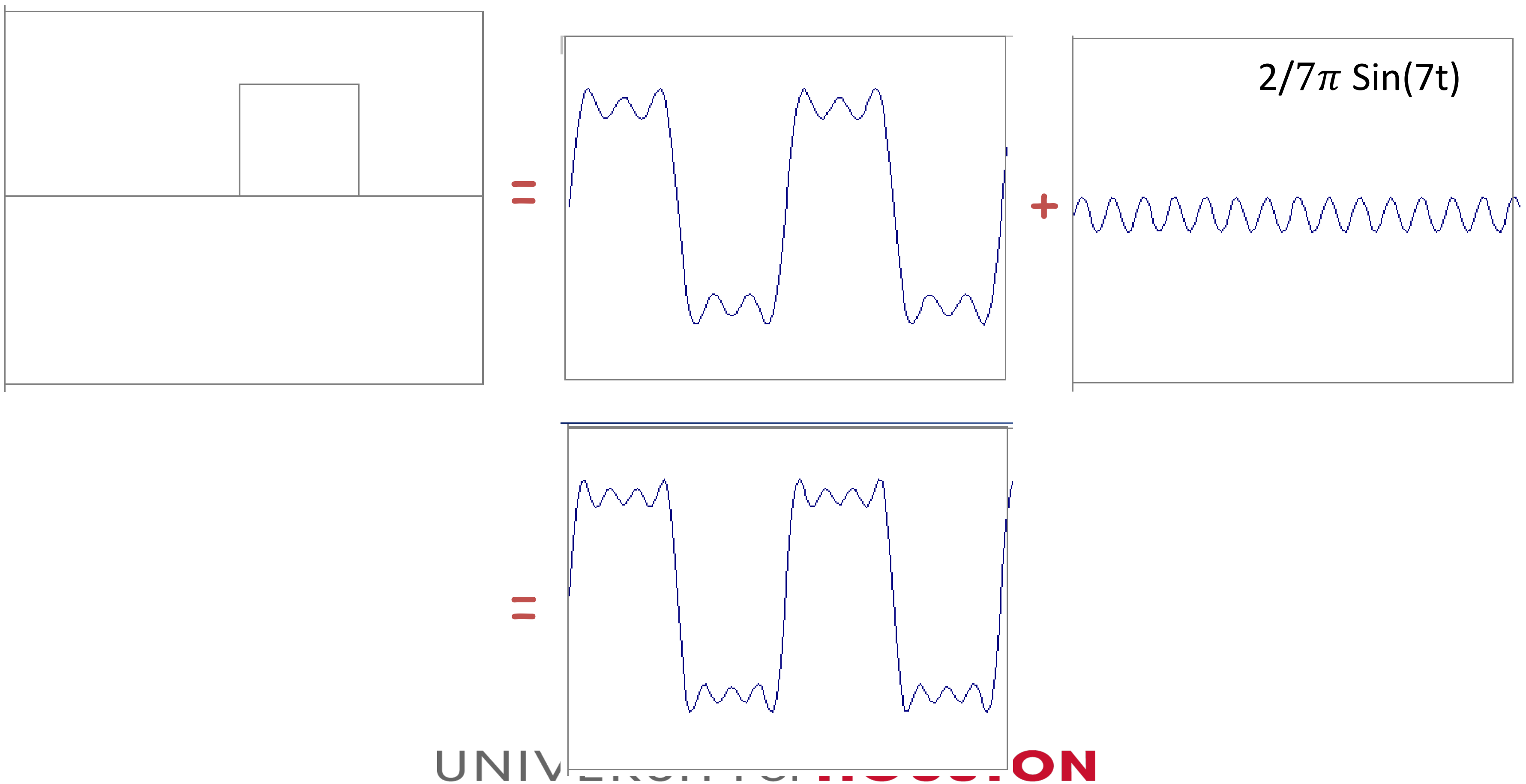




Frequency Spectra



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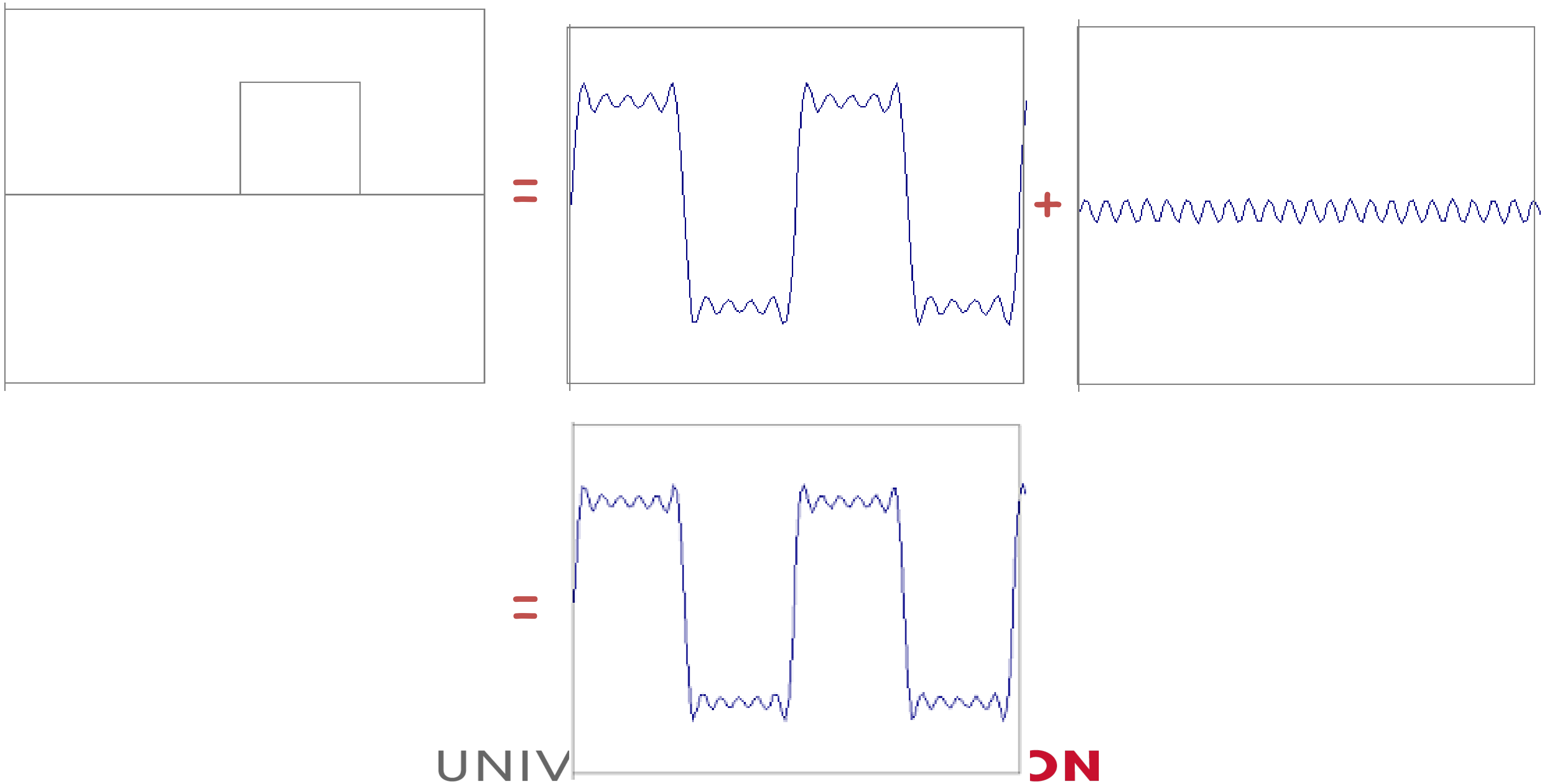

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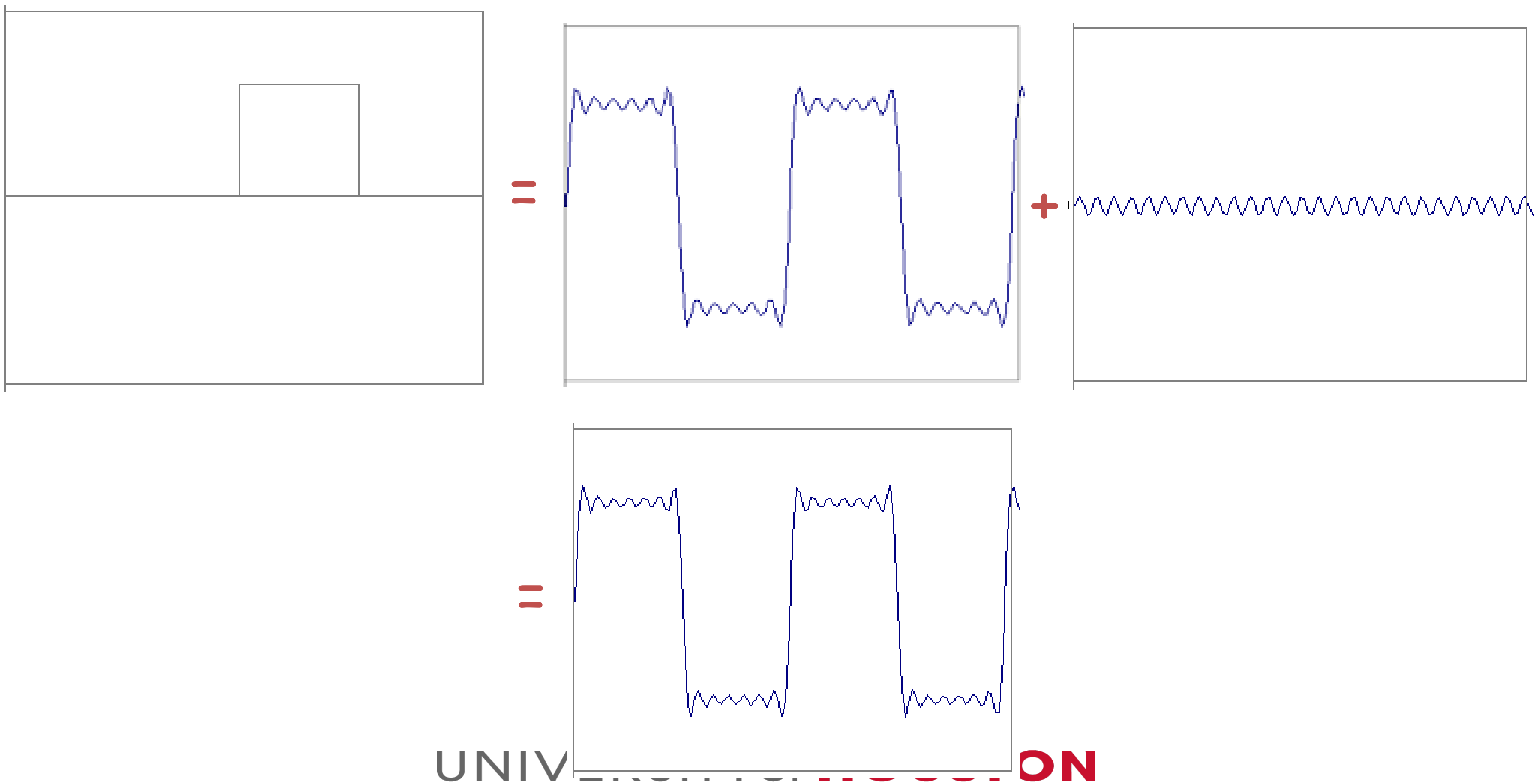

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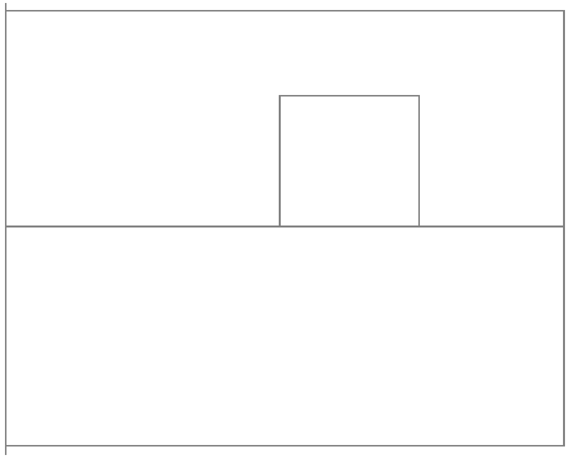
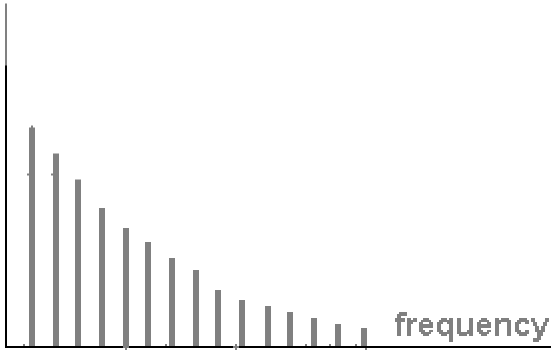
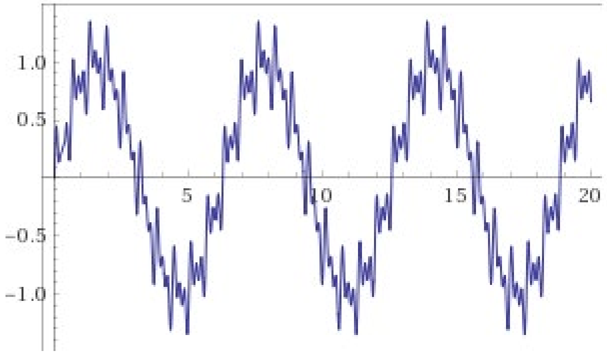


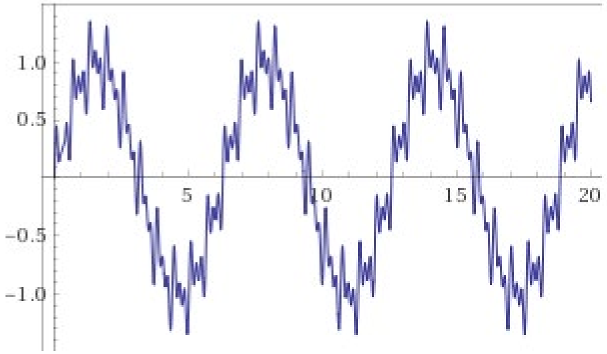
$$= \boxed{A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)}$$



Example

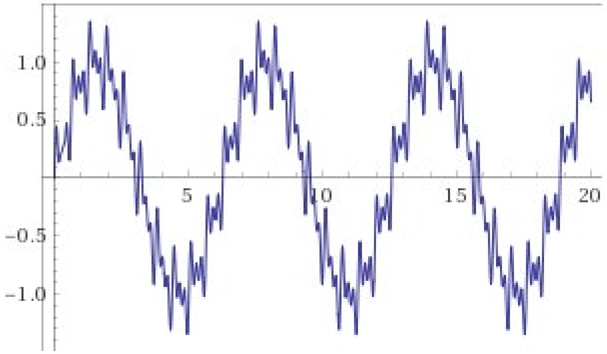


Example



f(t): A sine wave, with noise

Example

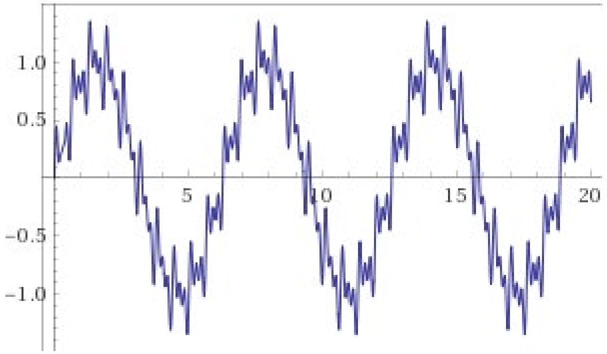


FT

Coefficients of the sine and cosine waves

t	0	1	…	10	…	20	….	….	30
cos	0	0	.	.	.	.	.	.	0
sin	0	1	0	0.15	0	0.15	.	0	0.15

Example



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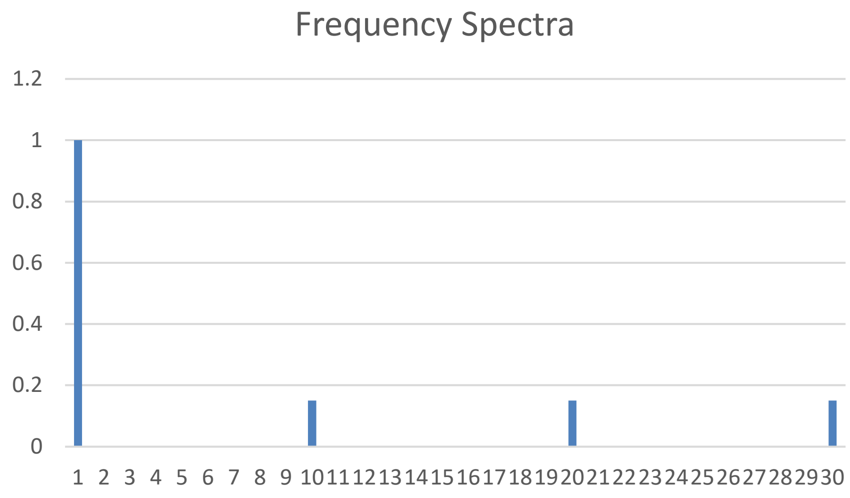
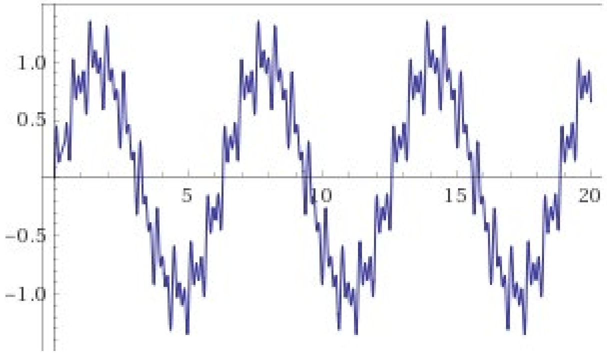


Example



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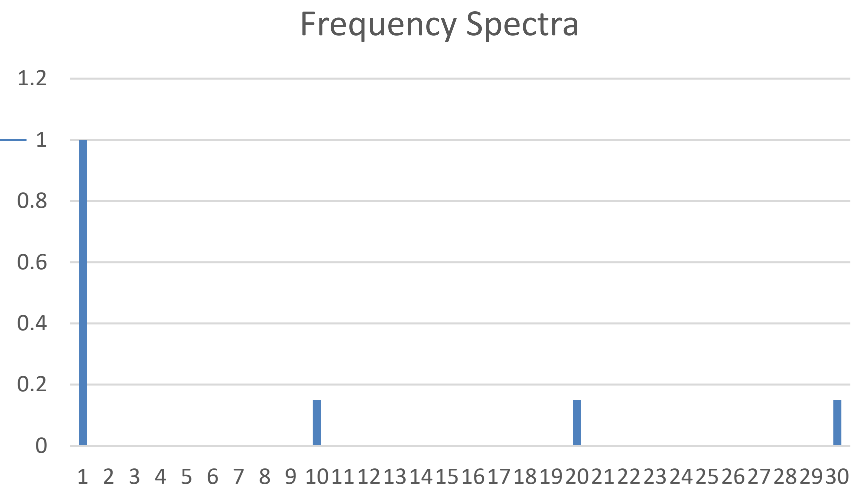
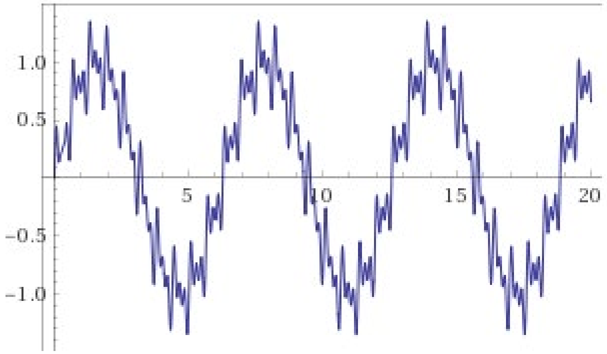
IFT



Example



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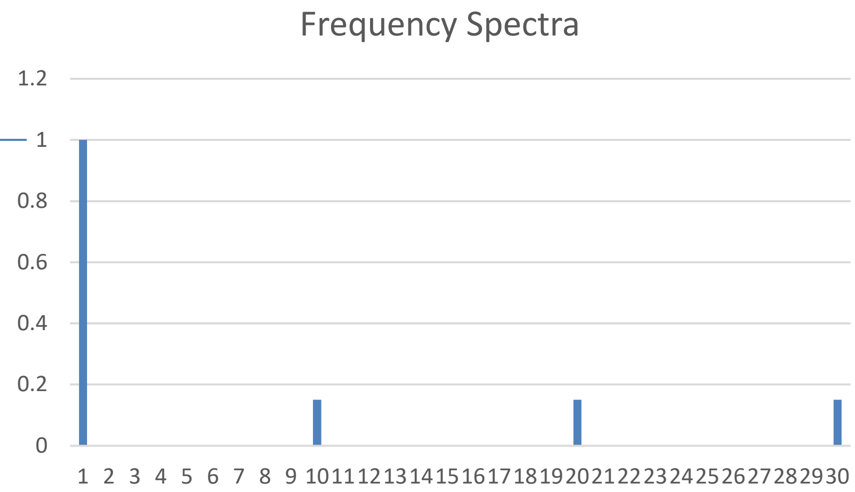
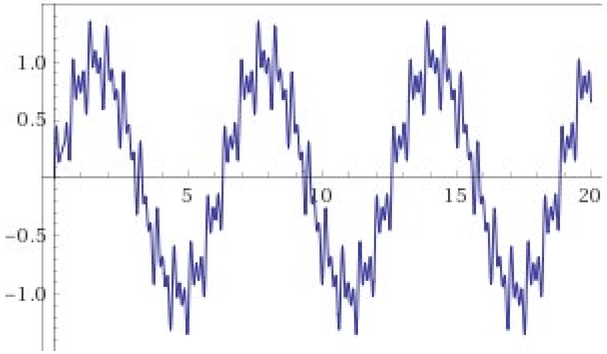


$$f(t) = \sin t + 0.15\sin 10t + 0.15\sin 20t + 0.15\sin 30t)$$

Example

Filter: Remove Noise



FT

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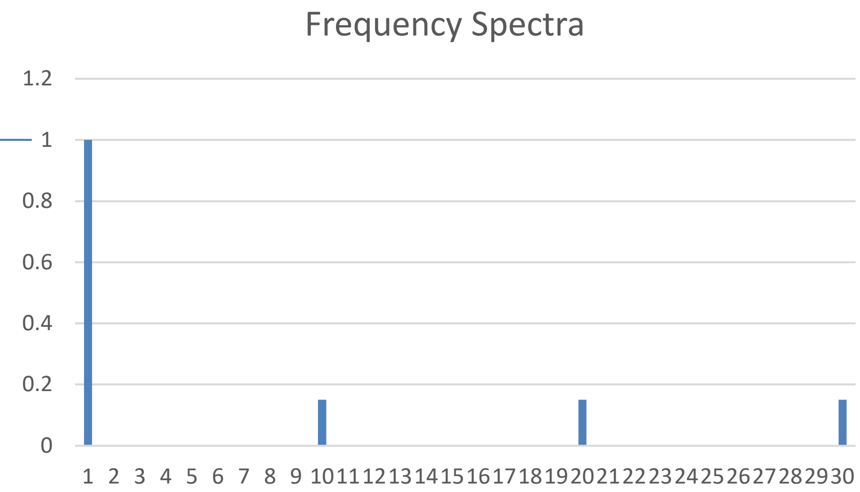
sin	0	1	0	0.15	0	0.15	.	0	0.15

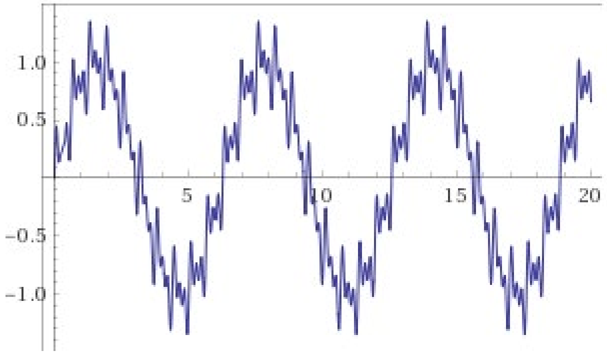


IFT

$$
\begin{aligned}
&\mathrm{f}(t) \\
&= \ sin\ t\ +\ 0.15 sin\ 10t \\
&+\ 0.15 sin\ 20t\ +\ 0.15 sin\ 30t)
\end{aligned}
$$

Example

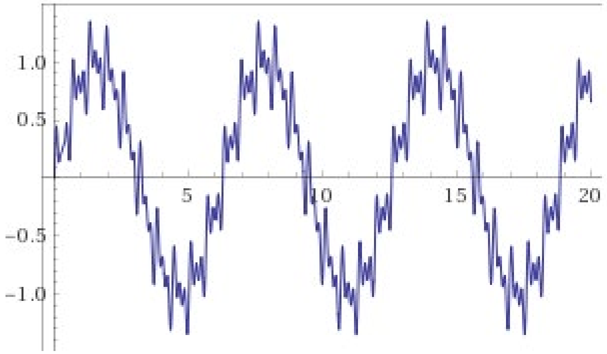
Filter: Remove Noise



FT

Example

Filter: Remove Noise (Remove high frequencies)



FT

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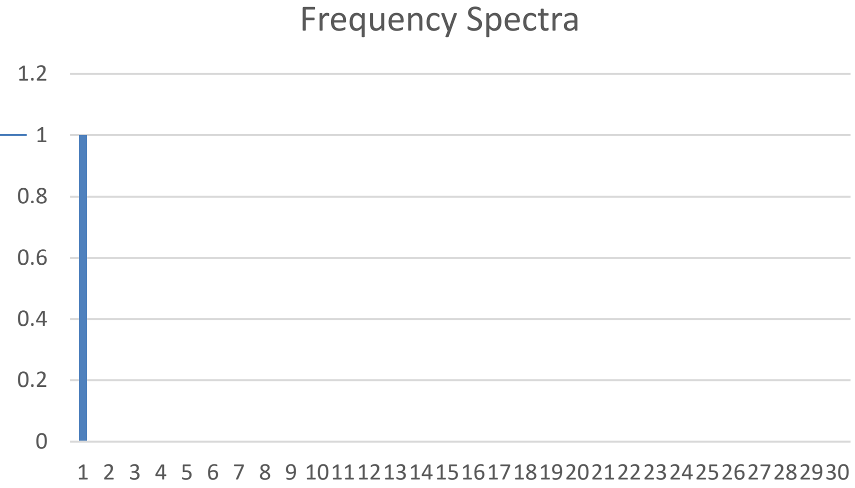
sin	0	1	0	**0**	0	**0**	.	0	**0**

IFT



Example

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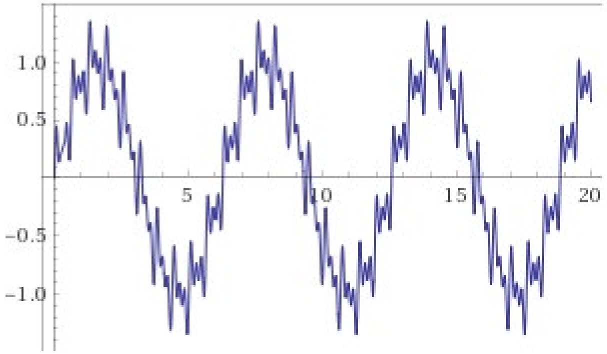

FT

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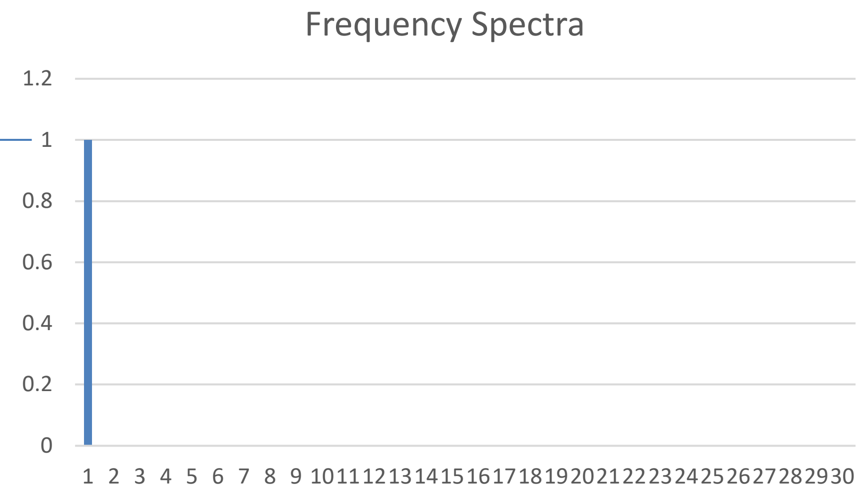
sin	0	1	0	**0**	0	**0**	.	0	**0**

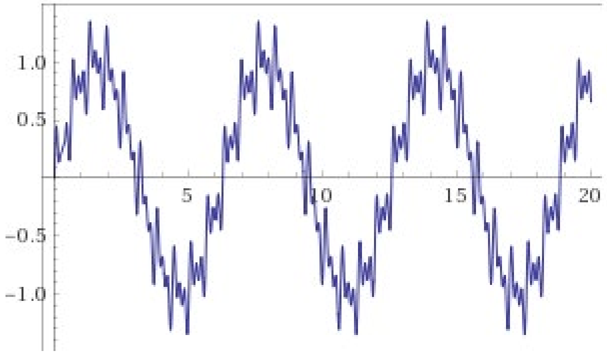


IFT

$$f(t) = \sin t + (0)\sin 10t + (0)\sin 20t + (0)\sin 30t) = \sin t$$

Example

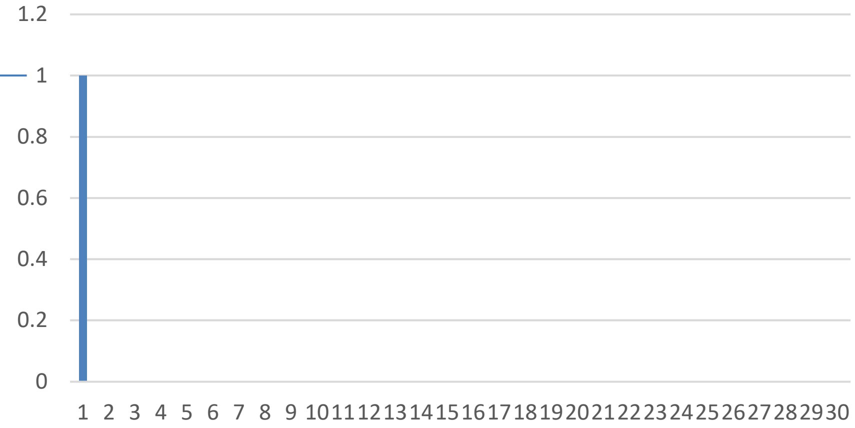
Filter: Remove Noise (Remove high frequencies)



FT

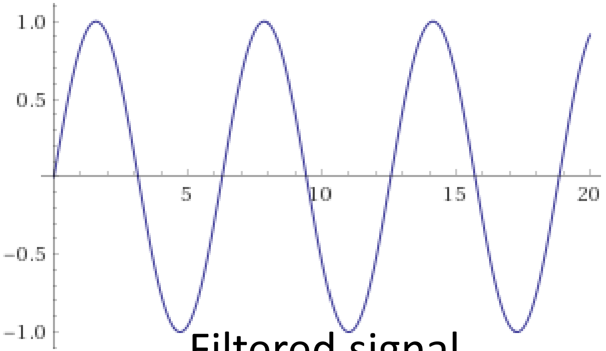
Coefficients of the sine and cosine waves

t	0	1	…	10	…	20	….	….	30
cos	0	0	.	.	.	.	.	.	0
sin	0	1	0	**0**	0	**0**	.	0	**0**



$$f(t) = \sin t + (0)\sin 10t + (0)\sin 20t + (0)\sin 30t) = \sin t$$

IFT

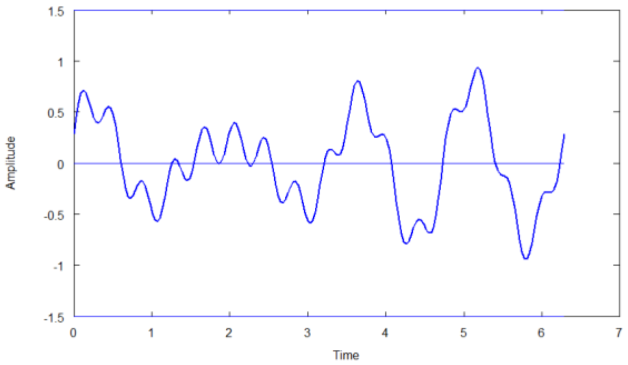


Filtered signal

DFT

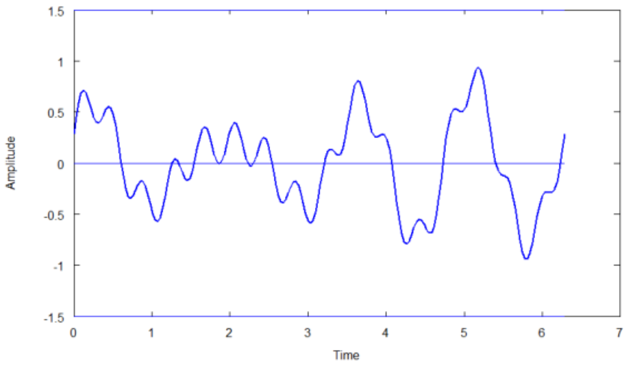
1. How to represent both the coefficients (sine and cos) of frequency t together (Complex Numbers)
2. How to compute DFT for 2D signals
3. Image as 2D discrete signals
4. DFT image
    1. Filtering
    2. .
    3. .

Frequency spectra



$$f(t) = a_0 + a_1 \cos(t) + a_2 \cos(2t) + \cdots$$
$$b_1 \sin(t) + b_2 \sin(2t) + \cdots$$

Frequency spectra



$$f(t) = a_0 + a_1\cos(t) + a_2\cos(2t) + \cdots$$
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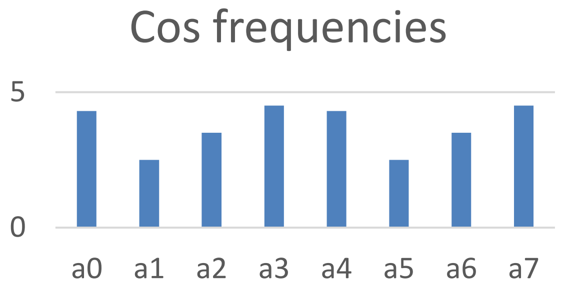
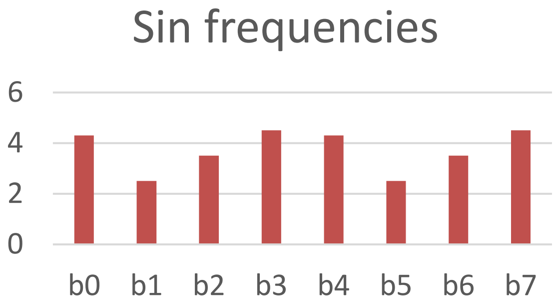
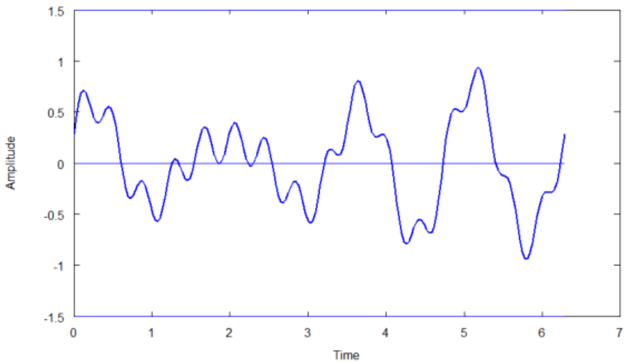




Frequency spectra



$$f(t) = a_0 + a_1 \cos(t) + a_2 \cos(2t) + \cdots$$
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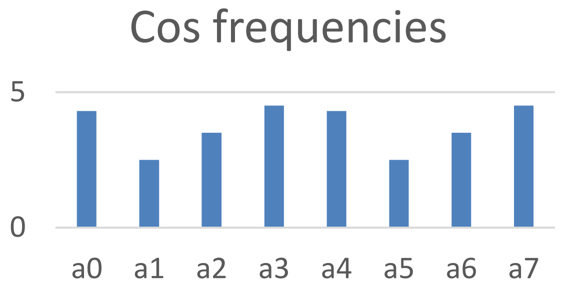

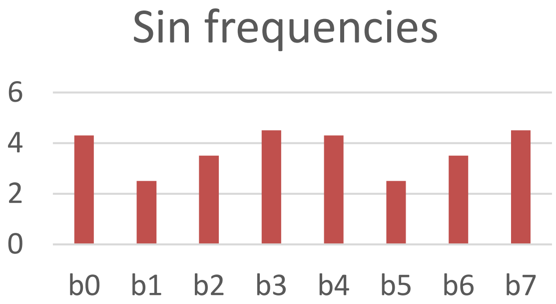

(a_0, b_0) $-$ *Corresponds to frequency* 0
(a_1, b_1) $-$ *Corresponds to frequency* 1
….
(a_n, b_n) $-$ *Corresponds to frequency* n

Combine them for compact representation

Complex Form of Fourier Series

- Compact Form easier to represent and integrate

Complex Form of Fourier Series

- Compact Form easier to represent and integrate

$$f(t) = a_0 + a_1 \cos(t) + a_2 \cos(2t) + \cdots$$
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Complex Form of Fourier Series

- Compact Form easier to represent and integrate

$$f(t) = a_0 + a_1 \cos(t) + a_2 \cos(2t) + \cdots$$
$$b_1 \sin(t) + b_2 \sin(2t) + \cdots$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt)$$

Complex Form of Fourier Series

- Compact Form easier to represent and integrate

$$f(t) = a_0 + a_1 \cos(t) + a_2 \cos(2t) + \cdots$$
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Euler's Rule:

$$e^{\sqrt{-1}\,nt} = \cos(nt) + \sqrt{-1}\,sin(nt),$$
$$e^{-\sqrt{-1}\,nt} = \cos(nt) - \sqrt{-1}\,sin(nt)$$

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$$\cos(nt) = \frac{1}{2}\left(e^{\sqrt{-1}\,nt} + e^{-\sqrt{-1}nt}\right)$$
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Complex Form of Fourier Series

- Compact Form easier to represent and integrate

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$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \frac{1}{2}\left(e^{\sqrt{-1}nt} + e^{-\sqrt{-1}nt}\right) + b_n \frac{\sqrt{-1}}{2}(e^{-\sqrt{-1}nt} - e^{\sqrt{-1}nt})]$$

$$\cos(nt) = \frac{1}{2}\left(e^{\sqrt{-1}\,nt} + e^{-\sqrt{-1}nt}\right)$$

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$$f(t) = a_0 + \frac{1}{2}\sum_{n=1}^{\infty}[(a_n - \sqrt{-1}b_n)e^{\sqrt{-1}nt} + (a_n + \sqrt{-1}b_n)\, e^{-\sqrt{-1}nt}]$$

Complex Form of Fourier Series

- Compact Form easier to represent and integrate

$$f(t) = a_0 + a_1 \cos(t) + a_2 \cos(2t) + \cdots$$
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Fourier Transform

- We want to understand the frequency n of our signal. So, let's reparametrize the signal by x instead of t:

$$f(x) \longrightarrow \boxed{\text{Fourier Transform}} \longrightarrow c_n$$

$$c_n = \frac{1}{2}(a_n - \sqrt{-1}b_n) \ \forall \ n \geq 1$$

Periodic Function

$$f(t) = a_0 + a_1 \cos(t) + a_2 \cos(2t) + \cdots$$
$$b_1 \sin(t) + b_2 \sin(2t) + \cdots$$

Sum of sine and cosine waves:

$$\Rightarrow a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)\,dt$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)\,dt$$

Fourier Transform

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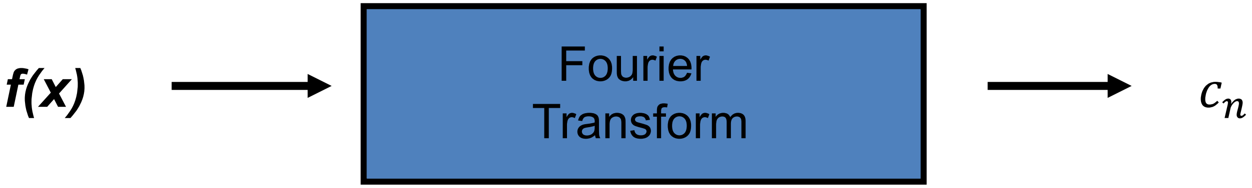

$$c_n = \frac{1}{2}(a_n - \sqrt{-1} b_n)\ \forall\ n \geq 1$$

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)(\cos nx)dx - \frac{1}{2\pi}\sqrt{-1}\int_0^{2\pi} f(x)(\quad \sin nx)dx$$

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$$f(x) \longrightarrow \boxed{\text{Fourier Transform}} \longrightarrow c_n$$

$$c_n = a_n - jb_n$$

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)(\cos nx)dx - \frac{1}{2\pi}j\int_0^{2\pi} f(x)(\quad \sin nx)dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(x)\left(\cos nx - \sqrt{-1}\sin nx\right)dx$$

Fourier Transform

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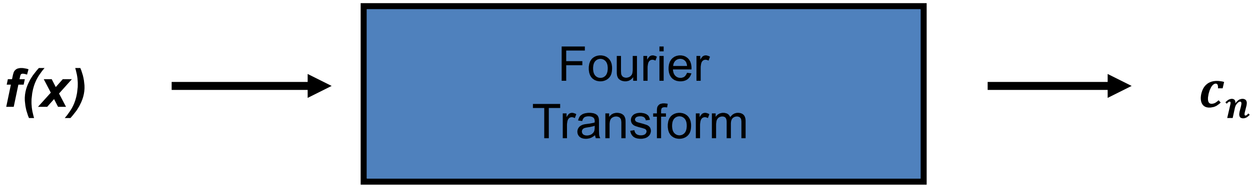

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Note: $e^{-\sqrt{-1}nx} = \cos(nx) - \sqrt{-1}sin(nx)$

Fourier Transform

- We want to understand the frequency n of our signal. So, let's reparametrize the signal by x instead of t:

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Note: $e^{-\sqrt{-1}nx} = \cos(nx) - j\sin(nx)$

$$\boxed{\boldsymbol{F(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-\sqrt{-1}nx}dx}}$$

Fourier Transform

- We want to understand the frequency $\boldsymbol{u}$ of our signal. So, let's reparametrize the signal by x instead of t:

$$\boldsymbol{f(x)} \longrightarrow \boxed{\text{Fourier Transform}} \longrightarrow \boldsymbol{F(u)}$$

$$F(u) = c_u$$

$$\boxed{\boldsymbol{F(u) = \int_{-\infty}^{\infty} f(x) e^{-\sqrt{-1}ux} dx}}$$

Spatial Domain (x) $\longrightarrow$ Frequency Domain (u)

(Frequency Spectrum $F(u)$)

Inverse Fourier Transform (IFT)

Frequency Domain (u) $\longrightarrow$ Spatial Domain (x)

$$\boldsymbol{f(x) = \int_{-\infty}^{\infty} c_u e^{\sqrt{-1}ut} du}$$
$$\boldsymbol{= \int_{-\infty}^{\infty} F(u)\, e^{\sqrt{-1}ut} du}$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt)$$

Discrete Fourier Transform

Spatial Domain (x) ⟶ Frequency Domain (u)

Fourier Transform

$$\boldsymbol{F(u) = \int_{-\infty}^{\infty} f(x) e^{-\sqrt{-1}ux} dx}$$

Discrete Fourier Transform

$$\boxed{F(u) = \sum_{x=-\infty}^{\infty} f(x) e^{-\sqrt{-\mathbf{1}}ux}}$$

$$e^{-\sqrt{-\mathbf{1}}x} = cosx \; - \sqrt{-\mathbf{1}} sinx$$

Frequency Domain (u) ⟶ Spatial Domain (x)

Inverse Fourier Transform

$$\boldsymbol{f(x) = \int_{-\infty}^{\infty} F(u) e^{\sqrt{-1}ux} du}$$

Inverse Discrete Fourier Transform

$$\boxed{f(x) = \sum_{u=-\infty}^{\infty} F(u) e^{\sqrt{-\mathbf{1}}ux}}$$

$$e^{\sqrt{-\mathbf{1}}x} = cosx + \sqrt{-\mathbf{1}} sinx$$