Digital Image Processing

COSC 6380/4393

Lecture – 15

Mar. 7th, 2023

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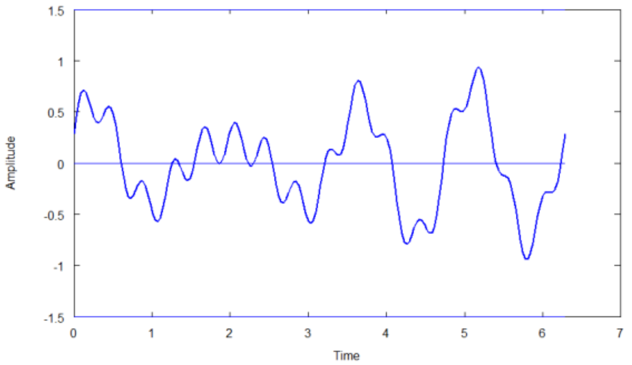
Slides from Dr. Shishir K Shah and S. Narasimhan

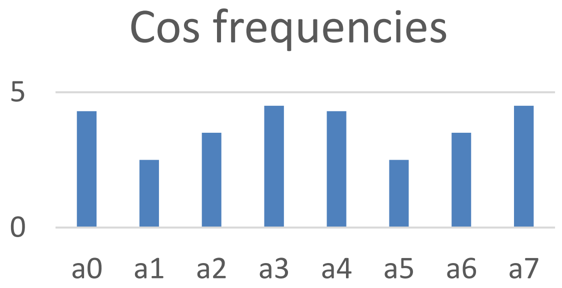
DFT

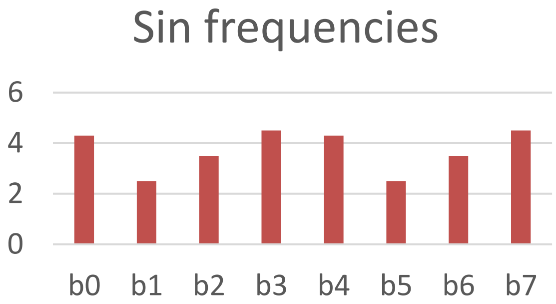
1. How to represent both the coefficients (sine and cos) of frequency t together (Complex Numbers)
2. How to compute DFT for 2D signals
3. Image as 2D discrete signals
4. DFT image
    1. Filtering
    2. .
    3. .

Frequency spectra



$$f(t) = a_0 + a_1\cos(t) + a_2\cos(2t) + \cdots$$
$$b_1\sin(t) + b_2\sin(2t) + \cdots$$





Discrete Fourier Transform

Spatial Domain (x) $\longrightarrow$ Frequency Domain (u)

Fourier Transform

$$\boldsymbol{F(u) = \int_{-\infty}^{\infty} f(x) e^{-\sqrt{-1}ux} dx}$$

Discrete Fourier Transform

$$\boxed{F(u) = \sum_{x=-\infty}^{\infty} f(x) e^{-\sqrt{-1}ux}}$$

$$e^{-\sqrt{-1}x} = cosx - \sqrt{-1}sinx$$

Frequency Domain (u) $\longrightarrow$ Spatial Domain (x)

Inverse Fourier Transform

$$\boldsymbol{f(x) = \int_{-\infty}^{\infty} F(u) e^{\sqrt{-1}ux} du}$$

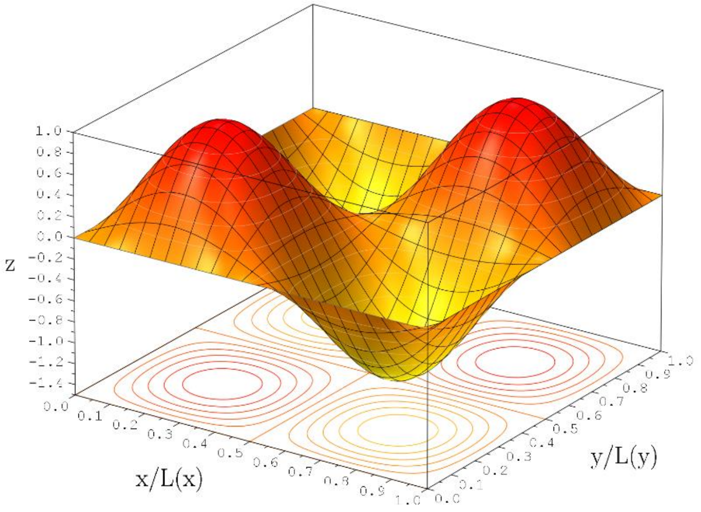
Inverse Discrete Fourier Transform

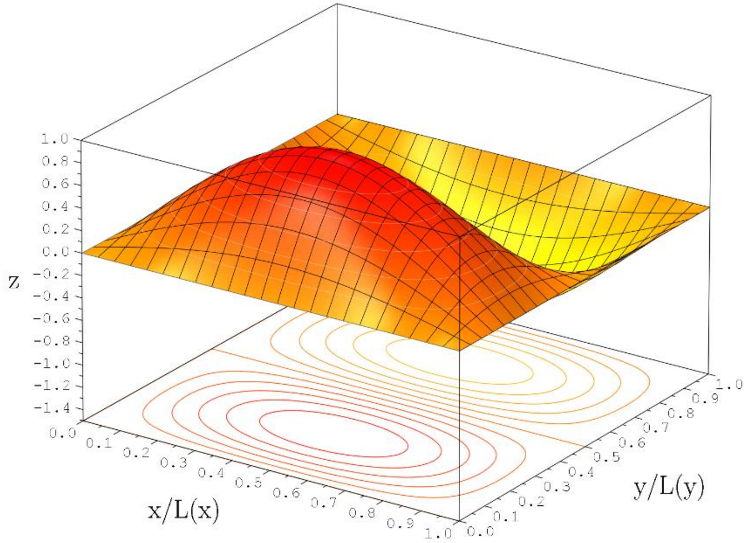
$$\boxed{f(x) = \sum_{u=-\infty}^{\infty} F(u) e^{\sqrt{-1}ux}}$$

$$e^{\sqrt{-1}x} = cosx + \sqrt{-1}sinx$$

From 1D → 2D

- One dimension (x) →frequency (u)
- Two dimensions →(i, j)
- Frequencies along (I,j) →(u,v)





Sinusoidal Images

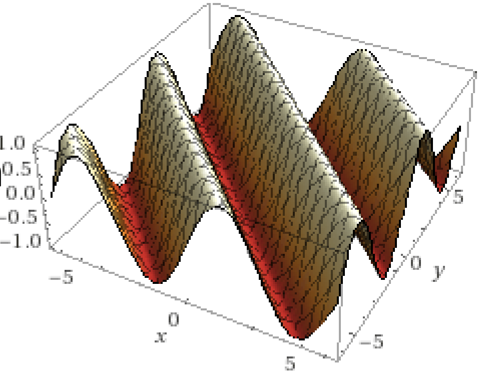
$2D\ sine\ wave \Rightarrow \sin(ui + vj)(u\ and\ v\ are\ frequencies\ along\ i\ and\ j)$

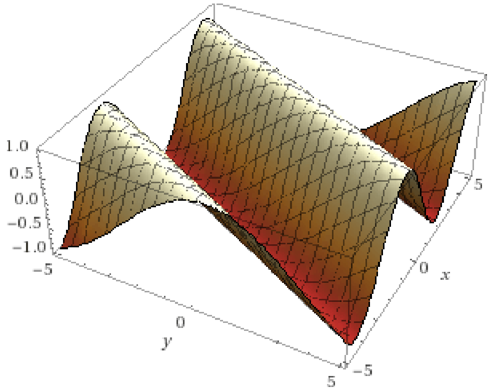
$\sin(i + j)(u = 1, v = 1)$

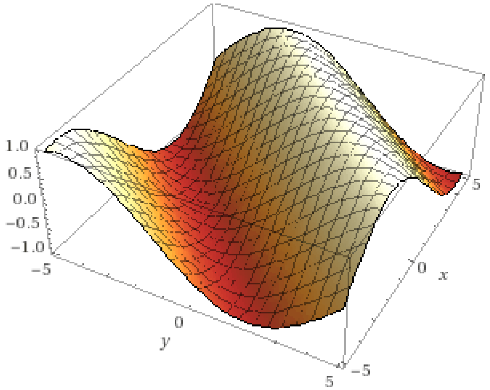
$\sin(i + 0.5j)(u = 1, v = 0.5)$

$\sin(0.5i + 0.5j)$
$u = v = 0.5$

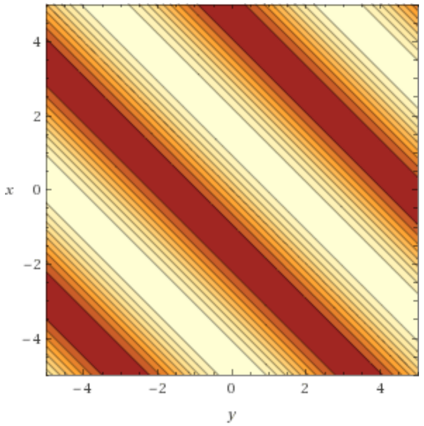
Waveform

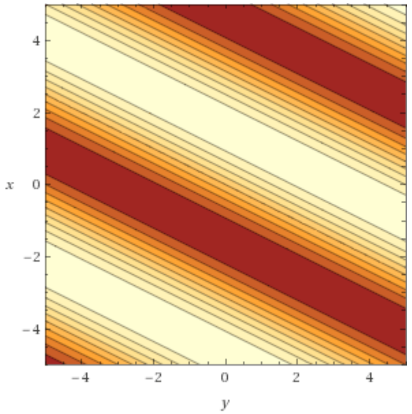


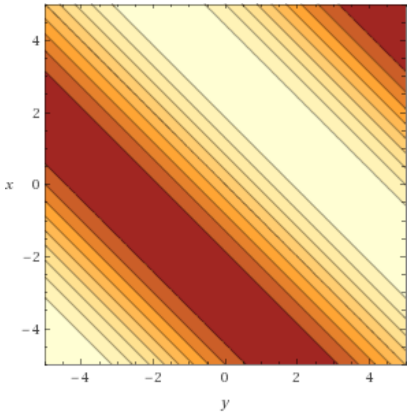




Contour plots







From Wolframalpha

2D Discrete Fourier Transform

Spatial Domain (i,*j*) $\longrightarrow$ Frequency Domain (*u*,*v*)

Fourier Transform

$$\boldsymbol{F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(i,j)e^{-\sqrt{-1}(ui+vj)}\,di\,dj}$$

Discrete Fourier Transform

$$\boxed{F(u,v) = \sum_{x=-\infty}^{\infty}\sum_{y=-\infty}^{\infty} f(i,j)e^{-\sqrt{-1}(ui+vj)}}$$

Frequency Domain (*u*,*v*) $\longrightarrow$ Spatial Domain (*i*,*j*)

Inverse Fourier Transform

$$\boldsymbol{f(i,j) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{\sqrt{-1}(ui+vj)}\,du\,dv}$$

Inverse Discrete Fourier Transform

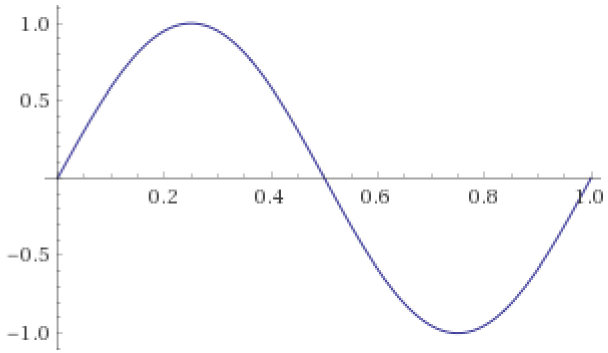
$$\boxed{f(i,j) = \sum_{u=-\infty}^{\infty}\sum_{v=-\infty}^{\infty} F(u,v)e^{\sqrt{-1}(ui+vj)}}$$

Images as 2D waves

- Are Images 2D Waves?
  - Continuous or discrete?
- Are they periodic?
- Can we apply DFT on images?

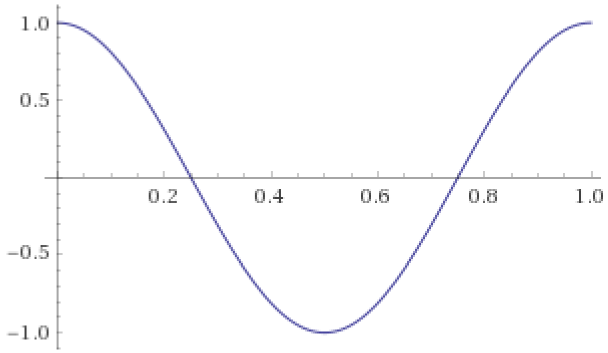
Recap: Sin and Cos

$\sin(2\pi t)$



Period = 1

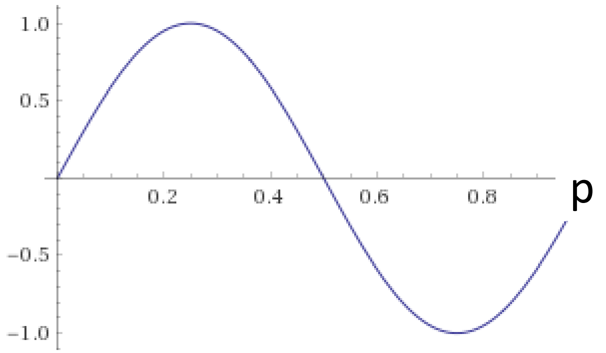
$\cos(2\pi t)$



Period = 1

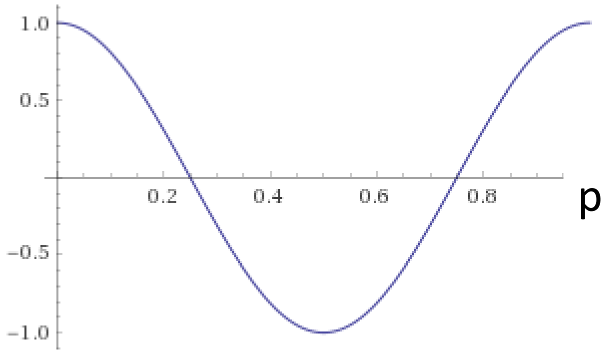
Recap: Sin and Cos

$\sin\left(2\pi\left(\frac{1}{p}\right)t\right)$



Period = p

$\cos\left(2\pi\left(\frac{1}{p}\right)t\right)$



Period = p

Sinusoidal Images

- We shall make frequent discussion in this module of the **frequency content** of an image.
- First consider images having the **simplest** frequency content.
- A **digital sine image I** is an image having elements

  $$I_1(i, j) = \sin\left[\frac{2\pi}{N}(ui + vj)\right] \text{ for } 0 \leq i, j \leq N\text{-}1$$

  and a **digital cosine image** has elements

  $$I_2(i, j) = \cos\left[\frac{2\pi}{N}(ui + vj)\right] \text{ for } 0 \leq i, j \leq N\text{-}1$$

  where u and v are **integer frequencies** in the i- and j-directions (measured in cycles/image; **notice** division by N).

2D Discrete Fourier Transform

Spatial Domain (i,*j*) ⟶ Frequency Domain (*u*,*v*)

Fourier Transform

$$\boldsymbol{F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(i,j)e^{-\sqrt{-1}(ui+vj)}\,di\,dj}$$

Discrete Fourier Transform

$$\boxed{F(u,v) = \sum_{x=-\infty}^{\infty}\sum_{y=-\infty}^{\infty} f(i,j)e^{-\sqrt{-1}(ui+vj)}}$$

Frequency Domain (*u*,*v*) ⟶ Spatial Domain (*i*,*j*)

Inverse Fourier Transform

$$\boldsymbol{f(i,j) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{\sqrt{-1}(ui+vj)}\,du\,dv}$$

Inverse Discrete Fourier Transform

$$\boxed{f(i,j) = \sum_{u=-\infty}^{\infty}\sum_{v=-\infty}^{\infty} F(u,v)e^{\sqrt{-1}(ui+vj)}}$$

2D Discrete Fourier Transform

- If I is an image of size N then

Sin image $I_1(i, j) = \sin\left[\frac{2\pi}{N}(ui + vj)\right]$ for $0 \leq i, j \leq N\text{-}1$

Cos image $I_2(i, j) = \cos\left[\frac{2\pi}{N}(ui + vj)\right]$ for $0 \leq i, j \leq N\text{-}1$

- Let $\tilde{I}$ be the DFT of the I

$$\tilde{I}(u, v) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j)\, e^{-\sqrt{-1}\frac{2\pi}{N}(ui+vj)}$$

$$F(u, v) = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(i, j) e^{-\sqrt{-1}(ui+vj)}$$

2D Inverse Discrete Fourier Transform

- Let $\tilde{I}$ be the DFT of the I

$$I(i,j) = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{I}(u,v)\, e^{\sqrt{-1}\frac{2\pi}{N}(ui+vj)}$$

$$f(i,j) = \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} F(u,v) e^{\sqrt{-1}(ui+vj)}$$

Example

$I =$

5	7
8	3

$\tilde{I} =$

?	
	?

Example

$$I = \begin{array}{|c|c|}\hline 5 & 7 \\ \hline 8 & 3 \\ \hline\end{array}$$

$$\tilde{I}(u,v) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j)\, e^{-\sqrt{-1}\frac{2\pi}{N}(ui+vj)}$$

$$\tilde{I} = \begin{array}{|c|c|}\hline ? & \\ \hline & ? \\ \hline\end{array}$$

Example

$$I = \begin{array}{|c|c|}\hline 5 & 7 \\ \hline 8 & 3 \\ \hline\end{array}$$

$$\tilde{I}(u,v) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j)\, e^{-\sqrt{-1}\frac{2\pi}{N}(ui+vj)}$$

$$\tilde{I}(0,0) = \sum_{i=0}^{2-1} \sum_{j=0}^{2-1} I(i,j)\, e^{-\sqrt{-1}\frac{2\pi}{2}(0*i+0*j)}$$

$$= \sum_{i=0}^{1} \sum_{j=0}^{1} I(i,j) =$$

$$\tilde{I} = \begin{array}{|c|c|}\hline ? & \\ \hline & ? \\ \hline\end{array}$$

Example

$$I = \begin{array}{|c|c|}\hline 5 & 7 \\ \hline 8 & 3 \\ \hline\end{array}$$

$$\tilde{I}(u,v) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)\, e^{-\sqrt{-1}\frac{2\pi}{N}(ui+vj)}$$

$$\tilde{I}(0,0) = \sum_{i=0}^{2-1}\sum_{j=0}^{2-1} I(i,j)\, e^{-\sqrt{-1}\frac{2\pi}{2}(0*i+0*j)}$$

$$= \sum_{i=0}^{1}\sum_{j=0}^{1} I(i,j) = 23$$

23	
	?

Example

$$I = \begin{array}{|c|c|}\hline 5 & 7 \\ \hline 8 & 3 \\ \hline \end{array}$$

$$\tilde{I}(u,v) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j)\, e^{-\sqrt{-1}\frac{2\pi}{N}(ui+vj)}$$

$$\tilde{I}(0,0) = \sum_{i=0}^{2-1} \sum_{j=0}^{2-1} I(i,j)\, e^{-\sqrt{-1}\frac{2\pi}{2}(0*i+0*j)}$$

$$= \sum_{i=0}^{1} \sum_{j=0}^{1} I(i,j) = 21$$

$\tilde{I}(0,1) = 3.+0.\sqrt{-1}$

$\tilde{I}(1,0) = 1.+0.\sqrt{-1}$ $\tilde{I}(1,1) = -7.+0.\sqrt{-1}$

23	
	?

Example

$I =$

5	7
8	3

$$\tilde{I}(u,v) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)\, e^{-\sqrt{-1}\frac{2\pi}{N}(ui+vj)}$$

$$\tilde{I}(0,0) = \sum_{i=0}^{2-1}\sum_{j=0}^{2-1} I(i,j)\, e^{-\sqrt{-1}\frac{2\pi}{2}(0*i+0*j)}$$

$$= \sum_{i=0}^{1}\sum_{j=0}^{1} I(i,j) = 21$$

$\tilde{I}(0,1) = 3.+0.\sqrt{-1}$

$\tilde{I}(1,0) = 1.+0.\sqrt{-1}$ $\quad$ $\tilde{I}(1,1) =$-7.+0. $\sqrt{-1}$

23	
	-7.+0.j