

Spatial Factor Models for Multivariate Spatial Data

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- We anticipate dependence between measurements
 - at a particular location
 - across locations

Multivariate spatial generalized linear model

- Spatial generalized linear model for h -variate spatial data for $j = 1, 2, \dots, h$ and $i = 1, \dots, n$:

$$y_j(\mathbf{s}_i) \sim f(\mu_j(\mathbf{s}_i), \tau_j)$$

$$\mu_j(\mathbf{s}_i) = g^{-1}(\eta_j(\mathbf{s}_i)) = \mathbf{x}(\mathbf{s}_i)^\top \beta_j + \mathbf{w}_j^*(\mathbf{s}_i)$$

- We can imagine modeling $\mathbf{w}^*(\mathbf{s}_i) = (w_1^*(\mathbf{s}_i), w_2^*(\mathbf{s}_i), \dots, w_h^*(\mathbf{s}_i))'$ as an h -variate Gaussian process

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- Could model using Multivariate NNGP as discussed previously with SVCs, works well when $h < 5$.
- But what about when h is large (e.g., 10, 100)?

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- We represent the $h \times 1$ vector $\mathbf{w}^*(\mathbf{s}_i)$ as a linear combination of latent spatial factors and factor loadings:

$$\mathbf{w}^*(\mathbf{s}_i) = \mathbf{\Lambda} \mathbf{w}(\mathbf{s}_i)$$

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- In traditional factor analysis, $\mathbf{w}(\mathbf{s}_i)$ are realizations from independent standard normal random variables.

Spatial Factor Model

- Choosing $q \ll m$ leads to substantial computational reductions.
- Simple to code: just sample from q independent GPs as with basic univariate models.
- Yields a non-separable multivariate cross-covariance function given by $\sum_{k=1}^q \mathbf{R}_k(\phi_k) \boldsymbol{\lambda}_k \boldsymbol{\lambda}_k^\top$
- Can simply replace the q full GPs with their corresponding NNGPs to yield a spatial factor NNGP model
- Identifiability constraints on $\mathbf{\Lambda}$: fix upper triangle to 0 and diagonal to 1. See Ren and Banerjee (2013) *Biometrics*

- Standard normal priors for the lower triangle of $\mathbf{\Lambda}$
- We like to model response-specific regression coefficients β_j hierarchically. For each $r = 1, \dots, p$ covariate, we model $\beta_{j,r}$ following

$$\beta_{j,r} \sim N(\mu_{\beta_r}, \tau_{\beta_r}^2)$$

- Gaussian hyperpriors for μ_{β_r} and IG or half-Cauchy priors for $\tau_{\beta_r}^2$
- Independent uniform priors for spatial decay parameters ϕ

- Full conditionals are in closed form for all parameters except ϕ .
- Update ϕ with an Adaptive Metropolis-within-Gibbs algorithm (Roberts and Rosenthal 2009)
- See Taylor-Rodriguez et al. 2019 for Gaussian sampler, spOccupancy website for Pólya-Gamma sampler

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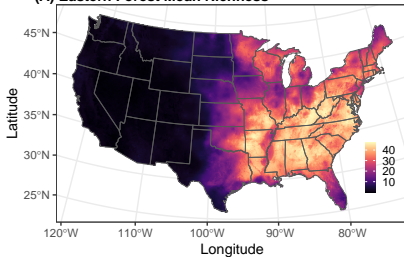
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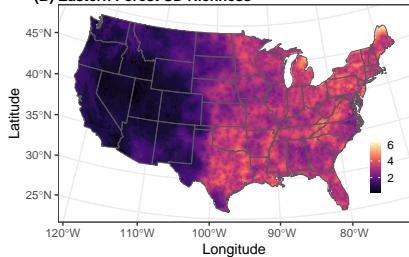
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- Straightforward extensions to spatially-varying coefficient models.

Example: bird communities across the continental US

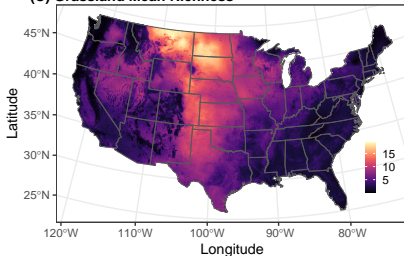
(A) Eastern Forest Mean Richness



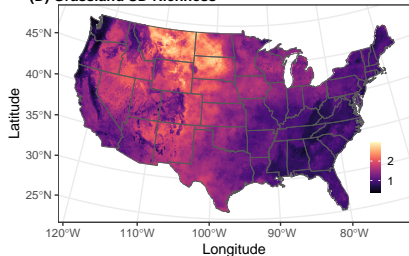
(B) Eastern Forest SD Richness



(C) Grassland Mean Richness

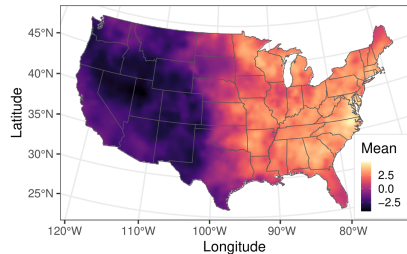
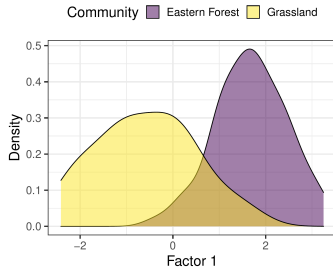


(D) Grassland SD Richness



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Visualization of the first spatial factor and corresponding factor loadings



Some downsides to spatial factor models

- Convergence assessment is not always straightforward
- Sensitivity to initial values
- Order of the first q species has important implications for convergence and mixing.
- Assume a multivariate stochastic process can be represented as a linear combination of independent univariate processes

- `spOccupancy`: spatial NNGP and non-spatial factor models for binary data
- `spAbundance`: Gaussian, Poisson, and NB spatial NNGP and non-spatial factor models.
- `boral`: many distributions for non-spatial and spatial factor models (Hui 2015 *MEE*; spatial use full GPs fit in JAGS)
- `Hmsc`: spatial models using NNGPs (Tikhonov et al. 2019; *MEE*)
- `spBFA`: a variety of spatial models with some nifty priors (Berchuck et al. 2022 *Bayesian Analysis*)

Modeling the distribution of 10 tree species across Vermont