Spatial Factor Models for Multivariate Spatial Data

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- We anticipate dependence between measurements
  - at a particular location
  - across locations

Multivariate spatial generalized linear model

- Spatial generalized linear model for m-variate spatial data for $j = 1, 2, \ldots, m$ and $i = 1, \ldots, n$:

$$y_j(\mathbf{s}_i) \sim f(\mu_j(\mathbf{s}_i), \tau_j)$$
$$\mu_j(\mathbf{s}_i) = g^{-1}(\eta_j(\mathbf{s}_i)) = \mathbf{x}(\mathbf{s}_i)^\top \boldsymbol{\beta}_j + \mathrm{w}_j^*(\mathbf{s}_i)$$

- We can imagine modeling $\mathbf{w}^*(\mathbf{s}_i) = (\mathrm{w}_1^*(\mathbf{s}_i), \mathrm{w}_2^*(\mathbf{s}_i), \ldots, \mathrm{w}_m^*(\mathbf{s}_i))'$ as an m-variate Gaussian process

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- Could model using Multivariate NNGP as disussed previously with SVCs, works well when $m < 5$.
- But what about when m is large (e.g,. 10, 100)?

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- We represent the $m \times 1$ vector $\mathbf{w}^*(\mathbf{s}_i)$ as a linear combination of latent spatial factors and factor loadings:

$$\mathbf{w}^*(\mathbf{s}) = \boldsymbol{\Lambda}\mathbf{w}(\mathbf{s}_i)$$

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- In traditional factor analysis, $\mathbf{w}(\mathbf{s}_i)$ are realizations from independent standard normal random variables.

Spatial Factor Model

- Choosing $q << m$ leads to substantial computational reductions.
- Simple to code: just sample from q independent GPs as with basic univariate models.
- Yields a non-separable multivariate cross-covariance function given by $\sum_{k=1}^{q} \mathbf{R}_k(\phi_k)\boldsymbol{\lambda}_k\boldsymbol{\lambda}_k^\top$
- Can simply replace the q full GPs with their corresponding NNGPs to yield a spatial factor NNGP model
- Identifiability constraints on $\boldsymbol{\Lambda}$: fix upper triangle to 0 and diagonal to 1. See Ren and Banerjee (2013) *Biometrics*

Priors

- Standard normal priors for the lower triangle of $\boldsymbol{\Lambda}$
- We like to model response-specific regression coefficients $\boldsymbol{\beta}_j$ hierarchically. For each $r = 1, \ldots, p$ covariate, we model $\beta_{j,r}$ following

$$\beta_{j,r} \sim N(\mu_{\beta_r}, \tau^2_{\beta_r})$$

- Gaussian hyperpriors for μ_{β_r} and IG or half-Cauchy priors for $\tau^2_{\beta_r}$
- Independent uniform priors for spatial decay parameters ϕ

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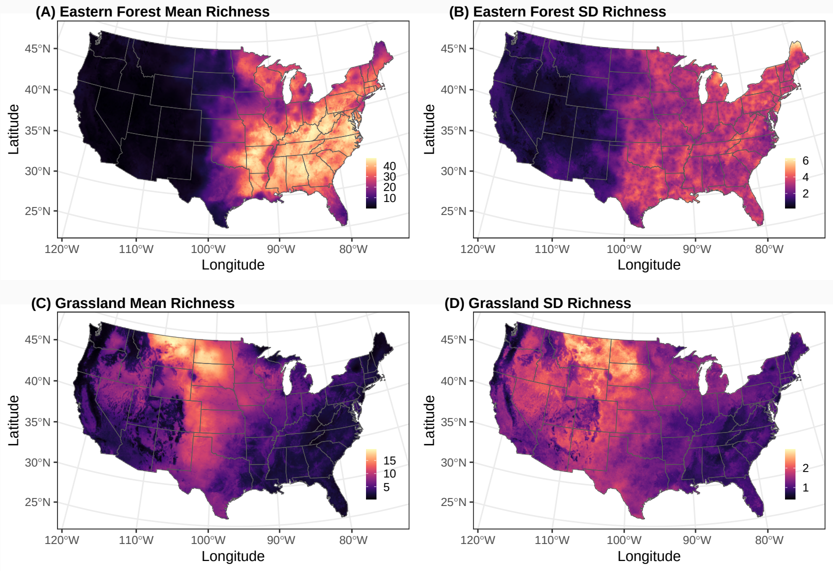
- Relatively fast and efficient (well, at least for Gaussian and Binomial).
- Factors and factor loadings can be used for model-based ordination.
- Straightforward extensions to spatially-varying coefficient models.

Example: bird communities across the continental US



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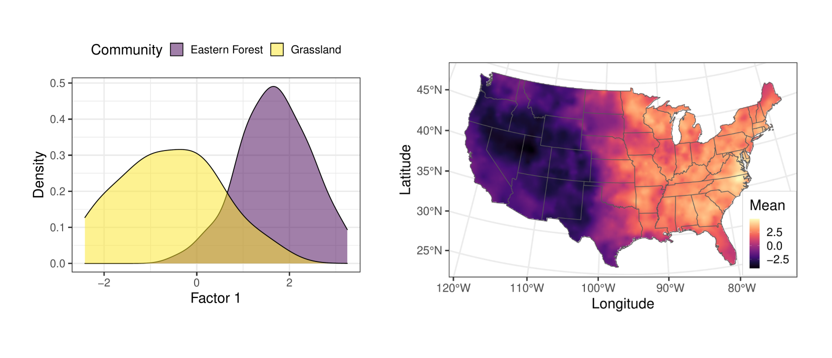
Visualization of the first spatial factor and corresponding factor loadings



Some downsides to spatial factor models

- Convergence assessment is not always straightforward
- Sensitivity to initial values
- Order of the first q species has important implications for convergence and mixing.
- Assume a multivariate stochastic process can be represented as a linear combination of independent univariate processes

Software

- `spOccupancy`: spatial NNGP and non-spatial factor models for binary data
- `spAbundance`: Gaussian, Poisson, and NB spatial NNGP and non-spatial factor models.
- `boral`: many distributions for non-spatial and spatial factor models (Hui 2015 *MEE*; spatial use full GPs fit in JAGS)
- `Hmsc`: spatial models using NNGPs (Tikhonov et al. 2019; *MEE*)
- `spBFA`: a variety of spatial models with some nifty priors (Berchuck et al. 2022 *Bayesian Analysis*)

Exercise

Modeling the distribution of 10 tree species across Vermont