Modeling non-Gaussian spatial data

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Non-Gaussian spatial data

- Often data sets preclude Gaussian modeling: $y(\mathbf{s})$ may not even be continuous
- Examples:
  - Binary: presence or absence of a species at location s.
  - Count: abundance of a species at location s.
  - Categorical: counts of trees by size class at location s.
- Replace Gaussian likelihood by exponential family member (Diggle, Tawn, and Moyeed (1998)).

Hierarchical Bayesian approach

- **First stage**: $y(\mathbf{s}_i)$ are conditionally independent given β and $\mathrm{w}(\mathbf{s}_i)$. Here we use a canonical link function, say $g(E[y(\mathbf{s}_i)]) = \eta(\mathbf{s}_i) = \mathbf{x}(\mathbf{s}_i)'\beta + \mathrm{w}(\mathbf{s}_i)$.

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- **Third stage**: Priors and hyperpriors.

MCMC sampling for spatial GLMMs

- Additional GLMM flexibility comes at a computational cost: lose conjugacy of $\boldsymbol{\beta}, \mathbf{w}$
- Requires more Metroplis steps. Particularly costly for $\mathbf{w}$
- Practical consequence: slower, less efficient algorithms
- Prediction and interpolation proceed as with the Gaussian case

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- **See Polson, Scott, Windle (2013) JASA**

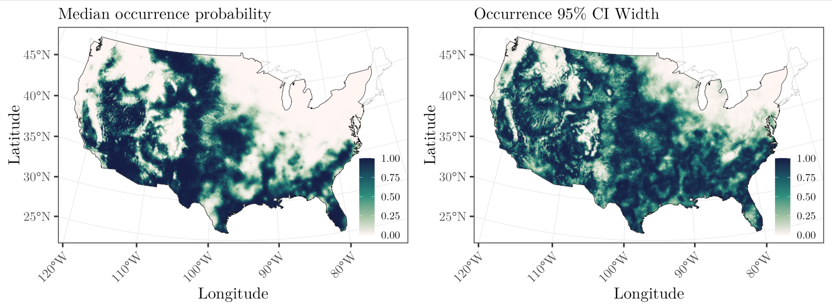
Example: Binary spatial regression

- Objective: predict the distribution of Loggerhead Shrike across the US

$$
\begin{aligned}
y(\mathbf{s}_i) &\sim \text{Bernoulli}(\psi(\mathbf{s}_i)) \\
\text{logit}(\psi(\mathbf{s}_i)) &= \mathbf{x}(\mathbf{s}_i)^\top \boldsymbol{\beta} + w(\mathbf{s}_i) \\
\mathbf{w} &\sim N(\mathbf{0}, \sigma^2 \mathbf{R}(\phi)) \\
\boldsymbol{\beta} &\sim N(\boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta) \\
\sigma^2 &\sim IG(a_\sigma, b_\sigma) \\
\phi &\sim \text{Uniform}(l, u)
\end{aligned}
$$

- For binomial data, we can use a Pólya-Gamma data augmentation approach

Example: Binary spatial regression



Some practical considerations

- Priors for σ^2 and ϕ may need to be more informative, particularly for binary data.
- Be careful with non-identity link functions when thinking about priors.
- Pólya-Gamma data augmentation works really for binomial data. Computational cost increases as Binomial weights increases.
- Pólya-Gamma data augmentation also applicable for Negative Binomial count data, but slow for large counts and can be unstable.

Software

- `spBayes`
  - Univariate and multivariate, full GPs or predictive processes
  - Gaussian, Binomial (no Pólya-Gamma data augmentation), Poisson
- `spNNGP`
  - Univariate, NNGPs
  - Gaussian, Binomial
- `spOccupancy`
  - Univariate and multivariate, focus on modeling wildlife distributions, full GPs or NNGPs
  - Bernoulli
- `spAbundance` (`https://github.com/doserjef/spAbundance`)
  - Univariate and multivariate, focus on modeling wildlife/plant abundance, NNGPs
  - Gaussian, Poisson, Negative Binomial