Introduction to Convex Optimization

Lec 5: Convex Optimization Problems

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Lecture Overview

In this lecture, we focus on several subclasses of convex optimization.

1. Convex functions.
2. Operations that preserve convexity.
3. Conjugate functions.
4. Quasiconvex functions.
5. Operations that preserve quasiconvexity.
6. Log-concave functions.
7. Convexity by generalized inequality.

We put some proofs in appendix.

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Standard Form of an Optimization Problem

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& h_i(x) = 0, \quad i = 1, \ldots, p
\end{array} \tag{1}
$$

- $x \in \mathbb{R}^n$ is the optimization variable.
- $f_0 : \mathbb{R}^n \to \mathbb{R}$ is the objective or cost function.
- $f_i(x) : \mathbb{R}^n \to \mathbb{R}, i = 1, ..., m$ are the inequality constraint functions.
- $h_i : \mathbb{R}^n \to \mathbb{R}, i = 1, ..., p$ are the equality constraint functions.
- The domain of the optimization problem

$$\mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom} f_i \cap \bigcap_{i=1}^{p} \mathbf{dom} h_i$$

  the domain of the optimization problem.
- Optimal value:

$$p^* = \inf \{ f_0(x) \mid f_i(x) \leq 0, i = 1, ..., m, h_i(x) = 0, i = 1, ..., p \}$$

- $p^* = \infty$ if the problem is infeasible (no x satisfies the constraints). $p^* = -\infty$ is problem is unbounded below.

Optimal and Locally Optimal Points

- x is feasible if $x \in \mathbf{dom} f_0$ and it satisfies the constraints ($x \in \mathcal{D}$).
- A feasible x is optimal if $f_0(x) = p^*$; X_{opt} is the set of optimal points.
- x is locally optimal if there is an $R > 0$ such that x is optimal for

$$\begin{array}{ll} \text{minimize (over } z) & f_0(z) \\ \text{subject to} & f_i(z) \leq 0, \quad i = 1, \dots, m, \quad h_i(z) = 0, \quad i = 1, \dots, p \\ & \|z - x\|_2 \leq R \end{array}$$

Examples (with $n = 1, m = p = 0$)

- $f_0(x) = 1/x, \mathbf{dom} f_0 = \mathbb{R}_{++} : p^\star = 0$, no optimal point
- $f_0(x) = -\log x, \mathbf{dom} f_0 = \mathbb{R}_{++} : p^\star = -\infty$
- $f_0(x) = x \log x, \mathbf{dom} f_0 = \mathbb{R}_{++} : p^\star = -1/e, x = 1/e$ is optimal
- $f_0(x) = x^3 - 3x, p^\star = -\infty$, local optimum at $x = 1$

Standard Form Convex Optimization Problem I

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& a_i^\top x = b_i, \quad i = 1, \ldots, p
\end{array}
\tag{2}
$$

Compared with the general standard form problem (Eq. 1), the convex problem has three additional requirements:

1. The objective function f_0 is convex.
2. The inequality constraint functions $f_1, ..., f_m$ must be convex.
3. The equality constraint functions $h_i(x)$ must be affine.

- If $f_0(x)$ is quasiconvex, then the problem is a quasiconvex optimization problem.
- Important Property: feasible set of a convex optimization problem is convex.
- Many problems can be reformulated into the convex optimization form.

Standard Form Convex Optimization Problem II

$$
\begin{array}{ll}
\text{minimize} & f_0(x) = x_1^2 + x_2^2 \\
\text{subject to} & f_1(x) = x_1/\left(1 + x_2^2\right) \leq 0 \\
& h_1(x) = (x_1 + x_2)^2 = 0
\end{array}
$$

- f_0 is convex; feasible set $\{(x_1, x_2) \mid x_1 = -x_2 \leq 0\}$ is convex
- not a convex problem (according to our definition): f_1 is not convex, h_1 is not affine
- equivalent (but not identical) to the convex problem

$$
\begin{array}{ll}
\text{minimize} & x_1^2 + x_2^2 \\
\text{subject to} & x_1 \leq 0 \\
& x_1 + x_2 = 0
\end{array}
$$

Local and Global Optimization Theorem

Theorem 1 (Local and Global Optimization Theorem)

Any local optimal solution of a convex optimization problem is also a global optimal solution.

- Suppose x is locally optimal, but there exists a feasible y with $f_0(y) \leq f_0(x)$
- x is locally optimal means there is an $R > 0$ such that

$$\forall z \text{ is feasible, } \|z - x\|_2 \leq R \Rightarrow f_0(z) \geq f_0(x)$$

- Consider $z = \theta y + (1 - \theta)x$ with $\theta = R/(2\|y - x\|_2)$
- z is a convex combination of two feasible points, hence also feasible
- $\|z - x\|_x = R/2$ and

$$f_0(z) = f_0(\theta y + (1 - \theta)x) \leq \theta f_0(y) + (1 - \theta) f_0(x) < f_0(x)$$

  which contradicts the assumption that x is locally optimal.
- The first inequality is because of the convexity of f_0, and the second inequality is because of the assumption $f_0(y) < f_0(x)$.

Optimality Criterion for Differentiable f_0 I

Suppose that the objective f_0 in a convex optimization problem is differentiable, so that for all $x, y \in \mathbf{dom} f_0$,

$$f_0(y) \geq f_0(x) + \nabla f_0(x)^\top (y - x)$$

Then x is optimal if and only if it is feasible ($x \in X$) and

$$\nabla f_0(x)^\top (y - x) \geq 0, \quad \text{for all feasible } y \tag{3}$$

If $\nabla f_0(x) \neq 0$, $-\nabla f_0(x)$ defines a supporting hyperplane to feasible set X at x; see Figure 1.

Optimality Criterion for Differentiable f_0 II

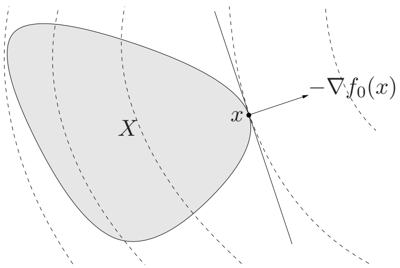

Fig. 1: The feasible set X is shown shaded. Some level curves of f_0 are shown as dashed lines. The point x is optimal: $-\nabla f_0(x)$ defines a supporting hyperplane (shown as a solid line) to X at x.

Optimality Criterion for Differentiable f_0 III

Proof. (By contradiction)

- Suppose $x \in X$ and satisfies Eq. 3. Then if $y \in X$ we have $f_0(y) \geq f_0(x)$, which shows x is optimal.
- Suppose x is optimal but Eq. 3 does not hold, i.e., for some $y \in X$ we have

$$\nabla f_0(x)^\top (y - x) < 0$$

- Consider $z(t) = ty + (1-t)x$, where $t \in [0,1]$ is a parameter. $z(t)$ is feasible since it is on the line segment between x and y.
-

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} f_0(z(t)) \bigg|_{t=0} &= \nabla f(z(t))^\top (y - x) \bigg|_{t=0} \\ &= \nabla f(x)^\top (y - x) \leq 0 \end{aligned}$$

  So $f_0(z(t)) < f_0(x)$ for t is small enough, which contradicts with x being optimal.

Next, we examine a few simple examples.

Unconstrainted Problems I

For an unconstrainted problem, the condition (Eq. 3) reduces to

$$\nabla f_0(x) = 0$$

for x to be optimal.

- Suppose x is optimal $\Rightarrow x \in \mathbf{dom} f_0$ and for all feasible y we have $\nabla f_0(x)^\top (y - x) \geq 0$
- f_0 is differentiable, so all y sufficiently close to x are feasible.
- Take $y = x - t\nabla f_0(x)$ where $t \in \mathbb{R}$ is a parameter.
- For t small and positive, y is feasible, and so

  $$\nabla f_0(x)^\top (y - x) = -t\|\nabla f_0(x)\|_2^2 \geq 0$$

  for which we conclude $\nabla f_0(x) = 0$.

Unconstrainted Problems II

Unconstrainted quadratic optimization

- Consider the problem of minimizing the quadratic function

$$f_0(x) = (1/2)x^\top Px + q^\top x + r$$

  where $P \in \mathcal{S}_+^n$ (which makes f_0 convex). The necessary and sufficient condition for x to be a minimizer of f_0 is

$$\nabla f_0(x) = Px + q = 0.$$

- If $q \notin \mathcal{R}(P)$, then there is no solution. In this case f_0 is unbounded below.
- If $P \succ 0$ (which is the condition for f_0 to be strictly convex), then there is a unique minimizer, $x^\star = -P^{-1}q$.
- If P is singular, but $q \in \mathcal{R}(P)$, then the set of optimal points is the (affine) set $X_{\text{opt}} = -P^+q + \mathcal{N}(P)$, where P^+ denotes the pseudo-inverse of P.

Problems with Equality Constraints Only

Consider the probelm with equality constraints only, i.e.,

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & Ax = b \end{array}$$

x is optimal iff $\exists u$, such that $Ax = b, \nabla f_0(x) - A^\top u = 0$

- The optimality condition for a feasible x is that

$$\nabla f_0(x)^\top (y - x) \geq 0$$

  hold for all y satisfying $Ay = b$.

- Since x is feasible, $A(x - y) = 0, (x - y) \in \mathcal{N}(A)$.
- $2x - y$ is also feasible $(A(2x - y) = b)$, so

$$\nabla f_0(x)^\top (x - y) \geq 0$$

  which means $\nabla f_0(x)(x - y) = 0$ for all $(x - y) \in \mathcal{N}(A)$.

- In other words, $\nabla f_0(x) \perp \mathcal{N}(A)$. Therefore, $\nabla f_0(x) \in \mathcal{R}\left(A^\top\right)$. $(\mathcal{N}(A)^\perp = \mathcal{R}\left(A^\top\right))$
- $\nabla f_0(x) = A^\top u$ for some u.

Minimization over Nonnegative Orthant

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & x \preceq 0 \end{array}$$

- The optimality condition is

$$x \succeq 0, \nabla f_0(x)^\top (y - x) \geq 0 for all y \succeq 0$$

- $\nabla f_0(x)^\top y$ is unbounded below on $y \succeq 0$ unless $\nabla f_0(x) \succeq 0$
- The condition reduces to $-\nabla f_0(x)^\top x \geq 0$.
- Note that $x \succeq 0$ and $\nabla f_0(x) \succeq 0$. We must have $\nabla f_0(x)^\top x = 0$, i.e.,

$$\sum_{i=1}^{n} (\nabla f_0(x))_i \, x_i = 0$$

- Since $(\nabla f_0(x))_i \geq 0, x_i \geq 0$, then

$$(\nabla f_0(x))_i \, x_i = 0, i = 1, ..., n$$

- x is optimal if and only if

$$x \in \mathbf{dom} f_0, \quad x \succeq 0, \quad \begin{cases} \nabla f_0(x)_i \geq 0, & x_i = 0 \\ \nabla f_0(x)_i = 0, & x_i > 0 \end{cases}$$

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Equivalent Convex Problems I

Definition 1 (Equivalent Convex Problems)

Two problems are (informally) equivalent if the solution of one is readily obtained from the solution of the other, and vice versa.

Eliminating equality constraint

$$\begin{array}{lll} \text{minimize}_x & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0, & i = 1, \ldots, p \\ & Ax = b, & A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \end{array}$$

is equivalent to

$$\begin{array}{lll} \text{minimize}_z & f_0(Fz + x_0) & \\ \text{subject to} & f_i\left(Fz + x_0\right) \leq 0, & i = 1, \ldots, p, F \in \mathbb{R}^{n \times r}, r = \operatorname{rank}(F) \end{array}$$

where the range of F is the nullspace of A, i.e., $AF = 0$, and $Ax_0 = b$.

Equivalent Convex Problems II

- Introducing equality constraints

$$\begin{array}{ll} \operatorname{minimize}_z & f_0\left(A_0 x+b_0\right) \\ \text{subject to} & f_i\left(A_i x+b_i\right) \leq 0, \quad i=1, \ldots, m \end{array}$$

  is equivalent to

$$\begin{array}{ll} \operatorname{minimize}\text{(over } x, y_i) & f_0\left(y_0\right) \\ \text{subject to} & f_i\left(y_i\right) \leq 0, \quad i=1, \ldots, m \\ & y_i=A_i x+b_i, \quad i=0,1, \ldots, m \end{array}$$

- Introducing slack variables for *linear inequalities*

$$\begin{array}{ll} \operatorname{minimize} & f_0(x) \\ \text{subject to} & a_i^{\top} x \leq b_i, \quad i=1, \ldots, m \end{array}$$

  is equivalent to

$$\begin{array}{ll} \operatorname{minimize}\text{(over } x, s) & f_0(x) \\ \text{subject to} & a_i^{\top} x+s_i=b_i, \quad i=1, \ldots, m \\ & s_i \geq 0, \quad i=1, \ldots m \end{array}$$

Equivalent Convex Problems III

- Epigraph Form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

is equivalent To

$$\begin{array}{ll} \text{minimize}_{x,t} & t \\ \text{subject to} & f_0(x) - t \leq 0 \\ & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

- Minimizing over some variables

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & f_i(x_1) \leq 0, \quad i = 1, \ldots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize} & \tilde{f}_0(x_1) \\ \text{subject to} & f_i(x_1) \leq 0, \quad i = 1, \ldots, m \end{array}$$

where $\tilde{f}_0(x_1) = \inf_{x_2} f_0(x_1, x_2)$

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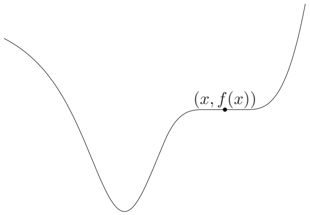
Standard Form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array} \tag{4}$$

with $f_0 : \mathbb{R}^n \to \mathbb{R}$ quasiconvex, $f_1, ..., f_m$ convex.

- A quasiconvex optimization problem can have locally optimal solutions that are not (globally) optimal.
- Solving a quasiconvex optimization problem can be reduced to solving a sequence of convex optimization problems.



Optimality Condition

Let X be the feasible set for the quasiconvex optimization probelm (Eq. 4). It follows from the first-order condition for quasiconvexity that x is optimal if

$$x \in X, \quad \nabla f_0(x)^\top (y - x) > 0 \text{ for all } y \in X \setminus \{x\}$$

- The condition is only sufficient for optimality, which needs not hold for an optimal point.
- The condition requires the gradient $\nabla f_0(x) \neq 0$, whereas the condition in the convex case does not.

Convex Representation of Sublevel Sets of f_0

If f_0 is quasiconvex, there exists a family of functions ϕ_t such that:

- $\phi_t(x)$ is convex in x for fixed t
- t-sublevel set of f_0 is 0-sublevel set of ϕ_t, i.e.,

$$f_0(x) \leq t \Longleftrightarrow \phi_t(x) \leq 0$$

For example, consider

$$f_0(x) = \frac{p(x)}{q(x)}$$

with p convex, q concave, and $p(x) \geq 0$, $q(x) > 0$ on $\mathbf{dom} f_0$

- It's easy to verify that $f_0(x)$ is quasiconvex. Note that $f_0(x) \geq 0$.

$$f_0(x) \leq t \Leftrightarrow \frac{p(x)}{q(x)} \leq t \Leftrightarrow p(x) - tq(x) \leq 0$$

  When $t \geq 0$, $\{x \mid p(x) - tq(x) \leq 0\}$ is convex.
- $\phi_t(x) = p(x) - tq(x)$ is convex in x for $t \geq 0$.
- $f_0(x) \leq t$ if and only if $\phi_t(x) \leq 0$.

Quasiconvex Optimization via Convex Feasibility Problems I

Let p^* denote the optimal value of the quasiconvex optimization problem (Eq. 4). If the following problem

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & \phi_t(x) \leq 0 \\ & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array} \tag{5}$$

is feasible, then $p^* \leq t$. Conversely, if the problem is infeasible, then $p^* \geq t$. We can solve a quasiconvex optimization problem using bisection, solving a convex feasibility problem at each step.

Quasiconvex Optimization via Convex Feasibility Problems II

Algorithm 1 Bisection method for quasiconvex optimization

Require: $l \leq p^*, u \geq p^*$, tolerance $\epsilon > 0$

```
repeat
    t := (l + u)/2
    Solve the convex feasiblity problem (Eq. 5) at t
    if feasible then
        u := t
    else
        l := t
until u − l ≤ ϵ
```

Complexity: requires exactly $\lceil \log_2((u-l)/\epsilon) \rceil$ iterations.

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Linear Programming I

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

Standard form linear programming (LP)

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & Ax = b \\ & x \succeq 0 \end{array}$$

Convert LP to standard forms

- Introduce slack variables s_i for the inequality constraints.
- Express the variable x as the difference of two nonnegative variables x^+ and x^-, i.e., $x = x^+ - x^-$

Linear Programming II

Diet problem: choose quantities $x_1, \dots, x_n$ of n foods

- one unit of food j costs c_j. contains amount $a_{i,j}$ of nutrient i
- healthy diet requires nutrient i in quantity at least b_i

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & Ax \succeq b, \quad x \succeq 0 \end{array}$$

Piecewise-linear minimization

$$\text{minimize} \quad \max_{i=1,\dots,m} \left(a_i^\top x + b_i\right)$$

equivalent to an LP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & a_i^\top x + b_i \leq t, \quad i = 1, ..., m \end{array}$$

Chebyshev center of a polyhedron

Find the largest Euclidean ball that lies in a polyhedron

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n \mid a_i^\top x \le b_i, i = 1, ..., m \right\}$$

The center of the optimal ball is called the Chebyshev center of the polyhedron. We represent the ball as

$$\mathcal{B} = \{ x_c + u \mid \|u\|_2 \le r \}$$

$\mathcal{B}$ in the halfspace $a_i^\top x \le b_i$ if and only if

$$a_i^\top (x_c + u) \le b_i, \quad \|u\|_2 \le r$$

Note the dual norm of $\| \cdot \|_2$ is also Euclidean norm, i.e.,

$$\|a_i\|_2 = \sup \left\{ a_i^\top x \mid \|x\|_2 \le 1 \right\}$$

Therefore, $\sup \left\{ a_i^\top u \mid \|u\|_2 \le r \right\} = r\|a_i\|_2$. We can solve the LP to get x_c, r.

$$\begin{array}{ll} \text{minimize} & r \\ \text{subject to} & a_i^\top x_c + r\|a_i\|_2 \le b_i, \quad i = 1, ..., m \end{array}$$