GENE638 - Homework 2

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1. For this system of equations:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 20 \\ 25 \end{bmatrix}$$

   (a) Determine the rank of the coefficient matrix

```
> as.numeric(Matrix::rankMatrix(coefficientMatrix))
[1] 2
```

   (b) Express any linearly dependent columns in the matrix form $\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} \\ A'_{12} \end{bmatrix} L$

$$\begin{aligned} A &= \left[\begin{array}{c:cc} 1 & 0 & 1 \\ \hdashline 0 & 2 & 2 \\ 1 & 2 & 3 \end{array}\right] \\ A_{\begin{bmatrix} 11 \\ 12 \end{bmatrix}} &= \begin{bmatrix} 0 & 1 \\ 2 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ A_{\begin{bmatrix} 12 \\ 12 \end{bmatrix}} &= \begin{bmatrix} (0 \times -1 + 1 \times 1) \\ (2 \times -1 + 2 \times 1) \\ (2 \times -1 + 3 \times 1) \end{bmatrix} \\ A_{\begin{bmatrix} 12 \\ 12 \end{bmatrix}} &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \end{aligned} \tag{1}$$

   (c) Find a generalized inverse of the coefficient matrix. Prove that it is a generalized inverse

```
> gInverse <- MASS::ginv(coefficientMatrix)
> gInverse
           [,1]          [,2]         [,3]
[1,]  0.5000000 -3.333333e-01 1.666667e-01
[2,] -0.3333333  3.333333e-01 1.942890e-16
[3,]  0.1666667  6.938894e-17 1.666667e-01

> round(gInverse %*% coefficientMatrix)
     [,1] [,2] [,3]
[1,]    1    0    0
[2,]    0    1    0
[3,]    0    0    1
```

```
> round(coefficientMatrix %*% gInverse %*% coefficientMatrix)
     [,1] [,2] [,3]
[1,]    1    0    1
[2,]    0    2    2
[3,]    1    2    3
```

(d) Using the inverse from part (c), solve for x. Prove your solution satisfies the equations

```
> x <- gInverse %*% c(5, 20, 25)
> round(x)
     [,1]
[1,]    0
[2,]    5
[3,]    5
> coefficientMatrix %*% x
     [,1]
[1,]    5
[2,]   20
[3,]   25
```

(e) What do your solutions estimate? *A set of numerical values in function of x_1 that satisfies the system of equations*

(f) Based in what you did in part (e): Can you estimate x_1? *Not numerically, but yes in function of another variable.* Can you estimate $x_1 - x_2$? *Yes, is -5*

2. Using the partitioned matrix inverse procedure on page 14 in the notes

(a) Find the inverse of: $\left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hdashline 1 & 1 & 3 \end{array}\right]$

$$\left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hdashline 1 & 1 & 3 \end{array}\right]^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A^{-1}B \\ I \end{bmatrix} (D - CA^{-1}B)^{-1} \begin{bmatrix} -CA^{-1} & I \end{bmatrix}$$

$$A^{-1} = \frac{1}{1 \times 1 - 0 \times 0} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hdashline 1 & 1 & 3 \end{array}\right]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} A^{-1}B \\ I \end{bmatrix} (D - CA^{-1}B)^{-1} \begin{bmatrix} -CA^{-1} & I \end{bmatrix}$$

$$\left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hdashline 1 & 1 & 3 \end{array}\right]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ 1 \end{bmatrix} (D - CA^{-1}B)^{-1} \begin{bmatrix} -CA^{-1} & I \end{bmatrix}$$

$$\left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hdashline 1 & 1 & 3 \end{array}\right]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \left(3 - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}\right)^{-1} \begin{bmatrix} -CA^{-1} & I \end{bmatrix}$$

$$\left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hdashline 1 & 1 & 3 \end{array}\right]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} (1)^{-1} \begin{bmatrix} -\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hdashline 1 & 1 & 3 \end{array}\right]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} (1)^{-1} \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hdashline 1 & 1 & 3 \end{array}\right]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hdashline 1 & 1 & 3 \end{array}\right]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hdashline 1 & 1 & 3 \end{array}\right]^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

(b) Prove that your answer in (a) is an inverse

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

(c) Solve $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

(d) What is the solution in (c) an estimate of? *The true values of* x_1, x_2 *and* x_3

(e) Can you estimate x_3? *Yes, it is zero.* With what? Can you estimate $6x_1 + 4x_2 - 8x_3$? *Yes, it is* $6 \times 1 + 4 \times 2 + 8 \times 0 = 14$. With what?

3. Assuming the linear model $y_i = \mu + \epsilon_i$ where y_i is and observed first lactation milkfat production, μ is the population mean milkfat production, and ϵ_i is the deviation of an individual cow's production from the mean. Write this model as

$$\underline{y} = 1\mu + \underline{\epsilon}$$

Cow	1^{st} lactation milkfat (lb)
1	300
2	290
3	405
4	360
5	315

(a) Calculate:

i. $X'X$

```
> X <- rep(1,length(y))
> t(X) %*% X
     [,1]
[1,]    5
```

ii. $(X'X)^{-1}$

```
> solve(t(X) %*% X)
     [,1]
[1,]  0.2
```

iii. $X'\underline{y}$

```
> t(X) %*% y
     [,1]
[1,] 1670
```

(b) Solve $\hat{\mu} = (X'X)^{-1}X'\underline{y}$

```
> muHat <- as.numeric(solve(t(X) %*% X) %*% X %*% y)
> muHat
[1] 334
```

(c) Show that $\hat{\mu} = \overline{y}$

```
> meanY <- mean(y)
> meanY
[1] 334
> all.equal(muHat, meanY)
[1] TRUE
```

(d) Find the predicted deviations $\underline{\hat{\epsilon}} = \underline{y} - \underline{1}\hat{\mu}$

```
> epsilonHat <- (y - muHat)
> epsilonHat
[1] -34 -44  71  26 -19
```

(e) Find $\frac{\underline{\hat{\epsilon}}'\underline{\hat{\epsilon}}}{n-rank(X)}$ and show that this is s^2 the sample variance.

```
> s2Hat <- as.numeric((t(epsilonHat) %*% epsilonHat) / (length(y) - 1))
> s2Hat
[1] 2292.5
> var(y)
[1] 2292.5
> all.equal(s2Hat, var(y))
[1] TRUE
```