GENE638 - Homework 5

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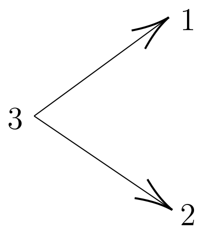
Data:

COW	YEAR	TICK COUNT
1	1	80
2	1	70
1	2	86
2	2	72
1	3	92
2	3	78

Pedrigree:



1. Construct the numerator relationship and its inverse

$$A = \begin{bmatrix} 1.00 & 0.25 & 0.50 \\ 0.25 & 1.00 & 0.50 \\ 0.50 & 0.50 & 1.00 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1.33 & 0.00 & \text{-0.67} \\ 0.00 & 1.33 & \text{-0.67} \\ \text{-0.67} & \text{-0.67} & 1.67 \end{bmatrix}$$

2. The predicted breeding value of the unidentified parent of animal 1 is $\frac{2}{3}(\hat{A}_1 - \frac{1}{2}\hat{A}_3)$ and that for the unidentified parent of animal 2 is $\frac{2}{3}(\hat{A}_2 - \frac{1}{2}\hat{A}_3)$. Identify the base generation animals and show that $1'A^{-1}\hat{u} = 0$ is consistent with the sum of breeding values of base generation being 0.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2.00 & 0.00 & \text{-1.00} & \text{-1.00} & \text{-0.00} \\ 0.00 & 2.00 & \text{-1.00} & \text{-0.00} & \text{-1.00} \\ \text{-1.00} & \text{-1.00} & 2.00 & 0.50 & 0.50 \\ \text{-1.00} & \text{-0.00} & 0.50 & 1.50 & 0.00 \\ \text{-0.00} & \text{-1.00} & 0.50 & 0.00 & 1.50 \end{bmatrix} \begin{bmatrix} 4.38 \\ \text{-4.38} \\ 0.00 \\ 2.92 \\ \text{-2.92} \end{bmatrix} = -1.332268e - 15$$

A1	A2	**A3**	**M1**	**M2**
5.84	-5.84	0.00	0.00	0.00

3. Write the data in general matrix terms $y = X\beta + Zu + e$ for the model $Y_{ij} = \text{Year}_i + \text{Animal}_j + e_{ij}$ where $\text{Year}_i = \mu + \text{Year}_i$. Include animal 3 in $\hat{u}$ as in (2)

$$\begin{bmatrix} 80 \\ 70 \\ 86 \\ 72 \\ 92 \\ 78 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} e_{11} \\ e_{21} \\ e_{12} \\ e_{22} \\ e_{13} \\ e_{23} \end{bmatrix}$$

4. Assuming $R = I\sigma_e^2$ and $G = A\sigma_a^2$ and $\lambda = \frac{\sigma_e^2}{\sigma_a^2} = 3$, write the MME for model in (3)

$$\begin{bmatrix} 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 7 & 0 & -2 \\ 1 & 1 & 1 & 0 & 7 & -2 \\ 0 & 0 & 0 & -2 & -2 & 5 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 150 \\ 158 \\ 170 \\ 258 \\ 220 \\ 0 \end{bmatrix}$$

5. From the row of the MME corresponding to $\hat{A}_3$ show that $\hat{A}_3 = \frac{2}{5}(\hat{A}_1 + \hat{A}_2)$

$$\begin{aligned} 0 &= -2\hat{A}_1 - 2\hat{A}_2 + 5\hat{A}_3 \\ -5\hat{A}_3 &= -2\hat{A}_1 - 2\hat{A}_2 \\ 5\hat{A}_3 &= 2\hat{A}_1 + 2\hat{A}_2 \\ \hat{A}_3 &= \frac{2\hat{A}_1 + 2\hat{A}_2}{5} \\ \hat{A}_3 &= \frac{2}{5}(\hat{A}_1 + \hat{A}_2) \end{aligned}$$

6. Absorb the year equations into the animal equations to obtain a system of equations involving only the unknown breeding values $\hat{A}_1$, $\hat{A}_2$, and $\hat{A}_3$

$$\begin{bmatrix} 0.280 & 0.137 & 0.167 \\ 0.137 & 0.280 & 0.167 \\ 0.167 & 0.167 & 0.333 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 19.0 \\ -19.0 \\ 0.0 \end{bmatrix}$$

7. What are the effective numbers of observations on all three animals?

```
> diag(t(Z) %*% M %*% Z)

 A1  A2  A3
1.5 1.5 0.0
```

8. From the appropiate row of the absorbed MME, once again show that $\hat{A}_3 = \frac{2}{5}(\hat{A}_1 + \hat{A}_2)$

$$C_{22} = \begin{bmatrix} 5.5 & -1.5 & -2.0 \\ -1.5 & 5.5 & -2.0 \\ -2.0 & -2.0 & 5.0 \end{bmatrix}$$

$$\begin{aligned} 0 &= -2\hat{A}_1 - 2\hat{A}_2 + 5\hat{A}_3 \\ -5\hat{A}_3 &= -2\hat{A}_1 - 2\hat{A}_2 \\ 5\hat{A}_3 &= 2\hat{A}_1 + 2\hat{A}_2 \\ \hat{A}_3 &= \frac{2\hat{A}_1 + 2\hat{A}_2}{5} \\ \hat{A}_3 &= \frac{2}{5}(\hat{A}_1 + \hat{A}_2) \end{aligned}$$

9. Using ordinary Gauss-Seidel iteration, find two succesive aproximations to the predicted breeding values in (6)

$$L = \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ -1.5 & 0.0 & 0.0 \\ -2.0 & -2.0 & 0.0 \end{bmatrix} \qquad D = \begin{bmatrix} 5.5 & 0.0 & 0.0 \\ 0.0 & 5.5 & 0.0 \\ 0.0 & 0.0 & 5.0 \end{bmatrix} \qquad b = \begin{bmatrix} 19.0 \\ -19.0 \\ 0.0 \end{bmatrix}$$

```
> L <- D <- matrix(0,nrow = nrow(C22), ncol = ncol(C22))
> L[lower.tri(L)] <- C22[lower.tri(C22)]
> diag(D) <- diag(C22)
> x0 <- solve(D) %*% b
> for(i in 1:2){
+   Xi <- solve(L + D) %*% (b - t(L) %*% x0)
+   rownames(Xi) <- c("A1","A2","A3")
+   print(Xi)
+   x0 <- Xi
+ }

          [,1]
A1  2.5123967
A2 -2.7693464
A3 -0.1027799
          [,1]
A1  2.6618947
A2 -2.7659487
A3 -0.0416216
```

10. Show that $\hat{A}_1 = 2.7143$, $\hat{A}_2 = -2.7143$ and $\hat{A}_3 = 0$, provide a solution to the absorbed equations in (6)

$$\begin{bmatrix} 0.280 & 0.137 & 0.167 \\ 0.137 & 0.280 & 0.167 \\ 0.167 & 0.167 & 0.333 \end{bmatrix} \begin{bmatrix} 2.7143 \\ -2.7143 \\ 0 \end{bmatrix} = \begin{bmatrix} 19.0 \\ -19.0 \\ 0.0 \end{bmatrix}$$

11. Using the breeding values in (10), backsolve the solutions to $\hat{Y}_1$, $\hat{Y}_2$ and $\hat{Y}_3$

```
> beta <- solve(t(X) %*% X) %*% t(X)%*%(y - Z %*% u)
> beta

   [,1]
Y1   75
Y2   79
Y3   85
```

12. The inverse of the coefficient matrix in (6) for animals in the order $\hat{A}_3, \hat{A}_1, \hat{A}_2$, is

$$\begin{bmatrix} 0.33330 & 0.16667 & 0.16667 \\ 0.16667 & 0.27976 & 0.13690 \\ 0.16667 & 0.1369 & 0.27976 \end{bmatrix} = (Z'MZ + A^{-1}\lambda)^{-1}$$

Which also is a submatrix of the inverse

$$\begin{bmatrix} X'X & X'Z \\ Z'X & Z'Z + A^{-1}\lambda \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

Calculate the first aproximations:

$$\sigma_e^2 = \frac{y'y - \underline{\hat{\beta}'}X'y - \hat{u}'Z'\underline{y}}{N - p}$$

$$\sigma_a^2 = \frac{\hat{u}'A^{-1}\hat{u} + \sigma_e^2 tr\left[A^{-1}C_{22}\right]}{q}$$

$$\lambda = \frac{\sigma_e^2}{\sigma_a^2}$$

```
> N <- length(y)
> p <- Matrix::rankMatrix(X)
> q <- ncol(Z)
> sigma2E <-(t(y) %*% y - t(beta) %*% t(X) %*%
+              y - t(u) %*% t(Z) %*% y) / as.numeric(N - p)
> sigma2E

         [,1]
[1,] 47.61905

> sigma2A <- (t(u) %*% Ainv %*% u +
+               sigma2E * sum(diag(Ainv %*% solve(C22))))/q
> sigma2A

        [,1]
[1,] 20.1542

> sigma2E/sigma2A

         [,1]
[1,] 2.362736
```