

GENE638 - Homework 2

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1. For this system of equations:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 20 \\ 25 \end{bmatrix}$$

- (a) Determine the rank of the coefficient matrix

```
> as.numeric(Matrix::rankMatrix(coefficientMatrix))
[1] 2
```

- (b) Express any linearly dependent columns in the matrix form $\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} \\ A'_{12} \end{bmatrix} L$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$A \begin{bmatrix} 11 \\ 12 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$A \begin{bmatrix} 12 \\ 12 \end{bmatrix} = \begin{bmatrix} (0 \times -1 + 1 \times 1) \\ (2 \times -1 + 2 \times 1) \\ (2 \times -1 + 3 \times 1) \end{bmatrix} \quad (1)$$

$$A \begin{bmatrix} 12 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

- (c) Find a generalized inverse of the coefficient matrix. Prove that it is a generalized inverse

```
> gInverse <- MASS::ginv(coefficientMatrix)
> gInverse
      [,1]      [,2]      [,3]
[1,] 0.5000000 -3.333333e-01 1.666667e-01
[2,] -0.3333333 3.333333e-01 1.942890e-16
[3,] 0.1666667 6.938894e-17 1.666667e-01
> round(gInverse %*% coefficientMatrix)
      [,1] [,2] [,3]
[1,] 1 0 0
[2,] 0 1 0
[3,] 0 0 1
```

```
> round(coefficientMatrix %*% gInverse %*% coefficientMatrix)
```

```
      [,1] [,2] [,3]
[1,]     1     0     1
[2,]     0     2     2
[3,]     1     2     3
```

(d) Using the inverse from part (c), solve for x . Prove your solution satisfies the equations

```
> x <- gInverse %*% c(5, 20, 25)
> round(x)
```

```
      [,1]
[1,]     0
[2,]     5
[3,]     5
```

```
> coefficientMatrix %*% x
```

```
      [,1]
[1,]     5
[2,]    20
[3,]    25
```

(e) What do your solutions estimate? *One of the possible solutions for the linear system of equations.*

(f) Based in what you did in part (e): Can you estimate x_1 ? Can you estimate $x_1 - x_2$?

2. Using the partitioned matrix inverse procedure on page 14 in the notes

(a) Find the inverse of: $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hline 1 & 1 & 3 \end{array} \right]$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hline 1 & 1 & 3 \end{array} \right]^{-1} = \left[\begin{array}{cc} A^{-1} & 0 \\ 0 & 0 \end{array} \right] + \left[\begin{array}{c} A^{-1}B \\ I \end{array} \right] (D - CA^{-1}B)^{-1} \left[\begin{array}{cc} -CA^{-1} & I \end{array} \right]$$

$$A^{-1} = \frac{1}{1 \times 1 - 0 \times 0} \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hline 1 & 1 & 3 \end{array} \right]^{-1} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] + \left[\begin{array}{c} A^{-1}B \\ I \end{array} \right] (D - CA^{-1}B)^{-1} \left[\begin{array}{cc} -CA^{-1} & I \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hline 1 & 1 & 3 \end{array} \right]^{-1} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] + \left[\begin{array}{cc} \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] & \left[\begin{array}{c} 1 \\ 1 \end{array} \right] \\ 1 \end{array} \right] (D - CA^{-1}B)^{-1} \left[\begin{array}{cc} -CA^{-1} & I \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hline 1 & 1 & 3 \end{array} \right]^{-1} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] + \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right] \left(3 - \left[\begin{array}{c} 1 \\ 1 \end{array} \right] \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \left[\begin{array}{cc} 1 & 1 \end{array} \right] \right)^{-1} \left[\begin{array}{cc} -CA^{-1} & I \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hline 1 & 1 & 3 \end{array} \right]^{-1} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] + \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right] (1)^{-1} \left[\begin{array}{cc} -\left[\begin{array}{cc} 1 & 1 \end{array} \right] \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] & 1 \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hline 1 & 1 & 3 \end{array} \right]^{-1} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] + \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right] (1)^{-1} \left[\begin{array}{ccc} -1 & -1 & 1 \end{array} \right]$$

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 3 & 3 \end{array} \right]^{-1} &= \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] + \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right] \begin{bmatrix} -1 & -1 & 1 \end{bmatrix} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 3 & 3 \end{array} \right]^{-1} &= \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] + \left[\begin{array}{ccc} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{array} \right] \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 3 & 3 \end{array} \right]^{-1} &= \left[\begin{array}{ccc} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{array} \right] \end{aligned}$$

(b) Prove that your answer in (a) is an inverse

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

(c) Solve $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

(d) What is the solution in (c) an estimate of? *The unique possible solution for the linear system of equations.*

(e) Can you estimate x_3 ? With what? Can you estimate $6x_1 + 4x_2 - 8x_3$? With what?

3. Assuming the linear model $y_i = \mu + \epsilon_i$ where y_i is and observed first lactation milkfat production, μ is the population mean milkfat production, and ϵ_i is the deviation of an individual cow's production from the mean. Write this model as

$$\underline{y} = 1\mu + \underline{\epsilon}$$

Cow	1 st lactation milkfat (lb)
1	300
2	290
3	405
4	360
5	315

(a) Calculate:

- i. $X'X$

```
> X <- rep(1,length(y))
> t(X) %*% X
      [,1]
[1,]      5
```
- ii. $(X'X)^{-1}$

```
> solve(t(X) %*% X)
      [,1]
[1,] 0.2
```

```

iii.  $X'y$ 
      > t(X) %*% y
      [,1]
      [1,] 1670

```

(b) Solve $\hat{\mu} = (X'X)^{-1}X'y$

```

> muHat <- as.numeric(solve(t(X) %*% X) %*% X %*% y)
> muHat
[1] 334

```

(c) Show that $\hat{\mu} = \bar{y}$

```

> meanY <- mean(y)
> meanY
[1] 334
> all.equal(muHat, meanY)
[1] TRUE

```

(d) Find the predicted deviations $\hat{\epsilon} = y - \underline{1}\hat{\mu}$

```

> epsilonHat <- (y - muHat)
> epsilonHat
[1] -34 -44 71 26 -19

```

(e) Find $\frac{\hat{\epsilon}'\hat{\epsilon}}{n-\text{rank}(X)}$ and show that this is s^2 the sample variance.

```

> s2Hat <- as.numeric((t(epsilonHat) %*% epsilonHat) / (length(y) - 1))
> s2Hat
[1] 2292.5
> var(y)
[1] 2292.5
> all.equal(s2Hat, var(y))
[1] TRUE

```