STAT 636, Fall 2018 - Assignment 1

1. Consider the matrix
$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$$
Without using a computer:
   (a) Find the eigenvalues and normalized eigenvectors of $\mathbf{A}$.
   (b) Write the spectral decomposition of $\mathbf{A}$.
   (c) Verify that the determinant of $\mathbf{A}$ equals the product of its eigenvalues.
   (d) The trace of a square matrix equals the sum of its diagonal elements. Verify that the trace of $\mathbf{A}$ equals the sum of its eigenvalues.
   (e) Is $\mathbf{A}$ orthogonal? Why or why not?
   (f) Is $\mathbf{A}$ positive definite? Why or why not?
   (g) Find $\mathbf{A}^{-1}$ and determine its eigenvalues and normalized eigenvectors.

2. Consider the matrices
$$\mathbf{A} = \begin{bmatrix} 4.000 & 4.001 \\ 4.001 & 4.002 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4.000 & 4.001 \\ 4.001 & 4.002001 \end{bmatrix}$$
These matrices are identical except for a small difference in the (2, 2) position. Also, the columns of $\mathbf{A}$ and $\mathbf{B}$ are nearly linearly dependent. Show that $\mathbf{A}^{-1} \approx (-3)\mathbf{B}^{-1}$. So, small changes - perhaps due to rounding - can result in substantially different inverses.

3. Derive expressions for the means and variances of the following linear combinations in terms of the means and covariances of the random variables X_1, X_2, and X_3.
   (a) $2X_1 - X_2$.
   (b) $X_1 + X_2 - 2X_3$.
   (c) $4X_1 - 3X_2$ if X_1 and X_2 are independent (so, $\sigma_{12} = 0$).

4. Consider the random vector $\mathbf{X}' = [X_1, X_2, X_3, X_4]$ with mean vector $\boldsymbol{\mu}' = [1, 2, 3, 4]$ and covariance matrix
$$\boldsymbol{\Sigma} = \begin{bmatrix} 4 & 0 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 3 & 1 & 9 & -2 \\ 1 & 0 & -2 & 4 \end{bmatrix}$$
Partition $\mathbf{X}$ as
$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \hdashline X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \hdashline \mathbf{X}^{(2)} \end{bmatrix}$$
Let
$$\mathbf{A} = \begin{bmatrix} 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix}$$
and consider the linear combinations $\mathbf{AX}^{(1)}$ and $\mathbf{BX}^{(2)}$. Find the following:

(a) $E\left(\mathbf{X}^{(1)}\right)$.

(b) $E\left(\mathbf{B}\mathbf{X}^{(2)}\right)$.

(c) $\operatorname{Cov}\left(\mathbf{A}\mathbf{X}^{(1)}\right)$.

(d) $\operatorname{Cov}\left(\mathbf{X}^{(1)}, \mathbf{X}^{(2)}\right)$.

(e) $\operatorname{Cov}\left(\mathbf{A}\mathbf{X}^{(1)}, \mathbf{B}\mathbf{X}^{(2)}\right)$.