STAT636 - Homework 1

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1. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$$

Without using a computer:

a) Find the eigenvalues and normalized eigenvectors of $\mathbf{A}$.

$$\left| \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| \tag{1}$$

$$\left| \begin{bmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix} \right| \tag{2}$$

$$(2-\lambda)(-1-\lambda) - 4 \tag{3}$$

$$-2 - 2\lambda + \lambda + \lambda^2 - 4 \tag{4}$$

$$\lambda^2 - \lambda - 6 \tag{5}$$

$$(\lambda+3)(\lambda-2) = 0 \tag{6}$$

$$(\lambda-3) = 0 : \lambda = 3 \tag{7}$$

$$(\lambda+2) = 0 : \lambda = -2 \tag{8}$$

The eigenvalues are: $[3, -2]$
Solving for $\lambda = 3$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \tag{9}$$

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{12}$$

$$x - 2y = 0 \tag{13}$$

$$x = 2y \tag{14}$$

Setting $y = 1$ then **the eigenvectors for** $\lambda = 3$ **are** $[2, 1]$

$$\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \times 2 + 1 \times 1 = 5 \tag{15}$$

The normalized eigenvectors for $\lambda = 3$ are $\left[\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right] = \begin{bmatrix} 0{,}8944272 & 0{,}4472136 \end{bmatrix}$
Solving for $\lambda = -2$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} - -2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{18}$$

$$\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{19}$$

$$x + \frac{1}{2}y = 0 \tag{20}$$

$$x = -\frac{1}{2}y \tag{21}$$

Setting $y = 1$ then **the eigenvectors for** $\lambda = -2$ **are** $[-\frac{1}{2}, 1]$

$$\begin{bmatrix} -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} = -\frac{1}{2} \times -\frac{1}{2} + 1 \times 1 = 1{,}25 \tag{22}$$

The normalized eigenvectors for $\lambda = 3$ are $\left[\frac{-\frac{1}{2}}{\sqrt{1{,}25}}, \frac{1}{\sqrt{1{,}25}}\right] = \begin{bmatrix} -0{,}4472136 & 0{,}8944272 \end{bmatrix}$

b) Write the spectral decomposition of $\mathbf{A}$.

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} = \left(3\begin{bmatrix} 0{,}894 \\ 0{,}447 \end{bmatrix}\begin{bmatrix} 0{,}894 & 0{,}447 \end{bmatrix}\right) + \left(-2\begin{bmatrix} -0{,}447 \\ 0{,}894 \end{bmatrix}\begin{bmatrix} -0{,}447 & 0{,}894 \end{bmatrix}\right) \tag{23}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} = \left(3\begin{bmatrix} 0{,}8 & 0{,}4 \\ 0{,}4 & 0{,}2 \end{bmatrix}\right) + \left(-2\begin{bmatrix} 0{,}2 & -0{,}4 \\ -0{,}4 & 0{,}8 \end{bmatrix}\right) \tag{24}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2{,}4 & 1{,}2 \\ 1{,}2 & 0{,}6 \end{bmatrix} + \begin{bmatrix} -0{,}4 & 0{,}8 \\ 0{,}8 & -1{,}6 \end{bmatrix} \tag{25}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \tag{26}$$

c) Verify that the determinant of $\mathbf{A}$ equals the product of its eigenvalues.

$$\left|\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}\right| = 3 \times -2 \tag{27}$$

$$(2 \times -1) - (2 \times 2) = -6 \tag{28}$$

$$-6 = -6 \tag{29}$$

d) The trace of a square matrix equals the sum of its diagonal elements. Verify that the trace of $\mathbf{A}$ equals the sum of its eigenvalues.

$$tr\left(\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}\right) = 3 + -2 \tag{30}$$

$$(2 + -1) = 1 \tag{31}$$

$$1 = 1 \tag{32}$$

e) Is $\mathbf{A}$ orthogonal? Why or why not?

f) Is $\mathbf{A}$ positive definite? Why or why not?

g) Find $\mathbf{A}^{-1}$ and determine its eigenvalues and normalized eigenvectors.

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \frac{1}{\left|\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}\right|}\begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix} \tag{33}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \frac{1}{(2 \times -1) - (2 \times 2)}\begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix} \tag{34}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \frac{1}{-6}\begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1}{3} \end{bmatrix} \tag{35}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 0{,}17 & 0{,}33 \\ 0{,}33 & -0{,}33 \end{bmatrix} \tag{36}$$

$$\left|\begin{bmatrix} 0{,}17 & 0{,}33 \\ 0{,}33 & -0{,}33 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| \tag{37}$$

$$\left|\begin{bmatrix} 0{,}17 - \lambda & 0{,}33 \\ 0{,}33 & -0{,}33 - \lambda \end{bmatrix}\right| \tag{38}$$

$$((0{,}17 - \lambda) \times (-0{,}33 - \lambda)) - (0{,}33 \times 0{,}33) \tag{39}$$

$$((0{,}17 - \lambda) \times (-0{,}33 - \lambda)) - 0{,}1089 \tag{40}$$

$$-0{,}0561 + 0{,}16\lambda + \lambda^2 - 0{,}1089 \tag{41}$$

$$\lambda^2 + 0{,}16\lambda - 0{,}165 = 0 \tag{42}$$

$$\left(\lambda^2 + 0{,}16\lambda - 0{,}165\right) \times 1000 = 0 \times 1000 \tag{43}$$

$$1000\lambda^2 + 160\lambda - 165 = 0 \tag{44}$$

For $a = 1000$, $b = 160$, $c = -165$:

$$\lambda = \frac{-160 \pm \sqrt{160^2 - 4 \times 1000\,(-165)}}{2 \times 1000} \tag{45}$$

$$\lambda_1 = \frac{-160 + \sqrt{160^2 - 4 \times 1000\,(-165)}}{2 \times 1000} = \frac{\sqrt{1714} - 8}{100} = 0{,}33 \tag{46}$$

$$\lambda_2 = \frac{-160 - \sqrt{160^2 - 4 \times 1000\,(-165)}}{2 \times 1000} = -\frac{8 + \sqrt{1714}}{100} = -0{,}5 \tag{47}$$

The eigenvalues are: $\begin{bmatrix} 0{,}33 & -0{,}55 \end{bmatrix}$
Solving for $\lambda = 0{,}33$

$$\begin{bmatrix} 0{,}17 & 0{,}33 \\ 0{,}33 & -0{,}33 \end{bmatrix} - 0{,}33 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0{,}16 & 0{,}33 \\ 0{,}33 & -0{,}66 \end{bmatrix} \tag{48}$$

$$\begin{bmatrix} -0{,}16 & 0{,}33 \\ 0{,}33 & -0{,}66 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{49}$$

$$\begin{bmatrix} 0{,}33 & -0{,}66 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{50}$$

$$\begin{bmatrix} 1 & -2{,}01 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{51}$$

$$x - 2{,}01y = 0 \tag{52}$$

Setting $x = 2{,}01$ then the eigenvalues for $\lambda = 0{,}3 = \begin{bmatrix} 2{,}01 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2{,}01 & 1 \end{bmatrix} \begin{bmatrix} 2{,}01 \\ 1 \end{bmatrix} = 5{,}04 \tag{53}$$

The normalized eigenvalues for $\lambda = 0{,}3 = \begin{bmatrix} \frac{2{,}01}{\sqrt{5{,}04}} & \frac{1}{\sqrt{5{,}04}} \end{bmatrix} = \begin{bmatrix} 0{,}895 & 0{,}445 \end{bmatrix}$
Solving for $\lambda = -0{,}5$

$$\begin{bmatrix} 0{,}17 & 0{,}33 \\ 0{,}33 & -0{,}33 \end{bmatrix} - -0{,}5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{54}$$

$$\begin{bmatrix} 0{,}66 & 0{,}33 \\ 0{,}33 & 0{,}16 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{55}$$

$$\begin{bmatrix} 0{,}66 & 0{,}33 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{56}$$

$$\begin{bmatrix} 1 & 0{,}5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{57}$$

$$x + 0{,}5y = 0 \tag{58}$$

Setting $x = -0{,}5$ then the eigenvalues for $\lambda = 0{,}3 = \begin{bmatrix} -0{,}5 & 1 \end{bmatrix}$

$$\begin{bmatrix} -0{,}5 & 1 \end{bmatrix} \begin{bmatrix} -0{,}5 \\ 1 \end{bmatrix} = 1{,}25 \tag{59}$$

The normalized eigenvalues for $\lambda = 0{,}3 = \begin{bmatrix} \frac{-0{,}5}{\sqrt{1{,}25}} & \frac{1}{\sqrt{1{,}25}} \end{bmatrix} = \begin{bmatrix} -0{,}445 & 0{,}895 \end{bmatrix}$

2. Consider the matrices These matrices are identical except for a small difference in the (2, 2) position. Also, the columns of A and B are nearly linearly dependent. Show that A-1 $\approx$ (-3)B-1. So, small changes - perhaps due to rounding - can result in substantially different inverses.