STAT636 - Homework 1

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1. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$$

Without using a computer:

(a) Find the eigenvalues and normalized eigenvectors of $\mathbf{A}$.

$$\left| \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| \tag{1}$$

$$\left| \begin{bmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix} \right| \tag{2}$$

$$(2-\lambda)(-1-\lambda) - 4 \tag{3}$$

$$-2 - 2\lambda + \lambda + \lambda^2 - 4 \tag{4}$$

$$\lambda^2 - \lambda - 6 \tag{5}$$

$$(\lambda + 3)(\lambda - 2) = 0 \tag{6}$$

$$(\lambda - 3) = 0 : \lambda = 3 \tag{7}$$

$$(\lambda + 2) = 0 : \lambda = -2 \tag{8}$$

The eigenvalues are: $[3, -2]$
Solving for $\lambda = 3$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \tag{9}$$

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{12}$$

$$x - 2y = 0 \tag{13}$$

$$x = 2y \tag{14}$$

Setting $y = 1$ then **the eigenvectors for** $\lambda = 3$ **are** $[2, 1]$

$$\begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \times 2 + 1 \times 1 = 5 \tag{15}$$

The normalized eigenvectors for $\lambda = 3$ are $\left[\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right] = \begin{bmatrix} 0.8944272 & 0.4472136 \end{bmatrix}$
Solving for $\lambda = -2$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} - -2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{18}$$

$$\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{19}$$

$$x + \frac{1}{2}y = 0 \tag{20}$$

$$x = -\frac{1}{2}y \tag{21}$$

Setting $y = 1$ then **the eigenvectors for** $\lambda = -2$ **are** $[-\frac{1}{2}, 1]$

$$\begin{bmatrix} -\frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} = -\frac{1}{2} \times -\frac{1}{2} + 1 \times 1 = 1.25 \tag{22}$$

The normalized eigenvectors for $\lambda = 3$ are $\left[\frac{-\frac{1}{2}}{\sqrt{1.25}}, \frac{1}{\sqrt{1.25}}\right] = \begin{bmatrix} -0.4472136 & 0.8944272 \end{bmatrix}$

(b) Write the spectral decomposition of $\mathbf{A}$.

$$\mathbf{A} = \mathbf{CDC}' \tag{23}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0.894 & 0.447 \\ -0.447 & 0.894 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0.894 & -0.447 \\ 0.447 & 0.894 \end{bmatrix} \tag{24}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} = \left(3 \begin{bmatrix} 0.894 \\ 0.447 \end{bmatrix} \begin{bmatrix} 0.894 & 0.447 \end{bmatrix}\right) + \left(-2 \begin{bmatrix} -0.447 \\ 0.894 \end{bmatrix} \begin{bmatrix} -0.447 & 0.894 \end{bmatrix}\right) \tag{25}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} = \left(3 \begin{bmatrix} 0.8 & 0.4 \\ 0.4 & 0.2 \end{bmatrix}\right) + \left(-2 \begin{bmatrix} 0.2 & -0.4 \\ -0.4 & 0.8 \end{bmatrix}\right) \tag{26}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2.4 & 1.2 \\ 1.2 & 0.6 \end{bmatrix} + \begin{bmatrix} -0.4 & 0.8 \\ 0.8 & -1.6 \end{bmatrix} \tag{27}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \tag{28}$$

(c) Verify that the determinant of $\mathbf{A}$ equals the product of its eigenvalues.

$$\left| \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \right| = 3 \times -2 \tag{29}$$

$$(2 \times -1) - (2 \times 2) = -6 \tag{30}$$

$$-6 = -6 \tag{31}$$

(d) The trace of a square matrix equals the sum of its diagonal elements. Verify that the trace of $\mathbf{A}$ equals the sum of its eigenvalues.

$$tr\left(\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}\right) = 3 + -2 \tag{32}$$

$$(2 + -1) = 1 \tag{33}$$

$$1 = 1 \tag{34}$$

(e) Is $\mathbf{A}$ orthogonal? Why or why not? *No, because* $\mathbf{A}'\mathbf{A} \neq \mathbf{I}$

(f) Is $\mathbf{A}$ positive definite? Why or why not? $\mathbf{A}$ *is not positive definite because their eigenvalues are not all positive.*

(g) Find $\mathbf{A}^{-1}$ and determine its eigenvalues and normalized eigenvectors.

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \frac{1}{\left| \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \right|} \begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix} \tag{35}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \frac{1}{(2 \times -1) - (2 \times 2)} \begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix} \tag{36}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \frac{1}{-6} \begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1}{3} \end{bmatrix} \tag{37}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.17 & 0.33 \\ 0.33 & -0.33 \end{bmatrix} \tag{38}$$

$$\left| \begin{bmatrix} 0.17 & 0.33 \\ 0.33 & -0.33 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| \tag{39}$$

$$\left|\begin{bmatrix} 0.17-\lambda & 0.33 \\ 0.33 & -0.33-\lambda \end{bmatrix}\right| \tag{40}$$

$$((0.17-\lambda)\times(-0.33-\lambda)) - (0.33\times 0.33) \tag{41}$$

$$((0.17-\lambda)\times(-0.33-\lambda)) - 0.1089 \tag{42}$$

$$-0.0561 + 0.16\lambda + \lambda^2 - 0.1089 \tag{43}$$

$$\lambda^2 + 0.16\lambda - 0.165 = 0 \tag{44}$$

$$\left(\lambda^2 + 0.16\lambda - 0.165\right)\times 1000 = 0\times 1000 \tag{45}$$

$$1000\lambda^2 + 160\lambda - 165 = 0 \tag{46}$$

For $a = 1000$, $b = 160$, $c = -165$:

$$\lambda = \frac{-160 \pm \sqrt{160^2 - 4\times 1000\,(-165)}}{2\times 1000} \tag{47}$$

$$\lambda_1 = \frac{-160 + \sqrt{160^2 - 4\times 1000\,(-165)}}{2\times 1000} = \frac{\sqrt{1714}-8}{100} = 0.33 \tag{48}$$

$$\lambda_2 = \frac{-160 - \sqrt{160^2 - 4\times 1000\,(-165)}}{2\times 1000} = -\frac{8+\sqrt{1714}}{100} = -0.5 \tag{49}$$

The eigenvalues are: $\begin{bmatrix} 0.33 & -0.5 \end{bmatrix}$
Solving for $\lambda = 0.33$

$$\begin{bmatrix} 0.17 & 0.33 \\ 0.33 & -0.33 \end{bmatrix} - 0.33\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.16 & 0.33 \\ 0.33 & -0.66 \end{bmatrix} \tag{50}$$

$$\begin{bmatrix} -0.16 & 0.33 \\ 0.33 & -0.66 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{51}$$

$$\begin{bmatrix} 0.33 & -0.66 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{52}$$

$$\begin{bmatrix} 1 & -2.01 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{53}$$

$$x - 2.01y = 0 \tag{54}$$

Setting $x = 2.01$ then the eigenvalues for $\lambda = 0.3 = \begin{bmatrix} 2.01 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2.01 & 1 \end{bmatrix}\begin{bmatrix} 2.01 \\ 1 \end{bmatrix} = 5.04 \tag{55}$$

The normalized eigenvalues for $\lambda = 0.3 = \begin{bmatrix} \frac{2.01}{\sqrt{5.04}} & \frac{1}{\sqrt{5.04}} \end{bmatrix} = \begin{bmatrix} 0.895 & 0.445 \end{bmatrix}$
Solving for $\lambda = -0.5$

$$\begin{bmatrix} 0.17 & 0.33 \\ 0.33 & -0.33 \end{bmatrix} - -0.5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{56}$$

$$\begin{bmatrix} 0.66 & 0.33 \\ 0.33 & 0.16 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{57}$$

$$\begin{bmatrix} 0.66 & 0.33 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{58}$$

$$\begin{bmatrix} 1 & 0.5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{59}$$

$$x + 0.5y = 0 \tag{60}$$

Setting $x = -0.5$ then the eigenvalues for $\lambda = 0.3$ are $= \begin{bmatrix} -0.5 & 1 \end{bmatrix}$

$$\begin{bmatrix} -0.5 & 1 \end{bmatrix} \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} = 1.25 \tag{61}$$

The normalized eigenvalues for $\lambda = 0.3$ are $= \begin{bmatrix} \frac{-0.5}{\sqrt{1.25}} & \frac{1}{\sqrt{1.25}} \end{bmatrix} = \begin{bmatrix} -0.445 & 0.895 \end{bmatrix}$

2. Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 4.000 & 4.001 \\ 4.001 & 4.002 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 4.000 & 4.001 \\ 4.001 & 4.002001 \end{bmatrix}$$

These matrices are identical except for a small difference in the (2, 2) position. Also, the columns of $\mathbf{A}$ and $\mathbf{B}$ are nearly linearly dependent. Show that $\mathbf{A}^{-1} \approx$ (-3)$\mathbf{B}^{-1}$. So, small changes - perhaps due to rounding - can result in substantially different inverses.

$$\begin{gathered}
\frac{1}{\left|\begin{bmatrix} 4.000 & 4.001 \\ 4.001 & 4.002 \end{bmatrix}\right|} \begin{bmatrix} 4.002 & -4.001 \\ -4.001 & 4.000 \end{bmatrix} = -3 \times \left[\frac{1}{\left|\begin{bmatrix} 4.000 & 4.001 \\ 4.001 & 4.002001 \end{bmatrix}\right|} \begin{bmatrix} 4.002001 & -4.001 \\ -4.001 & 4.000 \end{bmatrix} \right] \\
\frac{1}{-0.000001} \times \begin{bmatrix} 4.002 & -4.001 \\ -4.001 & 4.000 \end{bmatrix} = -3 \times \left[\frac{1}{0.000003} \times \begin{bmatrix} 4.002001 & -4.001 \\ -4.001 & 4.000 \end{bmatrix} \right] \\
-1000000 \times \begin{bmatrix} 4.002 & -4.001 \\ -4.001 & 4.000 \end{bmatrix} = -1000000 \times \begin{bmatrix} 4.002001 & -4.001 \\ -4.001 & 4.000 \end{bmatrix} \\
\begin{bmatrix} -4002000 & 4001000 \\ 4001000 & -4000000 \end{bmatrix} = \begin{bmatrix} -4002001 & 4001000 \\ 4001000 & -4000000 \end{bmatrix}
\end{gathered} \tag{62}$$

3. Derive expressions for the means and variances of the following linear combinations in terms of the means and covariances of the random variables X_1, X_2, and X_3.

(a) $2X_1 - X_2$

$$E(2X_1 - X_2) = 2 \times E(X_1) - E(X_2) \tag{63}$$

$$\begin{aligned} Var(2X_1 - X_2) &= 2^2 \times Var(X1) + Var(X2) - 2 \times 2 \times Cov(X_1, X_2) \\ &= 4 \times E((X1 - E(X1))^2) + E((X2 - E(X2))^2) \\ &- 4 \times \sum((X1 - E(X1)) \times (X2 - E(X2))) \end{aligned} \tag{64}$$

(b) $X_1 + X_2 - 2X_3$

$$E(X_1 + X_2 - 2X_3) = E(X_1) + E(X_2) - 2 \times E(X_3) \tag{65}$$

$$\begin{aligned} Var(X_1 + X_2 - 2X_3) = &(Var(X_1) + Var(X_2) + 2 \times Cov(X_1, X_2)) \\ &+2^2 \times Var(X_3) - 2 \times 2 \times Cov((X_1 + X_2), X_3) \end{aligned} \tag{66}$$

(c) $4X_1 - 3X_2$ if X_1 and X_2 are independent (so, $\sigma_{12} = 0$)

$$E(4X_1 - 3X_2) = 4 \times E(X_1) - 3 \times E(X_2) \tag{67}$$

$$\begin{aligned} Var(4X_1 - 3X_2) &= 4^2 \times Var(X_1) + 3^2 \times Var(X_2) \\ &= 16 \times E((X1 - E(X1))^2) + 9 \times E((X2 - E(X2))^2) \end{aligned} \tag{68}$$

4. Consider the random vector $\mathbf{X}' = [X_1, X_2, X_3, X_4]$ with mean vector $\mu' = [1, 2, 3, 4]$ and covariance matrix

$$\Sigma = \begin{bmatrix} 4 & 0 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 3 & 1 & 9 & -2 \\ 1 & 0 & -2 & 4 \end{bmatrix}$$

Partition $\mathbf{X}$ as

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \hdashline X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \hdashline \mathbf{X}^{(2)} \end{bmatrix}$$

Let

$$A = \begin{bmatrix} 2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix}$$

and consider the linear combinations $\mathrm{AX}^{(1)}$ and $\mathrm{BX}^{(2)}$. Find the following:

(a) $E(\mathbf{X}_1)$

$$\begin{aligned} E(\mathbf{X}_1) &= E(\begin{bmatrix} 1 & 2 \end{bmatrix}) \\ &= \frac{1+2}{2} \\ &= 1.5 \end{aligned} \tag{69}$$

(b) $E(\mathbf{BX}^{(2)})$

$$\begin{aligned} E(\mathbf{BX}^{(2)}) &= E\left(\begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) \\ &= E\left(\begin{bmatrix} (-1\times 3)+(-2\times 4) \\ (2\times 3)+(1\times 4) \end{bmatrix}\right) \\ &= E\left(\begin{bmatrix} -11 \\ 10 \end{bmatrix}\right) \\ &= \frac{-11+10}{2} \\ &= -0.5 \end{aligned} \tag{70}$$

(c) $Cov(\mathbf{AX}^{(1)})$

$$\begin{aligned} Cov(\mathbf{AX}^{(1)}) &= \mathbf{A} \times Cov(\mathbf{X}^{(1)}) \\ &= \begin{bmatrix} 2 & 1 \end{bmatrix} \times 0 \\ &= 0 \end{aligned} \tag{71}$$

(d) $Cov(\mathbf{X}^{(1)}, \mathbf{X}^{(2)})$

$$\begin{aligned} Cov(\mathbf{X}^{(1)}, \mathbf{X}^{(2)}) &= \sum (\mathbf{X}^{(1)} - E(\mathbf{X}^{(1)}))(\mathbf{X}^{(2)} - E(\mathbf{X}^{(2)})) \\ &= \sum (\begin{bmatrix} 1 & 2 \end{bmatrix} - 1.5)(\begin{bmatrix} 3 & 4 \end{bmatrix} - 3.5) \\ &= \begin{bmatrix} -0.5 & 0.5 \end{bmatrix} \times \begin{bmatrix} -0.5 & 0.5 \end{bmatrix} \\ &= 0.5 \end{aligned} \tag{72}$$

(e) $Cov(\mathbf{AX}^{(1)}, \mathbf{BX}^{(2)})$

$$\begin{aligned}
Cov(\mathbf{AX}^{(1)}, \mathbf{BX}^{(2)}) &= \mathbf{AB} \times Cov(\mathbf{X}^{(1)}, \mathbf{X}^{(2)}) \\
&= \left[\begin{array}{cc} 2 & 1 \end{array}\right] \left[\begin{array}{cc} -1 & -2 \\ 2 & 1 \end{array}\right] \times Cov(\mathbf{X}^{(1)}, \mathbf{X}^{(2)}) \\
&= -3 \times Cov(\mathbf{X}^{(1)}, \mathbf{X}^{(2)}) \\
&= -3 \times 0.5 \\
&= -1.5
\end{aligned} \tag{73}$$