

STAT636 - Homework 1

Daniel Osorio - dcosorih@tamu.edu

Department of Veterinary Integrative Biosciences

Texas A&M University

1. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$$

Without using a computer:

(a) Find the eigenvalues and normalized eigenvectors of \mathbf{A} .

$$\left| \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| \quad (1) \quad \text{Setting } y = 1 \text{ then the eigenvectors for } \lambda = 3 \text{ are } [2, 1]$$

$$\left| \begin{bmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix} \right| \quad (2) \quad \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \times 2 + 1 \times 1 = 5 \quad (15)$$

$$(2-\lambda)(-1-\lambda) - 4 \quad (3) \quad \text{The normalized eigenvectors for } \lambda = 3 \text{ are } \left[\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right] = [0.8944272 \quad 0.4472136]$$

$$-2 - 2\lambda + \lambda + \lambda^2 - 4 \quad (4) \quad \text{Solving for } \lambda = -2$$

$$\lambda^2 - \lambda - 6 \quad (5) \quad \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} - -2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \quad (16)$$

$$(\lambda + 3)(\lambda - 2) = 0 \quad (6) \quad \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (17)$$

$$(\lambda - 3) = 0 : \lambda = 3 \quad (7) \quad \begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (18)$$

$$(\lambda + 2) = 0 : \lambda = -2 \quad (8) \quad \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (19)$$

The eigenvalues are: $[3, -2]$

Solving for $\lambda = 3$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \quad (9) \quad x + \frac{1}{2}y = 0 \quad (20)$$

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (10) \quad x = -\frac{1}{2}y \quad (21)$$

$$\begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (11) \quad \text{Setting } y = 1 \text{ then the eigenvectors for } \lambda = -2 \text{ are } \left[-\frac{1}{2}, 1 \right]$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (12) \quad \begin{bmatrix} -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} = -\frac{1}{2} \times -\frac{1}{2} + 1 \times 1 = 1.25 \quad (22)$$

$$x - 2y = 0 \quad (13) \quad \text{The normalized eigenvectors for } \lambda = 3 \text{ are } \left[\frac{-\frac{1}{2}}{\sqrt{1.25}}, \frac{1}{\sqrt{1.25}} \right] = [-0.4472136 \quad 0.8944272]$$

$$x = 2y \quad (14)$$

(b) Write the spectral decomposition of \mathbf{A} .

$$\mathbf{A} = \mathbf{C}\mathbf{D}\mathbf{C}', \quad (23)$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0.894 & 0.447 \\ -0.447 & 0.894 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0.894 & -0.447 \\ 0.447 & 0.894 \end{bmatrix} \quad (24)$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} = \left(3 \begin{bmatrix} 0.894 \\ 0.447 \end{bmatrix} \begin{bmatrix} 0.894 & 0.447 \end{bmatrix} \right) + \left(-2 \begin{bmatrix} -0.447 \\ 0.894 \end{bmatrix} \begin{bmatrix} -0.447 & 0.894 \end{bmatrix} \right) \quad (25)$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} = \left(3 \begin{bmatrix} 0.8 & 0.4 \\ 0.4 & 0.2 \end{bmatrix} \right) + \left(-2 \begin{bmatrix} 0.2 & -0.4 \\ -0.4 & 0.8 \end{bmatrix} \right) \quad (26)$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2.4 & 1.2 \\ 1.2 & 0.6 \end{bmatrix} + \begin{bmatrix} -0.4 & 0.8 \\ 0.8 & -1.6 \end{bmatrix} \quad (27)$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \quad (28)$$

(c) Verify that the determinant of \mathbf{A} equals the product of its eigenvalues.

$$\left| \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \right| = 3 \times -2 \quad (29)$$

$$(2 \times -1) - (2 \times 2) = -6 \quad (30)$$

$$-6 = -6 \quad (31)$$

(d) The trace of a square matrix equals the sum of its diagonal elements. Verify that the trace of \mathbf{A} equals the sum of its eigenvalues.

$$\text{tr} \left(\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \right) = 3 + -2 \quad (32)$$

$$(2 + -1) = 1 \quad (33)$$

$$1 = 1 \quad (34)$$

(e) Is \mathbf{A} orthogonal? Why or why not? *No, because $\mathbf{A}'\mathbf{A} \neq \mathbf{I}$*

(f) Is \mathbf{A} positive definite? Why or why not? *\mathbf{A} is not positive definite because their eigenvalues are not all positive.*

(g) Find \mathbf{A}^{-1} and determine its eigenvalues and normalized eigenvectors.

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \frac{1}{\left| \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \right|} \begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix} \quad (35)$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \frac{1}{(2 \times -1) - (2 \times 2)} \begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix} \quad (36)$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \frac{1}{-6} \begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \quad (37)$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.17 & 0.33 \\ 0.33 & -0.33 \end{bmatrix} \quad (38)$$

$$\left| \begin{bmatrix} 0.17 & 0.33 \\ 0.33 & -0.33 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| \quad (39)$$

$$\left| \begin{bmatrix} 0.17 - \lambda & 0.33 \\ 0.33 & -0.33 - \lambda \end{bmatrix} \right| \quad (40)$$

$$((0.17 - \lambda) \times (-0.33 - \lambda)) - (0.33 \times 0.33) \quad (41)$$

$$((0.17 - \lambda) \times (-0.33 - \lambda)) - 0.1089 \quad (42)$$

$$-0.0561 + 0.16\lambda + \lambda^2 - 0.1089 \quad (43)$$

$$\lambda^2 + 0.16\lambda - 0.165 = 0 \quad (44)$$

$$(\lambda^2 + 0.16\lambda - 0.165) \times 1000 = 0 \times 1000 \quad (45)$$

$$1000\lambda^2 + 160\lambda - 165 = 0 \quad (46)$$

For $a = 1000$, $b = 160$, $c = -165$:

$$\lambda = \frac{-160 \pm \sqrt{160^2 - 4 \times 1000(-165)}}{2 \times 1000} \quad (47)$$

$$\lambda_1 = \frac{-160 + \sqrt{160^2 - 4 \times 1000(-165)}}{2 \times 1000} = \frac{\sqrt{1714} - 8}{100} = 0.33 \quad (48)$$

$$\lambda_2 = \frac{-160 - \sqrt{160^2 - 4 \times 1000(-165)}}{2 \times 1000} = -\frac{8 + \sqrt{1714}}{100} = -0.5 \quad (49)$$

The eigenvalues are: $\begin{bmatrix} 0.33 & -0.5 \end{bmatrix}$

Solving for $\lambda = 0.33$

$$\begin{bmatrix} 0.17 & 0.33 \\ 0.33 & -0.33 \end{bmatrix} - 0.33 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.16 & 0.33 \\ 0.33 & -0.66 \end{bmatrix} \quad (50)$$

$$\begin{bmatrix} -0.16 & 0.33 \\ 0.33 & -0.66 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (51)$$

$$\begin{bmatrix} 0.33 & -0.66 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (52)$$

$$\begin{bmatrix} 1 & -2.01 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (53)$$

$$x - 2.01y = 0 \quad (54)$$

Setting $x = 2.01$ then the eigenvalues for $\lambda = 0.3 = \begin{bmatrix} 2.01 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2.01 & 1 \end{bmatrix} \begin{bmatrix} 2.01 \\ 1 \end{bmatrix} = 5.04 \quad (55)$$

The normalized eigenvalues for $\lambda = 0.3 = \begin{bmatrix} \frac{2.01}{\sqrt{5.04}} & \frac{1}{\sqrt{5.04}} \end{bmatrix} = \begin{bmatrix} 0.895 & 0.445 \end{bmatrix}$

Solving for $\lambda = -0.5$

$$\begin{bmatrix} 0.17 & 0.33 \\ 0.33 & -0.33 \end{bmatrix} - -0.5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (56)$$

$$\begin{bmatrix} 0.66 & 0.33 \\ 0.33 & 0.16 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (57)$$

$$\begin{bmatrix} 0.66 & 0.33 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (58)$$

$$\begin{bmatrix} 1 & 0.5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (59)$$

$$x + 0.5y = 0 \quad (60)$$

Setting $x = -0.5$ then the eigenvalues for $\lambda = 0.3$ are $= \begin{bmatrix} -0.5 & 1 \end{bmatrix}$

$$\begin{bmatrix} -0.5 & 1 \end{bmatrix} \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} = 1.25 \quad (61)$$

The normalized eigenvalues for $\lambda = 0.3$ are $= \begin{bmatrix} \frac{-0.5}{\sqrt{1.25}} & \frac{1}{\sqrt{1.25}} \end{bmatrix} = \begin{bmatrix} -0.445 & 0.895 \end{bmatrix}$

2. Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 4.000 & 4.001 \\ 4.001 & 4.002 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 4.000 & 4.001 \\ 4.001 & 4.002001 \end{bmatrix}$$

These matrices are identical except for a small difference in the (2, 2) position. Also, the columns of \mathbf{A} and \mathbf{B} are nearly linearly dependent. Show that $\mathbf{A}^{-1} \approx (-3)\mathbf{B}^{-1}$. So, small changes - perhaps due to rounding - can result in substantially different inverses.

$$\frac{1}{\begin{vmatrix} 4.000 & 4.001 \\ 4.001 & 4.002 \end{vmatrix}} \begin{bmatrix} 4.002 & -4.001 \\ -4.001 & 4.000 \end{bmatrix} = -3 \times \left[\frac{1}{\begin{vmatrix} 4.000 & 4.001 \\ 4.001 & 4.002001 \end{vmatrix}} \begin{bmatrix} 4.002001 & -4.001 \\ -4.001 & 4.000 \end{bmatrix} \right] \quad (62)$$

$$\frac{1}{-0.000001} \times \begin{bmatrix} 4.002 & -4.001 \\ -4.001 & 4.000 \end{bmatrix} = -3 \times \left[\frac{1}{0.000003} \times \begin{bmatrix} 4.002001 & -4.001 \\ -4.001 & 4.000 \end{bmatrix} \right] \quad (63)$$

$$-1000000 \times \begin{bmatrix} 4.002 & -4.001 \\ -4.001 & 4.000 \end{bmatrix} = -1000000 \times \begin{bmatrix} 4.002001 & -4.001 \\ -4.001 & 4.000 \end{bmatrix} \quad (64)$$

$$\begin{bmatrix} -4002000 & 4001000 \\ 4001000 & -4000000 \end{bmatrix} = \begin{bmatrix} -4002001 & 4001000 \\ 4001000 & -4000000 \end{bmatrix} \quad (65)$$

3. Derive expressions for the means and variances of the following linear combinations in terms of the means and covariances of the random variables X_1 , X_2 , and X_3 .

(a) $2X_1 - X_2$

$$E(2X_1 - X_2) = 2 \times E(X_1) - E(X_2) \quad (66)$$

$$Var(2X_1 - X_2) = 2^2 \times Var(X_1) + Var(X_2) - 2 \times 2 \times Cov(X_1, X_2) \quad (67)$$

$$Var(2X_1 - X_2) = 4 \times E((X_1 - E(X_1))^2) + E((X_2 - E(X_2))^2) - 4 \times (E((X_1 - E(X_1))^2) \times E((X_2 - E(X_2))^2)) \quad (68)$$

(b) $X_1 + X_2 - 2X_3$

$$E(X_1 + X_2 - 2X_3) = E(X_1) + E(X_2) - 2 \times E(X_3) \quad (69)$$

$$Var(X_1 + X_2 - 2X_3) = Var(Var(X_1) + Var(X_2) + 2 \times Cov(X_1, X_2)) + 2^2 \times Var(X_3) \quad (70)$$

$$-2 \times 2 \times Cov(Var(X_1) + Var(X_2) + 2 \times Cov(X_1, X_2), X_3)$$

(c) $4X_1 - 3X_2$ if X_1 and X_2 are independent (so, $\sigma_{12} = 0$)

$$E(4X_1 - 3X_2) = 4 \times E(X_1) - 3 \times E(X_2) \quad (71)$$

$$Var(4X_1 - 3X_2) = 4^2 \times Var(X_1) + 3^2 \times Var(X_2) \quad (72)$$

$$Var(4X_1 - 3X_2) = 16 \times E((X_1 - E(X_1))^2) + 9 \times E((X_2 - E(X_2))^2) \quad (73)$$

4. Consider the random vector $\mathbf{X}' = [X_1, X_2, X_3, X_4]$ with mean vector $\mu' = [1, 2, 3, 4]$ and covariance matrix

$$\Sigma =$$