STAT636 - Homework 1

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1. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$$

Without using a computer:

(a) Find the eigenvalues and normalized eigenvectors of $\mathbf{A}$.

$$\left| \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| \tag{1}$$

$$\left| \begin{bmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix} \right| \tag{2}$$

$$(2-\lambda)(-1-\lambda) - 4 \tag{3}$$

$$-2 - 2\lambda + \lambda + \lambda^2 - 4 \tag{4}$$

$$\lambda^2 - \lambda - 6 \tag{5}$$

$$(\lambda + 3)(\lambda - 2) = 0 \tag{6}$$

$$(\lambda - 3) = 0 : \lambda = 3 \tag{7}$$

$$(\lambda + 2) = 0 : \lambda = -2 \tag{8}$$

The eigenvalues are: $[3, -2]$
Solving for $\lambda = 3$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \tag{9}$$

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{12}$$

$$x - 2y = 0 \tag{13}$$

$$x = 2y \tag{14}$$

Setting $y = 1$ then **the eigenvectors for** $\lambda = 3$ **are** $[2, 1]$

$$\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \times 2 + 1 \times 1 = 5 \tag{15}$$

The normalized eigenvectors for $\lambda = 3$ are $\left[\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right] = \begin{bmatrix} 0.8944272 & 0.4472136 \end{bmatrix}$
Solving for $\lambda = -2$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} - -2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{18}$$

$$\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{19}$$

$$x + \frac{1}{2}y = 0 \tag{20}$$

$$x = -\frac{1}{2}y \tag{21}$$

Setting $y = 1$ then **the eigenvectors for** $\lambda = -2$ **are** $[-\frac{1}{2}, 1]$

$$\begin{bmatrix} -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} = -\frac{1}{2} \times -\frac{1}{2} + 1 \times 1 = 1.25 \tag{22}$$

The normalized eigenvectors for $\lambda = 3$ are $\left[\frac{-\frac{1}{2}}{\sqrt{1.25}}, \frac{1}{\sqrt{1.25}}\right] = \begin{bmatrix} -0.4472136 & 0.8944272 \end{bmatrix}$

(b) Write the spectral decomposition of $\mathbf{A}$.

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} = \left(3\begin{bmatrix} 0.894 \\ 0.447 \end{bmatrix}\begin{bmatrix} 0.894 & 0.447 \end{bmatrix}\right) + \left(-2\begin{bmatrix} -0.447 \\ 0.894 \end{bmatrix}\begin{bmatrix} -0.447 & 0.894 \end{bmatrix}\right) \tag{23}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} = \left(3\begin{bmatrix} 0.8 & 0.4 \\ 0.4 & 0.2 \end{bmatrix}\right) + \left(-2\begin{bmatrix} 0.2 & -0.4 \\ -0.4 & 0.8 \end{bmatrix}\right) \tag{24}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2.4 & 1.2 \\ 1.2 & 0.6 \end{bmatrix} + \begin{bmatrix} -0.4 & 0.8 \\ 0.8 & -1.6 \end{bmatrix} \tag{25}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \tag{26}$$

(c) Verify that the determinant of $\mathbf{A}$ equals the product of its eigenvalues.

$$\left|\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}\right| = 3 \times -2 \tag{27}$$

$$(2 \times -1) - (2 \times 2) = -6 \tag{28}$$

$$-6 = -6 \tag{29}$$

(d) The trace of a square matrix equals the sum of its diagonal elements. Verify that the trace of $\mathbf{A}$ equals the sum of its eigenvalues.

$$tr\left(\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}\right) = 3 + -2 \tag{30}$$

$$(2 + -1) = 1 \tag{31}$$

$$1 = 1 \tag{32}$$

(e) Is $\mathbf{A}$ orthogonal? Why or why not? *No, because* $\mathbf{A}'\mathbf{A} \neq \mathbf{I}$

(f) Is $\mathbf{A}$ positive definite? Why or why not? $\mathbf{A}$ *is not positive definite because their eigenvalues are not all positive.*

(g) Find $\mathbf{A}^{-1}$ and determine its eigenvalues and normalized eigenvectors.

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \frac{1}{\left|\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}\right|}\begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix} \tag{33}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \frac{1}{(2 \times -1) - (2 \times 2)}\begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix} \tag{34}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \frac{1}{-6}\begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1}{3} \end{bmatrix} \tag{35}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.17 & 0.33 \\ 0.33 & -0.33 \end{bmatrix} \tag{36}$$

$$\left|\begin{bmatrix} 0.17 & 0.33 \\ 0.33 & -0.33 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| \tag{37}$$

$$\left|\begin{bmatrix} 0.17-\lambda & 0.33 \\ 0.33 & -0.33-\lambda \end{bmatrix}\right| \tag{38}$$

$$((0.17-\lambda) \times (-0.33-\lambda)) - (0.33 \times 0.33) \tag{39}$$

$$((0.17-\lambda)\times(-0.33-\lambda))-0.1089 \tag{40}$$
$$-0.0561+0.16\lambda+\lambda^2-0.1089 \tag{41}$$
$$\lambda^2+0.16\lambda-0.165=0 \tag{42}$$
$$\left(\lambda^2+0.16\lambda-0.165\right)\times 1000=0\times 1000 \tag{43}$$
$$1000\lambda^2+160\lambda-165=0 \tag{44}$$

For $a=1000$, $b=160$, $c=-165$:

$$\lambda=\frac{-160\pm\sqrt{160^2-4\times 1000\,(-165)}}{2\times 1000} \tag{45}$$
$$\lambda_1=\frac{-160+\sqrt{160^2-4\times 1000\,(-165)}}{2\times 1000}=\frac{\sqrt{1714}-8}{100}=0.33 \tag{46}$$
$$\lambda_2=\frac{-160-\sqrt{160^2-4\times 1000\,(-165)}}{2\times 1000}=-\frac{8+\sqrt{1714}}{100}=-0.5 \tag{47}$$

The eigenvalues are: $\begin{bmatrix} 0.33 & -0.55 \end{bmatrix}$
Solving for $\lambda=0.33$

$$\begin{bmatrix} 0.17 & 0.33 \\ 0.33 & -0.33 \end{bmatrix}-0.33\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} -0.16 & 0.33 \\ 0.33 & -0.66 \end{bmatrix} \tag{48}$$
$$\begin{bmatrix} -0.16 & 0.33 \\ 0.33 & -0.66 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{49}$$
$$\begin{bmatrix} 0.33 & -0.66 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{50}$$
$$\begin{bmatrix} 1 & -2.01 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{51}$$
$$x-2.01y=0 \tag{52}$$

Setting $x=2.01$ then the eigenvalues for $\lambda=0.3=\begin{bmatrix} 2.01 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2.01 & 1 \end{bmatrix}\begin{bmatrix} 2.01 \\ 1 \end{bmatrix}=5.04 \tag{53}$$

The normalized eigenvalues for $\lambda=0.3=\begin{bmatrix} \frac{2.01}{\sqrt{5.04}} & \frac{1}{\sqrt{5.04}} \end{bmatrix}=\begin{bmatrix} 0.895 & 0.445 \end{bmatrix}$
Solving for $\lambda=-0.5$

$$\begin{bmatrix} 0.17 & 0.33 \\ 0.33 & -0.33 \end{bmatrix}--0.5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{54}$$
$$\begin{bmatrix} 0.66 & 0.33 \\ 0.33 & 0.16 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{55}$$
$$\begin{bmatrix} 0.66 & 0.33 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{56}$$
$$\begin{bmatrix} 1 & 0.5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{57}$$
$$x+0.5y=0 \tag{58}$$

Setting $x=-0.5$ then the eigenvalues for $\lambda=0.3$ are $=\begin{bmatrix} -0.5 & 1 \end{bmatrix}$

$$\begin{bmatrix} -0.5 & 1 \end{bmatrix}\begin{bmatrix} -0.5 \\ 1 \end{bmatrix}=1.25 \tag{59}$$

The normalized eigenvalues for $\lambda=0.3$ are $=\begin{bmatrix} \frac{-0.5}{\sqrt{1.25}} & \frac{1}{\sqrt{1.25}} \end{bmatrix}=\begin{bmatrix} -0.445 & 0.895 \end{bmatrix}$

2. Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 4.000 & 4.001 \\ 4.001 & 4.002 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 4.000 & 4.001 \\ 4.001 & 4.002001 \end{bmatrix}$$

These matrices are identical except for a small difference in the (2, 2) position. Also, the columns of $\mathbf{A}$ and $\mathbf{B}$ are nearly linearly dependent. Show that $\mathbf{A}^{-1} \approx$ (-3)$\mathbf{B}^{-1}$. So, small changes - perhaps due to rounding - can result in substantially different inverses.

$$\frac{1}{\left|\begin{bmatrix} 4.000 & 4.001 \\ 4.001 & 4.002 \end{bmatrix}\right|} \begin{bmatrix} 4.002 & -4.001 \\ -4.001 & 4.000 \end{bmatrix} = -3 \times \left[\frac{1}{\left|\begin{bmatrix} 4.000 & 4.001 \\ 4.001 & 4.002001 \end{bmatrix}\right|} \begin{bmatrix} 4.002001 & -4.001 \\ -4.001 & 4.000 \end{bmatrix} \right] \tag{60}$$

$$\frac{1}{-0.000001} \times \begin{bmatrix} 4.002 & -4.001 \\ -4.001 & 4.000 \end{bmatrix} = -3 \times \left[\frac{1}{0.000003} \times \begin{bmatrix} 4.002001 & -4.001 \\ -4.001 & 4.000 \end{bmatrix} \right] \tag{61}$$

$$-1000000 \times \begin{bmatrix} 4.002 & -4.001 \\ -4.001 & 4.000 \end{bmatrix} = -1000000 \times \begin{bmatrix} 4.002001 & -4.001 \\ -4.001 & 4.000 \end{bmatrix} \tag{62}$$

$$\begin{bmatrix} -4002000 & 4001000 \\ 4001000 & -4000000 \end{bmatrix} = \begin{bmatrix} -4002001 & 4001000 \\ 4001000 & -4000000 \end{bmatrix} \tag{63}$$

3. Derive expressions for the means and variances of the following linear combinations in terms of the means and covariances of the random variables X_1, X_2, and X_3.

   (a) $2X_1 - X_2$

   (b) $X_1 + X_2 - 2X_3$

   (c) $4X_1 - 3X_2$ if X_1 and X_2 are independent (so, $\sigma_{12} = 0$)