

STAT636 - Homework 1

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1. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$$

Without using a computer:

a) Find the eigenvalues and normalized eigenvectors of \mathbf{A} .

$$\left| \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| \quad (1) \quad \text{Setting } y = 1 \text{ then the eigenvectors for } \lambda = 3 \text{ are } [2, 1]$$

$$\left| \begin{bmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix} \right| \quad (2) \quad \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \times 2 + 1 \times 1 = 5 \quad (15)$$

$$(2-\lambda)(-1-\lambda) - 4 \quad (3) \quad \text{The normalized eigenvectors for } \lambda = 3 \text{ are } \left[\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right] = [0,8944272 \quad 0,4472136]$$

$$-2 - 2\lambda + \lambda + \lambda^2 - 4 \quad (4) \quad \text{Solving for } \lambda = -2$$

$$\lambda^2 - \lambda - 6 \quad (5) \quad \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} - -2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \quad (16)$$

$$(\lambda + 3)(\lambda - 2) = 0 \quad (6) \quad \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (17)$$

$$(\lambda - 3) = 0 : \lambda = 3 \quad (7) \quad \begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (18)$$

$$(\lambda + 2) = 0 : \lambda = -2 \quad (8) \quad \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (19)$$

The eigenvalues are: $[3, -2]$

Solving for $\lambda = 3$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \quad (9) \quad x + \frac{1}{2}y = 0 \quad (20)$$

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (10) \quad x = -\frac{1}{2}y \quad (21)$$

$$\begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (11) \quad \text{Setting } y = 1 \text{ then the eigenvectors for } \lambda = -2 \text{ are } \left[-\frac{1}{2}, 1 \right]$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (12) \quad \begin{bmatrix} -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} = -\frac{1}{2} \times -\frac{1}{2} + 1 \times 1 = 1,25 \quad (22)$$

$$x - 2y = 0 \quad (13) \quad \text{The normalized eigenvectors for } \lambda = 3 \text{ are } \left[\frac{-\frac{1}{2}}{\sqrt{1,25}}, \frac{1}{\sqrt{1,25}} \right] = [-0,4472136 \quad 0,8944272]$$

$$x = 2y \quad (14)$$

b) Write the spectral decomposition of \mathbf{A} .

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} = \left(3 \begin{bmatrix} 0,894 \\ 0,447 \end{bmatrix} \begin{bmatrix} 0,894 & 0,447 \end{bmatrix} \right) + \left(-2 \begin{bmatrix} -0,447 \\ 0,894 \end{bmatrix} \begin{bmatrix} -0,447 & 0,894 \end{bmatrix} \right) \quad (23)$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} = \left(3 \begin{bmatrix} 0,8 & 0,4 \\ 0,4 & 0,2 \end{bmatrix} \right) + \left(-2 \begin{bmatrix} 0,2 & -0,4 \\ -0,4 & 0,8 \end{bmatrix} \right) \quad (24)$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2,4 & 1,2 \\ 1,2 & 0,6 \end{bmatrix} + \begin{bmatrix} -0,4 & 0,8 \\ 0,8 & -1,6 \end{bmatrix} \quad (25)$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \quad (26)$$

c) Verify that the determinant of \mathbf{A} equals the product of its eigenvalues.

$$\left| \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \right| = 3 \times -2 \quad (27)$$

$$(2 \times -1) - (2 \times 2) = -6 \quad (28)$$

$$-6 = -6 \quad (29)$$

d) The trace of a square matrix equals the sum of its diagonal elements. Verify that the trace of \mathbf{A} equals the sum of its eigenvalues.

$$tr \left(\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \right) = 3 + -2 \quad (30)$$

$$(2 + -1) = 1 \quad (31)$$

$$1 = 1 \quad (32)$$

e) Is \mathbf{A} orthogonal? Why or why not?

f) Is \mathbf{A} positive definite? Why or why not?

g) Find \mathbf{A}^{-1} and determine its eigenvalues and normalized eigenvectors.

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \frac{1}{\left| \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \right|} \begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix} \quad (33)$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \frac{1}{(2 \times -1) - (2 \times 2)} \begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix} \quad (34)$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \frac{1}{-6} \begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \quad (35)$$

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 0,17 & 0,33 \\ 0,33 & -0,33 \end{bmatrix} \quad (36)$$

$$\left| \begin{bmatrix} 0,17 & 0,33 \\ 0,33 & -0,33 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| \quad (37)$$

$$\left| \begin{bmatrix} 0,17 - \lambda & 0,33 \\ 0,33 & -0,33 - \lambda \end{bmatrix} \right| \quad (38)$$

$$((0,17 - \lambda) \times (-0,33 - \lambda)) - (0,33 \times 0,33) \quad (39)$$

$$((0,17 - \lambda) \times (-0,33 - \lambda)) - 0,1089 \quad (40)$$

$$-0,0561 + 0,16\lambda + \lambda^2 - 0,1089 \quad (41)$$

$$\lambda^2 + 0,16\lambda - 0,165 = 0 \quad (42)$$

$$(\lambda^2 + 0,16\lambda - 0,165) \times 1000 = 0 \times 1000 \quad (43)$$

$$1000\lambda^2 + 160\lambda - 165 = 0 \quad (44)$$

For $a = 1000$, $b = 160$, $c = -165$:

$$\lambda = \frac{-160 \pm \sqrt{160^2 - 4 \times 1000(-165)}}{2 \times 1000} \quad (45)$$

$$\lambda_1 = \frac{-160 + \sqrt{160^2 - 4 \times 1000(-165)}}{2 \times 1000} = \frac{\sqrt{1714} - 8}{100} = 0,33 \quad (46)$$

$$\lambda_2 = \frac{-160 - \sqrt{160^2 - 4 \times 1000(-165)}}{2 \times 1000} = -\frac{8 + \sqrt{1714}}{100} = -0,5 \quad (47)$$

The eigenvalues are: $\begin{bmatrix} 0,33 & -0,55 \end{bmatrix}$

Solving for $\lambda = 0,33$

$$\begin{bmatrix} 0,17 & 0,33 \\ 0,33 & -0,33 \end{bmatrix} - 0,33 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0,16 & 0,33 \\ 0,33 & -0,66 \end{bmatrix} \quad (48)$$

$$\begin{bmatrix} -0,16 & 0,33 \\ 0,33 & -0,66 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (49)$$

$$\begin{bmatrix} 0,33 & -0,66 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (50)$$

$$\begin{bmatrix} 1 & -2,01 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (51)$$

$$x - 2,01y = 0 \quad (52)$$

Setting $x = 2,01$ then the eigenvalues for $\lambda = 0,3 = \begin{bmatrix} 2,01 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2,01 & 1 \end{bmatrix} \begin{bmatrix} 2,01 \\ 1 \end{bmatrix} = 5,04 \quad (53)$$

The normalized eigenvalues for $\lambda = 0,3 = \begin{bmatrix} \frac{2,01}{\sqrt{5,04}} & \frac{1}{\sqrt{5,04}} \end{bmatrix} = \begin{bmatrix} 0,895 & 0,445 \end{bmatrix}$

Solving for $\lambda = -0,5$

$$\begin{bmatrix} 0,17 & 0,33 \\ 0,33 & -0,33 \end{bmatrix} - -0,5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (54)$$

$$\begin{bmatrix} 0,66 & 0,33 \\ 0,33 & 0,16 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (55)$$

$$\begin{bmatrix} 0,66 & 0,33 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (56)$$

$$\begin{bmatrix} 1 & 0,5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (57)$$

$$x + 0,5y = 0 \quad (58)$$

Setting $x = -0,5$ then the eigenvalues for $\lambda = 0,3 = \begin{bmatrix} -0,5 & 1 \end{bmatrix}$

$$\begin{bmatrix} -0,5 & 1 \end{bmatrix} \begin{bmatrix} -0,5 \\ 1 \end{bmatrix} = 1,25 \quad (59)$$

The normalized eigenvalues for $\lambda = 0,3 = \begin{bmatrix} \frac{-0,5}{\sqrt{1,25}} & \frac{1}{\sqrt{1,25}} \end{bmatrix} = \begin{bmatrix} -0,445 & 0,895 \end{bmatrix}$

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2. Consider the matrices These matrices are identical except for a small difference in the (2, 2) position. Also, the columns of A and B are nearly linearly dependent. Show that $A^{-1} \approx (-3)B^{-1}$. So, small changes - perhaps due to rounding - can result in substantially different inverses.