STAT636 - Homework 2

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1. Find the maximum likelihood estimates of the 2×1 mean vector μ and the 2×2 covariance matrix Σ based on the random sample

$$X = \begin{bmatrix} 3 & 6 \\ 4 & 4 \\ 5 & 7 \\ 4 & 7 \end{bmatrix}$$

```
> X <- matrix(data = c(3,6,4,4,5,7,4,7), ncol = 2, byrow = TRUE)
> n <- nrow(X)
> X_hat <- 1/n * rep(1,n) %*% X
> X_hat

     [,1] [,2]
[1,]    4    6

> S_hat <- 1/n * (t(X) - drop(X_hat)) %*% t(t(X) - drop(X_hat))
> S_hat

     [,1] [,2]
[1,] 0.50 0.25
[2,] 0.25 1.50
```

2. Let $X_1, X_2, \ldots, X_{60}$ be a random sample of size $n = 60$ from a $N_6(\mu, \Sigma)$ population. Specify each of the following.

   (a) The distribution of $(X_1 - \mu)'\Sigma^{-1}(X_1 - \mu)$.

$$(X_1 - \mu)'\Sigma^{-1}(X_1 - \mu) \sim \mathcal{X}_6^2$$

   (b) The distributions of $\bar{X}$ and $\sqrt{n}(\bar{X} - \mu)$.

$$\bar{X} \sim \mathcal{N}_6\left(\mu, \frac{1}{60}\Sigma\right)$$

$$\sqrt{n}(\bar{X} - \mu) \dot{\sim} \mathcal{N}_6\left(0, \Sigma\right)$$

   (c) The distribution of $n(\bar{X} - \mu)'\Sigma^{-1}(\bar{X} - \mu)$

$$n(\bar{X} - \mu)'\Sigma^{-1}(\bar{X} - \mu) \sim \mathcal{X}_6^2$$

   (d) The approximate distribution of $n(\bar{X} - \mu)'S^{-1}(\bar{X} - \mu)$

$$n(\bar{X} - \mu)'S^{-1}(\bar{X} - \mu) \dot{\sim} \frac{354}{54}\mathcal{F}_{60,54}$$

3. Consider the `used_car` data. For each of 10 used cars, we have the numeric variables Age (age of the car) and Price (sale price of car, in \$1,000s)

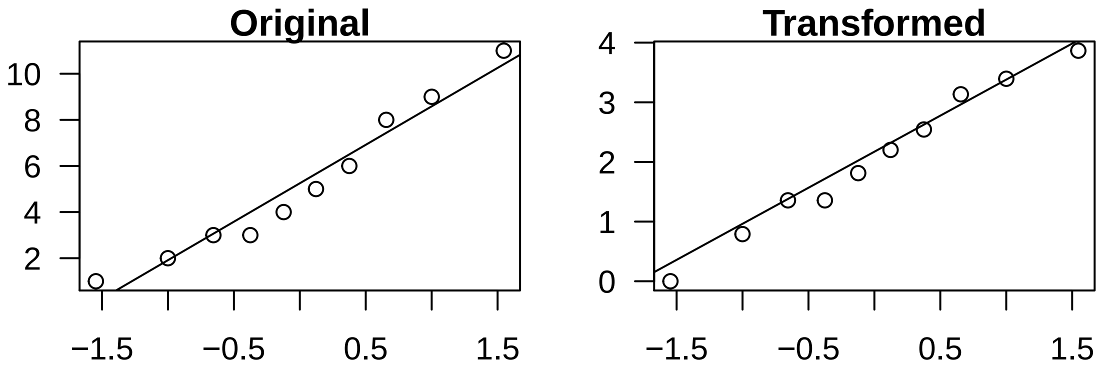
```
> used_car <- read.csv("used_cars.csv")
```

(a) Determine the power transformation $\hat{\lambda}_1$ that makes the x_1 values approximately normal. Construct a Q-Q plot for the transformed data.

```
> lambda <- as.numeric(car::powerTransform(used_car[,1])$lambda)
> lambda
```

```
[1] 0.3708906
```

```
> par(mfrow = c(1,2), mar=c(2.5,2.5,1,1))
> qqnorm(used_car[,1], main = "Original", las=1)
> qqline(used_car[,1])
> transformed <- ((used_car[,1] ^ lambda) - 1)/lambda
> qqnorm(transformed, main = "Transformed", las=1)
> qqline(transformed)
```

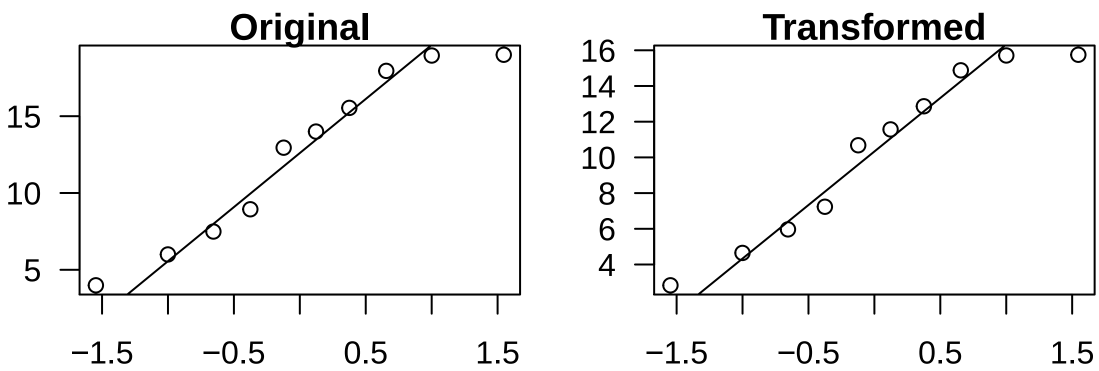


(b) Determine the power transformation $\hat{\lambda}_2$ that makes the x_2 values approximately normal. Construct a Q-Q plot for the transformed data.

```
> lambda <- as.numeric(car::powerTransform(used_car[,2])$lambda)
> lambda
```

```
[1] 0.9361967
```

```
> par(mfrow = c(1,2),mar=c(2.5,2.5,1,1))
> qqnorm(used_car[,2], main = "Original", las=1)
> qqline(used_car[,2])
> transformed <- ((used_car[,2] ^ lambda) - 1)/lambda
> qqnorm(transformed, main = "Transformed", las=1)
> qqline(transformed)
```



(c) Determine the power transformations $\hat{\lambda}' = \left[\hat{\lambda}_1, \hat{\lambda}_2\right]$ that make the $[x_1, x_2]$ values approximately multivariate normal. Compare the results with those from above.

```
> car::powerTransform(used_car)
```

```
Estimated transformation parameters
      Age     Price
1.2732157 0.0310405
```

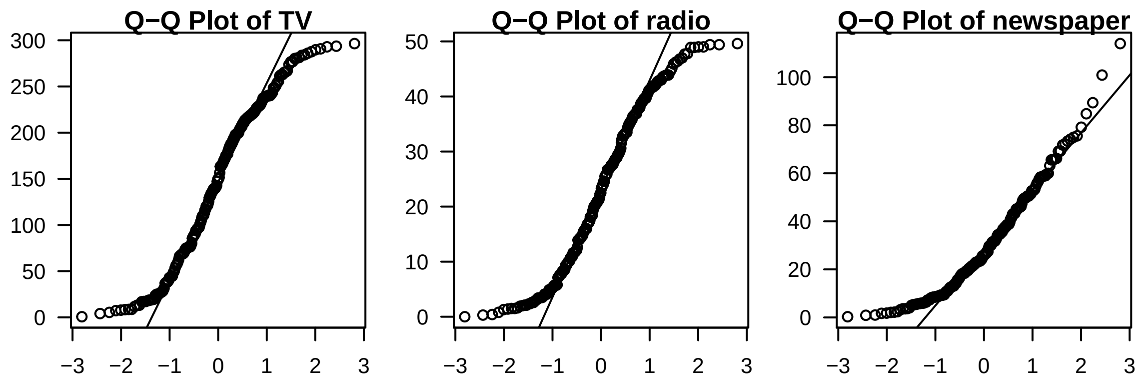
The values of λ required to approximate to the multinormal distribution are different to those computed independently for each variable.

4. Consider the `advertising` data. For each of 200 strategies, we have three numeric variables that influence the sales: `TV`, `radio`, and `Newspaper`.

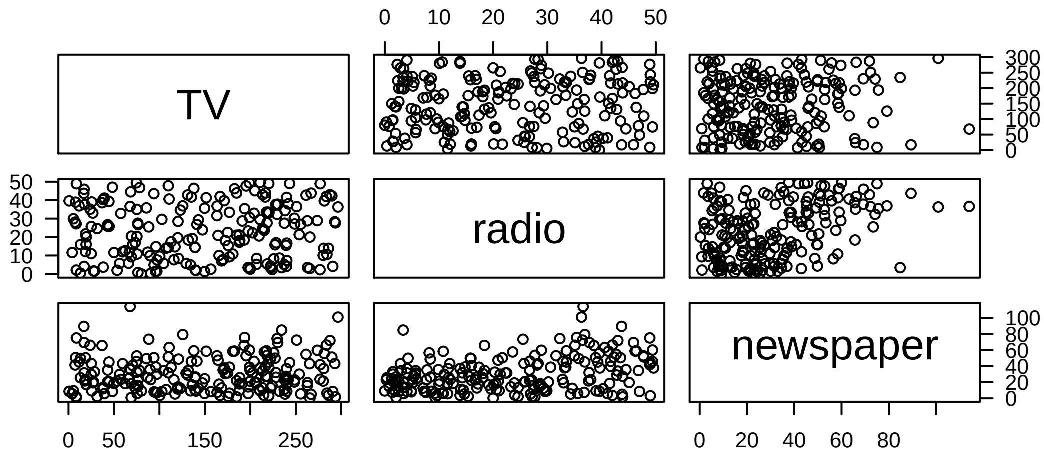
```
> advertising <- read.csv("advertising.csv", row.names = 1)
```

(a) Construct univariate Q-Q plots for each of the three variables. Also make the three pairwise scatterplots. Does the multivariate normal assumption seem reasonable?

```
> par(mfrow = c(1,3), mar=c(2.5,2.5,1,1))
> out <- sapply(colnames(advertising[,1:3]), function(x){
+   qqnorm(advertising[,x], main = paste0("Q-Q Plot of ",x), las = 1)
+   qqline(advertising[,x])
+ })
```




```
> par(mar=c(1,1,1,1))
> plot(advertising[,1:3], las = 1)
```



The multivariable normal distribution does not look reasonable for this dataset because of not all the variables are independently normal

(b) Determine the 95% confidence ellipsoid for μ.

```
> X_bar <- colMeans(advertising[,1:3])
> alpha <- 0.05
> S <- var(advertising[,1:3])
> e <- eigen(S)
> lambda <- e$values
> e <- e$vectors
> p <- ncol(advertising[,1:3])
> n <- nrow(advertising[,1:3])
> lb <- X_bar - abs(drop(sqrt(lambda) * sqrt(((p*(n-1))/(n*(n-p)))*
+                                    qf(1-alpha, df1 = p, df2 = (n-p)))) %*% e)
> ub <- X_bar + abs(drop(sqrt(lambda) * sqrt(((p*(n-1))/(n*(n-p)))*
+                                    qf(1-alpha, df1 = p, df2 = (n-p)))) %*% e)
> t(rbind(lb,ub))
```

```
          [,1]      [,2]
[1,] 129.75061 164.33439
[2,]  19.46324  27.06476
[3,]  27.31906  33.78894
```

Where is it centered?

```
> colMeans(advertising[,1:3])
```

```
       TV     radio newspaper
 147.0425   23.2640   30.5540
```

What are its axes and corresponding half-lengths?

```
> abs(drop(sqrt(lambda) * sqrt(((p*(n-1))/(n*(n-p)))*
+                                 qf(alpha, df1 = p, df2 = (n-p)))) %*% e)
```

```
         [,1]      [,2]      [,3]
[1,] 3.634489 0.7988623 0.6799346
```

(c) Compute 95% T2 simultaneous confidence intervals for the three mean components.

```
> alpha <- 0.05
> c2 <- (n - 1) * p * qf(1 - alpha, p, n - p) / (n - p)
> a <- c(1,0,0)
> t(a) %*% X_bar + c(-1, 1) * sqrt(c2 * t(a) %*% S %*% a / n)
```

```
[1] 129.8373 164.2477
```

```
> a <- c(0,1,0)
> t(a) %*% X_bar + c(-1, 1) * sqrt(c2 * t(a) %*% S %*% a / n)
```

```
[1] 20.2887 26.2393
```

```
> a <- c(0,0,1)
> t(a) %*% X_bar + c(-1, 1) * sqrt(c2 * t(a) %*% S %*% a / n)
```

```
[1] 26.18956 34.91844
```

(d) Compute 95% Bonferroni simultaneous confidence intervals for the three mean components.

```
> a <- c(1,0,0)
> t(a) %*% X_bar + c(-1, 1) * qt(1 - 0.05 / (2 * p), n - 1) *
+   sqrt(t(a) %*% S %*% a  / n)
```

```
[1] 132.3852 161.6998
```

```
> a <- c(0,1,0)
> t(a) %*% X_bar + c(-1, 1) * qt(1 - 0.05 / (2 * p), n - 1) *
+   sqrt(t(a) %*% S %*% a / n)
```

```
[1] 20.72931 25.79869
```

```
> a <- c(0,0,1)
> t(a) %*% X_bar + c(-1, 1) * qt(1 - 0.05 / (2 * p), n - 1) *
+   sqrt(t(a) %*% S %*% a  / n)
```

```
[1] 26.83589 34.27211
```

(e) Carry out a Hotelling's T^2 test of the null hypothesis $\text{H0} : \mu' = [150.0, 20.0, 30.0]$ at $\alpha = 0.05$. What is the test statistic?

```
> X <- c(150, 20, 30)
> print(T2 <- n * (X_bar - t(X)) %*% solve(S) %*% t(X_bar - t(X)))
```

```
         [,1]
[1,] 10.68821
```

What is the critical value?

```
> ((p * (n - 1))/ (n - p)) * qf(p = 1-0.05, df1 = p, df2 = (n-p))
```

```
[1] 8.032049
```

What is the p-value?

```
> 1-pf(q = T2, df1 = p, df2 = (n-p))
```

```
             [,1]
[1,] 1.533739e-06
```

What is your conclusion regarding H0? *There is enough evidence suggesting that* $\mu \neq [150, 20, 30]$ *for that reason I reject the null hypothesis*

(f) Is $\mu' = [150.0, 20.0, 30.0]$ inside the 95% confidence ellipse you computed in part (b)? *Yes, it is* Is this consistent with your findings in part (e)? *Yes, the ellipse is located at* $\mu = [147, 23.3, 30.6]$ *which is different of the hyphotesys tested.*

(g) Use the bootstrap to test the same null hypothesis as in part (e), now using this as your test statistic

$$\Lambda = \left(\frac{|S|}{|S_0|} \right)^{n/2},$$

where

$$S = \frac{1}{n-1} \sum_{j=1}^{n} (x_j - \bar{x})(x_j - \bar{x})'$$

is the sample covariance matrix, and

$$S_0 = \frac{1}{n-1} \sum_{j=1}^{n} (x_j - \bar{x})(x_j - \bar{x})'$$

is the sample covariance matrix under the assumption that H_0 is true. So that all our answers match, first do `set.seed(2)`, and use $B = 500$ bootstrap iterations. What is the p-value?

```
> set.seed(2)
> B <- 500
> n <- nrow(advertising)
> H0 <- c(140, 20, 30)
> dSi <- det(var(advertising[,1:3]))
```

```
> dS0 <- det(var(t(t(advertising[,1:3]) - colMeans(advertising) + H0)))
> dS <- sapply(seq_len(B), function(S){
+   S <- var(advertising[sample(seq_len(n), replace = TRUE), 1:3])
+   return(det(S))
+ })
> mean((dS/dS0) ^ (n/2) >  (dSi/ dS0) ^(n/2))
[1] 0.388
```