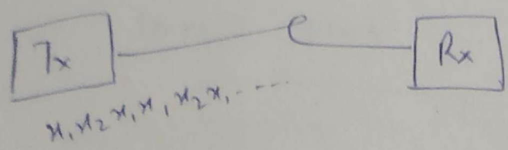
INFORMATION - THEORY

If the prob of occurance of a symbol is high than information associated with it will be less & vice-versa.



$x_1 \rightarrow 1000$ $\qquad$ $x_1 \rightarrow 999$

$x_2 \rightarrow 2$ $\qquad$ $x_2 \rightarrow 1$

$$I[x_i] \propto \frac{1}{P[x_i]}$$

$$I[x_i] = \log \frac{1}{P[x_i]}$$

$$\boxed{I[x_i] = -\log_b P(x_i)}$$

Base	Unit
2	bit
e	nat
10	~~decimal~~ decit

Q) A source is Txed 3- possible symb. with prob. of $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ resp. Find the information associated with each of the symbol.

$$I[x_1] = \log \frac{1}{P(x_1)} = \log_2 4 = 2 \text{ bits}.$$

$$I[x_2] = \log_2 2 = 1 \text{ bit}.$$

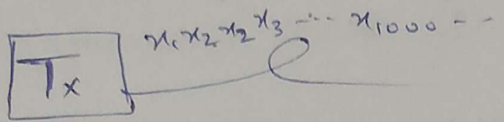
$$I[x_3] = \log_2 4 = 2 \text{ bits}.$$

Tx

$500 - x_1$
$500 - x_2$
1001
$P(x_1) = \frac{1}{2}$
$P(x_2) = \frac{1}{2}$

Avg. Information :-

(Entropy) :- → bits/symbol.

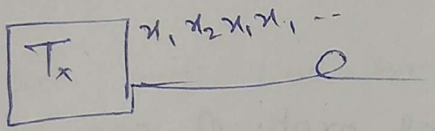


$$H = \sum_i I[x_i]\, P(x_i).$$

$$\boxed{H = -\sum_i P(x_i) \log_2 P(x_i).}$$

→ Entropy is identified as measure of uncertainity.

(I)



(i) $P(x_1) = P(x_2) = \frac{1}{2}$

(ii) $H = \sum_{i=1}^{2} P(x_i) \log_2 \frac{1}{P(x_i)} = \frac{1}{2}\log_2 2 + \frac{1}{2}\log_2 2$

$= \left(\frac{1}{2}\log_2 2\right) 2 = 1$ bit/symbol.

$$\boxed{H_{max} = 1 \frac{\text{bit}}{\text{Symbol}}.}$$ → max. uncertainity.

(II) $P(x_1) = 1$; $P(x_2) = 0$

$H = -\sum_i^{2} P(x_i) \log_2 P(x_i) = -[1 \times 0 + 0]$

$$\boxed{H = 0 \text{ bits/symbol}.}$$

→ no. uncertainity.

(III)

$$P(x_1) = P(x_2) = P(x_3) = \frac{1}{3}$$

$$H = \sum_{i=1}^{3} P(x_i) \log_2 \frac{1}{P(x_i)}$$

$$H_{max} = \log_2 3 \text{ bits/symbol}$$

(eg)

Case (I) $P(x_1) = 0.1$; $P(x_2) = 0.9$

Case (II) $P(x_1) = 0.51$; $P(x_2) = 0.49$

NOTE H is high in case 2.

→ IF all symbols are having equal prob. of occurance than entropy will be max. possible.

(eg) $P(x_1) = 1$ $P(x_2) = P(x_3) = 0$

$H = 0$

(eg) IF $P(x_1) = P(x_2) = \cdots = P(x_M) = \frac{1}{M}$

than $\boxed{H_{max} = \log_2 M}$ $\frac{\text{bits}}{\text{symbol}}$

Information Rate :-
(or) Data Rate :- bits/sec

$\boxed{\text{Symbol rate } (r)}$

$$R = \frac{\text{bits}}{\text{Symbol}} \cdot \frac{\text{Symbol}}{\text{sec}}$$

(*) $\boxed{R = H \times r}$

Q:- A source is Tx 4 possible symbol with the prob. of $\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}$ respectivily (1) Find entropy (2) information rate (3) if the source source Txed $\frac{1 \text{ symbol}}{\text{m sec}}$

$P(x_1) = 1/8 = P(x_2)$

$P(x_3) = \frac{1}{4}$ $\quad P(x_4) = 1/2$

$$H = \sum_{i=1}^{4} P(x_i) \log \frac{1}{P(x_i)}$$

$$H = \frac{7}{4} = 1.75 \text{ bits/symbol}$$

1 msec → 1 symbol

1 sec → 1000 symbol

$\gamma = 1000 \frac{\text{Symbols}}{\text{sec}}$

$R = \gamma H$

$= 1750$ bits/sec

Conditional & Joint prob. Matrix :→

Channel matrix

$\begin{matrix} 0 \\ 1 \end{matrix} \Big| X \text{ — } e \text{ — } Y \Big| \begin{matrix} 0 \\ 1 \end{matrix}$

Use of encriptor

$$\left\{P\left[Y/X\right]\right\} = \begin{array}{c|cccc} & Y_1 & Y_2 & \cdots & Y_n \\ \hline x_1 & P(Y_1/x_1) & P(y_2/x_1) & \cdots & P(y_n/x_1) \\ x_2 & P(y_1/x_2) & P(y_2/x_2) & \cdots & P(y_n/x_2) \\ \vdots & \vdots & & & \vdots \\ x_m & P(y_1/x_m) & P(y_2/x_m) & \cdots & P(y_n/x_m) \end{array}_{m \times n}$$

Cond^n prob. matrix (or) channel matrix.

$$[P(Y/X)] = \begin{array}{c|c} & y_1 \qquad\qquad y_2 \\ \hline x_1 & P(y_1/x_1) + P(y_2/x_1) = 1 \\ x_2 & P(y_1/x_2) + P(y_2/x_2) = 1 \end{array}$$

→ Sum of the element in each row of conditional prob. matrix is equal to one.

Info. $\xrightarrow{\text{encriptor}}$ $x_1 x_2 x_2 x_3 \cdots \xrightarrow{\text{SF/Hoffman}}$ 00 10 10 11 ... ... 00 10 10 11

$\downarrow$

$x_1 x_2 x_2 x_3 \cdots$

decriptor $\downarrow$

Info.

proper password of encriptor = decriptor.

$$[P(X)] = [P(x_1) \; P(x_2) \cdots P(x_n)]_{1 \times m}$$

$$[P(Y)] = [P(y_1) \; P(y_2) \cdots P(y_n)]_{1 \times n}$$

$$\boxed{[P(y)]_{1 \times n} = [P(x)]_{1 \times m} \, [P(y/x)]_{m \times n}}$$

00 → x_1
10 → x_2
⋮
x_n

$\xrightarrow{\quad e \quad}$

y_1 → 00
y_2 → 10
⋮
y_n

Joint prob. Matrix

$$[P(x,y)] = \begin{array}{c} \\ x_1 \\ x_2 \\ \vdots \\ x_m \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & \cdots & y_n \end{array} \\ \begin{bmatrix} P(x_1,y_1) & P(x_1,y_2) & \cdots & P(x_1,y_n) \\ P(x_2,y_1) & P(x_2,y_2) & \cdots & P(x_2,y_n) \\ \vdots & & & \vdots \\ P(x_m,y_1) & \cdots & \cdots & P(x_m,y_n) \end{bmatrix} \end{array}$$

$P(y_k/x_j)$ → prob. of y_k to be received, provided x_j was Txed.

$P(x_j, y_k)$ → prob. of x_j to be Txed & Rxed as y_k

$$[P(x,y)]_{m\times n} = [P(X)]_{d\,m\times m}\,[P(Y/X)]_{m\times n}$$

$$(P(x))_d = \begin{bmatrix} P(x_1) & 0 & \cdots & 0 \\ 0 & P(x) & 0 & \vdots \\ 0 & \cdots & \ddots & 0 \\ 0 & \cdots & \cdots & P(x) \end{bmatrix}$$

$$[P(Y/X)] = \frac{[P(x,y)]}{[P(X)]_d}$$

$$\boxed{[P(x/y)] = \frac{[P(x,y)]}{[P(y)]_d}}$$

Binary Symmetric channel (BSC) :-

$$\boxed{P(1/0) = P(0/1)}$$

$$P_{e0} = P_{e1}$$

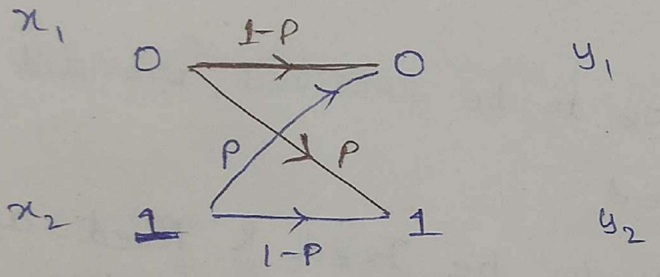
Bit error prob.
(or) transition prob.
(or) crossover prob.

P.T.R

Channel diagram :-

Assume $P(1/0) = P(0/1) = P$.

$P(y_1) = 1 - P$



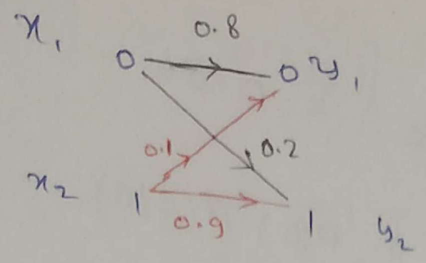
$$[P(y/x)] = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} P(y_1/x_1) & P(y_2/x_1) \\ P(y_1/x_2) & P(y_2/x_2) \end{bmatrix} \end{array}$$

$$[P(y/x)] = \begin{bmatrix} 1-p & P \\ p & 1-P \end{bmatrix}.$$

Binary Asymmetric channel :-

$$P(0/1) \neq P(1/0)$$

$$\Rightarrow P_{e1} \neq P_{e0}$$



$P_{Y/X}(0/1) = 0.1$

$P_X(0) = 0.4$

$P_{Y|X}(1/0) = 0.2$

$P_X(1) = 0.6.$

$$[P(y/x)] = \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & 0.8 & 0.2 \\ x_2 & 0.1 & 0.9 \end{array}$$

$$[P(x)] = [P_x(0) \quad P_x(1)] = [0.4 \quad 0.6]$$

$$[P(y)] = [P(x)][P(y|x)]$$

$$= [0.4 \quad 0.6]\begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix}$$

$$[P(Y)] = [0.38 \quad 0.62]$$

$$P_Y(0) = 0.38 \quad ; \quad P_Y(1) = 0.62.$$

$$[P(x,y)] = [P(x)]_d \; P[(y/x)]$$

$$= \begin{bmatrix} 0.4 & 0 \\ 0 & 0.6 \end{bmatrix}\begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix}.$$

Imp

$$[P(x,y)] = \begin{array}{c|cc|c} & y_1 & y_2 & [P(x)] \\ \hline x_1 & 0.32 + & 0.08 = & 0.4 \\ & + & + & \\ x_2 & 0.06 + & 0.54 = & 0.6 \\ \hline & \| & \| & \\ & 0.38 & 0.62 & [P(y)] \end{array}$$

(e9) Find overall prob. of error for above system.

BER $\rightarrow$ $P_e = P(1)\, P(0/1) \overset{\text{or}}{+} P(0)\, P(1/0).$

$$P_e = 0.6\,(0.1) + 0.4\,(0.2) = 0.14$$

(eg) For no error :

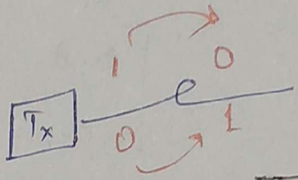
$$= 1 - P_e = 0.86$$

or

$$= P(1)\,P(1/1) + P(0)\,P(0/0)$$

$$= 0.6\,(0.9) + 0.4\,(0.8)$$

$$= 0.86.$$

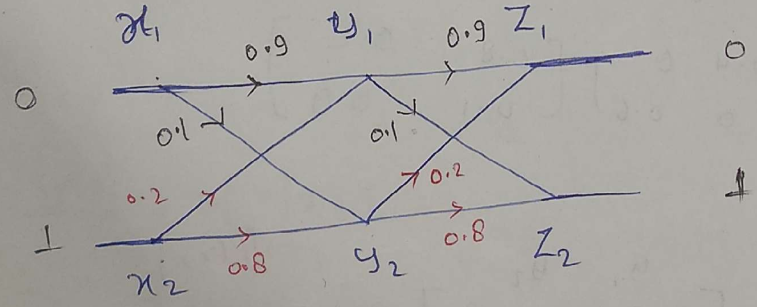


Bit error rate (BER).

BER → $\boxed{P_e = P(1)\,P(0/1) + P(0)\,P(1/0)}$

(eg) 2-binary channel are connected in cascade as shown below. Find overall channel matrix and plot equivalent diagram.

(2) Find $P(Z_1)$ & $P(Z_2)$, given that $P(x_1) = P(x_2) = 0.5$



$$[P(Y/X)] = \begin{array}{c|cc} & y_1 & y_2 \\ x_1 & 0.9 & 0.1 \\ x_2 & 0.2 & 0.8 \end{array}$$

$$[P(Z/Y)] = \begin{array}{c|cc} & Z_1 & Z_2 \\ Y_1 & 0.9 & 0.1 \\ Y_2 & 0.2 & 0.8 \end{array}$$

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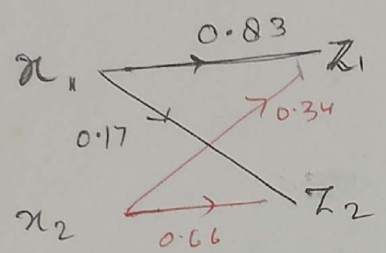
$$\boxed{[P(Z/X)] = [P(Z/Y)] \cdot [P(Y/X)]}$$

$$= \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

$$[P(Z/X)] = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix} \end{array}$$



$$[P(Z)] = [P(X)][P(Z/X)]$$

$$= \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix}$$

$$[P(Z)] = \begin{bmatrix} 0.585 & 0.415 \end{bmatrix}$$

$P(z_1) = 0.585$ $\quad$ $P(z_2) = 0.415$

→ we can find $H(Y)$ + $H(Z)$ also.

Conditional & joint Entropy :-

$$H(X) = -\sum_{j}^{m} P(x_j) \log_2 P(x_j)$$

$$H(Y) = -\sum_{K=1}^{n} P(y_K) \log_2 (y_K)$$

$$H(x/y) = -\sum_{j=1}^{m} \sum_{K=1}^{n} P(x_j, y_K) \log_2 P\left(\frac{x_j}{y_K}\right)$$

$$H(Y/x) = -\sum_{j=1}^{m} \sum_{K=1}^{n} P(x_j, y_K) \log_2 P\left(\frac{y_K}{x_j}\right).$$

$$H(x,y) = -\sum_{j=1}^{m}\sum_{k=1}^{n} P(x_j, y_k)\log_2 P(x_j, y_k)$$

eg Case ① $P(x_1) = 0.52$ $P(x_2) = 0.48$

Case ② $P(x_1) = 0.02$ $P(x_2) = 0.98$

(i) $H(x)$ ↑ in case (I)

eg (I) $P(x_1/y_1) = 0.02$ $P(x_2/y_1) = 0.98$

$P(x_1/y_2) = 0.01$ $P(x_2/y_2) = 0.99$

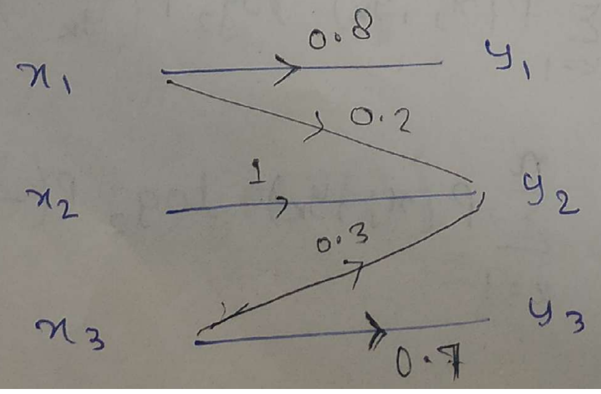
(II) $P(x_1/y_1) = 0.49$ $P(x_2/y_1) = 0.51$ $H(X|Y)$ ↑

$P(x_1/y_2) = 0.47$ $P(x_2/y_2) = 0.53$

$\boxed{H(X,Y) = H(X/Y) + H(Y)}$

$= H(Y|X) + H(X)$

eg A discrete source is Txing 3-possible symbols with the prob. of 0.3, 0.4, & 0.3 respectively. Find all possible entropies of the given system.



$[P(x)] = [0.3 \quad 0.4 \quad 0.3]$.

$$[P(Y/X)] = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 1 & 0 \\ 0 & 0.3 & 0.7 \end{bmatrix} \end{array}$$

$$[P(x,y)] = [P(x)]_d \, [P(Y/x)]$$

$$= \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0.24 & 0.06 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0.09 & 0.21 \end{bmatrix} \\ \begin{array}{ccc} 0.24 & 0.55 & 0.2 \end{array} \end{array} \begin{array}{c} \\ 0.3 \\ 0.4 \\ 0.3 \\ \to [P(x)] \end{array}$$

$[P(y)] \rightarrow$

$$[P(y)] = [0.24 \quad 0.55 \quad 0.21]$$

$$P(x_1) = \sum_{k=1}^{3} P(x_1, y_k) \qquad P(y_1) = \sum_{j=1}^{3} P(x_j, y_1).$$

$$[P(x_1)] = \sum_{k=1}^{3} P(x_1, y_k)$$

$$[P(y_1)] = \sum_{j=1}^{3} P(x_j, y_1)$$

$$[P(X/y)] = \frac{[P(x,y)]}{[P(Y)]_d}$$

Dividing each coloumn of $P(x,y)$ by $P(y_1)$, $P(y_2)$ & $P(y_3)$ respectivily gives →

$$[P(x/y)] = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 1 & 0.109 & 0 \\ x_2 & 0 & 0.727 & 0 \\ x_3 & 0 & 0.168 & 1 \end{array}$$

$P(x_2/y_2)$ = If y_2 was received what is the prob of x_2 is transmitted.

$P(x_1/y_1) = 1$ $\quad P(x_2/y_2) = 0.727$ $\quad P(x_1/y_2) = 0.109$

$P(y_2/x_2) = 1$ $\quad P(x_3/y_3) = 1.$ $\quad \cancel{P(y_2/x_1) = 0.109}$

$$\boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B)}} \quad (\ast\ast)$$

$$= \frac{1 \times 0.4}{0.55} = \underline{0.727}.$$

$$H(X) = -\sum_{j=1}^{3} P(x_j)\log_2 P(x_j) = 1.57 \text{ bits/symbol.}$$

$$H(Y) = -\sum_{k=1}^{3} P(Y_k)\log_2 P(Y_k) = 1.44 \text{ bits/symbol.}$$

$$H(X,Y) = -\sum_{j=1}^{3}\sum_{k=1}^{3} P(x_j, y_k)\log_2 P(x_j, y_k) = 2.05 \text{ bits/symbol.}$$

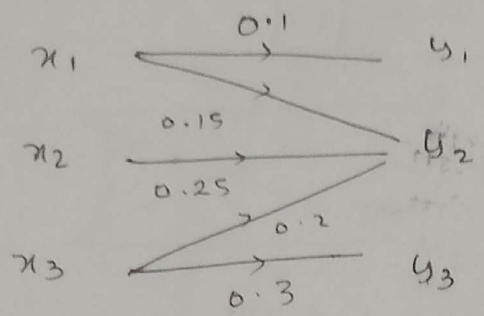
$$H(X/Y) = -\sum_{j=1}^{3}\sum_{k=1}^{3} P(x_j, y_k)\log_2 P\left(\frac{x_j}{y_k}\right) = H(X,Y) - H(Y)$$

$$= 0.61 \text{ bits/symbol.}$$

$H(Y/x) = H(x,y) - H(x)$

$= 0.48$ bits/symbol

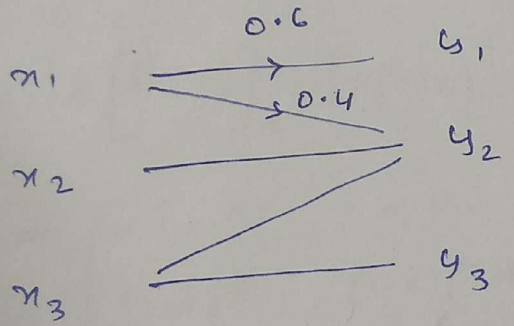
(eg)



$$\boxed{\text{Sum of all entries in } P(x,y) = 1}$$

So above matrix corresponds to $P(x,y)$.

(eg)



NOTE :- If n bits are Tx-ed than Prob. of getting error in 'r' bits is given by :-

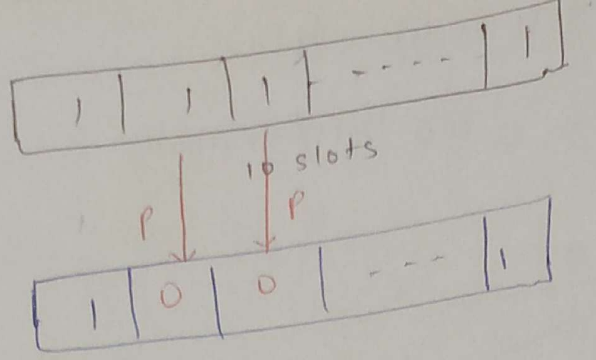
$$\boxed{{}^{n}C_{r} \, p^{r} (1-p)^{n-r}}$$ For binary.

$p \rightarrow$ bit error prob.

$\Rightarrow P(0/1) = P(1/0) = P$

eg

$n=10$



$r=2$

$$^{10}C_2 \; \underbrace{p^2}_{\text{both independent event}} (1-p)^8$$

$\frac{7}{20}$

No error + 1 bit error

$(r=0)$ $(r=1)$

$$(1-p)^n + np(1-p)^{n-1}$$

Mutual Information :→

$$I(x,y) = H(x) - H(X/y)$$

$$= H(Y) - H(Y/x)$$

$$= H(x) + H(y) - H(x,y)$$

→ Mutual information is identified as net reduction in uncertainity.

Channel capacity per symbol (C_s) :-

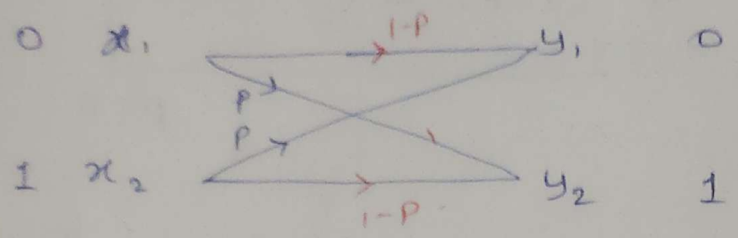
$$\boxed{C_s = Max[I(x,y)]} \text{ bits/Symbol.}$$

$$\text{Channell eff} = \frac{I[x,y]}{C_s}$$

(eg) For a Binary symmetric channel given below.

① Show that $I(x,y) = H(Y) + p\log_2 p + (1-p)\log_2(1-p)$

② Find channel capacity per symbol.



$P(x_1) = \alpha$ $\qquad P(x) = [\alpha \quad 1-\alpha]$

$P(x_2) = 1-\alpha$ $\qquad I(x,y) = H(y) - H(Y|x)$.

$$[P(Y|x)] = \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & 1-p & p \\ x_2 & p & 1-p \end{array}$$

$$[P(x,y)] = [P(x)]_d \cdot [P(Y|x)]$$

$$= \begin{bmatrix} \alpha(1-p) & \alpha p \\ (1-\alpha)p & (1-\alpha)(1-p) \end{bmatrix}$$

$$H(Y/x) = \sum_{j=1}^{2}\sum_{k=1}^{2} P(x_j, y_k)\log_2 P(y_k/x_j).$$

$$= -\{\alpha(1-p)\log_2(1-p) + \alpha p\log_2 p + (1-\alpha)p\log_2 p + (1-\alpha)(1-p)\log_2(1-p)]$$

$$H(Y/x) = -\{p\log_2 p + (1-p)\log_2(1-p)\}.$$

$$I(x;y) = H(Y) - H(Y/x)$$

$$= H(Y) + p\log_2 p + (1-p)\log_2(1-p).$$

(ii)

$$C = Max\,[I(x,y)]$$

$$= Max\{H(Y)\} + p\log_2 p + (1-p)\log_2(1-p) \qquad \log_2 2$$

$$\boxed{C = 1 + p\log_2 p + (1-p)\log_2(1-p)}$$

PTR.

Channel Capacity :→ $\frac{\text{bits}}{\text{sec}}$.

R bits/sec

T_x ——— c bits/sec

For no-information loss $\Rightarrow C \geq R$.

→ Capacity of AWGN channel :→

(or) Shannon-Hartley Law -:

$$\boxed{C = B\log_2\left(1 + \frac{S}{N}\right)}$$ →valid only for AWGN.

C → channel capacity; $\frac{\text{bit}}{\text{sec}}$.

B → channel BW, Hz.

S → Signal power at channel o/p watts.

N → noise power.

$\left(\frac{S}{N}\right)_{dB}$	$\frac{S}{N}$
10 dB	$10^1 = 10$
20 dB	$10^2 = 100$
15 dB	$10^{1.5}$

$$\left(\frac{S}{N}\right)_{dB} = 10\log_{10}\frac{S}{N}$$

$$10 = 10\log_{10} S/N$$

$$S/N = 10^1.$$

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